2 Entropy and evolution equations

In this section we introduce an entropy functional E with the nice property that its stationary points are precisely solutions to the static Einstein vacuum equations (1.15). Furthermore, we construct a parabolic flow which, in some sense, is the gradient flow to E. In the following we will work on a smooth Riemannian manifold Σ of dimension three for notational simplicity. Nevertheless all the forthcoming arguments are also valid for dimensions $n \geq 3$ with different constants depending only on n. A sketch of this is given in section 2.3.

2.1 Entropy

We consider configurations (g, u, f) where g is a smooth Riemannian metric (corresponding to the spatial part of the static Lorentz metric), u is a smooth function (such that e^{2u} corresponds to the lapse function), and f is a smooth function which can be viewed as a potential for suitable diffeomorphisms. Given such a configuration, we define:

Definition 2.1 *Let Σ be a smooth Riemannian manifold. Then the* entropy *of a configuration $(g, u, f) \in \mathcal{M}(\Sigma) \times C^\infty(\Sigma) \times C^\infty(\Sigma)$ is defined as follows:*

$$E(g, u, f) := \int_\Sigma \left(R - 2|du|^2 + |df|^2\right) e^{-f} dV \tag{2.1}$$

where R, $|\cdot|$, and dV are with respect to g.

Since we are interested in the critical points of E, we compute the Euler-Lagrange equations. Let $(v, w, h) \in \mathrm{Sym}_2(\Sigma)_0 \times C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$ be a variation of (g, u, f) with compact support. We want to compute

$$\delta E[g, u, f](v, w, h) := DE[g, u, f](v, w, h) := \frac{d}{d\varepsilon} E(g + \varepsilon v, u + \varepsilon w, f + \varepsilon h)_{|\varepsilon=0}$$

where DE is the Frechet derivative of E at the point $[g, u, f]$ in direction (v, w, h). We use the abbreviation δ for this operation in the following. The first step of the calculation is

$$\delta E[g, u, f](v, w, h) = \int_\Sigma \left(\delta R - 2\delta|du|^2 + \delta|df|^2\right) e^{-f} dV + \int_\Sigma \left(R - 2|du|^2 + |df|^2\right) \delta\left(e^{-f} dV\right) \ .$$

We compute all variations individually and find, using the variation formula for R from Lemma 1.4 and some partial integrations, that

$$\begin{aligned}
\int_\Sigma (\delta R)\, e^{-f} dV &= \int_\Sigma \left(-\Delta(\operatorname{tr} v) + \nabla_i \nabla_j v_{ij} - R_{ij} v_{ij}\right) e^{-f} dV \\
&= \int_\Sigma \left(-(\operatorname{tr} v)\Delta e^{-f} + v_{ij} \nabla_i \nabla_j e^{-f} - R_{ij} v_{ij} e^{-f}\right) dV \\
&= \int_\Sigma \left(\tfrac{\operatorname{tr} v}{2} \cdot \left\{2\Delta f - 2|df|^2\right\} + v_{ij} \cdot \left\{-R_{ij} - \nabla_i \nabla_j f + \partial_i f \partial_j f\right\}\right) e^{-f} dV \ .
\end{aligned}$$

Using Lemma 1.4 again, the variation of the second term is given by

$$\begin{aligned}\int_\Sigma (\delta|\nabla f|^2)\, e^{-f} dV &= \int_\Sigma \delta\left(g^{ij}\partial_i f \partial_j f\right)\, e^{-f} dV = \int_\Sigma (-v_{ij}\partial_i f\partial_j f + 2\partial_i f \partial_i h)\, e^{-f} dV \\ &= \int_\Sigma \left(v_{ij}\cdot\left\{-\partial_i f\partial_j f\right\} - h\cdot\left\{2\Delta f - 2|df|^2\right\}\right)\, e^{-f} dV\ .\end{aligned}$$

The third term yields in a similar way

$$\begin{aligned}\int_\Sigma -2\delta(|du|^2)\, e^{-f} dV &= \int_\Sigma (2v_{ij}\partial_i u\partial_j u - 4\partial_i u\partial_i w)\, e^{-f} dV \\ &= \int_\Sigma \left(v_{ij}\cdot 2\partial_i u\partial_j u + 8w\cdot\left\{\tfrac{1}{2}\Delta u - \tfrac{1}{2}\langle du, df\rangle\right\}\right)\, e^{-f} dV\ .\end{aligned}$$

We calculate the variation of the volume form $e^{-f} dV$ and get again from Lemma 1.4:

$$\int_\Sigma \delta\left(e^{-f} dV\right) = \int_\Sigma \left(\frac{\operatorname{tr} v}{2} - h\right)\, e^{-f} dV\ .$$

The idea now is to combine several terms in the variation of the integrand into a term where we can pull out a factor $\left(\frac{\operatorname{tr} v}{2} - h\right)$ as in the evolution of the volume form. Following an idea of Perelman in [Per02, §1], we then define the variation of f such that these terms vanish. We collect from above:

$$\begin{aligned}\delta E[g,u,f](v,w,h) &= \int_\Sigma \left(v_{ij}\cdot\left\{-R_{ij} - \nabla_i\nabla_j f + 2\partial_i u\partial_j u\right\} + 8w\cdot\left\{\tfrac{1}{2}\Delta u - \tfrac{1}{2}\langle du, df\rangle\right\}\right)\, e^{-f} dV \\ &\quad + \int_\Sigma \left(\tfrac{\operatorname{tr} v}{2} - h\right)\cdot\left\{2\Delta f - |df|^2 + R - 2|du|^2\right\} e^{-f} dV\ .\end{aligned}$$

By defining $h := (\operatorname{tr} v)/2$ (which fixes the volume form $e^{-f} dV$ and therefore couples the variation of f to the variation of g) we finally arrive at

$$\delta E[g,u,f](v,w) = \int_\Sigma \left(v_{ij}\cdot\left\{-R_{ij} - \nabla_i\nabla_j f + 2\partial_i u\partial_j u\right\} + 8w\cdot\left\{\tfrac{1}{2}\Delta u - \tfrac{1}{2}\langle du, df\rangle\right\}\right)\, e^{-f} dV \tag{2.2}$$

where E is now a functional of g and u alone since f is determined by g and u. From this expression we can extract the gradient flow equations for the pair (g,u). To this end we introduce the following weighted scalar product on the configuration space $H := \mathcal{M}(\Sigma)\times C^\infty(\Sigma)$ to be able to define the gradient of E:

$$\langle (g_1,u_1),(g_2,u_2)\rangle_H := \int_\Sigma \left(\langle g_1, g_2\rangle + 8\langle u_1, u_2\rangle\right)\, e^{-f} dV$$

where $\langle\cdot,\cdot\rangle$ denotes the pointwise Euclidean scalar product. Using the defining equation

$$DE[g,u](v,w) = \langle \operatorname{grad} E(g,u), (v,w)\rangle_H$$

for the gradient of E, we deduce the following flow equations from (2.2) and the definition of h

$$\begin{aligned}\partial_t g_{ij} &:= (\operatorname{grad}_1 E)_{ij} = -2R_{ij} + 4\partial_i u\partial_j u - 2\nabla_i\nabla_j f \\ \partial_t u &:= (\operatorname{grad}_2 E) = \Delta u - du(\nabla f) \\ \partial_t f &:= h = \tfrac{\operatorname{tr} v}{2} = -\Delta f - R + 2|du|^2\end{aligned} \tag{2.3}$$

after a multiplication by 2 for convenience. Since we aim for a system for g and u alone, we have to eliminate the terms involving the function f. A second glance at (2.3) shows that we can rewrite the equations for g and u as follows:

$$\begin{aligned}\partial_t g_{ij} &= -2R_{ij} + 4\partial_i u \partial_j u - (\mathcal{L}_{\nabla f} g)_{ij} \\ \partial_t u &= \Delta u - (\mathcal{L}_{\nabla f} u) \ .\end{aligned}$$

Therefore these terms just describe the infinitesimal action of the 1-parameter family of diffeomorphism $\Phi(t)$ generated by the gradient vector field $(\nabla f)(t)$ on g and u. Applying these diffeomorphisms yields the equivalent system

$$\begin{aligned}\partial_t g_{ij} &= -2R_{ij} + 4\partial_i u \partial_j u \\ \partial_t u &= \Delta u \\ \partial_t f &= -\Delta f + |df|^2 - R + 2|du|^2\end{aligned} \tag{2.4}$$

since $\mathcal{L}_{\nabla f} f = df(\nabla f) = |df|^2$. This follows from the general formula

$$\partial_t(\Phi^* g_{ij}) = \Phi^* \left(\partial_t g_{ij} + (\mathcal{L}_X g)_{ij}\right)$$

which in our case is given by

$$\partial_t(\Phi^* g_{ij}) = \Phi^* \left(-2R_{ij} + 4\partial_i u \partial_j u - 2\nabla_i \nabla_j f + 2\nabla_i \nabla_j f\right) = -2\Phi^* R_{ij} + 4\Phi^* \left(\partial_i u \partial_j u\right) \ .$$

