7 Controlling the injectivity radius

In general solutions to the flow will only exist for a certain finite time due to curvature blowup. Therefore we have to deal with solutions that are singular at some time T. To better understand these solutions, we need to control the geometry at such a singularity. There are two things to care for, the first being the curvature of the solution. From Theorem 6.15 we can deduce smoothness of the solution as long as the curvature stays bounded. We are going to prove that this is a sufficient condition for long time existence of the flow in the next section. Therefore we have to rescale the flow at a singular time in order to keep control on the curvature. The second geometric quantity we have to care for is the injectivity radius of the solution. As long as the metrics stay equivalent this is not a problem, but it could happen that a solution collapses geometrically in the following sense. Roughly speaking, the decrease of volume of balls under the flow compared to the radius is too large as we are approaching the singular time:

Definition 7.1 *Let $(g,u)(t)$ be a solution to (2.5) on $[0,T)\times\Sigma$. We say that $(g,u)(t)$ is* locally collapsing *at T, if there is a sequence of times $(t_k), t_k\to T$ and a sequence of balls $(B_k):=(B_{r_k}(p_k))$ at time t_k, such that r_k^2/t_k is bounded, the curvature satisfies $|Rm|(t_k)\le r_k^{-2}$ on B_k, and the volume decreases like $r_k^{-n}\,\mathrm{vol}(B_k)\to 0$.*

If this happens, a rescaling limit cannot be taken. To avoid this problem, we prove that such a collapse cannot occur with similar arguments as in [Per02, §4] for the Ricci Flow.

Theorem 7.2 *Suppose M is closed and $T<\infty$. Then a solution $(g,u)(t)$ of (2.5) on $[0,T)$ is noncollapsed on $[0,T]\times M$.*

Proof:
This is clear for $t<T$ from the equivalence of the metrics $g(t)$ from Lemma 2.8. For $t=T$ assume to the contrary that there are sequences (t_k) and (B_k) as above. Letting $\phi:=e^{-f/2}$ as in the proof of Proposition 5.8, we see that $\mu(g,u,\tau)(t)$ is the infimum of

$$\tilde{W}(g,u,w,\tau):=\int_M\left[\tau\left(4|d\phi|^2+S\phi^2\right)-\phi^2\ln\phi^2-n\phi^2\right](4\pi\tau)^{-\frac{n}{2}}\,dV \tag{7.1}$$

with the constraint

$$\int_M\phi^2(4\pi\tau)^{-\frac{n}{2}}\,dV=1\ . \tag{7.2}$$

Set $\tau(t):=(t_k+r_k^2)-t$ and define at time t_k:

$$\phi_k(x):=e^{C_k}\xi\left(r_k^{-1}d(x,p_k)\right) \tag{7.3}$$

where $\xi\in C^\infty(\mathbb{R}^+)$ has the profile

$$\xi=\begin{cases}1 & \text{on}\quad [0,\frac{1}{2}]\\ \searrow & \text{on}\quad [\frac{1}{2},1]\\ 0 & \text{on}\quad [1,\infty)\ .\end{cases}$$

Choose C_k such that the normalization condition (7.2) for ϕ_k is satisfied. Noting that ϕ_k vanishes outside B_k and that $\tau(t_k) = r_k^2$, we can compute

$$(4\pi r_k^2)^{\frac{n}{2}} = (4\pi\tau)^{\frac{n}{2}} = \int_M \phi_k^2 dV = e^{2C_k} \int_{B_k} \xi\big(r_k^{-1} d(x,p_k)\big)^2 dV \ .$$

Thus we conclude

$$(4\pi)^{\frac{n}{2}} = e^{2C_k} r_k^{-n} \int_{B_k} \underbrace{\xi^2}_{\leq 1} dV \leq e^{2C_k} \underbrace{r_k^{-n} \operatorname{vol}(B_k)}_{\longrightarrow 0}$$

which forces $C_k \to +\infty$ for $k \to \infty$. We insert ϕ_k and r_k^2 into (7.1) and get

$$\begin{aligned}\tilde{W}(g,u,\phi_k,r_k^2) &= (4\pi)^{-\frac{n}{2}} r_k^{-n} e^{2C_k} \int_{B_k} \Big(4r_k^2 \cdot |\xi'\big(r_k^{-1} d(x,p_k)\big)|^2 \cdot (r_k^{-2} \underbrace{|\nabla d(x,p_k)|^2}_{=1}) - 2\xi^2 \ln \xi\Big) dV \\ &\quad + r_k^2 \int_{B_k} T\phi_k^2 (4\pi)^{-\frac{n}{2}} r_k^{-n} dV - n - 2C_k \\ &\leq (4\pi)^{-\frac{n}{2}} r_k^{-n} e^{2C_k} \int_{B_k} \big(4|\xi'|^2 - 2\xi^2 \ln \xi\big) dV + r_k^2 \max_{B_k} S - n - 2C_k \ . \end{aligned} \tag{7.4}$$

