8 Finite-time singularities

8.1 A compactness property

Using the interior estimates in Theorem 6.15, we can prove compactness of a set of solutions satisfying natural geometric conditions. We first define what is meant by a converging sequence of solutions to (2.5).

Definition 8.1 *Let $(g_k, u_k)(t)$ be a family of solutions to (2.5) on $[T_A, T_O) \times \Sigma_k$ where Σ_k is complete. Let x_k be a base point in Σ_k. Furthermore let Σ_∞ be a complete Riemannian manifold, $(g_\infty, u_\infty)(t)$ a solution to (2.5), and $x_\infty \in \Sigma_\infty$ be a base point. Then the sequence $(\Sigma_k, (g_k, u_k)(t), x_k)$ converges to $(\Sigma_\infty, (g_\infty, u_\infty)(t), x_\infty)$, if there exists a sequence of open sets $U_k \subset \Sigma_\infty$ containing x_∞, and a sequence of diffeomorphisms $F_k : U_k \to V_k$ where $V_k \subset \Sigma_k$ is open, satisfying $F_k(x_\infty) = x_k$ such that any compact set in Σ_∞ eventually lies in all U_k and the pullbacks $\tilde{g}_k(t) := F_k^* g_k(t)$ and $\tilde{u}_k(t) := F_k^* u_k(t)$ converge to $(g_\infty, u_\infty)(t)$ on every compact subset of $(T_A, T_O) \times \Sigma_\infty$ uniformly together with all their derivatives.*

The compactness theorem is given by:

Theorem 8.2 *Let T_A, T_O be given such that $-\infty \le T_A < T_O \le \infty$ and fix $t_0 \in (T_A, T_O)$. Let $(\Sigma_k, g_k(t), u_k(t), x_k)$ be a pointed sequence of complete solutions to (2.5) for $t \in [T_A, T_O)$ satisfying*

$$\sup_{\Sigma_k} |Rm_k|_{g_k(t)}(t) \le C_0 \qquad \forall t \in (T_A, T_O) \tag{8.1}$$

$$\sup_{\Sigma_k} |u_k|(T_A) \le C_0' \tag{8.2}$$

where C_0 and C_0' are independent of k. Assume in addition that $(g_k, u_k)(t_0)$ is κ-noncollapsed for some $\kappa > 0$ independent of k. Then there exists a subsequence

$$(\Sigma_k, g_k(t), u_k(t), x_k) \xrightarrow{C^\infty} (\Sigma_\infty, g_\infty(t), u_\infty(t), x_\infty) \ ,$$

converging in the sense of Definition 8.1 to a complete, κ-noncollapsed solution of (2.5). All derivatives of the curvature Rm_∞ and of u_∞ are bounded above and there is a lower bound on the injectivity radius of g_∞.

To prove the theorem, we first show an auxiliary lemma which corresponds to [Ham95a, Lemma 2.4]. We estimate the solutions $(g_k, u_k)(t)$ with respect to the limit metric $g_\infty(t)$ on compact subsets of $(T_A, T_O) \times \Sigma_\infty$ using bounds at $t = t_0$ and bounds for $g_k(t), u_k(t)$ with respect to the individual metrics $g_k(t)$. We carefully distinguish to which metric the norms and derivatives belong.

Lemma 8.3 *Let (Σ, g) be a Riemannian manifold and $K \subset \Sigma$ be a compact subset. Assume $(g_k, u_k)(t)$ is a collection of solutions to (2.5), defined on neighborhoods of $[\alpha, \omega] \times K$ such that $\alpha < 0 < \omega$. Let $\nabla, |\cdot|$ denote covariant differentiation and length with respect to g and ${}^k\nabla, |\cdot|_k$ with respect to g_k. Furthermore suppose that at time $t = 0$ on K we have the bounds*

$$\begin{aligned} &(a) && cg(X,X) \leq g_k(X,X) \leq Cg(X,X) \quad \text{for all } X \in \mathcal{X}(\Sigma) \\ &(b) && |\nabla^p g_k| \leq \hat{C}_p \quad \text{for all } p \geq 1 \\ &(c) && |\nabla^p u_k| \leq \hat{C}'_p \quad \text{for all } p \geq 0\ , \end{aligned}$$

and in addition

$$\begin{aligned} &(d) && \sup_{[\alpha,\omega]\times K} |{}^k\nabla^p Rm_k|_k \leq C_p \quad \text{for all } p \geq 0 \\ &(e) && \sup_{[\alpha,\omega]\times K} |{}^k\nabla^p u_k|_k \leq C'_p \quad \text{for all } p \geq 0 \end{aligned}$$

