1 Concepts of Linear Elasticity

In linear elasticity, one is concerned with the deformation of an elastic body $\mathscr{B} \subset \mathbb{R}^d$, $d = 2, 3$, subjected to applied forces. Starting with the early works of Euler and Cauchy in the 18. and 19. century, [Eul57, Eul71, Cau23, Cau27], the theory of elasticity has been extensively developed. Here, we only give a short introduction to the basic concepts of linear elasticity and refer the reader to the monographs [dV79, MH94, Cia88, Gur81] for details . In this section, we follow the lines of [Cia88] and [dV79].

As is standard in the literature on elasticity here and throughout this work, we use boldface letters for vectors and tensors and normal typeface letters for scalar quantities. We follow the summation convention, i.e., summation is implicitly taken over indices occurring twice, and we assume Latin indices $i, j, k, l, \dots$ to be in the range from 1 to d, where, $d = 2, 3$ is the spatial dimension. The partial derivative is denoted by $w_{,j} = \frac{\partial w}{\partial x_j}$. As we are concerned with tensors, we note that we do not distinguish between covariant and contravariant quantities by using, e.g., upper and lower indices.

1.1 Kinematics and Strain

Three main ingredients are used within the theory of linear elasticity: kinematics, equilibrium conditions and constitutive laws.

Let us start with some kinematical considerations. An elastic body in its undeformed configuration might occupy the closure of the domain $\mathscr{B} \subset \mathbb{R}^d$, $d = 1, 2$, and we identify every material particle of the elastic body with a point $x = (x_1, \dots, x_d)^T \in \mathbb{R}^d$. That is, we do not distinguish between the body and its particles and the set $\bar{\mathscr{B}}$ and the points $x \in \bar{\mathscr{B}} \subset \mathbb{R}^d$, respectively. Subjected to volume and traction forces, the body will undergo some *deformation* $\boldsymbol{\varphi}$,

$$\boldsymbol{\varphi} \colon \bar{\mathscr{B}} \longrightarrow \mathbb{R}^d , \qquad \bar{\mathscr{B}} \ni x \mapsto \boldsymbol{\varphi}(x) ,$$

Denoting the final position of a particle x by $x^{\boldsymbol{\varphi}} := \boldsymbol{\varphi}(x)$, we can define the *displacement field* $\boldsymbol{u} = u_i \boldsymbol{e}_i = (u_i)_{1 \le i \le d}$ by

$$\boldsymbol{\varphi} = \boldsymbol{u} + \mathbf{id} .$$

Here, $\{\boldsymbol{e}_i\}_{1 \le i \le d}$ is the canonical basis of the linear space $\mathbb{R}^d$. Let us furthermore introduce the *deformation gradient*

$$\boldsymbol{\nabla}\boldsymbol{\varphi} = \left(\boldsymbol{\varphi}_{i,j}\right)_{1 \le i,j \le d} .$$

By physical considerations, the deformation $\boldsymbol{\varphi}$ is assumed to be injective and orientation preserving and for all $x \in \bar{\mathscr{B}}$ we have $\det \boldsymbol{\nabla}\boldsymbol{\varphi}(x) > 0$.

For the remainder of this section, we only deal with the case of full linear elasticity and set $d = 3$. Let us consider two particles $x \in \mathscr{B}$ and $x + dx \in \mathscr{B}$ with infinitesimal distance dx, changing their position to $\boldsymbol{\varphi}(x) \in \bar{\mathscr{B}}$ and $\boldsymbol{\varphi}(x) + dx^{\boldsymbol{\varphi}} \in \bar{\mathscr{B}}$, respectively. Enforcing the summation convention, the square of the final distance $dx^{\boldsymbol{\varphi}} = dx + d\boldsymbol{u}$ is given by

$$
\begin{aligned}
(dx^{\boldsymbol{\varphi}})^2 &= dx_i^{\boldsymbol{\varphi}} dx_i^{\boldsymbol{\varphi}} \\
&= (\delta_{ki} + u_{i,k})(\delta_{li} + u_{i,l})\, dx_k dx_l \\
&= (\delta_{ki}\delta_{li} + \delta_{ki}u_{i,l} + \delta_{li}u_{i,k} + u_{i,k}u_{i,l})\, dx_k dx_l \\
&= (u_{k,l} + u_{l,k} + u_{i,k}u_{i,l})\, dx_k dx_l + dx^2 \,.
\end{aligned}
\tag{1.1}
$$

Here, δ_{ki} denotes the Kronecker-symbol, i.e., $\delta_{ki} = 1$, if $k = i$, and 0 otherwise. These considerations give rise to the definition of the *Green St.–Venant strain tensor* $\boldsymbol{E}$,

$$E_{ij} = \frac{1}{2}(u_{j,k} + u_{i,k} + u_{i,k}u_{j,k}) \,.$$

To give an interpretation of the tensor $\boldsymbol{E}$, let us call a deformation *rigid*, if it can be obtained as a combination of a rotation around the origin and a translation. Since the tensor $\boldsymbol{E}$ can be seen to vanish iff the deformation is rigid (cf. see [Cia88, Theorem 1.8-1] or [Gur81, Theorem, p.56]), we can interpret $\boldsymbol{E}$ as measure of the *true deformation* of the body, that is, a deformation being orthogonal to the space of rigid deformations.

