2 Signorini's Problem

Signorini's problem in linear elasticity models the linearized contact of an elastic body with a rigid foundation. It has been originally introduced in 1933 by Signorini in [Sig33]. Although the body being in contact with the foundation is assumed to be linear elastic, Signorini's problem is nonlinear and nondifferentiable at the contact boundary. This is due to the a priori unknown contact boundary, since the *change of phase* at the contact boundary, i.e., contact or no contact, is highly nonlinear with respect to the displacements.

Before formulating Signorini's problem, we have to deal with some kinematical considerations concerning the non-penetration condition at the possible contact boundary of our body $\mathscr{B}$ in Section 2.1. Then, in Section 2.2 we give both, the variational formulation of the Signorini problem as well as its "classical" formulation as a boundary value problem. Moreover, we state results concerning the existence and regularity of the solution. In contrast to the variational formulation (1.14), the corresponding variationl formulation is not a variational equation, but a *variational inequality*. This is due to the unknown contact boundary. At the end of this section, in Section 2.4, we discretize Signorini's problem by finite elements und give a priori error estimates for the discretization error.

2.1 Linearized Contact

In the previous chapter, we have developed the boundary value problem of linear elasticity for hyperelastic materials. Let us recall, that one of the major assumptions is that of *small displacements*, which gives rise to the geometric linearization of the strain tensor. The assumption of small displacements plays also a major role in formulating the non-penetration condition at the possible contact boundary as we are going see in this section. Here and throughout this work, we assume that the domain $\mathscr{B}$, the body under consideration is identified with, is Lipschitzian, i.e., its boundary may locally be represented by a Lipschitz continuous parametrization, see, e.g., [Wlo82, Definition 2.4].

Let $\mathscr{B} \subset \mathbb{R}^3$ be the body under consideration and let the body's surface $\partial\mathscr{B}$ be decomposed into three disjoint parts

$$\partial\mathscr{B} = \Gamma_D \ \cup \ \Gamma_F \ \cup \ \Gamma_S$$

with Γ_D having positive measure. We assume Dirichlet boundary conditions to be prescribed at Γ_D and surface forces acting on the Neumann part of the boundary Γ_F. Finally, Γ_S denotes the possible contact boundary or *Signorini boundary*. Let furthermore the displacement of the body be constrained by a rigid foundation, or obstacle $\mathcal{G}$, as depicted in Figure 2.1. Then, our goal is to find a condition, which models the intuitive idea of $\mathscr{B}$ non-penetrating $\mathcal{G}$ and which is easy to handle. Assuming also the obstacle being a Lipschitz domain, for any point $y \in \partial\mathcal{G}$ there exists a neighbourhood U_y, such that we can represent the boundary segment $\gamma_y := \partial\mathcal{G} \cap U_y$ as

$$\gamma_y = \{(y_1, y_2, y_3) : \ y_3 = \eta_y(y_1, y_2)\, , |y_1|, |y_2| < \alpha_y \, , \alpha_y \ \text{ sufficiently small}\} \, .$$

Here, the coordinates (y_1, y_2, y_3) are given with respect to a possibly rotated coordinate system. Proceeding as before with an adjacent segment γ_x of the body's surface, we can

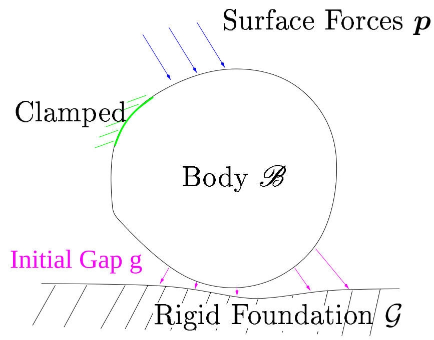

Figure 2.1: Contact with a rigid foundation

represent locally the points $(x_1, x_2, x_3) \in \gamma_x$ by

$$x_3 = \eta_x(x_1, x_2)\,,$$

again with respect the possibly rotated coordinate system used for the parametrization of γ_y. Here, we have implicitly made use of the assumption, that the outer normals are close to each other.

Any deformation $\boldsymbol{u}$ being admissible with respect to the non-penetration constraint has to satisfy

$$\begin{aligned}\eta_x(x_1, x_2) + u_3(x_1, x_2, \eta_x(x_1, x_2)) \;\leq\; \eta_y(x_1 + u_1(x_1, x_2, \eta_x(x_2, x_3)),\\ x_2 + \varphi_2(x_1, x_2, \eta_x(x_2, x_3)))\,.\end{aligned} \quad (2.1)$$

Here of course, $\boldsymbol{\varphi}$ is written with respect to the rotated coordinate system introduced above. Inequality (2.1) is referred to as *kinematical contact condition* for finite displacements. Taking a closer look at conditon (2.1), we see that not only the possibly unknown parametrization of both, the body's and the obstacle's surface is required but also the unknown displacement $\boldsymbol{\varphi}$. Recalling the assumptions of "small" displacements, we seek for an approximation of the kinematical contact condition in terms of the initial coordinates of the undeformed configuration $\mathscr{B}$. Following [KO88], we derive the sought approximation by linearizing the kinematical contact condition.