with constants $c, C, \hat{C}_p, \hat{C}'_p, C_p, C'_p$ independent of k. Then the following holds:

$$\begin{aligned} &(i) && \tilde{c}g(X,X) \leq g_k(X,X) \leq \tilde{C}g(X,X) \quad \text{on } [\alpha,\omega]\times K \\ &(ii) && \sup_{[\alpha,\omega]\times K} |\nabla^p g_k| \leq \tilde{C}_p \quad \text{for all } p \geq 1 \\ &(iii) && \sup_{[\alpha,\omega]\times K} |\nabla^p u_k| \leq \tilde{C}'_p \quad \text{for all } p \geq 0 \end{aligned}$$

for constants $\tilde{c}, \tilde{C}, \tilde{C}_p, \tilde{C}'_p$ independent of k, α, ω and K.

Proof:
Since $(g_k, u_k)(t)$ solves (2.5), we get for all vector fields $X \in \mathcal{X}(\Sigma)$ and all k that

$$\partial_t g_k(X,X) = -2Rc_k(X,X) + 4du_k(X) \otimes du_k(X).$$

Using (d) and (e), we can estimate

$$\begin{aligned} 2\big|Rc_k(X,X) - 2du_k(X) \otimes du_k(X)\big| &\leq 4(|Rc_k|_k + |du_k|_k^2)|X|_k^2 \\ &\leq (nC_0 + (C'_1)^2)g_k(X,X) =: A_0 g_k(X,X) \end{aligned}$$

with a constant $A_0 = A_0(n, C_0, C'_1)$, implying that

$$\partial_t g_k(X,X) \leq A_0 g_k(X,X)\ .$$

Reorganizing terms gives

$$\partial_t\big(\ln g_k(X,X)\big) = \frac{1}{g_k(X,X)}\partial_t g_k(X,X) \leq A_0$$

such that we get

$$\big|\partial_t\big(\ln g_k(X,X)\big)\big| \leq A_0\ .$$

This can be integrated

$$\begin{aligned}\ln g_k(X,X)(t) &= \ln g_k(X,X)(0) + \int_0^t \partial_\tau\big(\ln g_k(X,X)(\tau)\big)d\tau \\ &\le \ln g_k(X,X)(0) + \sup_{\tau\in[0,t]} \big|\partial_\tau\big(\ln g_k(X,X)(\tau)\big)\big|(t-0) \\ &\le \ln g_k(X,X)(0) + A_0 \cdot T\end{aligned}$$

and exponentiated

$$g_k(X,X)(t) \le \exp\big(\ln g_k(X,X)(0) + A_0\cdot T\big) = e^{A_0 T} g_k(X,X)(0).$$

From this we conclude

$$g_k(t) \le e^{A_0T} g_k(0) \le e^{A_0 T} C g =: \tilde{C} g$$

where $\tilde{C} = \tilde{C}(n, C_0, C_1', T, C)$ and (a) was used. Using (a), we analogously get the lower bound

$$g_k(t) \ge e^{-A_0T} g_k(0) \ge e^{-A_0T} c g =: \tilde{c} g$$

for a constant $\tilde{c} = \tilde{c}(n, C_0, C_1', T, c)$. This proves (i). Denote by Z from now on all constants depending only on $\tilde{c}, \tilde{C}$ which control the equivalence of the metrics.

Recall the evolution equation (2.9) for the connection Γ_k. Since Γ (of g) is time independent, we have

$$\partial_t\big(\Gamma_k - \Gamma\big) = {}^k\nabla Rc_k + du_k * {}^k\nabla^2 u_k \ .$$

Making use of (d) and (e), this implies

$$\big|\partial_t\big(\Gamma_k - \Gamma\big)\big|_k \le c(n)|{}^k\nabla Rc_k|_k + c(n)|du_k|_k|{}^k\nabla^2 u_k|_k \le C_1 + C_1'C_2' =: A_1$$

for a constant $A_1 = A_1(n, C_1, C_1', C_2')$. Since

$$\nabla g_k \simeq \Gamma_k - \Gamma \simeq {}^k\nabla - \nabla, \tag{8.3}$$

we deduce

$$|\partial_t \nabla g_k| \le c(n)\big|\partial_t\big(\Gamma_k - \Gamma\big)\big| \le c(n) Z \big|\partial_t\big(\Gamma_k - \Gamma\big)\big|_k \le Z\cdot A_1$$

where the constant Z comes from (i) which is already proven. Using the bounds (b) on ∇g_k at $t = 0$, integration gives

$$\big|\nabla g_k\big|(t) = \left|\nabla g_k(0) + \int_0^t \partial_\tau \nabla g_k(\tau) d\tau\right| \le |\nabla g_k|(0) + Z A_1 T \le \hat{C}_1 + Z A_1 T =: \tilde{C}_1 \ . \tag{8.4}$$

