Topology and Geometry of Deformation Spaces of G-trees

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Sebastian Meinert

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To my parents Rita and Bernd

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Introduction

The subject of this dissertation lies in the fields of group theory and topology. For a finitely generated group G, we are interested in the outer automorphism group

$$\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$$

where $\mathrm{Aut}(G)$ is the group of automorphisms of G and $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$ is the normal subgroup of inner automorphisms. One seeks to exhibit $\mathrm{Out}(G)$ as a group of symmetries of topological and geometric objects, and we will contribute to the theory of deformation spaces of G-trees, which provide natural and rich examples of such.

Motivation

Denote by F_n the free group of rank $n \geq 2$. Much progress has been made in the study of $\mathrm{Out}(F_n)$ by considering its action on *Culler-Vogtmann's Outer space* $\mathcal{PX}_n$ (see [Vog02] for a survey). Outer space is the space of F_n-equivariant isometry classes of free minimal metric F_n-trees (see below) with covolume 1, and $\mathrm{Out}(F_n)$ acts on $\mathcal{PX}_n$ by precomposing the F_n-actions on the metric trees.

Outer space with the action of $\mathrm{Out}(F_n)$ is an analogue of the Teichmüller space of a surface S with the action of the mapping class group $\mathrm{Mod}(S)$. Much of the theory developed in the context of Teichmüller space has been imitated in the setting of Outer space, allowing new insights into the structure of $\mathrm{Out}(F_n)$. In fact, "Outer spaces" can be defined in much greater generality, raising the question of which analogies extend to automorphism groups of more general groups:

Deformation spaces of G-trees Let G be a finitely generated group. A *metric G-tree* is a metric simplicial tree on which G acts by simplicial isometries without inversions of edges. Starting from a metric graph of groups decomposition of G, we obtain a metric G-tree by considering the associated Bass-Serre covering tree (see [Ser80] or [Bas93]). We restrict our attention to G-trees that are *minimal*, i.e., that do not contain a proper G-invariant subtree. To a nontrivial minimal metric G-tree T we associate its *deformation space* $\mathcal{D}$ consisting of the G-equivariant isometry classes of all minimal metric G-trees that have the same *elliptic subgroups* as T, where a subgroup $H \leq G$ is an *elliptic subgroup of T* if it fixes a point in T. Equivalently, a minimal metric G-tree T' lies in $\mathcal{D}$ if there exist G-equivariant (not necessarily simplicial) maps $T \to T'$ and $T' \to T$ (Proposition 1.24). If instead of G-equivariant isometry classes we consider G-equivariant homothety classes of

metric G-trees, we obtain the *projectivized deformation space* $\mathcal{PD} := \mathcal{D}/\mathbb{R}_{>0}$. The projectivized deformation space can naturally be given the structure of a simplicial complex with missing faces that carries the *weak topology* (see Section 1.2).

The subgroup $\mathrm{Aut}_\mathcal{D}(G) \leq \mathrm{Aut}(G)$ of automorphisms that leave the set of elliptic subgroups of T invariant acts on $\mathcal{D}$ and $\mathcal{PD}$ by precomposing the G-actions on the metric trees. Under certain assumptions on $\mathcal{D}$ we have $\mathrm{Aut}_\mathcal{D}(G) = \mathrm{Aut}(G)$ (see, for instance, Proposition 1.34). The inner automorphism group $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}_\mathcal{D}(G)$ acts trivially on $\mathcal{D}$ and $\mathcal{PD}$ and we obtain an induced action of

$$\mathrm{Out}_\mathcal{D}(G) := \mathrm{Aut}_\mathcal{D}(G)/\mathrm{Inn}(G) \leq \mathrm{Out}(G).$$

Statement of results

Our discussion of deformation spaces of metric G-trees is organized into 3 chapters. The central themes and main results in each chapter are the following:

Chapter 1: Topology of deformation spaces of G-trees

After deformation spaces of metric G-trees were introduced by Forester [For02], Clay [Cla05] and Guirardel-Levitt [GL07a] initiated their systematic study by showing that they are, like Outer space, contractible. Their argument involves a technique due to Skora [Sko89] of folding metric G-trees along suitable equivariant maps. Beyond, Guirardel-Levitt showed that certain fixed point sets under the action of $\mathrm{Out}_\mathcal{D}(G)$ on $\mathcal{D}$ and $\mathcal{PD}$ are contractible as well.

Important objects in algebraic topology are *classifying spaces for families of subgroups* (see Definition 1.53). These are CW-complexes with a cellular group action such that all subgroups in a family of subgroups have fixed points, all other subgroups do not, and all nonempty fixed point sets are contractible. They are unique up to equivariant homotopy equivalence.

Indeed, contractibility of certain fixed point sets under the action of $\mathrm{Out}_\mathcal{D}(G)$ on $\mathcal{PD}$ follows from the results mentioned above, but the natural simplicial structure with missing faces on $\mathcal{PD}$ is *a priori* not a genuine CW-structure. After reviewing preliminary notions and thoroughly defining deformation spaces of metric G-trees in Sections 1.1 and 1.2, we will proceed in Section 1.3 by showing that the natural simplicial structure with missing faces on $\mathcal{PD}$ can be refined to a genuine simplicial structure that defines the same weak topology on $\mathcal{PD}$ and with respect to which $\mathrm{Out}_\mathcal{D}(G)$ acts cellularly (Proposition 1.56). We will also give a detailed account of Clay's and Guirardel-Levitt's contractibility results. The main result in Chapter 1 is then the following, where a minimal metric G-tree is *irreducible* if G contains a free subgroup of rank 2 acting freely and a group is *slender* if all of its subgroups are finitely generated:

Theorem 1.63. *Let $\mathcal{PD}$ be a projectivized deformation space of irreducible metric G-trees, equipped with the weak topology. If $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ with slender point stabilizers then $\mathcal{PD}$ is a model for the classifying space of $\mathrm{Out}_{\mathcal{D}}(G)$ for the family of subgroups of isotropy groups.*

If $\mathcal{PD}$ is a model for the classifying space of $\mathrm{Out}_{\mathcal{D}}(G)$ for a family of subgroups $\mathcal{F}$, we say that $\mathcal{PD}$ is a *model for $E(\mathrm{Out}_{\mathcal{D}}(G), \mathcal{F})$* for short.

We obtain the following examples:

Example 1.66. Let G be a finitely generated virtually nonabelian free group, i.e., suppose that G contains a finitely generated nonabelian free subgroup of finite index. Let $\mathcal{PD}$ be the projectivized deformation space of minimal metric G-trees with finite vertex stabilizers (Example 1.30). We have $\mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Out}(G)$ and $\mathcal{PD}$ is a finite-dimensional model for $E(\mathrm{Out}(G), \mathcal{F}in)$, where $\mathcal{F}in$ is the family of finite subgroups.

A *generalized Baumslag-Solitar (GBS) group* is a finitely generated group that acts on a simplicial tree with infinite cyclic vertex and edge stabilizers. Among these groups are the classical *Baumslag-Solitar groups*

$$\mathrm{BS}(p, q) = \langle x, t \mid tx^p t^{-1} = x^q \rangle$$

with $p, q \in \mathbb{Z} \setminus \{0\}$. A GBS group is *nonelementary* if it is not isomorphic to $\mathbb{Z}$, $\mathrm{BS}(1, 1) \cong \mathbb{Z}^2$, or the Klein bottle group $\mathrm{BS}(1, -1)$.

Example 1.67. Let G be a nonelementary GBS group that is not isomorphic to a solvable Baumslag-Solitar group $\mathrm{BS}(1, q)$, $q \neq \pm 1$. Let $\mathcal{PD}$ be the projectivized deformation space of minimal metric G-trees with infinite cyclic vertex and edge stabilizers (Example 1.31). We have $\mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Out}(G)$ and $\mathcal{PD}$ is a model for $E(\mathrm{Out}(G), \mathcal{F})$, where $\mathcal{F}$ is a family of finitely generated virtually free abelian subgroups with bounded rank that is closed under taking finite-index supergroups. If G does not contain a solvable Baumslag-Solitar group $\mathrm{BS}(1, n)$ with $n \geq 2$ then $\mathcal{PD}$ is finite-dimensional.

Remark. In general, however, it is very complicated to describe the family of subgroups of isotropy groups algebraically (see, for instance, [BJ96]).

Chapter 2: The Lipschitz metric on deformation spaces of G-trees

In Chapter 2, we study deformation spaces of metric G-trees from a geometric point of view. The results in this chapter have already been released in [Mei13].

Unlike the topology of deformation spaces, the geometry of deformation spaces of metric G-trees has previously only been addressed in the special case of Outer space [FM11][1], which admits a description as a space of finite marked metric

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[1]And recently also in the case of the Outer space of a free product [FM13].

graphs. Here one studies the asymmetric *Lipschitz metric*, an analogue of the asymmetric Thurston metric on the Teichmüller space of a surface [Thu98]. Our main motivation in Chapter 2 is to introduce an asymmetric pseudometric on projectivized deformation spaces of metric G-trees that generalizes the asymmetric Lipschitz metric on Outer space. For this, we think of the metric G-trees in $\mathcal{PD}$ as their covolume-1-representatives in the unprojectivized deformation space $\mathcal{D}$ and for $T, T' \in \mathcal{PD}$ we define

$$d_{Lip}(T, T') := \log \left(\inf_f \sigma(f) \right)$$

where f ranges over all G-equivariant Lipschitz maps $T \to T'$ and $\sigma(f)$ denotes the Lipschitz constant of f. In general we have $d_{Lip}(T, T') \neq d_{Lip}(T', T)$, and $d_{Lip}(T, T') = 0$ does generally not imply that T and T' are G-equivariantly isometric (see Example 2.4). Nevertheless, the Lipschitz metric turns out to have useful properties. If $\mathcal{PD}$ consists of irreducible metric G-trees then the symmetrized Lipschitz metric

$$d_{Lip}^{sym}(T, T') := d_{Lip}(T, T') + d_{Lip}(T', T)$$

is an actual metric on $\mathcal{PD}$ (Proposition 2.5).

An important feature of the Lipschitz metric on Outer space is that the distance between two marked metric graphs is always realized by a map with minimal Lipschitz constant and that the minimum Lipschitz constant equals the maximum ratio of lengths of immersed loops in the corresponding quotient graphs [FM11, Proposition 3.15]. This reflects a theorem of Thurston that the Lipschitz distance between two hyperbolic surfaces in Teichmüller space is always realized by a minimal stretch map and that the extremal Lipschitz constant equals the maximum ratio of lengths of essential simple closed curves [Thu98, Theorem 8.5]. In the same spirit, we will show the following:

Theorems 2.6 and 2.14. *Let $\mathcal{PD}$ be a projectivized deformation space of irreducible metric G-trees. For all $T, T' \in \mathcal{PD}$ there exists*

(1) a G-equivariant Lipschitz map $f \colon T \to T'$ such that $d_{Lip}(T, T') = \log\left(\sigma(f)\right)$;

(2) a hyperbolic group element $\xi \in G$ such that

$$d_{Lip}(T, T') = \log \left(\frac{l_{T'}(\xi)}{l_T(\xi)} \right) = \log \left(\sup_g \frac{l_{T'}(g)}{l_T(g)} \right)$$

where g ranges over all hyperbolic group elements of G and by $l_T(g)$ we denote the translation length $\inf_{x \in T} d(x, gx)$ *of g in T.*

Francaviglia-Martino [FM11] showed, making use of Skora's folding technique

[Sko89], that the Lipschitz metric on Outer space is geodesic. We will apply the folding construction in the general context to show:

Theorem 2.23. *If $\mathcal{PD}$ is a projectivized deformation space of irreducible metric G-trees then for all $T, T' \in \mathcal{PD}$ there exists a d_{Lip}-geodesic (see Definition 2.21) $\gamma \colon [0,1] \to \mathcal{PD}$ with $\gamma(0) = T$ and $\gamma(1) = T'$.*

An automorphism $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ is *reducible* if there exists a metric G-tree $T \in \mathcal{D}$ and a G-equivariant map $f \colon T \to T\Phi$ that leaves an essential proper G-invariant subforest of T invariant, where a subforest $S \subset T$ is *essential* if it contains the hyperbolic axis of some hyperbolic group element. We say that $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ is *represented by a train track map* if there exists a metric G-tree $T \in \mathcal{D}$ and an extremal G-equivariant Lipschitz map $f \colon T \to T\Phi$ such that, loosely speaking, every iterate of f maps certain immersed paths in T to immersed paths (see Definition 2.35). Bestvina [Bes11] classified free group automorphisms Φ by studying associated displacement functions $T \mapsto d_{Lip}(T, T\Phi)$ on Outer space. By doing so, he gave an alternative proof of Bestvina-Handel's celebrated train track theorem [BH92, Theorem 1.7] that every irreducible automorphism of a free group is represented by a train track map. Generalizing Bestvina's approach, we will study displacement functions on projectivized deformation spaces of metric G-trees to classify automorphisms of more general groups and show:

Theorem 2.37. *Let $\mathcal{PD}$ be a projectivized deformation space of irreducible metric G-trees. If $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices then every irreducible automorphism $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ is represented by a train track map.*

Example 2.38. Let G be a finitely generated virtually nonabelian free group and $\mathcal{D}$ the deformation space of minimal metric G-trees with finite vertex stabilizers. Every irreducible automorphism $\Phi \in \mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Out}(G)$ is represented by a train track map. This generalizes [BH92, Theorem 1.7] to virtually free groups.

Example 2.39. Let G be a nonelementary GBS group that contains no solvable Baumslag-Solitar group $\mathrm{BS}(1,n)$ with $n \geq 2$. Let $\mathcal{D}$ be the deformation space of minimal metric G-trees with infinite cyclic vertex and edge stabilizers. Then every irreducible automorphism $\Phi \in \mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Out}(G)$ is represented by a train track map.

Chapter 3: Higher holomorphs

In Chapter 3, we study *higher holomorphs* of a finitely generated group G, which we define as the semidirect products

$$\mathrm{Aut}(G, k) := G^{k-1} \rtimes \mathrm{Aut}(G), \ k \in \mathbb{N}$$

with multiplication given by

$$((g_2, \ldots, g_k), \phi) \cdot ((h_2, \ldots, h_k), \psi) = ((g_2\phi(h_2), \ldots, g_k\phi(h_k)), \phi \circ \psi).$$

We also define $\mathrm{Aut}(G, 0) := \mathrm{Out}(G)$. The family of holomorphs $\mathrm{Aut}(G, k)$, $k \in \mathbb{N}_0$ forms a natural continuation of the sequence of groups

$$\mathrm{Aut}(G, 0) = \mathrm{Out}(G)$$
$$\mathrm{Aut}(G, 1) = \mathrm{Aut}(G)$$
$$\mathrm{Aut}(G, 2) = \mathrm{Hol}(G)$$

where $\mathrm{Hol}(G)$ is the classical *holomorph* of G (for a discussion on the holomorph of a group, see [Rot95, p. 164 and Example 7.9]).

Higher holomorphs of free groups have played an important role in the study of $\mathrm{Out}(F_n)$. As remarked in Example 3.1, they appear in the construction of a bordification of Outer space [BF00], they were used to prove homological stability of $\mathrm{Out}(F_n)$ [HV04], and they are building blocks for point stabilizers in the boundary of the free splitting complex [HM13b]. With regard to this, it seems worthwhile to study higher holomorphs in a more general context. We will take a step in this direction by constructing "higher spines" for higher holomorphs:

Let $\mathcal{PD}$ be a projectivized deformation space of metric G-trees. Since $\mathcal{PD}$ has the structure of a simplicial complex with missing faces, even if the group $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices, it does not act cocompactly. However, we will explain in Section 1.2.3 that $\mathcal{PD}$ deformation retracts onto its $\mathrm{Out}_{\mathcal{D}}(G)$-invariant *spine* $\mathcal{S}(\mathcal{PD}) \subset \mathcal{PD}$, which is a genuine simplicial complex. If the group $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices then it acts on the spine $\mathcal{S}(\mathcal{PD})$ cocompactly. Since $\mathcal{PD}$ is contractible (Theorem 1.57), the spine $\mathcal{S}(\mathcal{PD})$ is contractible as well.

Following the construction of the spine $\mathcal{S}(\mathcal{PD})$, we will construct simplicial complexes $\mathcal{S}(\mathcal{PD}, k)$, $k \in \mathbb{N}_0$ ("higher spines") such that $\mathcal{S}(\mathcal{PD}, 0) = \mathcal{S}(\mathcal{PD})$ and for $k \in \mathbb{N}_0$ the subgroup

$$\mathrm{Aut}_{\mathcal{D}}(G, k) := \begin{cases} \mathrm{Out}_{\mathcal{D}}(G) & \text{if } k = 0 \\ G^{k-1} \rtimes \mathrm{Aut}_{\mathcal{D}}(G) & \text{if } k \geq 1 \end{cases}$$

of $\mathrm{Aut}(G, k)$ acts on $\mathcal{S}(\mathcal{PD}, k)$ by simplicial automorphisms. Concretely, we will define $\mathcal{S}(\mathcal{PD}, k)$ as the geometric realization of a poset $\mathcal{C}ol(\mathcal{PD}, k)$ whose elements are equivalence classes of metric G-trees $T \in \mathcal{PD}$ with k basepoints $x_1, \ldots, x_k \in T$ and where the partial order is given by a forest collapse relation.

For $k \in \mathbb{N}$, the map $f_k \colon \mathcal{C}ol(\mathcal{PD}, k) \to \mathcal{C}ol(\mathcal{PD}, k-1)$ that sends a k-pointed G-tree $(T, x_1, \ldots, x_k)$ to the $(k-1)$-pointed G-tree $(T, x_1, \ldots, x_{k-1})$ preserves the

partial order and induces a continuous map $|f_k| \colon \mathcal{S}(\mathcal{PD}, k) \to \mathcal{S}(\mathcal{PD}, k-1)$ on geometric realizations. As our main result in Chapter 3, we will show:

Theorem 3.8. *For all $k \in \mathbb{N}$, the map $|f_k| \colon \mathcal{S}(\mathcal{PD}, k) \to \mathcal{S}(\mathcal{PD}, k-1)$ is a homotopy equivalence.*

Since the 0-th spine $\mathcal{S}(\mathcal{PD}, 0) = \mathcal{S}(\mathcal{PD})$ is contractible, we conclude:

Corollary 3.9. *For all $k \in \mathbb{N}_0$, the k-th spine $\mathcal{S}(\mathcal{PD}, k)$ is contractible.*

Chapter 1

Topology of deformation spaces of G-trees

1.1 G-trees

A *simplicial tree* is a contractible 1-dimensional simplicial complex T. We denote by $V(T)$ the set of vertices and by $E(T)$ the set of edges of T. It is sometimes convenient to endow each edge $e \in E(T)$ with an orientation, and we denote by $\iota(e)$ the initial vertex and by $\tau(e)$ the terminal vertex of e. A *metric simplicial tree* is a simplicial tree together with a positive length assigned to every edge. Every simplicial tree can be viewed as a metric simplicial tree by assigning length 1 to all of its edges. Every metric simplicial tree T carries a natural path metric $d = d_T$ and we equip T with the metric topology, which is generally coarser (i.e., has fewer open sets) than the CW-topology; the two topologies agree if and only if T is locally finite [Chi01, Lemma 2.2.6]. If T is a metric simplicial tree, any two points $x, y \in T$ are joined by a unique compact geodesic segment $[x, y] \subseteq T$ and between any two disjoint closed connected subsets $A, B \subset T$ there exists a unique compact connecting segment $[a, b] \subseteq T$ such that $A \cap [a, b] = a$ and $B \cap [a, b] = b$. In particular, T is a simplicial $\mathbb{R}$-tree (see [Chi01] for an introduction to $\mathbb{R}$-trees) and, in fact, every simplicial $\mathbb{R}$-tree arises this way [Chi01, Theorem 2.2.10].

Let G be a finitely generated group.

Definition 1.1. A *G-tree* is a simplicial tree on which G acts by simplicial automorphisms without inversions of edges. A *metric G-tree* is a metric simplicial tree on which G acts by simplicial isometries without inversions of edges.

Every metric G-tree has an underlying G-tree, and every G-tree can be viewed as a metric G-tree by assigning length 1 to all of its edges. We will sometimes speak of metric G-trees just as "G-trees" when it is understood that the trees carry metrics or when the metric is irrelevant. Unless mentioned otherwise, we will always assume that every G-tree T comes equipped with a *natural simplicial structure* that is not a subdivision of a coarser simplicial structure with respect to which the action of G on T would still be simplicial and without inversions of edges (i.e., T has no *redundant vertices*). For a vertex or edge $x \in V(T) \cup E(T)$, we denote by $G_x \leq G$ its stabilizer.

Bass-Serre theory gives a correspondence between (metric) G-trees and (metric) graph of groups decompositions of G; see [Ser80] or [Bas93].

Definition 1.2. Let T be a G-tree. A group element $g \in G$ is *elliptic in T* if it fixes a point in T and *hyperbolic* if not. Likewise, a subgroup $H \leq G$ is *elliptic in T* (or an *elliptic subgroup of T*) if it fixes a point in T.

The finite-order elements of G are always elliptic [Ser80, Proposition 19]. A finitely generated subgroup $H \leq G$ is elliptic in T if and only if every element of H is elliptic in T [Ser80, Corollary 6.5.3]. For instance, the finite subgroups of G are elliptic in every G-tree. However, an infinitely generated subgroup of G all of whose elements are elliptic need not be elliptic; it then fixes a unique end of T [Tit70, Proposition 3.4].

Translation lengths   The following are well-known facts about translation lengths in metric G-trees; for details see [CM87] or [Pau89].

Definition 1.3. Let (T, d) be a metric G-tree. For a group element $g \in G$, define the *translation length* of g in T by

$$l(g) = l_T(g) := \inf_{x \in T} d(x, gx) \in \mathbb{R}_{\geq 0}$$

and its *characteristic set* in T by

$$C_g = C_T(g) := \{x \in T \mid d(x, gx) = l_T(g)\} \subseteq T.$$

Conjugate group elements have the same translation length. The characteristic set C_g is always nonempty (i.e., metric trees admit no parabolic isometries) and g-invariant. The translation length function $l_T : G \to \mathbb{R}$ defines a point in $\mathbb{R}^{\mathcal{C}(G)}$, where $\mathcal{C}(G)$ denotes the set of conjugacy classes of G. If two metric G-trees are G-equivariantly isometric then their translation length functions agree (for a partial converse see Proposition 1.6). If T has finitely many G-orbits of edges then its translation length function has discrete image in $\mathbb{R}$.

A group element $g \in G$ is elliptic in T if and only if it has $l(g) = 0$. Its characteristic set is then its fixed point set and for all $x \in T$ the midpoint of the segment $[x, gx]$ is fixed by g. A group element $g \in G$ is hyperbolic in T if and only if it has $l(g) > 0$. Its characteristic set is then isometric to $\mathbb{R}$, the group element g acts on C_g by translations of length $l(g)$, and for all $k \in \mathbb{Z} \setminus \{0\}$ we have $l(g^k) = |k| \cdot l(g)$ and $C_{g^k} = C_g$. The characteristic set of a hyperbolic group element g is the unique g-invariant line in T. We will denote it by A_g instead of C_g and call it the *hyperbolic axis* of g.

Proposition 1.4 ([CM87, 1.3] and [Pau89, Propositions 1.6 and 1.8])**.** *Let T be a metric G-tree and $g, h \in G$.*

(1) *For all $x \in T$ we have $d(x, gx) = l(g) + 2d(x, C_g)$.*

(2) *Suppose that g and h are elliptic. Then $l(gh) = 2d(C_g, C_h)$. In particular, if the fixed point sets of g and h are disjoint then gh and hg are hyperbolic.*

(3) *Suppose that g and h are hyperbolic. If $A_g \cap A_h = \emptyset$ then*

$$l(gh) = l(hg) = l(g) + l(h) + 2d(A_g, A_h)$$

and, in particular, gh and hg are hyperbolic. The hyperbolic axes of gh and hg then both intersect each A_g and A_h.

1.1.1 Minimal G-trees

A G-tree is *minimal* if it does not contain a proper G-invariant subtree. Minimal G-trees are cocompact, i.e., their quotient graphs by the action of G are finite (see [Bas93, Proposition 7.9]), and G-equivariant maps between minimal G-trees are always surjective; both properties will be used frequently and without further notice. The *covolume* of a minimal metric G-tree T, denoted by $\mathrm{covol}(T)$, is the volume of the finite metric quotient graph $G\backslash T$. Every G-tree without a global fixed point contains a unique nontrivial minimal G-invariant subtree, given by the union of all hyperbolic axes [CM87, Proposition 3.1]. In the following, all minimal G-trees will be assumed to be nontrivial. In particular, the group G will always be infinite.

Definition 1.5. There are four types of nontrivial minimal G-trees (we adopt the naming convention from [GL07a]; see [CM87] for equivalent characterizations):
A nontrivial minimal G-tree T is

- *dihedral* if it is a line and the action of G does not preserve the orientation. The quotient graph $G\backslash T$ is then a single edge e with $\iota(e) \neq \tau(e)$ and the action factors through an action of the infinite dihedral group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

- *linear abelian* if it is a line and the action of G is by translations. The quotient graph $G\backslash T$ is then a single edge e with $\iota(e) = \tau(e)$ and the action factors through an action of $\mathbb{Z}$.

- *genuine abelian* if G fixes an end of T (i.e., an equivalence class of rays in T, where two rays are equivalent if their intersection is again a ray) and T is not a line. The quotient graph $G\backslash T$ is then homeomorphic to a circle.

- *irreducible* if G contains a free subgroup of rank 2 acting freely on T.

If two metric G-trees are G-equivariantly isometric then they have the same translation length function. As for the converse, we have the following:

Proposition 1.6 ([CM87, Theorem 3.7]). *Let T and T' be two minimal metric G-trees that are not genuine abelian. If for all $g \in G$ we have $l_T(g) = l_{T'}(g)$ then T and T' are G-equivariantly isometric. If T and T' are not linear abelian then the G-equivariant isometry is unique.*

Remark. Genuine abelian minimal metric G-trees, however, are not determined by their translation length functions. For instance, consider the graph of groups decompositions Γ and Γ' of $G = \mathrm{BS}(1,6) = \langle x, t \mid txt^{-1} = x^6 \rangle$ shown in Figure 1.1, where all edge group inclusions are by multiplication as suggested. If we give the edges of Γ and Γ' positive lengths such that Γ and Γ' have the same volume, the corresponding Bass-Serre trees T and T' have the same translation length function. However, T and T' are not homeomorphic and *a fortiori* not G-equivariantly isometric, as all vertices of T have valence 7, whereas the vertices of T' have valence 3 and 4.
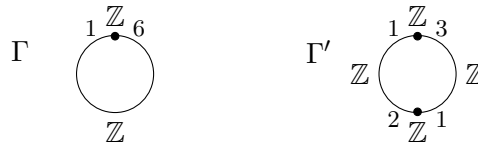

Figure 1.1: The Bass-Serre covering trees of Γ and Γ' have the same translation length function but are not G-equivariantly isometric.

Maps between minimal G-trees   We equip metric G-trees with the metric topology, but maps between them are sometimes more easily seen to be continuous in the CW-topology. Because the two topologies agree on finite subtrees, continuity in the one topology often relates with continuity in the other:

Proposition 1.7. *Let T and T' be minimal metric G-trees. A G-equivariant map $f \colon T \to T'$ that is continuous in the CW-topologies on T and T' is also continuous in the metric topologies on T and T'.*

Proof. Because f is continuous in the CW-topologies, for every closed edge $\bar{e}$ of T the image $f(\bar{e}) \subseteq T'$ lies in a finite subtree of T'. Thus, the restriction of f to any closed edge of T is continuous also in the metric topologies. Metric continuity of the G-equivariant map f at a branch point $v \in V(T)$ follows from this and the fact that T is minimal and hence the edges adjacent to v fall into finitely many G-orbits. $\qquad\square$

As for the converse, we have the following:

Proposition 1.8. *Let T and T' be minimal metric G-trees and let $f\colon T \to T'$ be a G-equivariant map that is continuous in the metric topologies on T and T'. If for every closed edge $\bar{e}$ of T the image $f(\bar{e}) \subseteq T'$ meets only finitely many edges of T' then f is also continuous in the CW-topologies on T and T'.*

If $f(\bar{e})$ meets infinitely many edges of T' then f must have "backtracks". It is then G-equivariantly homotopic relative to the vertices of T to a G-equivariant map $f'\colon T \to T'$ that is continuous in the CW-topologies on T and T'.

Proof. The map f is continuous in the CW-topologies if its restriction to every closed edge is continuous. Since $f(\bar{e})$ lies in a finite subtree of T', the two topologies agree on $\bar{e}$ and its image $f(\bar{e})$, and continuity of f in the CW-topologies follows. $\quad\square$

Corollary 1.9. *If T and T' are minimal metric G-trees then a G-equivariant map $f\colon T \to T'$ is a homeomorphism in the metric topologies on T and T' if and only if it is a homeomorphism in the CW-topologies on T and T'.*

Remark. The statement is wrong for general metric simplicial trees. For instance, let T and T' be the metric trees given by the one-point unions $T = \bigvee_{n \in \mathbb{N}} [0, 1]$ and $T' = \bigvee_{n \in \mathbb{N}} [0, \frac{1}{n}]$, where the basepoint of each interval is 0 and the length of each interval is its Euclidean length. Then T and T' are homeomorphic in the CW-topologies on T and T' but not in the metric topologies.

A G-equivariant homeomorphism between minimal G-trees $T \to T'$ is *simplicial* if it maps each edge to an edge, isometrically with respect to the natural metrics on T and T' that assign to all edges length 1. Every G-equivariant isometry between minimal metric G-trees that maps each edge to an edge is simplicial. Every G-equivariant homeomorphism between minimal G-trees $T \to T'$ that maps each edge to an edge is G-equivariantly isotopic relative to the vertices of T to a simplicial homeomorphism. On a dihedral, genuine abelian, or irreducible minimal G-tree T (which has reflection points and/or branch points), there is a prescribed simplicial structure coming from the topology of T and the action of G on T. Thus, a G-equivariant homeomorphism between dihedral, genuine abelian, or irreducible minimal G-trees with no redundant vertices always maps each edge to an edge.

Proposition 1.10. *Let T be a dihedral, genuine abelian, or irreducible minimal G-tree. If $f\colon T \to T$ is a G-equivariant simplicial automorphism then $f = id_T$.*

Equivalently, a G-equivariant simplicial homeomorphism between two minimal G-trees that are not linear abelian is always unique. The statement is clearly wrong for linear abelian minimal G-trees, as, for instance, the universal cover of the circle has many $\mathbb{Z}$-equivariant automorphisms given by translations.

Proof. We will first show that if T is a general minimal G-tree and $f \colon T \to T$ a G-equivariant simplicial automorphism then $f = id_T$ if f has a fixed point. For this, let $x \in T$ be a fixed point of f. By the G-equivariance of f, for all $g \in G$ we have $f(gx) = gf(x) = gx$. If we let $T_x \subseteq T$ be the subtree spanned by the G-orbit of x then T_x is G-invariant and f is the identity on T_x, as it fixes all spanning points gx, $g \in G$. Since T is assumed minimal, we may conclude that the G-invariant subtree $T_x \subseteq T$ agrees with T, whence $f = id_T$.

In order to prove the claim, it now suffices to show that f has a fixed point. First assume that T is dihedral. Any group element $g \in G$ that acts as reflection on T fixes a single point $p \in T$, and every G-equivariant automorphism of T must map p to itself.

Assume now that T is genuine abelian. Since G fixes an end of T, the intersection of the axes of any two hyperbolic elements is unbounded [CM87, Theorem 2.2]. By minimality and since T is not a line, there exist two hyperbolic elements $g, h \in G$ whose axes A_g and A_h are not equal and hence intersect in a ray. The G-equivariant automorphism f leaves both A_g and A_h invariant and restricts to an automorphism of $A_g \cap A_h$ that leaves the initial point of the ray fixed.

If T is irreducible, there exist two hyperbolic elements $g, h \in G$ whose axes A_g and A_h intersect in a nonempty compact segment [CM87, Theorem 2.7]. The G-equivariant automorphism f restricts to an automorphism of the finite subtree $A_g \cap A_h$ and thus has a fixed point by [Ser80, Corollary to Proposition 10]. $\square$

Corollary 1.11. *Let T be a minimal G-tree. Two metric G-trees (T, d_1) and (T, d_2) with underlying G-tree T are G-equivariantly isometric if and only if we have $d_1 = d_2$.*

Proof. If T is a minimal G-tree without redundant vertices that is not linear abelian then any G-equivariant isometry $(T, d_1) \to (T, d_2)$ is simplicial and the claim follows from Proposition 1.10. If T is linear abelian, all edges of T have the same length and the translation length functions of (T, d_1) and (T, d_2) only agree if $d_1 = d_2$ (this argument also applies in the dihedral case). $\square$

1.1.2 Forest collapses

Definition 1.12. If T is a minimal G-tree and $A \subseteq T$ a G-invariant simplicial subforest, we denote by T_A the minimal G-tree obtained from T by collapsing each connected component of A to a point. We say that T_A is obtained from T by a *forest collapse* and we let

$$k_A \colon T \to T_A$$

be the natural projection.

We may always assume that A has no trivial components, as collapsing each connected component of a G-orbit of vertices has no effect on the G-tree.

If $A \subset T$ is a proper G-invariant subforest then T_A is nontrivial. Moreover, the minimal G-trees T and T_A then have the same type in the sense of Definition 1.5, which can be seen as follows: If T is dihedral or linear abelian then T has only one G-orbit of edges and the only proper G-invariant subforest with no trivial components is the empty one, whence $T_A = T$. If T is genuine abelian then any ray representing the fixed end contains infinitely many representatives of each G-orbit of edges of T and the fixed end remains a fixed end after collapsing A. At the same time, there exist hyperbolic axes that intersect in a ray before and after collapsing A, whence T_A is not a line. If T is irreducible, this is [GL10, Lemma 3.18].

