Chapter 1

Preparation

Throughout this chapter let κ be an infinite regular cardinal.

1.1 κ-embeddings

1.1.1. Definition. A partially ordered structure $(P, \leq)$ is an algebraic structure P together with a partial order $\leq$. Of course, P itself may have no functions or relations. In this case $(P, \leq)$ is just a partial order. Typically, I will omit $\leq$ and write P instead of $(P, \leq)$. Likewise, I will rarely distinguish between an algebraic structure and its underlying set. As a subset X of a partially ordered structure P is cofinal in P iff it contains an upper bound of every element of P, X is coinitial in P iff it contains a lower bound of every element of P. The cofinality of P is the minimal cardinality of a cofinal subset of P and is denoted by $\mathrm{cf}(P)$. Similarly, the coinitiality of P is the minimal size of a coinitial subset of P and is denoted by $\mathrm{ci}(P)$.

Let P and Q be partially ordered structures such that $P \leq Q$, i.e. such that P is a substructure of Q. Then for $x \in Q$ the initial segment $\{a \in P : a \leq x\}$ is denoted by $P \downarrow x$ and the final segment $\{a \in P : a \geq x\}$ by $P \uparrow x$. P is called a *κ-substructure* of Q iff for each $x \in Q$ the initial segment $P \downarrow x$ and the final segment $P \uparrow x$ have cofinality respectively coinitiality $< \kappa$. In this case I write $P \leq_\kappa Q$. $P \leq_\sigma Q$ means $P \leq_{\aleph_1} Q$. The word 'substructure' can be replaced by 'suborder' or 'subalgebra', depending on the type of objects I am dealing with. An isomorphism between a partially ordered structure P and a κ-substructure P' of a partially ordered structure Q is called a κ-embedding.

Now let A and B be Boolean algebras such that A is a subalgebra of B and let $x \in B$. I write $A \upharpoonright x$ instead of $A \downarrow x$. $A \upharpoonright x$ can be regarded as an ideal of A or, if x is an element of A, as a Boolean algebra, namely the relative algebra of A with respect to x. The intended meaning will always be clear from the context. A is a *relatively complete* subalgebra of B iff $A \leq_{\aleph_0} B$. In this case I write $A \leq_{\mathrm{rc}} B$. Note that $A \leq_{\mathrm{rc}} B$ iff for every $x \in B$ the ideal $A \upharpoonright x$ is generated by a single element. The mapping lpr_A^B assigning to $x \in B$ the generator of $A \upharpoonright x$ is called the *lower projection* from B to A. □

In the following the letters A, B, and C will refer to Boolean algebras unless stated differently. Thus $A \leq B$ means that A is a subalgebra of B. Note that $A \leq_\kappa B$ iff $A \leq B$ and for every $x \in B$ the ideal $A \upharpoonright x$ has cofinality $< \kappa$. Also note that $A \leq_\kappa B$ iff $A \leq B$ and for every ideal I of B which has cofinality $< \kappa$ the ideal $I \cap A$ also has cofinality $< \kappa$.

The following two lemmas collect some frequently used facts on $\leq_\kappa$.

1.1.2. Lemma. *Let A and B be Boolean algebras such that $A \leq B$ and $x \in B$. Then $A \leq_\kappa A(x)$ iff $A \upharpoonright x$ and $A \upharpoonright -x$ both have cofinality $< \kappa$.*

Proof. The direction from the left to the right is trivial. For the other direction let $E \subseteq A$ and $F \subseteq A$ be sets of size $< \kappa$ which are cofinal in $A \upharpoonright x$ and $A \upharpoonright -x$ respectively. Suppose $y \in A(x)$. Then there are $v, w \in A$ such that $y = (v + x) \cdot (w + (-x))$. Let $z \in A$ be such that $z \leq y$. Then $z - v \leq x$ and $z - w \leq -x$. Hence $z - v \leq a$ and $z - w \leq b$ for some $a \in E$ and some $b \in F$. It follows that $z \leq (v + a) \cdot (w + b)$. Clearly, $(v + a) \cdot (w + b) \leq y$ for every $a \in E$ and every $b \in F$. Hence $\{(v + a) \cdot (w + b) : a \in E \wedge b \in F\}$ is cofinal in $A \upharpoonright y$. □

