4 The monotonicity formula

An important tool to control solutions of evolution equations in general are monotone quantities. Such a monotonicity formula also exists for the flow (2.5), which we will show for closed M in this section. Although there is the monotone entropy E given by (2.1), it will turn out that this is not sufficient for all our purposes.

We want to replace the entropy E by a scaling invariant integral and to this end introduce explicitly a scale parameter τ into the formula as it is done in [Per02, §3]. One should think of τ as backwards time, measured back from some fixed time. M will be a closed Riemannian manifold for the rest of this section.

Definition 4.1 *Let $\tau \in \mathbb{R}$ be a positive real number. Then the entropy W of a configuration*

$$(g, u, f, \tau) \in \mathcal{M}(M) \times C^{\infty}(M) \times C^{\infty}(M) \times \mathbb{R}^{+}$$

is defined to be

$$W(g, u, f, \tau) := \int_M \big[\tau(S + |df|^2) + f - n\big](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV \ . \tag{4.1}$$

From this definition we see that W is scaling invariant in the following sense:

Lemma 4.2 *Let $\alpha > 0$ be a constant and φ be a diffeomorphism of M. Then the entropy W is invariant under simultaneous scaling of g and τ by α in the sense that*

$$W(\alpha g, u, f, \alpha\tau) = W(g, u, f, \tau)$$

and invariant under diffeomorphisms

$$W(\varphi^* g, \varphi^* u, \varphi^* f, \tau) = W(g, u, f, \tau) \ .$$

Proof:
This is a short computation:

$$\begin{aligned}
&W(\alpha g, u, f, \alpha\tau) \\
&\quad = \int_M [\alpha\tau\big(R(\alpha g) - 2(\alpha g)^{ij}\partial_i u \partial_j u + (\alpha g)^{ij}\partial_i f \partial_j f\big) + f - n](4\pi\alpha\tau)^{-\frac{n}{2}} e^{-f}\sqrt{\det(\alpha g)}dx \\
&\quad = \int_M [\alpha\tau\big(\alpha^{-1}R - 2\alpha^{-1}|du|^2 + \alpha^{-1}|df|^2\big) + f - n]\alpha^{-\frac{n}{2}}(4\pi\tau)^{-\frac{n}{2}} e^{-f}\alpha^{\frac{n}{2}} dV \\
&\quad = W(g, u, f, \tau) \ .
\end{aligned}$$

The invariance under diffeomorphisms is clear since we are dealing with geometric quantities. One can use coordinates induced by φ for a proof.

□

We choose a variation vector (v, w, h, σ) as we did for E and compute the first variation of W, abbreviating $(4\pi\tau)^{-\frac{n}{2}} e^{-f} dV$ by dm in the following. The first component gives:

$$\begin{aligned}
\delta W[g,u,f,\tau](v,0,0,0) &= \int_M \big(\tau[\delta R(g)](v) - 2\tau v^{ij}\partial_i u \partial_j u + \tau v^{ij}\partial_i f \partial_j f\big)\, dm \\
&\quad + \int_M \big[\tau(S + |df|^2) + f - n\big](4\pi\tau)^{-\frac{n}{2}} e^{-f}[\delta dV(g)](v) \\
&= \int_M \tau \cdot (-\Delta(\operatorname{tr} v) + \nabla_i\nabla_j v_{ij} - R_{ij}v_{ij} + 2v_{ij}\partial_i u\partial_j u - v_{ij}\partial_i f \partial_j f)\, dm \\
&\quad + \int_M \big[\tau(S + |df|^2) + f - n\big] \cdot \tfrac{\operatorname{tr} v}{2} dm \\
&= \int_M v_{ij} \cdot \{-\tau S_{ij} - \tau\nabla_i\nabla_j f\} + \tfrac{\operatorname{tr} v}{2} \cdot \{2\tau\Delta f - 2\tau|df|^2\} dm \\
&\quad + \int_M \big[\tau(S + |df|^2) + f - n\big] \cdot \tfrac{\operatorname{tr} v}{2} dm
\end{aligned}$$

