Chapter 2

On Tightly κ-Filtered Boolean Algebras

Again, in this chapter I assume that κ is regular and infinite.

2.1 The number of tightly σ-filtered Boolean algebras

By a result by Koppelberg ([29]), there are only $2^{<\lambda}$ pairwise non-isomorphic projective Boolean algebras of size λ for every regular uncountable cardinal λ and there are 2^{λ} pairwise non-isomorphic projective Boolean algebras of size λ for every singular infinite cardinal λ. However, a similar statement does not hold for tightly σ-filtered Boolean algebras.

2.1.1. Theorem. *For every infinite cardinal λ there are 2^{λ} pairwise non-isomorphic tightly σ-filtered Boolean algebras of size λ satisfying the c.c.c.* $\square$

The proof of the theorem uses the following lemma, which says that stationary sets consisting of ordinals of countable cofinality can be coded by tightly σ-filtered Boolean algebras.

2.1.2. Lemma. *Let λ be an uncountable regular cardinal and let S be a subset of λ consisting of ordinals of cofinality $\aleph_0$. Then there are a Boolean algebra A of size λ and a tight σ-filtration $(A_\alpha)_{\alpha<\lambda}$ of A such that the following hold:*

a) $A_\alpha \not\leq_{\mathrm{rc}} A$ *for all* $\alpha \in S$

b) $A_\alpha \leq_{\mathrm{rc}} A$ *for all* $\alpha \in \lambda \setminus S$.

Proof. For every $\alpha \in S$ let $(\delta_n^\alpha)_{n\in\omega}$ be a strictly increasing sequence of ordinals with least upper bound α and $S \cap \{\delta_n^\alpha : n \in \omega\} = \emptyset$. I will construct $(A_\alpha)_{\alpha<\lambda}$ together with a sequence $(x_\alpha)_{\alpha<\lambda}$ such that

(i) $A_0 = 2$,

(ii) $A_{\alpha+1} = A_\alpha(x_\alpha)$ for all $\alpha < \lambda$,

(iii) x_α is independent over A_α whenever $\alpha \notin S$,

(iv) $A_\alpha \upharpoonright x_\alpha$ is generated by $\{x_{\delta_n^\alpha} : n \in \omega\}$ and $A_\alpha \upharpoonright -x_\alpha = \{0\}$ whenever $\alpha \in S$,

(v) $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ holds for all limit ordinals $\beta < \lambda$.

Clearly, the construction can be done and is uniquely determined. I have to show that a) and b) of the lemma hold for $(A_\alpha)_{\alpha<\lambda}$.

For a) let $\alpha \in S$. Then $A_\alpha \upharpoonright x_\alpha$ is non-principal. For suppose $a \in A_\alpha$ is such that $a \leq x_\alpha$. Since $(\delta_n^\alpha)_{n\in\omega}$ is cofinal in α, there is $n \in \omega$ such that $a \in A_{\delta_n^\alpha}$. Since $\delta_n^\alpha \notin S$, $x_{\delta_n^\alpha}$ is independent over $A_{\delta_n^\alpha}$ by construction. Hence $a + x_{\delta_n^\alpha}$ is strictly larger than a, but still smaller than x_α. So a does not generate $A_\alpha \upharpoonright x_\alpha$.

For b) let $\alpha \notin S$. By induction on $\gamma < \lambda$, I show that $A_\alpha \leq_{\mathrm{rc}} A_\gamma$ holds for every $\gamma \geq \alpha$. $A_\alpha \leq_{\mathrm{rc}} A_\alpha$ holds trivially. Suppose γ is a limit ordinal and $A_\alpha \geq_{\mathrm{rc}} A_\beta$ holds for all $\beta < \gamma$ such that $\alpha \leq \beta$. Then $A_\alpha \leq_{\mathrm{rc}} A_\gamma$ follows from Lemma 1.1.3. Now suppose $\gamma = \beta + 1$ for some $\beta \geq \alpha$. There are two cases:

I. $\beta \notin S$. In this case $A_\beta \leq_{\mathrm{rc}} A_\gamma$ by construction. By hypothesis, $A_\alpha \leq_{\mathrm{rc}} A_\beta$. By Lemma 1.1.3, this implies $A_\alpha \leq_{\mathrm{rc}} A_\gamma$.

II. $\beta \in S$. This is the non-trivial case. I *claim* that $A_\delta \leq_{\mathrm{rc}} A_\delta(x_\beta)$ holds for every $\delta < \beta$. This can be seen as follows: By Lemma 1.1.2, it is sufficient to show that both $A_\delta \upharpoonright x_\beta$ and $A_\delta \upharpoonright -x_\beta$ are principal. But $A_\delta \upharpoonright -x_\beta \subseteq A_\beta \upharpoonright -x_\beta = \{0\}$ by construction. Let $a \in A_\delta$ be such that

$a \le x_\beta$. Let $m := \{n \in \omega : x_{\delta_n^\beta} \in A_\delta\}$. Clearly $m \in \omega$. Let $T \in [\omega]^{<\omega}$ be such that $a \le \sum\{x_{\delta_n^\beta} : n \in T\}$. Then

$$a \le \sum\{x_{\delta_n^\beta} : n \in T \cap m\} + \sum\{x_{\delta_n^\beta} : n \in T \setminus m\}.$$

Since $\sum\{x_{\delta_n^\beta} : n \in T \setminus m\}$ is independent over A_δ by construction,

$$a \le \sum\{x_{\delta_n^\beta} : n \in T \cap m\} \le \sum\{x_{\delta_n^\beta} : n < m\} \le x_\beta.$$

This shows that $A_\delta \restriction x_\beta$ is generated by $\sum\{x_{\delta_n^\beta} : n < m\}$ and the claim holds. Now $A_\gamma = A_\beta(x_\beta) = \bigcup_{\alpha \le \delta < \beta} A_\delta(x_\beta)$. Hence $A_\alpha \le_{\mathrm{rc}} A_\gamma$ follows from the claim together with Lemma 1.1.3.

This shows b). $\square$

In order to show that the Boolean algebra A constructed in the lemma above satisfies the c.c.c., I use an argument which was used by Soukup ([15]) to prove that, modulo the consistency of the existence of a supercompact cardinal, it is consistent with ZFC+GCH that there is a complete c.c.c. Boolean algebra without the WFN.

2.1.3. Lemma. *The Boolean algebra A constructed in the proof of Lemma 2.1.2 satisfies the c.c.c.*

Proof. Assume A does not satisfy the c.c.c. Let $C \subset A$ be an uncountable antichain. Let $X := \{x_\alpha : \alpha < \lambda\}$. For $x \in X$ let $x^0 := x$ and $x^1 := -x$. I may assume that each $a \in C$ is an elementary product of elements of X, i.e. there is $X_a \in [X]^{<\aleph_0}$ and $f_a : X_a \to 2$ such that $a = \prod_{x \in X_a} x^{f_a(x)}$. After thinning out C if necessary, I may assume that $\{X_a : a \in C\}$ is a Δ-system with root R, there is $f : R \to 2$ such that $f_a \restriction R = f$ for all $a \in C$, and all X_a are of the same size, say n.

Claim. Let $Y \in [X]^{<\omega}$ and $g : Y \to 2$ be such that $\prod_{x \in Y} x^{g(x)} = 0$. Then there are $\alpha \in S$ and $i \in \omega$ with $x_\alpha, x_{\delta_i^\alpha} \in Y$ such that $g(x_\alpha) = 1$ and $g(x_{\delta_i^\alpha}) = 0$.

First note that for $y, z \in X$, $y^{g(y)} \cdot z^{g(z)} = 0$ holds iff there are $\alpha \in S$ and $i \in \omega$ with $\{y, z\} = \{x_\alpha, x_{\delta_i^\alpha}\}$ such that $g(x_\alpha) = 1$ and $g(x_{\delta_i^\alpha}) = 0$. Now I show the claim by in induction on $\max\{\alpha<\lambda : x_\alpha \in Y\}$. The case $|Y|< 3$ is trivial.

Assume the claim has been proved for $\max\{\alpha<\lambda : x_\alpha\in Y\}<\beta$. Suppose $\max\{\alpha<\lambda : x_\alpha\in Y\}=\beta$ and for no two elements $y,z\in Y$, $y^{g(y)}\cdot z^{g(z)}=0$. For $\beta\notin S$ the argument is easy. By assumption, $b:=\prod_{x\in Y\setminus\{x_\beta\}} x^{g(x)}\neq 0$. By construction, x_β and b are independent. Thus $\prod_{x\in Y} x^{g(x)}\neq 0$.

Now suppose $\beta\in S$ and $\prod_{x\in Y} x^{g(x)}=0$. By construction, $A_\beta\restriction -x_\beta=\{0\}$. Thus $b:=\prod_{x\in Y\setminus\{x_\beta\}} x^{g(x)}\not\leq -x_\beta$. Therefore $g(x_\beta)=1$ and $b\leq x_\beta$. By construction, there is $m\in\omega$ such that $b\leq\sum_{i<m} x_{\delta_i^\beta}$. It follows from the inductive hypothesis that $b\cdot\prod_{i<m} -x_{\delta_i^\beta}\neq 0$. This contradicts the choice of m and the claim is proved.

For each $a\in C$ let $X_a=\{x_{a,i} : i<n\}$. Clearly, I may assume that C has size $\aleph_1$. Let $\leq$ be a wellorder on C of ordertype ω_1. For each $\{a,b\}\in[C]^2$ choose a color $c(\{a,b\})\in n^2$ such that

$$\forall(i,j)\in n^2(c(\{a,b\})=(i,j)\wedge a\leq b\Rightarrow x_{a,i}^{f_a(x_{a,i})}\cdot x_{b,j}^{f_b(x_{b,j})}=0).$$

It follows from the claim that c can be defined. Clearly, for all $\{a,b\}\in[C]^2$, if $c(\{a,b\})=(i,j)$ and $a\leq b$, then $x_{a,i},x_{b,j}\notin R$. Baumgartner and Hajnal ([3]) established the following partition result:

$$\forall m\in\omega\forall\alpha<\omega_1(\omega_1\to(\alpha)^2_m).$$

In particular, $\omega_1\to(\omega+2)^2_{n^2}$ holds. That is, there are $(i,j)\in n^2$ and a subset C' of C of ordertype $\omega+2$ such that for all $\{a,b\}\in[C']^2$, $c(\{a,b\})=(i,j)$. Let a and b be the last two elements of C'. Assume $x_{a,j}=x_\alpha$ for some $\alpha\in S$. By construction of A, for all $c\in C'\setminus\{a,b\}$, $x_{c,i}=x_{\delta_k^\alpha}$ for some $k\in\omega$. By the Δ-system assumption, all the $x_{c,i}$'s are different. This implies $x_{a,j}=x_{b,j}$, contradicting the Δ-system assumption.

