Appendix C

Units, Notation, and Other Conventions

In this work, the notation and most conventions of Bjørken and Drell [BD66] are used whenever possible. Some important definitions and conventions are given in this chapter for the convenience of the reader.

C.1 System of units and physical constants

If not stated otherwise, relativistic natural units are used for which the vacuum velocity of light c, the reduced Planck constant $\hbar$ and the electron mass m_e all take the numerical value 1. Then the unit of length is the reduced Compton wave length $\lambda\!\!\!^{-}_\mathrm{c} = \frac{\hbar}{m_\mathrm{e}c}$ of the electron, the unit of time is $\lambda\!\!\!^{-}_\mathrm{c}/c = \frac{\hbar}{m_\mathrm{e}c^2}$, and the unit of energy is equal to $m_\mathrm{e}c^2$. Therefore, the relativistic natural units of length and time have the following values in in MKSA units [MT00]:

quantity	numerical value	unit
$\frac{\hbar}{m_\mathrm{e}c}$	3.86×10^{-13}	m
$\frac{\hbar}{m_\mathrm{e}c^2}$	1.29×10^{-21}	s

For the electrical charge Gaussian units are used, which are commonly preferred in atomic physics. This means that the unit of electrical charge is chosen such that the potential energy of two charges Q and q at a distance r is equal to,

$$\frac{Qq}{r}.$$

In the Gaussian system the elementary charge e is related to the fine-structure constant α by:

$$\alpha = \frac{e^2}{\hbar c}.$$

The fine-structure constant α is the only *physical constant* which directly enters the numerical calculations. The numerical value which has been used to obtain the results of this thesis is [EM95],

$$\alpha^{-1} = 137.0359895,$$

deviating insignificantly from the recently recommended value of $\alpha^{-1} = 137.0359998$ [MT00]. In the system of units adopted here, we have $\alpha = e^2$. Cross sections are conventionally given in barn. Regarding the conversion from relativistic natural units to barn, note that the area of 1 barn corresponds to the $10^{-28}\,\mathrm{m}^2$ in MKSA units and to the area,

$$0.670605 \times 10^{-3}\,\mathrm{r.u.},$$

in relativistic natural units. The Lorentz factor γ associated with the kinetic energy T of a heavy-ion collision given in GeV/u in a fixed target frame is defined as:

$$\gamma = \frac{T}{m_{\text{a.m.u.}}c^2} + 1. \tag{C.1}$$

For the purpose of conversion between these two quantities the value

$$m_{\text{a.m.u}} = 0.931494\,\text{GeV}/c^2.$$

has been used for the the atomic mass unit $m_{\text{a.m.u}}$ [MT00].

C.2 Dirac matrices and discrete symmetry transformations

In terms of the Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ the standard Pauli–Dirac representation of the Dirac-matrices $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and β is given by the following 4×4-matrices [BD66, Tha92]:

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}. \tag{C.2}$$

For numerical calculations the standard representation has been used. However, analytical considerations of the present work do not refer to some particular representation, if not noted otherwise.

We use the convention,

$$g_{\mu\nu} = \text{diag}(1, -1 - 1 - 1),$$

for the signature of the Minkowski metric, following [BD66, BLP82, EM95, Sch95, Jac99] and others. Since the Dirac-matrices are mutually anti-commuting and the γ-matrices are required to satisfy,

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \qquad \text{for } \mu, \nu = 0, \dots, 3,$$

the two sets of matrices are related by:

$$\gamma^0 = \beta, \qquad \gamma^i = \beta\alpha_i, \qquad \text{for } i = 1, 2, 3.$$

This relation is independent of the particular representation. However, it depends on the signature of the Minkowski metric, which is sometimes chosen differently [Wei95]. The definition of the matrix $\gamma^5 = \gamma_5$ adopted in this work is:

$$\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = -\mathrm{i}\alpha_1\alpha_2\alpha_3.$$

We refer to the following definitions of the operators of charge conjugation $\mathcal{C}$, time-reversal $\mathcal{T}$ and parity $\mathcal{P}$ acting on a classical Dirac field $\Psi(t, \boldsymbol{x})$ [Sch95]:

$$(\mathcal{C}\Psi)(t, \boldsymbol{x}) = \gamma^0 C\, \Psi^*(t, \boldsymbol{x}), \tag{C.3}$$

$$(\mathcal{T}\Psi)(t, \boldsymbol{x}) = \gamma_5 C\, \Psi^*(-t, \boldsymbol{x}), \tag{C.4}$$

$$(\mathcal{P}\Psi)(t, \boldsymbol{x}) = \gamma^0\, \Psi(t, -\boldsymbol{x}). \tag{C.5}$$