Correspondingly, the *right Cauchy strain tensor* $\boldsymbol{C}$ defined by

$$C_{ij} = \varphi_{i,k}\varphi_{j,k} \,.$$

is the identity, if and only if the deformation is rigid and we have

$$\boldsymbol{C} = \mathbf{id} + 2\boldsymbol{E} = \boldsymbol{\nabla\varphi}^T\boldsymbol{\nabla\varphi} \,.$$

Introducing the 1-form $dl = (dx_1, dx_2, dx_3)$, i.e. the *length element*, the change of length can be symbolically written as

$$(dx^{\boldsymbol{\varphi}})^2 = dx^T\boldsymbol{C}dx \,,$$

see, e.g., [Cia88, Sect. 1.8]. Thus, the Cauchy strain tensor $\boldsymbol{C}$ is describing the change of length under the deformation $\boldsymbol{\varphi}$.

Remark 1.1 *The change in the area element is governend by the first Piola–Kirchhoff transformation, which will be described in the following.*

Assuming the deformation $\boldsymbol{\varphi}$ to be sufficiently small, we can neglect the quadratic terms of the derivatives of $\boldsymbol{u}$ in (1.1). This linearization is referred to as *geometric linearization.* Now, retaining only linear terms, we can approximate in (1.1) the change of length by

$$(dx^{\boldsymbol{\varphi}})^2 - dx^2 = 2\,\varepsilon_{ij}\, dx_i dx_j \,. \tag{1.2}$$

Here, the *strain tensor* $\boldsymbol{\varepsilon}$ is defined by

$$\varepsilon_{ij} \;=\; \frac{1}{2}(u_{j,i} + u_{i,j}) \,.$$

We remark, that by definition the strain tensor $\boldsymbol{\varepsilon}$ is symmetric in i and j, i.e.,

$$\varepsilon_{ij} = \varepsilon_{ji} \,, \qquad 1 \le i, j \le 3 \,.$$

Consequently, six degrees of freedom are necessary to describe the change of relative length. By means of these degrees of freedom, the elastic deformation can be described completely up to terms of higher order. The three missing degrees of freedom needed to characterize the change of displacements are due to local rotations, which do not cause a change of length. They can be described using the antisymmetric *rotation tensor* $\boldsymbol{\omega}$,

$$\omega_{ij} = \frac{1}{2}(u_{j,i} - u_{i,j}),$$

leading to

$$u_{i,j} = \varepsilon_{ij} + \omega_{ij}.$$

For details, we refer to, e.g.,[dV79, MH94, Cia88, Ant95].

1.2 Stress and the Equilibrium Conditions

Up to now, we have been concerned with only kinematical considerations. In this section, we introduce the concept of stress and give the well-known Theorem of Cauchy, by means of which the stresses are connected with the so called stress tensor. Stress itself is introduced axiomatically as smooth a vectorfield. Since the equilibrium conditions are formulated with respect to the deformed configuration $\mathscr{B}^{\varphi}$, we additionally introduce the *Piola–Kirchhoff* transformation, by means of which the equilibrium conditions are transformed to the reference configuration. This is especially usefull for the numerical simulation of elastic behaviour, since the final configuration $\mathscr{B}^{\varphi}$ is in general unknown.

To distinguish the quantities given with respect to the deformed configuration from the ones given with respect to the undeformed configuration, all quantities associated with the deformed configuration are attached with the superscript $^{\varphi}$.

In continuum mechanics, one assumes there are only two kinds of forces the body might be subjected to, *volume forces* and *surface forces*. The volume forces are assumed to be proportional to mass and act at a distance. They are identified with a vector field $\boldsymbol{f}^{\varphi} : \mathscr{B}^{\varphi} \longrightarrow \mathbb{R}$, which gives the *density of volume force per unit volume*. The surface forces are assumed to have a short radius of action and are thus defined by a vector field $\boldsymbol{p}^{\varphi} : \Gamma \longrightarrow \mathbb{R}^3$, acting only on a subset Γ_N^{φ} of the body's boundary $\Gamma^{\varphi} := \partial\mathscr{B}^{\varphi}$. The vector field $\boldsymbol{p}^{\varphi}$, or the *surface traction vector*, gives the *density of surface force per unit area.*

At this point the question arises, how to describe the forces caused by the deformation $\boldsymbol{\varphi}$, which are acting in the interior of the body. This is answered by the *stress principle of Euler and Cauchy*, cf. [Cia88, Axiom 2.2-1]. It *states* the existence of a vector field, the so called *stress field* $\boldsymbol{t}^{\varphi}$, which is given by

$$\boldsymbol{t}^{\varphi} : \ \bar{\mathscr{B}}^{\varphi} \times S^2 \ni (x^{\varphi}, \boldsymbol{n}) \mapsto \mathbb{R}^3, \qquad S^2 = \{\boldsymbol{v} \in \mathbb{R}^3 : \ \|v\|_2 = 1\}.$$