Now assume that for all $\alpha\in S$, $x_{a,j}\neq x_\alpha$. In this case, for all $c\in C'\setminus\{a,b\}$, $x_{c,i}=x_\alpha$ for some $\alpha\in S$. Let d and e be the first two elements of C'. Now for all $c\in C'\setminus\{d,e\}$, $x_{c,j}=x_{\delta_k^\alpha}$ for some $k\in\omega$. By the Δ-system assumption, all the $x_{c,j}$'s are different. This implies $x_{d,i}=x_{e,i}$, contradicting the Δ-system assumption. This finishes the proof of the lemma. $\square$

Proof of the theorem. Let λ be an infinite cardinal. If $\lambda = \aleph_0$, then there are 2^λ pairwise non-isomorphic Boolean algebras of size λ and all of them are projective, hence tightly σ-filtered. Also, if λ is singular, then there are 2^λ pairwise non-isomorphic projective Boolean algebras by the result of Koppelberg mentioned before. Projective Boolean algebras satisfy the c.c.c.

For regular uncountable λ let $\mathcal{P}$ be a disjoint family of stationary subsets of $\{\alpha < \lambda : \mathrm{cf}(\alpha) = \aleph_0\}$ of size λ. Such a family exists by the wellknown results of Ulam and Solovay. For every subset $\mathcal{T}$ of $\mathcal{P}$ let $A^{\mathcal{T}}$ be the Boolean algebra which is constructed in the lemma from the set $S := \bigcup \mathcal{T}$ and let $(A^{\mathcal{T}}_\alpha)_{\alpha<\lambda}$ be its associated tight σ-filtration. Then for $\mathcal{T}, \mathcal{T}' \subseteq \mathcal{P}$ with $\mathcal{T} \neq \mathcal{T}'$ the Boolean algebras $A^{\mathcal{T}}$ and $A^{\mathcal{T}'}$ are non-isomorphic.

For suppose $h : A^{\mathcal{T}} \longrightarrow A^{\mathcal{T}'}$ is an isomorphism. W.l.o.g. I may assume that $\mathcal{T} \setminus \mathcal{T}'$ is nonempty. The set $\{\alpha < \lambda : h[A^{\mathcal{T}}_\alpha] = A^{\mathcal{T}'}_\alpha\}$ is club in λ. Since $\bigcup(\mathcal{T} \setminus \mathcal{T}')$ is stationary, there is $\alpha \in \bigcup(\mathcal{T} \setminus \mathcal{T}')$ such that $h[A^{\mathcal{T}}_\alpha] = A^{\mathcal{T}'}_\alpha$. But $A^{\mathcal{T}}_\alpha \not\leq_{\mathrm{rc}} A^{\mathcal{T}}$ and $A^{\mathcal{T}'}_\alpha \leq_{\mathrm{rc}} A^{\mathcal{T}'}$, a contradiction.

By Lemma 2.1.3, the Boolean algebras $A^{\mathcal{T}}$ satisfy the c.c.c. $\square$

The two lemmas above give even more:

2.1.4. Theorem. *Let λ be an uncountable and regular cardinal. Then there is a family of size 2^λ of tightly σ-filtered c.c.c. Boolean algebras of size λ such that no member of this family is embeddable into another one as an rc-subalgebra.*

Proof. Suppose $\mathcal{T}$ and $\mathcal{T}'$ are subsets of $\mathcal{P}$, where P is as in the proof of the theorem above. Assume there is an embedding $e : A^{\mathcal{T}} \to A^{\mathcal{T}'}$ such that $e[A^{\mathcal{T}}] \leq_{\mathrm{rc}} A^{\mathcal{T}'}$. Let $C \subseteq \lambda$ be a club such that $e[A^{\mathcal{T}}_\alpha] = A^{\mathcal{T}'}_\alpha \cap e[A^{\mathcal{T}}]$ and $\mathrm{lpr}^{A^{\mathcal{T}'}}_{e[A^{\mathcal{T}}]}[A^{\mathcal{T}'}_\alpha] \subseteq A^{\mathcal{T}'}_\alpha$ hold for every $\alpha \in C$. Let $\alpha \in C \cap \bigcup \mathcal{T}$. Then $e[A^{\mathcal{T}}_\alpha] \not\leq_{\mathrm{rc}} e[A^{\mathcal{T}}]$ and hence $e[A^{\mathcal{T}}_\alpha] \not\leq_{\mathrm{rc}} A^{\mathcal{T}'}$. Since $A^{\mathcal{T}'}_\alpha$ is closed under $\mathrm{lpr}^{A^{\mathcal{T}'}}_{e[A^{\mathcal{T}}]}$, $e[A^{\mathcal{T}}_\alpha] \leq_{\mathrm{rc}} A^{\mathcal{T}'}_\alpha$. Hence $A^{\mathcal{T}'}_\alpha \not\leq_{\mathrm{rc}} A^{\mathcal{T}'}$. Therefore $C \cap \bigcup \mathcal{T} \subseteq C \cap \bigcup \mathcal{T}'$. Thus, since $\mathcal{P}$ consists of stationary sets, $\mathcal{T} \subseteq \mathcal{T}'$. Now let I be an independent family of subsets of $\mathcal{P}$ of size 2^λ. In particular, the elements of I are pairwise

$\subseteq$-incomparable. Thus the family $\{A^{\mathcal{T}} : \mathcal{T} \in I\}$ consists of pairwise non-rc-embeddable tightly σ-filtered c.c.c. Boolean algebras of size λ. $\square$

2.2 Characterizations of Tightly κ-Filtered Boolean Algebras

In this section I give characterizations of tightly κ-filtered Boolean algebras which are similar to the characterizations known for projective Boolean algebras. For these characterizations I have to assume that κ is uncountable, simply because some of the proofs given below do not work for $\kappa = \aleph_0$. However, some of the characterizations given below are parallel to those of projective Boolean algebras. The main difference to the projective case is that projective Boolean algebras are exactly the retracts of free Boolean algebras. A similar characterization of tightly κ-filtered Boolean algebras does not seem to be available. For the characterization of tightly κ-filtered Boolean algebras I will use the concept of commuting subalgebras of a Boolean algebra.

2.2.1. Definition. Let A and B be subalgebras of the Boolean algebra C. Then A and B *commute* iff for every $a \in A$ and every $b \in B$ such that $a \cdot b = 0$ there is $c \in A \cap B$ such that $a \leq c$ and $b \leq -c$.

A family $\mathcal{F}$ of subsets of a Boolean algebra A is called *commutative* iff it consists of pairwise commuting subalgebras. $\square$

The connection between κ-subalgebras and commutative families is given by

2.2.2. Lemma. *Let $\mathcal{F}$ be a commutative family of subalgebras of A such that every $a \in A$ is contained in some $B \in \mathcal{F}$ of size $< \kappa$. Then $\mathcal{F}$ consists of κ-subalgebras of $\mathcal{F}$.*

Proof. Let $C \in \mathcal{F}$ and $a \in A$. Then there is $B \in \mathcal{F}$ such that $a \in B$. I *claim* that B contains a cofinal subset of $C \restriction a$. Let $c \in C \restriction a$. Now $-a \cdot c = 0$. Since B and C commute, there is $b \in B \cap C$ such that $c \leq b$ and $-a \leq -b$. But now $c \leq b \leq a$. $\square$

This lemma is implicitly contained in the book by Heindorf and Shapiro ([23]) for the case $\kappa = \aleph_1$.

It turns out that additivity of skeletons is what separates tight κ-filteredness from κ-filteredness.

2.2.3. Definition. A $< \kappa$-skeleton (respectively κ-skeleton) $\mathcal{S}$ of a Boolean algebra A is called *additive* iff for every subset $T \subseteq \mathcal{S}$ the Boolean algebra $\langle \bigcup T \rangle$ generated in A by $\bigcup T$ is a member of $\mathcal{S}$. $\square$

In order to make the similarities between projective Boolean algebras and tightly κ-filtered Boolean algebras apparent, I quote the following from Heindorf and Shapiro ([23]):

2.2.4. Theorem. *The following are equivalent for a Boolean algebra A:*

(i) A is projective.

(ii) For some ordinal δ, A is the union of a continuous chain $(A_\alpha)_{\alpha<\delta}$ consisting of rc-subalgebras such that $A_{\alpha+1}$ is countably generated over A_α for every $\alpha < \delta$ and A_0 is countable.

(iii) A has a tight rc-filtration.

(iv) A has an additive commutative skeleton.

(v) A has an additive skeleton consisting of rc-embedded subalgebras.

(vi) A is the union of a family $\mathcal{C}$ of countable subsets of A such that $\langle \bigcup S \rangle \leq_{\mathrm{rc}} A$ for every $S \subseteq \mathcal{C}$. $\square$

The characterization of tightly κ-filtered Boolean algebras is the following:

2.2.5. Theorem. *Let κ be an uncountable regular cardinal. The following are equivalent for a Boolean algebra A:*

(i) For some ordinal δ, A is the union of a chain $(A_\alpha)_{\alpha<\delta}$ of κ-subalgebras which is continous at limit ordinals of cofinality $\geq \kappa$ such that $A_{\alpha+1}$ is $\leq \kappa$-generated over A_α for every $\alpha < \delta$ and A_0 has size $\leq \kappa$.

(ii) A has a tight κ-filtration.

(iii) A has an additive commutative $< \kappa$-skeleton.

(iv) A has an additive $< \kappa$-skeleton consisting of κ-embedded subalgebras.

(v) A has an additive κ-skeleton consisting of κ-embedded subalgebras.