Here we use the linearity of the pullback and the formula for the Lie derivative of a Riemannian metric $(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i$ with $X = \nabla f$. Since the Ricci tensor is equivariant with respect to the action of the diffeomorphism group of Σ and d commutes with pullbacks, we obtain

$$\partial_t(\Phi^* g_{ij}) = -2R_{ij}(\Phi^* g) + 4\partial_i(\Phi^* u)\partial_j(\Phi^* u).$$

This is (2.4) in the coordinates induced by $\Phi(t)$. As u and f are functions, the application of Φ to the other equations gives with a similar formula as above

$$\begin{aligned}\partial_t(\Phi^* u) &= \Phi^* \left(\partial_t u + (\mathcal{L}_X u)\right) = \Phi^* \big(\Delta u - du(\nabla f) + du(\nabla f)\big) = (\Delta u) \circ \Phi \\ \partial_t(\Phi^* f) &= (-\Delta f + |df|^2 - R + 2|du|^2) \circ \Phi \ .\end{aligned}$$

Now that we calculated the gradient flow, we can examine its stationary solutions which correspond to critical points of the entropy E. The first two equations in (2.4) imply that a stationary solution must satisfy

$$\begin{aligned}R_{ij} &= 2\partial_i u \partial_j u \\ \Delta u &= 0 \ .\end{aligned}$$

From (1.15) we know that these are the equations that characterize static Einstein vacuum metrics given by (1.16).

Note that the results of this section are in perfect agreement with the work of Perelman in [Per02] when we set $u \equiv const.$

2.2 The flow equations

Motivated by the entropy E in the last section, we study the system

$$\begin{aligned} \partial_t g &= -2Rc(g) + 4du \otimes du \\ \partial_t u &= \Delta^g u \end{aligned} \tag{2.5}$$

for a Riemannian metric $g(t)$ and a smooth function $u(t)$ where some initial data $(\tilde{g}, \tilde{u}) \in \mathcal{M}(\Sigma) \times C^\infty(\Sigma)$ is given. We want to consider manifolds Σ which either are closed or complete and noncompact. Here $du \otimes du$ is the tensor $\partial_i u \partial_j u \, dx^i \otimes dx^j \in \mathrm{Sym}_2(\Sigma)$.

In the following we switch between (2.5) and the gradient flow (2.4). This is fine since the two systems are equivalent as proven in the last section. Moreover, the entropy is invariant under diffeomorphisms. We have $E(g, u, f) = E(\Phi^* g, u \circ \Phi, f \circ \Phi)$ for all diffeomorphisms Φ of Σ. This allows us to use the variational structure on the one hand and to work with the easier to analyze system (2.5) on the other hand.

We make the following general convention: *Whenever in the following a pair $(g, u)(t)$ is called a* solution *then it will be a smooth solution to the system (2.5) for smooth initial data $(\tilde{g}, \tilde{u})$.*

For convenience we introduce the symmetric tensor field $Sy \in \mathrm{Sym}_2(\Sigma)$ and its trace $S := g^{ij} S_{ij}$:

$$S_{ij} := R_{ij} - 2\partial_i u \partial_j u \quad \text{and} \quad S := R - 2|du|^2 \; .$$

The evolution of the metric can then be written as

$$\partial_t g_{ij} = -2S_{ij} \quad \text{and} \quad \partial_t g^{ij} = 2S_{ij} \; .$$

Currently the following solutions are known to the author:

- If $u \equiv c = const$ and $g(t)$ is an arbitrary solution to Hamilton's Ricci flow, then the pair $(g(t), c)$ solves (2.5).
- An arbitrary solution (g, u) of the static Einstein vacuum equations (1.15) is automatically a stationary solution of the flow equations.

Special solutions of the system (2.5), where the geometry does not change substantially, are called soliton solutions in the spirit of the definitions for the Ricci Flow [Ham95b, §2(e)].

Definition 2.2 *A solution $(g, u)(t)$ of (2.5) on $[0, T] \times \Sigma$ is called a* soliton solution*, if it varies only along a 1-parameter family of diffeomorphisms or by scaling. It therefore satisfies*

$$\begin{aligned} \partial_t g(t) &= \mathcal{L}_{X(t)} g(t) + c(t) g(t) \\ \partial_t u(t) &= \mathcal{L}_{X(t)} u(t), \end{aligned} \tag{2.6}$$

where $X \in \mathcal{X}([0, T] \times \Sigma)$ is the generator of the diffeomorphisms and $c : [0, T] \to \mathbb{R}$ is the scaling factor, depending on time only. If $X = \nabla h$ is the gradient of a function h, the soliton is called a gradient soliton. *If $c \neq 0$, the soliton is a* homothetic *(gradient) soliton. For $c < 0, = 0, > 0$ the soliton is shrinking, steady or expanding respectively.*

Lemma 2.3 *Let $(g,u)(t)$ be a soliton solution to (2.5). Then at each time it satisfies the (in general weakly elliptic) system*

$$\begin{aligned} 2R_{ij} - 4\partial_i u \partial_j u &= -\nabla_i X_j - \nabla_j X_i - c g_{ij} \\ \Delta u &= du(X) \ . \end{aligned} \tag{2.7}$$

Vice versa, we can construct a solution to (2.6) from a solution (g,u) satisfying (2.7) and a given time dependent $X(t) \in \mathcal{X}([0,T]\times\Sigma)$.

Proof:
The relation is given as follows. A solution $(g,u)(t)$ to (2.6) satisfies at an arbitrary time t_0

$$\begin{aligned} -2Rc(t_0) + 4du(t_0)\otimes du(t_0) &= \partial_t g(t)_{|t=t_0} = \mathcal{L}_{X(t_0)} g(t_0) + c(t_0) g(t_0) \\ \Delta u(t_0) &= \partial_t u(t)_{|t=t_0} = \mathcal{L}_{X(t_0)} u(t_0), \end{aligned}$$

which is (2.7) with $g := g(t_0)$, $u := u(t_0)$, $X := X(t_0)$ and $c := c(t_0)$.

On the other hand, assume $X(t)$ is a given time dependent vector field and $(g,u,X(0))$ satisfy (2.7). Setting $g(t) := \varphi_t^*\big((1+ct)g\big)$ and $u(t) := \varphi_t^* u$, we can construct a solution to (2.6) on the time interval $[0, \frac{1}{-c})$ if $c<0$, or on $[0,\infty)$ if $c \geq 0$. Denoting by φ_t the one-parameter family of diffeomorphisms generated by $X(t)$ satisfying $\varphi_0 \equiv id$, we compute:

$$\begin{aligned} \partial_t g(t) &= \varphi_t^*\big(\partial_t[(1+ct)g] + \mathcal{L}_{X(t)}[(1+ct)g]\big) = \frac{c}{1+ct} g(t) + \mathcal{L}_{X(t)} g(t) \\ \partial_t u(t) &= \varphi_t^* u = \mathcal{L}_{X(t)} u(t), \end{aligned}$$

which is (2.6) with $c(t) = \frac{c}{1+ct}$.

$\square$

We will use both descriptions of soliton solutions in the following without further reference.

2.3 Entropy and evolution in arbitrary dimensions

Instead of specializing to dimension $n=3$, we can write down a dimension dependent entropy E_n for a Riemannian metric g on an n-dimensional manifold Σ^n and functions $u, f \in C^\infty(\Sigma^n)$:

$$E_n(g,u,f) := \int_{\Sigma^n} \left(R - \frac{n-1}{n-2}|du|^2 + |df|^2 \right) e^{-f} dV \ .$$

Repeating all calculations of the previous chapter and using the following dimension dependent weighted scalar product on the configuration space H_n:

$$\langle (g_1,u_1),(g_2,u_2)\rangle_{H_n} := \int_{\Sigma^n} \left(\langle g_1, g_2\rangle + \frac{4(n-1)}{n-2}\langle u_1,u_2\rangle \right) e^{-f} dV,$$

the gradient flow for E_n is given by:

$$\begin{aligned} \partial_t g_{ij} &= -2R_{ij} + \tfrac{2(n-1)}{n-2}\partial_i u \partial_j u \\ \partial_t u &= \Delta u \\ \partial_t f &= -\Delta f + |df|^2 + \big(R - \tfrac{n-1}{n-2}|du|^2\big) \ . \end{aligned}$$

Critical points of E_n satisfy the system (1.15) which was derived from the static Einstein vacuum equations via the conformal transformation $\tilde{g} := e^{\frac{2u}{n-2}} \cdot g$.