These definitions are valid for any representation of the γ-matrices that is unitarily equivalent to the chiral representation or the standard representation. The matrix C

occurring in equations (C.3) and (C.4) is always unitary but depends on the representation. It satisfies

$$C^{-1}\gamma^{\mu}C = -\gamma^{\mu T}, \qquad C^T = -C \qquad \text{and} \qquad C^*C = -1, \tag{C.6}$$

and exits for any representation which is unitarily equivalent to the chiral representation as a consequence of Pauli's fundamental theorem on the representation of the γ-matrices. Furthermore, C is uniquely defined up to a complex phase, which may be proved by means of Schur's lemma [Goo55, Tha92, Sch95]. The commutation properties of the discrete symmetry operators are summarised by the equation:

$$\{\mathcal{C},\mathcal{P}\} = [\mathcal{P},\mathcal{T}] = [\mathcal{T},\mathcal{C}] = 0 \tag{C.7}$$

The operators $\mathcal{C}$ and $\mathcal{P}$ are involutions,

$$\mathcal{C}^2 = \mathcal{P}^2 = 1,$$

whereas the double application of the time reversal operation changes the sign of a wave function,

$$\mathcal{T}^2 = -1.$$

Note that the operators defined in equations (C.3–C.5) belong to a particular representation of the covering group of the Poincaré group and that non-isomorphic representations of this group exist. Nevertheless, all different possibilities yield the same projective representation of the Poincaré group [Tha92, pp. 76, 104–105].

Due to the hermitian properties of the γ-matrices, namely $\gamma^{0\dagger} = \gamma^0$ and $\gamma^{i\dagger} = -\gamma^i$, the following useful 'commutation relations' are easily obtained:

$$-C\gamma^{0*} = \gamma^0 C, \tag{C.8}$$

$$C\gamma^{i*} = \gamma^i C, \qquad i = 1,2,3, \tag{C.9}$$

$$C\gamma_5^* = \gamma_5 C. \tag{C.10}$$

For the standard representation (C.2) a common choice for C is [BD66, (5.6)],

$$C = \mathrm{i}\gamma^2\gamma^0,$$

where the phase of C has been chosen such that C becomes a real-valued matrix.

C.3 Symbols and Notation

Table C.1: Table of symbols

$a_{\Delta l,\Gamma k}$	Transition amplitude from an initial configuration (Γ, k) to a final configuration (Δ, l).
$c_{\Gamma,k}(t)$ and $c_i(t)$	Coefficients of a coupled channel expansion.
$d_{\mathrm{A}}(t, \boldsymbol{x})$	Distance between the centres A and B as measured in a rest frame of centre A (section 2.1).
$d_{\mathrm{B}}(t, \boldsymbol{x})$	The same as $d_{\mathrm{A}}(t, \boldsymbol{x})$, but with respect to a rest frame of centre B.

e	The physical unit charge $e > 0$. For natural relativistic and Gaussian units related to the fine structure constant by $e^2 = \alpha$.
$g(t, \boldsymbol{x})$	Gauge function.
$r_{\mathrm{A}}(t, \boldsymbol{x}), r_{\mathrm{B}}(t, \boldsymbol{x})$	Distance from a the centres A and B respectively in their respective rest frames (section 2.1).
$\boldsymbol{v}_{\mathrm{A}}, \boldsymbol{v}_{\mathrm{B}}$	Velocities of the centres A and B respectively.
$v_{\mathrm{A}}, v_{\mathrm{B}}$	Scalar three-velocities, which may be either positive or negative, in a frame where the centres move in the same direction. Hence, *not* the moduli of $\boldsymbol{v}_{\mathrm{A}}$ and $\boldsymbol{v}_{\mathrm{B}}$.
$\boldsymbol{x} = (x^1, x^2, x^3)$	Three-vectors. (In appendix A describing the numerical code $\boldsymbol{x} = (x, y, z)$ is used.)
x^μ, x^i	When using Einstein's summation convention, Greek indices $\mu, \nu, \sigma, \rho, \dots$ are running from 0 to 3 and Latin indices $i, j, k, l, \dots$ running from 1 to 3 only.
$(t, \boldsymbol{x})$ and $(A^0, \boldsymbol{A})$	Four-vectors.
$\mathbb{C}, \mathbb{R}$	The complex and real numbers respectively.
$D_0, D_i, \boldsymbol{D}$	A notation for partial differential operators, useful in calculations using the chain rule of differentiation: Partial differentiation with respect to the i-th argument of some function.
$H_0 = -\mathrm{i}\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta$	Free Dirac-Hamiltonian in relativistic units and in general for an arbitrary representation of the γ-matrices.
$P(b)$	Impact parameter dependent probability.
$P_{\mathrm{A}}(t)$ and $P_{\mathrm{B}}(t)$	Projectors onto the subspace spanned by the bound states of centre A and B respectively (section 3.1).
$\Re$ and $\Im$	Real and imaginary parts respectively of some complex number.
T	Collision energy in GeV/u.
$\mathcal{T}_i, \mathcal{P}_i$	Discrete symmetry operators (section 2.4).
$V(t)$	Interaction matrix in the matrix notation of the coupled channel equation (section 4.2).
$V_\Gamma(r)$	Spherically symmetric electrostatic potential in a rest frame of centre Γ (sec. 2.2).
$W(t, \boldsymbol{x})$	A hermitian 4×4-matrix-valued function, acting as an external potential matrix of a Dirac equation. (Not necessarily an external electromagnetic field minimally coupled to the Dirac field [Tha92].)