where we used that by partial integration

$$\int_M (-\Delta(\operatorname{tr} v) + \nabla_i\nabla_j v_{ij})\, e^{-f} dV = \int_M \big[\operatorname{tr} v \cdot (\Delta f - |df|^2) + v_{ij}(\partial_i f \partial_j f - \nabla_i \nabla_j f)\big]\; e^{-f} dV$$

holds. The variation of u is given by

$$\begin{aligned}
\delta W[g,u,f,\tau](0,w,0,0) &= \int_M -2\tau[\delta|du|^2](w) dm = \int_M (-2\tau \cdot 2\partial_i w \partial_i u) dm \\
&= \int_M 4\tau w \cdot (\Delta u - \langle du, df\rangle) dm = \int_M 8w \cdot \{\tfrac{\tau}{2}\Delta u - \tfrac{\tau}{2}\langle du, df\rangle\}\, dm \;,
\end{aligned}$$

and for f we find

$$\begin{aligned}
&\delta W[g,u,f,\tau](0,0,h,0) \\
&\quad = \int_M \big(\tau[\delta|df|^2](h) + h\big)\, dm + \int_M \big[\tau(S + |df|^2) + f - n\big](4\pi\tau)^{-\frac{n}{2}}[\delta e^{-f}](h) dV \\
&\quad = \int_M (\tau \cdot 2\partial_i h \partial_i f + h)\, dm + \int_M \big[\tau(S + |df|^2) + f - n\big](-h) dm \\
&\quad = \int_M \big(h \cdot \{-2\tau\Delta f + 2\tau|df|^2\} + h\big)\, dm + \int_M \big[\tau(S + |df|^2) + f - n\big](-h) dm \;.
\end{aligned}$$

Varying τ, we compute

$$\begin{aligned}
\delta W[g,u,f,\tau](0,0,0,\sigma) &= \int_M \sigma\big(S + |df|^2\big) dm - \int_M \big[\tau(S + |df|^2) - f - n\big][\delta(4\pi\tau)^{-\frac{n}{2}}](\sigma)\, e^{-f} dV \\
&= \int_M \sigma \cdot \{S + |df|^2\}\, dm + \int_M \big[\tau(S + |df|^2) + f - n\big] \cdot \big(-\tfrac{n\sigma}{2\tau}\big)\, dm \;.
\end{aligned}$$

Putting this together gives

$$\begin{aligned}
\delta W[g,u,f,\tau](v,w,h,\sigma) &= \int_M v_{ij} \cdot \{-\tau S_{ij} - \tau\nabla_i\nabla_j f\} + 8w \cdot \{\tfrac{\tau}{2}\Delta u - \tfrac{\tau}{2}\langle du, df\rangle\} dm \\
&\quad + \int_M \Big[\big(\tfrac{\operatorname{tr} v}{2} - h\big) \cdot \{2\tau\Delta f - 2\tau|df|^2\} + h + \sigma \cdot \{S + |df|^2\}\Big] dm \\
&\quad + \int_M \big(\tfrac{\operatorname{tr} v}{2} - h - \tfrac{n\sigma}{2\tau}\big)\, \big[\tau(S + |df|^2) + f - n\big]\, dm \;.
\end{aligned}$$

We will think of τ as a backward time and therefore set the variation of τ to be $\sigma \equiv -1$. In the same way as for E, we can choose the variation of f such that the measure is kept fixed:

$$h := \tfrac{\operatorname{tr} v}{2} + \tfrac{n}{2\tau} \quad \Rightarrow \quad \tfrac{\operatorname{tr} v}{2} - h + \tfrac{n}{2\tau} = 0 \ .$$