Moreover, it states the balance of momentum

$$\int\limits_{A^{\varphi}} \boldsymbol{f}^{\varphi}(x)\, dx^{\varphi} + \int\limits_{\partial A^{\varphi}} \boldsymbol{t}^{\varphi}(x^{\varphi}, \boldsymbol{n}^{\varphi})\, dx^{\varphi} = \boldsymbol{0}$$

and the balance of angular momentum

$$\int_{A^{\varphi}} \boldsymbol{x}^{\varphi} \times \boldsymbol{f}^{\varphi}(x)\, dx^{\varphi} + \int_{\partial A^{\varphi}} \boldsymbol{x}^{\varphi} \times \boldsymbol{t}^{\varphi}(x^{\varphi}, \boldsymbol{n}^{\varphi})\, dx^{\varphi} = \boldsymbol{0}$$

for any subdomain A^{φ} of $\bar{\mathscr{B}}^{\varphi}$. Let us take a closer look at the stress principle. Defining the stress vector $\boldsymbol{\tau}$ by

$$\tau_i = t^{\varphi}_{ij}(x^{\varphi}, n_j)\,,$$

it can be regarded as a vector representing interior intermolecular forces acting on an infinitesimal surface da^{φ} containing the point x^{φ} and having outer normal $\boldsymbol{n}$, see Figure 1.1. The stress principle now asserts the existence of the surface traction vector $\boldsymbol{\tau}$ and

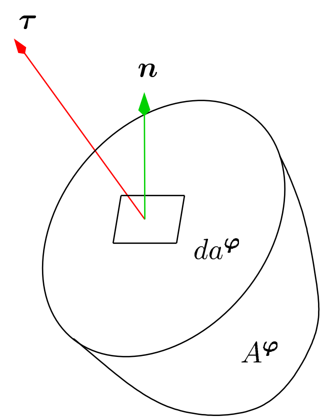

Figure 1.1: Surface traction vector $\boldsymbol{\tau}$

it states, that the surface traction $\boldsymbol{\tau}$ depends on the outer normal of the surface da^{φ} only. Moreover, the stress principle asserts static equilibrium of the body in the deformed configuration. Cauchy's theorem now asserts that under the assumptions given above, the surface traction vector $\boldsymbol{\tau}$ depends *linearly* on $\boldsymbol{n}$. In addition, the balance of angular momentum implies the symmetry of the Cauchy stress tensor $\boldsymbol{\sigma}$. We take the formulation of Cauchy's theorem from [Cia88, p. 62], see also [Gur81, dV79].

Theorem 1.2 (Cauchy's Theorem)
Assume that the applied body force density $\boldsymbol{f}^{\varphi} : \bar{\mathscr{B}}^{\varphi} \longrightarrow \mathbb{R}^3$ is continuous and that the Cauchy stress vector field

$$\boldsymbol{t}^{\varphi} : \bar{\mathscr{B}}^{\varphi} \times S^2 \ni (x^{\varphi}, \boldsymbol{n}) \mapsto \boldsymbol{t}^{\varphi}(x^{\varphi}, \boldsymbol{n}) \in \mathbb{R}^3$$

is continuously differentiable with respect to the variable $x^{\varphi} \in \bar{\mathscr{B}}^{\varphi}$ for each $\boldsymbol{n} \in S^2$ and continuous with respect to the variable $\boldsymbol{n} \in S^2$ for each $x^{\varphi} \in \bar{\mathscr{B}}^{\varphi}$. Then the axioms of *force and momentum balance imply that there exists a continuously differentiable tensor field*

$$\boldsymbol{T}^{\varphi} : \bar{\mathscr{B}}^{\varphi} \ni x^{\varphi} \mapsto \boldsymbol{T}^{\varphi}(x^{\varphi}) \in \mathbb{M}^3 ,$$

such that the Cauchy stress vector satisfies

$$\boldsymbol{t}^{\varphi}(\boldsymbol{x}^{\varphi}, \boldsymbol{n}) = \boldsymbol{T}^{\varphi}(x^{\varphi})\boldsymbol{n} , \qquad x^{\varphi} \in \bar{\mathscr{B}} , \quad \boldsymbol{n} \in S^2 ,$$

and such that

$$\begin{array}{rcll} -\mathbf{div}\, \boldsymbol{T}^{\varphi}(x^{\varphi}) & = & \boldsymbol{f}^{\varphi} & , x^{\varphi} \in \bar{\mathscr{B}} , \\ \boldsymbol{T}^{\varphi}(x^{\varphi}) & = & \boldsymbol{T}^{\varphi}(x^{\varphi})^T & , x^{\varphi} \in \bar{\mathscr{B}} , \\ \boldsymbol{T}^{\varphi}(x^{\varphi})\boldsymbol{n}^{\varphi} & = & \boldsymbol{p}^{\varphi}(x^{\varphi}) & , x^{\varphi} \in \Gamma_N^{\varphi} . \end{array} \tag{1.3}$$

We remark, that the symmetry of the Cauchy stress tensor is a consequence of the axiom of angular momentum and that equations (1.3) are called the *equations of equilibrium* in the deformed configuration.

For a geometrical interpretation of the Cauchy stress tensor see Figure 1.2. Stress itself is the intensity of force per unit area and therefore has the character of a pressure. Taking a closer look at equations (1.3), we see that they constitute a boundary value

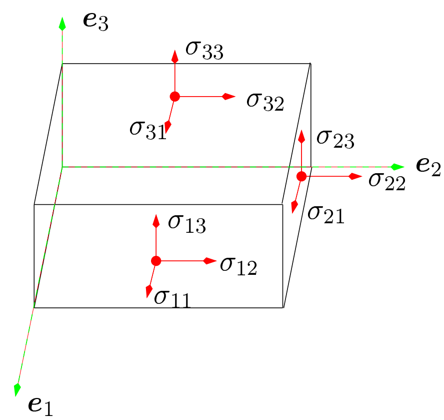

Figure 1.2: Geometric interpretation of the Cauchy stress Tensor

problem in divergence form. Naturally, the question arises, wether this boundary value problem can be rewritten as a variational equation. This is possible and in linear elasticity the resulting variational equations are referred to as *principal of virtual work*.