Here $\tilde{C}_1$ depends only on $n, \tilde{c}, \tilde{C}, \hat{C}_1, C_1, C_1', C_2', n,$ and T. Since u_k is bounded on $[\alpha,\omega]\times K$ by (e) and $|\cdot| = |\cdot|_k$ on functions, we easily obtain

$$|u_k|(t) = |u_k|_k(t) \le C_0' =: \tilde{C}_0' \ . \tag{8.5}$$

Similarly using (e) and (i), we calculate for the differential:

$$|\nabla u_k|(t) = |du_k|(t) = |{}^k\nabla u_k|(t) \le Z\cdot |{}^k\nabla u_k|_k(t) \le Z\cdot C_1' := \tilde{C}_1' \tag{8.6}$$

where $\tilde{C}_1'$ only depends on $\tilde{c}, \tilde{C}$ and C_1'.

Since we already have a bound for ∇g_k on $[\alpha, \omega] \times K$, we can estimate $\nabla^2 u$ at time t using the equivalence (8.3):

$$\partial_t \nabla^2 u_k = \nabla^2 f_k = (\nabla - {}^k\nabla) df_k + {}^k\nabla df_k = \nabla g_k * {}^k\nabla f_k + {}^k\nabla^2 f_k$$

where $f_k(t) := \Delta^k u_k(t) \in C^\infty([T_A, T_O) \times \Sigma)$ is just an abbreviation. Therefore we can estimate

$$|\partial_t \nabla^2 u_k| \leq |\nabla g_k| Z |{}^k\nabla f_k|_k + Z|{}^k\nabla^2 f_k|_k \leq \tilde{C}_1 Z C(n) C_3' + Z C(n) C_4' =: B_2$$

by (e) with $B_2 = B_2(n, \tilde{c}, \tilde{C}, \tilde{C}_1, C_3', C_4')$. Using (c), an integration gives

$$|\nabla^2 u_k|(t) \leq |\nabla^2 u_k|(0) + \int_0^t |\partial_\tau \nabla^2 u_k(\tau)| d\tau \leq \hat{C}_2' + B_2 T =: \tilde{C}_2'$$

with $\tilde{C}_2'$ depending on $n, \tilde{c}, \tilde{C}, \tilde{C}_1, \hat{C}_2', C_3', C_4'$, and T.

Higher derivatives of (g_k, u_k) with respect to g can be estimated in pairs $(\nabla^p g_k, \nabla^{p+1} u_k)$ for all $p \geq 2$. The technique is similar for all $p \geq 2$, so we only state the case $p = 2$ as reference. Since ∇ commutes with ∂_t we get an expression for $\partial_t \nabla^p g_k$ from the flow equations (2.5):

$$\partial_t \nabla^2 g_k = \nabla^2 \partial_t g_k = \nabla^2 \big(-2Rc_k + 4du_k \otimes du_k \big) = \nabla^2 Rc_k + \nabla^3 u_k * du_k + \nabla^2 u_k * \nabla^2 u_k \ .$$

Using (8.3), this can be rewritten in the following way:

$$\begin{aligned} \nabla^2 Rc_k &= (\nabla - {}^k\nabla)\nabla Rc_k + {}^k\nabla(\nabla - {}^k\nabla) Rc_k + {}^k\nabla^2 Rc_k \\ &= \nabla g_k * \big((\nabla - {}^k\nabla) Rc_k + {}^k\nabla Rc_k \big) + {}^k\nabla(\nabla g_k * Rc_k) + {}^k\nabla^2 Rc_k \\ &= \nabla g_k * \nabla g_k * Rc_k + \nabla g_k * {}^k\nabla Rc_k + {}^k\nabla\nabla g_k * Rc_k + \nabla g_k * {}^k\nabla Rc_k + {}^k\nabla^2 Rc_k \ . \end{aligned}$$