(vi) A is the union of a family $\mathcal{C}$ of subsets of size $< \kappa$ of A such that for all $S, T \subseteq \mathcal{C}$ the algebras $\langle \bigcup S \rangle$ and $\langle \bigcup T \rangle$ commute.

(vii) A is the union of a family $\mathcal{C}$ of subsets of size $< \kappa$ of A such that for every $S \subseteq \mathcal{C}$, $\langle \bigcup S \rangle \leq_\kappa A$.

(viii) A is the union of a family $\mathcal{C}$ of subsets of size $\leq \kappa$ of A such that for every $S \subseteq \mathcal{C}$, $\langle \bigcup S \rangle \leq_\kappa A$.

Proof. (i)$\Rightarrow$(ii) was proved by Koppelberg ([28]) for $\kappa = \aleph_1$. The proof for arbitrary regular κ is exactly the same. Let $(A_\alpha)_{\alpha<\delta}$ be a filtration of A as in (i). First make the sequence continuous by inserting the appropriate unions at those limit stages which lack continuity. Since this only happens at limits of cofinality $< \kappa$, the filtration remains a κ-filtration by part d) of Lemma 1.1.3. For $\alpha \leq \lambda$ let $X \in [A_{\alpha+1}]^{\leq\kappa}$ be such that $A_\alpha(X) = A_{\alpha+1}$. Let $X = \{x_\delta : \delta < \kappa\}$. Now insert $(A_\alpha(\{x_\gamma : \gamma < \beta\}))_{\beta<\kappa}$ between A_α and $A_{\alpha+1}$. Similarly, insert a continuous tight filtration of A_0 below A_0. The new filtration is a κ-filtration by part c) of Lemma 1.1.3 and it is tight by construction.

(iii)$\Rightarrow$(iv) follows from Lemma 2.2.2.

(iv)$\Rightarrow$(v) is trivial.

(iii)$\Rightarrow$(vi), (iv)$\Rightarrow$(vii), and (v)$\Rightarrow$(viii) can be seen using the same argument: Let the $\mathcal{C}$ consist of the elements of the $< \kappa$-skeleton (κ-skeleton) of size $< \kappa$ (of size $\leq \kappa$). Then additivity of the $< \kappa$-skeleton (κ-skeleton) yields the desired property of $\mathcal{C}$.

(vi)$\Rightarrow$(vii) follows from Lemma 2.2.2 applied to the family $\mathcal{F}$ of all subalgebras of A generated by a union of elements of $\mathcal{C}$.

(vii)$\Rightarrow$(i) and (viii)$\Rightarrow$(i) are easily seen using the following argument: Let $A = \{a_\alpha : \alpha <|A|\}$. For every $\alpha <|A|$ choose $B_\alpha \in \mathcal{C}$ such that $a_\alpha \in B_\alpha$. Let $A_\alpha := \langle \bigcup_{\beta<\alpha} B_\beta \rangle$ for every $\alpha <|A|$. $(A_\alpha)_{\alpha<|A|}$ works for (i).

(ii)$\Rightarrow$(iii) is the only part that requires some work. Let $(x_\alpha)_{\alpha<\delta} \in {}^\delta A$ be such that $(\langle \{x_\beta : \beta < \alpha\} \rangle)_{\alpha<\delta}$ is a tight κ-filtration of A. For every $S \subseteq \delta$ let $A_S := \langle \{x_\beta : \beta \in S\} \rangle$. With this notation the filtration is simply $(A_\alpha)_{\alpha<\delta}$. Choose $f : \delta \longrightarrow [\delta]^{<\kappa}$ such that for every $\alpha < \delta$ the ideals $A_\alpha \restriction x_\alpha$ and $A_\alpha \restriction -x_\alpha$ are generated by $(A_\alpha \restriction x_\alpha) \cap A_{f(\alpha)}$ and $(A_\alpha \restriction -x_\alpha) \cap A_{f(\alpha)}$ respectively and such that $f(\alpha) \subseteq \alpha$. Let $\mathcal{S} := \{A_T : T \subseteq \delta \wedge \bigcup f[T] \subseteq T\}$. $\mathcal{S}$ is an additive $< \kappa$-skeleton:

Clearly, every subset of A of size at least κ is included in a member of $\mathcal{S}$ of the same size. Moreover, any subset of A of size $< \kappa$ is included in an element of $\mathcal{S}$ of size $< \kappa$. Suppose $\mathcal{T} \subseteq \mathcal{S}$. Let $\mathcal{U} \subseteq \mathfrak{P}(\delta)$ be such that $\mathcal{T} = \{A_T : T \in \mathcal{U}\}$. Then $\langle \bigcup \mathcal{T} \rangle = A_{\bigcup \mathcal{U}} \in \mathcal{S}$ since $\bigcup \mathcal{U}$ is closed under f. In particular, $\mathcal{S}$ is closed under unions of subchains.

It remains to show that $\mathcal{S}$ is commutative.

Suppose $S, T \subset \kappa$ are closed under f. It is sufficient to show that $A_{S\cap\alpha}$ and $A_{T\cap\alpha}$ commute for every $\alpha < \delta$. I will do so by induction on α. The limit stages of the induction are trivial. Suppose $\alpha = \beta+1$. W.l.o.g. I may assume $\beta \in S$. Let $u \in A_{S\cap\alpha}$ and $v \in A_{T\cap\alpha}$ be such that $u \cdot v = 0$. W.l.o.g. I may assume that u is of the form $a \cdot x_\beta$ for some $a \in A_{S\cap\beta}$. The case $u = a - x_\beta$ is completely analogous. Only the following cases are interesting:

I. $v = b - x_\beta$ for some $b \in A_{T\cap\beta}$ and $\beta \in T$. Then $x_\beta \in A_S \cap A_T$, $u \leq x_\beta$ and $v \leq -x_\beta$.

II. $v = b \cdot x_\beta$ for some $b \in A_{T\cap\beta}$ and $\beta \in T$. Then $a \cdot b \cdot x_\beta = 0$. Hence $a \cdot b \leq -x_\beta$. Take $c \in A_{f(\beta)}$ such that $a \cdot b \leq c \leq -x_\beta$. Then $(a-c)\cdot(b-c) = 0$, $a \cdot x_\beta \leq a - c$ and $b \cdot x_\beta \leq b - c$. Now $a - c \in A_{S\cap\beta}$ and $b - c \in A_{T\cap\beta}$. By hypothesis, there is $r \in A_{T\cap\beta} \cap A_{S\cap\beta}$ such that $a - c \leq r$ and $b - c \leq -r$. r is as required.

III. $v \in A_{T\cap\beta}$. Then $a \cdot v \leq -x_\beta$. Choose $c \in A_{f(\beta)}$ such that $a \cdot v \leq c \leq -x_\beta$. Then $a \cdot v - c = 0$ and $u = a \cdot x_\beta \leq a - c$. Since $a - c \in A_{S\cap\beta}$, there is $r \in A_{S\cap\beta} \cap A_{T\cap\beta}$ such that $a - c \leq r$ and $v \leq -r$.

This completes the induction and (ii)$\Rightarrow$(iii) of the theorem follows. $\square$

2.2.6. Remark. It follows from the proof of this theorem that A is tightly κ-filtered iff it has a tight κ-filtration indexed by $|A|$. $\square$

The assumption $\kappa > \aleph_0$ was only needed for this theorem. From now on I only assume κ to be regular and infinite. The following corollary is very useful when one wants to show that some Boolean algebra is not tightly κ-filtered.

2.2.7. Corollary. *Let κ be an infinite regular cardinal. If a Boolean algebra A is tightly κ-filtered, then there is a function $f : A \to [A]^{<\kappa}$ such that for any two sets $X, Y \subseteq A$ which are closed under f, $\langle X \cup Y \rangle \leq_\kappa A$.*

Proof. By Theorem 2.2.5 respectively Theorem 2.2.4, there is a subset $\mathcal{C}$ of $[A]^{<\kappa}$ such that $A = \bigcup \mathcal{C}$ and for each $\mathcal{S} \subseteq \mathcal{C}$, $\langle \bigcup S \rangle \leq_\kappa A$. For each $a \in A$ choose $f(a) \in \mathcal{C}$ such that $a \in f(a)$. f works for the corollary. $\square$

The characterization of tight κ-filteredness also gives

2.2.8. Corollary. *a) Every Boolean algebra A of size κ is tightly κ-filtered.*

b) Every Boolean algebra of size κ^+ which has the κ-FN is tightly κ-filtered.

c) Every tightly κ-filtered Boolean algebra has the κ-FN.

d) If a Boolean algebra A is a retract of a tightly κ-filtered Boolean algebra B, then A is tightly κ-filtered, too.

Proof. a) follows immediately from (i) in Theorem 2.2.5 respectively from (ii) in Theorem 2.2.4.

For b) let A be a Boolean algebra of size κ^+ which has the κ-FN. By Lemma 1.4.4, A is κ-filtered. Let $\mathcal{S}$ be a κ-skeleton of A consisting of κ-subalgebras. In $\mathcal{S}$ choose a strictly increasing sequence $(A_\alpha)_{\alpha<\kappa^+}$ such that $A = \bigcup_{\alpha<\kappa^+} A_\alpha$ and for all $\alpha < \kappa^+$, $| A_\alpha |= \kappa$. By (i) of Theorem 2.2.5 respectively (ii) of Theorem 2.2.4, A is tightly κ-filtered.

c) follows easily from (v) of Theorem 2.2.5 respectively (v) of Theorem 2.2.4.

For d) let $p : B \to A$ and $e : A \to B$ be homomorphisms such that $p \circ e = \mathrm{id}_A$. By Theorem 2.2.5 respectively Theorem 2.2.4, B has an additive κ-skeleton $\mathcal{T}$ consisting of κ-subalgebras. Let $\mathcal{T}'$ be the set of those elements of $\mathcal{T}$ which are closed under $e \circ p$. It is easy to see that $\mathcal{T}'$ is an additive κ-skeleton for B as well. Now let

$$\mathcal{S} := \{p[C] : C \in \mathcal{T}'\}.$$

Again, it is easy to see that $\mathcal{S}$ is an additive κ-skeleton for A. I *claim* that $\mathcal{S}$ consists of κ-subalgebras of A.