Proposition 1.13. *Let T be a minimal G-tree and $A, B \subseteq T$ two G-invariant subforests with no trivial components. The G-trees T_A and T_B are G-equivariantly homeomorphic if and only if $A = B$.*

This result generalizes [SV87, Lemma 1.3], [Cla09, Lemma 1.8], and [HM13a, Lemma 1.3] to arbitrary minimal G-trees.

Proof. The "if" direction is trivial. As for the "only if" direction, a linear abelian or dihedral minimal G-tree with no redundant vertices has a single G-orbit of edges and the only G-invariant subforests with no trivial components are the empty one and the whole G-tree. In the genuine abelian or irreducible case, suppose that $A \neq B$ and let $e \in E(T)$ be an edge that is contained in, say, A but not in B. We will show that T_A and T_B are not G-equivariantly homeomorphic, whence the claim.

Assume first that T is genuine abelian. Since a nontrivial minimal G-tree is exhausted by its hyperbolic axes [CM87, Proposition 3.1], the edge e is contained in the axis of some hyperbolic group element $g \in G$. Both the initial and terminal vertex of e must have valence at least 3, as T has no redundant vertices and G fixes an end of T (so that there cannot exist reflection points of valence 2). There exists a ray $R \subset T$ such that for all hyperbolic group elements $\xi \in G$ the intersection $A_\xi \cap R$ contains a subray of R [CM87, Theorem 2.2]. We can thus find hyperbolic group elements $h_1, h_2 \in G$ such that the hyperbolic axes of h_1, h_2, and g intersect as in Figure 1.2. On the one hand, the edge $k_B(e) \in E(T_B)$ lies in the characteristic sets $C_{T_B}(g)$ and $C_{T_B}(h_2)$ but not in $C_{T_B}(h_1)$. On the other hand, the characteristic sets $C_{T_A}(g)$ and $C_{T_A}(h_2)$ do not share an edge that does not lie in $C_{T_A}(h_1)$, and we conclude that T_A and T_B are not G-equivariantly homeomorphic.

Suppose now that T is irreducible; our arguments will be similar to those in the proof of [GL10, Lemma 3.18]. Suppose for a moment that the initial vertex of e
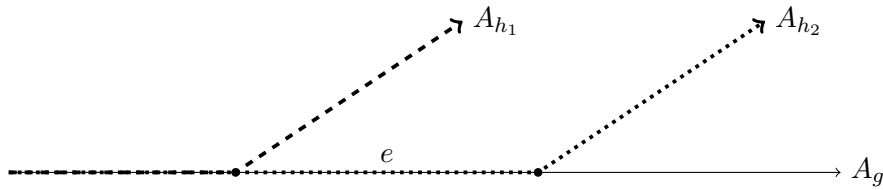
Figure 1.2: Hyperbolic axis in the genuine abelian case.

has valence 2. Since T has no redundant vertices, the second edge e' adjacent to $\iota(e)$ must lie in the same G-orbit as e. We denote by S the closure of $e \cup e'$ in T. Both endpoints of S must have valence at least 3, as T would otherwise have redundant vertices or be a line (this can be most easily seen from the quotient graph of groups of T). If the initial and terminal vertex of e both have valence at least 3, we let S be the closure of e in T. In either case, the segment S is collapsed in T_A but not in T_B. In an irreducible minimal G-tree, every compact segment is contained in the axis of a hyperbolic group element [Pau89, Lemma 4.3]. Therefore, we can find hyperbolic group elements $g_1, g_2 \in G$ whose hyperbolic axes lie as in Figure 1.3. Observe that the characteristic sets $C_{T_B}(g_1)$ and $C_{T_B}(g_2)$ in T_B
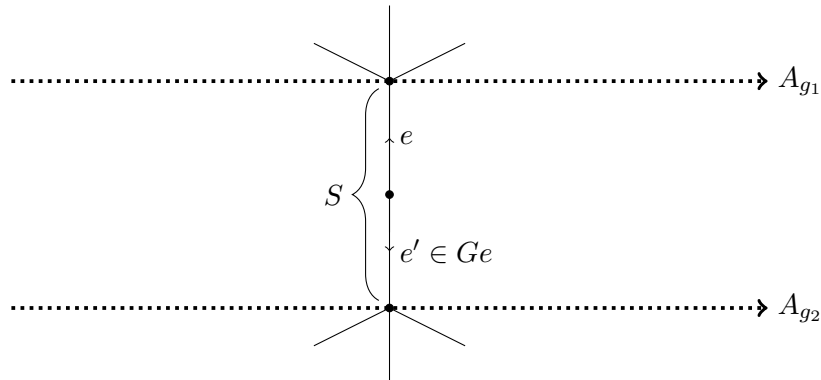
Figure 1.3: Hyperbolic axes in the irreducible case when $\iota(e)$ has valence 2.

are disjoint, whereas the characteristic sets $C_{T_A}(g_1)$ and $C_{T_A}(g_2)$ in T_A intersect. Hence, T_A and T_B are not G-equivariantly homeomorphic. $\qquad\square$

Elementary collapses If T is a minimal G-tree and $A \subseteq T$ a G-invariant subforest then the elliptic subgroups of T are also elliptic in T_A. Conversely:

Definition 1.14. We say that a G-invariant subforest $A \subset T$ is *collapsible* and that the corresponding forest collapse $k_A \colon T \to T_A$ is *elementary* if T and T_A have in fact the same elliptic subgroups.

If T is nontrivial then a collapsible G-invariant subforest $A \subset T$ is necessarily a proper subforest. Consequently, if T is a dihedral or linear abelian minimal G-tree then the only collapsible G-invariant subforest with no trivial components is the empty one.

The G-orbit of a single edge $e \in E(T)$ is collapsible if and only if $\iota(e)$ and $\tau(e)$ lie in distinct G-orbits and either $G_e = G_{\iota(e)}$ or $G_e = G_{\tau(e)}$. If T is a genuine abelian minimal G-tree with at least two G-orbits of edges then it is easily seen from the quotient graph of groups of T that every G-orbit of edges of T is collapsible.

Collapsing each connected component of a collapsible G-orbit of edges is called an *elementary collapse* for short. A G-tree is *reduced* if it admits no elementary collapses. Every minimal G-tree can be made reduced by performing finitely many elementary collapses. An *elementary expansion* is the reverse of an elementary collapse. A finite sequence of elementary collapses and expansions is an *elementary deformation*. Two minimal G-trees T and T' are *related by an elementary deformation* if T' is G-equivariantly homeomorphic to a G-tree that is obtained from T by an elementary deformation.

Theorem 1.15 ([For02, Theorem 4.2])**.** *Two minimal G-trees T and T' are related by an elementary deformation if and only if they have the same elliptic subgroups.*

1.2 The universe of metric G-trees

Let $\mathcal{T} = \mathcal{T}(G)$ be the set of G-equivariant isometry classes of nontrivial minimal metric G-trees. Whenever we speak of a "metric G-tree" $T \in \mathcal{T}$, we mean its G-equivariant isometry class. We call $\mathcal{T}$ the *universe of metric G-trees* and, as in [GL07a, Section 5], we consider three topologies on $\mathcal{T}$:

Axes topology The *axes topology* is the coarsest topology on $\mathcal{T}$ that makes the assignment of translation length functions $\mathcal{T} \to \mathbb{R}^{\mathcal{C}(G)}$, $T \mapsto l_T$ continuous.

A sequence of metric G-trees $(T_k)_{k \in \mathbb{N}}$ in $\mathcal{T}$ converges to $T \in \mathcal{T}$ in the axes topology if and only if for all $g \in G$ we have $\lim_{k \to \infty} l_{T_k}(g) = l_T(g)$ (pointwise convergence of translation length functions).

If we denote by $\mathcal{T}_{irr} \subset \mathcal{T}$ the subset of irreducible metric G-trees, the assignment of length functions $l \colon \mathcal{T}_{irr} \to \mathbb{R}^{\mathcal{C}(G)}$, $T \mapsto l_T$ is injective by [CM87, Theorem 3.7] and the axes topology on $\mathcal{T}_{irr} \subset \mathcal{T}$ agrees with the subspace topology defined by this inclusion.

Gromov topology   Let $T \in \mathcal{T}$. A neighborhood basis for T in the *Gromov topology* is given by subsets $V_T(X, A, \varepsilon) \subset \mathcal{T}$ defined as follows: Let $X \subset T$ and $A \subset G$ be finite subsets and $\varepsilon > 0$. A metric G-tree $T' \in \mathcal{T}$ lies in $V_T(X, A, \varepsilon)$ if there exists a map $X \to T'$, $x \mapsto \tilde{x}$ such that

(1) if $x, y \in X$ satisfy $y = gx$ for some $g \in A$ then $\tilde{y} = g\tilde{x}$;

(2) for all $x, y \in X$ and $g \in A \cup \{1\}$ we have $|d(x, gy) - d'(\tilde{x}, g\tilde{y})| < \varepsilon$.

By [Pau89], the Gromov topology and the axes topology agree on the subset of irreducible metric G-trees $\mathcal{T}_{irr} \subset \mathcal{T}$.

Weak topology   The *weak topology* describes $\mathcal{T}$ as a union of open cones:

Definition 1.16. The *open cone* $C(T) \subseteq \mathcal{T}$ spanned by a metric G-tree $T \in \mathcal{T}$ is the set of metric G-trees $T' \in \mathcal{T}$ that are G-equivariantly homeomorphic to T. Equivalently, a metric G-tree $T' \in \mathcal{T}$ lies in $C(T)$ if it is represented by a metric G-tree with underlying G-tree T. If such a representative exists, it is unique by Corollary 1.11. The *closed cone* $\overline{C}(T) \subseteq \mathcal{T}$ spanned by T is the union of open cones $\bigcup_{A \subset T} C(T_A)$, where A ranges over all proper G-invariant subforests of T and T_A denotes the metric G-tree obtained from T by collapsing A (see Section 1.1.2).

Since minimal G-trees are cocompact, T has only finitely many G-orbits of edges, say $\{[e_1], \ldots, [e_k]\}$. A metric G-tree structure (T, d) on T is determined by the finitely many edge lengths $\text{length}_{(T,d)}(e_i) > 0$, $i = 1, \ldots, k$. If we allow that $\text{length}_{(T,d)}(e) = 0$ for all edges in some proper G-invariant subforest $A \subset T$, we can naturally view the pseudometric d on T as a metric on T_A.

Let

$$\overline{C}_{\mathbb{E}}^k = \{(l_1, \ldots, l_k) \mid l_i \geq 0, \; i = 1, \ldots, k\} \setminus \{0, \ldots, 0\}$$

be the punctured first orthant in $\mathbb{R}^k$. Given a k-tuple of nonnegative edge lengths $(l_1, \ldots, l_k) \in \overline{C}_{\mathbb{E}}^k$, we denote by $d(l_1, \ldots, l_k)$ the pseudometric on T that assigns to the G-orbit of edges $[e_i]$ length l_i, where $i = 1, \ldots, k$. The following result is often used tacitly in the literature:

Proposition 1.17. *The natural surjection*

$$h \colon \overline{C}_{\mathbb{E}}^k \to \overline{C}(T) \subseteq \mathcal{T}$$
$$(l_1, \ldots, l_k) \mapsto (T, d(l_1, \ldots, l_k))$$

is injective.

Proof. It follows from Proposition 1.13 that if two points in $\overline{C}_{\mathbb{E}}^k$ lie in different open faces then their images under h lie in different open cones and hence are

distinct. Points in the same open face of $\overline{C}_{\mathbb{E}}^k$ get mapped to metric G-trees with the same underlying G-tree, in which case we may deduce injectivity of h from Corollary 1.11. $\qquad\square$

We push the subspace topology on $\overline{C}_{\mathbb{E}}^k \subset \mathbb{R}^k$ along this bijection to topologize $\overline{C}(T)$. Note that the coordinate map h itself depends on the choice of representative of the G-equivariant homeomorphism class of T and on the order of its G-orbits of edges $\{[e_1], \ldots, [e_k]\}$; however, the topology on $\overline{C}(T)$ does not depend on either of these choices:

Proposition 1.18. *The coordinates on $\overline{C}(T)$ are well-defined up to permutation.*

Proof. Let $T, T' \in \mathcal{T}$ be G-equivariantly homeomorphic G-trees with G-orbits of edges $\{[e_1], \ldots, [e_k]\}$ and $\{[e_1'], \ldots, [e_k']\}$ respectively. Given a G-equivariant homeomorphism $f \colon T \to T'$ (which we may choose to be simplicial), for some permutation $\pi \in S_k$ we have $f([e_i]) = [e_{\pi(i)}']$, $i = 1, \ldots, k$. Consider the permutation of coordinates

$$\pi \colon \overline{C}_{\mathbb{E}}^k \to \overline{C}_{\mathbb{E}}^k, \ (l_1, \ldots, l_k) \mapsto (l_{\pi(1)}, \ldots, l_{\pi(k)})$$

and the change of representatives

$$\overline{C}(T) \xrightarrow{id} \overline{C}(T'), \ (T, d) \mapsto (T', d_f)$$

where d_f is the unique metric on T' such that the G-equivariant homeomorphism $f \colon (T, d) \to (T', d_f)$ becomes an isometry, i.e., we define

$$\text{length}_{(T', d_f)}([e_i']) := \text{length}_{(T,d)}([f^{-1}(e_i')]) = \text{length}_{(T,d)}([e_{\pi^{-1}(i)}]).$$

With these definitions, the following diagram commutes:

$$
\begin{array}{ccc}
\overline{C}_{\mathbb{E}}^k & \xrightarrow{\ h_T\ } & \overline{C}(T) \\
{\scriptstyle \pi}\uparrow & & \downarrow{\scriptstyle id} \\
\overline{C}_{\mathbb{E}}^k & \xrightarrow[\ h_{T'}\ ]{} & \overline{C}(T')
\end{array}
$$

$\qquad\square$

Finally, we define a subset of $\mathcal{T}$ to be closed in the weak topology if its intersection with every closed cone is closed.

Proposition 1.19. *The weak topology on $\mathcal{T}$ is finer than the Gromov topology, which is finer than the axes topology.*

Thus, a weakly converging sequence in $\mathcal{T}$ also converges in the Gromov topology and *a fortiori* in the axes topology.

Proof. The Gromov topology on $\mathcal{T}$ is finer than the axes topology by [Pau89, Theorem 4.2] (although the theorem is stated only for the subset of irreducible metric G-trees, the proof does not make use of this assumption; the converse [Pau89, Theorem 4.4], however, is only valid under the irreducibility assumption).

In order to prove that the weak topology on $\mathcal{T}$ is finer than the Gromov topology, it suffices to show that every sequence $(T_k)_{k\in\mathbb{N}}$ in $\mathcal{T}$ that converges in the weak topology, say to $T \in \mathcal{T}$, also converges in the Gromov topology (because the weak topology is first-countable and therefore sequential). For this, fix an open neighborhood $V_T(X, A, \varepsilon)$ of T in the Gromov topology. We will show that for large $k \in \mathbb{N}$ we have $T_k \in V_T(X, A, \varepsilon)$.

Since the sequence $(T_k)_{k\in\mathbb{N}}$ converges in the weak topology, it meets only finitely many open simplices of $\mathcal{T}$ and we may in fact assume that it meets each of the finitely many open simplices infinitely often. After decomposing the sequence into subsequences (for each of which we will obtain the same result), we may assume that the metric G-trees $(T_k)_{k\in\mathbb{N}}$ are in fact all G-equivariantly homeomorphic, or even equal as nonmetric G-trees. The underlying G-tree T_k is then for all $k \in \mathbb{N}$ either equal to T or obtained from T by a fixed sequence of elementary expansions, and the condition that $(T_k)_{k\in\mathbb{N}}$ converges to T in the weak topology is equivalent to the condition that for all $e \in E(T_k)$ we have

$$\lim_{k\to\infty} \operatorname{length}_{T_k}(e) = \begin{cases} 0 & \text{if } e \text{ is collapsed in } T \\ \operatorname{length}_T(e) & \text{if } e \text{ is not collapsed in } T. \end{cases}$$

For $k \in \mathbb{N}$, define a (noncontinuous) map $T \to T_k$, $x \mapsto \tilde{x}$ as follows: For each oriented edge $e \in E(T)$, there exists a unique oriented edge $\tilde{e}_k \in E(T_k)$ that maps to e under the forest collapse $T_k \to T$. If $x \in T$ lies in the interior of an edge $e \in E(T)$, define $\tilde{x} \in T_k$ as the point in the interior of $\tilde{e}_k$ with the same linear parameter that x has in e (i.e., let $\tilde{x} = h(x)$ under the unique orientation-preserving homothety $h\colon e \to \tilde{e}_k$). Likewise, for each vertex $v \in V(T)$ there exists a unique (possibly degenerate) subtree $(S_v)_k \subset T_k$ that collapses to v under the forest collapse $T_k \to T$. If $x \in T$ is a vertex, define $\tilde{x} \in T_k$ as any point in $(S_x)_k$ and make this choice G-equivariantly such that the assignment $T \to T_k$, $x \mapsto \tilde{x}$ becomes G-equivariant.

With this construction, for all $x, y \in T$ we have $\lim_{k\to\infty} d_{T_k}(\tilde{x}, \tilde{y}) = d_T(x, y)$ (as suggested in Figure 1.4) and we conclude that for all $x, y \in X \subset T$ and $g \in A \subset G$ we have

$$\lim_{k\to\infty} d_{T_k}(\tilde{x}, g\tilde{y}) = \lim_{k\to\infty} d_{T_k}(\tilde{x}, \widetilde{gy}) = d_T(x, gy).$$

Since $X \subset T$ and $A \subset G$ are finite subsets, there exists a uniform threshold $K \in \mathbb{N}$ such that for all $x, y \in X$ and $g \in A \cup \{1\}$ we have $|d_T(x, gy) - d_{T_k}(\tilde{x}, g\tilde{y})| < \varepsilon$ whenever $k \geq K$, which proves the claim. $\square$
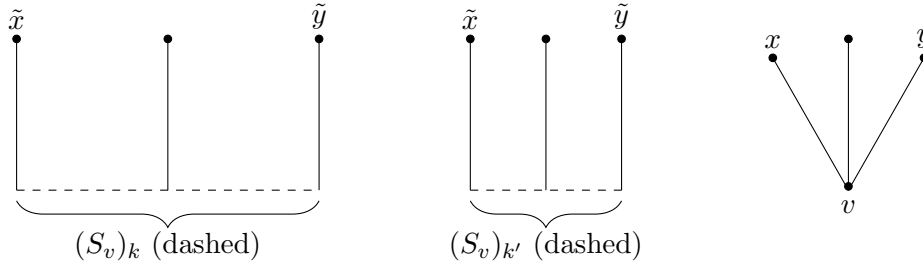
Figure 1.4: Local picture of a weakly convergent sequence of metric G-trees in $\mathcal{T}$ (from left to right). As the edge lengths of S_v approach 0 as k tends to ∞, the distances $d_k(\tilde{x}, \tilde{y})$ converge to $d(x, y)$.

The projectivized universe of metric G-trees

Definition 1.20. The multiplicative group of positive real numbers $\mathbb{R}_{>0}$ acts on $\mathcal{T}$ by scaling the metrics on the G-trees. The *projectivized universe of metric G-trees* is the quotient

$$\mathcal{PT} := \mathcal{T}/\mathbb{R}_{>0}$$

of $\mathcal{T}$ by this action, endowed with the quotient topology.

Whenever we speak of a "metric G-tree" $T \in \mathcal{PT}$, we mean its G-equivariant homothety class. Every metric G-tree in $\mathcal{PT}$ has a unique representative in $\mathcal{T}$ that has covolume 1 and, as a set, we will think of $\mathcal{PT}$ as the covolume-1-section in $\mathcal{T}$. In fact, if we endow $\mathcal{T}$ with the weak topology then the covolume function

$$\mathcal{T} \to (0, \infty), \ T \mapsto \mathrm{covol}(T)$$

is continuous and the natural projection of the covolume-1-section in $\mathcal{T}$ to the quotient $\mathcal{PT}$ is a homeomorphism (but the covolume function generally fails to be continuous in the Gromov or axes topology, as was shown in [MM96, Section 8.3]). Thus, if we equip $\mathcal{T}$ with the weak topology then the quotient $\mathcal{PT}$ inherits the structure of a simplicial complex coming from the natural simplicial structure on the covolume-1-section in $\mathcal{T}$.

A sequence $(T_k)_{k \in \mathbb{N}}$ of metric G-trees in $\mathcal{PT}$ converges to $T \in \mathcal{PT}$ in the projectivized axes topology if and only if there exists a sequence of positive real numbers $(C_k)_{k \in \mathbb{N}}$ such that for all $g \in G$ we have $\lim_{k \to \infty} C_k \cdot l_{T_k}(g) = l_T(g)$ (pointwise convergence of projectivized translation length functions).

1.2.1 Action of the automorphism group

The automorphism group $\mathrm{Aut}(G)$ acts on $\mathcal{T}$ from the right by precomposing the G-actions on the metric trees. Explicitly:

Definition 1.21. For $T \in \mathcal{T}$ with isometric G-action $\rho\colon G \to \mathrm{Isom}(T)$ and an automorphism $\Phi \in \mathrm{Aut}(G)$, we define $T\Phi \in \mathcal{T}$ as the metric G-tree with underlying metric simplicial tree T and isometric G-action $\rho \circ \Phi\colon G \to \mathrm{Isom}(T)$. For a point $x \in T$ and a group element $g \in G$, we will sometimes write $g \bullet_T x$ and $g \bullet_{T\Phi} x = \Phi(g) \bullet_T x$ for the translate in T and $T\Phi$ respectively.

The normal subgroup of inner automorphisms $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$ acts trivially on $\mathcal{T}$, as for all conjugation automorphisms $c_g \in \mathrm{Inn}(G)$ the metric G-trees T and Tc_g are G-equivariantly isometric via

$$T \to Tc_g, \ x \mapsto g \bullet_{Tc_g} x = g \bullet_T x.$$

Thus, the action of $\mathrm{Aut}(G)$ on $\mathcal{T}$ induces an action of the outer automorphism group $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ on $\mathcal{T}$.

Proposition 1.22.

(1) *The group $\mathrm{Out}(G)$ acts on $\mathcal{T}$ by mapping open (resp. closed) cones to open (resp. closed) cones of the same dimension, while preserving the covolume of each metric G-tree.*

(2) *If we choose coordinates on a closed cone and its image (as in Section 1.2), the action preserves these coordinates up to permutation. In particular, the action is linear with respect to the coordinates on a cone and its image.*

(3) *The action of $\mathrm{Out}(G)$ on $\mathcal{T}$ commutes with the action of $\mathbb{R}_{>0}$ on $\mathcal{T}$ and thus descends to an action on $\mathcal{PT}$.*

(4) *For every metric G-tree $T \in \mathcal{T}$, only finitely many metric G-trees in the $\mathrm{Out}(G)$-orbit of T lie in the open cone spanned by T.*

Proof. Let $T, T' \in \mathcal{T}$ and $\phi \in \mathrm{Out}(G)$. If T and T' are G-equivariantly homeomorphic then $T\phi$ and $T'\phi$ are G-equivariantly homeomorphic as well, whence $\mathrm{Out}(G)$ acts on $\mathcal{T}$ by mapping open cones to open cones. If $A \subset T$ is a proper G-invariant subforest then we have $(T_A)\phi = (T\phi)_A$ and thus the closed cone spanned by T maps to the closed cone spanned by $T\phi$. Since T and $T\phi$ have the same underlying metric simplicial tree, their cones have the same dimension and T and $T\phi$ have the same covolume. In order to show (2), observe that if we choose the same order

for the G-orbits of edges of T and $T\phi$ then the diagram

$$
\begin{array}{ccc}
 & & \overline{C}(T) \\
 & \nearrow^{h_T} & \Big| \\
\overline{C}^k_{\mathbb{E}} & & (T,d) \\
 & \searrow_{h_{T\phi}} & \updownarrow \\
 & & (T\phi,d) \\
 & & \overline{C}(T\phi)
\end{array}
$$

commutes and the map $(T,d) \mapsto (T\phi,d)$ preserves coordinates. Since these are well-defined up to permutation by Proposition 1.18, assertion (2) follows.

The fact that T and $T\phi$ have the same underlying metric simplicial tree also implies (3). In order to prove (4), suppose that T and $T\phi$ lie in the same open cone, i.e., are G-equivariantly homeomorphic. Then $T\phi$ is G-equivariantly isometric to (T,d) for some metric d on T obtained by permuting the lengths of the G-orbits of edges, of which there are only finitely many. $\qquad\square$

1.2.2 Deformation spaces

The universe of metric G-trees $\mathcal{T}$ decomposes into subspaces of metric G-trees similar to each other, in the following sense:

Definition 1.23. The *deformation space* associated to a metric G-tree $T \in \mathcal{T}$ is the subspace $\mathcal{D} = \mathcal{D}(T) \subseteq \mathcal{T}$ of metric G-trees that have the same elliptic subgroups as T. We endow $\mathcal{D}$ with the subspace topology coming from $\mathcal{T}$.

If two metric G-trees $T, T' \in \mathcal{T}$ lie in the same deformation space then the translation length functions l_T and $l_{T'}$ vanish on the same elements of G.

Proposition 1.24. *Two metric G-trees $T, T' \in \mathcal{T}$ lie in the same deformation space if and only if there exist G-equivariant maps $T \to T'$ and $T' \to T$.*

Proof. For the "if" direction, suppose that $H \leq G$ is an elliptic subgroup of T. Any G-equivariant map $T \to T'$ maps H-fixed points to H-fixed points, whence the claim. For the "only if" direction, suppose that T and T' have the same elliptic subgroups and let $\{v_1, \ldots, v_k\}$ be a choice of one representative from each G-orbit of vertices of T. Since T and T' have the same elliptic subgroups, for each $i = 1, \ldots, k$ the vertex stabilizer G_{v_i} also fixes a point in T'. We define a map $f \colon V(T) \to T'$ by mapping each v_i into the fixed point set of G_{v_i} in T' and extending it G-equivariantly to all of $V(T)$. Finally, we extend f linearly to the edges of T and we obtain a G-equivariant map $f \colon T \to T'$. $\qquad\square$

By Theorem 1.15, two metric G-trees $T, T' \in \mathcal{T}$ have the same elliptic subgroups if and only if their underlying G-trees are related by an elementary deformation (which is why $\mathcal{D}$ is called a "deformation space"). It follows from this and the remarks made in Section 1.1.2 that all G-trees in a deformation space $\mathcal{D}$ have the same type in the sense of Definition 1.5, and we say that $\mathcal{D}$ is *dihedral*, *linear abelian*, *genuine abelian*, or *irreducible* respectively.

If $\mathcal{D}$ is dihedral or linear abelian then all G-trees in $\mathcal{D}$ are G-equivariantly homeomorphic, which is clear by the fact that they are all related by elementary deformations but have only one G-orbit of edges. Therefore, the only interesting deformation spaces of metric G-trees are the genuine abelian and irreducible ones.

If we endow $\mathcal{T}$ with the weak topology, the deformation space $\mathcal{D}$ inherits a description as a union of open cones. However, beware that if the deformation space is irreducible then closed cones in $\mathcal{D}$ might have missing open cones (and not be closed in the ambient space $\mathcal{T}$), corresponding to nonelementary forest collapses (see for instance Figure 1.5). Namely, for a metric G-tree $T \in \mathcal{D}$, the intersection of the closed cone $\overline{C}(T) = \bigcup_{A \subset T} C(T_A) \subset \mathcal{T}$ with the deformation space $\mathcal{D}(T)$ is the union of only those open cones $C(T_A)$ for which the proper G-invariant subforest $A \subset T$ is collapsible. The following observation implies that closed cones in $\mathcal{D}$ will in fact have missing *closed* cones (or *missing faces* for short), which will be crucial in the proof of Proposition 1.56:

Proposition 1.25. *Let $\mathcal{D}$ be deformation space of metric G-trees and $T \in \mathcal{D}$. If for a proper G-invariant subforest $A \subset T$ the open cone $C(T_A) \subset \mathcal{T}$ does not lie in $\mathcal{D}$ then for all proper G-invariant subforests $B \subset T$ with $A \subset B \subset T$ the open cone $C(T_B)$ does also not lie in $\mathcal{D}$.*

Proof. The open cone $C(T_A)$ does not lie in $\mathcal{D}$ if and only if the forest collapse $k_A \colon T \to T_A$ creates new elliptic subgroups. These new elliptic subgroups are also elliptic in T_B, as the forest collapse $k_B \colon T \to T_B$ induces a G-invariant map $T_A \to T_B$. $\qquad\square$

Proposition 1.26 ([GL07a, Proposition 5.2])**.** *Let $\mathcal{D}$ be deformation space of metric G-trees. The Gromov topology and the weak topology agree on any finite union of open cones of $\mathcal{D}$.*

Projectivized deformation spaces The action of the multiplicative group of positive real numbers $\mathbb{R}_{>0}$ on $\mathcal{T}$ by scaling the metrics on the G-trees restricts to an action on any deformation space of metric G-trees $\mathcal{D} \subset \mathcal{T}$.

Definition 1.27. The *projectivized deformation space* is the quotient

$$\mathcal{PD} := \mathcal{D}/\mathbb{R}_{>0}$$

of $\mathcal{D}$ by this action, endowed with the quotient topology. Equivalently, it is the image of $\mathcal{D} \subset \mathcal{T}$ under the natural projection $\mathcal{T} \to \mathcal{PT}$.

If $\mathcal{D}$ is dihedral or linear abelian then all metric G-trees in $\mathcal{D}$ are G-equivariantly homeomorphic and have only one G-orbit of edges, whence $\mathcal{PD}$ is a point.

As a set, we will think of $\mathcal{PD}$ as the covolume-1-section in $\mathcal{D}$. If we equip $\mathcal{D}$ with the weak topology, the natural projection of the covolume-1-section in $\mathcal{D}$ to the quotient $\mathcal{PD}$ is a homeomorphism and the projectivized deformation space $\mathcal{PD}$ inherits the structure of a simplicial complex with missing faces.

Local finiteness Let $\mathcal{D}$ be a deformation space of metric G-trees. If some G-tree in $\mathcal{D}$ is locally finite then all G-trees in $\mathcal{D}$ are locally finite, as they are all related by elementary deformations. We then say that $\mathcal{D}$ is *locally finite*, and all vertex and edge stabilizers of all G-trees in $\mathcal{D}$ are then commensurable as subgroups of G. If $\mathcal{D}$ is locally finite then the projectivized deformation space $\mathcal{PD}$ endowed with the weak topology is a locally finite complex: As observed in [Lev07, Section 5], "closed simplices containing [a given metric G-tree] T [in $\mathcal{PD}$] consist of simplicial trees obtained from T by expansion moves. Performing such moves on T amounts to blowing up each vertex v of T into a subtree. Since v has finite valence, there are only finitely many ways of expanding (not taking the metric into account)."

Proposition 1.28 ([GL07a, Proposition 5.4])**.** *Let $\mathcal{D}$ be a deformation space of locally finite metric G-trees with finitely generated vertex stabilizers. The Gromov topology and the weak topology agree on $\mathcal{D}$.*

Examples

Example 1.29. Let F_n be the free group of rank $n \geq 2$. The deformation space $\mathcal{X}_n$ of minimal metric F_n-trees that are acted on freely is locally finite and irreducible, and the three topologies on $\mathcal{X}_n$ agree. The projectivized deformation space $\mathcal{PX}_n$ is better known as *Culler-Vogtmann's Outer space* [CV86].
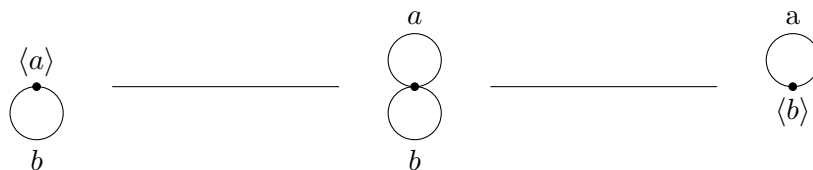

Figure 1.5: The Bass-Serre tree of the middle graph of groups decomposition of $F_2 = \langle a, b \rangle$ spans an open 1-simplex in Outer space $\mathcal{PX}_2$ whose 0-faces do not lie in $\mathcal{PX}_2$ (they correspond to nonfree F_2-trees).

Example 1.30. More generally, let G be a finitely generated virtually nonabelian free group, i.e., G contains a finitely generated nonabelian free subgroup of finite index. It is a standard result that G admits a minimal action on a simplicial tree T such that all vertex and edge stabilizers are finite (see [SW79, Theorem 7.3]). The finite subgroups of G are elliptic in every G-tree and hence all minimal metric G-trees with finite vertex stabilizers lie in the same deformation space $\mathcal{D}$. Since the finite-index nonabelian free subgroup of G acts freely on T, the deformation space $\mathcal{D}$ is irreducible. It is locally finite and the three topologies on $\mathcal{D}$ agree.

Example 1.31. Let G be a nonelementary GBS group (as defined in the introduction). By [For02, Corollary 6.10] and [For03, Lemma 2.6], all minimal metric G-trees with infinite cyclic vertex and edge stabilizers lie in the same deformation space $\mathcal{D}$, which is always locally finite. If G is a solvable Baumslag-Solitar group $\mathrm{BS}(1, q)$ with $q \neq \pm 1$ then the deformation space is genuine abelian. In all other cases, it is irreducible and the three topologies on $\mathcal{D}$ agree.

Action of the automorphism group

If an automorphism $\Phi \in \mathrm{Aut}(G)$ leaves the set of elliptic subgroups of a metric G-tree $T \in \mathcal{T}$ invariant then $T\Phi \in \mathcal{T}$ lies in the same deformation space as T. In general, however, the metric G-tree $T\Phi$ might lie in a different deformation space.