Since the second derivatives can be compared as follows

$${}^k\nabla\nabla g_k = \nabla^2 g_k + ({}^k\nabla - \nabla)\nabla g_k = \nabla^2 g_k + \nabla g_k * \nabla g_k \ ,$$

we get altogether:

$$\partial_t \nabla^2 g_k = \nabla^2 g_k * Rc_k + \nabla^3 u_k * \nabla u_k + \nabla g_k * \nabla g_k * Rc_k + \nabla g_k * {}^k\nabla Rc_k + {}^k\nabla^2 Rc_k + \nabla^2 u_k * \nabla^2 u_k \ .$$

Therefore it is necessary to control $\nabla^3 u_k$ to get an estimate for $\nabla^2 g_k$. Keeping this in mind, we estimate with the help of (d) and (e):

$$\begin{aligned} |\partial_t \nabla^2 g_k| &\leq Z C_0 |\nabla^2 g_k| + Z \tilde{C}_1' |\nabla^3 u_k| + Z \big\{ \tilde{C}_1^2 C_0 + \tilde{C}_1 C_1 + C_2 + (\tilde{C}_2')^2 \big\} \\ &\leq A_1 (|\nabla^2 g_k| + |\nabla^3 u_k|) + A_2 \ , \end{aligned} \tag{8.7}$$

defining $A_1 := Z \max\{C_0, \tilde{C}_1'\}$ and $A_2 = A_2(n, \tilde{c}, \tilde{C}, C_0, C_1, C_2, \tilde{C}_1, \tilde{C}_2')$. Doing the same calculation for $\nabla^3 u_k$, we get:

$$\begin{aligned} \partial_t \nabla^3 u_k &= (\nabla - {}^k\nabla)\nabla df_k + {}^k\nabla(\nabla - {}^k\nabla) df_k + {}^k\nabla^2 df_k \\ &= \nabla g_k * \nabla^2 f_k + {}^k\nabla(\nabla g_k * df_k) + {}^k\nabla^2({}^k\nabla f_k) \\ &= \nabla g_k * (\nabla - {}^k\nabla) df_k + {}^k\nabla\nabla g_k * df_k + \nabla g_k * {}^k\nabla^2 f_k + {}^k\nabla^3 f_k \\ &= \nabla g_k * \nabla g_k * {}^k\nabla f_k + \nabla^2 g_k * {}^k\nabla f_k + \nabla g_k * \nabla g_k * {}^k\nabla f_k + \nabla g_k * {}^k\nabla^2 f_k + {}^k\nabla^3 f_k \ . \end{aligned}$$

This leads to the estimate (again using (e)):

$$|\partial_t \nabla^3 u_k| \le ZC(n)C_3'|\nabla^2 g_k| + C(n)Z\{\tilde{C}_1^2 C_3' + \tilde{C}_1 C_4' + C_5'\} \le A_3|\nabla^2 g_k| + A_4 \tag{8.8}$$

where $A_3 = A_3(n, \tilde{c}, \tilde{C}, C_3')$ and $A_4 = A_4(n, \tilde{c}, \tilde{C}, \tilde{C}_1, C_3', C_4', C_5')$. Putting (8.7) and (8.8) together and realizing that $|\cdot|$ is independent of time, we arrive at

$$\begin{aligned}\big|\partial_t\big(|\nabla^2 g_k| + |\nabla^3 u_k|\big)\big| = \big||\partial_t \nabla^2 g_k| + |\partial_t \nabla^3 u_k|\big| &\le A_1\big(|\nabla^2 g_k| + |\nabla^3 u_k|\big) + A_2 + A_3|\nabla^2 g_k| + A_4 \\ &\le B_1\big(|\nabla^2 g_k| + |\nabla^3 u_k|\big) + B_2 \ .\end{aligned}$$

Since we know by (b) and (c) that

$$|\nabla^2 g_k|(0) + |\nabla^3 g_k|(0) \le \hat{C}_2 + \hat{C}_3' \ ,$$

we can integrate in time to obtain

$$|\nabla^2 g_k|(t) + |\nabla^3 u_k|(t) \le \tilde{C}_2 = \tilde{C}_3' \ . \tag{8.9}$$

Here both constants depend only on $n, \tilde{c}, \tilde{C}, \hat{C}_2, \hat{C}_3', \tilde{C}_1, C_0, C_1, C_2, C_3', C_4', C_5'$, and T.