Let $C \in \mathcal{T}'$ and $a \in A$. Let Y be a cofinal subset of $C \restriction e(a)$ of size $< \kappa$. Then $p[Y]$ is a cofinal subset of $p[C] \restriction a$ of size $< \kappa$. This proves the claim.

By Theorem 2.2.5 respectively Theorem 2.2.4, A is tightly κ-filtered. $\square$

2.3 Stone spaces of tightly κ-filtered Boolean algebras

The implication (i)$\Rightarrow$(viii) and the proof of (viii)$\Rightarrow$(i) of Theorem 2.2.5 show that for a tightly κ-filtered Boolean algebra there is a lot of freedom in the choice of a tight κ-filtration of A. This fact allows it to generalize certain results by Koppelberg ([29]) on Stone spaces of projective Boolean algebras to Stone spaces of tightly κ-filtered Boolean algebras. Let A be a tightly κ-filtered Boolean algebra of size λ and X be its Stone space. I am interested in the subspace of X of points of small character.

2.3.1. Definition. Let M_λ be the subspace of X that consists of the ultrafilters of A which have character $< \lambda$. For Boolean algebras $B \leq C$ an ultrafilter p of B *splits* in C iff there are distinct ultrafilters q and r of C both extending p. $\square$

Note that p splits in C iff there is $c \in C$ such that $p \cup \{c\}$ and $p \cup \{-c\}$ both have the finite intersection property.

2.3.2. Theorem. *Let A be a tightly κ-filtered Boolean algebra of size λ, where $\kappa < \lambda$, λ is regular, and $|\delta|^{<\kappa} < \lambda$ holds for every $\delta < \lambda$. Let X and M_λ be as above. Then M_λ is an intersection of subsets of X which are unions* *of less than κ clopen sets and is determined by a subalgebra B of A of size $< \lambda$, i.e. there is $B \leq A$ such that $|B| < \lambda$ and $p \cap B$ does not split in A for any $p \in M_\lambda$.*

Proof. For the first assertion it is enough to show that for every point p in the complement of M_λ, there is a set $a_p \subseteq X \setminus M_\lambda$ such that $p \in a_p$ and a_p is the intersection of less than κ clopen subsets of X.

Let $p \in X \setminus M_\lambda$. Then there is a κ-filtration $(A'_\alpha)_{\alpha<\lambda}$ pf A such that the following hold for all $\alpha < \lambda$:

a) $p \cap A'_\alpha$ splits in $A'_{\alpha+1}$

b) $A'_{\alpha+1}$ is κ-generated, but not $< \kappa$-generated over A'_α.

This filtration can be constructed as in the proof of (viii)⇒(i) of Theorem 2.2.5 using the fact $\chi(p) = \lambda$ to get a) together with some extra care to get b). Now this filtration can easily be refined to a tight κ-filtration $(A_\alpha)_{\alpha<\lambda}$ such that $p \cap A_\alpha$ splits in $A_{\alpha+1}$ for every ordinal $\alpha < \lambda$ of cofinality $\geq \kappa$.

A moment's reflection shows that for all $\alpha < \lambda$ the set a_α of ultrafilters of A_α which split in $A_{\alpha+1}$ is an intersection of less than κ clopen sets in the Stone space of A_α. More exactly: Let $x \in A_{\alpha+1}$ be such that $A_\alpha(x) = A_{\alpha+1}$. An ultrafilter q of A_α splits in $A_{\alpha+1}$ iff $q \cup \{x\}$ and $q \cup \{-x\}$ both are centered. Let I_x and I_{-x} be cofinal subsets of size $< \kappa$ of $A_\alpha \restriction x$ and $A_\alpha \restriction -x$ respectively. Now $q \cup \{x\}$ and $q \cup \{-x\}$ both are centered iff q is disjoint from $I_x \cup I_{-x}$. But this holds iff the point q in the Stone space of A_α is contained in the intersection of the clopen sets corresponding to complements of elements of $I_x \cup I_{-x}$.

For every $\alpha < \lambda$ let I_α be a subset of A_α of size $< \kappa$ which generates the filter corresponding to a_α.

W.l.o.g. I may assume that the underlying set of A is λ. Let

$$S := \{\alpha < \lambda : \alpha \text{ is a limit ordinal of cofinality } \geq \kappa \\ \text{and the underlying set of } A_\alpha \text{ is } \alpha\}.$$

Since λ is a regular cardinal larger than κ, S is a stationary subset of λ. Let $f : \lambda \longrightarrow \lambda$ be the mapping which assigns to each $\alpha < \lambda$ the least upper bound of I_α. Then f is regressive on S. Hence there is a stationary subset T of S such that f is constant on T. Let δ be the value of f on T. Since δ has less than λ subsets of size $< \kappa$, there is a stationary subset U of T such that the mapping $F : \alpha \longmapsto I_\alpha$ is constant on U. Let I be the value of F on U and let a_p be the corresponding closed subset of X which is an intersection of less than κ clopen sets. For every ultrafilter $q \in a_p$ and every $\alpha \in U$, $q \cap A_\alpha$ splits in $A_{\alpha+1}$. Therefore each $q \in a_p$ has character λ. Hence $a_p \subseteq X \setminus M_\lambda$. Finally, $p \in a_p$ by construction. This proves the first assertion of the theorem.

For the second assertion suppose that M_λ is not determined by a subalgebra of A of size less than λ. By a similar argument as above, get a tight κ-filtration $(A_\alpha)_{\alpha<\lambda}$ such that for every ordinal $\alpha < \lambda$ of cofinality $\geq \kappa$ there is an ultrafilter $p \in M_\lambda$ such that $p \cap A_\alpha$ splits in $A_{\alpha+1}$. As above, there is a stationary subset U of λ consisting of ordinals of cofinality $\geq \kappa$ and a subset I of A of size $< \kappa$ such that for every $\alpha \in U$ the filter generated by I in A_α corresponds to the closed subset of the Stone space of A_α of those ultrafilters which split in $A_{\alpha+1}$. Let a be the closed subset of X corresponding to I. a is an intersection of less than κ clopen sets. By construction, $a \cap M_\lambda$ is non-empty. But all points in M_λ have character less than λ and all points in a have character λ because λ is regular. Thus M_λ and a are disjoint. This contradicts the choice of the filtration. □

2.4 Boolean algebras that are rc-filtered, but not tightly κ-filtered

In this section the arguments will be mainly topological. Let me collect some topological characterizations of the Stonean duals of κ-embeddings.

2.4.1. Lemma. *Let A be a subalgebra of the Boolean algebra B. Let X and Y be the Stone spaces of A and B respectively. Let $\phi : Y \to X$ be the Stonean dual of the inclusion of A into B. The following statements are equivalent:*

(i) $A \leq_\kappa B$

(ii) For each clopen set $b \subseteq Y$, $\chi(\phi[b], X) < \kappa$.

(iii) For each closed set $b \subseteq Y$ such that $\chi(b, Y) < \kappa$, $\chi(\phi[b], X) < \kappa$.

Proof. Stone duality. □

Recall that for a closed subset a of topological space X the pseudo-character of a is the minimal size of an open family $\mathcal{F}$ in X such that $\bigcap \mathcal{F} = a$. For a Boolean space it sufficient to consider clopen families $\mathcal{F}$. The pseudo-character of a equals the character of a if X is compact.

The concept of a symmetric power of a topological space was used by Ščepin in order to get an openly generated space that is not Dugundji or, in terms of Boolean algebras, to get a Boolean algebra that is rc-filtered but not projective. I will give a slight generalizion of his result.

2.4.2. Definition. Let X be a topological space. Let $\sim_X$ be the equivalence relation on X^2 that identifies (x, y) and (y, x) for all $x, y \in X$. Let $\mathrm{SP}^2(X) := X^2/\sim$. If X is the Stone space of the Boolean algebra A, then $\mathrm{SP}^2(X)$ is also a Boolean space and the algebra of clopen subsets of $\mathrm{SP}^2(X)$ corresponds to the subalgebra $\mathrm{SP}^2(A)$ of $A \oplus A$ consisting of those elements which are fixed by the automorphism of $A \oplus A$ that interchanges the two copies of A. □

2.4.3. Lemma. *(Ščepin, see [23])* SP^2 *is a covariant functor from the category of Boolean algebras into itself where the definition of* SP^2 *on homomorphisms is the natural one. Let A be a Boolean algebra. Then the embedding* $\mathrm{SP}^2(A) \to A \oplus A$ *is relatively complete.* SP^2 *is continuous, i.e. if $(A_\alpha)_{\alpha<\lambda}$ is an ascending chain of subalgebras of A, then*

$$\mathrm{SP}^2\Big(\bigcup_{\alpha<\lambda} A_\alpha\Big) = \bigcup_{\alpha<\lambda} \mathrm{SP}^2(A_\alpha).$$

SP^2 *preserves cardinalities, i.e. if A is infinite, then $|A| = |\mathrm{SP}^2(A)|$.* $\mathrm{SP}^2(A)$ *is rc-filtered provided that A is.* □

It turns out that $\mathrm{SP}^2(\mathrm{Fr}(\lambda))$ is not tightly κ-filtered if λ is large enough. This will follow easily from

2.4.4. Lemma. *Let A, B, and C be infinite Boolean algebras such that the Stone space of A has character $\geq \kappa$.*

Then

$$\langle \mathrm{SP}^2(A \oplus B) \cup \mathrm{SP}^2(A \oplus C) \rangle \not\leq_\kappa \mathrm{SP}^2(A \oplus B \oplus C).$$