Definition 1.32. For a metric G-tree $T \in \mathcal{T}$ with associated deformation space $\mathcal{D} \subset \mathcal{T}$, we denote by $\mathrm{Aut}_{\mathcal{D}}(G) \leq \mathrm{Aut}(G)$ the subgroup of automorphisms that leave the set of elliptic subgroups of T invariant. The inner automorphism group $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}_{\mathcal{D}}(G)$ and we define

$$\mathrm{Out}_{\mathcal{D}}(G) := \mathrm{Aut}_{\mathcal{D}}(G)/\mathrm{Inn}(G) \leq \mathrm{Out}(G).$$

The action of $\mathrm{Out}(G)$ on $\mathcal{T}$ restricts to an action of $\mathrm{Out}_{\mathcal{D}}(G)$ on $\mathcal{D}$ that descends to an action on $\mathcal{PD}$.

The modular homomorphism If $\mathcal{D}$ is a deformation space of locally finite metric G-trees then all vertex and edge stabilizers of all G-trees in $\mathcal{D}$ are commensurable as subgroups of G. As in [For02, Remark 3.14], we then define the *modular homomorphism* $\mu = \mu(\mathcal{D}) \colon G \to (\mathbb{Q}_{>0}, \times)$ by

$$\mu(g) := \frac{[H : (H \cap gHg^{-1})]}{[gHg^{-1} : (H \cap gHg^{-1})]}$$

where H is any subgroup of G commensurable with a vertex or edge stabilizer of a G-tree in $\mathcal{D}$. Indeed, μ does not depend on the choice of H. We say that $\mathcal{D}$ has *no nontrivial integral modulus* if $\mathrm{im}(\mu) \cap \mathbb{Z} = \{1\}$.

Lemma 1.33 ([Lev07, Lemma 2.4]). *Let G be a nonelementary GBS group. The deformation space $\mathcal{D}$ of minimal metric G-trees with infinite cyclic vertex and edge stabilizers has no nontrivial integral modulus if and only if G contains no solvable Baumslag-Solitar group $\mathrm{BS}(1, n)$ with $n \geq 2$.*

Remark. The group $\mathrm{BS}(1, -n)$ contains a subgroup isomorphic to $\mathrm{BS}(1, n^2)$. Hence, if G contains no solvable Baumslag-Solitar group $\mathrm{BS}(1, n)$ with $n \geq 2$ then it contains no solvable Baumslag-Solitar group $\mathrm{BS}(1, q)$ with $q \neq \pm 1$ and, in particular, the deformation space $\mathcal{D}$ is irreducible.

A subgroup $H \leq G$ is *small in G* (as in [GL07a, Section 8]) if there does not exist a G-tree in which the axes of any two hyperbolic group elements of H intersect in a compact set. Being small in G is a commensurability invariant and stable under taking subgroups.

Proposition 1.34 ([GL07a, Proposition 8.6]). *Let $\mathcal{D}$ be a deformation space of locally finite irreducible metric G-trees whose vertex and edge stabilizers are all commensurable with a finitely generated subgroup $H \leq G$.*

(1) *If H is small in G then $\mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Out}(G)$.*

(2) *If every subgroup of G commensurable with H has finite outer automorphism group and $\mathcal{D}$ has no nontrivial integral modulus then $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{D}$ with finitely many orbits of open cones (and on the projectivized deformation space $\mathcal{PD}$ with finitely many orbits of open simplices).*

If $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices then, in particular, $\mathcal{PD}$ is finite-dimensional.

Example 1.35. The unprojectivized Outer space $\mathcal{X}_n$ (Example 1.29) is locally finite and irreducible, and all vertex and edge stabilizers of the F_n-trees in $\mathcal{X}_n$ are trivial. We clearly have $\mathrm{Out}_{\mathcal{X}_n}(F_n) = \mathrm{Out}(F_n)$, and $\mathrm{Out}(F_n)$ acts on Outer space $\mathcal{PX}_n$ with finitely many orbits of simplices.

Example 1.36. More generally, let G be a finitely generated virtually nonabelian free group. The deformation space $\mathcal{D}$ of minimal metric G-trees with finite vertex stabilizers is locally finite and irreducible (Example 1.30). Since the elliptic subgroups of $\mathcal{D}$ are precisely the finite subgroups of G, we have $\mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Out}(G)$. Choosing $H = \{1\}$, we see that $\mu(\mathcal{D}) \equiv 1$ and $\mathrm{Out}(G)$ acts on the projectivized deformation space $\mathcal{PD}$ with finitely many orbits of simplices.

Example 1.37. Let G be a nonelementary GBS group that contains no solvable Baumslag-Solitar group $\mathrm{BS}(1, n)$ with $n \geq 2$. The deformation space $\mathcal{D}$ of minimal metric G-trees with infinite cyclic vertex and edge stabilizers is locally finite and irreducible (see Example 1.31 and the remark made after Lemma 1.33). Let H

be any vertex or edge stabilizer of any G-tree in $\mathcal{D}$. If G acts on a tree such that H is not elliptic then all nontrivial elements of H have the same hyperbolic axis (because H is infinite cyclic), whence H is small in G and $\mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Out}(G)$. Every subgroup of G commensurable with H, being virtually cyclic, has finite outer automorphism group. By Lemma 1.33, the deformation space $\mathcal{D}$ has no nontrivial integral modulus and we conclude that $\mathrm{Out}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices.

Remark. Let G be a nonelementary GBS group and $\mathcal{D}$ the deformation space of metric G-trees with infinite cyclic vertex and edge stabilizers. By the above arguments, if G contains no solvable Baumslag-Solitar group $\mathrm{BS}(1,n)$ with $n \geq 2$ then the projectivized deformation space $\mathcal{PD}$ is finite-dimensional. However, it need not be finite-dimensional in general (see, for instance, [Cla09, Example 2.2]).

1.2.3 Simplicial completion and spine

Let $\mathcal{D}$ be a deformation space of metric G-trees, endowed with the weak topology. Because of its missing faces, the projectivized deformation space $\mathcal{PD}$ with its natural decomposition into open simplices fails to be a genuine simplicial complex, and even if the group $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of open simplices, it does not act cocompactly. In this section, we will associate to $\mathcal{PD}$ two genuine simplicial complexes to which the action of $\mathrm{Out}_{\mathcal{D}}(G)$ extends respectively restricts. If $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of open simplices then the action on both complexes will be cocompact.

Simplicial completion

Recall that the projectivized universe of metric G-trees $\mathcal{PT}$ endowed with the weak topology is a genuine simplicial complex.

Definition 1.38. Define the *simplicial completion* $\mathcal{PD}^*$ of $\mathcal{PD}$ as its closure in $\mathcal{PT}$ in the weak topology. It is the smallest simplicial subcomplex of $\mathcal{PT}$ that contains $\mathcal{PD}$.

The simplicial complex $\mathcal{PD}^*$ is obtained from $\mathcal{PD}$ by adding those (missing) faces obtained by collapsing a G-invariant proper subforest that is noncollapsible (see Definition 1.14). It is easily seen that the action of $\mathrm{Out}_{\mathcal{D}}(G)$ on $\mathcal{PD}$ extends to a simplicial action on the simplicial completion $\mathcal{PD}^*$: If $T' \in \mathcal{T}$ is obtained from $T \in \mathcal{T}$ by a (possibly nonelementary) forest collapse then for all $\phi \in \mathrm{Out}_{\mathcal{D}}(G)$ the metric G-tree $T'\phi$ is obtained from $T\phi$ by collapsing the same subforest.

If $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of open simplices then it acts on $\mathcal{PD}^*$ with finitely many orbits of *closed* simplices and the quotient $\mathcal{PD}^*/\mathrm{Out}_{\mathcal{D}}(G)$ is compact.

Example 1.39. The simplicial completion of Outer space $\mathcal{PX}_n$ is better known as the *free splitting complex* [HM13a] or Hatcher's *sphere complex* [Hat95].

With regard to Example 1.39, we ask the following question:

Question 1.40. If the projectivized deformation space $\mathcal{PD}$ is finite-dimensional (for example, if $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of open simplices such as in Examples 1.36 and 1.37) then by [BH99, Theorem I.7.19] the simplicial completion $\mathcal{PD}^*$ carries a natural complete geodesic metric, the *simplicial metric*. The simplicial metric on the simplicial completion of Outer space is known to be Gromov-hyperbolic [HM13a], and the following question arises naturally: *Under which assumptions on $\mathcal{PD}$ is the simplicial metric on $\mathcal{PD}^*$ Gromov-hyperbolic?*

Spine

Denote by $\mathcal{PD}_1^*$ the first barycentric subdivision of the simplicial completion $\mathcal{PD}^*$.

Definition 1.41. Define the *spine* $\mathcal{S}(\mathcal{PD})$ of $\mathcal{PD}$ as the maximal subcomplex of $\mathcal{PD}_1^*$ that is contained in $\mathcal{PD}$.

The spine $\mathcal{S}(\mathcal{PD})$ can naturally be viewed as the geometric realization of the poset $\mathcal{C}ol(\mathcal{PD})$ that is defined as follows:

- The elements of $\mathcal{C}ol(\mathcal{PD})$ are the open simplices of $\mathcal{PD}$ in the weak topology, where for two open simplices σ_1 and σ_2 of $\mathcal{PD}$ we have $\sigma_1 \leq \sigma_2$ if σ_1 is contained in the closure of σ_2 (see Figure 1.6).

- Equivalently, the elements of $\mathcal{C}ol(\mathcal{PD})$ are G-equivariant homeomorphism classes of the G-trees in $\mathcal{PD}$, where for $T_1, T_2 \in \mathcal{C}ol(\mathcal{PD})$ we have $T_1 \leq T_2$ if T_1 is obtained from T_2 by an elementary forest collapse.

closed simplex in $\mathcal{PD}$ closed simplex in $\mathcal{PD}^*$ simplices in $\mathcal{S}(\mathcal{PD})$
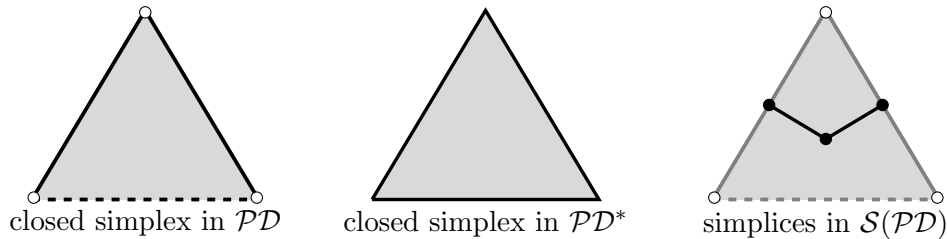
Figure 1.6: A closed simplex in $\mathcal{PD}$ and in the simplicial completion $\mathcal{PD}^*$, and the corresponding simplices in the spine $\mathcal{S}(\mathcal{PD})$.

The action of $\mathrm{Out}_{\mathcal{D}}(G)$ on $\mathcal{PD}$ induces an order-preserving action on $\mathcal{C}ol(\mathcal{PD})$ and restricts to an action on the spine $\mathcal{S}(\mathcal{PD}) \subset \mathcal{PD}$. If $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$

with finitely many orbits of open simplices then the elements of $\mathcal{C}ol(\mathcal{PD})$ fall into finitely many $\mathrm{Out}_{\mathcal{D}}(G)$-orbits and the action of $\mathrm{Out}_{\mathcal{D}}(G)$ on $\mathcal{S}(\mathcal{PD})$ is cocompact.

Proposition 1.42. *The projectivized deformation space $\mathcal{PD}$ equipped with the weak topology $\mathrm{Out}_{\mathcal{D}}(G)$-equivariantly deformation retracts onto its spine $\mathcal{S}(\mathcal{PD})$.*

Proof. In the language of [MM96, §8], the projectivized deformation space $\mathcal{PD}$ is an *ideal simplicial complex*, which deformation retracts onto its barycentric spine $\mathcal{S}(\mathcal{PD})$ by [MM96, Proposition 8.1]. In particular, there exists a deformation retraction that commutes with all automorphisms of $\mathcal{PD}$ that are linear with respect to the barycentric coordinates of open simplices and their images, and hence is $\mathrm{Out}_{\mathcal{D}}(G)$-equivariant by Proposition 1.22. □

Surviving spine In general, the projectivized deformation space $\mathcal{PD}$ and its spine $\mathcal{S}(\mathcal{PD})$ need not be finite-dimensional, but it was shown in [GL07a, Section 7] and [Cla09, Section 1] that $\mathcal{S}(\mathcal{PD})$ deformation retracts onto an $\mathrm{Out}_{\mathcal{D}}(G)$-invariant simplicial subcomplex that is finite-dimensional in certain cases where the spine itself is infinite-dimensional. We will briefly review their construction, for we will describe a generalization of it in Section 3.2.3. For this, we think of the spine $\mathcal{S}(\mathcal{PD})$ as the geometric realization of the poset $\mathcal{C}ol(\mathcal{PD})$ of G-equivariant homeomorphism classes of G-trees in $\mathcal{PD}$.

Definition 1.43. Given a G-tree $T \in \mathcal{C}ol(\mathcal{PD})$, an edge $e \in E(T)$ is *surviving* if it is noncollapsible (see Definition 1.14) or may be made noncollapsible by collapsing other collapsible edges of T. Equivalently, e is surviving if T can be made reduced without collapsing e. The union of all nonsurviving edges of T forms a collapsible G-invariant subforest that we will always denote by W. The G-tree T_W obtained from T by collapsing W has the same elliptic subgroups as T and the set

$$\mathcal{C}ol_W(\mathcal{PD}) := \{T_W \mid T \in \mathcal{C}ol(\mathcal{PD})\}$$

is a subposet of $\mathcal{C}ol(\mathcal{PD})$. Its geometric realization $\mathcal{S}_W(\mathcal{PD}) := |\mathcal{C}ol_W(\mathcal{PD})|$ is a subcomplex of the spine $\mathcal{S}(\mathcal{PD})$ that we call the *surviving spine* of $\mathcal{PD}$.

Example 1.44. In the case of Outer space $\mathcal{PX}_n$, the subcomplex $\mathcal{S}_W(\mathcal{PX}_n)$ of $\mathcal{S}(\mathcal{PX}_n)$ is the spine of *reduced Outer space* $\mathcal{PX}_n^R \subset \mathcal{PX}_n$, the subspace of marked metric metric graphs with no separating edges (see [CV86]).

One readily sees that the subset $\mathcal{C}ol_W(\mathcal{PD}) \subseteq \mathcal{C}ol(\mathcal{PD})$ is $\mathrm{Out}_{\mathcal{D}}(G)$-invariant. The action of $\mathrm{Out}_{\mathcal{D}}(G)$ on the spine $\mathcal{S}(\mathcal{PD})$ thus restricts to an action on the surviving spine $\mathcal{S}_W(\mathcal{PD}) \subseteq \mathcal{S}(\mathcal{PD})$. In fact, we have the following:

Proposition 1.45 ([Cla09, Section 1.2])**.** *The spine $\mathcal{S}(\mathcal{PD})$ deformation retracts $\mathrm{Out}_{\mathcal{D}}(G)$-equivariantly onto the surviving spine $\mathcal{S}_W(\mathcal{PD}) \subseteq \mathcal{S}(\mathcal{PD})$.*

Proof. The natural projection $f\colon \mathcal{C}ol(\mathcal{PD}) \to \mathcal{C}ol_W(\mathcal{PD}) \subseteq \mathcal{C}ol(\mathcal{PD})$, $T \mapsto T_W$ is $\mathrm{Out}_\mathcal{D}(G)$-equivariant and order-preserving, and for all $T \in \mathcal{C}ol(\mathcal{PD})$ we have $f(T) \leq T$. It follows from the *homotopy property* of poset maps described in [Qui78, 1.3] that the induced $\mathrm{Out}_\mathcal{D}(G)$-equivariant map on geometric realizations $|f|\colon \mathcal{S}(\mathcal{PD}) \to \mathcal{S}_W(\mathcal{PD}) \subseteq \mathcal{S}(\mathcal{PD})$ is homotopic to the identity on $\mathcal{S}(\mathcal{PD})$, and the suggested homotopy is easily seen to be $\mathrm{Out}_\mathcal{D}(G)$-equivariant and the identity on the subcomplex $\mathcal{S}_W(\mathcal{PD}) \subseteq \mathcal{S}(\mathcal{PD})$. $\qquad\square$

Definition 1.46. A deformation space of metric G-trees $\mathcal{D}$ is *nonascending* if it is irreducible and for all metric G-trees $T \in \mathcal{D}$ the following condition is satisfied: For $e \in E(T)$, if the initial vertex $\iota(e)$ and the terminal vertex $\tau(e)$ lie in the same G-orbit then the inclusions $G_e \hookrightarrow G_{\iota(e)}$ and $G_e \hookrightarrow G_{\tau(e)}$ are either both proper inclusions or both isomorphisms. On the level of quotient graph of groups, this means that for no G-tree $T \in \mathcal{D}$ the quotient graph of groups $G\backslash T$ contains a strict ascending loop, i.e., an edge e with $\iota(e) = \tau(e)$ such that exactly one of the two inclusions $G_e \hookrightarrow G_{\iota(e)}$ and $G_e \hookrightarrow G_{\tau(e)}$ is an isomorphism.

Definition 1.47. For a deformation space of metric G-trees $\mathcal{D}$, let $b_1(\mathcal{D}) \in \mathbb{N}_0$ be the first Betti number of the quotient graph $G\backslash T$, where T is any G-tree in $\mathcal{D}$. The number $b_1(\mathcal{D})$ does not depend on the choice of T, as all G-trees in $\mathcal{D}$ are related by elementary deformations.

Theorem 1.48 ([GL07a, Theorem 7.6] and [Cla09, Theorem 1.18])**.** *Let $\mathcal{D}$ be an irreducible deformation space of metric G-trees. If either*

- *$\mathcal{D}$ is nonascending; or*
- *$\mathcal{D}$ is locally finite and has $b_1(\mathcal{D}) \leq 1$*

then the surviving spine $\mathcal{S}_W(\mathcal{PD}) \subseteq \mathcal{S}(\mathcal{PD})$ is finite-dimensional.

Example 1.49. Let $G = \mathrm{BS}(2,4)$ and $\mathcal{PD}$ be the projectivized deformation space of minimal metric G-trees with infinite cyclic vertex and edge stabilizers (Example 1.31), which is irreducible and locally finite. By [Cla09, Example 2.1] we have $b_1(\mathcal{PD}) = 1$ and hence the surviving spine $\mathcal{S}_W(\mathcal{PD})$ is finite-dimensional, though the spine $\mathcal{S}(\mathcal{PD})$ itself is infinite-dimensional.

1.3 Classifying spaces for families of subgroups

Let G be a discrete (not necessarily finitely generated) group.

Definition 1.50. A *G-CW-complex* is a G-space X together with a G-invariant filtration $\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq \bigcup_{n\geq 0} X_n = X$ such that

- X carries the colimit topology with respect to this filtration;

- for each $n \geq 0$ the space X_n is obtained from X_{n-1} by attaching n-dimensional *G-equivariant cells*, i.e., there exists a *G*-pushout

$$
\begin{array}{ccc}
\amalg_{i \in I_n} \left((G/H_i) \times S^{n-1} \right) & \longrightarrow & X_{n-1} \\
\downarrow & & \downarrow \\
\amalg_{i \in I_n} \left((G/H_i) \times D^n \right) & \longrightarrow & X_n.
\end{array}
$$

Definition 1.51. An action of G on a CW-complex X is *cellular* if for each $g \in G$ and each open cell E of X the translate gE is again an open cell of X and if $gE = E$ implies that the induced map $E \to E$, $x \mapsto gx$ is the identity. (This definition also makes sense if X is a simplicial complex with missing faces.)

For instance, if a group G acts on a simplicial complex X simplicially then it acts on the first barycentric subdivision of X cellularly (the easy argument given in the proof of Lemma 1.55 applies in general).

Proposition 1.52 ([tD87, Proposition II.1.15])**.** *Let X be a CW-complex with a cellular G-action. Then X is a G-CW-complex with n-skeleton X_n.*

A *family of subgroups* of G is a collection of subgroups of G that is closed under conjugation and taking subgroups. Examples of families of subgroups are

- $\mathcal{A}ll$, the family of all subgroups;
- $\mathcal{VC}yc$, the family of virtually cyclic subgroups;
- $\mathcal{F}in$, the family of finite subgroups;
- $\{1\}$, the family that consists of only the trivial subgroup.

Definition 1.53. Let $\mathcal{F}$ be a family of subgroups of G. A *model for the classifying space of G for the family $\mathcal{F}$* (or a *model for $E(G, \mathcal{F})$* for short) is a G-CW-complex X such that all isotropy groups belong to $\mathcal{F}$ and for all $H \in \mathcal{F}$ the H-fixed point set X^H is contractible (and in particular nonempty).

There exists a model for the classifying space of G for any family $\mathcal{F}$ and it is unique up to G-equivariant homotopy equivalence (see [Lüc05]). Classifying spaces for families of subgroups are important objects in algebraic topology that play a central role in ongoing research. For instance, classifying spaces for the family $\mathcal{VC}yc$ appear in the statement of the Farrell-Jones conjecture in algebraic K- and L-theory (see [BLR08] for a survey). The conjecture predicts that for any ring R the algebraic K- and L-theory of the group ring $R[G]$ can be computed in terms of group homology of G and the K- and L-theory of group rings $R[V]$, where V ranges over the virtually cyclic subgroups of G. Because of the *transitivity principle*

[BL07, Theorem 1.5], one is also interested in classifying spaces for other families of subgroups. We will show that for a finitely generated group G certain projectivized deformation spaces of metric G-trees $\mathcal{PD}$ are models for the classifying spaces of $\mathrm{Out}_{\mathcal{D}}(G)$ for a family of subgroups (Theorem 1.63). We will proceed by

(1) showing that $\mathcal{PD}$ can be given the structure of an $\mathrm{Out}_{\mathcal{D}}(G)$-CW-complex (Proposition 1.56);

(2) reviewing Clay's [Cla05] and Guirardel-Levitt's [GL07a] geometric proof that $\mathcal{PD}$ is contractible (Theorem 1.57);

(3) discussing Guirardel-Levitt's argument that certain fixed point sets under the action of $\mathrm{Out}_{\mathcal{D}}(G)$ on $\mathcal{PD}$ are contractible as well (Theorem 1.59).

1.3.1 An $\mathrm{Out}_{\mathcal{D}}(G)$-CW-structure on $\mathcal{PD}$

Let G be a finitely generated group and $\mathcal{PD}$ a projectivized deformation space of metric G-trees. The group $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ by mapping open simplices to open simplices (Proposition 1.22), but the action is generally not cellular, as the following example demonstrates:

Example 1.54. The Bass-Serre trees of the metric graph of groups decompositions of $F_2 = \langle a, b \rangle$ shown in Figure 1.7 lie in the same open simplex of Outer space $\mathcal{PX}_2$. The latter is the image of the first under the automorphism $a \mapsto b,\ b \mapsto a$, but the two metric F_2-trees are not F_2-equivariantly isometric.

Figure 1.7: The Bass-Serre trees of the graph of groups shown above lie in the same $\mathrm{Out}(F_2)$-orbit and in the same open simplex of $\mathcal{PX}_2$.

Recall from Section 1.2.3 that the action of $\mathrm{Out}_{\mathcal{D}}(G)$ on $\mathcal{PD}$ extends to an action on the simplicial completion $\mathcal{PD}^*$. Clearly, as $\mathrm{Out}_{\mathcal{D}}(G)$ does not act cellularly on $\mathcal{PD}$, the action on $\mathcal{PD}^*$ fails to be cellular as well. Denote by $\mathcal{PD}_1^*$ the first barycentric subdivision of $\mathcal{PD}^*$. Since the action of $\mathrm{Out}_{\mathcal{D}}(G)$ on the simplicial completion $\mathcal{PD}^*$ is linear with respect to the barycentric coordinates of an open simplex and its image (Proposition 1.22), the group $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}_1^*$ again by mapping open simplices to open simplices. In fact, we have the following:

Lemma 1.55. *The action of* $\mathrm{Out}_{\mathcal{D}}(G)$ *on* $\mathcal{PD}_1^*$ *is cellular.*

Hence, the simplicial complex $\mathcal{PD}_1^*$ is an $\mathrm{Out}_{\mathcal{D}}(G)$-CW-complex.

Proof. We need to show that if an automorphism $\phi \in \mathrm{Out}_{\mathcal{D}}(G)$ leaves an open simplex of $\mathcal{PD}_1^*$ invariant then it fixes it pointwise. Recall that (in the notation of Section 1.2) the open simplex spanned by a metric G-tree T in $\mathcal{PD}^*$ consists of the metric G-trees $(T, d(l_1, \ldots, l_k))$ with $l_i > 0$ for all $i = 1, \ldots, k$ and $\sum_{i=1}^k l_i = 1$. The metric G-trees in the open simplex spanned by T in $\mathcal{PD}_1^*$ additionally satisfy

(1.1) $$l_1 \leq \ldots \leq l_k$$

up to some fixed permutation of the indices, where each "$\leq$" is either a strict inequality or an equality. If $\phi \in \mathrm{Out}_{\mathcal{D}}(G)$ leaves an open simplex of $\mathcal{PD}_1^*$ invariant then by Proposition 1.22 it permutes its barycentric coordinates, and (1.1) is only preserved if ϕ fixes the open simplex pointwise. $\square$

Proposition 1.56. *The natural simplicial structure with missing faces on $\mathcal{PD}$ can be refined to a genuine simplicial structure that defines the same weak topology on $\mathcal{PD}$ and with respect to which $\mathrm{Out}_{\mathcal{D}}(G)$ acts cellularly.*

Proof. Since $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}_1^*$ cellularly, it suffices to refine the induced simplicial structure with missing faces on $\mathcal{PD} \subset \mathcal{PD}_1^*$ $\mathrm{Out}_{\mathcal{D}}(G)$-equivariantly to a genuine simplicial structure that defines the same topology.

Let X_1 be the maximal genuine simplicial subcomplex of $\mathcal{PD}_1^*$ that is contained in $\mathcal{PD}$. For $k \in \mathbb{N}$, we inductively define

(1) $\mathcal{PD}_{k+1}^*$ as the *first barycentric subdivision of $\mathcal{PD}_k^*$ holding X_k fixed* (as defined in [Mun84, §16]);

(2) $X_{k+1} \subset \mathcal{PD}_{k+1}^*$ as the maximal genuine simplicial subcomplex that is contained in $\mathcal{PD}$ (see Figure 1.8).
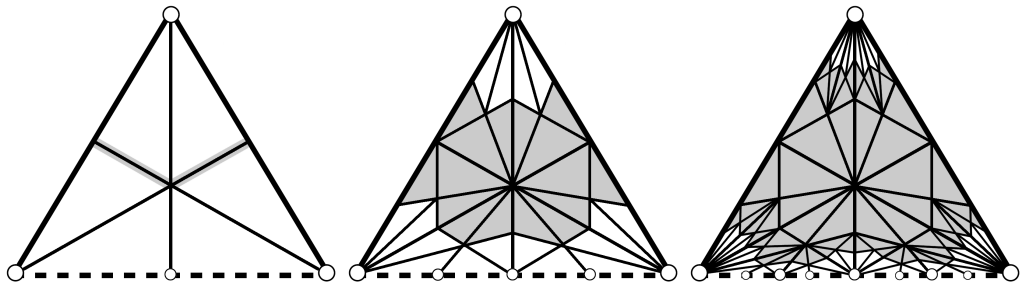

Figure 1.8: A closed simplex with missing faces in $\mathcal{PD}$ after the first, second, and third subdivision step. The genuine simplicial subcomplexes X_1, X_2, and X_3 are colored gray.

Since the group $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}^*$ linearly with respect to the barycentric coordinates of a simplex and its image (Proposition 1.22), the subdivision process is $\mathrm{Out}_{\mathcal{D}}(G)$-equivariant.

We claim that the subspaces $X_k \subset \mathcal{PD}$, $k \in \mathbb{N}$ fully exhaust $\mathcal{PD}$, i.e., that for all $x \in \mathcal{PD}$ there exists $k \in \mathbb{N}$ such that $x \in X_k$. Since each X_k is a simplicial subcomplex of X_l for all $l \geq k$, we then obtain a genuine simplicial structure on $\mathcal{PD} = \bigcup_{k \in \mathbb{N}} X_k$ that, as we will show in the second step, defines the same weak topology on $\mathcal{PD}$ as the natural simplicial structure with missing faces.

If $x \in \mathcal{PD}$ is contained in a closed simplex of $\mathcal{PD}^*$ all of whose faces lie in $\mathcal{PD}$ then x will lie in X_1 and we are done. If, instead, the closed simplex $\sigma(x) \subset \mathcal{PD}^*$ spanned by x has faces that do not belong to $\mathcal{PD}$, we will call these faces *missing faces* and we argue as follows: Since $\mathcal{PD}$ has missing *closed* simplices (Proposition 1.25), the point x does not lie in the boundary of a missing open simplex (cf. Figure 1.9). Hence, under the natural identification of $\sigma(x)$ with a Euclidean simplex as in Section 1.2, the point x has positive Euclidean distance $d > 0$ to the missing boundary of $\sigma(x)$. For $k \in \mathbb{N}$, the intersection $X_k \cap \sigma(x)$ is
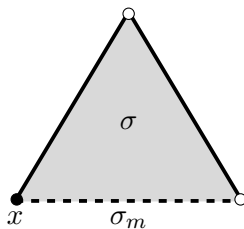

Figure 1.9: The above situation, in which a point in $\mathcal{PD}$ lies in the boundary of a missing open simplex σ_m, cannot occur.

the complement in $\sigma(x)$ of the open simplices of $\mathcal{PD}_k^* \cap \sigma(x)$ that have a face in the missing boundary of $\sigma(x)$. For large k, the diameter of these simplices is smaller than d and we conclude that $x \in X_k$. As a set, we thus have $\mathcal{PD} = \bigcup_{k \in \mathbb{N}} X_k$.

Denote by $\mathcal{PD}_\infty$ the projectivized deformation space $\mathcal{PD}$ equipped with the weak topology defined by the genuine simplicial structure $\bigcup_{k \in \mathbb{N}} X_k$. In order to prove that the weak topology on $\mathcal{PD}$ agrees with the weak topology on its simplicial refinement $\mathcal{PD}_\infty$, it suffices to show that every convergent sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{PD}$, say with $\lim_{n \to \infty} x_n = x \in \mathcal{PD}$, also converges in $\mathcal{PD}_\infty$. Suppose to the contrary that $(x_n)_{n \in \mathbb{N}}$ does not converge in $\mathcal{PD}_\infty$. Since the topology on any finite union of simplices of $\mathcal{PD}_\infty$ agrees with the subspace topology coming from $\mathcal{PD}$, the sequence $(x_n)_{n \in \mathbb{N}}$ must meet infinitely many open simplices of $\mathcal{PD}_\infty$. On the other hand, convergence in $\mathcal{PD}$ implies that the sequence meets only finite many open simplices of $\mathcal{PD}$. After passing to a subsequence, we may assume that the sequence stays within a single open simplex $\sigma \subset \mathcal{PD}$ and that the sequence

members lie in pairwise distinct open simplices of $\mathcal{PD}_\infty$. The limit point $x \in \mathcal{PD}$ (which lies in the closure of σ in $\mathcal{PD}$) has again positive Euclidean distance to the missing boundary of σ and, by the arguments given above, for large k we have $x \in X_k$. In fact, we can choose k large enough such that x lies in the interior of X_k and the sequence $(x_n)_{n\in\mathbb{N}}$ eventually lies in X_k as well. However, the intersection $X_k \cap \sigma$ consists of only finitely many simplices of $\mathcal{PD}_\infty$, which contradicts our assumption that the sequence $(x_n)_{n\in\mathbb{N}}$ does not converge in $\mathcal{PD}_\infty$. $\qquad\square$

1.3.2 Contractibility of deformation space of G-trees

Theorem 1.57 ([Cla05, Theorem 6.7] and [GL07a, Theorem 6.1]). *Let $\mathcal{D}$ be a deformation space of metric G-trees.*

- *$\mathcal{D}$ is contractible in the weak topology.*

- *If $\mathcal{D}$ contains a metric G-tree with finitely generated vertex stabilizers then the deformation space is also contractible in the Gromov topology.*

The same results hold for the projectivized deformation space $\mathcal{PD}$.

Recall from Section 1.2.2 that every dihedral or linear abelian deformation space $\mathcal{D}$ is homeomorphic to $\mathbb{R}_{>0}$ and that $\mathcal{PD}$ is then a single point. Therefore, we are only interested in the genuine abelian and the irreducible case.

Culler-Vogtmann's original proof [CV86] of the contractibility of Outer space $\mathcal{PX}_n$ (Example 1.29) was combinatorial and formulated in the language of marked metric graphs. An alternative, geometric proof was given by Skora [Sko89] in the language of metric F_n-trees. The proof of Theorem 1.57, of which we will now review an outline, is a generalization of Skora's arguments and relies on the idea of folding metric G-trees along suitable G-equivariant maps:

Definition 1.58. Let $\mathcal{D}$ be a deformation space of metric G-trees and $T, T' \in \mathcal{D}$. A G-equivariant map $f\colon T \to T'$ is *simplicial* if it maps each edge of T isometrically to an edge of T'. A G-equivariant map $f\colon T \to T'$ is a *morphism* if it is an isometry on edges or, equivalently, if the simplicial structures on T and T' may be subdivided (allowing redundant vertices) such that f becomes simplicial.