For higher derivatives we compute the following equations:

$$\begin{aligned}\partial_t \nabla^p g_k &= Rc_k * \nabla^p g_k + \sum_{i=1}^{p} {}^k\nabla^i Rc_k * P(\nabla^0 g_k, \dots, \nabla^{p-i} g_k) + Rc_k * P(\nabla^0 g_k, \dots, \nabla^{p-1} g_k) \\ &\quad + {}^k\nabla u_k * \nabla^{p+1} u_k + \sum_{i=1}^{p-1} \nabla^{1+i} u_k * \nabla^{1+p-i} u_k \\ \partial_t \nabla^{p+1} u_k &= {}^k\nabla f_k * \nabla^p g_k + \sum_{i=2}^{p+1} {}^k\nabla^i f_k * P(\nabla^0 g_k, \dots, \nabla^{p+1-i} g_k) + {}^k\nabla f_k * P(\nabla^0 g_k, \dots, \nabla^{p-1} g_k)\end{aligned} \tag{8.10}$$

where in both cases $p \ge 2$ and P is a polynomial in the components of the derivatives of g_k of the designated order. The equations can be proven by induction on p. This allows us to estimate

$$\begin{aligned}|\partial_t \nabla^p g_k| &\le ZC_0|\nabla^p g_k| + Z\sum_{i=1}^{p} C_i \cdot C(\tilde{c}, \tilde{C}, \tilde{C}_1, \dots, \tilde{C}_{p-i}) + ZC_0 \cdot C(\tilde{c}, \tilde{C}, \tilde{C}_1, \dots, \tilde{C}_{p-1}) \\ &\quad + Z\tilde{C}_1'|\nabla^{p+1} u_k| + Z\sum_{i=1}^{p-1} \tilde{C}_{1+i}' \cdot \tilde{C}_{1+p-i}' \\ &\le C(Z, C_0, C_1')\big(|\nabla^p g_k| + |\nabla^{p+1} u_k|\big) + C(n, \tilde{c}, \tilde{C}, C_0, \dots, C_p, \tilde{C}_1, \dots, \tilde{C}_{p-1}, \tilde{C}_1', \dots, \tilde{C}_p') \\ |\partial_t \nabla^{p+1} u_k| &\le ZC_3'|\nabla^p g_k| + Z\sum_{i=2}^{p+1} C_{i+2}' \cdot C(\tilde{c}, \tilde{C}, \tilde{C}_1, \dots, \tilde{C}_{p+1-i}) + ZC_3' C(\tilde{c}, \tilde{C}, \tilde{C}_1, \dots, \tilde{C}_{p-1}) \\ &\le C(Z, C_3')|\nabla^p g_k| + C(n, \tilde{c}, \tilde{C}, \tilde{C}_1, \dots, \tilde{C}_{p-1}, C_3', \dots, C_{p+3}') \ ,\end{aligned}$$

using that for all $p \ge 0$

$$|{}^k\nabla^p f_k| = |{}^k\nabla^p \Delta^k u_k| \le C(n)Z|{}^k\nabla^{p+2} u_k|_k \le C(n, \tilde{c}, \tilde{C})C_{p+2}' \ .$$

Setting $F_k^p := |\nabla^p g_k| + |\nabla^{p+1} u_k|$, we can calculate as before

$$|\partial_t F_k^p| \leq (A_1 + A_3)|\nabla^p g_k| + A_1|\nabla^{p+1} u_k| + A_2 + A_4 \leq A_5 F_k^p + A_6$$

which again can be integrated in time. Thus we get from the bounds (b) and (c) at time $t = 0$ the estimate

$$|\nabla^p g_k|(t) + |\nabla^{p+1} u_k|(t) \leq \tilde{C}_p = \tilde{C}'_{p+1}$$

where $\tilde{C}_p$, $\tilde{C}'_{p+1}$ depend on $n, \tilde{c}, \tilde{C}, \hat{C}_p, \hat{C}'_{p+1}, C_0, \ldots, C_p, C'_3, \ldots, C'_{p+3}, \tilde{C}_1, \ldots, \tilde{C}_{p-1}, \tilde{C}'_1, \ldots, \tilde{C}'_p$, and T. Together with (8.4), (8.5), (8.6), and (8.9), this shows (ii) and (iii) and therefore completes the proof of the lemma.

□

Using the lemma we can prove the theorem as follows:

Proof: (of Theorem 8.2)
We want to use the convergence theorem as stated in Theorem 7.4. Assume for the proof that $T_A, T_O < \infty$ and that $t_0 = 0$ without loss of generality.