2.4 Warped product flows

Since (2.5) looks similar to the Ricci Flow of a warped product, this section is committed to show that there is no direct connection. Although we restrict to $n = 3$ here, the argument is the same for dimensions $n \geq 3$. At the end of the section we comment on the case of dimension $n = 2$ where the situation is different.

Consider two compact Riemannian manifolds (B^3, h) and (F^1, δ).

Given a smooth positive warping function $\psi : B^3 \to \mathbb{R}$, one can form the warped product manifold $M^4 := B^3 \times_\psi F^1$ with metric $g = h + \psi^2\delta$. The Ricci tensor of g is given by

$$Rc(g) = \begin{pmatrix} & & & 0 \\ & Rc(h) - \psi^{-1}\nabla_h^2\psi & & 0 \\ & & & 0 \\ 0 & 0 & 0 & -\psi^{-1}\Delta^h\psi \end{pmatrix}$$

where the derivatives are with respect to h. Assume that g is a solution to

$$\partial_t g = -2Rc(g)$$

on M^4. This induces the following coupled system for h and ψ on B^3:

$$\begin{aligned} \partial_t h &= -2Rc(h) + 2\psi^{-1}\nabla_h^2\psi \\ \partial_t\psi &= 2\psi^{-1}\Delta^h\psi \end{aligned} \tag{2.8}$$

which looks similar to the system (2.5). In fact, stationary solutions of this system can be interpreted as solutions to the static Einstein vacuum equations [EK62, Theorem 2-3.3]. We check if (2.8) and (2.5) are equivalent. Substituting $\omega := \beta^{-1} \cdot \ln\psi$ with a constant β to be determined, we get

$$\begin{aligned} \partial_t h &= -2Rc(h) + 2\beta\nabla_h^2\omega + 2\beta^2 d\omega \otimes d\omega \\ \partial_t\omega &= 2e^{-\beta\omega}\big(\Delta^h\omega + \beta|d\omega|_h^2\big), \end{aligned}$$

considering that ω is time dependent. As in the elliptic case (1.14), we try to remove the Hessian of ω by a conformal transformation. For a second constant α, we make the ansatz $\tilde{h} := e^{2\alpha\omega} \cdot h$ and get

$$\begin{aligned} \partial_t\tilde{h} = \partial_t\big(e^{2\alpha\omega} \cdot h\big) &= e^{2\alpha\omega}\partial_t h + 2\alpha e^{2\alpha\omega}\partial_t\omega \cdot h \\ &= e^{2\alpha\omega}\big(-2Rc(\tilde{h}) + (2\beta - 2\alpha)\nabla_h^2\omega + (2\alpha^2 + 2\beta^2)d\omega \otimes d\omega \\ &\quad + (4\alpha e^{-\beta\omega} - 2\alpha)\Delta^h\omega + (4\alpha\beta e^{-\beta\omega} - 2\alpha^2)|d\omega|_h^2 \cdot h\big) \ . \end{aligned}$$

We are forced to set $\alpha = \beta$ to eliminate the Hessian term. However, we are unable to get rid of the last line since its vanishing would imply $\alpha = \ln 2 \cdot \omega$ and that ω would have to be constant. The equation for ω is given by

$$\partial_t\omega = 2e^{-\alpha\omega}\big(\Delta^h\omega + \alpha|d\omega|_h^2\big) = 2e^{-\alpha\omega}\big(e^{2\alpha\omega}\Delta^{\tilde{h}}\omega - e^{-\alpha\omega}h(\nabla e^{\alpha\omega}, \nabla\omega) + \alpha|d\omega|_h^2\big) = 2e^{\alpha\omega}\Delta^{\tilde{h}}\omega$$

and we end up with the system

$$\begin{aligned}
\partial_t \tilde{h} &= -2e^{2\alpha\omega}\big(Rc(\tilde{h}) - 2\alpha^2 d\omega \otimes d\omega\big) + e^{2\alpha\omega}(4\alpha e^{-\alpha\omega} - 2\alpha)(\Delta^h \omega + \alpha |d\omega|_h^2) \cdot h \\
\partial_t \omega &= 2e^{\alpha\omega}\Delta^{\tilde{h}}\omega \ .
\end{aligned}$$

This is different from (2.5). Clearly the time dependence of the conformal transformation does not allow the manipulation we did in the static case.

In dimension $n = 2$ the conformal transformation $\tilde{h} := e^{\frac{2}{n-2}\omega} \cdot h$ turning the Hessian term into the squared gradient term does not exist. Therefore we cannot use this idea to show equivalence of (2.5) and (2.8).

Since (2.8) is a Ricci flow though, it is possible to obtain an entropy for this system from Perelman's construction in dimension $n = 4$.

2.5 Evolution equations

The aim of the next two sections is to provide the evolution equations of all tensors related to the solution $(g, u)(t)$ which we are going to need later on. We begin with the scalar curvature:

Lemma 2.4 *Let $(g, u)(t)$ be a solution to (2.5). Then the scalar curvature of the evolving metric $g(t)$ satisfies the evolution equation*

$$\partial_t R = \Delta R + 2|Rc|^2 + 4|\Delta u|^2 - 4|\nabla^2 u|^2 - 8\langle Rc, du \otimes du\rangle \ .$$

Proof:
We use the variation formula from Lemma 1.4 to compute in normal coordinates

$$\begin{aligned}
\partial_t R &= -\Delta(-2R + 4|du|^2) + \nabla_i\nabla_j(-2R_{ij} + 4\partial_i u \partial_j u) - R_{ij}(-2R_{ij} + 4\partial_i u\partial_j u) \\
&= 2\Delta R - 4\nabla_i\nabla_i(\partial_j u \partial_j u) - 2\nabla_i\nabla_j R_{ij} + 4\nabla_i(\nabla_j\nabla_i u \partial_j u + \partial_i u \nabla_j\nabla_j u) \\
&\quad + 2R_{ij}R_{ij} - 4R_{ij}\partial_i u \partial_j u \ .
\end{aligned}$$

The contracted second Bianchi identity (1.10) implies $-\nabla_i(2\nabla_j R_{ij}) = -\nabla_i(\nabla_i R) = -\Delta R$. In addition, we can commute the second derivatives of u from (1.7) and find:

$$\begin{aligned}
\partial_t R &= 2\Delta R - 8(\nabla_i\nabla_i\nabla_j u\partial_j u + \nabla_i\nabla_j u\nabla_i\nabla_j u) - \Delta R + 4\nabla_i\nabla_i\nabla_j u\partial_j u \\
&\qquad + 4\nabla_j\nabla_i u\nabla_i\nabla_j u + 4\Delta u\Delta u + 4\partial_i u(\partial_i\Delta u) + 2|Rc|^2 - 4R_{ij}\partial_i u\partial_j u \\
&= \Delta R - 4(\partial_i\Delta u)\partial_i u - 4R_{jijp}\partial_p u\partial_i u - 4\nabla_i\nabla_j u\nabla_i\nabla_j u + 4|\Delta u|^2 + 4\partial_i u(\partial_i\Delta u) \\
&\qquad + 2|Rc|^2 - 4R_{ij}\partial_i u\partial_j u
\end{aligned}$$

where we interchanged covariant derivatives using (1.6). Fortunately the third order terms in u cancel. Remembering $R_{jijp} = R_{ip}$ from (1.5), we get the desired result.