Given a morphism $f\colon T \to T'$, we may "fold T along f" to obtain a 1-parameter family of metric G-trees $(T_t)_{t\in[0,\infty]}$ in $\mathcal{D}$ together with morphisms $\phi_t\colon T \to T_t$ and $\psi_t\colon T_t \to T'$ such that

(1) $T_0 = T$ and $T_\infty = T'$;

(2) $\phi_0 = id_T$, $\phi_\infty = \psi_0 = f$, and $\psi_\infty = id_{T'}$;

(3) for all $t \in [0, \infty]$ the following diagram commutes:

$$
\begin{array}{ccc}
T & \xrightarrow{\quad f \quad} & T' \\
\phi_t \searrow & & \nearrow \psi_t \\
& T_t &
\end{array}
$$

For this, for $t \in [0, \infty]$ we let $\sim_t$ be the G-equivariant equivalence relation on T defined by

$$
x \sim_t y \quad \text{if} \quad f(x) = f(y) \text{ and } f([x, y]) \subseteq D_t(f(x))
$$

where $D_t(f(x))$ denotes the closed ball of radius t around $f(x) \in T'$. The quotient $T/\sim_t$ is then a simplicial tree [GL07b, Proposition 3.6] that carries an induced G-action. We define

- T_t as the unique minimal G-invariant subtree of $T/\sim_t$ (the quotient itself need not be minimal);

- $\phi_t \colon T \to T_t$ as the G-equivariant quotient map $T \to T/\sim_t$ composed with the G-equivariant projection of $T/\sim_t$ onto its G-invariant subtree T_t;

- $\psi_t \colon T_t \to T'$ as the restriction of the induced G-equivariant map $T/\sim_t \to T'$ to the minimal G-invariant subtree $T_t \subset T/\sim_t$.

- Finally, we equip T_t with the maximal metric making ϕ_t 1-Lipschitz. Both ϕ_t and ψ_t are then morphisms [GL07b, Lemma 3.3].

The minimal metric G-tree T_t lies in $\mathcal{D}$, for there exist G-equivariant maps $T \to T_t$ and $T_t \to T'$. The folding path $[0, \infty] \to \mathcal{D}$, $t \to T_t$ is continuous in the Gromov topology (see [GL07b, Section 3.2]) and therefore also in the axes topology. The metric G-trees $(T_t)_{t \in [0, \infty]}$ are contained in a finite union of open cones [GL07a, Lemma 6.5] because of which the folding path is also continuous in the weak topology (see Proposition 1.26).

Note that the construction is equivariant with respect to the action of $\mathbb{R}_{>0}$ on $\mathcal{D}$. That is, if we scale the metrics on T and T' by a positive factor $\lambda > 0$ then $f \colon \lambda T \to \lambda T'$ remains a morphism and we have $(\lambda T)_t = \lambda(T_t)$ for all $t \in [0, \infty]$.

Outline of the proof of Theorem 1.57. We will describe a deformation retraction $r \colon \mathcal{D} \times [0, \infty] \to \mathcal{D}$ of the deformation space $\mathcal{D}$ onto the open cone spanned by a suitable metric G-tree $T_0 \in \mathcal{D}$. As open cones are contractible both in the weak topology and the Gromov topology (see Proposition 1.26), this then proves the claim. The deformation retraction will be equivariant with respect to the action of $\mathbb{R}_{>0}$ on $\mathcal{D}$ and we thus obtain an induced deformation retraction of the projectivized deformation space $\mathcal{PD}$ onto the open simplex spanned by $T_0 \in \mathcal{PD}$, proving contractibility of $\mathcal{PD}$.

Choose a metric G-tree $T_0 \in \mathcal{D}$ that is reduced (i.e., admits no elementary collapses) and, when proving the second assertion, has finitely generated vertex stabilizers. Denote by $C(T_0) \subset \mathcal{D}$ the open cone spanned by T_0 and assume for a moment that the metric G-trees in $\mathcal{D}$ have distinguished basepoints. To every $T \in \mathcal{D}$ with distinguished basepoint $P \in T$ we will associate a metric G-tree $T_0(T)$ inside the open cone $C(T_0)$ and a morphism $f_T \colon T_0(T) \to T$. If we then traverse the folding path defined by f_T in the opposite direction, we obtain a continuous path $\gamma_T \colon [0, \infty] \to \mathcal{D}$ with $\gamma_T(0) = T$ and $\gamma_T(\infty) \in C(T_0)$. In order to construct f_T, fix a representative $v_i \in V(T_0)$ in each G-orbit of vertices of T_0 and define $f_T(v_i) \in T$ as the projection of the distinguished basepoint $P \in T$ to the fixed point set of G_{v_i} in T. Then extend f_T G-equivariantly to the vertices and linearly to the edges of T_0 to obtain a G-equivariant map $f_T \colon T_0 \to T$. Since T_0 was chosen reduced, no edge of T_0 is collapsed under f_T and there is a unique metric d_T on T_0 such that $f_T \colon (T_0, d_T) \to T$ is a morphism (i.e., an isometry on edges). Finally, we let $T_0(T) = (T_0, d_T)$.

We now choose for every metric G-tree $T \in \mathcal{D}$ a distinguished basepoint $P \in T$, and we proceed as follows: If $\mathcal{D}$ is irreducible, there exist hyperbolic group elements $g, h \in G$ whose commutator $[g, h] \in G$ is hyperbolic as well [CM87, Theorem 2.7]. In every G-tree $T \in \mathcal{D}$ the hyperbolic axes A_g and A_h then intersect in a (possibly empty) compact segment [GL07a, Proposition 5.6]. As in [GL07a, Section 6], "if A_g and A_h meet, we order A_g so that the action of g is by a positive translation and we let P be the largest element of the segment $A_g \cap A_h$. If A_g and A_h are disjoint, we define P as the point of A_g closest to A_h." If $\mathcal{D}$ is genuine abelian, we choose hyperbolic group elements $g, h \in G$ with distinct axes in some and therefore in every G-tree in the deformation space[1], and for $T \in \mathcal{D}$ we define the basepoint $P \in T$ as the initial point of the ray $A_g \cap A_h$.

It is shown in [GL07a, Section 6] that, with this particular choice of basepoints, the assignment

$$r \colon \mathcal{D} \times [0, \infty] \to \mathcal{D}$$
$$(T, t) \mapsto \gamma_T(t)$$

is continuous (the assumption that T_0 has finitely generated vertex stabilizers is used in the proof of continuity in the Gromov topology). Since for all $T \in \mathcal{D}$ we have $r(T, 0) = T$ and $r(T, \infty) \in C(T_0)$, this proves the theorem. $\qquad \square$

---

[1] As explained in the proof of [GL07a, Proposition 5.7], "the fact that two hyperbolic elements g, h have the same axis depends only on $\mathcal{D}$ (it is characterized by ellipticity of the group generated by the elements $[g^n, h], \ n \in \mathbb{Z}$)."

1.3.3 Contractibility of fixed point sets

Besides contractibility of $\mathcal{D}$ (Theorem 1.57), in the irreducible case certain fixed point sets under the action of $\mathrm{Out}_{\mathcal{D}}(G)$ on $\mathcal{D}$ are contractible as well:

Theorem 1.59 ([GL07a, Theorem 8.3])**.** *Let $\mathcal{D}$ be an irreducible deformation space of metric G-trees.*

- *If a finitely generated subgroup $H \leq \mathrm{Out}_{\mathcal{D}}(G)$ fixes a metric G-tree in $\mathcal{D}$ then the H-fixed point set $\mathcal{D}^H \subseteq \mathcal{D}$ is contractible in the weak topology.*

- *If H fixes a metric G-tree with finitely generated vertex stabilizers then $\mathcal{D}^H$ is also contractible in the Gromov topology.*

The same results hold for the action of $\mathrm{Out}_{\mathcal{D}}(G)$ on the projectivized deformation space $\mathcal{PD}$.

The proof of Theorem 1.59, which we now review, proceeds by showing that the H-fixed point set $\mathcal{D}^H \subseteq \mathcal{D}$ is a deformation space of metric $\widehat{H}$-trees, where $\widehat{H}$ is the (finitely generated) preimage of H under the natural projection of $\mathrm{Aut}(G)$ onto $\mathrm{Out}(G)$. If a metric G-tree in $\mathcal{D}^H$ has finitely generated vertex stabilizers, the corresponding $\widehat{H}$-tree will have finitely generated vertex stabilizers as well. Contractibility of $\mathcal{D}^H$ then follows from Theorem 1.57.

Remark. Using Skora's folding construction directly, it was previously shown by White [Whi93] that all nonempty fixed point sets under the action of $\mathrm{Out}(F_n)$ on Outer space $\mathcal{PX}_n$ are contractible.

The proof of Theorem 1.59 is based on operations of restricting and extending group actions on trees to normal subgroups and certain supergroups respectively:

Restriction Let $\widehat{G}$ be a finitely generated group and $G \trianglelefteq \widehat{G}$ a finitely generated normal subgroup. Every $\widehat{G}$-tree T naturally gives rise to a G-tree T_G by restricting the action. One readily sees that if T_G is minimal and irreducible then so is T. In fact, conversely, if T is an irreducible minimal $\widehat{G}$-tree and G is not elliptic in T then the G-tree T_G is also minimal and irreducible [GL07a, Lemma 8.1].

The elliptic G-subgroups of T_G are the elliptic $\widehat{G}$-subgroups of T intersected with G. Thus, if two metric $\widehat{G}$-trees T and T' lie in the same deformation space $\mathcal{D}_{\widehat{G}}$ then T_G and T'_G lie in the same deformation space of metric G-trees $\mathcal{D}_G$.

Lemma 1.60 ([GL07a, Lemma 8.2])**.** *If $\mathcal{D}_{\widehat{G}}$ is an irreducible deformation space of metric $\widehat{G}$-trees in which G is not elliptic then the restriction map*

$$\mathrm{res} \colon \mathcal{D}_{\widehat{G}} \to \mathcal{D}_G, \ T \mapsto T_G$$

is a homeomorphism onto its image, both in the weak topology and the Gromov topology. Every irreducible deformation space of metric G-trees $\mathcal{D}_G$ contains the image of at most one deformation space of metric $\widehat{G}$-trees.

Extension Let G be a finitely generated group. We observe the following:

Proposition 1.61. *If T is a minimal G-tree that is not linear abelian then the center $Z(G)$ of G acts trivially on T.*

Proof. This immediately follows from Proposition 1.10, as for any $g \in Z(G)$ the map $T \to T$, $x \mapsto gx$ is a G-equivariant simplicial automorphism. $\qquad\square$

The quotient $G/Z(G)$ is canonically isomorphic to the inner automorphism group $\mathrm{Inn}(G)$. Every irreducible minimal G-tree T can thus naturally be viewed as an $\mathrm{Inn}(G)$-tree with action

$$\mathrm{Inn}(G) \times T \to T, \ (c_g, x) \mapsto gx$$

where $c_g \in \mathrm{Inn}(G)$ denotes conjugation with $g \in G$.

For a subgroup $H \leq \mathrm{Out}(G)$, we denote by $\widehat{H}$ the preimage of H under the natural projection $\mathrm{Aut}(G) \to \mathrm{Out}(G)$. We remark that H and $\widehat{H}$ fit into the following commutative diagram of groups, whose rows are short exact sequences:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Inn}(G) & \longrightarrow & \mathrm{Aut}(G) & \longrightarrow & \mathrm{Out}(G) & \longrightarrow & 1 \\
& & \| & & \uparrow{\scriptstyle\leq} & & \uparrow{\scriptstyle\leq} & & \\
1 & \longrightarrow & \mathrm{Inn}(G) & \longrightarrow & \widehat{H} & \longrightarrow & H & \longrightarrow & 1
\end{array}
$$

If H is finitely generated then $\widehat{H}$ is finitely generated as well.

Lemma 1.62. *Let $\mathcal{D}$ be an irreducible deformation space of metric G-trees and $H \leq \mathrm{Out}_{\mathcal{D}}(G)$ a finitely generated subgroup. A metric G-tree $T \in \mathcal{D}$ is fixed by H if and only if the action of $\mathrm{Inn}(G)$ on T given by $(c_g, x) \mapsto gx$ extends to an isometric action of $\widehat{H}$ on T.*

Proof. This observation is hinted at in the proof of [GL07a, Theorem 8.3]. We take the opportunity to give an easy argument:

If T is fixed by H then, by definition, for all $\phi \in \widehat{H}$ there exists a G-equivariant isometry $f_\phi\colon T \to T\phi$, which is unique by Proposition 1.10. The group $\widehat{H}$ then acts on T via $\widehat{H} \times T \to T$, $(\phi, x) \mapsto f_\phi(x)$, where for $c_g \in \mathrm{Inn}(G) \leq \widehat{H}$ the unique G-equivariant isometry $f_{c_g}\colon T \to Tc_g$ is given by $x \mapsto g \bullet_{Tc_g} x = g \bullet_T x$. We conclude that the action of $\widehat{H}$ on T extends the action of $\mathrm{Inn}(G)$.

Conversely, assume that the action of $\mathrm{Inn}(G)$ on T extends to an isometric action $\widehat{H} \times T \to T$, $(\phi, x) \mapsto \phi \bullet x$. By assumption, for all $c_g \in \mathrm{Inn}(G)$ we have $c_g \bullet x = gx$. Given $\phi \in \widehat{H}$, consider the (nonequivariant) isometry $f_\phi \colon T \to T$ given by $f_\phi(x) := \phi \bullet x$. For all $g \in G$ and $x \in X$ we have

$$
\begin{aligned}
f_\phi(gx) &= \phi \bullet gx = \phi \bullet c_g \bullet x = (\phi \circ c_g) \bullet x \\
&= (c_{\phi(g)} \circ \phi) \bullet x = c_{\phi(g)} \bullet \phi \bullet x = \phi(g)\,(\phi \bullet x) = \phi(g) f_\phi(x)
\end{aligned}
$$

and thus f_ϕ is G-equivariant as an isometry from T to $T\phi$, whence the metric G-tree T is fixed by H. $\qquad\square$

Proof of Theorem 1.59. Suppose that a finitely generated subgroup $H \leq \mathrm{Out}_{\mathcal{D}}(G)$ fixes a metric G-tree $T \in \mathcal{D}$. By Lemma 1.62, the action of $\mathrm{Inn}(G)$ on T extends to an isometric action of $\widehat{H}$ on T and we denote the corresponding irreducible minimal metric $\widehat{H}$-tree by $T_{\widehat{H}}$. In order to prove Theorem 1.59, we will argue that the H-fixed point set $\mathcal{D}^H \subseteq \mathcal{D}$ is homeomorphic to the deformation space $\mathcal{D}_{\widehat{H}}$ of metric $\widehat{H}$-trees associated to $T_{\widehat{H}}$. The claim then follows from Theorem 1.57.

As the $\mathrm{Inn}(G)$-tree T is irreducible, so is $\mathcal{D}_{\widehat{H}}$. The group $\mathrm{Inn}(G)$ does not have a fixed point in T and thus is not elliptic in $\mathcal{D}_{\widehat{H}}$. By Lemma 1.60, the restriction map $\mathrm{res} \colon \mathcal{D}_{\widehat{H}} \to \mathcal{D}$, $X \mapsto X_{\mathrm{Inn}(G)}$ is a homeomorphism onto its image, both in the weak topology and the Gromov topology, and all metric $\widehat{H}$-trees whose restriction to $\mathrm{Inn}(G) \leq \widehat{H}$ lies in $\mathcal{D}$ lie in the same deformation space $\mathcal{D}_{\widehat{H}}$. Therefore, $\mathrm{res}(\mathcal{D}_{\widehat{H}}) \subset \mathcal{D}$ is precisely the subspace of all metric G-trees whose $\mathrm{Inn}(G)$-action extends to an isometric action of $\widehat{H}$, i.e., the H-fixed point set. Finally, if the G-tree T has finitely generated vertex stabilizers then so does the $\widehat{H}$-tree $T_{\widehat{H}}$, as $\widehat{H}$ is an extension of $\mathrm{Inn}(G)$ by a finitely generated group. Contractibility of $\mathcal{D}^H$ now follows from Theorem 1.57. $\qquad\square$

1.3.4 A model for $E(\mathrm{Out}_{\mathcal{D}}(G), \mathcal{F})$

Let G be a finitely generated group. The discussion in Sections 1.3.1 through 1.3.3 proves the following, where a group is *slender* (or *Noetherian*) if all of its subgroups are finitely generated:

Theorem 1.63. *Let $\mathcal{PD}$ be a projectivized deformation space of irreducible metric G-trees, equipped with the weak topology. If $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ with slender point stabilizers then $\mathcal{PD}$ is a model for the classifying space of $\mathrm{Out}_{\mathcal{D}}(G)$ for the family of subgroups of isotropy groups.*

Below, we will give examples in which the family of subgroups of isotropy groups can be described algebraically. In general, however, this is very complicated (see, for instance, [BJ96]).

Proof. The projectivized deformation space $\mathcal{PD}$ can be given the structure of a genuine $\text{Out}_{\mathcal{D}}(G)$-CW-complex by Proposition 1.56. Contractibility of fixed points sets follows from Theorem 1.59, as all subgroups of point stabilizers are assumed finitely generated. $\qquad\square$

Examples

Recall that a group G is *polycyclic* if it admits a finite subnormal series

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$$

such that for each $i = 0, \ldots, n-1$ the quotient G_{i+1}/G_i is cyclic. Polycyclic groups are slender. The *Hirsch length* of a polycyclic group G is the number of infinite cyclic quotients in the above filtration and it is an invariant of G. A group is *virtually polycyclic* if it contains a finite-index subgroup that is polycyclic. For $n \in \mathbb{N}_0$, let $\text{GVP}(n)$ be the class of finitely generated groups G for which there exists a locally finite irreducible minimal G-tree T whose edge stabilizers are virtually polycyclic of Hirsch length n. (Since T is locally finite, its vertex stabilizers are then also virtually polycyclic of Hirsch length n.)

It follows from [Cla07, Lemma 2.1] that if G belongs to $\text{GVP}(n)$ for some $n \in \mathbb{N}_0$ then all locally finite irreducible minimal metric G-trees whose edge stabilizers are virtually polycyclic of Hirsch length n lie in the same deformation space $\mathcal{D}_G$ and we have $\text{Out}_{\mathcal{D}_G}(G) = \text{Out}(G)$.

Example 1.64.

(1) The class $\text{GVP}(0)$ is the class of finitely generated virtually nonabelian free groups. If G is such a group then $\mathcal{D}_G$ is the deformation space of minimal metric G-trees with finite vertex stabilizers from Example 1.30.

(2) The torsion-free groups in $\text{GVP}(1)$ are the nonelementary GBS groups that are not isomorphic to a solvable Baumslag-Solitar group $\text{BS}(1, q)$, $q \neq \pm 1$. If G is such a group then $\mathcal{D}_G$ is the deformation space of minimal metric G-trees with infinite cyclic vertex and edge stabilizers from Example 1.31.

Theorem 1.65 ([Cla07])**.** *Let $G \in \text{GVP}(n)$ for some $n \in \mathbb{N}_0$. If $K \leq \text{Out}(G)$ is a polycyclic subgroup that fixes a point in $\mathcal{D}_G$ then every subgroup $H \leq \text{Out}(G)$ commensurable with K also fixes a point in $\mathcal{D}_G$.*

We may deduce from this the following applications of Theorem 1.63, where all deformation spaces are equipped with the weak topology:

Example 1.66. Let G be a finitely generated virtually nonabelian free group. The projectivized deformation space $\mathcal{PD}$ of minimal metric G-trees with finite

vertex stabilizers is a finite-dimensional model for $E(\mathrm{Out}(G), \mathcal{F}in)$, where $\mathcal{F}in$ is the family of finite subgroups.

Proof. Every finite subgroup of $\mathrm{Out}(G)$ is commensurable with the trivial group and hence fixes a point in $\mathcal{PD}$ by Theorem 1.65. At the same time, it follows from [GL07a, Proposition 8.6] that $\mathrm{Out}(G)$ acts on $\mathcal{PD}$ with finite point stabilizers. Since $\mathrm{Out}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices (Example 1.36), the projectivized deformation space is finite-dimensional. $\square$

Example 1.67. Let G be a nonelementary GBS group that is not isomorphic to a solvable Baumslag-Solitar group $\mathrm{BS}(1, q)$, $q \neq \pm 1$. The projectivized deformation space $\mathcal{PD}$ of minimal metric G-trees with infinite cyclic vertex and edge stabilizers is a model for $E(\mathrm{Out}(G), \mathcal{F})$, where $\mathcal{F}$ is a family of finitely generated virtually free abelian subgroups with bounded rank that is closed under taking finite-index supergroups. If G does not contain a solvable Baumslag-Solitar group $\mathrm{BS}(1, n)$ with $n \geq 2$ then $\mathcal{PD}$ is finite-dimensional.

Proof. By [Lev07, Theorem 3.10], the group $\mathrm{Out}(G)$ acts on $\mathcal{PD}$ with point stabilizers virtually isomorphic to $\mathbb{Z}^k$, where $k = b_1(\mathcal{D})$ or $b_1(\mathcal{D}) - 1$ (see Definition 1.47) depending on G. Theorem 1.65 implies that the family of subgroups of isotropy groups is closed under taking finite-index supergroups.

If G does not contain a solvable Baumslag-Solitar group $\mathrm{BS}(1, n)$ with $n \geq 2$ then $\mathrm{Out}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices (Example 1.37) and the projectivized deformation space is finite-dimensional. $\square$

Chapter 2

The Lipschitz metric on deformation spaces of G-trees

The results in this chapter have already been released in [Mei13].

2.1 The Lipschitz metric

Let $\mathcal{D}$ be a deformation space of metric G-trees and $T, T' \in \mathcal{D}$. By Proposition 1.24, there exists a G-equivariant map $f\colon T \to T'$, which we may choose to be Lipschitz continuous. We denote by $\sigma(f)$ its Lipschitz constant.

Every G-equivariant Lipschitz map $f\colon T \to T'$ is G-equivariantly homotopic relative to the vertices of T to a G-equivariant Lipschitz map $f'\colon T \to T'$ that is linear (i.e., either constant or an immersion with constant slope) on edges. The Lipschitz constant $\sigma(f')$ is then given by the maximal slope of f' on the finitely many G-orbits of edges of T and we have $\sigma(f') \leq \sigma(f)$. We may therefore always assume every G-equivariant Lipschitz map $f\colon T \to T'$ to be linear on edges without increasing its Lipschitz constant.

Definition 2.1. Define $\sigma(T, T') := \inf_f \sigma(f)$, where f ranges over all G-equivariant Lipschitz maps from T to T'.

Recall that, as a set, we think of the projectivized deformation space $\mathcal{PD}$ as the covolume-1-section in $\mathcal{D}$. With this convention, we can assign to each pair of projectivized metric G-trees $(T, T') \in \mathcal{PD} \times \mathcal{PD}$ the well-defined value $\sigma(T, T')$.

Proposition 2.2. *The function*

$$d_{Lip}\colon \mathcal{PD} \times \mathcal{PD} \to \mathbb{R}, \ (T, T') \mapsto \log\big(\sigma(T, T')\big)$$

is an asymmetric pseudometric on $\mathcal{PD}$. That is, for all $T, T', T'' \in \mathcal{PD}$ we have

(1) $d_{Lip}(T, T') \geq 0$;

(2) if T and T' are G-equivariantly isometric then $d_{Lip}(T, T') = 0$;

(3) $d_{Lip}(T, T'') \leq d_{Lip}(T, T') + d_{Lip}(T', T'')$.

We call d_{Lip} the Lipschitz metric.

Proof. To prove (1), let $f\colon T \to T'$ be a G-equivariant Lipschitz map. We will show that $\sigma(f)$ is bounded below by 1. Since the metric G-trees T and T' are minimal, both f and the induced map on metric quotient graphs $G\backslash f\colon G\backslash T \to G\backslash T'$ are surjective. We have $\sigma(G\backslash f) = \sigma(f)$ and $\mathrm{vol}(G\backslash T) = \mathrm{vol}(G\backslash T') = 1$. If now $\sigma(f) < 1$ then

$$\mathrm{vol}(\mathrm{im}(G\backslash f)) \leq \sigma(f) \cdot \mathrm{vol}(G\backslash T) < 1$$

contradicting surjectivity of $G\backslash f$.

Statement (2) is immediate. In order to show (3), observe that for any sequence of G-equivariant Lipschitz maps $T \xrightarrow{f} T' \xrightarrow{f'} T''$ we have $\sigma(T,T'') \leq \sigma(f' \circ f)$ and $\sigma(f' \circ f) \leq \sigma(f) \cdot \sigma(f')$, whence

$$
\begin{aligned}
\log\left(\sigma(T,T'')\right) \leq \inf_{f,f'} \log\left(\sigma(f' \circ f)\right) &\leq \inf_{f,f'} \log\left(\sigma(f) \cdot \sigma(f')\right) \\
&= \inf_{f,f'}\left(\log\left(\sigma(f)\right) + \log\left(\sigma(f')\right)\right) \\
&= \inf_{f} \log\left(\sigma(f)\right) + \inf_{f'} \log\left(\sigma(f')\right) \\
&= \log\left(\sigma(T,T')\right) + \log\left(\sigma(T',T'')\right). \quad \square
\end{aligned}
$$

Proposition 2.3. *The group $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $(\mathcal{PD}, d_{Lip})$ by isometries, i.e., for all $T,T' \in \mathcal{PD}$ and $\phi \in \mathrm{Out}_{\mathcal{D}}(G)$ we have $d_{Lip}(T\phi,T'\phi) = d_{Lip}(T,T')$.*

Proof. Every G-equivariant map from T to T' is also G-equivariant with respect to the actions twisted along ϕ, and vice versa. $\square$

The following example demonstrates why we speak of the Lipschitz metric as an "asymmetric pseudometric":

Example 2.4. In general we have $d_{Lip}(T,T') \neq d_{Lip}(T',T)$ (see [AKB12] for examples in the case of Outer space; see also the remark made after Proposition 2.5). Moreover, $d_{Lip}(T,T') = 0$ does generally not imply that T and T' are G-equivariantly isometric (see Proposition 2.16 for an exception in the case of Outer space; see also Section 2.2.1):

Let $G = F_2 * (\mathbb{Z}/2\mathbb{Z})$ and consider the metric graph of groups decompositions Γ and Γ' of G as in Figure 2.1, where all edge group inclusions are the obvious ones and all edges have length $\frac{1}{3}$. The corresponding Bass-Serre trees T and T' lie in the same deformation space, as they are related by an elementary collapse followed by an elementary expansion (the intermediate graph of groups is given by Γ_{int}). The vertices of T have valence 3 and 6, whereas the vertices of T' all have valence 5. Consequently, T and T' are not homeomorphic and in particular not G-equivariantly isometric. Still, the natural morphism of graphs of groups (in the sense of [Bas93]) from Γ to Γ' lifts to a G-equivariant map from T to T' that is an isometry on edges and thus has Lipschitz constant 1, whence $d_{Lip}(T,T') = 0$.
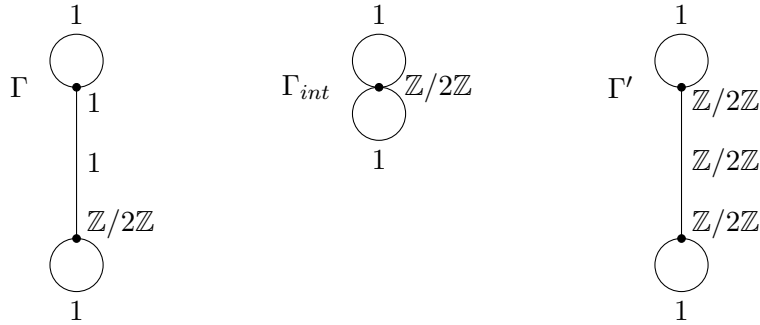
Figure 2.1: The Bass-Serre trees of the above graphs of groups lie in the same deformation space of metric G-trees. They are irreducible and locally finite. Since the group G acts on them cocompactly and with finite point stabilizers, it is virtually free.

The symmetrized Lipschitz metric   A standard way to overcome these issues is to consider the symmetrized Lipschitz metric

$$d_{Lip}^{sym} \colon \mathcal{PD} \times \mathcal{PD} \to \mathbb{R}, \ (T, T') \mapsto d_{Lip}(T, T') + d_{Lip}(T', T)$$

which turns out to be an actual metric on projectivized deformation spaces of irreducible metric G-trees (in Section 2.1.3 we discuss its convergent sequences):

Proposition 2.5. *If the projectivized deformation space $\mathcal{PD}$ is irreducible then for all $T, T' \in \mathcal{PD}$ we have $d_{Lip}^{sym}(T, T') = 0$ if and only if T and T' are G-equivariantly isometric.*

Proof. By Proposition 2.2(2) it suffices to show the "only if" direction. Suppose that we have $d_{Lip}^{sym}(T, T') = 0$, equivalently $d_{Lip}(T, T') = 0$ and $d_{Lip}(T', T) = 0$. Then for all $\varepsilon > 0$ there exist G-equivariant $(1 + \varepsilon)$-Lipschitz maps $f \colon T \to T'$ and $f' \colon T' \to T$. Let $g \in G$ be a hyperbolic group element in T and $p \in A_g \subset T$ a point in its hyperbolic axis. We have

$$l_{T'}(g) \leq d(f(p), gf(p)) = d(f(p), f(gp))$$
$$\leq \sigma(f) \cdot d(p, gp) = \sigma(f) \cdot l_T(g) \leq (1 + \varepsilon) \cdot l_T(g)$$

and, analogously, $l_T(g) \leq (1 + \varepsilon) \cdot l_{T'}(g)$. As ε was arbitrary, we conclude that $l_T = l_{T'}$ and hence, by [CM87, Theorem 3.7], that the irreducible metric G-trees T and T' are G-equivariantly isometric. $\qquad\square$

Remark. Thus, for T and T' as in Example 2.4 we have $d_{Lip}(T', T) > 0$, since $d_{Lip}(T, T') = 0$ but T and T' are not G-equivariantly isometric.

Nevertheless, the arguments in Section 2.2 are specific for the asymmetric pseudometric d_{Lip}. Besides, in contrast to d_{Lip}, the symmetrization d_{Lip}^{sym} fails to be geodesic, as was shown in [FM11, Section 6] in the special case of Outer space (see Section 2.1.4 for the existence of d_{Lip}-geodesics).

2.1.1 Minimal stretch maps

Theorem 2.6. *Let $\mathcal{D}$ be a deformation space of irreducible metric G-trees. For all $T, T' \in \mathcal{D}$ there exists a G-equivariant Lipschitz map $f : T \to T'$ such that $\sigma(f) = \sigma(T, T')$.*

The proof of Theorem 2.6 will involve an argument of Horbez [Hor14] that uses nonprincipal ultrafilters and ultralimits of metric spaces, which are defined as follows:

Definition 2.7. A *nonprincipal ultrafilter* ω on an infinite set I is a finitely additive probability measure with values in $\{0, 1\}$ such that all subsets $S \subseteq I$ are ω-measurable and $\omega(S) = 0$ if S is finite.

Existence of nonprincipal ultrafilters follows from the axiom of choice. Given a nonprincipal ultrafilter ω on the set of natural numbers $\mathbb{N}$, for every bounded sequence $(c_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ there exists a unique point $\lim_\omega c_n \in \mathbb{R}$ such that for all open neighborhoods U of $\lim_\omega c_n$ we have $\omega(\{n \in \mathbb{N} \mid c_n \in U\}) = 1$ (see, for instance, [Kap01, 9.1]). In particular, if the sequence $(c_n)_{n \in \mathbb{N}}$ converges then we have $\lim_\omega c_n = \lim_{n \to \infty} c_n$.

Definition 2.8. Let ω be a nonprincipal ultrafilter on $\mathbb{N}$. For a sequence of metric spaces $(X_n, d_n)_{n \in \mathbb{N}}$ with basepoints $(p_n)_{n \in \mathbb{N}}$ let X_∞ be the set of all sequences $(x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} X_n$ for which the sequence $(d_n(x_n, p_n))_{n \in \mathbb{N}} \subset \mathbb{R}$ is bounded. Let $\sim$ be the equivalence relation on X_∞ defined by

$$(x_n)_{n \in \mathbb{N}} \sim (y_n)_{n \in \mathbb{N}} \quad \text{if} \quad \lim_\omega d_n(x_n, y_n) = 0.$$

Define the *ω-ultralimit* X_ω of $(X_n, d_n, p_n)_{n \in \mathbb{N}}$ as the quotient $X_\infty / \sim$ endowed with the metric $d_\omega((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) := \lim_\omega d_n(x_n, y_n)$.

If each (X_n, d_n), $n \in \mathbb{N}$ is a complete $\mathbb{R}$-tree then (X_ω, d_ω) is again a complete $\mathbb{R}$-tree [Sta09, Lemma 4.6]. Moreover, if a group G acts on each (X_n, d_n) by isometries and for all $g \in G$ the sequence $(d_n(gp_n, p_n))_{n \in \mathbb{N}} \subset \mathbb{R}$ is bounded then (X_ω, d_ω) carries a natural isometric G-action: For $g \in G$ and $(x_n)_{n \in \mathbb{N}} \in X_\omega$ we define $g(x_n)_{n \in \mathbb{N}} := (gx_n)_{n \in \mathbb{N}}$. Since for all $g \in G$ and $n \in \mathbb{N}$ we have

$$d_n(gx_n, p_n) \leq d_n(gx_n, gp_n) + d_n(gp_n, p_n)$$

$$= d_n(x_n, p_n) + d_n(gp_n, p_n)$$

and the sequences of real numbers $(d_n(x_n, p_n))_{n \in \mathbb{N}}$ and $(d_n(gp_n, p_n))_{n \in \mathbb{N}}$ are bounded, so is the sequence $(d_n(gx_n, p_n))_{n \in \mathbb{N}}$. For all $g \in G$ we have

$$
\begin{aligned}
d_\omega(g(x_n)_{n \in \mathbb{N}}, g(y_n)_{n \in \mathbb{N}}) &= \lim_\omega d_n(gx_n, gy_n) \\
&= \lim_\omega d_n(x_n, y_n) = d_\omega((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}})
\end{aligned}
$$

and the action of G on (X_ω, d_ω) is by isometries.

Proof of Theorem 2.6. Let $T, T' \in \mathcal{D}$ and $C = \sigma(T, T')$. We wish to construct a G-equivariant C-Lipschitz map $f \colon T \to T'$.