Proof. I prove the topological dual. Let X, Y, and Z be the Stone spaces of A, B, and C respectively. To commence I introduce names for several mappings. Let π^2_{XY} and π^2_{XZ} denote the projections of $(X \times Y \times Z)^2$ onto $(X \times Y)^2$ and $(X \times Z)^2$ respectively. Let π denote the quotient map from $(X \times Y \times Z)^2$ onto $\mathrm{SP}^2(X \times Y \times Z)$. It follows from Lemma 2.4.3 that π is open. Let π_{XY} and π_{XZ} denote the projections of $X \times Y \times Z$ onto $X \times Y$ and $X \times Z$ respectively. Now $\mathrm{SP}^2(\pi_{XY})$ and $\mathrm{SP}^2(\pi_{XZ})$ are also defined. Let

$$\begin{aligned} \phi : \mathrm{SP}^2(X \times Y \times Z) \to \mathrm{SP}^2(X \times Y) \times \mathrm{SP}^2(X \times Z); \\ p \mapsto (\mathrm{SP}^2(\pi_{XY})(p), \mathrm{SP}^2(\pi_{XZ})(p)) \end{aligned}$$

and $P := \operatorname{Im} \phi$. Note that ϕ is the Stonean dual of the inclusion from

$$\langle \mathrm{SP}^2(A \oplus B) \cup \mathrm{SP}^2(A \oplus C) \rangle$$

into $\mathrm{SP}^2(A \oplus B \oplus C)$. The picture looks like this:

$$\begin{tikzcd}
 & (X \times Y \times Z)^2 \arrow[dl, "\pi^2_{XY}"'] \arrow[dr, "\pi^2_{XZ}"] \arrow[dd, "\pi"] & \\
(X \times Y)^2 \arrow[dd] & & (X \times Z)^2 \arrow[dd] \\
 & \mathrm{SP}^2(X \times Y \times Z) \arrow[dl, "\mathrm{SP}^2(\pi^2_{XY})"'] \arrow[d, "\phi"] \arrow[dr, "\mathrm{SP}^2(\pi^2_{XZ})"] & \\
\mathrm{SP}^2(X \times Y) & P \arrow[d, "\subseteq"] & \mathrm{SP}^2(X \times Z) \\
 & \mathrm{SP}^2(X \times Y) \times \mathrm{SP}^2(X \times Z) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Here the mappings that are not labeled are the natural ones.

Now let $U_1, U_2 \subseteq Y$ and $V_1, V_2 \subseteq Z$ be non-empty, clopen, and disjoint.

Claim 1: $\pi[X \times U_1 \times V_1 \times X \times U_2 \times V_2]$ is clopen in $\mathrm{SP}^2(X \times Y \times Z)$ but $(\phi \circ \pi)[X \times U_1 \times V_1 \times X \times U_2 \times V_2]$ has character $\geq \kappa$ in P.

This claim together with Lemma 2.4.1 proves the lemma. For its proof I need

Claim 2:

$$
\begin{aligned}
&W := (\phi^{-1} \circ \phi \circ \pi)[X \times U_1 \times V_1 \times X \times U_2 \times V_2] \\
&= \pi[X \times U_1 \times V_1 \times X \times U_2 \times V_2] \cup \bigcup_{x \in X} \pi[\{x\} \times U_1 \times V_2 \times \{x\} \times U_2 \times V_1].
\end{aligned}
$$

Proof of Claim 2: Let $(a_1, b_1, c_1, a_2, b_2, c_2)$ be such that $\pi(a_1, b_1, c_1, a_2, b_2, c_2)$ is contained in W but not in $\pi[X \times U_1 \times V_1 \times X \times U_2 \times V_2]$. Then there is $(a_1', b_1', c_1', a_2', b_2', c_2') \in X \times U_1 \times V_1 \times X \times U_2 \times V_2$ s.t.

$$(\phi \circ \pi)(a_1, b_1, c_1, a_2, b_2, c_2) = (\phi \circ \pi)(a_1', b_1', c_1', a_2', b_2', c_2').$$

I may assume $a_1 = a_1'$ and $a_2 = a_2'$. Now the following holds: $\{b_1, b_2\} = \{b_1', b_2'\}$, $\{c_1, c_2\} = \{c_1', c_2'\}$, $b_1' \neq b_2'$, $c_1' \neq c_2'$, and hence $c_1 \neq c_2$ and $b_1 \neq b_2$.

Suppose $a_1 \neq a_2$. In this case

$$((a_1, b_1), (a_2, b_2)) \sim_{X \times Y} ((a_1', b_1'), (a_2', b_2'))$$

and

$$((a_1, c_1), (a_2, c_2)) \sim_{X \times Z} ((a_1', c_1'), (a_2', c_2')).$$

Moreover, $b_i = b_i'$ and $c_i = c_i'$ for $i = 1, 2$, and hence

$$\pi(a_1, b_1, c_1, a_2, b_2, c_2) \in \pi[X \times U_1 \times V_1 \times X \times U_2 \times V_2],$$

a contradiction. Thus, $a_1 = a_2$. Since $\{b_1, b_2\} = \{b_1', b_2'\}$ and $\{c_1, c_2\} = \{c_1', c_2'\}$,

$$(a_1, b_1, c_1, a_2, b_2, c_2) \sim_{X \times Y \times Z} (a_1', b_1', c_2', a_2', b_2', c_1').$$

Therefore

$$(a_1, b_1, c_1, a_2, b_2, c_2) \in \bigcup_{x \in X} \pi[\{x\} \times U_1 \times V_2 \times \{x\} \times U_2 \times V_1].$$

Conversely, let $a \in X$, $b_i \in U_i$, and $c_i \in V_i$ for $i = 1, 2$. Now

$$\begin{aligned}(\phi \circ \pi)(a, b_1, c_2, a, b_2, c_1) &= (\phi \circ \pi)(a, b_1, c_1, a, b_2, c_2) \\ &\in (\phi \circ \pi)[X \times U_1 \times V_1 \times X \times U_2 \times V_2].\end{aligned}$$

This finishes the proof of Claim 2.

Proof of Claim 1: $\pi[X \times U_1 \times V_1 \times X \times U_2 \times V_2]$ is clopen in $\mathrm{SP}^2(X \times Y \times Z)$ since

$$\begin{aligned}&(\pi^{-1} \circ \pi)[X \times U_1 \times V_1 \times X \times U_2 \times V_2] \\ &\quad = (X \times U_1 \times V_1 \times X \times U_2 \times V_2) \cup (X \times U_2 \times V_2 \times X \times U_1 \times V_1)\end{aligned}$$

is clopen in $(X \times Y \times Z)^2$.

For the character part of Claim 1 let $\Delta^2[X]$ be the diagonal $\{(x, x) : x \in X\}$ of X^2. Now

$$\begin{aligned}&\chi((\phi \circ \pi)[X \times U_1 \times V_1 \times X \times U_2 \times V_2], P) \\ &\quad \geq \chi\left(\bigcup_{x \in X} \pi[\{x\} \times U_1 \times V_2 \times \{x\} \times U_2 \times V_1], \mathrm{SP}^2(X \times Y \times Z)\right) \\ &\qquad \geq \chi\Bigg(\bigcup_{x \in X}(\{x\} \times U_1 \times V_2 \times \{x\} \times U_2 \times V_1) \\ &\qquad\quad \cup \bigcup_{x \in X}(\{x\} \times U_2 \times V_1 \times \{x\} \times U_1 \times V_2), (X \times Y \times Z)^2\Bigg) \\ &\qquad\qquad \geq \chi(\Delta^2[X], X^2) \geq \chi(X).\end{aligned}$$

Here the last inequality can be seen as follows. Let $\mu := \chi(\Delta^2[X], X^2)$ and let $\{U^\alpha : \alpha < \mu\}$ be a local base at $\Delta^2[X]$. For each $x \in X$ and each $\alpha < \mu$ pick an open set $U_x^\alpha \subseteq X$ containing x such that $(U_x^\alpha)^2 \subseteq U^\alpha$. Now

$(\bigcap_{\alpha<\mu} U_x^\alpha)^2 = \bigcap_{\alpha<\mu}(U_x^\alpha)^2 \subseteq \Delta^2[X]$. Hence $\bigcap_{\alpha<\mu} U_x^\alpha = \{x\}$. Thus x has pseudo-character $\leq \mu$. Since X is compact, x has character $\leq \mu$. □

Now I am ready to prove a theorem which yields the promised examples of rc-filtered Boolean algebras which are not tightly κ-filtered.

2.4.5. Theorem. *Let κ and λ be regular.* $\mathrm{SP}^2(\mathrm{Fr}\,\lambda)$ *is tightly κ-filtered iff $\lambda \leq \kappa^+$.*

Proof. $A := \mathrm{SP}^2(\mathrm{Fr}\,\lambda)$ is rc-filtered by Lemma 2.4.3. In particular, A is κ-filtered for every regular cardinal κ. For $\lambda \leq \kappa^+$, $|A| \leq \kappa^+$. Hence, by the characterization of tightly κ-filtered Boolean algebras, A is tightly κ-filtered. This proves the easy implication of the theorem.

Now let $\lambda > \kappa^+$. Suppose A is tightly κ-filtered. Then there is a function $f : A \to [A]^{<\kappa}$ as in Corollary 2.2.7. For $S \subseteq \lambda$ let $\mathrm{SP}(S) := \mathrm{SP}^2(\mathrm{Fr}\,S)$ and consider this algebra as a subalgebra of A in the obvious way. Since SP^2 is continuous and cardinal preserving, there are disjoint sets $S, T \in [\lambda]^{\kappa^+}$ such that $\mathrm{SP}(S)$ and $\mathrm{SP}(S \cup T)$ are closed under f. Choose $S' \subseteq S \cup T$ such that $\mathrm{SP}(S')$ is closed under f and $|S' \cap S| = |S' \cap T| = \kappa$. Let $S_0 := S' \cap S$ and $T_0 := S' \cap T$. Finally, choose $S_1 \in [S]^\kappa$ disjoint from S_0 such that $\mathrm{SP}(S_0 \cup S_1)$ is closed under f. Since $\mathrm{SP}(S_0 \cup S_1)$ and $\mathrm{SP}(S_0 \cup T_0)$ are closed under f and by the choice of f,

$$\langle \mathrm{SP}(S_0 \cup S_1) \cup \mathrm{SP}(S_0 \cup T_0) \rangle \leq_\kappa A.$$

This contradicts Lemma 2.4.4. □

Clearly, this theorem implies

2.4.6. Corollary. *For each regular cardinal κ there is a Boolean algebra A such that A is rc-filtered but not tightly κ-filtered.* □

2.5 Complete Boolean algebras and tight σ-filtrations

Fuchino and Soukup ([19]) have shown that there may be arbitrarily large complete Boolean algebras which are σ-filtered. More exactly, if CH holds and $0^\sharp$ does not exist, then all complete c.c.c. Boolean algebras are σ-filtered. In this section, I look at the stronger property of having a tight σ-filtration. It turns out that no infinite complete Boolean algebra of size larger than $(2^{\aleph_0})^+$ is tightly σ-filtered. It is sufficient to prove that the completion of the free Boolean algebra over $(2^{\aleph_0})^{++}$ generators has no tight σ-filtration, since the Balcar-Franěk Theorem implies that this algebra is a retract of every complete Boolean algebra of size larger than $(2^{\aleph_0})^+$. A similar argument will show that adding $\aleph_3$ Cohen reals to a model of CH yields a model where $\mathfrak{P}(\omega)$ is not tightly σ-filtered but still has the WFN.