□

We continue with the Ricci curvature:

Lemma 2.5 *Let $(g, u)(t)$ be a solution to (2.5). Then the Ricci tensor with respect to $g(t)$ evolves according to*

$$\partial_t R_{ij} = \Delta R_{ij} - 2R_{ip}R_{jp} + 2R_{piqj}\big(R_{pq} - 2\partial_p u\partial_q u\big) + 4\Delta u\nabla_i\nabla_j u - 4\nabla_p\nabla_i u\nabla_p\nabla_j u \ .$$

Proof:
We use the variation formula

$$\partial_t R_{ij} = -\tfrac{1}{2}\Delta v_{ij} - \tfrac{1}{2}\nabla_i\nabla_j v + \tfrac{1}{2}g^{pq}(\nabla_p\nabla_j v_{iq} + \nabla_p\nabla_i v_{jq})$$

for the Ricci tensor from Lemma 1.4 and compute:

$$-\tfrac{1}{2}\Delta(\partial_t g_{ij}) = \Delta R_{ij} - 2\Delta(\partial_i u\partial_j u) = \Delta R_{ij} - 2(\Delta\nabla_i u)\partial_j u - 2\partial_i u(\Delta\nabla_j u) - 4\nabla_p\nabla_i u\nabla_p\nabla_j u \ .$$

The second term gives

$$-\tfrac{1}{2}\nabla_i\nabla_j(g^{pq}\partial_t g_{pq}) = \nabla_i\nabla_j R - 2\nabla_i\nabla_j(\partial_p u\partial_p u) = \nabla_i\nabla_j R - 4\nabla_i\nabla_j\nabla_p u\partial_p u - 4\nabla_p\nabla_i u\nabla_p\nabla_j u \ .$$

For the next term we need the Bianchi identity (1.10) to find

$$\begin{aligned}
\tfrac{1}{2}g^{pq}\nabla_p\nabla_j(\partial_t g_{iq}) &= -\nabla_p\nabla_j R_{ip} + 2\nabla_p(\nabla_j\nabla_i u\partial_p u + \partial_i u\nabla_j\nabla_p u)\\
&= (-\nabla_j\nabla_p R_{ip} - R_{pjiq}R_{qp} - R_{pjpq}R_{iq}) + 2\nabla_p\nabla_i\nabla_j u\partial_p u + 2\nabla_i\nabla_j u\Delta u\\
&\quad + 2\nabla_p\nabla_i u\nabla_p\nabla_j u + 2\partial_i u\nabla_p\nabla_p\nabla_j u\\
&= -\tfrac{1}{2}\nabla_j\nabla_i R - R_{pijq}R_{pq} - R_{jq}R_{iq} + (2\nabla_i\nabla_j\nabla_p u\partial_p u + 2R_{pijq}\partial_q u\partial_p u)\\
&\quad + 2\Delta u\nabla_i\nabla_j u + 2\nabla_p\nabla_i u\nabla_p\nabla_j u + 2\partial_i u(\Delta\nabla_j u)
\end{aligned}$$

where we also used (1.6). The same works for the fourth term

$$\begin{aligned}
\tfrac{1}{2}g^{pq}\nabla_p\nabla_i(\partial_t g_{jp}) &= -\tfrac{1}{2}\nabla_i\nabla_j R - R_{pjiq}R_{pq} - R_{iq}R_{jq} + (2\nabla_j\nabla_i\nabla_p u\partial_p u + 2R_{pjiq}\partial_q u\partial_p u)\\
&\quad + 2\Delta u\nabla_j\nabla_i u + 2\nabla_p\nabla_j u\nabla_p\nabla_i u + 2\partial_j u(\Delta\nabla_i u) \ .
\end{aligned}$$

All but the following terms cancel:

$$\partial_t R_{ij} = \Delta R_{ij} - 2R_{ip}R_{jp} - 2R_{pijq}R_{pq} + 4\Delta u\nabla_i\nabla_j u - 4\nabla_p\nabla_i u\nabla_p\nabla_j u + 4R_{pijq}\partial_p u\partial_q u \ .$$

This proves the Lemma.

$\square$

We also compute the evolution of the Christoffel symbols:

Lemma 2.6 *Let $(g, u)(t)$ be a solution to (2.5). Then the Christoffel symbols associated to $g(t)$ in a coordinate chart evolve according to*

$$\partial_t\Gamma^k_{ij} = g^{kl}\left(-\nabla_i R_{jl} - \nabla_j R_{il} + \nabla_l R_{ij} + 4\nabla_i\nabla_j u\partial_l u\right) \ . \tag{2.9}$$

Proof:
In normal coordinates the definition (1.3) and the variation formula for Γ from Lemma 1.4 yield:

$$\begin{aligned}
\partial_t\Gamma^k_{ij} &= \tfrac{1}{2}g^{kl}\left(\partial_i\partial_t g_{jl} + \partial_j\partial_t g_{il} - \partial_l\partial_t g_{ij}\right)\\
&= \tfrac{1}{2}g^{kl}\big(\partial_i\left\{-2R_{jl} + 4\partial_j u\partial_l u\right\} + \partial_j\left\{-2R_{il} + 4\partial_i u\partial_l u\right\} - \partial_l\left\{-2R_{ij} + 4\partial_i u\partial_j u\right\}\big)\\
&= g^{kl}\big(-\nabla_i R_{jl} + 2\nabla_i\nabla_j u\partial_l u + 2\partial_j u\nabla_i\nabla_l u - \nabla_j R_{il} + 2\nabla_j\nabla_i u\partial_l u + 2\partial_i u\nabla_j\nabla_l u + \nabla_l R_{ij}\\
&\qquad - 2\nabla_l\nabla_i u\partial_j u - 2\partial_i u\nabla_l\nabla_j u\big)\\
&= g^{kl}\big(-\nabla_i R_{jl} - \nabla_j R_{il} + \nabla_l R_{ij} + 4\nabla_i\nabla_j u\partial_l u\big)\ .
\end{aligned}$$

In the last step, several terms cancel because of (1.7).

□

The next thing to look after is the evolution of the norm of the derivative of u. This evolution equation will turn out to be very important in the subsequent considerations.

Lemma 2.7 *Suppose $(g,u)(t)$ is a solution to (2.5). Then the following evolution equations hold for the derivative of u:*

$$\partial_t\partial_i u = \Delta\partial_i u - R_{ip}\partial_p u \tag{2.10}$$

$$\partial_t|du|^2 = \Delta|du|^2 - 2|\nabla^2 u|^2 - 4|du|^4\ . \tag{2.11}$$

In particular, the evolution of the norm does not depend on the curvature of g.

Proof:
Since u is a function, we can switch the time and space derivative and get

$$\partial_t\partial_i u = \partial_i\partial_t u = \nabla_i\nabla_p\nabla_p u = \nabla_p\nabla_p\nabla_i u + R_{ippq}\partial_q u = \Delta\partial_i u - R_{ip}\partial_p u\ .$$

Knowing this, we calculate

$$\begin{aligned}
\partial_t|du|^2 &= (\partial_t g^{pq})\partial_p u\partial_q u + 2(\partial_t\partial_p u)\partial_p u = -(\partial_t g_{pq})\partial_p u\partial_q u + 2(\Delta\partial_p u)\partial_p u - 2R_{pq}\partial_p u\partial_q u\\
&= 2R_{pq}\partial_p u\partial_q u - 4\partial_p u\partial_q u\partial_p u\partial_q u + (\Delta|du|^2 - 2|\nabla^2 u|^2) - 2R_{pq}\partial_p u\partial_q u\\
&= \Delta|du|^2 - 2|\nabla^2 u|^2 - 4|du|^4\ .
\end{aligned}$$

Here we used that

$$\Delta|du|^2 = \nabla_q(2\nabla_q\nabla_p u\partial_p u) = 2(\Delta\partial_p u)\partial_p u + 2|\nabla^2 u|^2\ .$$

□

As an application for (2.11), we can prove the equivalence of the metrics $g(t)$ on closed manifolds M as long as the curvature stays bounded.

Lemma 2.8 *Let $(g,u)(t)$ be a solution to (2.5) on $[0,T)\times M$ for closed M with initial data $(\tilde g,\tilde u)$. Define $c_0 := \sup_M|d\tilde u|_0^2$. Assume furthermore that the curvature satisfies $|Rm|^2(t)\le k_0$ for $t\in[0,T)$. Then all metrics $g(t)$ are equivalent and we can estimate*

$$e^{-CT}\tilde g(x) \le g(t,x) \le e^{CT}\tilde g(x) \qquad \forall (t,x)\in[0,T)\times M$$

for a constant $C = C(n,k_0,c_0)$ depending only on n,k_0,c_0. In fact $C = 2n\sqrt{k_0} + 4c_0$.

Proof:
The length of a time independent vector field $V \in \mathcal{X}(M)$, measured by $g(t)$, satisfies the evolution equation

$$\partial_t |V|^2 = \partial_t g(V,V) = -2Sy(V,V) \ .$$

From (2.11) we get $|du|^2(t) \leq \sup_M |d\tilde{u}|_0^2 = c_0$ for all $t \in [0,T)$ by the maximum principle since M is closed. This implies the bound

$$|Sy| = |Rc - 2du \otimes du| \leq |Rc| + 2|du|^2 \leq n|Rm| + 2|du|^2 \leq n\sqrt{k_0} + 2c_0 \ ,$$

and we can estimate

$$|\partial_t g(V,V)| \leq 2|Sy(V,V)| \leq 2|Sy||V|^2 \leq 2n\sqrt{k_0} + 4c_0 g(V,V) \ .$$

Using Gronwall's inequality, this implies

$$e^{-CT}\tilde{g}(V,V) \leq e^{-Ct}\tilde{g}(V,V) \leq g(t)(V,V) \leq e^{Ct}\tilde{g}(V,V) \leq e^{CT}\tilde{g}(V,V) \ .$$

Therefore $g(t)(V,V)$ is uniformly bounded from above and below on $[0,T)$, and all metrics $g(t)$ are equivalent.