Let $(g_k, u_k)(t)$ be a sequence of solutions on $[T_A, T_O) \times \Sigma_k$ such that $|Rm_k|_k^2(t) \leq C_0$ from (8.1). Then we can bound the injectivity radius $\mathrm{inj}(g_k(0)) > \delta$ at time $t = 0$ uniformly in k from the κ-noncollapsing assumption (since κ is also uniform in k) using [CGT82, Theorem 4.7] and the uniform curvature bound.

The uniform bound $|u_k(T_A)| \leq C'_0$ from (8.2) implies not only a uniform bound $|u_k| \leq C'_0$ on $[T_A, T_O) \times \Sigma$ from Lemma 6.11, but also a uniform bound $|du_k|_k^2 \leq C'_1 = C'_1(C_0, T)$, using (6.13). Here C'_1 only depends on C'_0 and T. Therefore we can apply Corollary 6.16 to get uniform bounds on all derivatives of $Rm_k(t)$ and $u_k(t)$

$$\sup_{(T_A,T_O)\times\Sigma_k} \left(|{}^k\nabla^i Rm_k|_k + |{}^k\nabla^{i+2} u_k|_k \right) \leq C_i = C_i(n, T, C_0, C'_0) \tag{8.11}$$

for all $i \geq 0$ where the C_i are constants depending only on the curvature bound C_0, the initial bound on u_k given by C'_0, n, and T, but not on k.

Using these bounds at $t = 0$ and the lower injectivity radius bound, we can apply Theorem 7.4 to get a convergent subsequence of $(\Sigma_k, g_k(0), x_k)$, also denoted $(\Sigma_k, g_k(0), x_k)$, at time $t = 0$ to a limit $(\Sigma_\infty, G, x_\infty)$ in the sense of Definition 7.3. The convergence is with respect to the limit metric G, in particular we have

$$\lim_{k\to\infty} \left| {}^G\nabla^i \left(F_k^* g_k(0) \right) - {}^G\nabla^i G \right|_G = 0 \qquad \forall i \geq 0 \ . \tag{8.12}$$

The pullbacks $\tilde{g}_k := F_k^* g_k(t)$ and $\tilde{u}_k := F_k^* u_k(t)$ are defined for all times $t \in (T_A, T_O)$ though. To prove convergence of g_k, u_k for all t, we need uniform estimates for the derivatives of $\tilde{g}_k, \tilde{u}_k$ on $(T_A, T_O) \times \Sigma_\infty$. We use G as reference metric from now on.

Let $[\alpha, \omega] \subset (T_A, T_O)$, $0 \in [\alpha, \omega]$, and $K \subset \Sigma_\infty$ be compact. Since we have convergence of $(\tilde{g}_k(0))$ to the limit metric G at $t = 0$, the following is true:

1. $\tilde{g}_k(0)$ is equivalent to G on K, that is $\bar{c}G \leq \tilde{g}_k(0) \leq \bar{C}G$ holds for all k big enough and some constants $\bar{c}$ and $\bar{C}$ independent of k.

2. The covariant derivatives of $\tilde{g}_k(0)$ with respect to G are uniformly bounded on $\{0\} \times K$: From (8.12) we have $|^G\nabla^i \tilde{g}_k(0)|_G \leq \hat{C}_i$ for all $i \geq 1$ independent of k.

3. By assumption $|\tilde{u}_k(0)|_G = |u_k(0)| \leq \hat{C}_0 := C'_0$ at time $t = 0$ independent of k. Furthermore we also have $|^G\nabla \tilde{u}_k(0)|_G \leq \hat{C}'_1$ independent of k since

$$\begin{aligned} |^G\nabla \tilde{u}_k(0)|_G = |d\tilde{u}_k(0)|_G = |F_k^*\big(du_k(0)\big)|_G \leq C(\bar{c}, \bar{C}) \cdot |F_k^*\big(du_k(0)\big)|_{\tilde{g}_k(0)} \\ = C(\bar{c}, \bar{C}) \cdot |du_k(0)|_k = C(\bar{c}, \bar{C}) \cdot |^k\nabla u_k(0)|_k \leq \hat{C}'_1 \end{aligned}$$

   from 1. and the bound for $|^k\nabla u_k|_k$ above for k big enough.