Let the parametrization η_y and η_x have at least a continuous first and a bounded second derivative and let us denote in an abuse of notation our small displacement by

$$(u_1, u_2, u_3)^T = \boldsymbol{u}(x_1, x_2, \eta_x(x_1, x_2))\,.$$

Then, we can linearize (2.1) to obtain the simplified condition

$$\eta_x(x_1, x_2) + u_3 \;\leq\; \eta_y(x_1, x_2) + \big(\frac{\partial \eta_y}{\partial y_1}, \frac{\partial \eta_y}{\partial y_2}\big) \cdot (u_1, u_2)^T\,,$$

that is

$$\big(-\frac{\partial \eta_y}{\partial y_1}, -\frac{\partial \eta_y}{\partial y_2}, 1\big) \cdot (u_1, u_2, u_3)^T \;\leq\; \eta_y(x_1, x_2) - \eta_x(x_1, x_2)\,. \quad (2.2)$$

This equation gives a linearized contact condition with respect to the direction vector $\tilde{\boldsymbol{n}}_y = (-\frac{\partial \eta_y}{\partial y_1}, -\frac{\partial \eta_y}{\partial y_2}, 1)$, which is normal to the surface segment γ_y at the point y. We normalize equation (2.2) by dividing both sides by $\|\tilde{\boldsymbol{n}}_y\|_2$ and gain the equivalent normalized equation

$$\boldsymbol{n}_y \cdot (u_1, u_2, u_3)^T \leq G(x)\,, \tag{2.3}$$

denoting by $\boldsymbol{n}_y = \tilde{\boldsymbol{n}}_y / \|\tilde{\boldsymbol{n}}_y\|_2$ the outer normal to γ_y at y and by

$$G(x) = \frac{\eta_y(x_1, x_2) - \eta_x(x_1, x_2)}{\|\tilde{\boldsymbol{n}}_y\|_2}$$

the initial gap between the body $\mathscr{B}$ and the obstacle, normalized with respect to $\boldsymbol{n}_y$.

Equation (2.3) is much easier to handle than the kinematical contact condition (2.1), but involves the outer normal of the *obstacle*. However to verify the condition (2.3), it is necessary to compute the outer normal of some unknown point $y \in \partial\mathcal{G}$.

Exploiting the assumption of small deformations, the surfaces can be seen to differ only within terms of at least quadratic order, see, e.g., [Eck96]. That is, since both surfaces are known to be very close initially, within our linear model we can reformulate condition (2.3) with respect to the outer normal $\boldsymbol{n}_x(x)$ of $\mathscr{B}$ at a point $x \in \gamma_x \subset \partial\mathscr{B}$ as

$$\boldsymbol{n}_x(x) \cdot (u_1, u_2, u_3)^T \leq g(x)\,. \tag{2.4}$$

Here, we have set

$$g(x) = \frac{\eta_y(x_1, x_2) - \eta_x(x_1, x_2)}{\|\tilde{\boldsymbol{n}}_x(x)\|_2} \quad \text{and} \quad \tilde{\boldsymbol{n}}_x = (-\frac{\partial \eta_x}{\partial x_1}, -\frac{\partial \eta_x}{\partial x_2}, 1)\,.$$

Equation (2.4) constitutes our final contact condition. It can be computed easily in terms of the initial gap in normal direction and can be regarded as part of the initial data. We remark, that tangential displacements are not taken into acount by condition (2.4) and that penetration might occur due to tangential displacements.

2.2 Strong Formulation

In this section, we define the Signorini problem and give it's classical as well as variational formulation. We start with the classical formulation of Signorini's problem in terms of a differential equation and show its equivalence with the variational formulation, provided the solution is smooth. At the end of this section, we give some results concerning the existence and regularity of the solution.

In the previous section, we derived the linearized contact condition (2.4), which is a condition given with respect to the displacements. Let us now consider the stresses developed at the contact boundary. It is clear, that the stresses $\sigma_n(\boldsymbol{u})$ in normal direction developed on Γ_S have to be *compressive stresses* or have to vanish, i.e., that we have

$$\sigma_n(\boldsymbol{u}) \leq 0\,. \tag{2.5}$$

Moreover, we assume frictionless contact. Thus, the body is allowed to displace freely in tangential direction and for the tangential stresses $\boldsymbol{\sigma}_T(\boldsymbol{u})$ holds

$$\boldsymbol{\sigma}_T(\boldsymbol{u}) = 0\,. \tag{2.6}$$

In other words, the *primal* contact condition (2.4) for the displacements is accomponied by the *dual* conditions (2.5) and (2.6)for the boundary stresses at the possible contact boundary.