$\square$

We continue with the computation of the evolution of the Hessian of u.

Lemma 2.9 *Suppose $(g,u)(t)$ is a solution to (2.5). Then the Hessian of $u(t)$ satisfies the evolution equation*

$$\partial_t(\nabla_i\nabla_j u) = \Delta(\nabla_i\nabla_j u) + 2R_{ipjq}\nabla_p\nabla_q u - R_{ip}\nabla_j\nabla_p u - R_{jp}\nabla_i\nabla_p u - 4|du|^2\nabla_i\nabla_j u \ . \quad (2.12)$$

Proof:
We calculate

$$\partial_t(\nabla_i\nabla_j u) = \partial_t\big(\partial_i\partial_j u - \Gamma_{ij}^k\partial_k u\big) = \nabla_i\nabla_j\big(\partial_t u\big) - \big(\partial_t\Gamma_{ij}^k\big)\partial_k u$$

and consider the two terms separately. The first comes down to

$$\begin{aligned}
\nabla_i\nabla_j\big(\partial_t u\big) &= \nabla_i\nabla_j\nabla_k\nabla_k u = \nabla_i\big(\nabla_k\nabla_j\nabla_k u + R_{jkkp}\partial_p u\big)\\
&= \nabla_k\nabla_i\nabla_k\nabla_j u + R_{ikjp}\nabla_p\nabla_k u + R_{ikkp}\nabla_j\nabla_p u - \nabla_i R_{jp}\partial_p u - R_{jp}\nabla_i\nabla_p u\\
&= \nabla_k\nabla_k\nabla_i\nabla_j u + \nabla_k R_{ikjp}\partial_p u + R_{ikjp}\nabla_k\nabla_p u + R_{ikjp}\nabla_p\nabla_k u - R_{ip}\nabla_j\nabla_p u\\
&\quad - \nabla_i R_{jp}\partial_p u - R_{jp}\nabla_i\nabla_p u\\
&= \Delta(\nabla_i\nabla_j u) + \nabla_k R_{jpik}\partial_p u + 2R_{ikjp}\nabla_k\nabla_p u - R_{ip}\nabla_j\nabla_p u - \nabla_i R_{jp}\partial_p u - R_{jp}\nabla_i\nabla_p u
\end{aligned}$$

using (1.6), (1.5) and the symmetry of the Riemann tensor. We get for the second term:

$$\begin{aligned}
-\big(\partial_t\Gamma_{ij}^k\big)\partial_k u &= g^{kl}\big(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij} - 4\nabla_i\nabla_j u\partial_l u\big)\partial_k u\\
&= \nabla_i R_{jk}\partial_k u + \nabla_j R_{ik}\partial_k u - \nabla_k R_{ij}\partial_k u - 4\nabla_i\nabla_j u|du|^2 \ .
\end{aligned}$$

After some rearranging we obtain altogether

$$\begin{aligned}
\partial_t(\nabla_i\nabla_j u) &= \Delta(\nabla_i\nabla_j u) + 2R_{ipjq}\nabla_p\nabla_q u - R_{ip}\nabla_j\nabla_p u - R_{jp}\nabla_i\nabla_p u - 4\nabla_i\nabla_j u|du|^2\\
&\quad + \big(-\nabla_k R_{jpki} + \nabla_j R_{ip} - \nabla_p R_{ij}\big)\partial_p u
\end{aligned}$$

since two terms already cancel. Finally, due to the contracted Bianchi identity (1.9) the last line vanishes. This proves the claim.

□

To compute the evolution of Sy, we also need the equation for $du \otimes du$.

Lemma 2.10 *Let $(g, u)(t)$ be a solution to (2.5). Then the symmetric tensor $(du \otimes du)(t)$ satisfies the evolution equation*

$$\partial_t(\partial_i u \partial_j u) = \Delta(\partial_i u \partial_j u) - 2\nabla_p \nabla_i u \nabla_p \nabla_j u - R_{ip}\partial_j u \partial_p u - R_{jp}\partial_i u \partial_p u \ .$$

Proof:
Since

$$\Delta(\partial_i u \partial_j u) = (\Delta \partial_i u)\partial_j u + 2\nabla_p \nabla_i u \nabla_p \nabla_j u + \partial_i u(\Delta \partial_j u)$$

holds, we can use (2.10) to prove

$$\begin{aligned}\partial_t(\partial_i u \partial_j u) &= (\partial_t \partial_i u)\partial_j u + \partial_i u(\partial_t \partial_j u) = (\Delta \partial_i u + R_{ippq}\partial_q u)\partial_j u + \partial_i u(\Delta \partial_j u + R_{jppq}\partial_q u) \\ &= \Delta(\partial_i u \partial_j u) - 2\nabla_p \nabla_i u \nabla_p \nabla_j u - R_{ip}\partial_i u \partial_p u - R_{jp}\partial_j u \partial_p u\end{aligned}$$

as required.

□

We can now compute the evolution of Sy:

Lemma 2.11 *Let $(g, u)(t)$ be a solution of (2.5). Then the tensor Sy and its trace S evolve according to*

$$\partial_t S_{ij} = \Delta S_{ij} - R_{ip}S_{jp} - R_{jp}S_{ip} - 2R_{pijq}S_{pq} + 4\Delta u \nabla_i \nabla_j u \tag{2.13}$$

$$\partial_t S = \Delta S + 2|Sy|^2 + 4|\Delta u|^2 \ . \tag{2.14}$$

Proof:
We combine Lemma 2.5 and Lemma 2.10 for the two parts of Sy to get

$$\begin{aligned}\partial_t S_{ij} &= \Delta R_{ij} - 2R_{ip}R_{jp} - 2R_{pijq}R_{pq} + 4\Delta u \nabla_i \nabla_j u - 4\nabla_p \nabla_i u \nabla_p \nabla_j u + 4R_{pijq}\partial_p u \partial_q u \\ &\quad - 2\Delta(\partial_i u \partial_j u) + 4\nabla_p \nabla_i u \nabla_p \nabla_j u + 2R_{ip}\partial_j u \partial_p u + 2R_{jp}\partial_i u \partial_p u \\ &= \Delta(R_{ij} - 2\partial_i u \partial_j u) + 4\Delta u \nabla_i \nabla_j u - R_{ip}(R_{jp} - 2\partial_j u \partial_p u) - R_{jp}(R_{ip} - 2\partial_i u \partial_p u) \\ &\quad - 2R_{pijq}(R_{pq} - 2\partial_p u \partial_q u) \ .\end{aligned}$$

Taking the trace of (2.13) with the inverse of the evolving metric, (2.14) follows.

□

This implies the following result on closed manifolds:

Lemma 2.12 *Let $(g, u)(t)$ be a solution of (2.5) on a closed manifold M. If $S \geq 0$ holds at $t = 0$ then also for all $t > 0$ as long as the solution exists.*

Proof:
Since S satisfies the evolution equation

$$(\partial_t - \Delta)S = 2|Sy|^2 + 4|\Delta u|^2 \geq 0 \ ,$$

an application of the maximum principle shows that $S(t)$ is bounded from below by its initial value for all $t > 0$ as long as it exists.

□

2.6 Monotonicity of the entropy

We prove in this section that the entropy $E(t)$ of a solution is increasing in time. Furthermore, a solution with constant entropy must be a special geometry as described in Definition 2.2. Note that the following lemma is still true on a complete manifold Σ as long as the integration by parts can be justified.