1.1.3. Lemma. *Let P, Q, and R be partial orders.*

a) $P \leq_\kappa Q \leq_\kappa R \Rightarrow P \leq_\kappa R$.

b) If Q is the union of a family $\mathcal{Q}$ of suborders of Q and $P \leq_\kappa Q'$ for every $Q' \in \mathcal{Q}$, then $P \leq_\kappa Q$.

c) If $(P_\alpha)_{\alpha<\lambda}$ is an ascending chain of κ-suborders of Q and $\mathrm{cf}(\lambda) < \kappa$, then $\bigcup_{\alpha\leq\lambda} P_\alpha \leq_\kappa Q$.

Now let A, B, and C be Boolean algebras.

d) $A \leq_\kappa B$, $X \in [B]^{<\kappa} \Rightarrow A(X) \leq_\kappa B$.

e) $A \leq_{\mathrm{rc}} B$, $C \leq B$, *and* $\mathrm{lpr}^B_A[C] \subseteq C \Rightarrow A \cap C \leq_{\mathrm{rc}} C$.

Proof. a), b), and e) are easy. For c) let $R := \bigcup_{\alpha<\lambda} P_\alpha$. Fix a cofinal set $X \subseteq \lambda$ of size $< \kappa$. For $q \in Q$ and $\alpha \in X$ let Y^q_α be a cofinal subset of $P_\alpha \downarrow q$ of size $< \kappa$. Then $\bigcup_{\alpha \in X} Y^q_\alpha$ is cofinal in $R \downarrow q$ and has size $< \kappa$ by regularity of κ. By the same argument, $R \uparrow q$ has coinitiality $< \kappa$.

d) was shown by Koppelberg for $\kappa \leq \aleph_1$ ([29]). The proof for the general case is the same. Let C be the subalgebra of B generated by X. Suppose $b \in B$. For each $c \in C$ fix a set $Y_c \in [A]^{<\kappa}$ which is cofinal in $A \restriction -c + b$. I *claim* that the algebra $D \leq A(X)$ generated by $C \cup \bigcup_{c \in C} Y_c$ contains a cofinal subset of $A(X) \restriction b$.

Let $a \in A(X) \restriction b$. There are $n \in \omega$, $a_0, \dots, a_{n+1} \in A$, and $c_0, \dots, c_{n-1} \in C$ such that $a = \sum_{i<n} a_i c_i$. Since $a \leq b$, $a_i \leq -c_i + b$ for each $i < n$. Hence, for each $i < n$ there is $a'_i \in Y_{c_i}$ such that $a_i \leq a'_i \leq -c_i + b$. Now $a \leq \sum_{i<n} a'_i c_i \leq b$ and $\sum_{i<n} a'_i c_i \in D$. This proves the claim. By regularity of κ, $|D| < \kappa$. □

1.2 κ-filtrations

A partial order is κ-filtered iff it has many κ-suborders. In order to give a precise formulation of 'many', I introduce various notions of skeletons.

1.2.1. Definition. Let $\mathcal{S}$ be a family of suborders of a partial order P. $\mathcal{S}$ is called a $< \kappa$*-skeleton* of P iff the following conditions hold:

(i) $\mathcal{S}$ is closed under unions of subchains.

(ii) For every suborder Q of P there are $\mu < \kappa$ and $R \in \mathcal{S}$ such that $Q \subseteq R$ and $|R| \leq |Q| + \mu$.

$\mathcal{S}$ is called a κ*-skeleton* of P iff $\mathcal{S}$ satisfies (i) as above and instead of (ii) the following holds:

(ii)′ Every suborder Q of P is included in a member R of $\mathcal{S}$ such that $|R|=|Q|+\kappa$.