For stating Signorini's problem, we basically add the primal condition (2.4) and the dual conditions (2.5) and (2.6) to the equations of equilibrium. To be more precise, we assume the body is clamped at Γ_D, i.e., we have

$$\boldsymbol{u} = 0\,, \qquad \text{on } \Gamma_D \tag{2.7}$$

and surface forces $\boldsymbol{p}$ are applied at Γ_F, giving

$$\sigma_{ij}(\boldsymbol{u}) \cdot \boldsymbol{n}_j = p_i\,, \qquad \text{on } \Gamma_F\,. \tag{2.8}$$

As in the previous section, $\boldsymbol{n} = (n_1, \dots, n_d)$ denotes the outer normal on $\partial\mathscr{B}$. Let us remark, that for ease of presentation here and in the following we shall say *surface forces* and *volume forces*, instead of using the terms *density of surface forces* and *density of volume forces*, respectively. Denoting the volume forces by $\boldsymbol{f}$, we furthermore assume the deformed body is in equilibrium state such that

$$-\sigma_{ij}(\boldsymbol{u})_{,j} = f_i\,, \qquad \text{in } \mathscr{B}\,. \tag{2.9}$$

Considering hyperelastic, homogeneous and isotropic materials, the Cauchy stress tensor $\boldsymbol{\sigma}$ is given by Hooke's Law

$$\sigma_{kl}(\boldsymbol{u}) = H_{klij}\varepsilon_{ij}(u), \quad 1 \leq i,j,k,l \leq d\,, \tag{2.10}$$

in terms of the linearized strain tensor $\boldsymbol{\varepsilon}$, see Chapter 1 and Hooke's Tensor $\boldsymbol{H}$ has the following symmetry properties

$$H_{ijkl} = H_{jilk} = H_{ijlk} = H_{klij}\,. \tag{2.11}$$

The remaining part Γ_S is the *possible contact boundary* or *Signorini boundary.* That is, only Γ_S or some parts of it might come into contact with the rigid foundation $\mathcal{G}$. Given that, we carefully distinguish between the possible contact boundary Γ_S and the zone of actual contact, which is assumed to be a subset of Γ_S. Bearing contact condition (2.4) in mind, we impose the following boundary conditions on the Γ_S

$$\left.\begin{array}{rcl} \sigma_T(\boldsymbol{u}) & = & 0\,, \\ \sigma_n(\boldsymbol{u})(\boldsymbol{\varphi}\cdot\boldsymbol{n} - g) & = & 0\,, \\ \boldsymbol{\varphi}\cdot\boldsymbol{n} - g & \leq & 0\,, \\ \sigma_n(\boldsymbol{u}) & \leq & 0\,, \end{array}\right\} \text{on } \Gamma_S\,. \tag{2.12}$$

As has been defined in Section 2.1, the scalar function g denotes the initial gap between $\mathscr{B}$ and the rigid foundation $\mathcal{G}$, and we write for the normal component of the Cauchy stress tensor

$$\sigma_n = \sigma_{ij} n_i n_j \tag{2.13}$$

and for the tangential component

$$(\boldsymbol{\sigma}_T)_j = \sigma_{ij} n_i - \sigma_n n_j \,. \tag{2.14}$$

When referring to *boundary stresses* or *contact stresses*, we always mean the pair $(\sigma_n, \boldsymbol{\sigma}_T)$ or σ_n, respectively. Consequently, contact stresses act normal to the surface and boundary stresses in normal *and* tangential direction.

Now we are in a state to give the following

Definition 2.1 (Signorini's problem) *The boundary value problem given by equations (2.7), (2.8), (2.10), (2.12) is called* Signorini's problem in linear elasticity *or* Signorini's problem*.*

Although boundary conditions (2.12), (2.13), (2.14) make sense with respect to their physical meaning, we still have to show that Signorini's problem is a *well posed* boundary value problem. This is implicitly shown when stating the existence of a solution to Signorini's problem. Before doing so, let us take a closer look at equations (2.12). The first equation in (2.12), that is $\boldsymbol{\sigma}_T = 0$, means *frictionless* contact, i.e., the body is free to displace in tangential direction to minimize the total energy. The second and last equation of (2.12) state that there must be vanishing contact stress in case of no contact and that only compressive normal stress is allowed, respectively. Additionally, the third equation states that no penetration in normal direction occurs.

Remark 2.2 *The only primal constraint showing up in Signorini's problem is given with respect to the displacement in* normal *direction.*

2.3 Weak formulation

As in the linear case, there is a variational formulation of Signorini's problem, corresponding to (1.14). Due to the non-penetration condition (2.4), we cannot expect the energy functional $\bar{\mathcal{J}}$ to be associated with Problem (2.1) to be Gâteaux–differentiable, which was neccessary to give the minimizer of $\mathcal{J}$ as stationary point in terms of a variational equation. Instead, the variational formulation of Signorini's problem turns out to be a *variational inequality.* problem as variational inequality, existence and uniquenes of a solution can be obtained by using results from convex analysis. Moreover, the formulation of Signorini's problem in terms of a variational inequality will be usefull for the numerical algorithm to be developed, since the nonlinearity of the problem essentially can be resolved by solving *local* variational inclusions.