Set $V(r) := \operatorname{vol}(B_{r_k}(p_k))$. From the curvature bound on B_k we know that $Rc \geq -(n-1)C^2 r_k^{-2}$ on B_k. Thus we can compare $V(r)$ with the volume $\tilde{V}(r)$ of the corresponding ball in the model space of constant sectional curvature $-C^2 r_k^{-2}$. Since $\frac{\tilde{V}(r)}{\tilde{V}(r/2)}$ is bounded, we get from the Bishop Volume Comparison Theorem [SY94, Theorem 1.3] that

$$\frac{V(r_k)}{V(r_k/2)} \leq \frac{\tilde{V}(r_k)}{\tilde{V}(r_k/2)} \leq C'$$

holds, implying the inequality

$$V(r_k) - V(r_k/2) \leq (C'-1) V(r_k/2) \ .$$

Since $\xi \equiv 1$ on $B_{r_k/2}(p_k)$, this allows us to estimate

$$\begin{aligned}\int_{B_k} \big(4|\xi'|^2 - 2\xi^2 \ln \xi\big) dV &= 0 + \int_{r_k/2 \leq d(p_k,x) \leq r_k} \big(4|\xi'|^2 - 2\xi \ln \xi\big) dV \\ &\leq \max_{r_k/2 \leq d(p_k,x) \leq r_k} \big|4|\xi'|^2 - 2\xi^2 \ln \xi\big| \cdot \big[V(r_k) - V(r_k/2)\big] \\ &\leq C \cdot \big(V(r_k) - V(r_k/2)\big) \leq C'' V(r_k/2) = C'' \int_{B_{r_k/2}(p_k)} \xi^2 dV \\ &\leq C'' \int_{B_k} \xi^2 dV = C'' (4\pi)^{\frac{n}{2}} r_k^n e^{-2C_k} \ . \end{aligned}$$

Plugging this into (7.4), we conclude

$$\tilde{W}(g,u,\phi_k,r_k^2) \leq C'' + r_k^2 \max_{B_k} \big(R - 2|du|^2\big) - n - 2C_k \leq C'' + r_k^2 \max_{B_k} R - 2C_k \leq C - 2C_k \stackrel{k\to\infty}{\longrightarrow} -\infty$$

where we estimated $\max R \leq n^2 \cdot r_k^{-2}$ by the initial bound on Rm. Here C is independent of k. Choosing $\tau(t) = (t_k + r_k^2) - t$, the monotonicity of μ from Lemma 5.10 shows for $k \to \infty$ that

$$\mu(g(0), u(0), t_k + r_k^2) \leq \mu(g(t_k), u(t_k), r_k^2) \leq \tilde{W}(g(t_k), u(t_k), \phi_k, r_k^2) \longrightarrow -\infty \ .$$

But $\mu(0) > -\infty$ since $g(0)$ and $u(0)$ are fixed and smooth and $t_k + r_k^2$ is bounded.

□

In the following we need a convergence result for Riemannian manifolds proven in [Ham95b]. We first state the definition of convergence of Riemannian manifolds.

Definition 7.3 *A sequence (Σ_k, g_k, x_k) of marked Riemannian manifolds converges to a marked Riemannian manifold (Σ, g, x), if there exists a sequence of compact sets K_k exhausting Σ, and a sequence of diffeomorphisms $F_k : K_k \to \Sigma_k$, such that $F_k(x) = x_k$ and such that the pullbacks $\tilde{g}_k := F_k^* g$ converge to g on every compact subset of Σ uniformly with all their derivatives.*

We state Hamilton's convergence theorem:

Theorem 7.4 *[Ham95b, Theorem 16.1] Given any sequence $\Sigma_k = (\Sigma_k, g_k, x_k)$ satisfying the bounds*

$$|\nabla^p Rm_k|_k \leq B(s,p)$$

on balls of radius k for all $p \geq 0$ with constants B_p independent of k, and such that we have uniform lower bounds on the injectivity radii $\rho(x_k)$ of the manifolds Σ_k at the origins x_k

$$\rho(x_k) \geq \delta$$

for some $\delta > 0$ independent of k, we can find a convergent subsequence.