$W_\Gamma(t, \boldsymbol{x})$	External potential of the two-centre Dirac equation due to a static charge distribution in a rest frame of centre Γ (section 2.2).
$W_\Gamma^\infty(t, \boldsymbol{x})$	Residual external field caused by a long range force of centre Γ that remains for bound states at the other centre, even at large times (section 3.7).
$\partial_t, \partial_1, \partial_2, \partial_3$	Partial derivatives with respect to the variables t, x^1, x^2, x^3 respectively.
$\gamma^i = \beta\alpha_i$ and $\gamma^0 = \beta$	Dirac matrices. If not indicated otherwise the equations are stated without reference to a particular representation.
$\gamma_5 = \gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$	Definition of the matrix γ_5.
γ_A and γ_B	Lorentz factors, $\gamma_\Gamma = [1 - \boldsymbol{v}_\Gamma^2]^{-1/2}$ for $\Gamma = \mathrm{A}, \mathrm{B}$.
μ_A and μ_B	Inverse screening length of the model potential in section 2.2.
ϱ_A and ϱ_B	Nuclear radius in the model potential in section 2.2.
$\sigma_1, \sigma_2, \sigma_3$	Pauli matrices.
χ_A and χ_B	Rapidities of the respective centres in a frame of reference where the centres move along parallel trajectories.
Γ, Δ	Indices for the scattering channels, which may principally take the values A, B and C.
$\Phi_{\Gamma,k}(t, \boldsymbol{x})$	If not indicated otherwise, it denotes either a solution of a scattering-channel Dirac equation (an asymptotic configuration), or a basis function of the coupled-channel ansatz.
$\Psi(t, \boldsymbol{x})$	Usually denotes a solution of the two-centre Dirac equation.
$\Psi_{\Gamma,k}^\pm(t, \boldsymbol{x})$	Incoming $(+)$ and outgoing $(-)$ scattering states, which correspond to the asymptotic configuration $\Phi_{\Gamma,k}(t, \boldsymbol{x})$ (section 3.1).
$\Omega_\mathrm{A}(t, s), \Omega_\mathrm{B}(t, s), \Omega_\mathrm{C}(t, s)$	Product of time-evolution operators (section 3.1).
$\Omega_\mathrm{A}^\pm(s), \Omega_\mathrm{B}^\pm(s), \Omega_\mathrm{C}^\pm(s)$	Møller operators of the three scattering channels (section 3.1).
$\{\cdot, \cdot\}, [\cdot, \cdot]$	Anticommutator and commutator brackets.
$(\Psi_1(t), \Psi_2(t))$	Scalar product of wave functions.
$\lvert\boldsymbol{x}\rvert$	Modulus of a three-vector.
$\lVert\Psi(t)\rVert = \sqrt{(\Psi(t), \Psi(t))}$	Norm of the wave function $\Psi(t, \boldsymbol{x})$ at time t.
$\lVert v\rVert_2 = \sqrt{\sum_{i=1}^n v_i^* v_i}$	Norm of a finite vector $v \in \mathbb{C}^n$, as in [Kat80, DH93, GV96].
$\lVert M\rVert_2 = \sup_{v\in\mathbb{C}^n} \frac{\lVert Mv\rVert_2}{\lVert v\rVert_2}$	Matrix norm corresponding to the finite-vector norm $\lVert v\rVert_2$ [GV96].

$\lVert\Psi(t,\boldsymbol{x})\rVert_{L^2(\mathbb{R}^3,\mathrm{d}^3x)^4}$	The same as the Hilbert space norm $\lVert\Psi(t)\rVert$, i.e. the square root of the spatial integral over $\lVert\Psi(t,\boldsymbol{x})\rVert_2^2$. This alternative notation is helpful, if the variable over which is integrated, must appear for some reason.
$\lVert f\rVert_{L^p(\mathbb{R}^3)}$ and $\lVert f(\boldsymbol{x})\rVert_{L^p(\mathbb{R}^3,\mathrm{d}^3x)}$	L^p-norm of a function f, which is defined as, $\lVert f\rVert_{L^p(\mathbb{R}^3)} = (\int \lvert f(\boldsymbol{x})\rvert^p)^{1/p}$ [For84, RS80].
$\lVert f\rVert_{L^\infty(\mathbb{R}^n)}$	Suprenum norm of the function f [RS80].
$N^{\mathrm{T}}, \Psi^{\mathrm{T}}, \ldots$	Transposed of matrices, spinors, vectors etc.
$v^\dagger, \Psi^\dagger, N^\dagger$	Hermitian conjugates (i.e. the transposed and complex conjugated objects) of finite vectors, spinors, and finite matrices.
z^*, Ψ^*, N^*	Complex conjugate of a number z, Dirac spinor Ψ, matrix N.
$H^*, U(t,s)^*$	Adjoint of an operator acting on an infinite dimensional Hilbert space [RS80].