Remark 2.3 *In case of contact with friciton, standard theorems from convex analysis cannot be applied, since the energy functional is not convex and the existence theory for*

Signorini's problem with Coulomb friciton turns out to be rather demanding. We discuss this topic in more detail in Chapter 6.

By $H^1(\mathscr{B})$ we denote the usual space of weakly differentiable functions with derivative in $L^2(\mathscr{B})$ and we set

$$\boldsymbol{H}^1(\mathscr{B}) := (H^1(\mathscr{B}))^d\,, \qquad \boldsymbol{L}^2(\mathscr{B}) := (L^2(\mathscr{B}))^d$$

and son on. For details concerning Sobolev spaces we refer to, e.g., [Wlo82]. Let

$$\gamma_D\colon \boldsymbol{H}^1(\mathscr{B}) \longrightarrow \boldsymbol{H}^{1/2}(\Gamma_D)$$

be the trace operator and let $\boldsymbol{H} := \{\boldsymbol{v} \in \boldsymbol{H}^1(\mathscr{B}) \,|\, \gamma_D(\boldsymbol{v}) = 0\}$. Furthermore, let $v_n = \boldsymbol{v}\cdot\boldsymbol{n}$ be well defined on Γ_S. Then, for any given positive obstacle $g\colon \Gamma_S \longrightarrow \mathbb{R}$, we can define the set $\mathcal{K}$ of *admissible displacements* with respect to contact condition (2.4) by

$$\mathcal{K} := \{\boldsymbol{v} \in \boldsymbol{H} \,|\, v_n \le g \text{ on } \Gamma_S\}\,. \tag{2.15}$$

It can be seen immediately from the definition, that $\mathcal{K}$ is a convex subset of $\boldsymbol{H}$. Multiplying (2.9) with test function $\boldsymbol{v} - \boldsymbol{u} \in \mathcal{K}$ for arbitrary $\boldsymbol{v} \in \mathcal{K}$ and integrating by parts, we get for the virtual work produced by the displacements $\boldsymbol{v} - \boldsymbol{u}$

$$\int_{\mathscr{B}} \sigma_{ij}(\boldsymbol{u})(v_i - u_i)_{,j}\, dx \;=\; \int_{\mathscr{B}} -\sigma_{ij}(\boldsymbol{u})_{,j}(v_i - u_i)\, dx + \int_{\partial\mathscr{B}} \sigma_{ij}(\boldsymbol{u})n_j(v_i - u_i)\, dx\,. \tag{2.16}$$

Defining the symmetric (cf. (2.10) and (2.11)) bilinearform $a\colon \boldsymbol{H} \times \boldsymbol{H} \longrightarrow \mathbb{R}$

$$a(\boldsymbol{u}, \boldsymbol{v}) \;=\; \int_{\mathscr{B}} \sigma_{ij}(\boldsymbol{u})\, v_{i,j}\, dx\,,$$

equation (2.16) can be rewritten as

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) \;=\; (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_{\boldsymbol{L}^2(\mathscr{B})} + (\boldsymbol{\sigma}(\boldsymbol{u})\cdot\boldsymbol{n}, \boldsymbol{v} - \boldsymbol{u})_{\boldsymbol{L}^2(\partial\mathscr{B})}\,.$$

We note, that the boundary $\partial\mathscr{B}$ is decomposed into three disjoint parts Γ_D, Γ_F, Γ_S. We find for $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}$

$$(\boldsymbol{\sigma}(\boldsymbol{u})\cdot\boldsymbol{n}, \boldsymbol{w})_{\boldsymbol{L}^2(\partial\mathscr{B})} = (\boldsymbol{\sigma}(\boldsymbol{u})\cdot\boldsymbol{n}, \boldsymbol{w})_{\boldsymbol{L}^2(\Gamma_D)} + (\boldsymbol{\sigma}(\boldsymbol{u})\cdot\boldsymbol{n}, \boldsymbol{w})_{\boldsymbol{L}^2(\Gamma_F)} + (\boldsymbol{\sigma}(\boldsymbol{u})\cdot\boldsymbol{n}, \boldsymbol{w})_{\boldsymbol{L}^2(\Gamma_S)}\,.$$

The boundary condition (2.7) yields that the first term on the right side is zero, and (2.8) gives that $(\boldsymbol{\sigma}(\boldsymbol{u})\cdot\boldsymbol{n}, \boldsymbol{w})_{\boldsymbol{L}^2(\Gamma_F)}$ can be replaced by $(\boldsymbol{p}, \boldsymbol{w})_{\boldsymbol{L}^2(\Gamma_F)}$. In a next step, we decompose the boundary stress $\sigma_{ij}(\boldsymbol{u})n_j$ on the possible contact boundary Γ_S with respect to its normal and tangential component. In case of frictionless contact, on Γ_S holds