Theorem 1.3 (Virtual work) *The equilibrium equations (1.3) can be equivalently written as a variational problem:*

$$\int_{\mathscr{B}^{\varphi}} \boldsymbol{T}^{\varphi} : \boldsymbol{\nabla}^{\varphi}\boldsymbol{\vartheta}^{\varphi}\, dx^{\varphi} = \int_{\mathscr{B}^{\varphi}} \boldsymbol{f}^{\varphi} \cdot \boldsymbol{\vartheta}^{\varphi}\, dx^{\varphi} + \int_{\Gamma_N^{\varphi}} \boldsymbol{p}^{\varphi} \cdot \boldsymbol{\vartheta}^{\varphi}\, da^{\varphi} , \tag{1.4}$$

where $\boldsymbol{T}^\varphi$ *and* $\boldsymbol{\vartheta}^\varphi$ *are elements of appropriate function spaces. Moreover,* $\boldsymbol{T}^\varphi$ *and* $\boldsymbol{\vartheta}^\varphi$ *have to satisfy suitable boundary conditions.*

For a proof, we refer to [Cia88, Theorem2.4-1].

Within this setting, the test functions $\boldsymbol{\vartheta}$ are often called *virtual displacements*, since they are in essence mathematical objects and not to be viewed as real displacements the body might undergo.

As a matter of fact, Cauchy's theorem is given with respect to the deformed configuration, i.e., with respect to the Euler variable x^φ. We now introduce the *Piola–Kirchhoff* transformation, by means of which the boundary value problem (1.3) is transformed to the reference configuration, i.e., the resulting boundary value problem is given with respect to the Lagrangian variable x. To get the idea of the Piola–Kirchhoff transformation, let us consider the transformed volume element

$$dx^{\boldsymbol{\varphi}} = |\boldsymbol{\nabla}\boldsymbol{\varphi}|\, dx\,.$$

Since $\boldsymbol{f}$ gives the density of volume force per unit volume dx, we get

$$\boldsymbol{f}(x) = |(\boldsymbol{\nabla}\boldsymbol{\varphi})|\, \boldsymbol{f}^{\boldsymbol{\varphi}}(x^{\boldsymbol{\varphi}})\,,$$

if the volume forces $\boldsymbol{f}$ are *dead loads*, i.e., do not depend on the particular deformation $\boldsymbol{\varphi}$. Similarly, the stress tensor $\boldsymbol{T}^{\boldsymbol{\varphi}}$ can be transformed to the reference configuration. We define the *first Piola–Kirchhoff stress* $\boldsymbol{T}$ by

$$\boldsymbol{T}(x) \;=\; |\boldsymbol{\nabla}\boldsymbol{\varphi}(x)|\, \boldsymbol{T}^{\boldsymbol{\varphi}}\, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)^{-T}$$

and the symmetrized *second Piola–Kirchhoff stress* $\boldsymbol{\Sigma}$ by

$$\boldsymbol{\Sigma}(x) \;=\; |\boldsymbol{\nabla}\boldsymbol{\varphi}(x)|\, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)^{-1}\, \boldsymbol{T}^{\boldsymbol{\varphi}}(x^{\boldsymbol{\varphi}})\, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)^{-T}\,. \tag{1.5}$$

1.3 Constitutive Equations

In the previous sections, we have been concerned with the definition of the stress and the equilibrium conditions. Since the equilibrium conditions are designed to be valid regardless of the macroscopic continuum under consideration, they cannot be employed to state any mechanical properties of the material itself. Counting unknown functions, the equilibrium conditions provide us with three equations, whereas, taking the symmetry of the stress tensor into account, we have nine unknown functions, i.e., the displacements and the six components of the stress tensor. The six remaining degrees of freedom are determined by taking the specific properties of the material under consideration into account. This is done by means of a *constitutive equation* and a *response function*, characterizing the elastic media. In general, constitutive theory might by viewed as a way to construct well-posed problems describing the behaviour of the specific medium under consideration. For a detailed discussion of this topic, we refer to [MH94, Chapter 3].

In this section, we introduce both, the constitutive equation as well as the response function for elastic materials. Adding the axiom of *material frame indifference* or *objectivity*, the class of possible response functions can be simplified considerably. If the response function is *frame indifferent*, i.e., if it does not depend on the chosen orthonormal basis, it is determinend by its restriction to the set of symmetric, positive definite matrices. The mathematical analysis of elastic materials is considerably simplified by the physical requirements.

For convenience, let us introduce the subsets $\mathbb{M}^3_+$, $\mathbb{S}^3$ and $\mathbb{O}$ the set $\mathbb{M}^3$ of all real square matrices of order three. The set $\mathbb{M}^3_+$ consists of all matrices with positive determinant, $\mathbb{S}^3$ of all symmetric matrices and $\mathbb{O}$ of all rotations. We define what we understand to be an *elastic material.*

Definition 1.4 *A material is called elastic, if there exists a mapping*

$$\hat{\boldsymbol{T}}\colon \bar{\mathscr{B}} \times \mathbb{M}^3_+ \ni (x, \boldsymbol{F}) \mapsto \hat{\boldsymbol{T}}(x, \boldsymbol{F}) \in \mathbb{M}^3 ,$$

such that for the first Piola–Kirchhoff stress tensor $\boldsymbol{T}$ *holds*

$$\boldsymbol{T}(x) = \hat{\boldsymbol{T}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)) \tag{1.6}$$

for any deformed configuration and any point $x \in \bar{\mathscr{B}}$.