Lemma 2.13 *Let $(g, u, f)(t)$ be a solution to (2.3) for $t \in [0, T)$ on a a closed manifold M. Then the evolution of the entropy is given by*

$$\partial_t E(g, u, f)(t) = 2 \int_M \left(|Sy + \nabla^2 f|^2 + 2|\Delta u - du(\nabla f)|^2\right) e^{-f} dV \geq 0 \ .$$

In particular the entropy is nondecreasing. Equality holds if and only if the solution is a gradient soliton. In this case $(g, u, f)(t)$ satisfies at all times t:

$$Sy + \nabla^2 f = 0 \quad \textit{and} \quad \Delta u - du(\nabla f) = 0 \ .$$

Proof:
Using the variation formulas from Lemma 1.4, we compute

$$\begin{aligned} \partial_t E(t) &= \int_M \left(\partial_t R + \partial_t |\nabla f|^2 - 2\partial_t |du|^2\right) e^{-f} dV + \int_M \left(R + |\nabla f|^2 - 2|du|^2\right) \underbrace{\partial_t (\, e^{-f} dV)}_{=0} \\ &= \int_M \big(-\Delta\left[g^{ij}\partial_t g_{ij}\right] + \nabla_i \nabla_j (\partial_t g_{ij}) - R_{ij}(\partial_t g_{ij}) - \partial_t g_{ij} \partial_i f \partial_j f + 2\nabla_i (\partial_t f) \partial_i f \\ &\qquad + 2(\partial_t g_{ij}) \partial_i u \partial_j u - 4\nabla_i (\partial_t u) \partial_i u\big) \, e^{-f} dV \ . \end{aligned}$$

Integration by parts gives

$$\begin{aligned} \partial_t E(t) = \int_M \big(&-(g^{ij}\partial_t g_{ij})\Delta e^{-f} + \partial_t g_{ij} \nabla_i \nabla_j e^{-f} - R_{ij}(\partial_t g_{ij}) e^{-f} - \partial_t g_{ij} \partial_i f \partial_j f e^{-f} \\ &- 2\partial_t f(\Delta f e^{-f} + \partial_i f \partial_i e^{-f}) + 2(\partial_t g_{ij})\partial_i u \partial_j u e^{-f} + 4\partial_t u(\Delta u e^{-f} + \partial_i u \partial_i e^{-f})\big) dV. \end{aligned}$$

Inserting the evolution equations (2.3) yields

$$\begin{aligned}\partial_t E(t) = \int_M \Big[& 2(S+\Delta f)(|df|^2 - \Delta f) - 2\langle Sy, df\otimes df\rangle - 2\langle \nabla^2 f, df\otimes df\rangle + 2\langle Sy, \nabla^2 f\rangle \\ & + 2|\nabla^2 f|^2 + 2|Rc|^2 + 2\langle Rc, \nabla^2 f\rangle - 4\langle Rc, du\otimes du\rangle + 2\langle Sy, df\otimes df\rangle \\ & + 2\langle \nabla^2 f, df\otimes df\rangle + 2(\Delta f + S)(\Delta f - |df|^2) - 4\langle Rc, du\otimes du\rangle - 4\langle \nabla^2 f, du\otimes du\rangle \\ & + 8|du\otimes du|^2 + 4|\Delta u - du(\nabla f)|^2 \Big] e^{-f} dV \\ = \int_M \Big[& 2|Rc|^2 + 2|\nabla^2 f|^2 + 2|2du\otimes du|^2 + 4|\Delta u - du(\nabla f)|^2 \\ & + 4\langle Rc, \nabla^2 f\rangle - 4\langle Rc, 2du\otimes du\rangle - 4\langle \nabla^2 f, 2du\otimes du\rangle \Big] e^{-f} dV \\ = 2\int_M & \left[|Rc + \nabla^2 f - 2du\otimes du|^2 + 2|\Delta u - du(\nabla f)|^2 \right] e^{-f} dV \ .\end{aligned}$$

This is what we wanted to prove.

□

2.7 Evolution of the curvature tensor

We take upon ourselves the task to compute the evolution of the Riemann tensor. Since the calculation using Lemma 1.4 is lengthy, we rather use equation (1.4) and Lemma 2.6.

Lemma 2.14 *Let $(g,u)(t)$ be a solution to (2.5). Then the evolution of the curvature tensor as a $(1,3)$-tensor is given by*

$$\begin{aligned}\partial_t R_{ijl}^k = g^{kr}\big(& \nabla_i\nabla_r R_{jl} - \nabla_i\nabla_l R_{jr} - \nabla_j\nabla_r R_{il} + \nabla_j\nabla_l R_{ir} + 4\nabla_i\nabla_r u\nabla_j\nabla_l u - 4\nabla_i\nabla_l u\nabla_j\nabla_r u \\ & - g^{pq}(R_{ijlp}R_{qr} - R_{ijrp}R_{lq} + 4R_{ijlp}\partial_q u\partial_r u)\big) \ ,\end{aligned} \tag{2.15}$$

and as a $(0,4)$-tensor by

$$\begin{aligned}\partial_t R_{ijkl} = \ & \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) - R_{ijpl}R_{kp} + R_{ijkp}R_{lp} + R_{pjkl}R_{pi} + R_{ipkl}R_{pj} \\ & + 4(\nabla_i\nabla_k u\nabla_j\nabla_l u - \nabla_i\nabla_l u\nabla_j\nabla_k u) \ .\end{aligned} \tag{2.16}$$

Here the tensor $B = \{B_{ijkl}\}$ is defined as $B_{ijkl} := g^{pr}g^{qs}R_{piqj}R_{rksl}$. For its properties we refer to [Ham82, §7] where it was first introduced.

Proof:
From (1.4) we find that in normal coordinates

$$R_{ijl}^k = \partial_i\Gamma_{jl}^k - \partial_j\Gamma_{il}^k$$

holds. Using the evolution equation for the Christoffel symbols in Lemma 2.6, we compute

$$\begin{aligned}
\partial_t R_{ijl}^k &= g^{kr}\partial_i\big(-\nabla_j R_{lr} - \nabla_l R_{jr} + \nabla_r R_{jl} + 4\nabla_j\nabla_l u\partial_r u\big) \\
&\quad - g^{kr}\partial_j\big(-\nabla_i R_{lr} - \nabla_l R_{ir} + \nabla_r R_{il} + 4\nabla_i\nabla_l u\partial_r u\big) \\
&= g^{kr}\big(-\nabla_i\nabla_j R_{lr} - \nabla_i\nabla_l R_{jr} + \nabla_i\nabla_r R_{jl} + 4\nabla_i\nabla_j\nabla_l u\partial_r u + 4\nabla_j\nabla_l u\nabla_i\nabla_r u \\
&\qquad + \nabla_j\nabla_i R_{lr} + \nabla_j\nabla_l R_{ir} - \nabla_j\nabla_r R_{il} - 4\nabla_j\nabla_i\nabla_l u\partial_r u - 4\nabla_i\nabla_l u\nabla_j\nabla_r u\big)\ .
\end{aligned}$$

This is (2.15), taking into account that (1.6) implies

$$\begin{aligned}
\big(\nabla_i\nabla_j - \nabla_j\nabla_i\big)R_{lr} &= R_{ijlp}R_{pr} + R_{ijrp}R_{lp} \\
\big(\nabla_i\nabla_j - \nabla_j\nabla_i\big)(\partial_l u) &= R_{ijlp}\partial_p u\ .
\end{aligned}$$

To prove the equation for the $(0,4)$-Riemann tensor, we contract with the metric to find

$$\begin{aligned}
\partial_t R_{ijkl} = \partial_t(g_{kp}R_{ijl}^p) &= (\partial_t g_{kp})R_{ijl}^p + g_{kp}\partial_t R_{ijl}^p = -2R_{kp}R_{ijl}^p + 4\partial_k u\partial_p u R_{ijl}^p \\
&\quad + (\nabla_i\nabla_k R_{jl} - \nabla_i\nabla_l R_{jk} - \nabla_j\nabla_k R_{il} + \nabla_j\nabla_l R_{ik} - R_{ijlp}R_{pk} - R_{ijkp}R_{lp}) \\
&\quad + 4(R_{ijlp}\partial_p u\partial_k u + \nabla_i\nabla_k u\nabla_j\nabla_l u - \nabla_i\nabla_l u\nabla_j\nabla_k u) \\
&= \nabla_i\nabla_k R_{jl} - \nabla_i\nabla_l R_{jk} - \nabla_j\nabla_k R_{il} + \nabla_j\nabla_l R_{ik} - R_{ijpl}R_{kp} - R_{ijkp}R_{lp} \\
&\quad + 4(\nabla_i\nabla_k u\nabla_j\nabla_l u - \nabla_i\nabla_l u\nabla_j\nabla_k u)
\end{aligned}$$

since we have the equalities

$$\begin{aligned}
4R_{ijl}^p\partial_k u\partial_p u &= 4R_{ijpl}\partial_k u\partial_p u = -4R_{ijlp}\partial_k u\partial_p u \\
-R_{kp}R_{ijl}^p &= -R_{ijpl}R_{kp} = R_{ijlp}R_{pk}\ .
\end{aligned}$$

Using the identity

$$\begin{aligned}
&\nabla_i\nabla_k R_{jl} - \nabla_i\nabla_l R_{jk} - \nabla_j\nabla_k R_{il} + \nabla_j\nabla_l R_{ik} \\
&\qquad = \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) - R_{pjkl}R_{pi} - R_{ipkl}R_{pj}
\end{aligned}$$

from [Ham82, Lemma 7.2] which is valid for an arbitrary Riemannian metric, the desired conclusion follows. It should be noted that the right hand side of (2.16) only contains terms quadratic in the curvature and the Hessian of u. This fact will be useful later.