$\mathcal{S}$ is called a *skeleton* iff it is an $\aleph_0$-skeleton. □

The exact definition of κ-filteredness is the following:

1.2.2. Definition. A partial order P is κ-filtered iff it has a κ-skeleton $\mathcal{S}$ consisting of κ-suborders. P is *σ-filtered* iff it is $\aleph_1$-filtered. A Boolean algebra A is *rc-filtered* iff it is $\aleph_0$-filtered. □

Note that if $\mathcal{S}$ is a κ-skeleton of a Boolean algebra A, then it includes a κ-skeleton $\mathcal{S}'$ of A consisting of subalgebras of A. Thus a Boolean algebra A is κ-filtered iff it has a κ-skeleton consisting of κ-subalgebras. If κ is uncountable, then every $<\kappa$-skeleton of a Boolean algebra A contains a $<\kappa$-skeleton consisting of subalgebras of A. However, the latter is not true for $\kappa=\aleph_0$ since every infinite Boolean algebra A has a $<\aleph_0$-skeleton which contains no finite subalgebra of A.

The other notion, apart from κ-filteredness, that will be investigated in this thesis is tight κ-filteredness. At least using the definition given below, this notion only makes sense for Boolean algebras. While κ-filteredness and tight κ-filteredness seem to be unrelated at first sight, it will turn out later that tight κ-filteredness is stronger than κ-filteredness.

1.2.3. Definition. Let A be a Boolean algebra and δ an ordinal. A continuous ascending chain $(A_\alpha)_{\alpha<\delta}$ of subalgebras of A is called a (wellordered) *filtration* of A.

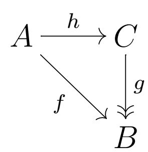
A filtration $(A_\alpha)_{\alpha<\delta}$ is called *tight* iff $A_0=2$ and there is a sequence $(x_\alpha)_{\alpha<\delta}$ in A such that $A_{\alpha+1}=A_\alpha(x_\alpha)$ holds for all $\alpha<\delta$.

A filtration $(A_\alpha)_{\alpha<\delta}$ is called a *κ-filtration* (*rc-filtration*, *σ-filtration*) iff $A_\alpha \leq_\kappa A_{\alpha+1}$ ($A_\alpha \leq_{\mathrm{rc}} A_{\alpha+1}$, $A_\alpha \leq_\sigma A_{\alpha+1}$) holds for all $\alpha<\delta$. A is *tightly κ-filtered* iff it has a tight κ-filtration. □

1.3 Universal properties

This section will not really be needed for the rest of this thesis, but it provides some motivation for studying tight κ-filteredness. Tightly κ-filtered Boolean algebras have properties similar to projectivity. While no infinite complete Boolean algebra is projective, in some models of set theory interesting complete Boolean algebras are for example tightly σ-filtered. This has nice applications concerning the existence of certain homomorphisms.

1.3.1. Definition. A Boolean algebra A is *projective* iff for any two Boolean algebras B and C, every epimorphism $g : C \to B$, and every homomorphism $f : A \to B$ there is a homomorphism $h : A \to C$ such that



commutes. □

While this definition works in every category, the following characterization provides more insight into the structure of projective Boolean algebras.

1.3.2. Definition and Lemma. *A is a* retract *of B iff there are homomorphisms $e : A \to B$ and $p : B \to A$ such that $p \circ e = \mathrm{id}_A$. A Boolean algebra A is projective iff it is a retract of a free Boolean algebra.*

Proof. Abstract nonsense. □

This lemma is true in every category with sufficiently many free objects. However, there are categories in which this lemma does not hold since there are non-trivial projective objects, but no non-trivial free objects. (See [20] for an example.)

By theorems by Haydon, Koppelberg, and Ščepin, the tightly rc-filtered Boolean algebras are exactly the projective Boolean algebras. (See [29] or [23].) The following theorem generalizes one direction of this to tightly κ-filtered Boolean algebras and was proved by Koppelberg ([28]) for $\kappa = \aleph_1$.

Her proof works for uncountable κ as well. Let me introduce some additional notions first.

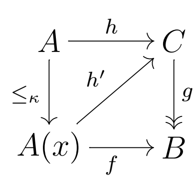
1.3.3. Definition. A Boolean algebra A has the *κ-separation property* (κ-s.p. for short) iff for any two subsets S and T of A of size $< \kappa$ with $S \cdot T := \{s \cdot t : s \in S \wedge t \in T\} = \{0\}$ there is $a \in A$ such that $s \leq a$ for all $s \in S$ and $t \leq -a$ for all $t \in T$. An ideal I of a Boolean algebra A is *κ-directed* iff every subset of I of size $< \kappa$ has an upper bound in I. □

In particular, every κ-complete Boolean algebra has the κ-s.p. Similarly, every κ-ideal, i.e. every ideal which is closed under sums of less than κ elements, is κ-directed.