Using the Piola–Kirchhoff transformation, this can be written equivalently in terms of the Cauchy stress tensor

$$\boldsymbol{T}^{\boldsymbol{\varphi}}(x^{\boldsymbol{\varphi}}) \;=\; \hat{\boldsymbol{T}}^D(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x))\,, \qquad x^{\boldsymbol{\varphi}} = \boldsymbol{\varphi}(x)\,,$$

where

$$\hat{\boldsymbol{T}}^D : \bar{\mathscr{B}} \times \mathbb{M}^3_+ \ni (x, \boldsymbol{F}) \mapsto \hat{\boldsymbol{T}}(x, \boldsymbol{F}) \in \mathbb{S}^3 .$$

Here, we have been attaching the superscript ^{D}, indicating we are dealing with quantities being defined on the deformed configuration. The function $\boldsymbol{T}$ is called the *response function* of the elastic material and relation (1.6) is called the *constitutive equation* of the material. We remark, that the value of the response function depends only on the deformation gradient $\boldsymbol{\nabla}\boldsymbol{\varphi}(x)$ and the point x, i.e., the stress is assumed to depend *locally* on the strain. In addition, the response function depends on the chosen basis and on the reference configuration.

If the value of the response function does not depend on the point x but only on the deformation gradient, the material is said to be *homogeneous* and we can write

$$\hat{\boldsymbol{T}}^D(x^{\boldsymbol{\varphi}}, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)) = \hat{\boldsymbol{T}}^D(\boldsymbol{\nabla}\boldsymbol{\varphi}(x))\,.$$

In contrast to the property of material frame indifference, homogeneity is not a physical requirement but a property of the material. This is the same for *isotropy*, which states that the stress response of the material should not depend on the direction, i.e., the material has no preferential directions. To be precise, a material is said to be *isotropic* at a point x, if its response function satisfies

$$\hat{\boldsymbol{T}}^D(x, \boldsymbol{F}\boldsymbol{Q}) = \hat{\boldsymbol{T}}^D(x, \boldsymbol{F})\,, \qquad \boldsymbol{F} \in \mathbb{M}^3_+\,, \boldsymbol{Q} \in \mathbb{O}^3_+\,. \tag{1.7}$$

Let us note, that in contrast to homogeneity, isotropy is a property of the material given with respect to the reference configuration. Before studying the implications of isotropy for the response function $\boldsymbol{T}$, we formulate the requirement of material frame indifference in terms of the response function $\hat{\boldsymbol{T}}$. This is also known as invariance under a change of observer, see [Gur81, Section VII.21]. We say, the response function $\hat{\boldsymbol{T}}^D \to \bar{\mathscr{B}} \times \mathbb{M}^3_+ \to \mathbb{S}^3$ satisifes the axiom of material indifference if and only if

$$\hat{\boldsymbol{T}}^D(x, \boldsymbol{Q}\boldsymbol{F}) = \boldsymbol{Q}\hat{\boldsymbol{T}}^D(x, \boldsymbol{F})\boldsymbol{Q}\,, \qquad \boldsymbol{F} \in \mathbb{M}^3_+\,, \boldsymbol{Q} \in \mathbb{O}^3_+ \tag{1.8}$$

holds for all $x \in \bar{\mathscr{B}}$. What makes this definition interesting is, that any material indifferent response function is completely determined by its restriction to the set $\mathbb{S}^+_>$ of symmetric and positive definite matrices. More precisely, we have the following theorem, which we formulate in terms of the symmetric second Piola–Kirchhoff stress (1.5).

Theorem 1.5 *Let* $\hat{\boldsymbol{\Sigma}} \to \bar{\mathscr{B}} \times \mathbb{S}^3_+ \to \mathbb{S}^3$ *be a response function for the second Piola–Kirchhoff stress satisfying the axiom of material frame indifference. Then there exists a mapping* $\tilde{\boldsymbol{\Sigma}}\colon \bar{\mathscr{B}} \times \mathbb{S}^3_> \to \mathbb{S}^3$ *such that*

$$\boldsymbol{\Sigma}(x, \boldsymbol{F}) \;=\; \tilde{\boldsymbol{\Sigma}}(x, \boldsymbol{F}^T\boldsymbol{F})\,, \qquad \boldsymbol{F} \in \mathbb{M}^3_+$$

holds for all $x \in \bar{\mathscr{B}}$.

Theorem 1.5 tells us the way rotations in the deformed configuration affect the stress response of the material. Correspondingly, multiplying $\boldsymbol{F}$ on the right by an orthogonal transformation $\boldsymbol{Q}$ can be interpreted as transformation in the reference configuration. Remembering that isotropy was defined as a property given in terms of rotations in the reference configuration, we can reformulate (1.7) equivalently as follows, see [Cia88, Theorem 3.4-1]: The response function $\hat{\boldsymbol{T}}^D$ is isotropic at $x \in \bar{\mathscr{B}}$, if and only if there exists a mapping $\bar{\boldsymbol{T}}^D(x, \cdot)\colon \mathbb{S}^3_> \to \mathbb{S}^3$ such that

$$\hat{\boldsymbol{T}}^D(x, \boldsymbol{F}) = \bar{\boldsymbol{T}}^D(x, \boldsymbol{F}\boldsymbol{F}^T)\,, \qquad \boldsymbol{F} \in \mathbb{M}^3\,, \boldsymbol{Q} \in \mathbb{O}^3_+ \tag{1.9}$$

Up to now we have only been concerned with the behaviour of the response function under orthogonal transformation, but we did not explicitely state any expression for the response function. Combining material frame indifference (1.8) and isotropy (1.9), one can show the following theorem, which can be found, e.g., in [Cia88, Theorem3.6-2] and [Gur81, Theorem on p.170].