$$\begin{aligned}
\sigma_{ij}n_j(v_i - u_i) \;&=\; \big((\boldsymbol{\sigma}_T(\boldsymbol{u}))_i + \sigma_n(\boldsymbol{u})n_i\big)(v_i - u_i)\\
&=\; 0 + \sigma_n(\boldsymbol{u})(v_n - u_n)\\
&=\; \sigma_n(\boldsymbol{u})(v_n - u_n + g - g)\\
&=\; \sigma_n(\boldsymbol{u})(v_n - g)\\
&\ge\; 0\,,
\end{aligned}$$

where we have used (2.12). Summing up, we have derived on Γ_S the variational inequality

$$\boldsymbol{u} \in \mathcal{K}: \quad a(\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) \quad \geq \quad f(\boldsymbol{v}-\boldsymbol{u}), \quad \boldsymbol{v} \in \mathcal{K}, \tag{2.17}$$

where we have set

$$f(\boldsymbol{v}) \;=\; (\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u})_{\boldsymbol{L}^2(\mathscr{B})} + (\boldsymbol{p}, \boldsymbol{v}-\boldsymbol{u})_{\boldsymbol{L}^2(\Gamma_F)}\,.$$

This kind of inequality is referred to as *variational inequality of the first kind.* For sufficiently smooth $\boldsymbol{u}$, inequality (2.17) and the strong formulation of Signorini's problem can be shown to be equivalent. This is done by integration by parts in (2.17) for suitable test functions $\boldsymbol{v}$, see, e.g., [HHNL88, Chapter 2.1.3].

Variational inequality (2.17) may also be formulated on the whole space $\boldsymbol{H}$. To this end, we introduce the *characteristic functional* $\varphi\colon \boldsymbol{H} \longrightarrow \mathbb{R}$ by

$$\varphi_{\mathcal{K}}(\boldsymbol{v}) = \begin{cases} \;\;0 & , \quad \boldsymbol{v} \in \mathcal{K}, \\ +\infty & , \quad \boldsymbol{v} \notin \mathcal{K}, \end{cases}$$

and we can rewrite (2.17) as

$$\boldsymbol{u} \in \boldsymbol{H}: \quad a(\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}) + \varphi_{\mathcal{K}}(\boldsymbol{v}) - \varphi_{\mathcal{K}}(\boldsymbol{u}) \quad \geq \quad f(\boldsymbol{v}-\boldsymbol{u}), \quad \boldsymbol{v} \in \boldsymbol{H}. \tag{2.18}$$

Using the functional $\varphi_{\mathcal{K}}$, we define the functional $\bar{\mathcal{J}}\colon H \longrightarrow \mathbb{R} \cup \{+\infty\}$ of total energy for any $\boldsymbol{u} \in \boldsymbol{H}$ by

$$\bar{\mathcal{J}}(\boldsymbol{u}) \;=\; \frac{1}{2} a(\boldsymbol{u}, \boldsymbol{u}) - f(\boldsymbol{u}) + \varphi_{\mathcal{K}}(\boldsymbol{u})\,.$$

Any displacement being not admissible with respect to the non penetration condition is penalized with infinite energy. The following theorem is a usefull characterization of the convexity of $\bar{\mathcal{J}}$

Theorem 2.4 (Theorem 4.7–9, [Cia88]) *The functional* $\bar{\mathcal{J}}\colon \boldsymbol{H} \longrightarrow \mathbb{R}\cup\{+\infty\}$ *is convex if and only if the set* $\mathcal{K}$ *is convex and the functional* $\mathcal{J}\colon \boldsymbol{H} \longrightarrow \mathbb{R}$ *is convex.*

Since the functional $\mathcal{J}$ of total energy is convex and the set of admissible displacements $\mathcal{K}$ is convex, we have that $\bar{\mathcal{J}}$ is convex.

Unfortunately, the energy functional $\bar{\mathcal{J}}$ is not Gâteaux–differentiable with respect to the displacement $\boldsymbol{u}$ at the contact boundary and we cannot apply Theorem 1.8, which shows the minimizer of a convex functional to be the solution of a variational equation. We overcome this difficulty by introducing the *subdifferential* of a function, which is a set-valued function. Then, variational equation (1.15), characterizing the solution as stationary point of the energy functional, can be rewritten in terms of a *variational inclusion.*

To fix ideas, let us remark, that for any given real Hilbert space H a Gâteaux–differentiable function $G\colon H \longrightarrow \mathbb{R}$ is convex if and only if for $u \in H$

$$G(u) - G(v) \;\leq\; \langle \frac{\partial G}{\partial u}(u), u - v \rangle\,, \qquad v \in H\,. \tag{2.19}$$

For a proof, see, e.g., [IS93, Cla83]. Taking $v = u + w$ in equation (2.19), we get

$$G(u+w) \geq G(u) + \langle \frac{\partial G}{\partial u}(u), w \rangle ,$$

that is, in linearizing we keep staying below (sub) the functional G. This observation forms the motivation for the following definition of the *subdifferential* of a function. Here, the set of all subsets of H is denoted by 2^H.