4. From (8.11) we can bound $|^G\nabla^i \tilde{u}_k(0)|_G$ for all $i \geq 2$ at time $t = 0$ as follows:

$$|^G\nabla^i \tilde{u}_k(0)|_G \leq C(\bar{c}, \bar{C})|^G\nabla^i \tilde{u}_k(0)|_{\tilde{g}_k(0)} \leq C \cdot |^{\tilde{g}_k(0)}\nabla^i \tilde{u}_k(0)|_{\tilde{g}_k(0)} = C \cdot |^k\nabla^i u_k(0)|_k \leq \hat{C}'_i$$

   for k big enough independent of k where we used the equivalence of G and $\tilde{g}_k(0)$ and the convergence $^{\tilde{g}_k(0)}\nabla^i \longrightarrow {}^G\nabla^i$ at time $t = 0$.

Together with (8.11) this allow us to apply Lemma 8.3. We get equivalence of $\tilde{g}_k(t)$ to G on $[\alpha, \omega] \times K$ and uniform bounds $|^G\nabla^i \tilde{g}_k(t)|_G \leq \tilde{C}_i$ for all $i \geq 1$ and $|^G\nabla^i \tilde{u}_k(t)|_G \leq \tilde{C}'_i$ for all $i \geq 0$ on $[\alpha, \omega] \times K$ where $\tilde{C}_i$ and $\tilde{C}'_i$ do not depend on k or the chosen domain $[\alpha, \omega] \times K$.

Therefore all derivatives of the pullbacks $\tilde{g}_k$ and $\tilde{u}_k$ are uniformly bounded with respect to the fixed metric G on $[\alpha, \omega] \times K$, and we can find a subsequence converging uniformly on every compact subset of $(T_A, T_O) \times \Sigma_\infty$. In addition the limit $g_\infty(t) := \lim_{k\to\infty} \tilde{g}_k(t)$ will agree at time $t = 0$ with G since it already converged there by construction. Defining $u_\infty(t) := \lim_{k\to\infty} \tilde{u}_k(t)$, we see that $(g_\infty, u_\infty)(t)$ is also a solution of (2.5) since the convergence is smooth and taking the limit commutes with all derivatives. Furthermore it satisfies the same bounds on derivatives and the injectivity radius.

If $T_A = \infty$ or $T_O = \infty$, we can apply the theorem for a sequence of times $T_{A_j} \to -\infty$ or $T_{O_j} \to \infty$ respectively on finite time intervals. A diagonalization argument yields a subsequence converging on the union of these intervals [Ham95a, §2].

□

8.2 Rescaling the flow near singularities

Due to the noncollapsing result in Theorem 7.2 we can rescale the solution at a singular time. This can be seen as a microscopic view on the solution when approaching the singularity. Then a comparison of the solution near the singular time and close enough to the singular point with the rescaling limit is possible. It is crucial to know what these regions look like to set up the delicate surgery procedures as described in [Ham97] or [Per03]. We first give some definitions.

Definition 8.4 *A solution $(g, u)(t)$ to (2.5) on a complete Riemannian manifold is called* ancient*, if it exists for all $t \in (-\infty, T]$ up to some time $T \geq 0$.*

Definition 8.5 *Let $(g,u)(t)$ be a maximal solution to (2.5) on $[0,T)\times\Sigma$ for some $T\in(0,\infty]$. A sequence $(t_k,x_k)\subset[0,T)\times\Sigma$ is called an* essential blowup sequence, *if $t_i\to T$, and there is a constant $C\geq 1$ such that*

$$\sup_{[0,t_k]\times\Sigma}|Rm|(t,x)\leq C|Rm|(t_k,x_k)\ .$$

Theorem 8.6 *Let $(g,u)(t)$ be a solution to (2.5) on $[0,T)\times M$ for M closed and T finite. Assume (t_k,x_k) is an essential blowup sequence as defined above and set $B_k:=|Rm|(t_k,x_k)$. Define the rescalings $g_k(s):=B_k\cdot g(\frac{s}{B_k}+t_k)$ and $u_k(s):=u(\frac{s}{B_k}+t_k)$. Then a subsequence of $(M,g_k(s),u_k(s),x_k)$ converges smoothly on all compact subsets of $[0,T)\times M$ to a complete ancient solution $(M_\infty,g_\infty(s),u_\infty(s),x_\infty)$ which is noncollapsed on all scales for some $\kappa>0$. Moreover $u_\infty(s)\equiv const$ and $g_\infty(s)$ is a solution to the Ricci Flow.*

Proof:
We want to apply Theorem 8.2 to the sequence of rescalings $(M,g_k(s),u_k(s),x_k)$.