Fix f and choose h as above. In the same way fix τ and choose σ as above. Considering W as a functional of g and u alone, we finally get:

$$\begin{aligned} \delta W[g,u,f,\tau](v,w) = & \int_M v_{ij} \cdot \{-\tau S_{ij} - \tau \nabla_i \nabla_j f\} + w \cdot 4\tau\{\Delta u - \langle du, df\rangle\}\, dm \\ & + \int_M \Big[\tau \cdot \frac{n}{2\tau} \underbrace{\big(2|df|^2 - 2\Delta f\big)}_{=0} + h - S - \underbrace{|df|^2}_{=\Delta f}\Big]\, dm \ . \end{aligned} \tag{4.2}$$

Since on closed M the following identity for the Laplacian and the norm squared of df is true

$$0 = (4\pi\tau)^{-\frac{n}{2}} \int_M \Delta e^{-f}\, dV = \int_M \big(|df|^2 - \Delta f\big)(4\pi\tau)^{-\frac{n}{2}} e^{-f} dV, \tag{4.3}$$

we can cancel one term in (4.2) and replace $|df|^2$ by Δf in the other. If we vary W along the variation given by the following evolution equations

$$\begin{aligned} v &:= \partial_t g := -2Sy - 2\nabla^2 f \\ w &:= \partial_t u := \Delta u - \langle du, df\rangle \\ h &:= \partial_t f := \tfrac{\operatorname{tr} v}{2} + \tfrac{n}{2\tau} = -\Delta f - S + \tfrac{n}{2\tau} \\ \sigma &:= \partial_t \tau := -1 \ , \end{aligned} \tag{4.4}$$

we calculate that

$$\partial_t W(g,u,f,\tau)(t) = \int_M \big[2\tau|Sy + \nabla^2 f|^2 + 4\tau|\Delta u - \langle du, df\rangle|^2 - 2\Delta f - 2S + \tfrac{n}{2\tau}\big]\, dm \ .$$

Since

$$2\tau|Sy + \nabla^2 f - \tfrac{1}{2\tau} g|^2 = 2\tau|Sy + \nabla^2 f|^2 + 2\tau\big(\tfrac{n}{4\tau^2} - \tfrac{1}{\tau}S - \tfrac{1}{\tau}\Delta f\big) \ ,$$

we finally conclude

$$\partial_t W(g,u,f,\tau)(t) = \int_M 2\tau|Sy + \nabla^2 f - \tfrac{1}{2\tau} g|^2 + 4\tau|\Delta u - \langle du, df\rangle|^2\, dm \ .$$

Remark 4.3 *Note that the following theorem is still true for a complete noncompact manifold Σ as long as the integrations by parts can be performed. This is possible for example by imposing falloff conditions on (g, u, f).*

So everything comes together to the following result:

Theorem 4.4 *Let M be a closed Riemannian manifold and assume that g, u, f and τ satisfy on $[0,T)\times M$ the evolution equations*

$$\begin{aligned}\partial_t g &= -2Sy\\ \partial_t u &= \Delta u\\ \partial_t f &= -\Delta f + |\nabla f|^2 - S + \tfrac{n}{2\tau}\\ \partial_t \tau &= -1\ .\end{aligned} \tag{4.5}$$

Then the following monotonicity formula holds:

$$\partial_t W(t) = \int_M \left[2\tau|Sy + \nabla^2 f - \tfrac{1}{2\tau}g|^2 + 4\tau|\Delta u - du(\nabla f)|^2\right] dm \geq 0\ . \tag{4.6}$$

In particular, the entropy W is nondecreasing and equality holds if and only if the solution is a homothetic shrinking gradient soliton. In this case $(g,u,f,\tau)(t)$ satisfies at every $t\in[0,T)$:

$$Sy + \nabla^2 f - \tfrac{1}{2\tau}g = 0 \quad \textit{and} \quad \Delta u - du(\nabla f) = 0\ .$$

Proof:
We can apply the diffeomorphisms generated by $\nabla f(t)$ to the system (4.4) in the same way as we did for (2.3). Then the result follows from the preceding calculations, considering that W is invariant under diffeomorphisms of M. Solitons have been introduced in Definition 2.2.

□

Applications for the monotonicity formula are the proofs of nonexistence of periodic shrinking solutions and of the noncollapse of solutions to (2.5) at finite times.