Definition 2.5 (Subdifferential) *Let H be a Hilbert space and let $G\colon H \longrightarrow (-\infty, +\infty]$. Then, the* subdifferential *of G at $u \in H$ is defined to be the set $\partial G(u) \subset H$ of all elements $\xi \in H$, such that*

$$(G(v) - G(u), v - u)_H \geq (\xi, v - u)_H \qquad v \in H,\, \xi \in \partial G(u)\,, \tag{2.20}$$

where $(\cdot,\cdot)_H$ denotes the inner product of the Hilbert space H. The multivalued mapping $\partial G\colon H \longrightarrow 2^H$ is called the subdifferential mapping *and the elements $\xi \in \partial G(u)$ are called* subgradients.

In the preceeding definition, we have made use of the fact, that a Hilbert space may be identified with its dual space. In particular, every subgradient ξ defines a linear mapping $(\xi, \cdot)_H$ on H, corresponding to $\frac{\partial G}{\partial u}(u)$ being a linear mapping on H for every fixed u. In fact, if G is convex and Gâteaux differentiable, then the subdifferential is a univalued operator on H and it holds $\{\frac{\partial G}{\partial u}(u)\} = \partial G(u)$. For a proof of this results and for a more detailed discussion of the subdifferential and its properties, we refer to, e.g., [IS93, Kor97a, Appendix, Chapter 2] and for generalizations of this concept to [Cla83].

Exploiting the definition of the subdifferential, we can reformulate the variational inequality (2.18) as a variational inclusion

$$a(\boldsymbol{u}, \cdot) - f(\cdot) \in \partial\varphi(\boldsymbol{u})\,. \tag{2.21}$$

Remark 2.6 *In case of Signorini's problem, the subdifferential is the boundary stress developed at the actual zone of contact. In (2.13), we have been defining the normal stresses $\sigma_n(\boldsymbol{u})$ in a strong sense as normal component of the Cauchy stress tensor. This definition can be generalized on the basis of Green's Theorem. Let $\boldsymbol{u}$ be a solution of Signorini's problem. Then, $\sigma_n(\boldsymbol{u})$ can be seen to satisfy for $\boldsymbol{v} \in \boldsymbol{H}$, $\boldsymbol{v}_T = 0$ on Γ_s,*

$$(\sigma_n(\boldsymbol{u}), v_n)_{\boldsymbol{L}^2(\Gamma_S)} = a(\boldsymbol{u}, \boldsymbol{v}) - f(\boldsymbol{v})\,. \tag{2.22}$$

Consequently, $\sigma_n(\boldsymbol{u}) \in H^{-1/2}(\Gamma_S)$, i.e., the boundary stresses are elements of the dual of the trace space $H^{1/2}(\Gamma_S)$, see, e.g., [KO88, Chapter 5]. For $\sigma_n(\boldsymbol{u})$ smooth enough (2.13) and (2.22) coincide. The dual interpretation of boundary stresses can also be taken advantage of in the context of elastic contact problems, see [KW00] and Chapter 7.

The following theorem, which can be found in, e.g. [Glo84, Lemma 4.1] or [DL72], states existence and uniqueness of a solution to Signorini's problem. Moreover, it also gives an equivalent formulation of the problem as as *convex minimization problem*, i.e., the variational inequality (2.17) can equivalently be rewritten as

$$\bar{\mathcal{J}}(\boldsymbol{u}) \leq \bar{\mathcal{J}}(\boldsymbol{v}) \qquad \boldsymbol{v} \in \boldsymbol{H}. \tag{2.23}$$

This formulation is going to be the starting point for the construction of our numerical method in Chapter 3, which is based on a successive minimization of the nondifferentiable functional $\bar{\mathcal{J}}$.

Before we can state the desired theorem, let us give the following

Definition 2.7 *Let H be a real Hilbert space. A functional $G\colon H \longrightarrow \mathbb{R}\cup\{+\infty\}$ is called* lower semicontinuous (l.s.c.)*, if*

$$\liminf G(v^k) \geq G(v)$$

for any sequence $v^k \to v$ and it is called proper*, if*

$$G \not\equiv +\infty \quad \textit{and} \quad G(v) > -\infty\,, \qquad v \in H\,. \tag{2.24}$$

For a non-empty set $\mathcal{K}$ of admissible displacements the characteristic functional $\varphi_{\mathcal{K}}$ is l.s.c. and proper. In addition, the ellipticity of the bilinearform $a(\cdot,\cdot)$ in case of $\mathrm{meas}(\Gamma_D) > 0$ is a consequence of *Korn's inequality*, see, e.g. [KO88]. Now, existence und uniqueness of a solution to Signorini's problem is guaranteed by the following theorem

Theorem 2.8 *Let $a\colon \boldsymbol{H} \times \boldsymbol{H} \longrightarrow \mathbb{R}$ be a symmetric continuous bilinear and $\boldsymbol{H}$-elliptic form, i.e.,*

$$a(\boldsymbol{v},\boldsymbol{v}) \geq \alpha\|\boldsymbol{v}\|_H\,. \tag{2.25}$$

Let $f \in V'$ and $\varphi\colon H \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a convex, l.s.c. proper functional. Let $\bar{\mathcal{J}}(\boldsymbol{v}) = \frac{1}{2}a(\boldsymbol{v},\boldsymbol{v}) + \varphi(\boldsymbol{v}) - f(\boldsymbol{v})$. Then, the minimization problem (2.23) has a unique solution which is also characterized by (2.17).