1.3.4. Theorem. *Let A be a tightly κ-filtered Boolean algebra. If B and C are Boolean algebras, C has the κ-s.p., $g : C \to B$ is an epimorphism such that the kernel of g is κ-directed, and $f : A \to B$ is a homomorphism, then there is a homomorphism $h : A \to C$ such that $g \circ h = f$.* □

The proof needs

1.3.5. Lemma. *Let A and A' be Boolean algebras such that A' is a simple extension of A, i.e. $A' = A(x)$ for some $x \in A'$. Assume that $A \leq_\kappa A(x)$, B and C are Boolean algebras, C has the κ-s.p., $g : C \to B$ is an epimorphism with κ-directed kernel, $f : A' \to B$ is a homomorphism, and $h : A \to C$ is a homomorphism such that $g \circ h = f \restriction A$. Then there is an extension $h' : A' \to C$ of h such that $g \circ h' = f$, i.e.*



commutes.

Proof. Let $S, T \in [A]^{<\kappa}$ be such that S is cofinal in $A \restriction x$ and T is cofinal in $A \restriction -x$. Fix some $z \in C$ such that $g(z) = f(x)$. Since the kernel of g is κ-directed, there is $i \in g^{-1}(0)$ such that for all $s \in S$, $h(s) \leq z + i$ and for all $t \in T$, $h(t) \leq -z + i$. Note that $\{i, z - i, -z - i\}$ is a partition of unity in C. By the κ-s.p. of C, there is $j \in C$ such that $j \leq i$, $h(s) \cdot i \leq j$ for all $s \in S$, and $h(t) \cdot i \leq -j$ for all $t \in T$. Let $z' := (z - i) + j$. Now it is a straightforward consequence of Sikorski's extension theorem that there is an extension $h' : A' \to C$ of h such that $h'(x) = z'$. Since $A' = A(x)$, this extension is unique. It is easy to see that h' works for the lemma. □

Proof of the theorem. Fix a tight κ-filtration of A and construct h by transfinite induction along this filtration, using Lemma 1.3.5 at the successor stages. □

In particular, this theorem gives that if A has the κ-s.p., $f : A \to B$ is an epimorphism with κ-directed kernel, and B is tightly κ-filtered, then there is an homomorphism $h : B \to A$ such that $f \circ h = \mathrm{id}_B$. h is called a *lifting* for f. Note that h is injective.

1.3.6. Definition. Let $\mathcal{M}$ be the ideal of meager subsets of the Cantor space ${}^\omega 2$ and let $\mathcal{N}$ be the ideal of subsets of ${}^\omega 2$ of measure zero. Here the measure on ${}^\omega 2$ is just the product measure induced by the measure on 2 mapping the singletons to $\frac{1}{2}$. Let $\mathrm{Bor}({}^\omega 2)$ be the σ-algebra of Borel subsets of ${}^\omega 2$ and let $\mathbb{C}(\omega) := \mathrm{Bor}({}^\omega 2)/\mathcal{M}$ and $\mathbb{R}(\omega) := \mathrm{Bor}({}^\omega 2)/\mathcal{N}$. $\mathbb{C}(\omega)$ is the *Cohen algebra* or *category algebra* and $\mathbb{R}(\omega)$ is the *measure algebra* or *random algebra*. Let $p : \mathrm{Bor}({}^\omega 2) \to \mathbb{R}(\omega)$ and $q : \mathrm{Bor}({}^\omega 2) \to \mathbb{C}(\omega)$ be the quotient mappings. A lifting for p is a *Borel lifting for measure* and a lifting for q is a *Borel lifting for category*. □

Using her version of Theorem 1.3.4, Koppelberg gave uniform proofs of several mostly known results about the existence of certain homomorphism into Boolean algebras with the countable separation property. Among other things, she observed that under CH and after adding $\aleph_2$ Cohen reals to a model of CH, $\mathbb{C}(\omega)$ and $\mathbb{R}(\omega)$ are tightly σ-filtered. This implies the existence of Borel liftings for measure and category in the respective models. (See [28].) Originally, the results on Borel liftings in these models were obtained by von Neumann, Stone, Carlson, Frankiewicz, and Zbierski.