Theorem 1.6 (Constitutive equation for an isotropic material) *Let there be given an elastic material whose response function is isotropic and frame indifferent at a point* $x \in \bar{\mathscr{B}}$. *Then, the response function* $\bar{\boldsymbol{\Sigma}}(x, \cdot)\colon \mathbb{S}^3_> \to \mathbb{S}^3$ *for the second Piola–Kirchhoff stress can be written in the form*

$$\bar{\boldsymbol{\Sigma}}(x, \boldsymbol{C}) = \beta_0(x, \mathscr{I}(\boldsymbol{C}))\boldsymbol{I} + \beta_1(x, \mathscr{I}(\boldsymbol{C}))\boldsymbol{C} + \beta_2(x, \mathscr{I}(\boldsymbol{C}))\boldsymbol{C}^2\,, \qquad \boldsymbol{C} \in \mathbb{S}^3_>\,, \tag{1.10}$$

where $\beta_0(x.\cdot), \beta_1(x, \cdot), \beta_2(x, \cdot)$ *are scalar functions of the three principal invariants* $\mathscr{I}(C)$ *of the matrix* $\boldsymbol{C}$.

Here, the *list of invariants* $\mathscr{I}(\boldsymbol{A})$ of a matrix $\boldsymbol{A}$ is given by

$$\mathscr{I}(\boldsymbol{A}) = (\mathscr{I}(\boldsymbol{A})_1, \mathscr{I}(\boldsymbol{A})_2, \mathscr{I}(\boldsymbol{A})_3)\,,$$

where, for any matrix $\boldsymbol{A}$ of order three, the invariants are defined to be the coefficients of the characteristic polynomial of $\boldsymbol{A}$,

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = -\lambda^3 + \mathscr{I}(\boldsymbol{A})_1 \lambda^2 - \mathscr{I}(\boldsymbol{A})_2 \lambda + \mathscr{I}(\boldsymbol{A})_3\,.$$

Alternatively, the invariants can be expressed in terms of the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $\boldsymbol{A}$

$$\begin{aligned} \mathscr{I}(\boldsymbol{A})_1 &= \lambda_1 + \lambda_2 + \lambda_3 = \operatorname{tr} \boldsymbol{A}\,, \\ \mathscr{I}(\boldsymbol{A})_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1\,, \\ \mathscr{I}(\boldsymbol{A})_3 &= \lambda_1\lambda_2\lambda_3 = \det(\boldsymbol{A})\,. \end{aligned}$$

As we have seen in Theorem 1.6, any frame indifferent response function for the second Piola–Kirchhoff stress for an isotropic material is determined completely by the right Cauchy–Green strain tensor. Unfortunately, the form of the response function given in equation (1.10) involves quantities as the determinant, which are in general costly to compute. For small deformations near the reference configuration we can make a Taylor expansion of (1.10), provided the scalar functions $\beta_0, \beta_1, \beta_2$ given in Theorem 1.6 are smooth enough. Assuming furthermore the reference configuration to be *stress free* or in *natural state* and assuming homogeneity, we can express the response function of an isotropic, homogeneous, elastic material, whose reference configuration is in a natural state, by

$$\boldsymbol{\Sigma}(\boldsymbol{E}) = \lambda(\operatorname{tr} \boldsymbol{E})\boldsymbol{I} + 2\mu\boldsymbol{E} + o(\boldsymbol{E})\,, \tag{1.11}$$

where $\boldsymbol{\Sigma}\colon \mathbb{M}^3_+ \to \mathbb{S}^3$. Here, λ and μ are two constants called the *Lamé constants* of the material. Neglecting now the terms of higher order, we can define the so called *St. Venant–Kirchhoff* materials, see also [MH94, Example 5.17], whose response function is simply given by

$$\boldsymbol{\Sigma}(\boldsymbol{E}) = \lambda(\operatorname{tr} \boldsymbol{E})\boldsymbol{I} + 2\mu\boldsymbol{E}\,.$$

Let us note, that all considerations made above are valid for the strain tensor $\boldsymbol{E}$. In the next section, we replace $\boldsymbol{E}$ by its linearization $\boldsymbol{\varepsilon}$, leading us to the linear equations of linear elasticity.

1.4 The Equations of Linear Elasticity

The equations of linear elasticity can be regarded as the result of *two* linearizations. The first one is the linearization of the response function, which gives rise to (1.11). The second one is the linearization of the strain tensor $\boldsymbol{E}$ in (1.2). Replacing $\boldsymbol{E}$ by $\boldsymbol{\varepsilon}$ and inserting in (1.11), we find

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \lambda(\operatorname{tr} \boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}\,. \tag{1.12}$$

Here, we have written $\boldsymbol{\sigma}$ for the resulting stress tensor, indicating that we are dealing with a linearized object.

Let us recall, what kind of linearizations we have been introducing so far and what quantities have been linearized

Type of linearization	Linearized quantity
geometric linearization (1.2)	strain tensor
linearization of the constitutive equation (1.11)	response function.