Let ω be a nonprincipal ultrafilter on $\mathbb{N}$ and $(f_n \colon T \to T')_{n \in \mathbb{N}}$ a sequence of G-equivariant C_n-Lipschitz maps with $C_n \leq 2C$ and $\lim_{n \to \infty} C_n = C$. We will first choose a distinguished basepoint $p \in T$ and show that for all $n \in \mathbb{N}$ the image $f_n(p)$ lies in a bounded subset of T' that does not depend on n. This then implies that for all $g \in G$ the sequence $d'(gf_n(p), f_n(p))_{n \in \mathbb{N}} \subset \mathbb{R}$ is bounded and that the ω-ultralimit $T'_\omega = (T', d', f_n(p))_\omega$ carries a natural isometric G-action. (Evidently, $T_\omega = (T, d, p)_\omega$ carries a natural isometric G-action as well.) Indeed, as the action of G on T is irreducible, G contains a free subgroup of rank 2 acting freely. Suppose that this free subgroup is generated by $g, h \in G$. Since T and T' have the same elliptic subgroups, the free subgroup $\langle g, h \rangle \leq G$ also acts freely on T'. If the hyperbolic axes A_g and A_h in T intersect, they must intersect in a compact segment, as we could otherwise find integers $k, l \in \mathbb{Z} \setminus \{0\}$ such that $g^k h^{-l}$ fixes a point in $A_g \cap A_h$. For the following arguments we will assume that they intersect; if they are disjoint, we replace the basis of the free subgroup with $\{g, hg\}$, whose associated axes then intersect by Proposition 1.4(3). Let $p \in A_g \cap A_h$ be a point that lies in both axes and denote the hyperbolic axes of g and h in T' by A'_g and A'_h respectively. By Proposition 1.4(1), and since f_n is G-equivariant and C_n-Lipschitz with $C_n \leq 2C$, for all $n \in \mathbb{N}$ we have

$$
\begin{aligned}
2C \cdot l_T(g) = 2C \cdot d(gp, p) &\geq d'(f_n(gp), f_n(p)) \\
&= d'(gf_n(p), f_n(p)) = l_{T'}(g) + 2d'(f_n(p), A'_g)
\end{aligned}
$$

and hence $d'(f_n(p), A'_g) \leq \frac{1}{2}(2C \cdot l_T(g) - l_{T'}(g)) \leq C \cdot l_T(g)$. Therefore, $f_n(p)$ lies within a $(C, l_T(g))$-bounded distance from A'_g and, analogously, within a $(C, l_T(h))$-bounded distance from A'_h. We thus conclude that $f_n(p)$ lies within a $(C, l_T(g), l_T(h))$-bounded distance from the compact segment $A'_g \cap A'_h$ if the two axes intersect and from the unique compact connecting segment between them if they are disjoint. In particular, $f_n(p)$ lies in a bounded subset of T' that does not depend on n. As remarked above, this implies that the ultralimits $T_\omega = (T, d, p)_\omega$

and $T'_\omega = (T', d', f_n(p))_\omega$ carry natural isometric G-actions.

The metric G-trees T and T' embed G-equivariantly and isometrically into T_ω and T'_ω respectively: Since for all $n \in \mathbb{N}$ the point $f_n(p) \in T'$ lies in a bounded subset that does not depend on n, for all $x \in T'$ the sequence $(d'(x, f_n(p)))_{n \in \mathbb{N}}$ is bounded and the constant sequence $(x)_{n \in \mathbb{N}}$ defines a point in T'_ω. One easily verifies that the natural inclusion $T' \hookrightarrow T'_\omega$, $x \mapsto (x)_{n \in \mathbb{N}}$ is indeed G-equivariant and isometric. We analogously obtain a G-equivariant isometric embedding $T \hookrightarrow T_\omega$.

Observe next that if $(d(x_n, p))_{n \in \mathbb{N}}$ is bounded then $(d'(f_n(x_n), f_n(p)))_{n \in \mathbb{N}}$ is bounded as well, since for all $n \in \mathbb{N}$ we have $d'(f_n(x_n), f_n(p)) \leq 2C \cdot d(x_n, p)$. Thus, the maps $(f_n)_{n \in \mathbb{N}}$ induce a natural map

$$f_\omega \colon T_\omega \to T'_\omega, \ (x_n)_{n \in \mathbb{N}} \mapsto (f_n(x_n))_{n \in \mathbb{N}}.$$

The map f_ω is easily seen to be G-equivariant, since for all $g \in G$ we have

$$\begin{aligned} f_\omega(g(x_n)_{n \in \mathbb{N}}) = f_\omega((gx_n)_{n \in \mathbb{N}}) &= (f_n(gx_n))_{n \in \mathbb{N}} \\ &= (gf_n(x_n))_{n \in \mathbb{N}} = g(f_n(x_n))_{n \in \mathbb{N}} = gf_\omega((x_n)_{n \in \mathbb{N}}). \end{aligned}$$

Moreover, f_ω is C-Lipschitz, since for all $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \in T_\omega$ we have

$$\begin{aligned} d'_\omega(f_\omega((x_n)_{n \in \mathbb{N}}), f_\omega((y_n)_{n \in \mathbb{N}})) &= \lim_\omega d'(f_n(x_n), f_n(y_n)) \\ &\leq \lim_\omega (C_n \cdot d(x_n, y_n)) \\ &= \lim_\omega C_n \cdot \lim_\omega d(x_n, y_n) = C \cdot d_\omega((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}). \end{aligned}$$

Finally, T'_ω is a complete $\mathbb{R}$-tree, being the ω-ultralimit of complete $\mathbb{R}$-trees (namely, metric simplicial trees). In particular, the metric simplicial tree T' embeds into T'_ω as a closed subspace, as complete subspaces of complete metric spaces are closed. By the nature of $\mathbb{R}$-trees, there exists a continuous nearest point projection of T'_ω onto the closed G-invariant subtree T', which is easily seen to be G-equivariant and 1-Lipschitz. We define $f \colon T \to T'$ as the composition of the G-equivariant isometric embedding $T \hookrightarrow T_\omega$ with $f_\omega \colon T_\omega \to T'_\omega$ and the nearest point projection $T'_\omega \to T'$, and we obtain a G-equivariant C-Lipschitz map from T to T'. □

Train tracks and optimal maps

We will be interested in particularly nice G-equivariant Lipschitz maps realizing $\sigma(T, T')$, so-called *optimal maps*. In order to define and construct optimal maps, we involve the concept of *train tracks*:

Definition 2.9. Let $\mathcal{D}$ be a deformation space of metric G-trees and $T \in \mathcal{D}$. A *direction* at a point $x \in T$ is a germ of isometric embeddings $\gamma\colon [0, \varepsilon) \to T$, $\varepsilon > 0$ with $\gamma(0) = x$. Given $g \in G$ with $gx \neq x$, we will denote the unique direction at x pointing towards gx by $\delta_{x,gx}$. Denote the set of directions at x by $D_x T$. A *train track structure* on T is a collection of equivalence relations, one on $D_v T$ for each vertex $v \in V(T)$, such that two directions $\delta_1, \delta_2 \in D_v T$ are equivalent (denoted $\delta_1 \sim \delta_2$) if and only if for all $g \in G$ the directions $g\delta_1, g\delta_2 \in D_{gv} T$ are equivalent as well. Equivalence classes of directions at a vertex $v \in V(T)$ are called *gates* at v. A *turn* at a vertex $v \in V(T)$ is a pair of directions at v. Given a train track structure on T, we say that a turn at a vertex is *illegal* if the two directions are equivalent, i.e., if they represent the same gate, and *legal* if not. Whenever a nondegenerate immersed path γ in T passes through a vertex v of T, we may locally reparametrize γ to an isometric embedding so that the incoming direction (with opposite orientation) and the outgoing direction of γ at v define a turn at v. A nondegenerate immersed path in T is *legal* if it only makes legal turns and *illegal* otherwise.

Definition 2.10. Let $\mathcal{D}$ be a deformation space of metric G-trees and $T, T' \in \mathcal{D}$. Let $f\colon T \to T'$ be a G-equivariant map that is linear on edges. We denote the union of all (closed) edges of T on which f attains its maximal slope by $\Delta(f) \subset T$ and we call it the *tension forest* of f. The tension forest $\Delta(f) \subset T$ is a G-invariant subforest.

Every G-equivariant map $f\colon T \to T'$ that is linear on edges defines a natural train track structure on its tension forest $\Delta = \Delta(f) \subset T$ as follows: For each vertex $v \in V(\Delta)$ we have a map $D_v f\colon D_v \Delta \to D_{f(v)} T'$ that maps the direction of $\gamma\colon [0, \varepsilon) \to \Delta$ with $\gamma(0) = v$ to the direction of the unique isometric embedding in the reparametrization class of $f \circ \gamma$ (since f does not collapse any edges in its tension forest, it has nonzero slope on the image of γ). We define two directions $\delta_1, \delta_2 \in D_v \Delta$ to be equivalent if $D_v f(\delta_1) = D_v f(\delta_2)$. By the G-equivariance of f, this collection of equivalence relations is indeed a train track structure on Δ.

The tension forest $\Delta(f)$ endowed with the train track structure defined by f might have vertices of valence 1 and, more generally, there might be vertices with only one gate.

Definition 2.11. A G-equivariant Lipschitz map $f\colon T \to T'$ that realizes $\sigma(T, T')$ and is linear on edges is an *optimal map* if its tension forest $\Delta(f)$ has at least 2 gates at every vertex.

Optimality of f implies that any legal path in $\Delta(f)$ may be extended in both directions to a longer legal path and, inductively, that there exists a legal line in $\Delta(f)$. This will be made use of in the proof of Theorem 2.14.

Proposition 2.12. *Let $\mathcal{D}$ be a deformation space of metric G-trees and $T, T' \in \mathcal{D}$. Every G-equivariant Lipschitz map $f \colon T \to T'$ that realizes $\sigma(T, T')$ and is linear on edges is G-equivariantly homotopic to an optimal map $f' \colon T \to T'$ with $\Delta(f') \subseteq \Delta(f)$. If f is not optimal to begin with then we have $\Delta(f') \neq \Delta(f)$.*

Theorem 2.6 and Proposition 2.12 imply that if $\mathcal{D}$ consists of irreducible metric G-trees then for all $T, T' \in \mathcal{D}$ there exists an optimal map $f \colon T \to T'$.

Proof. Let $\Delta = \Delta(f)$. If a vertex $v \in V(\Delta)$ has only one gate $\delta \in D_v\Delta$, slightly move $f(v)$ in the direction of $D_v f(\delta) \in D_{f(v)} T'$ (see Figure 2.2). Perform this
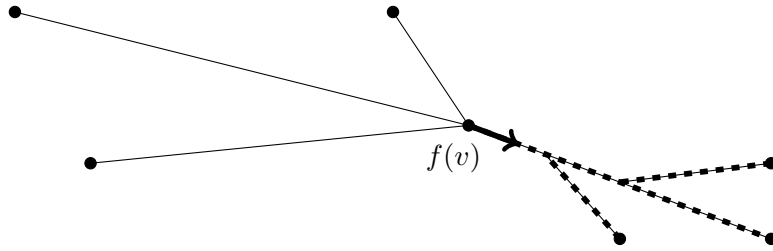

Figure 2.2: The image of Δ under f (dashed) and the direction in which we slightly move $f(v)$ (arrow).

perturbation G-equivariantly and keep the homotopy fixed on all other G-orbits of vertices of T. This decreases the slope of f on the G-orbits of all edges of Δ adjacent to v and we obtain a G-equivariant Lipschitz map $f' \colon T \to T'$ with $\Delta(f') \subset \Delta$ but $\Delta(f') \neq \Delta$. Keeping the perturbation small enough ensures that the (finitely many) G-orbits of the edges of $T \setminus \Delta$ adjacent to v, on which the slope is increased, do not become part of the new tension forest. As f is assumed to have minimal Lipschitz constant among all G-equivariant Lipschitz maps from T to T', we will not have removed all edges of Δ and started over with a new tension forest that corresponds to a strictly smaller maximal stretching factor. This process eventually terminates by the cocompactness of T. □

2.1.2 Witnesses

The results in this section will imply that the Lipschitz metric on projectivized deformation space of irreducible metric G-trees can be computed in terms of hyperbolic translation lengths. We begin with an easy observation:

Lemma 2.13. *Let $\mathcal{D}$ be a deformation space of metric G-trees and $T, T' \in \mathcal{D}$. For any G-equivariant Lipschitz map $f \colon T \to T'$ and any hyperbolic group element $g \in G$ we have $\sigma(f) \geq \frac{l_{T'}(g)}{l_T(g)}$. In particular, we have $\sigma(T, T') \geq \sup_g \frac{l_{T'}(g)}{l_T(g)}$, where g ranges over all hyperbolic group elements of G.*

Proof. Let $p \in A_g$. We have $l_{T'}(g) \leq d(gf(p), f(p)) \leq \sigma(f) \cdot d(gp, p) = \sigma(f) \cdot l_T(g)$, whence the claim. $\square$

Theorem 2.14. *Let $\mathcal{D}$ be a deformation space of irreducible metric G-trees. For all $T, T' \in \mathcal{D}$ there exists a hyperbolic group element $\xi \in G$ such that*

$$\sigma(T, T') = \frac{l_{T'}(\xi)}{l_T(\xi)} = \sup_g \frac{l_{T'}(g)}{l_T(g)}$$

where g ranges over all hyperbolic group elements of G. In fact, we can always arrange that some (and hence any) fundamental domain for the action of ξ on its hyperbolic axis $A_\xi \subset T$ meets each G-orbit of vertices of T at most 10 times.

We will call a hyperbolic group element $\xi \in G$ (or, depending on the context, its hyperbolic axis $A_\xi \subset T$) satisfying $\sigma(T, T') = \frac{l_{T'}(\xi)}{l_T(\xi)}$ a *witness* for the minimal stretching factor from T to T'. A hyperbolic group element $g \in G$ such that some (and hence any) fundamental domain for the action of g on its axis $A_g \subset T$ meets each G-orbit of vertices of T at most 10 times will be called a *candidate of T*. Theorem 2.14 asserts that there always exists a witness which is a candidate (our notion of candidates is nonstandard, as remarked below). We will denote by $\text{cand}(T) \subset G$ the set of candidates of T.

If we choose for each $g \in \text{cand}(T)$ a fundamental domain for the action of g on its axis $A_g \subset T$, these fundamental domains project to only finitely many different edge loops in the metric quotient graph $G \backslash T$. In particular, the set of translation lengths $\{l_T(g) \mid g \in \text{cand}(T)\} \subset \mathbb{R}$ is finite. At the same time, for any $T' \in \mathcal{D}$ the set $\{l_{T'}(g) \mid g \in \text{cand}(T)\} \subset \mathbb{R}$ is finite as well, since the image of $l_{T'}$ in $\mathbb{R}$ is discrete and we have $l_{T'}(g) \leq \sigma(T, T') \cdot l_T(g)$ for all $g \in G$. If $T, T' \in \mathcal{D}$ are G-equivariantly homeomorphic then $\text{cand}(T) = \text{cand}(T')$.

Remark. With significantly more effort, one can further show that there always exists a witness whose hyperbolic axis projects to a loop in $G \backslash T$ with certain topological properties, as was done in [FM11, Proposition 3.15] for free metric F_n-trees and in [FM13, Theorem 9.10] in the special case of irreducible metric G-trees with trivial edge stabilizers. However, the weaker finiteness properties of candidates discussed above will suffice for all our applications.

In the proof of Theorem 2.14 we will make use of the following characterization of witnesses:

Lemma 2.15. *Let $\mathcal{D}$ be a deformation space of metric G-trees and $T, T' \in \mathcal{D}$. For an optimal map $f : T \to T'$ and a hyperbolic group element $g \in G$, the following are equivalent:*

(1) $\sigma(f) = \frac{l_{T'}(g)}{l_T(g)}$;

(2) *the hyperbolic axis $A_g \subseteq T$ is contained in the tension forest $\Delta(f)$ and it is legal with respect to the train track structure defined by f;*

(3) *the hyperbolic axis $A_g \subseteq T$ is contained in the tension forest $\Delta(f)$ and $f(A_g) \subseteq T'$ equals A'_g, the hyperbolic axis of g in T'.*

Proof. $(2) \Rightarrow (3) \Rightarrow (1)$ Since f is G-equivariant and A_g is legal, the image $f(A_g)$ is a g-invariant line and thus equals A'_g. Consequently, and since A_g is assumed to lie in the tension forest $\Delta(f)$, for $p \in A_g$ we have

$$l_{T'}(g) = d(gf(p), f(p)) = d(f(gp), f(p)) = \sigma(f) \cdot d(gp, p) = \sigma(f) \cdot l_T(g)$$

and we conclude that $\sigma(f) = \frac{l_{T'}(g)}{l_T(g)}$.

$(1) \Rightarrow (2)$ We will argue by contradiction. First, suppose that A_g is not contained in the tension forest $\Delta(f)$. Then for any $p \in A_g$ the segment $[p, gp]$ is stretched by strictly less than $\sigma(f)$ and we have

$$\begin{aligned} l_{T'}(g) &\leq d(gf(p), f(p)) = d(f(gp), f(p)) \\ &< \sigma(f) \cdot d(gp, p) = \sigma(f) \cdot l_T(g) \end{aligned}$$

whence $\sigma(f) > \frac{l_{T'}(g)}{l_T(g)}$. On the other hand, if A_g is contained in $\Delta(f)$ but not legal with respect to the train track structure defined by f then there exists a vertex $v \in V(A_g)$ at which the two directions of A_g define the same gate. The images of $[g^{-1}v, v]$ and $[v, gv]$ under f then overlap in a segment of positive length and $l_{T'}(g)$ is strictly smaller than $\sigma(f) \cdot l_T(g)$, whence $\frac{l_{T'}(g)}{l_T(g)} < \sigma(f)$. $\qquad \square$

Proof of Theorem 2.14. Since T and T' are irreducible, there exists an optimal map $f \colon T \to T'$ (this is the only step in the proof that uses irreducibility). By Lemma 2.15, it suffices to find a hyperbolic group element $\xi \in G$ whose axis $A_\xi \subseteq T$ is contained in $\Delta = \Delta(f)$ and legal with respect to the train track structure defined by f. It will be clear from our construction of ξ that a fundamental domain for the action of ξ on A_ξ meets each G-orbit of vertices of T at most 10 times, i.e., ξ is a candidate.

Since Δ has at least 2 gates at every vertex, we can find a legal ray $R \subset \Delta$ based at some vertex $v_0 \in V(\Delta)$. There always exists a vertex $x \in V(R)$ such that $x = gx_0$ for some $x_0 \in [v_0, x)$ and some hyperbolic group element $g \in G$, which can be seen as follows: Since T is minimal and therefore cocompact, there are only finitely many G-orbits of vertices in T. We can thus find pairwise distinct vertices $x_0, x_1, x_2 \in V(R)$ and $g_1, g_2 \in G$ such that $x_1 = g_1x_0$ and $x_2 = g_2x_1$. If either g_1 or g_2 is hyperbolic, we are done. If both are elliptic, each g_i fixes only the midpoint of the segment $[x_{i-1}, x_i]$ and the product $g = g_2g_1$ maps x_0 to x_2. The fixed point sets of g_1 and g_2 being disjoint, g is hyperbolic by Proposition 1.4(2).

We choose x to be the first vertex of R with this property, for which the segment $[x_0, x]$ meets the G-orbit of x_0 at most 3 times (there could lie an elliptic translate of x_0 in between x_0 and gx_0) and each G-orbit of vertices other than that of x_0 at most 2 times. The segment $[x_0, x] \subset R$ is then a closed fundamental domain for the action of g on A_g and the stretching factor of f on any subsegment of A_g equals that of f on $[x_0, x] \subset \Delta$, whence $A_g \subseteq \Delta$. If A_g is legal, we are done. If not, since all turns of A_g in between x_0 and x are legal but A_g is assumed illegal, the turns at x_0 and x must be illegal. We then have $A_g \cap R = [x_0, x]$ and we continue moving along the legal ray R until we reach the first vertex $y \in V(R)$ with $y = hy_0$ for some $y_0 \in (x, y)$ and some hyperbolic group element $h \in G$. Analogously, the segment $[y_0, y]$ meets the G-orbit of y_0 at most 3 times and each G-orbit of vertices other than that of y_0 at most 2 times. Note that the open segment (x, y_0) meets each G-orbit of vertices of T at most 2 times.

If $A_h \subseteq \Delta$ is legal, we are done. If not, we have $A_g \cap A_h = \emptyset$ and the product hg is hyperbolic by Proposition 1.4(3). A closed fundamental domain for the action of hg on its axis A_{hg} is given by $[x_0, hx] = [x_0, x] \cup [x, y_0] \cup [y_0, y] \cup h[y_0, x] \subset \Delta$, since we have $hg[x_0, hx] \cup [x_0, hx] = \{hx\}$ (see Figure 2.3). We conclude that $A_{hg} \subseteq \Delta$. In particular, the fundamental domain $[x_0, hx)$ meets each G-orbit of vertices of T at most $3 + 2 + 3 + 2 = 10$ times and hg is a candidate of T.
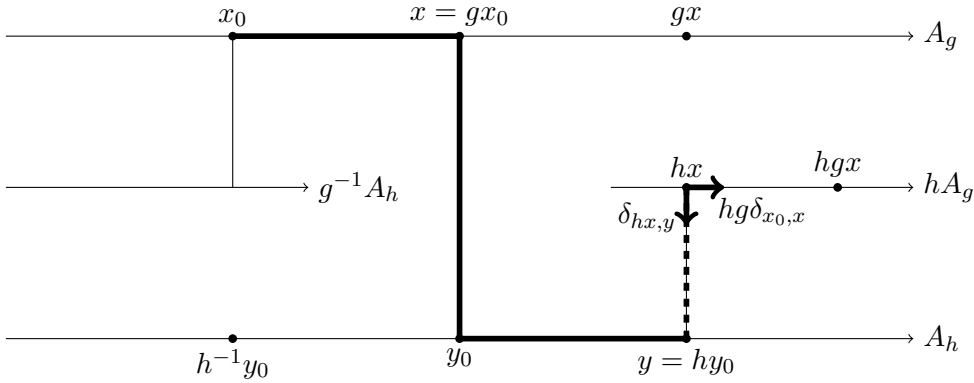

Figure 2.3: The segment $[x_0, hx]$ (both bold and dashed, where we know that the bold part lies in R) and the directions $hg\delta_{x_0,x}$ and $\delta_{hx,y}$ (arrows).

In order to show that A_{hg} is legal, it suffices to show that $[x_0, hx]$ does not make any illegal turns and that the directions $hg\delta_{x_0,x}$ and $\delta_{hx,y}$ are not equivalent. By the legality of R, it is clear that all turns of the subsegment $[x_0, y]$ are legal. The turn of $[x_0, hx]$ at y is legal if and only if $\delta_{y,y_0} \not\sim \delta_{y,hx}$. This is equivalent to $\delta_{y_0,h^{-1}y_0} \not\sim \delta_{y_0,x}$ but which is true since A_h is assumed illegal (i.e., $\delta_{y_0,h^{-1}y_0} \sim \delta_{y_0,y}$)

and $\delta_{y_0,y} \not\sim \delta_{y_0,x}$ by the legality of R. Lastly, we need to show that $hg\delta_{x_0,x} \not\sim \delta_{hx,y}$. We analogously observe that this is the case if and only if $\delta_{x,gx} \not\sim \delta_{x,y_0}$ but which is true as A_g is illegal (i.e., $\delta_{x,gx} \sim \delta_{x,x_0}$) and $\delta_{x,x_0} \not\sim \delta_{x,y_0}$ by the legality of R. $\qquad\square$

We give a proof of the following well-known fact in the language of trees:

Proposition 2.16. *The Lipschitz metric on Outer space $\mathcal{PX}_n$ (Example 1.29) is an asymmetric metric. That is, if two metric F_n-trees $T, T' \in \mathcal{PX}_n$ satisfy $d_{Lip}(T, T') = 0$ then they are F_n-equivariantly isometric.*

Proof. Let $f\colon T \to T'$ be an optimal map with $\sigma(f) = 1$. For $e \in E(T)$ we denote by $\sigma_e(f)$ the slope of f on e. Since f is surjective, the induced map on metric quotient graphs $F_n\backslash f\colon F_n\backslash T \to F_n\backslash T'$ is surjective as well and we have

$$1 = \mathrm{vol}(\mathrm{im}(F_n\backslash f)) = \left(\sum_{e \in E(F_n\backslash T)} \sigma_e(f) \cdot \mathrm{length}(e) \right) - C$$

where $C \geq 0$ measures overlaps of images of edges. Since T has covolume 1 and $\sigma_e(f) \leq \sigma(f) = 1$ for all $e \in E(T)$, we conclude that $1 \leq \sigma(f) - C = 1 - C$, whence $C = 0$. Consequently, we have $\sigma_e(f) = 1$ for all $e \in E(T)$ and hence $\Delta(f) = T$.

The F_n-trees in $\mathcal{PX}_n$ are irreducible, and in order to prove the claim it suffices to show that for all hyperbolic (here, nontrivial) group elements $g \in F_n$ we have $l_T(g) = l_{T'}(g)$. On the one hand, if $g \in F_n$ is hyperbolic and $p \in A_g \subset T$ a point in its hyperbolic axis, we have

$$l_{T'}(g) \leq d(f(p), gf(p)) = d(f(p), f(gp)) \leq \sigma(f) \cdot d(p, gp) = l_T(g).$$

On the other hand, suppose that there exists a hyperbolic group element $g \in F_n$ such that $l_{T'}(g)$ is strictly smaller than $l_T(g)$. Since the tension forest of f is all of T, by Lemma 2.15 the hyperbolic axis $A_g \subset T$ cannot be legal with respect to the train track structure defined by f. Hence, we can find a vertex $v \in V(A_g)$ at which the turn defined by A_g is not legal, i.e., at which the germs of two adjacent edges are mapped to the same germ under f. Since F_n acts on T freely, the two germs are not F_n-equivalent and we can find a fundamental domain $X \subset T$ for the action of F_n on T that contains the two germs and has volume 1. Its image $f(X) \subset T'$ is a fundamental domain for the action of F_n on T' whose volume is strictly smaller than 1, contradicting the fact that T' has covolume 1. $\qquad\square$

Remark. The proof of Proposition 2.16 is specific for free metric F_n-trees, as the two germs may otherwise be G-equivalent (their common vertex may be stabilized by a nontrivial group element that swaps the two adjacent edges). In that case, we can no longer find a fundamental domain of volume 1 that contains both germs.

2.1.3 Convergent sequences

In this section we relate topological convergence in projectivized deformation spaces of metric G-trees with convergence with respect to the (symmetrized) Lipschitz metric.

Proposition 2.17. *Let $\mathcal{PD}$ be a projectivized deformation space of irreducible metric G-trees and $(T_k)_{k\in\mathbb{N}}$ a sequence in $\mathcal{PD}$ that converges to $T \in \mathcal{PD}$ in the weak topology. Then $\lim_{k\to\infty} d_{Lip}^{sym}(T_k, T) = 0$.*

In the weak topology, $\mathcal{PD}$ is homeomorphic to the covolume-1-section in the unprojectivized deformation space $\mathcal{D}$. Thus, the sequence $(T_k)_{k\in\mathbb{N}}$ weakly converges to T also as covolume-1-representatives in $\mathcal{D}$. The weak topology being the finest of the three topologies, $(T_k)_{k\in\mathbb{N}}$ converges to T in all three topologies, where convergence in the unprojectivized axes topology means that for all $g \in G$ we have $\lim_{k\to\infty} l_{T_k}(g) = l_T(g)$ (pointwise convergence of translation length functions).

Proof. We will first show that $\lim_{k\to\infty} d_{Lip}(T_k, T) = 0$. Let $(f_k \colon T_k \to T)_{k\in\mathbb{N}}$ be a sequence of optimal maps. By Theorem 2.14, for all $k \in \mathbb{N}$ there exists a candidate $\xi_k \in \operatorname{cand}(T_k) \subset G$ such that

$$d_{Lip}(T_k, T) = \log\left(\frac{l_T(\xi_k)}{l_{T_k}(\xi_k)}\right).$$

Since the sequence $(T_k)_{k\in\mathbb{N}}$ converges weakly, it meets only finitely many open simplices of $\mathcal{PD}$ and the metric G-trees $(T_k)_{k\in\mathbb{N}}$ are of only finitely many G-equivariant homeomorphism types. After decomposing the sequence into subsequences (for each of which we will obtain the same result), we may assume that the metric G-trees are in fact all G-equivariantly homeomorphic, or even equal as nonmetric G-trees. The set of candidates $\operatorname{cand}(T_k) \subset G$ is then independent of k and $(l_T(\xi_k))_{k\in\mathbb{N}}$ takes only finitely many values. After decomposing $(T_k)_{k\in\mathbb{N}}$ into subsequences once more, we may assume that $(l_T(\xi_k))_{k\in\mathbb{N}}$ is constant, say $l_T(\xi_k) = C$ for all $k \in \mathbb{N}$.

By the remarks made above, the sequence $(T_k)_{k\in\mathbb{N}}$ converges also as covolume-1-representatives in the unprojectivized axes topology. Thus, for all $K \in \mathbb{N}$ we have $\lim_{k\to\infty} l_{T_k}(\xi_K) = l_T(\xi_K) = C$. Recall that the candidates $(\xi_K)_{K\in\mathbb{N}} \subset G$ give rise to only finitely many different edge loops in the quotient graph $G\backslash T_1$. In fact, if two candidates ξ_{K_1} and ξ_{K_2} give rise to the same edge loop in $G\backslash T_1$ then they give rise to the same edge loop in $G\backslash T_k$ for all $k \in \mathbb{N}$ (because the metric G-trees $(T_k)_{k\in\mathbb{N}}$ all have the same underlying nonmetric G-tree). Thus, the family of sequences $\{(l_{T_k}(\xi_K))_{k\in\mathbb{N}} \mid K \in \mathbb{N}\}$ is finite and for all $\varepsilon > 0$ there exists $N > 0$

such that for all $K \in \mathbb{N}$ we have

$$|C - l_{T_k}(\xi_K)| < \varepsilon$$

whenever $k \geq N$. In particular, we have $|C - l_{T_k}(\xi_k)| < \varepsilon$ whenever $k \geq N$ and we conclude that $\lim_{k\to\infty} l_{T_k}(\xi_k) = C$. Consequently,

$$\lim_{k\to\infty} d_{Lip}(T_k, T) = \log\left(\lim_{k\to\infty} \frac{l_T(\xi_k)}{l_{T_k}(\xi_k)}\right) = \log\left(\frac{C}{\lim_{k\to\infty} l_{T_k}(\xi_k)}\right) = \log(1) = 0.$$

Showing that $\lim_{k\to\infty} d_{Lip}(T, T_k) = 0$ is similar but easier, because it does not require that the sequence $(T_k)_{k\in\mathbb{N}}$ meets only finitely many open simplices of $\mathcal{PD}$. $\qquad\square$

As for the converse of Proposition 2.17, we have the following:

Proposition 2.18. *Let $\mathcal{PD}$ be a projectivized deformation space of irreducible metric G-trees and $(T_k)_{k\in\mathbb{N}}$ a sequence in $\mathcal{PD}$ such that for some $T \in \mathcal{PD}$ we have $\lim_{k\to\infty} d_{Lip}^{sym}(T_k, T) = 0$. Then $(T_k)_{k\in\mathbb{N}}$ converges to T in the axes topology.*

In contrast to convergence in the unprojectivized axes topology, the sequence $(T_k)_{k\in\mathbb{N}}$ converges to T in the projectivized axes topology if there exist positive real numbers $(C_k)_{k\in\mathbb{N}}$ such that for all $g \in G$ we have $\lim_{k\to\infty} C_k \cdot l_{T_k}(g) = l_T(g)$ (pointwise convergence of projectivized translation length functions).

Proof. We will argue as in the proof of [FM11, Theorem 4.11]. For any positive real-valued function f satisfying $\sup \frac{1}{f(x)} < \infty$ we have $\sup \frac{1}{f(x)} = \frac{1}{\inf f(x)}$. Therefore, since

$$\frac{1}{\frac{l_{T_k}(g)}{l_T(g)}} = \frac{l_T(g)}{l_{T_k}(g)} \leq \sigma(T_k, T) < \infty$$

for all hyperbolic group elements $g \in G$, we have

$$\lim_{k\to\infty} d_{Lip}^{sym}(T_k, T) = 0 \quad \Leftrightarrow \quad \lim_{k\to\infty} \frac{\sup_g \frac{l_{T_k}(g)}{l_T(g)}}{\inf_g \frac{l_{T_k}(g)}{l_T(g)}} = 1.$$

Assuming that $\lim_{k\to\infty} d_{Lip}^{sym}(T_k, T) = 0$, we conclude that for all $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that for all $k \geq K$ we have

$$(2.1) \qquad \inf_g \frac{l_{T_k}(g)}{l_T(g)} \leq \sup_g \frac{l_{T_k}(g)}{l_T(g)} \leq \inf_g \frac{l_{T_k}(g)}{l_T(g)} \cdot (1 + \varepsilon).$$

Clearly, for all hyperbolic group elements $\xi \in G$ we have

$$\inf_g \frac{l_{T_k}(g)}{l_T(g)} \leq \frac{l_{T_k}(\xi)}{l_T(\xi)} \leq \sup_g \frac{l_{T_k}(g)}{l_T(g)}.$$

Setting $I_k = \inf_g \frac{l_{T_k}(g)}{l_T(g)}$, inequality (2.1) implies that $I_k \leq \frac{l_{T_k}(\xi)}{l_T(\xi)} \leq I_k \cdot (1 + \varepsilon)$ whenever $k \geq K$. In particular, the unprojectivized translation length functions $(\frac{1}{I_k} l_{T_k})_{k \in \mathbb{N}}$ converge to l_T uniformly and *a fortiori* pointwise. We conclude that the metric G-trees $(T_k)_{k \in \mathbb{N}}$ converge to T in the projectivized axes topology. $\square$

Recall from Section 1.2.2 that if $\mathcal{PD}$ is a projectivized deformation space of locally finite irreducible metric G-trees with finitely generated vertex stabilizers then the Gromov topology, the axes topology, and the weak topology agree.