One may ask whether the existence of a Borel lifting implies the existence of a tight σ-filtration of the respective algebra. At least for measure, this it not the case. According to Burke ([9]), Veličkovič has shown that after adding $\aleph_2$ random reals to a model of CH, there is a Borel lifting for measure. It will turn out later that in that model $\mathbb{R}$ is not tightly σ-filtered.

I do not know whether tight κ-filteredness can be characterized by some property like the one in Theorem 1.3.4. However, there will be several internal characterizations of tight κ-filteredness in the second chapter.

1.4 The κ-Freese-Nation property

1.4.1. Definition. A partial order $(P, \leq)$ has the *κ-Freese-Nation property* (κ-FN for short) iff there is a function $f : P \to [P]^{<\kappa}$ such that for all $a, b \in P$ with $a \leq b$ there is $c \in f(a) \cap f(b)$ such that $a \leq c \leq b$. f is called a *κ-FN-function* for P. The $\aleph_0$-FN is the original Freese-Nation property (FN), which was used by Freese and Nation to characterize projective lattices ([13]). The $\aleph_1$-FN is called *weak* Freese-Nation property (WFN for short) and was introduced by Heindorf and Shapiro ([23]). WFN(P) denotes the statement 'P has the WFN'. □

It is easily seen that small partial orders have the κ-FN.

1.4.2. Lemma. *([16]) Every partial order P of size $\leq \kappa$ has the κ-FN.* □

By a result by Heindorf ([23]), a Boolean algebra is rc-filtered iff it has the FN. Similarly, in [23] it is proved that for Boolean algebras the WFN is the same as σ-filteredness. Fuchino, Koppelberg, and Shelah ([16]) have shown that for all regular infinite κ a partial order P has the κ-FN iff it is κ-filtered. However, they formulated κ-filteredness in terms of elementary submodels of some H_χ rather than in terms of skeletons. But these two formulation are easily seen to be equivalent.

When dealing with elementary submodels of some H_χ, I will usually assume that χ is 'large enough' or 'sufficiently large'. This simply means that χ is chosen so large that all the objects I am going to consider are contained in H_χ and all the properties of these objects I am going to use are absolute over H_χ. In the context of forcing sometimes a class M is considered which is a model of a 'sufficiently large fragment of ZFC'. This means that M satisfies enough of ZFC to carry out the arguments I am going to use. The problem is that typically, one cannot get suitable set models for all of ZFC. See [32] for these questions. I use ZFC* to abbreviate 'sufficiently large fragment of ZFC'.

The basic observations in order to get the desired characterization of partial orders with the κ-FN are the following:

1.4.3. Lemma. *a)([16]) If f is a κ-FN-function for a partial order P and $Q \subseteq P$ is closed under f, then $Q \leq_\kappa P$.*

b) If Q is a κ-suborder of a partial order P and P has the κ-FN, then Q has the κ-FN, too.

c)([16]) Let δ be a limit ordinal and let $(P_\alpha)_{\alpha\leq\delta}$ be an increasing continuous chain of partial orders such that $P_\alpha \leq_\kappa P_\delta$ for every $\alpha < \delta$. If P_α has the κ-FN for every $\alpha < \delta$, then P_δ has the κ-FN as well.

Proof. Only b) has not been proved in [16]. Let f be a κ-FN-function for P. For each $p \in P$ fix $X_p \in [Q]^{<\kappa}$ such that X_p is cofinal in $Q \downarrow p$. For each $q \in Q$ let $g(q) := \bigcup_{p\in f(q)} X_p$. g is a κ-FN-function for Q: By regularity of κ, $|g(q)| < \kappa$ for every $q \in Q$. Let $q, r \in Q$ be such that $q \leq r$. Now there is $p \in f(q) \cap f(r)$ such that $q \leq p \leq r$. Let $p' \in X_p$ be such that $q \leq p' \leq p$. Now $q \leq p' \leq r$ and $p' \in g(q) \cap g(r)$. □

From Lemma 1.4.3 one can obtain the following charaterization of partial orders with the κ-FN:

1.4.4. Theorem. *(Implicitly in [16]) Let $(P, \leq)$ be a partial order and χ large enough. The following are equivalent:*

(i) P has the κ-FN.