Although the above theorem provides us with a solution to Signorini's problem, it does not give any estimates of the boundary regularity of the solution $\boldsymbol{u}$ of Signorini's problem. Results concerning the regularity of the solution in the *interior* of the domain are well known, see, e.g., [KO88]. Here, for sufficiently smooth data and smooth domain, local $\boldsymbol{H}^2$–regularity of the solution on compact subsets of $\mathscr{B}$ is shown. Results estimating the regularity of the solution at the boundary may be found in [Sch88, Kin82b, Kin82a, NJH80, Eck96, HH80]. Assuming sufficiently smooth $\mathscr{B}$, in [Sch88] the solution is shown to belong to the space $C^{1+\alpha}$ of Hölder–continuous functions for some unknown $\alpha \in (0,1)$. For particular domains, estimates of the weak regularity of the solution to Signorini's problem with small friction in terms of the given data can be found in [NJH80].

2.4 Discretization and Error Estimates

In this section, we introduce the discretization used for the numerical approximation of Signorini's problem and give a priori estimates of the discretization error.

We consider a Galerkin approximation of the Hilbert space $\boldsymbol{H}$, that is, we consider a sequence $\left(\boldsymbol{S}^{(j)}\right)_{j\in\mathbb{N}_0}$ of finite dimensional subspaces $\boldsymbol{S}^{(j)} \subset \boldsymbol{H}$ with

$$\overline{\bigcup_{j=0}^{\infty} \boldsymbol{S}^{(j)}}^{\|\cdot\|_{\boldsymbol{H}}} = \boldsymbol{H} .$$

Here, the closure is taken with respect to the Hilbert–norm $\|\cdot\|_{\boldsymbol{H}}$.

The spaces $\boldsymbol{S}^{(j)}$, $j = 0, \ldots,$ are spanned by linearly independent functions $\boldsymbol{\lambda}_p^{(j)} \in H$, where $p \in \mathcal{N}^{(j)}$ and $\mathcal{N}^{(j)}$ is a suitable finite index set with $n_j = \#\mathcal{N}^{(j)} < \infty$ elements, i.e., we have

$$\boldsymbol{S}^{(j)} = \operatorname{span}\{\boldsymbol{\lambda}_p^{(j)} \,|\, p \in \mathcal{N}^{(j)}\} .$$

Moreover, we assume the basis functions $\boldsymbol{\lambda}_p^{(j)}$ to be bounded and *locally* supported.

As an additional property of the spaces $\boldsymbol{S}^{(j)}$, we require an *approximation property*, which is also known as Jackson inequality, see, e.g., [BS68, BL76]. Let

$$E_j(\boldsymbol{u}) = \inf_{\boldsymbol{v}\in\boldsymbol{S}^{(j)}} \|\boldsymbol{u} - \boldsymbol{v}\|_{\boldsymbol{H}} = \|\boldsymbol{u}_j - \boldsymbol{u}\|_{\boldsymbol{H}} ,$$

then for some *smoothness parameter* $s \geq 0$, the following inequality is assumed to hold with a generic constant $c > 0$ independent of j

$$E_j(\boldsymbol{u}) \leq c\, n_j^{-s/d} \|\boldsymbol{u}\|_{\boldsymbol{H}} . \tag{2.26}$$

To give an example for suitable spaces $\boldsymbol{S}^{(j)}$, let $\mathcal{T}_J$ be a given partition of $\mathscr{B}$ into triangles (tetrahedra) t with minimal diameter $h_J = \mathcal{O}(2^{-l})$ and let $\mathcal{N}^{(J)}$ denote the set of vertices contained in $\mathscr{B} \cup \Gamma_F \cup \Gamma_S$. Let $\lambda_p^{(j)}$ be the continuous piecewise linear nodal basis functions with $\lambda_p^{(j)}(q) = \delta_{pq}$ for any $p, q \in \mathcal{N}^{(j)}$and let $\{\boldsymbol{E}^i\}$ denote the canonical basis of the $\mathbb{R}^d$. Then, setting $\boldsymbol{\lambda}_p^{(j)} = (\lambda_p^{(j)} \boldsymbol{E}^i)^T$, we obtain low order finite elements

$$\boldsymbol{S}^{(j)} = \{\boldsymbol{v} \,|\, \boldsymbol{v} = (v_1, \ldots, v_d) \in C(\mathscr{B})^d \cap \boldsymbol{H} v_i|_t \text{ is linear, } i = 1, \ldots, d,\ t \in \mathcal{T}^j\} . \tag{2.27}$$

Here, the index set $\mathcal{N}^{(j)}$ can be identified with the vertices of a triangulation of the domain $\mathscr{B}$. Consequently, any degree of freedom is associated with a geometric object and for some fixed J, we can call $\boldsymbol{S}^{(J)}$ the *fine grid* space and the spaces $\boldsymbol{S}^{(j)}$, $j < J$ the *coarse grid* spaces. The approximation property (2.26) is well known for linear finite elements and can be found for example in [Hac85, Bra93, Xu89]. If not stated otherwise, in what follows we always assume the spaces $\boldsymbol{S}^{(j)}$ to be the spaces of linear finite elements given above.