□

Using the formula in this lemma, we we can deduce an evolution equation for the norm of the curvature tensor as defined in (1.2).

Lemma 2.15 *Let $(g,u)(t)$ be a solution of (2.5). Then the norm of the Riemann tensor satisfies the evolution inequality:*

$$\partial_t|Rm|^2 \le \Delta|Rm|^2 - 2|\nabla Rm|^2 + C|Rm|^3 + C|Rm||\nabla^2 u|^2 + C|du|^2|Rm|^2 \tag{2.17}$$

where C is a numerical constant depending only on the dimension.

Proof:
We will use the notation $A * B$ introduced earlier. Thus we can express (2.16) as

$$\partial_t R_{ijkl} = \Delta R_{ijkl} + Rm * Rm + \nabla^2 u * \nabla^2 u \ .$$

A computation in normal coordinates yields:

$$\partial_t |Rm|^2 = \big(\partial_t g^{ac}\big) R_{abij} R_{cbij} + \ldots + \big(\partial_t g^{jl}\big) R_{abij} R_{abil} + 2\big(\partial_t R_{abij}\big) R_{abij} \ .$$

Note that we replaced g by δ as soon as it is not differentiated and used the symmetry properties of Rm. The first term computes to

$$\begin{aligned}\big(\partial_t g^{ac}\big) R_{abij} R_{cbij} &= \big(2R_{ac} - 4\partial_a u \partial_c u\big) R_{abij} R_{cbij} = Rc * Rm * Rm + du * du * Rm * Rm \\ &\leq C|Rm|^3 + C|du|^2 |Rm|^2\end{aligned}$$

by the Cauchy-Schwarz inequality. The next three terms can be estimated analogously. The last one comes down to

$$\begin{aligned}2\big(\partial_t R_{abij}\big) R_{abij} &= 2\big(\Delta R_{abij}\big) R_{abij} + 2\big(Rm * Rm\big)_{abij} R_{abij} + 2\big(\nabla^2 u * \nabla^2 u\big)_{abij} R_{abij} \\ &= \Delta |Rm|^2 - 2|\nabla Rm|^2 + Rm * Rm * Rm + \nabla^2 u * \nabla^2 u * Rm \\ &\leq \Delta |Rm|^2 - 2|\nabla Rm|^2 + C|Rm|^3 + C|Rm||\nabla^2 u|^2 \ .\end{aligned}$$

Together these inequalities imply (2.17).

□

Doing a similar calculation for the Hessian of u we find:

Lemma 2.16 *Let $(g, u)(t)$ be a solution to (2.5). Then the norm of the Hessian of u satisfies the inequality*

$$\partial_t |\nabla^2 u|^2 \leq \Delta |\nabla^2 u|^2 - 2|\nabla^3 u|^2 + C|Rm||\nabla^2 u|^2 + C|du|^2 |\nabla^2 u|^2 \ . \tag{2.18}$$

Proof:
The proof is similar to the one of the previous lemma. A computation gives:

$$\partial_t |\nabla^2 u|^2 = \big(\partial_t g^{ik}\big) \nabla_i \nabla_j u \nabla_k \nabla_j u + \big(\partial_t g^{jl}\big) \nabla_i \nabla_j u \nabla_i \nabla_l u + 2\big(\partial_t \nabla_i \nabla_j u\big) \nabla_i \nabla_j u \ .$$

Using the evolution equation (2.12) for the Hessian

$$\partial_t (\nabla_i \nabla_j u) = \Delta(\nabla_i \nabla_j u) + Rm * \nabla^2 u + Rc * \nabla^2 u + |du|^2 * \nabla^2 u \ ,$$

we conclude from the Bochner formula that

$$\begin{aligned}\partial_t |\nabla^2 u|^2 \leq\ & Rc * \nabla^2 u * \nabla^2 u + du * du * \nabla^2 u * \nabla^2 u \\ &+ \Delta |\nabla^2 u|^2 - 2|\nabla^3 u|^2 + C|Rm||\nabla^2 u|^2 + C|du|^2 |\nabla^2 u|^2\end{aligned}$$

holds as required.

□

As a first application we prove:

Proposition 2.17 *Let $(g,u)(t)$ be a solution to (2.5) for $t \in [0,T)$ on a closed manifold M with initial data $(\tilde{g},\tilde{u})$. Assume that $|Rm|^2 \le k_0$ holds on $[0,T)\times M$. Define $c_0 := \max_M |d\tilde{u}|_0^2$ and $s_0 := \max_M |\tilde{\nabla}^2\tilde{u}|_0^2$ where $|\cdot|_0$ and $\tilde{\nabla}$ are with respect to $\tilde{g}$. Then there exists a constant $c = c(n)$ depending only on n such that the Hessian of $u(t)$ satisfies:*

$$|\nabla^2 u|^2(t) \le s_0 + c(n)\big(\sqrt{k_0} + c_0)\big)\cdot c_0 \qquad \forall t \in [0,T)\ .$$

Proof:
Recall from (2.11) that the exterior derivative of u satisfies the evolution equation

$$\big(\partial_t - \Delta\big)|du|^2 = -2|\nabla^2 u|^2 - 4|du|^4 \le 0\ .$$

Therefore, using the maximum principle, we can bound $|du|^2$ by its initial value:

$$|du|^2(t) \le |du|^2(0) \le \sup_M |d\tilde{u}|_0^2 = c_0 \qquad \forall t \in [0,T)\ . \tag{2.19}$$

Combining the evolution equations for $|du|^2$ and $|\nabla^2 u|^2$ from (2.18), we find that

$$\big(\partial_t - \Delta\big)\big(|\nabla^2 u|^2 + \lambda|du|^2\big) \le \big(C_1|Rm| + C_2|du|^2 - 2\lambda\big)|\nabla^2 u|^2$$

for constants C_1, C_2 depending on n. Choose the constant 2λ bigger than $C_1|Rm| + C_2|du|^2 \le C_1\cdot\sqrt{k_0} + C_2\cdot c_0$ using the curvature bound and (2.19). Then the right hand side is negative, and by the maximum principle we conclude for all $t \in [0,T)$ that

$$|\nabla^2 u|^2(t) \le |\nabla^2 u|^2(0) + \lambda|du|^2(0) \le s_0 + \lambda(n,k_0,c_0)\cdot c_0 \le C(n,k_0,c_0,s_0)\ .$$

□

To prove a priori estimates for solutions $(g,u)(t)$ of (2.5) it is very useful to collect the component functions of $Rm(g)$ and $\nabla^2 u$ in a vector valued function Φ as follows:

$$\Phi := (R_{ijkl}, \nabla_p\nabla_q u), \qquad i,j,k,l,p,q = 1\ldots n\ . \tag{2.20}$$

We estimate Φ using the Euclidean vector norm in a single point $p \in \Sigma$:

$$|\Phi|_p^2 := |Rm|_p^2 + |\nabla^2 u|_p^2$$

which is the representation of the norm (1.2) in normal coordinates around p. Combining exactly these two tensors is a result of examining the scaling properties of $|Rm|^2$ and $|\nabla^2 u|^2$ since both scale like λ^{-3} under the scaling $\tilde{g}(t) := \lambda g(\frac{t}{\lambda})$. We extend these definitions accordingly to higher derivatives of Rm and $\nabla^2 u$. This construction simplifies further calculations significantly. Since (2.17) and (2.18) have the same structure, we can combine them to an inequality for $\partial_t|\Phi|^2$.

Lemma 2.18 *Let $(g,u)(t)$ be a solution to (2.5) and let Φ be defined as in (2.20). Then its norm satisfies the inequality*

$$\partial_t|\Phi|^2 \le \Delta|\Phi|^2 - 2|\nabla\Phi|^2 + C|\Phi|^3 + C|du|^2|\Phi|^2 \tag{2.21}$$

where $\nabla\Phi$ is the collection of first derivatives of Rm and $\nabla^2 u$ and C depends only on n.

□

These results can be extended to higher derivatives of Φ. To this end we derive evolution inequalities for the derivatives of Rm and $\nabla^2 u$.