(ii) For every elementary submodel M of H_χ such that $(P, \leq), \kappa \in M$ and $\kappa \subseteq M$, $P \cap M \leq_\kappa P$ holds.

(iii) P is κ-filtered.

Proof. (i)⇒(ii) is proved in [16] for elementary submodels of size κ, but the same argument works here as well. Since M knows that P has the κ-FN, there is $f \in M$ such that $f : P \to [P]^{<\kappa}$ is a κ-FN-function for P. For each $p \in P \cap M$, $f(p) \in M$. Since $|f(p)| < \kappa$ and $\kappa \subseteq M$, $f(p) \subseteq M$. It follows that $P \cap M$ is closed under f. By Lemma 1.4.3, $P \cap M \leq_\kappa P$.

Now assume (ii). Fix a wellorder $\trianglelefteq$ of H_χ. (iii) is then witnessed by

$$\mathcal{S} := \{P \cap M : M \precsim (H_\chi, \trianglelefteq) \wedge (P, \leq), \kappa \in M \wedge \kappa \subseteq M\} :$$

Clearly, every subset X of P is included in some $Q \in \mathcal{S}$ such that $|Q| \leq |X| + \kappa$. By (ii), every $Q \in \mathcal{S}$ is a κ-suborder of P. Let $\mathcal{T} \subseteq \mathcal{S}$ be a chain. Since $\trianglelefteq$ is a wellordering of H_χ, $(H_\chi, \trianglelefteq)$ has definable Skolem functions. For each $Q \in \mathcal{T}$ let M_Q be the Skolem hull of Q in H_χ. By definition, every $Q \in \mathcal{T}$ has the form $P \cap M$ for some elementary submodel M of $(H_\chi, \trianglelefteq)$. Therefore $M_Q \cap P = Q$. It follows that $\{M_Q : Q \in \mathcal{T}\}$ is a chain of elementary submodels of H_χ. Thus $N := \bigcup_{Q \in \mathcal{T}} M_Q \precsim H_\chi$. Therefore $\bigcup \mathcal{T} = P \cap N \in \mathcal{S}$.

For (iii)⇒(i) let $\mathcal{S}$ be a κ-skeleton of P consisting of κ-suborders. Clearly $P \in \mathcal{S}$. Assume that P does not have the κ-FN. Let $Q \in \mathcal{S}$ be of minimal size such that Q does not have the κ-FN. By Lemma 1.4.2, $|Q| > \kappa$. By the properties of $\mathcal{S}$, there is a strictly increasing continuous chain $(Q_\alpha)_{\alpha<|Q|}$ in $\mathcal{S} \cap [P]^{<|Q|}$ such that $Q \subseteq \bigcup_{\alpha<|Q|} Q_\alpha$. By the choice of Q, every Q_α has the κ-FN. By part c) of Lemma 1.4.3, $\bigcup_{\alpha<\lambda} Q_\alpha$ has the κ-FN. This contradicts part b) of Lemma 1.4.3. □

A more advanced version of this theorem has been found by Fuchino and Soukup. In this theorem only very nice submodels of H_χ have to be considered.

1.4.5. Definition. Let χ be a cardinal such that $\kappa < \chi$. $M \precsim H_\chi$ is *V_κ-like* iff $M = \bigcup_{\alpha<\kappa} M_\alpha$ for a continuously increasing chain $(M_\alpha)_{\alpha<\kappa}$ of elementary submodels of M of size $< \kappa$ such that for each $\alpha < \kappa$, $(M_\beta)_{\beta\leq\alpha} \in M_{\alpha+1}$. □

It is easy to see that every subset of H_χ of size κ is a subset of some V_κ-like elementary submodel of H_χ. Fuchino and Soukup proved the following:

1.4.6. Theorem. *([19]) Let P be a partial order and let χ be large enough.*

a) If $0^\sharp$ does not exist, then P has the κ-FN iff for every V_κ-like elementary submodel M of H_χ such that $P \in M$, $P \cap M \leq_\kappa P$.

b) If $|P| < \aleph_\omega$, then P has the κ-FN iff for every V_κ-like elementary submodel M of H_χ such that $P \in M$, $P \cap M \leq_\kappa P$. □

Unfortunately, part a) of this theorem really needs some assumptions on the non-existence of certain large cardinals, as was also shown by Fuchino and Soukup ([19]). In the proof of this theorem, as well as in the proofs of of similar theorems that will be stated later, $\neg 0^\sharp$ is used in the following way:

The proof uses some transfinite induction on cardinals. There occurs a problem at uncountable cardinals of countable cofinality. In order to proceed with the induction at a stage λ with $\mathrm{cf}(\lambda) = \aleph_0$, some weak form of the $\square$-principle as well as some assumption like $\mathrm{cf}([\lambda]^{\aleph_0}) = \lambda^+$ is needed.

The following lemma comes in handy when one wants to find out whether or not certain complete Boolean algebras have the WFN. The κ-FN does not reflect to suborders in general, but to suborders which are retracts.

1.4.7. Definition. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be partial orders. A mapping $e : P \to Q$ is an *order embedding* iff for all $a, b \in P$, $a \leq_P b$ iff $e(a) \leq_Q e(b)$. P is an *order retract* of Q iff there are monotone mappings $e : P \to Q$ and $p : Q \to P$ such that $p \circ e = \mathrm{id}_P$. □

1.4.8. Lemma. *([16]) Let P and Q be partial orders. If P is an order retract of Q and Q has the κ-FN, then P has the κ-FN.* □

If P is order embeddable into Q and sufficiently complete, then P is an order retract of Q.

1.4.9. Corollary. *Let P and Q be partial orders and let $e : P \to Q$ be an order embedding. If Q has the κ-FN and in P every subset has a least upper bound, then P has the κ-FN.*

Proof. For each $q \in Q$ let $p(q) := \sup\{p \in P : e(p) \leq q\}$. $p : Q \to P$ is monotone and $p \circ e = \mathrm{id}_P$. Thus P is an order retract of Q and Lemma 1.4.8 applies. □

Since $\mathfrak{P}(\omega)$ embeds into every infinite complete Boolean algebra, $\mathfrak{P}(\omega)$ has the κ-FN iff any infinite complete Boolean algebra does. The most interesting case seems to be $\kappa = \aleph_1$. Fuchino, Koppelberg, and Shelah ([16]) noticed that $\mathfrak{P}(\aleph_1)$ does not have the WFN, i.e. $\neg\,\mathrm{WFN}(\mathfrak{P}(\aleph_1))$ is provable from ZFC. Therefore, again by the corollary above, no complete Boolean algebra without c.c.c. has the WFN. As mentioned earlier, for every partial order P of size $\aleph_1$, $\mathrm{WFN}(P)$ holds. Thus CH implies $\mathrm{WFN}(\mathfrak{P}(\omega))$. It is possible to enlarge the continuum by adding Cohen reals without destroying $\mathrm{WFN}(\mathfrak{P}(\omega))$. Here adding κ Cohen reals means forcing with $\mathrm{Fn}(\kappa, 2)$. In [16] and [19] the following facts about $\mathrm{WFN}(\mathfrak{P}(\omega))$ were established:

1.4.10. Theorem. *a)([16]) Adding less than $\aleph_\omega$ Cohen reals to a model of CH gives a model of* $\mathrm{WFN}(\mathfrak{P}(\omega))$.

b)([19]) Adding any number of Cohen reals to a model of CH+$\neg 0^\sharp$ gives a model of $\mathrm{WFN}(\mathfrak{P}(\omega))$.

c)([16]) $\mathrm{WFN}(\mathfrak{P}(\omega))$ *implies that the unboundedness number $\mathfrak{b}$ is $\aleph_1$.* □

It follows that the question whether there are any infinite complete Boolean algebras having the WFN cannot be settled in ZFC. It will turn out that the universe must be quite similar to a model obtained by adding Cohen reals to a model of CH if $\mathrm{WFN}(\mathfrak{P}(\omega))$ holds, at least as far as the reals are concerned. Note that the Cohen algebra $\mathbb{C}$ and $\mathfrak{P}(\omega)$ both are retracts of each other. Therefore one of them has WFN iff the other one does. This was noticed by Koppelberg ([28]).