By choice of the scale factor the rescaled solution exists for $s\in[-B_k\cdot t_k,0]$, and we can compute

$$\sup_{x\in M}|Rm_k|_{g_k}(s,x)=B_k^{-1}\sup_{x\in M}|Rm|(s/B_k+t_k,x)\leq B_k^{-1}\cdot CB_k=C$$

for all $s\in[-B_k\cdot t_k,0]$ from the scaling behavior of $|Rm|$. Since u is controlled on closed M from Lemma 6.11, we also get a uniform bound:

$$\sup_M|u_k|(s)\leq C'$$

for all $s\in[-B_k\cdot t_k,0]$ independent of k since $u\in C^\infty(M)$ is scaling invariant. Finally, because $(g,u)(t)$ is defined on a finite time interval and on closed M, we know from Theorem 7.2 that it is κ-noncollapsed on the scale $\sqrt{T}$ for some $\kappa>0$ depending only on the initial data. From Lemma 7.6 we see that all the rescaled solutions $(g_k,u_k)(s)$ are also κ-noncollapsed for the same κ but on larger and larger scales $\rho_k:=\sqrt{B_kT}$. (Remember that $B_k\to\infty$.) Therefore we can apply Theorem 8.2 on all time intervals $[-A,0], A>0$, to conclude the existence of a subsequence converging to a complete solution $(M_\infty,g_\infty(s),u_\infty(s),x_\infty)$. The limit is ancient since $-B_k\cdot t_k\to-\infty$ for $k\to\infty$ and κ-noncollapsed on all scales since $\rho_k\to\infty$.

It remains to show that the limit is in fact a solution to Ricci Flow. To this end, we will show that $|du_\infty|^2_{g_\infty}\equiv 0$. Recall the a priori estimate (6.14) for $|du|^2$ which is valid for all $t\in(0,T)$:

$$\sup_{x\in M}|du|^2(t,x)\leq\tfrac{1}{4}t^{-1}\ .$$

Since we have $g_k:=B_k\cdot g$, we get for all $s\in(-\infty,0]$ and k big enough that

$$|du_k|^2_{g_k}(s)=(g_k)^{ij}\partial_iu_k\partial_ju_k=B_k^{-1}\cdot|du|^2(s/B_k+t_k)\leq\tfrac{1}{4}B_k^{-1}\cdot\frac{B_k}{s+t_kB_k}=\tfrac{1}{4}\cdot\frac{1}{s+t_kB_k}$$

holds. Passing to the limit $k\to\infty$ for fixed s, we conclude that $|du_\infty|_{g_\infty}(s)\equiv 0$. Since s is arbitrary, the system (2.5) reduces to the Ricci Flow equation for g_∞ on $(-\infty,0]\times M_\infty$.

□

An immediate consequence is:

Corollary 8.7 *For every maximal solution $(g,u)(t)$ of the system (2.5) on $[0,T)\times M$ for $T<\infty$ and closed M, there is a sequence of dilations such that the limit is a complete, ancient solution to the Ricci flow which is κ-noncollapsed on all scales for some $\kappa>0$.*

Proof:
If a solution $(g,u)(t)$ is singular at time $t=T$, the curvature has to blow up as $t\nearrow T$ from Theorem 6.22 in the sense that $\lim_{t\nearrow T}\big(\sup_{x\in M}|Rm|(t,x)\big)=\infty$. We can pick an essential blowup sequence (t_k,x_k) satisfying $|Rm|(t_k,x_k)\geq C\sup_{[0,t_k]\times M}|Rm|(t,x)$ for some constant C independent of k since $[0,t_k]\times M$ is compact.

In particular, we can set $\Lambda:=\sup_M|Rm|^2(0)$. Then there is a first time t_k such that

$$\sup_{[0,t_k]\times M}|Rm|(t,x)=k\cdot\Lambda$$

for all $k\geq 1$. But since t_k is chosen minimally and M is compact, there exists a point x_k at time t_k where the supremum is attained. This allows us to choose $C=1$. Then the corollary follows from the previous theorem.

□

Remark 8.8 *Although we can always find an essential blowup sequence at a singular time T of a given maximal solution $(g,u)(t)$, we point out that there may be other singularities forming at the same time but with a higher blowup rate. These are called slowly-forming singularities. To fully understand the solution at time T, one also needs to understand these singularities. In the Ricci flow literature [Ham95b, §16] dilation limits at singularities are referred to as* singularity models *and are classified in two types I and II(a) for $T<\infty$. Following [Per02], we do not make this distinction here. Perelman instead uses the concept of "ancient κ-solutions" to understand the singularities in the Ricci flow [Per02, §11,§12].*