2.5.1. Definition. For a set X let the *Cohen algebra* $\mathbb{C}(X)$ over X be the completion of the free Boolean algebra $\mathrm{Fr}(X)$ over X. For $X \subseteq Y$, $\mathbb{C}(X)$ will be regarded as a complete subalgebra of $\mathbb{C}(Y)$ in the obvious way. $\square$

A technical lemma

Both results mentioned above depend heavily on the next lemma or rather on its more convenient second version, but neither one uses the full strength of the lemma. However, this seems to be approximately the weakest lemma that works for both proofs. It roughly says that the left-hand-side of the inequality $(*)$ only badly approximates the right-hand-side.

2.5.2. Lemma. *Let A, B, and C be Boolean algebras, $n \in \omega$, and $(a_i)_{i\leq n} \in A^{n+1}$ and $(b_i)_{i\leq n} \in B^{n+1}$ antichains with $a_i, b_i \neq 0$ for all $i \in n+1$. For each $k < n$ and each $i < n+1$ let $u_i^k, v_i^k \in C$ be such that*

$$(*) \quad \sum_{k<n} \left(\sum_{i,j<n+1} a_i u_i^k v_j^k b_j \right) \leq \sum_{i<n+1} a_i b_i$$

holds in $A \oplus B \oplus C$. Then for each $c \in C^+$ there are $d \in (C \restriction c)^+$ and $i < n+1$ such that

$$a_i b_i d \cdot \sum_{k<n} \left(\sum_{i,j<n+1} a_i u_i^k v_j^k b_j \right) = 0.$$

Proof. Since $(a_i)_{i<n+1}$ and $(b_i)_{i<n+1}$ are antichains without zero elements, by $(*)$, $u_i^k v_j^k = 0$ whenever $i \neq j$. Hence

$$\sum_{k<n}\left(\sum_{i,j<n+1} a_i u_i^k v_j^k b_j\right) = \sum_{k<n}\sum_{i<n+1} a_i u_i^k v_i^k b_i$$
$$= \sum_{i<n+1} a_i b_i \left(\sum_{k<n} u_i^k v_i^k\right).$$

Let $c \in C$. Let $P \subseteq C$ be the set of all atoms of the subalgebra of C that is generated by c together with the elements $u_i^k v_i^k$ for $k < n$ and $i < n+1$. Choose $d \in P$ such that $d \leq c$. Define the 2-valued matrix $(d_{ik})_{i<n+1,k<n}$ by letting $d_{ik} := 0$ iff $du_i^k v_i^k = 0$ and $d_{ik} := 1$ iff $d \leq u_i^k v_i^k$. This matrix is well defined since d was taken from P. For each $k \leq n$, $(u_i^k v_i^k)_{i<n+1}$ is an antichain. Therefore each column of $(d_{ik})_{i<n+1,k<n}$ contains at most one 1. Hence there is $i < n+1$ such that the i'th row contains no 1. i and d work for the lemma. $\square$

The following version of this lemma will be more convenient for the intended application. For a Boolean algebra A let $\overline{A} := \mathrm{ro}(A)$ and consider A as a subalgebra of $\overline{A}$ in the usual way.

2.5.3. Lemma. *Let A, B, and C be Boolean algebras, $n \in \omega$, and $(a_i)_{i\leq n} \in A^{n+1}$ and $(b_i)_{i\leq n} \in B^{n+1}$ antichains with $a_i, b_i \neq 0$ for all $i \in n+1$. Suppose $\{x_k : k < n\} \subseteq \overline{A \oplus C}$ and $\{y_k : k < n\} \subseteq \overline{B \oplus C}$ are s.t.*

$$\sum_{k<n} x_k y_k \leq \sum_{i<n+1} a_i b_i$$

in $\overline{A \oplus B \oplus C}$. Then for each $c \in C^+$ there are $d \in (C \restriction c)^+$ and $i < n+1$ such that $a_i b_i d \cdot \sum_{k<n} x_k y_k = 0$.

Proof. Let $(S^k)_{k<n}$ and $(T^k)_{k<n}$ be disjoint families of sets and for every $n < k$, $s \in S^k$, and $t \in T^k$ let $a_s \in A^+$, $v_s, w_t \in C^+$, and $b_t \in B^+$ such that $x_k = \sum_{s\in S^k} a_s v_s$ and $y_k = \sum_{t\in T^k} b_t w_t$. For $i < n+1$ and $k < n$ let $S_i^k := \{s \in S^k : a_s \leq a_i\}$ and $T_i^k := \{t \in T^k : b_t \leq b_i\}$. Then $(S_i^k)_{i<n+1}$ and

$(T_i^k)_{i<n+1}$ are partitions of S^k and T^k respectively. Moreover, if $i \neq j$, then for all $k < n$ and for all $s \in S_i^k$ and $t \in T_j^k$, $s \cdot t = 0$. Now

$$\begin{aligned}\sum_{k<n} x_k y_k &= \sum_{k<n}\left(\left(\sum_{s\in S^k} a_s v_s\right)\left(\sum_{s\in T^k} b_t w_t\right)\right)\\ &= \sum_{k<n}\sum_{s\in S^k, t\in T^k} a_s v_s w_t b_t = \sum_{k<n}\sum_{i<n+1}\sum_{s\in S_i^k, t\in T_i^k} a_s v_s w_t b_t\\ &\leq \sum_{k<n}\sum_{i<n+1}\sum_{s\in S_i^k, t\in T_i^k} a_i v_s w_t b_i = \sum_{k<n}\sum_{i<n+1}\left(a_i b_i\left(\sum_{s\in S_i^k, t\in T_i^k} v_s w_t\right)\right)\\ &\leq \sum_{i<n+1} a_i b_i.\end{aligned}$$

For each $k < n$ and $i < n+1$ let $v_i^k := \sum_{s\in S_i^k} v_s$ and $w_i^k := \sum_{t\in T_i^k} w_t$. Then $\sum_{s\in S_i^k, t\in T_i^k} v_s w_t = v_i^k w_i^k$ and thus

$$\sum_{k<n} x_k y_k \leq \sum_{k<n}\sum_{i<n+1} a_i v_i^k w_i^k b_i \leq \sum_{i<n+1} a_i b_i.$$

Now for each $c \in C^+$ suitable d and i exist by Lemma 2.5.2. $\square$

Complete Boolean algebras of size $\geq (2^{\aleph_0})^{++}$ have no tight σ-filtration

The essential observation in order to get the theorem for $\mathbb{C}((2^{\aleph_0})^{++})$ is

2.5.4. Lemma. *Let X, Y, and Z be disjoint infinite sets. Let $C_0 := \mathbb{C}(X \cup Z)$, $C_1 := \mathbb{C}(Y \cup Z)$, and $C := \mathbb{C}(X \cup Y \cup Z)$. Then $\langle C_0 \cup C_1\rangle \not\leq_\sigma C$.*

Proof. Let $X_0 \subset X$ and $Y_0 \subset X$ be countably infinite. Let $g : \omega \to \mathrm{Fr}(X_0 \cup Y_0)$ be a surjection, $f : \omega \times \omega \to \omega$ a bijection, and $f_0, f_1 : \omega \to \omega$ such that $f^{-1} = (f_0(\cdot), f_1(\cdot))$. Let $(c_i)_{i\in\omega}$ be an antichain in $\mathrm{Fr}(Z)$ without zero elements and put $x := \sum_{i\in\omega} c_i (g \circ f_0)(i)$. I *claim* that $\langle C_0 \cup C_1\rangle \restriction x$ is not countably generated.

Proof of the claim: Let $\{x_n : n \in \omega\} \subseteq \langle C_0 \cup C_1\rangle \restriction x$ be closed under finite joins. Let $n \in \omega$. Then there is $k \in \omega$ such that for each $i \in \omega$ there are $p^i_j \in C_0$ and $q^i_j \in C_1$, $j < k$, s.t.

$$c_i \cdot x_n = p^i_0 q^i_0 + \cdots + p^i_{k-1} q^i_{k-1}.$$

Now there is $m \in \omega$ such that $(g \circ f_0) f(m,n) = \sum_{l<k+1} a_l b_l$ for two antichains $(a_l)_{l<k+1}$ and $(b_l)_{l<k+1}$ without zero elements in $\mathrm{Fr}(X_0)$ and $\mathrm{Fr}(Y_0)$ respectively. Since $c_{f(m,n)} \cdot x_n \leq \sum_{l<k+1} a_l b_l$, one can use Lemma 2.5.3 to get $l < k$ and $d \in \mathrm{Fr}(Z)^+$ such that $d \leq c_{f(m,n)}$ and $a_l b_l d x_{f(m,n)} = 0$. Let $y_{f(m,n)} := a_l b_l d$ and let $y_{f(m',n)} := 0$ for $m' \neq m$. Finally let $y := \sum_{i\in\omega} y_i$. Note that for suitable $(a'_i)_{i\in\omega} \in {}^\omega C_0$ and $(b'_i)_{i\in\omega} \in {}^\omega C_1$, $y = \left(\sum_{i\in\omega} a'_i c_i\right) \cdot \left(\sum_{i\in\omega} b'_i c_i\right)$. Therefore $y \in \langle C_0 \cup C_1 \rangle \restriction x$. However, $y \not\leq x_n$ for any $n \in \omega$. This proves the claim and hence finishes the proof of the lemma. □

Now I am ready to prove

2.5.5. Theorem. $\mathbb{C}((2^{\aleph_0})^{++})$ *is not tightly σ-filtered.* □

But before embarking the proof of this theorem, let me deduce from it

2.5.6. Corollary. *No complete Boolean algebra of size strictly larger than $(2^{\aleph_0})^+$ has a tight σ-filtration.*