We did not only assume small displacements, i.e., $|u_{i,j}| \ll 1$, but also homogeneity and isotropy of the material under consideration. As additional property of the material, we required a *linear stress-strain relation.* Thus, from a nonlinear point of view, the term "linearized equations of linearized elasticty" would be more suitbale than the term "linear elasticity".

Nevertheless, linear stress response of many materials for small displacements is known from experiments. In Figure 1.4, the stress response of steel is depicted, showing a linear stress-strain relation of the material until some critical stress σ_e is reached. In between σ_e and σ_p, the stress-strain relation of the material is nonlinear, but the material is still elastic. If the critical *yield stress* σ_p is reached, the material is deformed plastically and the stress response of the material is highly nonlinear. To summarize, the combination

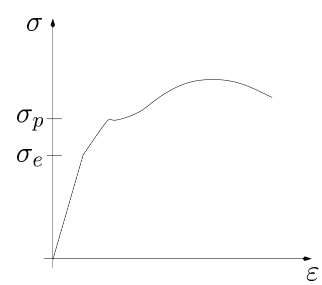

Figure 1.3: Stress response of steel

of both linearizations seems to be a reasonable simplification of even nonlinear material behaviour near a reference configuration of natural state.

1.5 Hyperelastic Materials

In this section, hyperelastic materials are defined to be materials, for which a smooth *stored energy function* $\widehat{W}(x, \boldsymbol{F})$ exists, such that $\boldsymbol{C} = \partial\widehat{W}/\partial\boldsymbol{F}$. Hyperelastic materials are not only well understood from the point of mathematical analysis, but form also an important class of materials for the numerical simulation of elastic materials. In particular, we exploit the existence of the functional $\mathcal{J}$ of *total energy* when analyzing the monotone multigrid method for contact problems presented in this work.

Despite the good theoretical properties of hyperelastic materials, there is also a more mechanical interpretation of hyperelasticity connected to the work in so called *closed* *processes.* This is described in more detail at the end of this section.

Remark 1.7 *Due to physical requirements, the stored energy function can be seen to be nonconvex with respect to it's argument* $\boldsymbol{F}$. *To close the resulting gap in existence theory, Ball has developed the theory of* polyconvex stored energy functions. *We will not discuss this topic here and refer to the monograph [MH94] and the references cited therein.*

An elastic material with response function $\hat{\boldsymbol{T}}$ is said to be *hyperelastic*, if there exists a *stored energy function* $\widehat{W}: \bar{\mathscr{B}} \times \mathbb{M}^3_+ \to \mathbb{R}$ such that

$$\hat{\boldsymbol{T}}(x, \boldsymbol{F}) = \frac{\partial \widehat{W}}{\partial \boldsymbol{F}}(x, \boldsymbol{F}), \qquad x \in \bar{\mathscr{B}}, \boldsymbol{F} \in \mathbb{M}^3_+ .$$

Here, for fixed $x \in \bar{\mathscr{B}}$ the derivative of $\widehat{W}$ with respect to $\boldsymbol{F}$ has to be understood as a function $\mathbb{M}^3_+ \to \mathscr{L}(\mathbb{M}^3_+, \mathbb{R})$, assigning the linear mapping $\frac{\partial \widehat{W}}{\partial \boldsymbol{F}}(x, \boldsymbol{F}) \in \mathscr{L}(\mathbb{M}^3_+, \mathbb{R})$ to $\widehat{W}(x, \cdot)$ at $\boldsymbol{F}$.

To fix ideas, let us take the principle of virtual work and the asociated variational equation (1.4) as starting point. Since the term

$$\int_{\mathscr{B}^{\varphi}} \boldsymbol{f}^{\varphi} \boldsymbol{\cdot} \boldsymbol{\vartheta}^{\varphi} \, dx^{\varphi}$$

can be regarded as a linear functional in $\boldsymbol{\vartheta}^{\varphi}$ we can assume for the moment, that there exists a Gâteaux differentiable functional F with

$$\int_{\mathscr{B}^{\varphi}} \boldsymbol{f}^{\varphi} \boldsymbol{\cdot} \boldsymbol{\vartheta}^{\varphi} \, dx^{\varphi} = \hat{F}'(\boldsymbol{\varphi}) \boldsymbol{\vartheta}^{\varphi} .$$

This is possible for example for dead loads. Equally rewriting the surface forces, we end up with the right hand side of (1.4) expressed as

$$\hat{F}'(\boldsymbol{\varphi}) \boldsymbol{\vartheta}^{\varphi} + \hat{G}'(\boldsymbol{\varphi}) \boldsymbol{\vartheta}^{\varphi} , \tag{1.13}$$

where G is a suitable Gâteaux differentiable functional representing the surface forces. To be able to reformulate the left hand side of (1.4) as well, we assume the material under consideration to be hyperelastic. Using the Piola–Kirchhoff transformation and defining the linear functional of *strain energy* as

$$W(\boldsymbol{\psi}) = \int_{\mathscr{B}} \widehat{W}(x, \boldsymbol{\nabla} \boldsymbol{\psi}(x) \, dx ,$$

we can conclude by means of the differentiability properties of $\widehat{W}$ that

$$W'(\boldsymbol{\psi}) \boldsymbol{\vartheta} = \int_{\mathscr{B}} \hat{\boldsymbol{T}}(x, \boldsymbol{\nabla} \boldsymbol{\psi}(x)) \boldsymbol{:} \boldsymbol{\nabla} \boldsymbol{\vartheta}(x)) \, dx .$$