Before we examine the injectivity radius of $g(t)$ we make the notion of noncollapse more precise.

Definition 7.5 *A metric g is said to be κ-*noncollapsed on the scale *ρ, if every metric ball $B_r(x_0)$ of radius $r < \rho$ such that* $\sup_{x \in B_r(x_0)} |Rm| \leq r^{-2}$ *has volume at least κr^n.*

Let $(g,u)(t)$ be a solution to (2.5) on $[0,T) \times M$ for closed M. Theorem 7.2 implies that $g(t)$ is κ-noncollapsed on the scale $\sqrt{T}$ for all $t \in [0,T)$ and some κ depending only on the initial data (in particular on $\sup_M |\tilde{Rm}|_0, \mathrm{inj}(\tilde{g})$), the dimension n, and T. For later use we note the following scaling property:

Lemma 7.6 *Assume a metric g is κ-noncollapsed on the scale ρ. Then the rescaled metric $h := c^2 \cdot g$ is κ-noncollapsed on the scale $c\rho$.*

□

Using the noncollapsing result, we can prove a lower bound for the injectivity radius of a solution with bounded curvature as follows:

Proposition 7.7 *Let* $(g,u)(t)$ *be a solution on a closed manifold* M *for* $t \in [0,T)$ *which is* κ*-noncollapsed for a* $\kappa > 0$*. Suppose there is a sequence of points* x_i *and times* t_i *and curvature bounds*

$$|Rm|(t_i,x) \leq CK_i \qquad \forall x \in B^{t_i}_{1/\sqrt{CK_i}}(x_i)$$

for some constant $C > 0$ *where we define* $K_i := |Rm|^2(t_i,x_i)$*. Then there exists a constant* $a = a(C,\kappa,T) > 0$ *such that the injectivity radius of the solution satisfies:*

$$\mathrm{inj}_{g(t_i)}(x_i) \geq \frac{a}{\sqrt{K_i}} \ .$$

Proof:
Suppose to the contrary that the assertion is wrong for all a and some constant $C < \infty$. Then there exists a subsequence $\big((g,u)(t_i),x_i\big)$ such that the curvature bound is satisfied, but for the injectivity radius

$$\varepsilon_i := \sqrt{CK_i} \cdot \mathrm{inj}_{g(t_i)}(x_i) \longrightarrow 0$$

holds. From Theorem 7.2 we obtain a lower bound on the volume

$$|B^{t_i}_r(x_i)| \geq \kappa r^n$$

for all $r \leq (CK_i)^{-\frac{1}{2}}$. Define $g_i := g(t_i)$. We rescale with respect to the injectivity radius, set $\delta_i := \mathrm{inj}_{g_i}(x_i)$ and work with the rescaled metrics $h_i := \delta_i^{-2} \cdot g_i$. Consider the sequence of pointed Riemannian manifolds $(B_{1/\varepsilon_i}(x_i), h_i, x_i)$ where the radius is measured by h_i. The curvature of the rescalings satisfies

$$|Rm(h_i)|_i = \big(\delta_i^{-2}\big)^{-1} \cdot |Rm(g_i)|_{g_i} \leq \delta_i^2 \cdot CK_i = \varepsilon_i^2$$

in $B_{1/\varepsilon_i}(x_i)$ where B is with respect to h_i. From Theorem 6.15 we have control on all derivatives of the curvature. Since the rescaled metrics have injectivity radius

$$\mathrm{inj}_{h_i}(x_i) = \sqrt{\delta_i^{-2}} \cdot \mathrm{inj}_{g_i}(x_i) = \delta_i^{-1} \cdot \delta_i = 1 \ ,$$

we can apply Theorem 7.4 to extract a converging subsequence of these pointed manifolds. In the limit we obtain a complete flat Riemannian manifold $(M_\infty, h_\infty, x_\infty)$. The limit is complete since all metrics h_i are complete on compact subsets of $B_{1/\varepsilon_i}(x_i)$ and $1/\varepsilon_i \to \infty$, and flat since the curvature satisfies $|Rm(h_i)|_{h_i} \leq \varepsilon_i^2 \to 0$ for $i \to \infty$. In addition the limit satisfies the volume condition

$$|B_r(x_\infty)| \geq \kappa r^n$$

for all $r < \infty$. This implies that it must be isometric to Euclidean space, contradicting the fact that $\mathrm{inj}_{h_\infty}(x_\infty) = 1$.

□