Remark 2.9 *In general, the basis functions* $\boldsymbol{\lambda}_p^{(j)}$ *do not have to be associated with a geometric object. For example, using* algebraic multigrid *methods, the crucial task is not building up suitable coarse grid* meshes *but suitable* spaces*, that is, spaces, for which approximation property (2.26) holds. From that point of view, standard geometric multigrid methods might be viewed as a reliable method to construct a sequence* $\{\boldsymbol{S}^{(j)}\}_{j\in\mathbb{N}_0}$ *of nested spaces with property (2.26). We emphasize, that our monotone multigrid method does not require any geometric information besides the obstacle and the initial configuration. For an implementation of the method within the framework of an algebraic multigrid method, we refer the reader to Section 4.6.*

In a next step, we discretize the convex set $\mathcal{K}$ of admissible displacements. We replace the set $\mathcal{K} \subset \boldsymbol{H}$ by its discrete analogue $\mathcal{K}_j \subset \boldsymbol{S}^{(j)}$,

$$\mathcal{K}_j = \left\{ \boldsymbol{v} \mid \boldsymbol{v} \in \boldsymbol{S}^{(j)}, \ \boldsymbol{v}(p) \cdot \boldsymbol{n}_j(p) \le g_j(p), \quad p \in \mathcal{N}^{(j)} \cap \Gamma_S \right\},$$

based on suitable approximations $\boldsymbol{n}_j$ and g_j of $\boldsymbol{n}$ and g, respectively. Note that in general $\mathcal{K}_j \not\subset \mathcal{K}$, since the constraints are given *pointwise*. Based on this approximation of $\mathcal{K}$, we obtain the discrete minimization problem

$$\boldsymbol{u}_j \in \mathcal{K}_j : \quad \mathcal{J}(\boldsymbol{u}_j) \le \mathcal{J}(\boldsymbol{v}), \qquad \boldsymbol{v} \in \mathcal{K}_j . \tag{2.28}$$

The existence and uniqueness of the solution follows from Theorem 2.8.

For any particular choice of spaces $\boldsymbol{S}^{(j)}$, the approximation property (2.26) has to be verified. Let us consider the case of linear finite elements, i.e., $\boldsymbol{S}^{(j)}$ as in (2.27). Here, special care has to be taken in case of non-polygonal domains $\mathscr{B}$, since then the partition $\mathcal{T}$ associated with $\boldsymbol{S}^{(j)}$ is only an approximation of $\mathscr{B}$. Following the lines of [KO88], we define the extension $\tilde{\boldsymbol{v}}$ of the finite element function $\boldsymbol{v} \in \boldsymbol{S}^{(j)}$ by constant extension in direction of the outer normal of the discrete boundary Γ_j. The discrete boundary Γ_j consists of the outer edges or faces of the partition $\mathcal{T}^j$.

For shape regular partitions $\mathcal{T}^j$ and sufficiently smooth solutions $\boldsymbol{u} \in \mathcal{K}$ of (2.17), the approximation property (2.26) is well known to hold for the spaces $\boldsymbol{S}^{(j)}$. Written in terms of the meshsize parameter h_j of $\boldsymbol{S}^{(j)}$, it takes the form

$$\|\boldsymbol{u}_j - \boldsymbol{u}\|_{\boldsymbol{H}} \le c\, h_j \, \|\boldsymbol{u}\|_{\boldsymbol{H}} ,$$

for the Hilbert space norm $\|\cdot\|_{\boldsymbol{H}}$. Using this approximation property, for sufficiently smooth domains $\mathscr{B} \in \mathcal{C}^{2,1}$ and obstacles $g \in H^{3/2}(\Gamma_S)$, the sequence $\left(\tilde{\boldsymbol{u}}_j\right)_{j\in\mathbb{N}_0}$ of extended solutions $\boldsymbol{u}_j$ of (2.28) can be shown to converge to the solution $\boldsymbol{u} \in \mathcal{K}$ of the variational inequality (2.17). That is, there exists a constant $c > 0$ independent of j, such that [KO88, Theorem 6.4]

$$\|\tilde{\boldsymbol{u}}_j - \boldsymbol{u}\|_{\boldsymbol{H}} \le c h_j^s \, \|\boldsymbol{u}\|_{\boldsymbol{H}^{1+s}} , \qquad \boldsymbol{u} \in \boldsymbol{H}^{1+s} \tag{2.29}$$

where $\boldsymbol{H}^{1+s}$ is a space containing suitable smooth functions, e.g., a Sobolov space and $0 \le s \le 1$.