Proof. Suppose A is a complete Boolean algebra of size $\geq (2^{\aleph_0})^{++}$. By the well-known Balcar-Franěk Theorem, $\mathrm{Fr}((2^{\aleph_0})^{++})$ embeds into A. By completeness of A, this embedding extends to $\mathbb{C}((2^{\aleph_0})^{++})$. Since the free algebra is dense in the Cohen algebra, this extension is an embedding as well. By completeness of $\mathbb{C}((2^{\aleph_0})^{++})$, $\mathbb{C}((2^{\aleph_0})^{++})$ is a retract of A. Since being tightly σ-filtered is hereditary with respect to retracts (Corollary 2.2.8) and by the theorem above, A is not tightly σ-filtered. □

Proof of the theorem. Suppose $A := \mathbb{C}((2^{\aleph_0})^{++})$ has a tight σ-filtration. Let $f : A \to [A]^{<\kappa}$ be a function as in Corollary 2.2.7. Since A satisfies c.c.c., every subalgebra of A of size $2^{\aleph_0}$ or $(2^{\aleph_0})^+$ is contained in a complete subalgebra of the same size. Hence, using the argument in the proof of Theorem 2.4.5, I can find non-empty disjoint sets $S_0, S_1, T_0 \subseteq \lambda$ of size $2^{\aleph_0}$ such that

$\mathbb{C}(S_0 \cup S_1)$ and $\mathbb{C}(S_0 \cup T_0)$ are closed under f. By the preceding lemma,

$$\langle \mathbb{C}(S_0 \cup S_1) \cup \mathbb{C}(S_0 \cup T_0) \rangle \not\leq_\sigma \mathbb{C}(S_0 \cup S_1 \cup T_0).$$

A contradiction. $\square$

After adding many Cohen reals, $\mathfrak{P}(\omega)$ is not tightly σ-filtered

The proof of this theorem is very similar to the proof of Theorem 2.5.5. The parallel of Lemma 2.5.4 is

2.5.7. Lemma. *Let A and B be complete Boolean algebras both adding Cohen reals such that any countable set of ordinals in a generic extension by $A \oplus B$ of the ground model M is contained in a countable set in M. Let G be $(A \oplus B)$-generic over M. Let $P_0 := \mathfrak{P}(\omega)^{M[G\cap A]}$ and $P_1 := \mathfrak{P}(\omega)^{M[G\cap B]}$. Then $\langle P_0 \cup P_1 \rangle \not\leq_\sigma P := \mathfrak{P}(\omega)^{M[G]}$.*

Proof. Since A and B both add Cohen reals, there are countable atomless regular subalgebras A_0 and B_0 of A and B respectively. Let $g : \omega \to A_0 \oplus B_0$ be onto, $f : \omega \times \omega \to \omega$ a bijection, and $f_0, f_1 : \omega \to \omega$ such that $f^{-1} = (f_0(\cdot), f_1(\cdot))$, like in the proof of Lemma 2.5.4. Let $\sigma := g \circ f_0$. Consider σ as an $\overline{A \oplus B}$-name for a subset of ω. I will show that $\langle P_0 \cup P_1 \rangle \restriction \sigma_G$ is not countably generated. Suppose $S \in M[G]$ is a countable subset of this ideal which is closed under finite joins. For every $a \in S$ there is a name $\tau^a : \omega \to \overline{A \oplus B}$ such that $a = \tau^a_G$. Let $T := \{\tau^a : a \in S\}$. Since $S \subseteq \langle P_0 \cup P_1 \rangle$, I may assume that

($*$) for each $\tau \in T$ there is $k_\tau \in \omega$ such that for all $m \in \omega$ there are $p^m_0, \ldots, p^m_{k_\tau - 1} \in A$ and $q^m_0, \ldots, q^m_{k_\tau - 1} \in B$ such that $\tau(m) = \sum_{i<k_\tau} p^m_i q^m_i$.

Here the exact reasoning is like this: Each a in S is some Boolean combination of elements from P_0 and P_1. Hence, if τ is a name for a, i.e. if $\tau_G = a$, then there are a condition r in G and $k_\tau \in \omega$ such that

$$r \Vdash \exists p_0, \ldots, p_{k_\tau - 1} \in P_0 \exists q_0, \ldots, q_{k_\tau - 1} \in P_1 \left(\tau = \sum_{i<k_\tau} p_i q_i \right).$$

By the maximal principle, there are names $\{(m, p_i^m) : m \in \omega\}$ and $\{(m, q_i^m) : m \in \omega\}$ for the p_i and q_i respectively. From these names I can construct a name τ for a which works for $(*)$.

Now for each $\tau \in T$ choose $p_\tau \in G$ such that $p_\tau \Vdash \tau \subseteq \sigma$. Note that $p \Vdash \tau \subseteq \sigma$ iff $\tau(m) \leq -p + \sigma(m)$ for all $m \in \omega$. This is equivalent to $\tau(m) \cdot p \leq \sigma(m)$ for all $m \in \omega$. Let $\tau \in T$. From $p_\tau \in G$ it follows that $a = \tau^a_G = (\tau^a \cdot p_{\tau^a})_G$, where $\tau \cdot p$ is the function that maps every $m \in \omega$ to $\tau(m) \cdot p$. Since $\{p \cdot q : p \in A, q \in B\}$ is dense in $\overline{A \oplus B}$, I may assume $p_\tau = p^\tau \cdot q^\tau$ for some $p^\tau \in A$ and $q^\tau \in B$ for each $\tau \in T$. This is handy, since replacing each $\tau \in T$ by $\tau \cdot p^\tau \cdot q^\tau$ preserves property $(*)$.

Therefore I may assume that $(*)$ holds and for every $\tau \in T$, $\tau \leq \sigma$, i.e. for all $m \in \omega$ the inequality $\tau(m) \leq \sigma(m)$ holds. By assumption, T is contained in a countable set T' of names in the ground model. Since only those names $\tau \in T'$ that do not spoil $(*)$ and for which $\tau \leq \sigma$ holds are relevant and since these properties are definable in the ground model, I may assume that $(*)$ holds for T' and $\tau \leq \sigma$ holds for every $\tau \in T'$. Moreover, I may assume that T' is closed under finite joins, in the sense that for all $\tau, \tau' \in T'$ the name $\{(m, \tau(m) + \tau'(m)) : m \in \omega\}$ is also an element of T'. Let $(\tau_n)_{n \in \omega} \in M$ be an enumeration of T'. Since $A_0 \oplus B_0$ is a regular subalgebra of $\overline{A \oplus B}$, I will be done if I can prove the following

Claim. There is a name $\rho : \omega \to \overline{A \oplus B}$ for an element of $\langle P_0 \cup P_1 \rangle$ such that for every $n \in \omega$ and every $r \in A_0 \oplus B_0$ there is $s \leq r$, $s \in A_0 \oplus B_0$, such that $s \Vdash \rho \not\leq \tau_n$.

Proof of the claim: Construct ρ as follows: For each $n \in \omega$ choose $k_n \in \omega$ and sequences $(p^m_{i,n})_{i<k_n, m \in \omega}$ in A and $(q^m_{i,n})_{i<k_n, m \in \omega}$ in B as promised in $(*)$ for τ_n. For $m, n \in \omega$ such that $\sigma(f(m,n)) = \sum_{i<k_n+1} a_i b_i$ for some antichains $(a_i)_{i<k_n}$ and $(b_i)_{i<k_n}$ in A_0^+ and B_0^+ respectively let $i < k_n + 1$ be such that $a_i b_i \tau_n(f(m,n)) = 0$. This is possible by Lemma 2.5.3. Note that in this case the algebra C mentioned in the lemma is trivial. Let $\rho(f(m,n)) := a_i b_i$. Now $a_i b_i \Vdash \rho \not\subseteq \tau_n$. In any other case let $\rho(f(m,n)) := 0$. Clearly, ρ is a name for an element of $\langle P_0 \cup P_1 \rangle$.

ρ works for the claim: Let $n \in \omega$ and $r \in \overline{A_0 \oplus B_0}$. W.l.o.g. I may assume $r = a \cdot b$ for some $a \in A_0$ and $b \in B_0$. Let $m \in \omega$ such that

$\sigma(f(m,n)) = \sum_{i<k_n+1} a_i b_i \leq a \cdot b$ for some antichains $(a_i)_{i\leq k_n+1}$ and $(b_i)_{i\leq k_n+1}$ in A_0^+ and B_0^+ respectively. Note that the a_i and b_i are uniquely determined by $\sigma(f(m,n))$, up to permutation of the common index set. This is not really important here, but it makes the argument somewhat shorter. Now $\rho(f(m,n)) = a_i b_i$ for some $i < k_n + 1$ and $\tau_n(f(m,n)) \cdot a_i b_i = 0$. Hence $s := a_i b_i \Vdash \rho \not\subseteq \tau_n$ and $s \leq r$. This finishes the proof of the claim and hence the proof of the lemma. □

With this lemma at hand, I can prove the announced result on Cohen forcing. In fact, I will prove a slightly more general theorem.