Corollary 2.19. *Let $\mathcal{PD}$ be a projectivized deformation space of locally finite irreducible metric G-trees with finitely generated vertex stabilizers. The symmetrized Lipschitz metric d_{Lip}^{sym} induces the standard topology on $\mathcal{PD}$.*

Proof. Since the locally finite complex $\mathcal{PD}$ is metrizable, it suffices to show that the two topologies have the same convergent sequences. This immediately follows from Propositions 2.17 and 2.18 and the fact that the Gromov topology, the axes topology, and the weak topology agree on $\mathcal{PD}$. $\square$

Example 2.20. The symmetrized Lipschitz metric induces the standard topology on the projectivized deformation spaces discussed in Examples 1.30 and 1.31.

2.1.4 Folding paths and geodesics

Let $\mathcal{D}$ be a deformation space of metric G-trees and $T, T' \in \mathcal{D}$. Recall from Section 1.3.2 that if we are given a morphism $f \colon T \to T'$, we may "fold T along f" to obtain a 1-parameter family of metric G-trees $(T_t)_{t \in [0, \infty]}$ in $\mathcal{D}$ together with morphisms $\phi_t \colon T \to T_t$ and $\psi_t \colon T_t \to T'$ such that

- $T_0 = T$ and $T_\infty = T'$;
- $\phi_0 = id_T$, $\phi_\infty = \psi_0 = f$, and $\psi_\infty = id_{T'}$;
- for all $t \in [0, \infty]$ the following diagram commutes:

$$
\begin{array}{ccc}
T & \xrightarrow{\quad f \quad} & T' \\
& \phi_t \searrow \quad \nearrow \psi_t & \\
& T_t &
\end{array}
$$

As remarked in Section 1.3.2, the folding path $[0, \infty] \to \mathcal{D}$, $t \to T_t$ and its projection to the projectivized deformation space $\mathcal{PD}$ are continuous in all three topologies.

Definition 2.21. Let $\mathcal{PD}$ be a projectivized deformation space of metric G-trees. A path $\gamma \colon [a, b] \to \mathcal{PD}$, $t \mapsto \gamma(t)$ with $a < b \in \mathbb{R}$ is d_{Lip}-*continuous* if for all convergent sequences $(x_n)_{n \in \mathbb{N}} \subset [a, b]$ with $\lim_{n \to \infty} x_n = x$ we have

$$\lim_{n \to \infty} d_{Lip}(\gamma(x_n), \gamma(x)) = 0 \quad \text{and} \quad \lim_{n \to \infty} d_{Lip}(\gamma(x), \gamma(x_n)) = 0.$$

We say that a d_{Lip}-continuous path $\gamma \colon [a, b] \to \mathcal{PD}$, $t \mapsto \gamma(t)$ with $a < b \in \mathbb{R}$ is a d_{Lip}-*geodesic* if for all $x < y < z \in [a, b]$ we have

$$d_{Lip}(\gamma(x), \gamma(y)) + d_{Lip}(\gamma(y), \gamma(z)) = d_{Lip}(\gamma(x), \gamma(z)).$$

Remark. In metric spaces, geodesics in the above sense can be reparametrized to have unit speed. However, since d_{Lip} is an asymmetric pseudometric, unit speed reparametrizations in $\mathcal{PD}$ need not always exist.

Lemma 2.22. *Let $\mathcal{PD}$ be a projectivized deformation space of irreducible metric G-trees and $\gamma \colon [a, b] \to \mathcal{PD}$ a d_{Lip}-continuous path with $a < b \in \mathbb{R}$. If for all $x < y < z \in [a, b]$ there exists a hyperbolic group element $\xi \in G$ such that*

$$(2.2) \qquad \sigma(\gamma(x), \gamma(y)) = \frac{l_{\gamma(y)}(\xi)}{l_{\gamma(x)}(\xi)} \quad \text{and} \quad \sigma(\gamma(y), \gamma(z)) = \frac{l_{\gamma(z)}(\xi)}{l_{\gamma(y)}(\xi)}$$

then γ is a d_{Lip}-geodesic.

Proof. We have

$$\sup_g \frac{l_{\gamma(z)}(g)}{l_{\gamma(x)}(g)} \geq \frac{l_{\gamma(z)}(\xi)}{l_{\gamma(x)}(\xi)} = \frac{l_{\gamma(y)}(\xi)}{l_{\gamma(x)}(\xi)} \cdot \frac{l_{\gamma(z)}(\xi)}{l_{\gamma(y)}(\xi)} = \sup_g \left(\frac{l_{\gamma(y)}(g)}{l_{\gamma(x)}(g)} \right) \cdot \sup_g \left(\frac{l_{\gamma(z)}(g)}{l_{\gamma(y)}(g)} \right)$$

and hence, by Theorem 2.14, $d_{Lip}(\gamma(x), \gamma(z)) \geq d_{Lip}(\gamma(x), \gamma(y)) + d_{Lip}(\gamma(y), \gamma(z))$. We conclude that $d_{Lip}(\gamma(x), \gamma(z)) = d_{Lip}(\gamma(x), \gamma(y)) + d_{Lip}(\gamma(y), \gamma(z))$. $\square$

By Proposition 2.17, if $\mathcal{PD}$ is irreducible then any path in $\mathcal{PD}$ that is continuous in the weak topology – such as the folding path $[0, \infty] \to \mathcal{PD}$, $t \to T_t$ described above – is d_{Lip}-continuous. As in [FM11] in the special case of Outer space, one can make use of folding paths to construct geodesics in projectivized deformation spaces of irreducible metric G-trees:

Theorem 2.23. *If $\mathcal{PD}$ is a projectivized deformation space of irreducible metric G-trees then for all $T, T' \in \mathcal{PD}$ there exists a d_{Lip}-geodesic $\gamma \colon [0, 1] \to \mathcal{PD}$ with $\gamma(0) = T$ and $\gamma(1) = T'$.*

Proof. Let $f\colon T \to T'$ be an optimal map and $\xi \in G$ a witness for the distance from T to T'. By Lemma 2.22, it suffices to construct a path $\gamma\colon [0,1] \to \mathcal{PD}$ from T to T' such that for all $x < y < z \in [0,1]$ we have (2.2). We will construct such a path in the unprojectivized deformation space $\mathcal{D}$, and since any witness for the minimal stretching factor between two metric G-trees remains a witness after scaling the metrics on the trees, the projection of the path to $\mathcal{PD}$ will still satisfy (2.2). In order to do so, we again regard T and T' as their covolume-1-representatives in $\mathcal{D}$. Let

$$C = \exp(d_{Lip}(T,T')) = \frac{l_{T'}(\xi)}{l_T(\xi)}$$

and let $\overline{T}$ be the metric G-tree obtained from T by G-equivariantly shrinking each edge of T that is mapped to a point under f to length 0 (collapsing these edges does not create any new elliptic subgroups, as the G-equivariant map $f\colon T \to T'$ factors through the quotient) and G-equivariantly shrinking all other edges so that they are stretched by the factor C under f. Note that we only shrink edges in the complement of the tension forest $\Delta(f)$. Then, homothete $\overline{T}$ to $C\overline{T}$ such that $f\colon C\overline{T} \to T'$ becomes an isometry on edges, i.e., a morphism. We may now fold $C\overline{T}$ along f to obtain a family of metric G-trees $(T_t)_{t \in [0,\infty]}$ that interpolate between $C\overline{T}$ and T' as explained above (see Figure 2.4 for a structural sketch). This produces a path $\gamma\colon [0,1] \to \mathcal{D}$ from T to T' that is continuous in all three
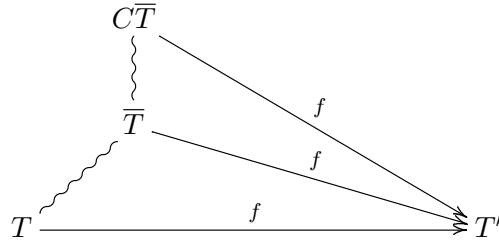
Figure 2.4: The path from T to $\overline{T}$ to $C\overline{T}$ to T' in $\mathcal{D}$ projects to a geodesic from T to T' in $\mathcal{PD}$.

topologies and also with respect to d_{Lip}.

We claim that for every metric G-tree S in between T and $C\overline{T}$ we have

$$\sigma(T,S) = \frac{l_S(\xi)}{l_T(\xi)} \quad \text{and} \quad \sigma(S,T') = \frac{l_{T'}(\xi)}{l_S(\xi)}.$$

Analogously, we claim that for every metric G-tree T_t in between $C\overline{T}$ and T' we

have

$$\sigma(T, T_t) = \frac{l_{T_t}(\xi)}{l_T(\xi)} \quad \text{and} \quad \sigma(T_t, T') = \frac{l_{T'}(\xi)}{l_{T_t}(\xi)}.$$

As for any $a \leq s \leq t \leq b$ the same construction of a path from $\gamma(s)$ to $\gamma(t)$ yields precisely the restriction of γ to $[s, t]$, this then proves that γ satisfies (2.2).

First, consider a metric G-tree S that lies in between T and $\overline{T}$. As S is obtained from T by shrinking edges of T, we have $\sigma(T, S) \leq 1$. However, as we only shrink edges outside of $\Delta(f)$, the hyperbolic axis $A_\xi \subset \Delta(f)$ is not touched and we have $\frac{l_S(\xi)}{l_T(\xi)} = 1$. We may immediately deduce from this that $\sigma(T, S) = \frac{l_S(\xi)}{l_T(\xi)}$, as for all hyperbolic group elements $g \in G$ we have $\sigma(T, S) \geq \frac{l_S(g)}{l_T(g)}$ (see Lemma 2.13). Likewise, the map $f \colon S \to T'$ still has Lipschitz constant C so that $\sigma(S, T') \leq C$. The axis $A_\xi \subset \Delta(f) \subset S$ remains legal and is stretched by the factor C, whence $\frac{l_{T'}(\xi)}{l_S(\xi)} = C$ and therefore $\sigma(S, T') = \frac{l_{T'}(\xi)}{l_S(\xi)}$.

Analogously, if S lies in between $\overline{T}$ and $C\overline{T}$, say $S = C'\overline{T}$ with $C' \in [1, C]$, then $\sigma(T, S) \leq C'$ and $\frac{l_S(\xi)}{l_T(\xi)} = C'$, whence $\sigma(T, S) = \frac{l_S(\xi)}{l_T(\xi)}$. The map $f \colon S \to T'$ has Lipschitz constant $\frac{C}{C'}$ and the axis $A_\xi \subset \Delta(f) \subset S$ is stretched by $\frac{C}{C'}$. We conclude that $\sigma(S, T') = \frac{l_{T'}(\xi)}{l_S(\xi)}$.

Consider now a metric G-tree T_t in between $C\overline{T}$ and T'. As the quotient map $\phi_t \colon C\overline{T} \to T_t$ is 1-Lipschitz, the composition $T \xrightarrow{id} C\overline{T} \xrightarrow{\phi_t} T_t$ is C-Lipschitz. The hyperbolic axis $A_\xi \subset \Delta(f) \subset T$ is legal with respect to f and hence does not get folded in $T_t = C\overline{T}/\sim_t$. We therefore have $\frac{l_{T_t}(\xi)}{l_T(\xi)} = C$, whence $\sigma(T, T_t) = \frac{l_{T_t}(\xi)}{l_T(\xi)}$. Analogously, the induced map $\psi_t \colon T_t \to T'$ is 1-Lipschitz and the hyperbolic axis $A_\xi \subset \Delta(\psi_t) \subset T_t$ is legal with respect to ψ_t. We conclude that $\frac{l_{T'}(\xi)}{l_{T_t}(\xi)} = 1$ and hence that $\sigma(T_t, T') = \frac{l_{T'}(\xi)}{l_{T_t}(\xi)}$. $\qquad\square$

2.2 Displacement functions

Let $\mathcal{PD}$ be a projectivized deformation space of metric G-trees and $\Phi \in \mathrm{Out}_\mathcal{D}(G)$. We equip $\mathcal{PD}$ with the Lipschitz metric d_{Lip} and define the *displacement function* associated to Φ as the function

$$\widetilde{\Phi} \colon \mathcal{PD} \to \mathbb{R}_{\geq 0}, \ T \mapsto d_{Lip}(T, T\Phi).$$

We call Φ *elliptic* if $\inf \widetilde{\Phi} = 0$ and the infimum is realized. We say that Φ is *hyperbolic* if $\inf \widetilde{\Phi} > 0$ and the infimum is realized. Lastly, Φ is *parabolic* if $\inf \widetilde{\Phi}$ is not realized.

2.2.1 Elliptic automorphisms

If $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ is elliptic then, by definition, there exists a metric G-tree $T \in \mathcal{PD}$ such that $d_{Lip}(T, T\Phi) = 0$. One would like to conclude that T lies in the fixed point set of Φ, but from Example 2.4 we know that the asymmetric pseudometric d_{Lip} fails to be an asymmetric metric, i.e., $d_{Lip}(T, T') = 0$ does generally not imply that T and T' are G-equivariantly isometric. The metric G-trees T and T' in the counterexample are not homeomorphic and thus they do not lie in the same $\mathrm{Out}_{\mathcal{D}}(G)$-orbit, for they would otherwise have the same underlying metric simplicial tree. Therefore, one may still ask whether d_{Lip} is an asymmetric metric on $\mathrm{Out}_D(G)$-orbits. As we will see, the general answer is "no" (Example 2.30) but it is "yes" in certain cases (Proposition 2.24). The arguments in this section arose out of discussions with Camille Horbez and Gilbert Levitt.

The separation property of d_{Lip} on $\mathrm{Out}_D(G)$-orbits Let $\mathcal{PD}$ be a projectivized deformation space of metric G-trees. Let $T \in \mathcal{PD}$ and $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ such that $d_{Lip}(T, T\Phi) = 0$. If T is irreducible then there exists an optimal map $f : T \to T\Phi$ with $\sigma(f) = 1$, and one easily shows (as in the proof of Proposition 2.16) that f has stretching factor 1 on all edges of T. After subdividing the simplicial structures on T and $T\Phi$ (independently of each other) by G-equivariantly adding redundant vertices, f becomes simplicial (see Definition 1.58). We will denote the subdivided metric G-trees again by T and $T\Phi$.

If all edge stabilizers of T are finitely generated then by [BF91, Section 2] the simplicial map f factors as a finite composition of G-equivariant simplicial quotient maps, so-called *folds*, which can be classified into types IA-IIIA, IB-IIIB, and IIIC (we refer the reader to [BF91] for definitions). All folds other than type IIA and IIB folds irreversibly decrease the metric covolume, so they cannot occur. After subdividing the simplicial structure on T once more, a type IIB fold is a composition of two type IIA folds (these subdivisions add only a finite number of G-orbits of vertices), so we may assume that f factors as a finite composition of type IIA folds. Explicitly, a type IIA fold is a simplicial quotient map $T \to T/\sim$, where $\sim$ is a G-equivariant equivalence relation on T that is of the following form: There are distinct edges $e_1, e_2 \in E(T)$ with $\iota(e_1) = \iota(e_2) \in V(T)$ and a group element $g \in G_{\iota(e_1)}$ such that $ge_1 = e_2$, and $\sim$ is the equivalence relation generated by $he_1 \sim he_2$ for all $h \in G$. Intuitively, on the level of quotient graphs of groups, performing a type IIA fold corresponds to pulling an element of a vertex stabilizer along an edge (see Figure 2.5).

A type IIA fold always enlarges but never reduces an edge group. We will make use of this behavior to confirm the separation property of d_{Lip} on $\mathrm{Out}_{\mathcal{D}}(G)$-orbits in the following special case:

$$G \bullet\!\!\!\!\xrightarrow{E}\!\!\!\!\bullet H \qquad \xrightarrow[\;g \in G,\; g \notin E\;]{\text{type IIA fold}} \qquad G \bullet\!\!\!\!\xrightarrow{\langle E, g\rangle}\!\!\!\!\bullet \langle H, g\rangle$$

Figure 2.5: The effect of a type IIA fold on the quotient graph of groups.

Proposition 2.24 (Levitt)**.** *Let $\mathcal{PD}$ be a projectivized deformation space of locally finite irreducible metric G-trees with finitely generated edge stabilizers. If $\mathcal{PD}$ has no nontrivial integral modulus (see Section 1.2.2) and if $T \in \mathcal{PD}$ and $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ satisfy $d_{Lip}(T, T\Phi) = 0$ then T and $T\Phi$ are G-equivariantly isometric.*

Before we turn to the proof of Proposition 2.24, we discuss the existence of *maximal elliptic subgroups*, i.e., elliptic subgroups that are not properly contained in any other elliptic subgroup. A maximal elliptic subgroup is always a vertex stabilizer.

Lemma 2.25. *Let $\mathcal{PD}$ be a projectivized deformation space of locally finite metric G-trees. If $\mathcal{PD}$ has no nontrivial integral modulus then for any G-tree $T \in \mathcal{PD}$ and any edge $e \in E(T)$ the edge group G_e is contained in a maximal elliptic subgroup of T.*

Proof. We first observe that, under these assumptions, for any vertex $v \in V(T)$ the vertex group $G_v \leq G$ is not properly contained in a conjugate of itself: Suppose to the contrary that there exists a vertex $v \in V(T)$ such that G_v is a proper subgroup of gG_vg^{-1} for some $g \in G$. We then have

$$\mu(g) = \frac{[G_v : (G_v \cap gG_vg^{-1})]}{[gG_vg^{-1} : (G_v \cap gG_vg^{-1})]} = \frac{1}{[gG_vg^{-1} : G_v]}$$

with $[gG_vg^{-1} : G_v] > 1$, in which case $\mu(g^{-1}) = \frac{1}{\mu(g)}$ is a nontrivial integral modulus, contradicting our assumptions. To prove the lemma, we again argue by contradiction: Suppose that the edge group G_e is not contained in a maximal elliptic subgroup. Each vertex group adjacent to e is then properly contained in another vertex group, which is again properly contained in yet another vertex group. Inductively, we obtain an infinite properly ascending chain of vertex groups that lie in only finitely many conjugacy classes by the cocompactness of T. We conclude that there exists a vertex $v \in V(T)$ and a group element $g \in G$ such that G_v is a proper subgroup of gG_vg^{-1}, which contradicts the first part of the proof. $\qquad\square$

The following lemma will be used in the proof of Proposition 2.24 as well:

Lemma 2.26 ([For06, Lemma 8.1]). *Let $Q \subset (\mathbb{Q}_{>0}, \times)$ be a finitely generated subgroup such that $Q \cap \mathbb{Z} = \{1\}$. Then for any $r \in \mathbb{Q}$ the set $rQ \cap \mathbb{Z}$ is finite.*

Proof of Proposition 2.24. Since T is irreducible and has finitely generated edge stabilizers, after subdividing the simplicial structures on T and $T\Phi$ there exists a G-equivariant simplicial map $f \colon T \to T\Phi$ that factors as a finite composition of type IIA folds. We claim that $T\Phi$ cannot be obtained from T by nontrivial type IIA folds, whence T and $T\Phi$ are G-equivariantly isometric:

By Lemma 2.25, the stabilizer G_e of any edge $e \in E(T)$ is contained in a maximal elliptic subgroup of T, which is always a vertex stabilizer. Let $M_i \leq G$, $i \in I$ be the maximal elliptic subgroups of T that contain G_e. Since T has only finitely many G-orbits of vertices, the vertex groups M_i, $i \in I$ fall into only finitely many conjugacy classes, and we assume for a moment that they are in fact all conjugate. Then, for a distinguished maximal elliptic subgroup M containing G_e, the image of the modular homomorphism $\mu \colon G \to (\mathbb{Q}_{>0}, \times)$ defined by

$$\mu(g) = \frac{[M : (M \cap gMg^{-1})]}{[gMg^{-1} : (M \cap gMg^{-1})]}$$

contains the values

$$\frac{[M_i : (M \cap M_i)]}{[M : (M \cap M_i)]} = \frac{[M_i : (M \cap M_i)]}{[M : (M \cap M_i)]} \cdot \frac{[(M \cap M_i) : G_e]}{[(M \cap M_i) : G_e]} = \frac{[M_i : G_e]}{[M : G_e]}, \ i \in I.$$

Since $\mathcal{PD}$ has no nontrivial integral modulus, Lemma 2.26 implies that the indices $[M_i : G_e]$, $i \in I$ can take only finitely many values. Consequently, there exists a maximum index $\mathcal{I}(G_e)$ of G_e in the maximal elliptic subgroups M_i, $i \in I$. If the maximal elliptic subgroups containing G_e are not all conjugate, we associate to each of their finitely many conjugacy classes the maximum index of G_e and define $\mathcal{I}(G_e)$ as the sum of these. One readily sees that for all $g \in G$ we have $\mathcal{I}(gG_eg^{-1}) = \mathcal{I}(G_e)$ so that we have $\mathcal{I}(G_e) = \mathcal{I}(G_{e'})$ if e and e' lie in the same G-orbit of edges of T. Finally, let

$$\mathcal{I}(T) = \sum_{e \in G \backslash T} \text{length}(e) \cdot \mathcal{I}(G_e)$$

where e ranges over the finitely many edges in the metric quotient graph of groups of T. The value $\mathcal{I}(T)$ is insensitive to simplicial subdivisions of T and for all $\Phi \in \text{Out}_{\mathcal{D}}(G)$ we have $\mathcal{I}(T\Phi) = \mathcal{I}(T)$. On the other hand, after performing a type IIA fold, for the enlarged edge group $\langle E, g \rangle$ we have $\mathcal{I}(\langle E, g \rangle) < \mathcal{I}(E)$, whereas all other edge groups are left invariant. Thus, if $T' \in \mathcal{PD}$ is obtained from T by a nontrivial sequence of type IIA folds then $\mathcal{I}(T') < \mathcal{I}(T)$, whence the claim. $\square$

Example 2.27. Let G be a finitely generated virtually nonabelian free group and $\mathcal{PD}$ the projectivized deformation space of minimal metric G-trees with finite vertex stabilizers (Example 1.30). We know from Example 1.36 that $\mathcal{PD}$ has no nontrivial integral modulus. Hence, if for $T \in \mathcal{PD}$ and $\Phi \in \operatorname{Out}_{\mathcal{D}}(G) = \operatorname{Out}(G)$ we have $d_{Lip}(T, T\Phi) = 0$ then T and $T\Phi$ are G-equivariantly isometric.

Corollary 2.28. *Let G be a finitely generated virtually nonabelian free group and $\mathcal{PD}$ the projectivized deformation space of minimal metric G-trees with finite vertex stabilizers. An automorphism $\Phi \in \operatorname{Out}_{\mathcal{D}}(G) = \operatorname{Out}(G)$ is elliptic with respect to d_{Lip} if and only if it has finite order.*

Proof. By Example 2.27, an automorphism $\Phi \in \operatorname{Out}(G)$ is elliptic with respect to d_{Lip} if and only if it has a fixed point in $\mathcal{PD}$, and we know from Example 1.66 that Φ has a fixed point in $\mathcal{PD}$ if and only if it has finite order. $\qquad\square$

Example 2.29. Let G be a nonelementary GBS group that contains no solvable Baumslag-Solitar group $\operatorname{BS}(1, n)$ with $n \geq 2$. Let $\mathcal{PD}$ be the projectivized deformation space of minimal metric G-trees with infinite cyclic vertex and edge stabilizers (Example 1.31). By Lemma 1.33, $\mathcal{PD}$ has no nontrivial integral modulus. Thus, if for $T \in \mathcal{PD}$ and $\Phi \in \operatorname{Out}_{\mathcal{D}}(G) = \operatorname{Out}(G)$ we have $d_{Lip}(T, T\Phi) = 0$ then T and $T\Phi$ are G-equivariantly isometric.

However, the asymmetric Lipschitz pseudometric d_{Lip} does not restrict to an asymmetric metric on $\operatorname{Out}_{\mathcal{D}}(G)$-orbits in general:

Example 2.30 (Horbez)**.** Let $G = \operatorname{BS}(1, 6) * F_2 = \langle x, t \mid txt^{-1} = x^6 \rangle * F_2$ and consider the metric graph of groups decompositions Γ and Γ' of G shown in Figure 2.6, where all edge group inclusions are the obvious ones and all edges have length $\frac{1}{3}$. Let T and T' be the corresponding metric G-trees. The automorphism

$$\varphi \colon \operatorname{BS}(1, 6) \xrightarrow{\;\cong\;} \operatorname{BS}(1, 6), \; x \mapsto x^3, \; t \mapsto t$$

induces an automorphism $\Phi = \varphi * id_{F_2} \in \operatorname{Aut}(G)$ for which we have $T' = T\Phi$. Similarly as in Example 2.4, the natural morphism of graphs of groups from Γ to Γ' lifts to a G-equivariant map from T to $T\Phi$ (namely, a type IIA fold) that is an isometry on edges and thus has Lipschitz constant 1, whence $d_{Lip}(T, T\Phi) = 0$. However, T and $T\Phi$ are not G-equivariantly isometric, as the group element $x \in G$ stabilizes an edge in $T\Phi$ but not in T (x is not a conjugate of x^3).

2.2.2 Nonparabolic automorphisms

Let $\mathcal{PD}$ be a projectivized deformation space of irreducible metric G-trees and $\Phi \in \operatorname{Out}_{\mathcal{D}}(G)$ a nonparabolic automorphism, i.e., $\inf \widetilde{\Phi}$ is realized. Let $T \in \mathcal{PD}$
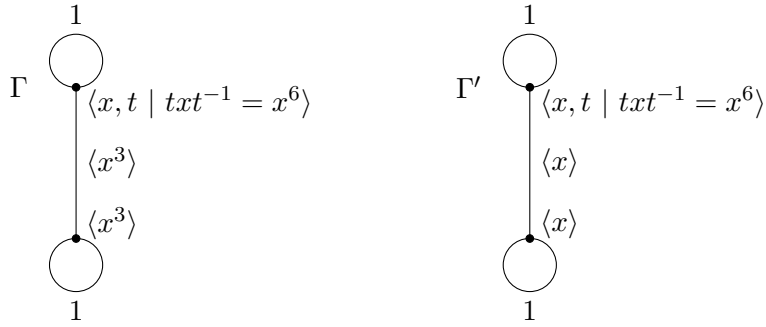
Figure 2.6: The Bass-Serre trees of the graphs of groups shown above lie in the same $\text{Out}_{\mathcal{D}}(G)$-orbit. They are irreducible but not locally finite.

such that $d_{Lip}(T, T\Phi) = \inf \widetilde{\Phi}$ and let $f \colon T \to T\Phi$ be an optimal map with tension forest $\Delta = \Delta(f) \subset T$. The following observation will be used in the proof of Theorem 2.37:

Proposition 2.31. *After a small perturbation of the metric on T, preserving the condition that $d_{Lip}(T, T\Phi) = \inf \widetilde{\Phi}$, the map $f \colon T \to T\Phi$ is G-equivariantly homotopic to an optimal map $f' \colon T \to T\Phi$ with $\Delta(f') \subseteq \Delta$ such that*

$$f'\left(\Delta(f')\right) \subseteq \Delta(f').$$

Proof. Suppose that $f(\Delta)$ is not contained in Δ and let $e \in E(\Delta)$ be an edge such that $f(e) \nsubseteq \Delta$. Slightly scale up the metric on Δ and down on $T \setminus \Delta$ while maintaining covolume 1. This lowers the stretching factor on e and produces a new tension forest, of the original map f made linear on edges, that is properly contained in the old one. Since $d_{Lip}(T, T\Phi)$ is minimal among all translation distances of Φ, we will not have removed all edges of Δ and started over with a new tension forest that corresponds to a strictly smaller maximal stretching factor. In particular, there always exists an edge $e' \in E(\Delta)$ such that $f(e') \subseteq \Delta$. The stretching factor of f on e' remains unchanged and we preserve the condition that $d_{Lip}(T, T\Phi) = \inf \widetilde{\Phi}$. As T has only finitely many G-orbits of edges, after finitely many repetitions we have $f(\Delta) \subseteq \Delta$. If at this point Δ has a vertex with only one gate, we perturb f to an optimal map f' as in the proof of Proposition 2.12. $\square$

2.2.3 Parabolic automorphisms

Let $\mathcal{PD}$ be a projectivized deformation space of metric G-trees and $T \in \mathcal{PD}$. We say that a G-invariant subforest $S \subseteq T$ is *essential* if it contains the hyperbolic axis of some hyperbolic group element. The notion of essential G-invariant subforests

generalizes the notion of homotopically nontrivial subgraphs of marked metric graphs in Outer space.

Definition 2.32. An automorphism $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ is *reducible* if there exists a metric G-tree $T \in \mathcal{PD}$ and a G-equivariant map $f\colon T \to T\Phi$ that leaves an essential proper G-invariant subforest of T invariant. If Φ is not reducible, it is *irreducible*.

As we will see, parabolic automorphisms are often reducible (Corollary 2.34). For this, let $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ be a parabolic automorphism (i.e., $\inf \widetilde{\Phi}$ is not realized) and $(T_k)_{k \in \mathbb{N}}$ a sequence of metric G-trees in $\mathcal{PD}$ such that

$$\lim_{k \to \infty} d_{Lip}(T_k, T_k \Phi) = \inf \widetilde{\Phi}.$$

For $\Theta > 0$ we denote by $\mathcal{PD}(\Theta)$ the $\mathrm{Out}_{\mathcal{D}}(G)$-invariant subspace of $\mathcal{PD}$ consisting of all metric G-trees $T \in \mathcal{PD}$ that satisfy $l_T(g) \geq \Theta$ for all hyperbolic group elements $g \in G$. We call $\mathcal{PD}(\Theta)$ the Θ-*thick part* of $\mathcal{PD}$.

Proposition 2.33. *If the projectivized deformation space $\mathcal{PD}$ is irreducible and $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices then for only finitely many $k \in \mathbb{N}$ we have $T_k \in \mathcal{PD}(\Theta)$.*

Proof. We will argue as in the proof of [Meu11, Claim 72]. Suppose that the proposition is false and that, after passing to a subsequence, we have $T_k \in \mathcal{PD}(\Theta)$ for all $k \in \mathbb{N}$. We will lead this to a contradiction.

Since $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ with finitely many orbit of simplices, it acts on the thick part $\mathcal{PD}(\Theta)$ cocompactly in all three topologies. In particular, the image of $(T_k)_{k \in \mathbb{N}}$ in the quotient $\mathcal{PD}(\Theta)/\mathrm{Out}_{\mathcal{D}}(G)$ has a weakly convergent subsequence. We can thus find a sequence of outer automorphisms $(\psi_k)_{k \in \mathbb{N}} \subset \mathrm{Out}_{\mathcal{D}}(G)$ such that, after passing to a subsequence, $(T_k \psi_k)_{k \in \mathbb{N}}$ weakly converges in $\mathcal{PD}(\Theta)$ to some $T \in \mathcal{PD}(\Theta)$. We have

$$d_{Lip}(T\psi_k^{-1}, T\psi_k^{-1}\Phi) \leq d_{Lip}(T\psi_k^{-1}, T_k) + d_{Lip}(T_k, T_k\Phi) + d_{Lip}(T_k\Phi, T\psi_k^{-1}\Phi)$$
$$= d_{Lip}(T, T_k\psi_k) + d_{Lip}(T_k, T_k\Phi) + d_{Lip}(T_k\psi_k, T)$$

where $\lim_{k \to \infty} d_{Lip}(T, T_k\psi_k) = \lim_{k \to \infty} d_{Lip}(T_k\psi_k, T) = 0$ by Proposition 2.17. Hence, $\lim_{k \to \infty} d_{Lip}(T, T\psi_k^{-1}\Phi\psi_k) = \lim_{k \to \infty} d_{Lip}(T_k, T_k\Phi) = \inf \widetilde{\Phi}$.

By Theorem 2.14, for all $k \in \mathbb{N}$ there exists a candidate $\xi_k \in \mathrm{cand}(T)$ such that

$$\sigma(T, T\psi_k^{-1}\Phi\psi_k) = \frac{l_{T\psi_k^{-1}\Phi\psi_k}(\xi_k)}{l_T(\xi_k)} = \frac{l_T(\psi_k^{-1}\Phi\psi_k(\xi_k))}{l_T(\xi_k)}.$$

The translation length function of T has discrete image in $\mathbb{R}$ and hence the numerator takes discrete values. Since the candidates of T have only finitely many

different translation lengths, the denominator takes only finitely many values and we conclude that the sequence $\big(\sigma(T, T\psi_k^{-1}\Phi\psi_k)\big)_{k\in\mathbb{N}}$ is discrete. For large k we thus have

$$d_{Lip}(T\psi_k^{-1}, T\psi_k^{-1}\Phi) = d_{Lip}(T, T\psi_k^{-1}\Phi\psi_k) = \inf\widetilde{\Phi}$$

contradicting the assumption that Φ is parabolic. $\qquad\square$

Corollary 2.34. *Under the assumptions of Proposition 2.33, for large k any optimal map $f\colon T_k \to T_k\Phi$ leaves an essential proper G-invariant subforest of T_k invariant up to G-equivariant homotopy. In particular, every parabolic automorphism $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ is reducible.*

If T is a minimal G-tree then a subforest $S \subseteq T$ with no trivial components is a *core subforest* if it does not have any vertices of valence 1. Every G-invariant subforest $S \subseteq T$ with no trivial components contains a unique (possibly empty) maximal G-invariant core subforest $\mathrm{core}(S) \subseteq S \subseteq T$, obtained by inductively removing G-orbits of edges whose terminal or initial vertex has valence 1. The process of removing G-orbits of edges terminates after finitely many steps by the cocompactness of T.