Lemma 2.19 *Let $(g,u)(t)$ be a solution to (2.5). Then the covariant derivatives of the curvature tensor satisfy for all $k \geq 0$ the evolution equations:*

$$\begin{aligned}\partial_t \nabla^k Rm = \Delta \nabla^k Rm &+ \sum_{\alpha+\beta=k} \nabla^\alpha Rm * \nabla^\beta Rm + \sum_{\alpha+\beta=k} \nabla^{2+\alpha} u * \nabla^{2+\beta} u \\ &+ \sum_{\alpha+\beta=k-1} du * \nabla^{2+\alpha} u * \nabla^\beta Rm + \sum_{\alpha+\beta+\gamma=k-2} \nabla^{2+\alpha} u * \nabla^{2+\beta} u * \nabla^\gamma Rm.\end{aligned} \tag{2.22}$$

Proof:
We prove (2.22) by induction. The case $k = 0$ is proven in (2.16). Assume for the induction step that the equation is already true for $\nabla^k Rm$. We then proceed

$$\begin{aligned}\partial_t \nabla^{k+1} Rm &= \nabla\big(\partial_t \nabla^k Rm\big) + \partial_t \Gamma * \nabla^k Rm \\ &= \nabla\Bigg(\Delta \nabla^k Rm + \sum_{\alpha+\beta=k} \nabla^\alpha Rm * \nabla^\beta Rm + \sum_{\alpha+\beta=k} \nabla^{2+\alpha} u * \nabla^{2+\beta} u \\ &\quad + \sum_{\alpha+\beta=k-1} du * \nabla^{2+\alpha} u * \nabla^\beta Rm + \sum_{\alpha+\beta+\gamma=k-2} \nabla^{2+\alpha} u * \nabla^{\beta+2} u * \nabla^\gamma Rm\Bigg) \\ &\quad + \nabla Rm * \nabla^k Rm + du * \nabla^2 u * \nabla^k Rm\ .\end{aligned}$$

Here we used (2.9) to compute the evolution of the connection and interchanged derivatives:

$$\begin{aligned}\partial_t(\nabla A) &= \partial_t(\partial A + \Gamma * A) = \nabla(\partial_t A) + \partial_t \Gamma * A \\ \nabla(\Delta A) &= \nabla_i \nabla_p \nabla_p A = \nabla_p(\nabla_i \nabla_p A) + Rm * \nabla A = \Delta(\nabla A) + \nabla(Rm * A) + Rm * \nabla A \\ &= \Delta(\nabla A) + \nabla Rm * A + Rm * \nabla A\ .\end{aligned}$$

These commutatation relations hold for an arbitrary tensor A. Using the product rule, we see the correctness of (2.22) for $\nabla^{k+1} Rm$.

□

Lemma 2.20 *Let $(g,u)(t)$ be a solution to (2.5). Then the norms of the derivatives of Rm satisfy for all $k \geq 0$ the evolution inequalities:*

$$\begin{aligned}\partial_t |\nabla^k Rm|^2 \leq \Delta |\nabla^k Rm|^2 - 2|\nabla^{k+1} Rm|^2 &+ C\Bigg\{ \sum_{\alpha+\beta=k} |\nabla^\alpha Rm||\nabla^\beta Rm||\nabla^k Rm| \\ &+ \sum_{\alpha+\beta=k} |\nabla^{2+\alpha} u||\nabla^{2+\beta} u||\nabla^k Rm| + \sum_{\alpha+\beta=k-1} |du||\nabla^{2+\alpha} u||\nabla^\beta Rm||\nabla^k Rm| \\ &+ \sum_{\alpha+\beta+\gamma=k-2} |\nabla^{2+\alpha} u||\nabla^{2+\beta} u||\nabla^\gamma Rm||\nabla^k Rm| + |du|^2 |\nabla^k Rm|^2 \Bigg\}\ .\end{aligned} \tag{2.23}$$

Proof:
Considering that

$$\partial_t|\nabla^k Rm|^2 = (2Rc - 4du \otimes du) * \nabla^k Rm * \nabla^k Rm + \partial_t \nabla^k Rm * \nabla^k Rm \ ,$$

the lemma follows directly from the previous lemma and the Cauchy-Schwarz inequality.

□

Lemma 2.21 *Let $(g,u)(t)$ be a solution to (2.5). Then for all $k \geq 0$ the evolution of $\nabla^{2+k}u$ is given by*

$$\begin{aligned}\partial_t \nabla^{2+k}u = \Delta\nabla^{2+k}u &+ \sum_{\alpha+\beta=k} \nabla^{2+\alpha}u * \nabla^\beta Rm + \sum_{\alpha+\beta=k-1} du * \nabla^{2+\alpha}u * \nabla^{2+\beta}u \\ &+ \sum_{\alpha+\beta+\gamma=k-2} \nabla^{2+\alpha}u * \nabla^{2+\beta}u * \nabla^{2+\gamma}u + |du|^2 \cdot \nabla^{k+2}u \ .\end{aligned} \tag{2.24}$$

Proof:
The idea is the same as above. We proof (2.24) by induction where the claim for $\nabla^{2+0}u$ is proven in (2.12). Plugging in the induction hypotheses for $\nabla^{2+k}u$, we compute

$$\begin{aligned}\partial_t\big(\nabla\nabla^{2+k}u\big) &= \nabla\big(\partial_t\nabla^{2+k}u\big) + \partial_t\Gamma * \nabla^{2+k}u = \Delta\nabla^{3+k}u + \nabla Rm * \nabla^{2+k}u + Rm * \nabla^{3+k}u \\ &+ \sum_{\alpha+\beta=k}\big(\nabla^{2+\alpha+1}u * \nabla^\beta Rm + \nabla^{2+\alpha}u * \nabla^{\beta+1}Rm\big) \\ &+ \sum_{\alpha+\beta=k-1}\big(\nabla^2 u * \nabla^{2+\alpha}u * \nabla^{2+\beta}u + du * \nabla^{2+\alpha+1}u * \nabla^{2+\beta}u \\ &+ du * \nabla^{2+\alpha}u * \nabla^{2+\beta+1}u\big) + \sum_{\alpha+\beta+\gamma=k-2}\big(\nabla^{2+\alpha+1}u * \nabla^{2+\beta}u * \nabla^{2+\gamma}u \\ &\quad + \nabla^{2+\alpha}u * \nabla^{2+\beta+1}u * \nabla^{2+\gamma}u + \nabla^{2+\alpha}u * \nabla^{2+\beta}u * \nabla^{2+\gamma+1}u\big) \\ &+ \nabla^2 u * du * \nabla^{2+k}u + |du|^2 \cdot \nabla^{3+k}u \ .\end{aligned}$$

This can be rearranged to yield the claim for $\nabla^{2+(k+1)}u$.

□

Using the Cauchy-Schwarz inequality again, we compute:

Lemma 2.22 *Let $(g,u)(t)$ be a solution to (2.5). Then the norms of the covariant derivatives of u satisfy for all $k \geq 0$ the inequality*

$$\begin{aligned}\partial_t|\nabla^{2+k}u|^2 \leq \Delta|\nabla^{2+k}u|^2 - 2|\nabla^{2+k+1}u|^2 &+ C\Big\{\sum_{\alpha+\beta=k}|\nabla^{2+\alpha}u||\nabla^\beta Rm||\nabla^{2+k}u| \\ &+ \sum_{\alpha+\beta=k-1}|du||\nabla^{2+\alpha}u||\nabla^{2+\beta}u||\nabla^{2+k}u| \\ &+ \sum_{\alpha+\beta+\gamma=k-2}|\nabla^{2+\alpha}u||\nabla^{2+\beta}u||\nabla^{2+\gamma}u||\nabla^{2+k}u| + |du|^2|\nabla^{2+k}u|^2\Big\} \ .\end{aligned} \tag{2.25}$$

□

Evidently, the equations (2.23) and (2.25) have the same structure. Therefore we can combine them into an inequality for $\nabla^k\Phi$.

Lemma 2.23 *Let $(g,u)(t)$ be a solution to (2.5) and $\nabla^k\Phi$ be as defined in (2.20). Then the derivatives of Φ satisfy for all $k \geq 0$ the inequality*

$$\partial_t|\nabla^k\Phi|^2 \leq \Delta|\nabla^k\Phi|^2 - 2|\nabla^{k+1}\Phi|^2 + C\Big\{\sum_{\alpha+\beta=k}|\nabla^\alpha\Phi||\nabla^\beta\Phi||\nabla^k\Phi| + \sum_{\alpha+\beta=k-1}|du||\nabla^\alpha\Phi||\nabla^\beta\Phi||\nabla^k\Phi|$$
$$+ \sum_{\alpha+\beta+\gamma=k-2}|\nabla^\alpha\Phi||\nabla^\beta\Phi||\nabla^\gamma\Phi||\nabla^k\Phi| + |du|^2|\nabla^k\Phi|^2\Big\}$$

where $C = C(n)$ is a constant depending only on the dimension.

$\square$