2.5.8. Theorem. *Let λ be a cardinal such that $\lambda^{\aleph_0} = \lambda$ in the ground model M. Let $(A_\alpha)_{\alpha<\lambda^{++}}$ be a sequence Boolean algebras in the ground model, each adding at most λ new reals, such that*

$$A := \bigoplus_{\alpha<\lambda^{++}} A_\alpha$$

satisfies c.c.c. Let G be A-generic over M. Then

$$M[G] \models \mathfrak{P}(\omega) \text{ has no tight } \sigma\text{-filtration.}$$

In particular, adding $\aleph_3$ Cohen reals to a model of CH gives a model in which $\mathfrak{P}(\omega)$ fails to be tightly σ-filtered, though $\mathrm{WFN}(\mathfrak{P}(\omega))$ *still holds.*

Proof. For $S \subseteq \lambda^{++}$ let $A_S := \bigoplus_{\alpha\in S} A_\alpha$, $G_S := G \cap A_S$, and $P_S := \mathfrak{P}(\omega)^{M[G_S]}$. Suppose $\mathfrak{P}(\omega)$ has a tight σ-filtration in $M[G]$. I may assume that this is already forced by 1_A. In $M[G]$ let $f : \mathfrak{P}(\omega) \to [\mathfrak{P}(\omega)]^{\aleph_0}$ be a function as in Corollary 2.2.7. Let $\phi \in M$ be an $A_{\lambda^{++}}$-name for such a function. Using c.c.c., one can construct a function $g : {}^{\omega}A_{\lambda^{++}} \to [\lambda^{++}]^{\aleph_0}$ such that for every name $\tau : \omega \to A_{\lambda^{++}}$, $\Vdash \phi(\tau) \subseteq P_{g(\tau)}$. Call a subset S of λ^{++} *good* iff $\bigcup g[{}^{\omega}A_S] \subseteq S$. Let S and T be disjoint subsets of λ^{++} of size λ^+ such that S and $S \cup T$ are good. This is possible since $(\lambda^+)^{\aleph_0} = \lambda^+$. Now let $S_0, S_1 \subseteq S$ and $T_0 \subseteq T$ be disjoint sets of size λ such that $S_0 \cup S_1$ and $S_0 \cup T_0$ are good. Applying the last lemma to the algebras $\overline{A}_{S_1}$ and $\overline{A}_{T_0}$ with $M[G_{S_0}]$ as the ground model, it follows that $\langle P_{S_0\cup S_1} \cup P_{S_0\cup T_0}\rangle \not\leq_\sigma P_{S_0\cup S_1\cup T_0}$

in $M[G_{S_0}][G_{S_1\cup T_1}]$. By c.c.c., $\langle P_{S_0\cup S_1} \cup P_{S_0\cup T_0}\rangle \not\leq_\sigma P_{\lambda^{++}}$ holds in $M[G]$. This is a contradiction since by the choice of g, the algebras $P_{S_0\cup S_1}$ and $P_{S_0\cup T_0}$ are closed under ϕ_G. $\square$

The pseudo product of Cohen forcings

While so far the only known way to obtain a model of $\neg\,\mathrm{CH} + \mathrm{WFN}(\mathfrak{P}(\omega))$ is to add Cohen reals to a model of CH, there is some freedom in the choice of the iteration used for adding the Cohen reals. In [18] Fuchino, Shelah, and Soukup introduced a new kind of side-by-side product of partial orders.

2.5.9. Definition. Let $(P_i)_{i\in X}$ be a family of partial orders where each P_i has a largest element 1_{P_i}. As usual, for $p \in \prod_{i\in X} P_i$ let $\mathrm{supp}(p) := \{i \in X : p(i) \neq 1_{P_i}\}$ be the *support* of p. Let $\prod^*_{i\in X} P_i := \{p \in \prod_{i\in X} P_i : |\mathrm{supp}(p)| \leq \aleph_0\}$ be ordered such that for all $p, q \in \prod^*_{i\in X} P_i$,

$$p \leq q \Leftrightarrow \forall i \in X(p(i) \leq q(i)) \wedge |\{i \in X : p(i) \neq q(i) \neq 1_{P_i}\}| < \aleph_0. \quad \square$$

Among other things, Fuchino, Shelah, and Soukup proved the following about this product:

2.5.10. Lemma. *Let $(P_i)_{i\in X}$ be as in the definition above.*
a) For every $Y \subseteq X$, $\prod^*_{i\in X} \cong \prod^*_{i\in Y} \times \prod^*_{i\in X\setminus Y}$.
b) Under CH, $\prod^*_{i\in X} \mathrm{Fn}(\omega, 2)$ satisfies the $\aleph_2$-c.c. and is proper. $\square$

I will show that $\mathfrak{P}(\omega)$ has the WFN after forcing with $\prod^*_{i\in X} \mathrm{Fn}(\omega, 2)$ over a model of CH, provided $|X|$ is smaller than $\aleph_\omega$. I will use the well-known

2.5.11. Lemma. *Suppose the partial order P is a union of an increasing chain $(P_\alpha)_{\alpha<\lambda}$ of completely embedded suborders. Let G be P-generic over the ground model M and for each $\alpha < \lambda$ let $G_\alpha := P_\alpha \cap G$. If λ has uncountable cofinality, then for every real $x \in M[G]$ there is $\alpha < \lambda$ such that $x \in M[G_\alpha]$.*

Proof. Let x be a real in $M[G]$. I may assume that x is a function from ω to 2. Let $\dot{x}$ be a P-name for x. For each $\alpha < \lambda$ let $\dot{x}_\alpha$ be a P_α-name for a function from ω to 2 such that

$$\forall n \in \omega \exists p \in P_\alpha \exists i \in 2((p \Vdash_P \dot{x}(n) = i) \Rightarrow (p \Vdash_{P_\alpha} \dot{x}_\alpha(n) = i)).$$

For each $n \in \omega$ let $\alpha_n < \lambda$ be such that there is $p \in G_\alpha$ deciding $\dot{x}(n)$. Let $\alpha := \sup_{n\in\omega} \alpha_n$. Now $\alpha < \lambda$ since λ has uncountable cofinality. Clearly, $(\dot{x}_\alpha)_{G_\alpha} = \dot{x}_G$. Thus $x \in M[G_\alpha]$. □

2.5.12. Theorem. *Let $\lambda < \aleph_\omega$ be an uncountable cardinal and suppose CH holds. Let $P := \prod^*_{\alpha<\lambda} \mathrm{Fn}(\omega, 2)$. Then*

$$\Vdash_P \mathrm{WFN}(\mathfrak{P}(\omega)) \textit{ and } 2^{\aleph_0} = \lambda.$$

Proof. Let M be the ground model satisfying CH and let G be P-generic over M. It follows from Lemma 2.5.10 that P is cardinal preserving and that the continuum is λ in $M[G]$. Throughout this proof I will use Lemma 2.5.10 without referring to it anymore. For each $X \subseteq \lambda$ with $X \in M$ consider $P_X := \prod^*_{\alpha\in X} \mathrm{Fn}(\omega, 2)$ as a suborder of P in the obvious way and let $G_X := P_X \cap G$ and $\mathfrak{P}_X := (\mathfrak{P}(\omega))^{M[G_X]}$. $(\mathfrak{P}_\alpha)_{\alpha\le\lambda}$ is continuous at limit ordinals of uncountable cofinality by Lemma 2.5.11.

Claim. In $M[G]$: For each $\alpha < \lambda$, $\mathfrak{P}_\alpha \le_\sigma \mathfrak{P}(\omega)$.

Proof of the claim: I argue in $M[G]$. Let $\alpha < \lambda$. Let $x \in \mathfrak{P}(\omega)$. By $\aleph_2$-c.c. of P, in M there is a subset X of λ of size $< \aleph_2$ such that $x \in \mathfrak{P}_X$. By Lemma 2.5.11, in M there is a countable subset Y of $X \setminus \alpha$ such that $x \in M[G_\alpha][G_Y]$. The set $D := \{p \in P_Y : \mathrm{supp}(p) = Y\}$ is dense in P_Y. Thus there is $p \in G_Y \cap D$. It is easy to see that $P_Y \downarrow p$ is isomorphic to $\mathrm{Fn}(\omega, 2)$. Thus there is a Cohen real r over $M[G_\alpha]$ in $M[G]$ such that $x \in M[G_\alpha][r]$. It was shown in [16] that

$$M[G_\alpha][r] \models (\mathfrak{P}(\omega) \cap M[G_\alpha]) \restriction x \text{ has countable cofinality.}$$

(This also follows from Theorem 3.1.4 in the next chapter.) By properness of P, $\mathfrak{P}_\alpha \restriction x$ really has countable cofinality. This finishes the proof of the claim.

Now it follows by induction on the size of λ that $\mathrm{WFN}(\mathfrak{P}(\omega))$ holds in

$M[G]$. The induction uses Lemma 1.4.3 and the fact that WFN($\mathfrak{P}(\omega)$) holds under CH. $\square$

Using the same argument as in the the proof of theorem 2.5.8, one can show that $\mathfrak{P}(\omega)$ is not tightly σ-filtered after forcing with $\prod^*_{\alpha<\omega_3} \mathrm{Fn}(\omega, 2)$ over a model of CH.

2.5.13. Theorem. *Assume CH and let $\lambda \geq \aleph_3$. Let $P := \prod^*_{\alpha<\lambda} \mathrm{Fn}(\omega, 2)$. Then*

$$\Vdash_P \mathfrak{P}(\omega) \text{ is not tightly } \sigma\text{-filtered.}$$

Proof. Again, in this proof I will use Lemma 2.5.10 without referring to it explicitly. Let G be P-generic over the ground model M. I argue in $M[G]$. Suppose that $\mathfrak{P}(\omega)$ is tightly σ-filtered. Let f be a function as in Corollary 2.2.7. For $X \subseteq \lambda$ with $X \in M$ let P_X, G_X, and $\mathfrak{P}_X$ be defined as in the proof of Theorem 2.5.12. Let $S \subseteq T \subseteq \lambda$ with $S, T \in M$ be such that $|S|{=}|T|{=}|T \setminus S|{=}\,\aleph_2$ and $\mathfrak{P}_S$ and $\mathfrak{P}_T$ are closed under f. This is possible by Lemma 2.5.11. In M choose disjoint sets $S_0, S_1 \subseteq S$ and a set $T_0 \subseteq T \setminus S$ such that $\mathfrak{P}_{S_0 \cup S_1}$ and $\mathfrak{P}_{S_0 \cup T_1}$ are closed under f. By Lemma 2.5.7,

$$M[G_{S_0 \cup S_1 \cup T_0}] \models \langle \mathfrak{P}(\omega) \cap (M[G_{S_0 \cup S_1}] \cup M[G_{S_0 \cup T_0}]) \rangle \not\leq_\sigma \mathfrak{P}(\omega).$$

Since $P_{\lambda \setminus (S_0 \cup S_1 \cup T_0)}$ is proper and $M[G] = M[G_{S_0 \cup S_1 \cup T_0}][G_{\lambda \setminus (S_0 \cup S_1 \cup T_0)}]$,

$$\langle \mathfrak{P}_{S_0 \cup S_1} \cup \mathfrak{P}_{S_0 \cup T_0} \rangle \not\leq_\sigma \mathfrak{P}(\omega).$$

This contradicts the choice of f. $\square$