Proof. For $T \in \mathcal{PD}$ and $\varepsilon > 0$, let $T^\varepsilon \subseteq T$ be the union of all subsets of the form $\bigcup_{k\in\mathbb{Z}} g^k[x, gx]$ with $g \in G$ hyperbolic and $x \in T$ such that $d(x, gx) \leq \varepsilon$. In particular, T^ε contains the axes of all hyperbolic group elements $g \in G$ with $l_T(g) \leq \varepsilon$. Although $T^\varepsilon \subseteq T$ is generally not a simplicial subcomplex of T, we will still speak of T^ε as a (nonsimplicial) subforest, as it becomes a subcomplex after subdividing the simplicial structure on T. In fact, T^ε has no trivial components and its maximal G-invariant core subforest $\mathrm{core}(T^\varepsilon) \subseteq T^\varepsilon$ will be a genuine simplicial subforest of T. Since G acts on T by isometries, if $\bigcup_{k\in\mathbb{Z}} g^k[x, gx]$ is contained in T^ε then for all $h \in G$ the translate

$$h(\bigcup_{k\in\mathbb{Z}} g^k[x, gx]) = \bigcup_{k\in\mathbb{Z}} (hgh^{-1})^k[hx, hgx]$$

is contained in T^ε as well. Thus, $T^\varepsilon \subseteq T$ is G-invariant.

Since $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices, the complex $\mathcal{PD}$ must be finite-dimensional, say of dimension $d \in \mathbb{N}$, and the number of G-orbits of edges of any $T \in \mathcal{PD}$ is bounded above by $d + 1$. Because the metric G-trees in $\mathcal{PD}$ have covolume 1, in any metric G-tree $T \in \mathcal{PD}$ there exists an orbit of edges with associated edge length $\geq \frac{1}{d+1}$. Therefore, for $\varepsilon < \frac{1}{d+1}$ the subforest $T^\varepsilon \subseteq T$ is a proper subforest. Given G-invariant simplicial subforests $S' \subseteq S$ of T with no trivial components, the subforest S' is a proper subforest of S if and only if $G\backslash S - G\backslash S'$ consists of at least one edge. Hence, as the G-trees in $\mathcal{PD}$ have at most $d + 1$ G-orbits of edges, the number $d + 1$ is a uniform bound for

the length of any chain of proper G-invariant simplicial subforests with no trivial components of any G-tree in $\mathcal{PD}$.

Let $D = \inf \widetilde{\Phi}$. Moreover, let $\varepsilon < \frac{1}{d+1}$ and $\Theta = \frac{\varepsilon}{e^{(D+1)(d+1)}}$. By Proposition 2.33, we can choose k so large that $T_k \notin \mathcal{PD}(\Theta)$ and $d_{Lip}(T_k, T_k\Phi) < D + 1$. For $i = 0, \dots, d+1$, define $\delta_i = \frac{\varepsilon}{e^{(D+1)i}}$ and consider the chain of G-invariant subforests

$$T_k^\varepsilon = T_k^{\delta_0} \supseteq T_k^{\delta_1} \supseteq \dots \supseteq T_k^{\delta_{d+1}} = T_k^\Theta$$

all of which are proper subforests of T_k. Note that $T_k^\Theta \neq \emptyset$, since $T_k \notin \mathcal{PD}(\Theta)$ and thus there exists a hyperbolic group element $g \in G$ with $l_{T_k}(g) < \Theta$ whose axis lies in T_k^Θ. The associated chain of core subforests is a chain of G-invariant simplicial subforests of T_k, whose number of proper inclusions is bounded by d by the arguments given above. Thus, there exists $i \in \{0, \dots, d\}$ for which we have $\mathrm{core}(T_k^{\delta_{i+1}}) = \mathrm{core}(T_k^{\delta_i})$. Since $d_{Lip}(T_k, T_k\Phi) < D + 1$, the Lipschitz constant of the optimal map $f \colon T_k \to T_k\Phi$ is smaller than e^{D+1} and we have

$$f(\mathrm{core}(T_k^{\delta_{i+1}})) \subseteq f(T_k^{\delta_{i+1}}) \subseteq T_k^{\delta_i}.$$

The subforest $\mathrm{core}(T_k^{\delta_i}) \subseteq T_k^{\delta_i}$ is a G-equivariant deformation retract of $T_k^{\delta_i} \subseteq T_k$ and the obvious deformation retraction extends to a G-equivariant self homotopy equivalence h of T_k (this is easily seen on the level of quotient graphs of groups). Now f is G-equivariantly homotopic to the G-equivariant map

$$T_k \xrightarrow{f} T_k\Phi \xrightarrow{h} T_k\Phi$$

that leaves the proper G-invariant subforest $\mathrm{core}(T_k^{\delta_{i+1}}) = \mathrm{core}(T_k^{\delta_i}) \subset T_k$ invariant. As remarked above, there exists a hyperbolic group element $g \in G$ whose hyperbolic axis lies in T_k^Θ and therefore also in $\mathrm{core}(T_k^{\delta_i})$, and we conclude that $\mathrm{core}(T_k^{\delta_i})$ is essential. $\qquad\square$

2.2.4 Train track representatives

Let $\mathcal{PD}$ be a projectivized deformation space of metric G-trees and $T \in \mathcal{PD}$, and let $\Phi \in \mathrm{Out}_\mathcal{D}(G)$.

Definition 2.35. An optimal map $f \colon T \to T\Phi$ is a *train track map* if it satisfies the following three conditions:

(i) $\Delta(f) = T$;

(ii) f maps edges to legal paths (see Definition 2.9);

(iii) If f maps a vertex $v \in V(T)$ to a vertex $f(v) \in V(T\Phi)$ then it maps legal turns at v to legal turns at $f(v)$. (If v has 2 gates then $f(v)$ could

alternatively lie in the interior of an edge. Since f is linear on edges, it then maps inequivalent directions at v to inequivalent directions at $f(v)$.)

If f is a train track map then for any legal line $L \subset T$ and every $k \in \mathbb{N}$ the image $f^k(L) \subset T\Phi^k$ is again a legal line. We say that an automorphism $\Phi \in \text{Out}_{\mathcal{D}}(G)$ is *represented by a train track map* if there exists a metric G-tree $T \in \mathcal{PD}$ and an optimal map $f \colon T \to T\Phi$ that is a train track map.

Proposition 2.36. *Let $\mathcal{PD}$ be a projectivized deformation space of metric G-trees. If an automorphism $\Phi \in \text{Out}_{\mathcal{D}}(G)$ is represented by a train track map $f \colon T \to T\Phi$ then $d_{Lip}(T, T\Phi) = \inf \widetilde{\Phi}$ and, in particular, Φ is nonparabolic.*

Proof. Our argument is a generalization of [Bes11, Remark 8]. Suppose that $f \colon T \to T\Phi$ is an optimal map that is a train track map. By Theorem 2.14 and Lemma 2.15, there exists a hyperbolic group element $\xi \in G$ whose axis $A_\xi \subset T$ lies in $\Delta(f)$ and is legal with respect to the train track structure defined by f (once we know that there exists an optimal map $f \colon T \to T\Phi$, Theorem 2.14 no longer requires T to be irreducible). Since f is a train track map, for all $k \in \mathbb{N}$ the image $f^k(A_\xi) \subset T\Phi^k$ is a ξ-invariant line – and thus equals the hyperbolic axis of ξ in $T\Phi^k$ – that lies in $\Delta(f)$ and is legal. We therefore have

$$\sigma(T, T\Phi^k) = \sup_g \frac{l_{T\Phi^k}(g)}{l_T(g)} \geq \frac{l_{T\Phi^k}(\xi)}{l_T(\xi)} = \frac{l_{T\Phi}(\xi)}{l_T(\xi)} \cdots \frac{l_{T\Phi^k}(\xi)}{l_{T\Phi^{k-1}}(\xi)}$$
$$= \sigma(T, T\Phi) \cdots \sigma(T\Phi^{k-1}, T\Phi^k) = \sigma(T, T\Phi)^k$$

from which we conclude that $\sigma(T, T\Phi^k) = \sigma(T, T\Phi)^k$. In order to show that $d_{Lip}(T, T\Phi) = \inf \widetilde{\Phi}$, let $T' \in \mathcal{PD}$ be any other G-tree. We have

$$k \cdot d_{Lip}(T, T\Phi) = d_{Lip}(T, T\Phi^k)$$
$$\leq d_{Lip}(T, T') + d_{Lip}(T', T'\Phi^k) + d_{Lip}(T'\Phi^k, T\Phi^k)$$
$$\leq d_{Lip}^{sym}(T, T') + k \cdot d_{Lip}(T', T'\Phi)$$

and hence $d_{Lip}(T, T\Phi) \leq \frac{1}{k} \cdot d_{Lip}^{sym}(T, T') + d_{Lip}(T', T'\Phi)$. Letting k go to infinity, we see that $d_{Lip}(T, T\Phi) \leq d_{Lip}(T', T'\Phi)$. $\square$

As for existence of train track representatives, we have the following:

Theorem 2.37. *Let $\mathcal{PD}$ be a projectivized deformation space of irreducible metric G-trees. If $\text{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices then every irreducible automorphism (see Definition 2.32) $\Phi \in \text{Out}_{\mathcal{D}}(G)$ is represented by a train track map.*

Proof. Since the automorphism Φ is irreducible, by Corollary 2.34 it is non-parabolic, i.e., $\inf \widetilde{\Phi}$ is realized. Let $T \in \mathcal{PD}$ such that $d_{Lip}(T, T\Phi) = \inf \widetilde{\Phi}$ and let $f \colon T \to T\Phi$ be an optimal map, which exists by the irreducibility of $\mathcal{PD}$. We claim that f already satisfies (i) and (ii) of Definition 2.9:

Assertion (i) immediately follows from Proposition 2.31, as we could otherwise slightly perturb the metric on T and find an optimal map $T \to T\Phi$ that leaves an essential proper G-invariant subforest of T invariant (the tension forest of an optimal map is always essential by Theorem 2.14 and Lemma 2.15), contradicting the assumption that Φ is irreducible.

As for (ii), suppose that an edge $e \in E(T)$ is mapped over an illegal turn. Slightly fold the illegal turn G-equivariantly and scale the metric on T back to covolume 1. The optimal map $f \colon T \to T\Phi$ naturally induces a G-equivariant map that we make linear on edges relative to the vertices of T. The performed perturbation lowers the stretching factor of f on the edge induced by e, which therefore drops out of the tension forest. Each witness $A_\xi \subset \Delta(f) = T$ is legal with respect to f and does not get folded, whence the stretching factor of f on A_ξ does not increase[1]. Hence, we preserve the condition that $d_{Lip}(T, T\Phi) = \inf \widetilde{\Phi}$ and the Lipschitz constant of f remains minimal among all G-equivariant Lipschitz maps from T to $T\Phi$. After perturbing f to an optimal map as in the proof of Proposition 2.12, we obtain an optimal map $T \to T\Phi$ whose tension forest is a proper subforest of T. By Proposition 2.31, this again contradicts the assumption that Φ is irreducible.

Finally, we may perturb T and f by an arbitrarily small amount, preserving the condition that $d_{Lip}(T, T\Phi) = \inf \widetilde{\Phi}$ and that $f \colon T \to T\Phi$ is an optimal map – and therefore also preserving conditions (i) and (ii) – such that (iii) of Definition 2.9 is satisfied as well: If f maps a legal turn at a vertex $v \in V(T)$ to an illegal turn, slightly fold the illegal turn G-equivariantly (see Figure 2.7). Again, each witness $A_\xi \subset T$ is legal with respect to f and does not get folded so that the stretching factor of f on A_ξ does not increase. Thus, we preserve the property that $d_{Lip}(T, T\Phi) = \inf \widetilde{\Phi}$ and that f is a minimal stretch map. The perturbation makes the legal turn at v illegal, but the induced map f made linear on edges is still optimal and v has still at least two gates, for f would otherwise give rise to an optimal map whose tension forest is a proper subforest of T. The folding decreases the number $G(T) = \sum_w \max\{0, G(w) - 2\}$, where w ranges over the finitely many G-orbits of vertices of T and $G(w)$ denotes the number of gates at w. After finitely many steps, we obtain an optimal map $f \colon T \to T\Phi$ that also satisfies condition (iii). $\qquad\square$

---

[1] In fact, there also exists a witness $A_\xi \subset T$ whose f-image $A_{\Phi(\xi)} \subset T$ does not get folded either, as the induced map would otherwise have strictly smaller Lipschitz constant, contradicting the fact that $d_{Lip}(T, T\Phi)$ is minimal among all translation distances of Φ.
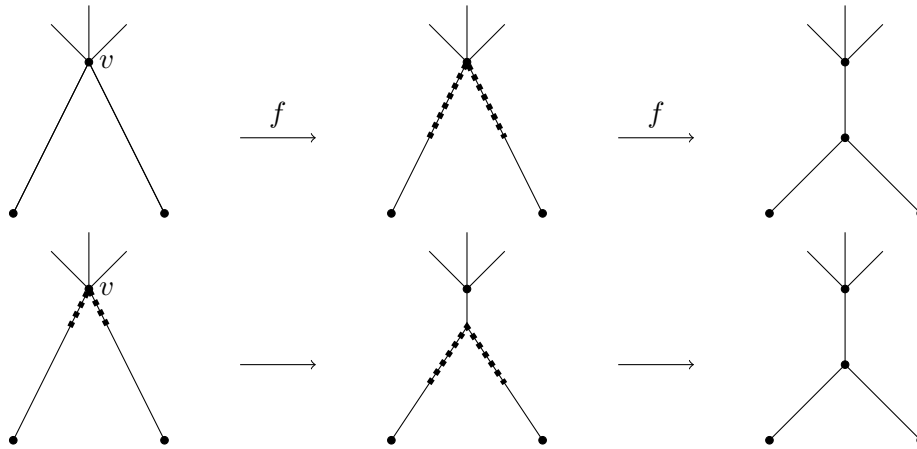
Figure 2.7: Legal and illegal (dashed) turns. The upper row shows turns before folding, the bottom row after folding. The number $G(T)$ in the upper row is $3 + 2 + (2 + 1) = 8$, whereas in the bottom row it has decreased to $2 + (2 + 0) + (2 + 1) = 7$.

Example 2.38. Let G be a finitely generated virtually nonabelian free group and $\mathcal{PD}$ the projectivized deformation space of minimal metric G-trees with finite vertex stabilizers (Example 1.30); it is irreducible and $\mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Out}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices (Example 1.36). Consequently, every irreducible automorphism of G is represented by a train track map. This generalizes [BH92, Theorem 1.7] to virtually free groups.

Example 2.39. Let G be a nonelementary GBS group that contains no solvable Baumslag-Solitar group $\mathrm{BS}(1, n)$ with $n \geq 2$. The projectivized deformation space $\mathcal{PD}$ of minimal metric G-trees with infinite cyclic vertex and edge stabilizers is irreducible (Example 1.31) and $\mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Out}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices (Example 1.37). Hence, every irreducible automorphism of G is represented by a train track map.

Chapter 3

Higher holomorphs

Let G be a finitely generated group. Recall from the introduction that for $k \in \mathbb{N}$ we define the *k-th holomorph* of G as the semidirect product

$$\operatorname{Aut}(G, k) := G^{k-1} \rtimes \operatorname{Aut}(G)$$

with multiplication given by

$$((g_2, \ldots, g_k), \phi) \cdot ((h_2, \ldots, h_k), \psi) = ((g_2\phi(h_2), \ldots, g_k\phi(h_k)), \phi \circ \psi).$$

We further define $\operatorname{Aut}(G, 0) := \operatorname{Out}(G)$. The family of holomorphs continues the sequence of groups

$$\operatorname{Aut}(G, 0) = \operatorname{Out}(G)$$
$$\operatorname{Aut}(G, 1) = \operatorname{Aut}(G)$$
$$\operatorname{Aut}(G, 2) = \operatorname{Hol}(G)$$

where $\operatorname{Hol}(G)$ is the classical *holomorph* of G (for a discussion on the holomorph of a group, see [Rot95, p. 164 and Example 7.9]).

Convention. It will be convenient to write the elements of $\operatorname{Aut}(G, k)$ in the opposite order, i.e., to write $\operatorname{Aut}(G, k) = \operatorname{Aut}(G) \ltimes G^{k-1}$ with multiplication given by

$$(\phi, (g_2, \ldots, g_k)) \cdot (\psi, (h_2, \ldots, h_k)) = (\phi \circ \psi, (g_2\phi(h_2), \ldots, g_k\phi(h_k)))$$

and we make the convention to do so.

Example 3.1. Higher holomorphs of free groups have played an important role in the study of $\operatorname{Out}(F_n)$. For instance:

- In [BF00, Section 2.5] Bestvina-Feighn construct a bordification of Outer space $\mathcal{P}\mathcal{X}_n$. In their construction, the k-th holomorph $\operatorname{Aut}(F_n, k)$ appears as "$\operatorname{Out}(n, k)$".

- The k-th holomorph $\operatorname{Aut}(F_n, k)$ is isomorphic to the group $G_{n,k} \cong \Gamma_{n,k}$ considered in [HV04], where Hatcher-Vogtmann prove homological stability

statements for $\Gamma_{n,k}$ and deduce from these homological stability of $\mathrm{Out}(F_n)$ (see also [HVW06]).

- For $k \geq 1$, the k-th holomorph $\mathrm{Aut}(F_n, k)$ is isomorphic to the group

$$\mathrm{Aut}^k(F_n) := \left\{ (\phi_1, \ldots, \phi_k) \in \mathrm{Aut}(F_n)^k \ \middle| \ [\phi_1] = \ldots = [\phi_k] \in \mathrm{Out}(F_n) \right\}$$

considered in [HM13b, Section 1.5]. The isomorphism is given by

$$\mathrm{Aut}(F_n, k) \to \mathrm{Aut}^k(F_n)$$
$$(\phi, (g_2, \ldots, g_k)) \mapsto (\phi, c_{g_2} \circ \phi, \ldots, c_{g_k} \circ \phi)$$

where $c_g \in \mathrm{Inn}(F_n)$ denotes conjugation with $g \in F_n$. Recall that $\mathrm{Out}(F_n)$ acts on Outer space $\mathcal{PX}_n$ with finite point stabilizers (see Example 1.66). Handel-Mosher show in [HM13b, Lemma 18] that the stabilizer of a point in the missing boundary of $\mathcal{PX}_n$ (i.e., in $\mathcal{PX}_n^* \setminus \mathcal{PX}_n$, where $\mathcal{PX}_n^*$ is the free splitting complex from Example 1.39) under the action of $\mathrm{Out}(F_n)$ is virtually isomorphic to a direct product of the form

$$\mathrm{Aut}^{k_1}(F_{n_1}) \times \ldots \times \mathrm{Aut}^{k_s}(F_{n_s})$$

with $s \geq 1$ and $\sum_{i=1}^{s} n_i \leq n$.

In the light of Example 3.1, it seems worthwhile to study higher holomorphs in a more general context. We will take a step in this direction by constructing "higher spines" for higher holomorphs, as advertised in the introduction.

3.1 k-pointed G-trees

Definition 3.2. For $k \in \mathbb{N}$, a *k-pointed G-tree* $(T, x_1, \ldots, x_k)$ is a G-tree T with k (not necessarily distinct) basepoints $x_1, \ldots, x_k \in T$. A *0-pointed* (or *unpointed*) G-tree is just a G-tree. We will always assume that T carries the natural simplicial structure relative to the distinguished basepoints $x_1, \ldots, x_k$, i.e., the coarsest G-invariant simplicial structure such that each x_i, $i = 1, \ldots, k$ is a vertex of T. We denote by $V(T, x_1, \ldots, x_k)$ the set of natural vertices and by $E(T, x_1, \ldots, x_k)$ the set of natural edges of $(T, x_1, \ldots, x_k)$.

Two k-pointed G-trees $(T, x_1, \ldots, x_k)$ and $(T', x_1', \ldots, x_k')$ are *G-equivariantly homeomorphic* if there exists a G-equivariant homeomorphism $h \colon T \to T'$ such that $h(x_i) = x_i'$ for all $i = 1, \ldots, k$. We say that a k-pointed G-tree is *minimal* and *dihedral, linear abelian, genuine abelian,* or *irreducible* if its underlying G-tree has these properties. Unlike on dihedral, genuine abelian, or irreducible minimal

G-trees (which have reflection points and branch points respectively), on a linear abelian minimal G-tree T there is no prescribed natural simplicial structure coming from the topology of T or the action of G on T. If for $k \geq 1$ a minimal k-pointed G-tree $(T, x_1, \ldots, x_k)$ is linear abelian, we declare the G-orbit of x_1 as the natural vertex set of the underlying G-tree T (so that (T, x_1) has only one G-orbit of edges). We have the following addition to Proposition 1.10:

Proposition 3.3. *For $k \geq 0$, a G-equivariant simplicial homeomorphism*

$$f \colon (T, x_1, \ldots, x_k) \to (T', x_1', \ldots, x_k')$$

between two minimal k-pointed G-trees that are not linear abelian is always unique. If T and T' are linear abelian, the same is true for $k \geq 1$.

Proof. If T and T' are not linear abelian, the assertion follows from Proposition 1.10. In the general case, if we assume that $k \geq 1$, any other G-equivariant simplicial homeomorphism f' must map x_1 to x_1' as well. Thus, the composition $(f')^{-1} \circ f \colon T \to T$ has a fixed point and equals the identity by the first part of the proof of Proposition 1.10. $\qquad\square$

3.1.1 k-pointed forest collapses

For $k \in \mathbb{N}_0$, given a k-pointed G-tree $(T, x_1, \ldots, x_k)$ and a G-invariant simplicial subforest $A \subseteq (T, x_1, \ldots, x_k)$, analogously to the classical forest collapses in Section 1.1.2 we let

$$k_A \colon (T, x_1, \ldots, x_k) \to (T_A, k_A(x_1), \ldots, k_A(x_k))$$

be the corresponding k-pointed forest collapse map. We have the following generalization of Proposition 1.13:

Proposition 3.4. *For $k \in \mathbb{N}_0$, let $(T, x_1, \ldots, x_k)$ be a minimal k-pointed G-tree and $A, B \subseteq (T, x_1, \ldots, x_k)$ two G-invariant subforests with no trivial components. The k-pointed G-trees $(T_A, k_A(x_1), \ldots, k_A(x_k))$ and $(T_B, k_B(x_1), \ldots, k_B(x_k))$ are G-equivariantly homeomorphic if and only if $A = B$.*

Proof. The statement holds for $k = 0$ by Proposition 1.13, so we may assume that $k \geq 1$. The "if" direction is trivial. For the "only if" direction, suppose that $A \neq B$ and let $e \in E(T, x_1, \ldots, x_k)$ be an edge that is contained in, say, A but not in B. We will show that $(T_A, k_A(x_1), \ldots, k_A(x_k))$ and $(T_B, k_B(x_1), \ldots, k_B(x_k))$ are not G-equivariantly homeomorphic, and we will consider three cases:

First, if neither $\iota(e)$ nor $\tau(e)$ lie in the G-orbit of a distinguished basepoint of $(T, x_1, \ldots, x_k)$ then both vertices are natural vertices of T and we can argue as in the proof of Proposition 1.13.

Second, suppose that both $\iota(e)$ and $\tau(e)$ lie in the G-orbit of a distinguished basepoint of $(T, x_1, \ldots, x_k)$, say $\iota(e) = g_1 x_{i_1}$ and $\tau(e) = g_2 x_{i_2}$ with $g_1, g_2 \in G$ and $i_1, i_2 \in \{1, \ldots, k\}$ (and possibly $i_1 = i_2$). If T is linear abelian then this is the only case that can occur. We then have $k_A(\iota(e)) = k_A(\tau(e))$ and therefore $k_A(x_{i_1}) = g_1^{-1} g_2 k_A(x_{i_2})$, whereas $k_B(\iota(e)) \neq k_B(\tau(e))$ and hence $k_B(x_{i_1}) \neq g_1^{-1} g_2 k_B(x_{i_2})$. Thus, there does not exist a G-equivariant homeomorphism $T_A \to T_B$ mapping $k_A(x_{i_1})$ to $k_B(x_{i_1})$ and $k_A(x_{i_2})$ to $k_B(x_{i_2})$.

Third, suppose that $\iota(e)$ lies in the G-orbit of a distinguished basepoint of $(T, x_1, \ldots, x_k)$, say $\iota(e) = g_1 x_1$ with $g_1 \in G$, and $\tau(e)$ does not. In particular, the vertex $\tau(e)$ is then a natural vertex of T. We only need to treat the dihedral, genuine abelian, and irreducible case, and our argument is a slight modification of the proof of Proposition 1.13:

If T is dihedral then there exists a group element $g \in G$ that acts on T as reflection at $\tau(e)$. We have $k_A(\iota(e)) = k_A(\tau(e))$ and hence $g k_A(\iota(e)) = k_A(\iota(e))$, whence $g_1^{-1} g g_1 k_A(x_1) = k_A(x_1)$. On the other hand, we have $k_B(\iota(e)) \neq k_B(\tau(e))$ and therefore $g_1^{-1} g g_1 k_B(x_1) \neq k_B(x_1)$. We conclude that there does not exist a G-equivariant homeomorphism $T_A \to T_B$ that maps $k_A(x_1)$ to $k_B(x_1)$.

If T is genuine abelian, we can find hyperbolic group elements $g, h \in G$ whose hyperbolic axes intersect in a ray as in one of the two cases depicted in Figure 3.1. Since A_g and A_h contain infinitely many G-translates of e, the group elements g
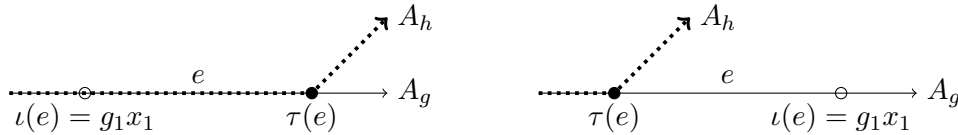

Figure 3.1: Hyperbolic axes in the genuine abelian case. The terminal vertex $\tau(e)$ is a natural vertex of T, the initial vertex $\iota(e)$ is not.

and h remain hyperbolic in T_B. If T_A and T_B were G-equivariantly homeomorphic then g and h would also be hyperbolic in T_A. Any G-equivariant homeomorphism $T_A \to T_B$ would map the initial point p_A of the ray $A_{T_A}(g) \cap A_{T_A}(h) \subset T_A$ to the initial point p_B of $A_{T_B}(g) \cap A_{T_B}(h) \subset T_B$, where $A_{T_A}(g)$ is the hyperbolic axis of g in T_A. However, we have $g_1 k_A(x_1) = k_A(\iota(e)) = k_A(\tau(e)) = p_A$, whereas $g_1 k_B(x_1) = k_B(\iota(e)) \neq k_B(\tau(e)) = p_B$. Hence, there does not exist a G-equivariant homeomorphism $T_A \to T_B$ that maps $k_A(x_1)$ to $k_B(x_1)$.

If T is irreducible, we can find a hyperbolic group element $g \in G$ whose axis lies as in Figure 3.2. The point $g_1 k_A(x_1) = k_A(\iota(e)) = k_A(\tau(e)) \in T_A$ lies in the characteristic set $C_{T_A}(g)$, whereas $g_1 k_B(x_1) = k_B(\iota(e)) \in T_B$ does not lie in
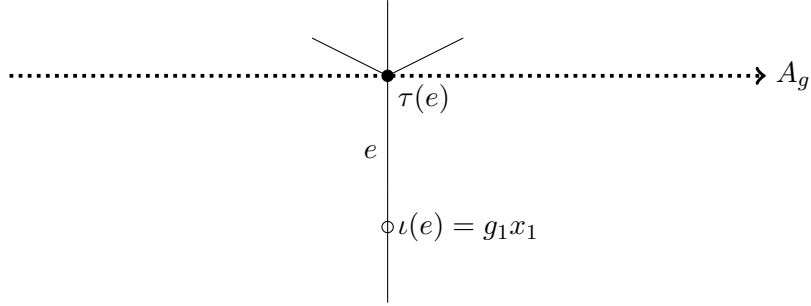
Figure 3.2: Hyperbolic axis in the irreducible case. The terminal vertex $\tau(e)$ is a natural vertex of T, the initial vertex $\iota(e)$ is not.

$C_{T_B}(g)$. Hence, there cannot exist a G-equivariant homeomorphism $T_A \to T_B$ mapping $k_A(x_1)$ to $k_B(x_1)$. $\qquad\square$

3.2 The k-th spine $\mathcal{S}(\mathcal{PD}, k)$

Let $\mathcal{PD}$ be a projectivized deformation space of metric G-trees. We will only be interested in the G-equivariant homeomorphism types of the metric G-trees in $\mathcal{PD}$ and we will therefore speak of the elements of $\mathcal{PD}$ just as "G-trees" (and not as "metric G-trees").

Definition 3.5. For $k \in \mathbb{N}_0$, we define $\mathcal{C}ol(\mathcal{PD}, k)$ as the set of G-equivariant homeomorphism classes of k-pointed G-trees $(T, x_1, \ldots, x_k)$ with $T \in \mathcal{PD}$. Given two k-pointed G-trees $(T, x_1, \ldots, x_k), (T', x'_1, \ldots, x'_k) \in \mathcal{C}ol(\mathcal{PD}, k)$, we write

$$(T, x_1, \ldots, x_k) \leq (T', x'_1, \ldots, x'_k)$$

if there exists a G-invariant subforest $F \subset (T', x'_1, \ldots, x'_k)$ with no trivial components such that the k-pointed G-trees $(T, x_1, \ldots, x_k)$ and $(T'_F, k_F(x'_1), \ldots, k_F(x'_k))$ are G-equivariantly homeomorphic, where $k_F \colon T' \to T'_F$ is the forest collapse map. By Proposition 3.4, the subforest F is uniquely determined by the G-equivariant homeomorphism type of $(T, x_1, \ldots, x_k)$.

The relation $\leq$ defines a partial order on $\mathcal{C}ol(\mathcal{PD}, k)$ and we define

$$\mathcal{S}(\mathcal{PD}, k) := |\mathcal{C}ol(\mathcal{PD}, k)|$$

where $|\mathcal{C}ol(\mathcal{PD}, k)|$ denotes the geometric realization of the poset $(\mathcal{C}ol(\mathcal{PD}, k), \leq)$. We call $\mathcal{S}(\mathcal{PD}, k)$ the *k-th spine* of $\mathcal{PD}$.

Recall that the projectivized deformation space $\mathcal{PD}$ is acted on by the subgroup

$\mathrm{Aut}_{\mathcal{D}}(G) \leq \mathrm{Aut}(G)$ of automorphisms that leave the set of elliptic subgroups of $\mathcal{PD}$ invariant. Since the inner automorphism group $\mathrm{Inn}(G) \leq \mathrm{Aut}_{\mathcal{D}}(G)$ acts trivially on $\mathcal{PD}$, we obtain an induced action of $\mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Aut}_{\mathcal{D}}(G)/\mathrm{Inn}(G) \leq \mathrm{Out}(G)$ on $\mathcal{PD}$.

Definition 3.6. For $k \in \mathbb{N}_0$, the subgroup

$$\mathrm{Aut}_{\mathcal{D}}(G, k) := \begin{cases} \mathrm{Out}_{\mathcal{D}}(G) & \text{if } k = 0 \\ \mathrm{Aut}_{\mathcal{D}}(G) \ltimes G^{k-1} & \text{if } k \geq 1 \end{cases}$$

of the k-th holomorph $\mathrm{Aut}(G, k)$ acts on $\mathcal{C}ol(\mathcal{PD}, k)$ from the right via

$$(T, x_1, x_2, \ldots, x_k) \cdot (\phi, g_2, \ldots, g_k) = (T\phi, x_1, g_2^{-1} \bullet_T x_2, \ldots, g_k^{-1} \bullet_T x_k).$$

The action preserves the partial order on $\mathcal{C}ol(\mathcal{PD}, k)$ and therefore induces a simplicial action of $\mathrm{Aut}_{\mathcal{D}}(G, k)$ on the k-th spine $\mathcal{S}(\mathcal{PD}, k) = |\mathcal{C}ol(\mathcal{PD}, k)|$. By definition, whenever we have $\mathrm{Aut}_{\mathcal{D}}(G) = \mathrm{Aut}(G)$ (as in Examples 1.36 and 1.37) then we also have $\mathrm{Aut}_{\mathcal{D}}(G, k) = \mathrm{Aut}(G, k)$.

Example 3.7. Let G be the free group F_n, $n \geq 2$ and $\mathcal{PD}$ be Outer space $\mathcal{PX}_n$. We have $\mathrm{Aut}_{\mathcal{X}_n}(F_n) = \mathrm{Aut}(F_n)$ and therefore $\mathrm{Aut}_{\mathcal{X}_n}(F_n, k) = \mathrm{Aut}(F_n, k)$. The k-th spine $\mathcal{S}(\mathcal{PX}_n, k)$ agrees with "the spine of Outer space for $\mathrm{Out}(n, k)$" constructed in [BF00, Section 2.5] and "the spine of k-pointed autre espace" in [HM13b, Section 1.5].

3.2.1 Contractibility

Let $\mathcal{PD}$ be a projectivized deformation space of metric G-trees. For all $k \in \mathbb{N}$, the projection

$$f_k \colon \mathcal{C}ol(\mathcal{PD}, k) \to \mathcal{C}ol(\mathcal{PD}, k - 1)$$
$$(T, x_1, \ldots, x_k) \mapsto (T, x_1, \ldots, x_{k-1})$$

preserves the partial order and thus induces a continuous simplicial map

$$|f_k| \colon \mathcal{S}(\mathcal{PD}, k) \to \mathcal{S}(\mathcal{PD}, k - 1)$$

on geometric realizations. In this section, we will generalize arguments developed in [Mei11] in the special case of Outer space to show the following:

Theorem 3.8. *For all $k \in \mathbb{N}$, the map $|f_k| \colon \mathcal{S}(\mathcal{PD}, k) \to \mathcal{S}(\mathcal{PD}, k - 1)$ is a homotopy equivalence.*

We know from Proposition 1.42 that $\mathcal{S}(\mathcal{PD}, 0) = \mathcal{S}(\mathcal{PD})$ is a deformation retract of the projectivized deformation space $\mathcal{PD}$ in the weak topology. Since $\mathcal{PD}$ and hence $\mathcal{S}(\mathcal{PD}, 0)$ is contractible (Theorem 1.57), Theorem 3.8 implies the following:

Corollary 3.9. *For all $k \in \mathbb{N}_0$, the k-th spine $\mathcal{S}(\mathcal{PD}, k)$ is contractible.*

Remark. As a special case, this gives an alternative proof of [BF00, Theorem 2.21] that "the spine of Outer space for $\mathrm{Out}(n, k)$" is contractible.