Putting everything together and setting $\mathcal{J} = W - (F + G)$, we see that we can write equation (1.4) as

$$\mathcal{J}'(\boldsymbol{\varphi})\boldsymbol{\vartheta} = 0 \tag{1.14}$$

for $\boldsymbol{\vartheta} \colon \bar{\mathscr{B}} \to \mathbb{R}$ being sufficiently smooth and vanishing on Γ_D. Equation (1.14) states that for hyperelastic materials the solution of the boundary value problem (1.3) can be found as stationary point of the functional $\mathcal{J}$ of *total energy*. Since stationary points are closely connected to local minima, this corresponds to the physical interpretation, that the deformation is such that some suitable measure of strain energy is minimized in the equilibrium state of the body. To state things more precisely, we have the following

Theorem 1.8 (see also [Cia88, Theorem 4.1-2]) *Assume the material under consideration is hyperelastic and assume that the volume and surface forces can be written as in (1.13). Let furthermore $\hat{W}$ be the stored energy function of the hyperelastic material and let $\boldsymbol{\Phi} = \{\boldsymbol{\psi} \colon \bar{\mathscr{B}} \to \mathbb{R}^3 : \boldsymbol{\psi} = \boldsymbol{\varphi}_0 \text{ on } \Gamma_D\}$. Then any smooth mapping $\boldsymbol{\varphi}$ satisfying*

$$\mathcal{J}(\boldsymbol{\varphi}) = \inf_{\boldsymbol{\psi} \in \boldsymbol{\Phi}} \mathcal{J}(\boldsymbol{\psi}) \tag{1.15}$$

for the functional $\mathcal{J}$ of total energy defined above solves the boundary value problem

$$\begin{aligned} -\mathbf{div}\, \frac{\partial \hat{W}}{\partial \boldsymbol{F}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)) &= \boldsymbol{f}(x, \boldsymbol{\varphi}(x)) \quad , x \in \bar{\mathscr{B}}\,, \\ \frac{\partial \hat{W}}{\partial \boldsymbol{F}}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x))\boldsymbol{n} &= \boldsymbol{p}(x, \boldsymbol{\nabla}\boldsymbol{\varphi}(x)), x \in \Gamma_N\,, \\ \boldsymbol{\varphi}(x) &= \boldsymbol{\varphi}_0(x) \qquad , x \in \Gamma_D\,. \end{aligned}$$

Unfortunately, this theorem cannot be applied directly to nonlinear contact problems, since in that case, the corresponding energy $\bar{\mathcal{J}}$ functional turns out to be nondifferentiable. We will overcome this difficulty by introducing the *subdifferential* of the functional $\bar{\mathcal{J}}$. This enables us to generalize condition (1.14) for a stationary point of the total energy to nondifferentiable energy functionals. In particular, the variational equation (1.4) becomes a *variational inclusion*, see Chapter 2.

Remark 1.9 *For a discussion of the relation between the stored energy functional and the* free energy functional *used in thermodynamics, we refer to [MH94, Section 3.2].*

It is now possible, to formulate frame indifference as well as isotropy in terms of the stored energy function, as it has been done in Theorem (1.9). Here, concerning this subject we do not go into detail, but present a more mechanical interpretation of the term "hyperelasticity" with respect to *closed processes*, see [Gur81]. Let us define a *motion* $\boldsymbol{\varphi}_{(\cdot)}(\cdot) \colon \bar{\mathscr{B}} \times \mathbb{R} \to \mathbb{R}^3$ to be a sufficiently smooth mapping, such that for any fixed t, $\boldsymbol{\varphi}_t(\cdot)$ is a deformation of $\mathscr{B}$. Thus a motion is a family of deformations, the time t being the parameter. Then, by a *dynamical process*, we mean the pair $(\boldsymbol{\varphi}_{(\cdot)}(\cdot), \boldsymbol{T}(\cdot, \boldsymbol{\nabla}\boldsymbol{\varphi}_{(\cdot)}(\cdot)))$ and we call the dynamical process *closed* during $[t_0, t_1]$, if

$$\begin{aligned} \boldsymbol{\varphi}_{t_0}(x) &= \boldsymbol{\varphi}_{t_1}(x)\,, \\ \tfrac{d}{dt}\boldsymbol{\varphi}_{t_0}(x) &= \tfrac{d}{dt}\boldsymbol{\varphi}_{t_1}(x)\,, \end{aligned}$$

where $\boldsymbol{T}$ is the first Pioal–Kirchhoff stress. Then, the following theorem holds.

Theorem 1.10 (see [Gur81, Theorem p.186]) *Let the work on any part $A \subset \mathscr{B}$ be defined by*

$$\int_{t_0}^{t_1} \int_{\partial A_t} \boldsymbol{T}(x) \frac{d}{dt}\big(\boldsymbol{\nabla} \boldsymbol{\varphi}_t(x)\big)\, dx\, dt\,.$$

Then, the body $\mathscr{B}$ is hyperelastic if and only if the work is nonnegative in closed processes, i.e., if for any closed process there holds

$$\int_{t_0}^{t_1} \int_{\partial A_t} \boldsymbol{T}(x) \frac{d}{dt}\big(\boldsymbol{\nabla} \boldsymbol{\varphi}_t(x)\big)\, dx\, dt \geq 0$$

Hyperelasticity can now be interpreted as the ability of the body $\mathscr{B}$ to store and release energy regardless of the deformations it is undergoing. For example, this is not the case if the material under consideration is plastic or even elastic.