The usual ways of refuting the κ-FN of some partial order P are either showing that P has an order retract which does not have the κ-FN or giving a counter-example to part a) of Lemma 1.4.3. Concerning $\mathrm{WFN}(\mathfrak{P}(\omega))$, I will only use the second method. The following lemma has probably never been stated explicitly, but it should be well-known.

1.4.11. Lemma. *Suppose either that M and N are transitive models of* ZFC^* *and $M \subseteq N$ such that M is a definable class in N, or that $N = V$ and M is an elementary submodel of some H_χ, where χ is a sufficiently large* *cardinal. Then for all* $P, Q \in \{({}^\omega\omega, \leq), ({}^\omega\omega, \leq^*), \mathfrak{P}(\omega), \mathfrak{P}(\omega)/fin\}$,

$$N \models (P \cap M \leq_\sigma P \Leftrightarrow Q \cap M \leq_\sigma Q).$$

Proof. I argue in N. The equivalence for $\mathfrak{P}(\omega)$ and $\mathfrak{P}(\omega)/fin$ follows easily from the fact that fin is a countable subset of M. Similarly, the equivalence holds for $({}^\omega\omega, \leq)$ and $({}^\omega\omega, \leq^*)$ since for each $f : \omega \to \omega$ the set $\{g \in {}^\omega\omega : g =^* f\}$ is a countable subset of M if $f \in M$.

Mapping each $x \subseteq \omega$ to its characteristic function gives an order embedding from $\mathfrak{P}(\omega)$ into $({}^\omega\omega, \leq)$. Since $\mathfrak{P}(\omega)$ is complete, it is an order retract of $({}^\omega\omega, \leq)$. The mappings proving this are elements of M if M is an elementary submodel of H_χ for some large χ. If M is a definable class, then the restrictions of these mappings to M are in M. It is easy to see that this implies

$$({}^\omega\omega \cap M, \leq) \leq_\sigma ({}^\omega\omega, \leq) \Rightarrow \mathfrak{P}(\omega) \cap M \leq_\sigma \mathfrak{P}(\omega).$$

Now suppose $\mathfrak{P}(\omega) \cap M \leq_\sigma \mathfrak{P}(\omega)$ and let $f \in {}^\omega\omega$. Let $x := \{(n,m) \in \omega \times \omega : m \leq f(n)\}$ and let C be an at most countable cofinal subset of $(\mathfrak{P}(\omega \times \omega) \restriction x) \cap M$. For each $c \in C$ and each $n \in \omega$ let $f_c(n) := \max\{m \in \omega : (n,m) \in c\}$. Now for each $c \in C$, $f_c \in M$. $\{f_c : c \in C\}$ is cofinal in $({}^\omega\omega \cap M, \leq) \downarrow f$.

Assume $({}^\omega\omega \cap M, \leq) \uparrow f$ is non-empty. Let D be a countable cofinal subset of $(\mathfrak{P}(\omega \times \omega) \restriction \omega \times \omega \setminus x) \cap M$. I may assume that for all $d \in D$ and all $n \in \omega$ there is some $m \in \omega$ such that $(n,m) \in d$ since there is $g \in {}^\omega\omega \cap M$ such that $f \leq g$ by assumption. For each $d \in D$ and each $n \in \omega$ let $g_d(n) := \min\{m \in \omega : (n,m) \in d\}$. Now for each $d \in D$, $g_d \in M$. $\{g_d : d \in D\}$ is coinitial in $({}^\omega\omega \cap M, \leq) \uparrow f$. □

From this lemma together with Lemma 1.4.4 it follows that WFN($\mathfrak{P}(\omega)$), WFN($\mathfrak{P}(\omega)/fin$), WFN($({}^\omega\omega, \leq)$), and WFN($({}^\omega\omega, \leq^*)$) are equivalent. This was partially observed by Koppelberg in [28].