The proof of Theorem 3.8

If the projectivized deformation space $\mathcal{PD}$ is linear abelian then it has only one element (see Section 1.2.2) and all 1-pointed G-trees (T, x) with $T \in \mathcal{PD}$ are G-equivariantly homeomorphic, as translations on T are G-equivariant homeomorphisms (cf. Proposition 1.10). Hence, the poset $\mathcal{C}ol(\mathcal{PD}, 1)$ has only one element. The assertion of Theorem 3.8 for the map $|f_1| \colon \{*\} = \mathcal{S}(\mathcal{PD}, 1) \to \mathcal{S}(\mathcal{PD}, 0) = \{*\}$ is then trivially true and we may rule out the case where $\mathcal{PD}$ is linear abelian and $k-1 = 0$, which enables us to apply Proposition 3.3 in the proof of Proposition 3.11 without further case distinctions.

For a poset $(P, \leq)$ and an element $p \in P$, we let $P_{\geq p} := \{p' \in P \mid p' \geq p\}$. The following is an immediate consequence of Quillen's celebrated "Theorem A" (see also [Qui78, Proposition 1.6]):

Lemma 3.10 ([Bab93, Lemma 1]). *Let P and Q be posets and $f \colon P \to Q$ an order-preserving map. If*

(1) for all $q \in Q$ the geometric realization $|f^{-1}(q)|$ is contractible;

(2) for all $q \in Q$ and $p \in P$ with $f(p) \leq q$ the geometric realization $|P_{\geq p} \cap f^{-1}(q)|$ is contractible

then the induced map on geometric realizations $|f| \colon |P| \to |Q|$ is a homotopy equivalence.

In the following, we will speak of $f^{-1}(q)$ and its geometric realization $|f^{-1}(q)|$ as *(pointwise) fibers* and of $P_{\geq p} \cap f^{-1}(q)$ and its geometric realization $|P_{\geq p} \cap f^{-1}(q)|$ as *Quillen fibers*. In order to prove Theorem 3.8, we will show that the order-preserving map of posets $f_k \colon \mathcal{C}ol(\mathcal{PD}, k) \to \mathcal{C}ol(\mathcal{PD}, k-1)$ satisfies the conditions of Lemma 3.10. We will proceed in two steps, first proving condition (1) and then proving condition (2):

Contractibility of the pointwise fiber Let $(T, x_1, \ldots, x_{k-1}) \in \mathcal{C}ol(\mathcal{PD}, k-1)$. The pointwise fiber $f_k^{-1}((T, x_1, \ldots, x_{k-1})) \subset \mathcal{C}ol(\mathcal{PD}, k)$ consists of k-pointed G-trees $(T, x_1, \ldots, x_{k-1}, x)$, where either

(1) x is a natural vertex of $(T, x_1, \ldots, x_{k-1})$ or

(2) x is not a natural vertex of $(T, x_1, \ldots, x_{k-1})$, i.e.,

$$V(T, x_1, \ldots, x_{k-1}, x) = V(T, x_1, \ldots, x_{k-1}) \amalg Gx.$$

We call k-pointed G-trees in the pointwise fiber *unextended* if they are of the first type and *extended* if they are of the second type. If $(T, x_1, \ldots, x_{k-1}, x)$ is extended then x lies in the interior of a natural edge of $(T, x_1, \ldots, x_{k-1})$.

Proposition 3.11. *The geometric realization of the fiber $f_k^{-1}((T, x_1, \ldots, x_{k-1}))$ is simplicially homeomorphic to the first barycentric subdivision of T with respect to the natural simplicial structure relative to $x_1, \ldots, x_{k-1}$.*

In particular, the geometric realization of $f_k^{-1}((T, x_1, \ldots, x_{k-1}))$ is contractible. This verifies (1) of Lemma 3.10 for the map $f_k \colon \mathcal{C}ol(\mathcal{PD}, k) \to \mathcal{C}ol(\mathcal{PD}, k-1)$.

Proof. The first barycentric subdivision of $(T, x_1, \ldots, x_{k-1})$ is simplicially homeomorphic to the geometric realization of the poset

$$P = V(T, x_1, \ldots, x_{k-1}) \amalg E(T, x_1, \ldots, x_{k-1})$$

where $x, y \in P$ satisfy $x \le y$ if $y \in E(T, x_1, \ldots, x_{k-1})$ and $x = \iota(y)$ or $x = \tau(y)$. In order to prove the claim, it suffices to find a bijective map

$$h \colon f_k^{-1}((T, x_1, \ldots, x_{k-1})) \to P$$

such that we have $h((T, x_1, \ldots, x_{k-1}, x)) \le h((T, x_1, \ldots, x_{k-1}, y))$ if and only if $(T, x_1, \ldots, x_{k-1}, x) \le (T, x_1, \ldots, x_{k-1}, y)$, i.e., an isomorphism of posets. Define

$$h((T, x_1, \ldots, x_{k-1}, x)) = \begin{cases} x & \text{if } (T, x_1, \ldots, x_{k-1}, x) \text{ is unextended} \\ e_x & \text{if } (T, x_1, \ldots, x_{k-1}, x) \text{ is extended} \end{cases}$$

where e_x denotes the natural edge of $(T, x_1, \ldots, x_{k-1})$ in whose interior x lies. This assignment is well-defined by Proposition 3.3, as the $(k-1)$-pointed minimal G-tree $(T, x_1, \ldots, x_{k-1})$ has no nontrivial G-equivariant simplicial automorphisms (remember that we ruled out the case where $\mathcal{PD}$ is linear abelian and $k = 1$). The map h is clearly bijective (for injectivity, keep in mind that all G-trees are considered up to G-equivariant homeomorphism) and it remains to show that h is in fact an isomorphism of posets. For the "if" direction, suppose that

$$(T, x_1, \ldots, x_{k-1}, x) \le (T, x_1, \ldots, x_{k-1}, y).$$

Since an elementary collapse reduces the number of G-orbits of vertices,

- $(T, x_1, \ldots, x_{k-1}, y)$ must be extended,
- $(T, x_1, \ldots, x_{k-1}, x)$ must be unextended, and
- $(T, x_1, \ldots, x_{k-1}, x)$ is obtained from $(T, x_1, \ldots, x_{k-1}, y)$ by collapsing a single G-orbit of edges.

Thus, y lies in the interior of a natural edge e_y of $(T, x_1, \ldots, x_{k-1})$ whose initial or terminal vertex is x. In particular, we have

$$h((T, x_1, \ldots, x_{k-1}, x)) \leq h((T, x_1, \ldots, x_{k-1}, y)).$$

Conversely, suppose we have $h((T, x_1, \ldots, x_{k-1}, x)) \leq h((T, x_1, \ldots, x_{k-1}, y))$, i.e., y lies in the interior of a natural edge e_y of $(T, x_1, \ldots, x_{k-1})$ whose, say, initial vertex is x. Denote by $(e_y)_1$ and $(e_y)_2$ the natural edges of $(T, x_1, \ldots, x_{k-1}, y)$ from $x = \iota(e)$ to y and from y to $\tau(e)$ respectively. The edge $(e_y)_1$ is clearly collapsible and collapsing all edges in its G-orbit is an elementary collapse from $(T, x_1, \ldots, x_{k-1}, y)$ to $(T, x_1, \ldots, x_{k-1}, x)$, whence

$$(T, x_1, \ldots, x_{k-1}, x) \leq (T, x_1, \ldots, x_{k-1}, y). \qquad \square$$

Contractibility of the Quillen fiber We will next verify condition (2) of Lemma 3.10 for the order-preserving map $f_k \colon \mathcal{C}ol(\mathcal{PD}, k) \to \mathcal{C}ol(\mathcal{PD}, k-1)$. For this, let $q = (T, x_1, \ldots, x_{k-1}) \in \mathcal{C}ol(\mathcal{PD}, k-1)$ and $p = (S, y_1, \ldots, y_k) \in \mathcal{C}ol(\mathcal{PD}, k)$ with

$$f_k((S, y_1, \ldots, y_k)) = (S, y_1, \ldots, y_{k-1}) \leq (T, x_1, \ldots, x_{k-1}).$$

By the definition of the partial order, there then exists a G-invariant subforest with no trivial components $F \subset (T, x_1, \ldots, x_{k-1})$ such that

$$(S, y_1, \ldots, y_{k-1}) = (T_F, k_F(x_1), \ldots, k_F(x_{k-1}))$$

where $k_F \colon T \to T_F$ is the forest collapse map. We know from Proposition 3.4 that the subforest F is uniquely determined by the G-equivariant homeomorphism type of $(S, y_1, \ldots, y_{k-1})$.

We are interested in the Quillen fiber $\mathcal{C}ol(\mathcal{PD}, k)_{\geq p} \cap f_k^{-1}(q)$, which consists of the k-pointed G-trees $(T, x_1, \ldots, x_{k-1}, x)$ with $x \in k_F^{-1}(y_k) \subset T$. (A structural overview over the pointed G-trees under consideration is given in Figure 3.3.)

In order to show that $|\mathcal{C}ol(\mathcal{PD}, k)_{\geq p} \cap f_k^{-1}(q)|$ is contractible, we consider two cases: First, if $y_k \in T_F$ does not lie in the image of $F \subset (T, x_1, \ldots, x_{k-1})$ under the quotient map $k_F \colon T \to T_F$ then the preimage $k_F^{-1}(y_k)$ consists of a single point and $|\mathcal{C}ol(\mathcal{PD}, k)_{\geq p} \cap f_k^{-1}(q)|$ is trivially contractible. For instance, this is always the case if y_k is not a natural vertex of $(T_F, k_F(x_1), \ldots, k_F(x_{k-1}))$. Second, if y_k lies in the image of F under k_F then $k_F^{-1}(y_k) \subset T$ is a connected component of F

$$(T, x_1, \ldots, x_{k-1}, x) \xrightarrow{\quad f_k \quad} q = (T, x_1, \ldots, x_{k-1})$$

with vertical maps k_F and bottom map

$$p = (T_F, k_F(x_1), \ldots, k_F(x_{k-1}), y_k) \xrightarrow{\quad f_k \quad} (T_F, k_F(x_1), \ldots, k_F(x_{k-1}))$$

Figure 3.3: Structural overview over the pointed G-trees under consideration.

and hence a simplicial subtree of $(T, x_1, \ldots, x_{k-1})$. We then have the following:

Proposition 3.12. *If $y_k \in T_F$ lies in the image of F under the quotient map $k_F \colon T \to T_F$ then $|\mathcal{C}ol(\mathcal{PD}, k)_{\geq p} \cap f_k^{-1}(q)|$ is simplicially homeomorphic to the first barycentric subdivision of the simplicial subtree $k_F^{-1}(y_k) \subset (T, x_1, \ldots, x_{k-1})$.*

In particular, $|\mathcal{C}ol(\mathcal{PD}, k)_{\geq p} \cap f_k^{-1}(q)|$ is always contractible, which completes the proof of Theorem 3.8.

Proof. The first barycentric subdivision of the subtree $k_F^{-1}(y_k) \subset (T, x_1, \ldots, x_{k-1})$ is simplicially homeomorphic to the geometric realization of the poset

$$P = V(k_F^{-1}(y_k)) \amalg E(k_F^{-1}(y_k))$$

where $x, y \in P$ satisfy $x \leq y$ if $y \in E(k_F^{-1}(y_k))$ and $x = \iota(y)$ or $x = \tau(y)$. Similarly as in the proof of Proposition 3.11, one now easily shows that the map $h \colon \mathcal{C}ol(\mathcal{PD}, k)_{\geq p} \cap f_k^{-1}(q) \to P$ given by

$$(T, x_1, \ldots, x_{k-1}, x) \mapsto \begin{cases} x & \text{if } (T, x_1, \ldots, x_{k-1}, x) \text{ is unextended} \\ e_x & \text{if } (T, x_1, \ldots, x_{k-1}, x) \text{ is extended} \end{cases}$$

is an isomorphism of posets, where e_x denotes the natural edge of $(T, x_1, \ldots, x_{k-1})$ in whose interior x lies. $\qquad\square$

3.2.2 Dimension and cocompactness

Let $\mathcal{PD}$ be a projectivized deformation space of metric G-trees.

Proposition 3.13. *If the spine $\mathcal{S}(\mathcal{PD}) = \mathcal{S}(\mathcal{PD}, 0)$ is of dimension $\leq b \in \mathbb{N}$ then for $k \in \mathbb{N}$ the k-th spine $\mathcal{S}(\mathcal{PD}, k)$ is of dimension $\leq b + k$.*

Proof. The spine $\mathcal{S}(\mathcal{PD})$ is of dimension $\leq b$ if and only if the number of G-orbits of edges of any collapsible G-invariant subforest of any G-tree in $\mathcal{PD}$ is uniformly

bounded above by b. Suppose this is the case and let $(T, x_1, \ldots, x_k) \in \mathcal{C}ol(\mathcal{PD}, k)$. Let M be a maximal collapsible G-invariant subforest of the unpointed G-tree T. The first basepoint $x_1 \in T$ lies as in one of the following cases:

(1) x_1 lies in the natural vertex set of T, the simplicial structures on T and (T, x_1) are the same, and the collapsible G-invariant subforest $M \subset (T, x_1)$ remains maximal; or

(2) x_1 lies in the interior of a natural edge of T that also lies in M, in which case the collapsible G-invariant subforest $M \subset (T, x_1)$ remains maximal but has one extra G-orbits of edges (one of the old ones being subdivided); or

(3) x_1 subdivides a natural edge e of T that does not lie in M into e_1 and e_2, in which case $M \cup Ge_i$ is a maximal collapsible G-invariant subforest of (T, x_1) for each $i \in \{1, 2\}$.

Thus, since a maximal collapsible G-invariant subforest of T has at most b G-orbits of edges, a maximal collapsible G-invariant subforest of (T, x_1) has at most $b + 1$ G-orbits of edges. Inductively, we conclude that a maximal collapsible G-invariant subforest of any k-pointed G-tree in $\mathcal{C}ol(\mathcal{PD}, k)$ has at most $b + k$ G-orbits of edges, proving that $\mathcal{S}(\mathcal{PD}, k)$ is of dimension $\leq b + k$. $\qquad\square$

Proposition 3.14. *If $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices then for all $k \in \mathbb{N}_0$ the k-holomorph $\mathrm{Aut}_{\mathcal{D}}(G, k)$ acts on the k-th spine $\mathcal{S}(\mathcal{PD}, k)$ with finitely many orbits of simplices.*

Proof. We must show that the k-pointed G-trees in $\mathcal{C}ol(\mathcal{PD}, k)$ fall into finitely many $\mathrm{Aut}_{\mathcal{D}}(G, k)$-orbits. By assumption, this is true for $k = 0$. That is, there are finitely many G-trees $T_1, \ldots, T_n \in \mathcal{C}ol(\mathcal{PD})$ such that for any $T \in \mathcal{C}ol(\mathcal{PD})$ there exists $i \in \{1, \ldots, n\}$ and $\phi \in \mathrm{Out}_{\mathcal{D}}(G)$ and for any representative $\Phi \in \mathrm{Aut}_{\mathcal{D}}(G)$ of ϕ a G-equivariant homeomorphism $h_\Phi \colon T \to T_i \Phi$.

Choose for each $i \in \{1, \ldots, n\}$ a fundamental domain $F_i \subset T_i$ for the action of G on T_i. Since T_i is minimal and therefore cocompact, F_i contains only finitely many vertices and edges. We will show that any $(T, x_1, \ldots, x_k) \in \mathcal{C}ol(\mathcal{PD}, k)$ lies in the $\mathrm{Aut}_{\mathcal{D}}(G, k)$-orbit of a k-pointed G-tree of the form $(T_i, y_1, \ldots, y_k)$ with $i \in \{1, \ldots, n\}$ and $y_1, \ldots, y_k \in F_i$. Since F_i has only finitely many vertices and edges, the k-pointed G-trees $(T_i, y_1, \ldots, y_k)$ with $y_1, \ldots, y_k \in F_i$ are of only finitely many G-equivariant homeomorphism types, whence the claim.

By assumption, for some $i \in \{1, \ldots, n\}$ and some $\Phi \in \mathrm{Aut}_{\mathcal{D}}(G)$ there exists a G-equivariant homeomorphism

$$h_\Phi \colon (T, x_1, \ldots, x_k) \to (T_i \Phi, h_\Phi(x_1), \ldots, h_\Phi(x_k))$$

and we conclude that $(T, x_1, \ldots, x_k)$ lies in the same $\mathrm{Aut}_{\mathcal{D}}(G, k)$-orbit as

$$(T_i, h_\Phi(x_1), \ldots, h_\Phi(x_k)).$$

For each $j = 1, \ldots, k$ there exists a group element $g_j \in G$ such that $g_j \bullet_{T_i} h_\Phi(x_j)$ lies in the fundamental domain F_i. The k-pointed G-tree $(T_i, h_\Phi(x_1), \ldots, h_\Phi(x_k))$ lies in the same $\mathrm{Aut}_{\mathcal{D}}(G, k)$-orbit as $(T_i c_{g_1^{-1}}, h_\Phi(x_1), \ldots, h_\Phi(x_k))$, and we have a G-equivariant homeomorphism

$$(T_i c_{g_1^{-1}}, h_\Phi(x_1), \ldots, h_\Phi(x_k)) \to (T_i, g_1 \bullet_{T_i} h_\Phi(x_1), \ldots, g_1 \bullet_{T_i} h_\Phi(x_k))$$

given by $x \mapsto g_1 \bullet_{T_i} x$. Finally, we have

$$
\begin{aligned}
&(T_i, g_1 h_\Phi(x_1), g_1 h_\Phi(x_2), \ldots, g_1 h_\Phi(x_k)) \cdot (id, (g_1 g_2^{-1}, \ldots, g_1 g_k^{-1})) \\
&= (T_i, g_1 h_\Phi(x_1), (g_1 g_2^{-1})^{-1} g_1 h_\Phi(x_2), \ldots, (g_1 g_k^{-1})^{-1} g_1 h_\Phi(x_k)) \\
&= (T_i, g_1 h_\Phi(x_1), \ldots, g_k h_\Phi(x_k)).
\end{aligned}
$$

Altogether, we conclude that $(T, x_1, \ldots, x_k)$ lies in the same $\mathrm{Aut}_{\mathcal{D}}(G, k)$-orbit as a k-pointed G-tree $(T_i, y_1, \ldots, y_k)$ with $y_1, \ldots, y_k \in F_i$. $\qquad\square$

Application to higher holomorphs of virtually free groups

Recall that if G is a finitely generated virtually nonabelian free group then all minimal metric G-trees with finite vertex stabilizers lie in the same deformation space $\mathcal{D}$ (Example 1.30) and we have $\mathrm{Aut}_{\mathcal{D}}(G) = \mathrm{Aut}(G)$ (Example 1.36).

Proposition 3.15. *Let G be a finitely generated virtually nonabelian free group and $\mathcal{PD}$ the projectivized deformation space of minimal metric G-trees with finite vertex stabilizers. For all $k \in \mathbb{N}_0$, the k-th holomorph $\mathrm{Aut}(G, k)$ acts on $\mathcal{S}(\mathcal{PD}, k)$ with finite point stabilizers.*

Proof. We show that $\mathrm{Aut}(G, k)$ acts on the poset $\mathcal{Col}(\mathcal{PD}, k)$ with finite stabilizers. Let $(T, x_1, \ldots, x_k) \in \mathcal{Col}(\mathcal{PD}, k)$ and $(\phi, g_2, \ldots, g_k) \in \mathrm{Aut}(G, k)$, and suppose that $(T, x_1, \ldots, x_k)$ is G-equivariantly homeomorphic to

$$(T, x_1, x_2 \ldots, x_k) \cdot (\phi, g_2, \ldots, g_k) = (T\phi, x_1, g_2^{-1} \bullet_T x_2, \ldots, g_k^{-1} \bullet_T x_k).$$

Then there exists a G-equivariant homeomorphism $f_\phi \colon T \to T\phi$ such that we have $f_\phi(x_1) = x_1$ and $f_\phi(x_i) = g_i^{-1} \bullet_T x_i$ for all $i = 2, \ldots, k$. By Proposition 3.3, the G-equivariant homeomorphism f_ϕ is unique up to G-equivariant isotopy relative to the natural vertices of $(T, x_1, \ldots, x_k)$, and we may arrange it to be simplicial.

We will first show that there are only finitely many possibilities for ϕ. It follows from [GL07a, Proposition 8.6] that $\mathrm{Out}(G)$ acts on $\mathcal{PD}$ with finite point stabilizers

and then from Proposition 1.22(4) that the stabilizers under the action of $\mathrm{Out}(G)$ on the poset $\mathcal{C}ol(\mathcal{PD})$ are finite as well. Therefore, there are only finitely many possibilities for the outer automorphism class of ϕ. If $\phi' = \phi \circ c_g$, where c_g denotes conjugation with $g \in G$, then the unique G-equivariant simplicial homeomorphism $f_{\phi'} \colon T \to T\phi \to T\phi' = T\phi c_g$ is given by $x \mapsto f_\phi(x) \mapsto g \bullet_{T\phi} f_\phi(x)$. Note that $f_{\phi'}(x_1) = g \bullet_{T\phi} f_\phi(x_1) = g \bullet_{T\phi} x_1$ equals x_1 if and only if g lies in the stabilizers of $x_1 \in T\phi$, which is finite. Hence, within each of the finitely many possible outer automorphism classes $[\phi]$, there are only finitely many possible representatives ϕ.

It remains to show that for fixed ϕ and each $i = 2, \ldots, k$, there are only finitely many possibilities for g_i. However, this is immediate, as we have $f_\phi(x_i) = g_i^{-1} \bullet_T x_i$ and the stabilizer of $x_i \in T$ is finite. $\square$

Recall that a group G is of *type F_∞* if it admits a CW-model for its classifying space that has finitely many G-orbits of cells in each dimension. Also, we denote by $cd_\mathbb{Q}(G)$ the rational cohomological dimension of G.

Corollary 3.16. *Let G be a finitely generated virtually nonabelian free group. For all $k \in \mathbb{N}_0$, the k-th holomorph $\mathrm{Aut}(G, k)$ has finite rational cohomological dimension and is of type F_∞. In fact, if we let $\mathcal{PD}$ be the projectivized deformation space of minimal metric G-trees with finite vertex stabilizers, we have*

$$cd_\mathbb{Q}(\mathrm{Aut}(G, k)) \leq dim(\mathcal{S}(\mathcal{PD})) + k.$$

Proof. The simplicial complex with missing faces $\mathcal{PD}$ and its spine $\mathcal{S}(\mathcal{PD})$ are finite-dimensional, as the group $\mathrm{Out}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices (Example 1.36). By Proposition 3.13, the k-th spine $\mathcal{S}(\mathcal{PD}, k)$ has dimension $dim(\mathcal{S}(\mathcal{PD}, k)) \leq dim(\mathcal{S}(\mathcal{PD})) + k$. The bound on $cd_\mathbb{Q}(\mathrm{Aut}(G, k))$ follows from [Pet07, Lemma 3.3], since $\mathrm{Aut}(G, k)$ acts on the contractible simplicial complex $\mathcal{S}(\mathcal{PD}, k)$ with finite point stabilizers.

Moreover, since $\mathrm{Out}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices, $\mathrm{Aut}(G, k)$ acts on the k-th spine $\mathcal{S}(\mathcal{PD}, k)$ with finitely many orbits of simplices (Proposition 3.14). It follows from [Bro87, Corollary 3.3] that $\mathrm{Aut}(G, k)$ is of type F_∞ (finitely presented and of type FP_∞). $\square$

3.2.3 A finite-dimensional deformation retract

Let $\mathcal{PD}$ be a projectivized deformation space of metric G-trees. Recall from Proposition 1.45 that the spine $\mathcal{S}(\mathcal{PD}) = \mathcal{S}(\mathcal{PD}, 0) = |\mathcal{C}ol(\mathcal{PD}, 0)|$ deformation retracts $\mathrm{Out}_\mathcal{D}(G)$-equivariantly onto the surviving spine $\mathcal{S}_W(\mathcal{PD}) \subset \mathcal{S}(\mathcal{PD})$. By Theorem 1.48, if $\mathcal{PD}$ is irreducible and

- nonascending; or
- locally finite and has $b_1(\mathcal{D}) \leq 1$

then $\mathcal{S}_W(\mathcal{PD})$ is finite-dimensional. Indeed, the surviving spine $\mathcal{S}_W(\mathcal{PD})$ turns out to be finite-dimensional in certain cases where the spine $\mathcal{S}(\mathcal{PD})$ itself is infinite-dimensional (see Example 1.49).

Analogously to the unpointed case, we say that for $k \in \mathbb{N}_0$ an edge of a k-pointed G-tree $(T, x_1, \ldots, x_k)$ is *surviving* if it is noncollapsible or may be made noncollapsible by collapsing other collapsible edges of $(T, x_1, \ldots, x_k)$. Let $W \subseteq (T, x_1, \ldots, x_k)$ be the G-invariant subforest consisting of all nonsurviving edges. It is collapsible and the set

$$\mathcal{C}ol_W(\mathcal{PD}, k) := \{(T_W, k_W(x_1), \ldots, k_W(x_k)) \mid (T, x_1, \ldots, x_k) \in \mathcal{C}ol(\mathcal{PD}, k)\}$$

is a subposet of $\mathcal{C}ol(\mathcal{PD}, k)$, where $k_W \colon T \to T_W$ is the forest collapse map. The geometric realization $\mathcal{S}_W(\mathcal{PD}, k) := |\mathcal{C}ol_W(\mathcal{PD}, k)|$ is a simplicial subcomplex of the k-th spine $\mathcal{S}(\mathcal{PD}, k)$ that we call the k-th *surviving spine* of $\mathcal{PD}$. We have the following generalization of Proposition 1.45:

Proposition 3.17. *For all $k \in \mathbb{N}_0$, the k-th spine $\mathcal{S}(\mathcal{PD}, k)$ deformation retracts $\mathrm{Out}_{\mathcal{D}}(G)$-equivariantly onto the k-th surviving spine $\mathcal{S}_W(\mathcal{PD}, k) \subseteq \mathcal{S}(\mathcal{PD}, k)$.*

Proof. The natural map

$$f \colon \mathcal{C}ol(\mathcal{PD}, k) \to \mathcal{C}ol_W(\mathcal{PD}, k) \subseteq \mathcal{C}ol(\mathcal{PD}, k)$$
$$(T, x_1, \ldots, x_k) \mapsto (T_W, k_W(x_1), \ldots, k_W(x_k))$$

is order-preserving and for all $\xi \in \mathcal{C}ol(\mathcal{PD}, k)$ we have $f(\xi) \leq \xi$. As explained in the proof of Proposition 1.45, the claim follows from [Qui78, 1.3]. $\square$

We conclude with an immediate generalization of Theorem 1.48:

Theorem 3.18. *Let $\mathcal{D}$ be an irreducible deformation space of metric G-trees. If either*

- *$\mathcal{D}$ is nonascending; or*
- *$\mathcal{D}$ is locally finite and has $b_1(\mathcal{D}) \leq 1$*

then for all $k \in \mathbb{N}_0$ the k-th surviving spine $\mathcal{S}_W(\mathcal{PD}, k) \subset \mathcal{S}(\mathcal{PD}, k)$ is finite-dimensional.

Proof. The k-th surviving spine $\mathcal{S}_W(\mathcal{PD}, k) \subset \mathcal{S}(\mathcal{PD}, k)$ is finite-dimensional if and only if the number of G-orbits of edges of any collapsible G-invariant subforest of any k-pointed G-tree in $\mathcal{C}ol_W(\mathcal{PD}, k)$ is uniformly bounded. By Theorem 1.48, the assertion holds for $k = 0$ and hence there exists such a uniform bound $b \in \mathbb{N}$ for the unpointed G-trees in $\mathcal{C}ol_W(\mathcal{PD})$. One easily sees that a k-pointed G-tree with no nonsurviving edges (relative to the finer simplicial structure) is the same

as a G-tree with no nonsurviving edges together with k basepoints. Thus, every k-pointed G-tree in $\mathcal{C}ol_W(\mathcal{PD}, k)$ is obtained from a G-tree $T \in \mathcal{C}ol_W(\mathcal{PD})$ by adding k basepoints, and we can argue as in the proof of Proposition 3.13. $\square$

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Abstract

For a finitely generated group G, we study deformation spaces of metric G-trees, which are analogues of the Teichmüller spaces of surfaces for group actions on trees. Deformation spaces of metric G-trees generalize *Culler-Vogtmann's Outer space*, the deformation space of free actions on trees, which has proven to be immensely useful in the study of $\mathrm{Out}(F_n)$, the outer automorphism group of the free group of rank $n \geq 2$.

Let $\mathcal{D}$ be a deformation space of metric G-trees. The group of positive real numbers $\mathbb{R}_{>0}$ acts on $\mathcal{D}$ by scaling the metrics on the trees and we define the *projectivized deformation space* as the quotient $\mathcal{PD} = \mathcal{D}/\mathbb{R}_{>0}$. The outer automorphism group $\mathrm{Out}(G)$ contains a certain subgroup $\mathrm{Out}_{\mathcal{D}}(G)$ that acts on $\mathcal{D}$ and $\mathcal{PD}$ by precomposing the G-actions on the trees. In Chapter 1, we present a complete argument that under certain assumptions the projectivized deformation space $\mathcal{PD}$ is a model for the classifying space of $\mathrm{Out}_{\mathcal{D}}(G)$ for a family of subgroups. In Chapter 2, we introduce an asymmetric pseudometric on $\mathcal{PD}$ that generalizes the asymmetric *Lipschitz metric* on Outer space and is an analogue of the Thurston metric on Teichmüller space. Making use of the Lipschitz metric on $\mathcal{PD}$, we prove existence of train track representatives for irreducible automorphisms of virtually free groups and nonelementary generalized Baumslag-Solitar groups that contain no solvable Baumslag-Solitar group $\mathrm{BS}(1, n)$ with $n \geq 2$. In Chapter 3, we define the *higher holomorphs* $\mathrm{Aut}(G, k)$, $k \in \mathbb{N}$, which are "higher-pointed" variants of the automorphism group $\mathrm{Aut}(G)$. Following the construction of the *spine of Outer space*, we construct a family of simplicial complexes $\mathcal{S}(\mathcal{PD}, k)$, $k \in \mathbb{N}$ on which certain subgroups $\mathrm{Aut}_{\mathcal{D}}(G, k) \leq \mathrm{Aut}(G, k)$ act and we show that these complexes are always contractible.

Zusammenfassung

Im Rahmen dieser Arbeit studieren wir Deformationsräume metrischer G-Bäume für endlich erzeugte Gruppen G. Deformationsräume metrischer G-Bäume sind Analoga zu Teichmüller-Räumen von Flächen für Gruppenwirkungen auf Bäumen. Sie verallgemeinern *Culler-Vogtmann's Outer space*, den Deformationsraum freier Gruppenwirkungen auf Bäumen, welcher im Studium von $\mathrm{Out}(F_n)$, der äußeren Automorphismengruppe der freien Gruppe vom Rang $n \geq 2$, eine zentrale Rolle spielt.

Sei $\mathcal{D}$ ein Deformationsraum metrischer G-Bäume. Die Gruppe der positiven reellen Zahlen $\mathbb{R}_{>0}$ operiert auf $\mathcal{D}$ durch Skalieren der Metriken auf den Bäumen. Der *projektivierte Deformationsraum* ist der Quotient $\mathcal{PD} = \mathcal{D}/\mathbb{R}_{>0}$. Die äußere Automorphismengruppe $\mathrm{Out}(G)$ enthält eine gewisse Untergruppe $\mathrm{Out}_{\mathcal{D}}(G)$, die auf $\mathcal{D}$ und $\mathcal{PD}$ durch Präkomposition der G-Wirkungen auf den Bäumen operiert. In Kapitel 1 präsentieren wir einen vollständigen Beweis, dass der projektivierte Deformationsraum $\mathcal{PD}$ in bestimmten Fällen ein Modell für den klassifizierenden Raum von $\mathrm{Out}_{\mathcal{D}}(G)$ für eine Familie von Untergruppen ist. In Kapitel 2 definieren wir eine asymmetrische Pseudometrik auf $\mathcal{PD}$, welche die asymmetrische *Lipschitz-Metrik* auf Outer space verallgemeinert und ein Analogon zur Thurston-Metrik auf dem Teichmüller-Raum einer Fläche darstellt. Mit Hilfe dieser zeigen wir, dass jeder irreduzible Automorphismus einer virtuell freien Gruppe oder einer nicht-elementaren verallgemeinerten Baumslag-Solitar-Gruppe, welche keine auflösbare Baumslag-Solitar-Gruppe $\mathrm{BS}(1,n)$ mit $n \geq 2$ enthält, von einer *Train-Track-Abbildung* repräsentiert wird. In Kapitel 3 definieren wir die *höheren Holomorphe* $\mathrm{Aut}(G,k)$, $k \in \mathbb{N}$, welche "höher-punktierte" Varianten der Automorphismen-gruppe $\mathrm{Aut}(G)$ darstellen. Analog zur Konstruktion des *Spine of Outer space* konstruieren wir eine Familie von simplizialen Komplexen $\mathcal{S}(\mathcal{PD},k)$, $k \in \mathbb{N}$, auf denen gewisse Untergruppen $\mathrm{Aut}_{\mathcal{D}}(G,k) \leq \mathrm{Aut}(G,k)$ operieren, und wir zeigen, dass diese Komplexe immer kontraktibel sind.

Curriculum Vitae

For privacy reasons, the curriculum vitae is not included in the electronic version of this thesis.

Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig angefertigt und ausschließlich die angegebenen Quellen und Hilfsmittel verwendet habe.

Berlin, 28. Juli 2014   Sebastian Meinert