1 Preliminaries

1.1 Notation and conventions

In this section we introduce some symbols and conventions which will be used throughout the whole paper without further comment.

We work on a smooth n-dimensional Riemannian manifold Σ or M, where M is always used to specify a closed manifold, meaning that M is compact without boundary. While the tangent bundle is denoted by $T\Sigma$ and the space of smooth vector fields by $\mathcal{X}(\Sigma)$ respectively, we write $\Omega^p(\Sigma) := \Omega^p(\Sigma, \mathbb{R})$ for the vector space of smooth real valued differential forms of grade p on Σ. Finally, the smooth functions on Σ will be denoted by $C^\infty(\Sigma)$. We also make the following definition:

Definition 1.1 *Let Σ be a smooth Riemannian manifold. We define the* space of smooth symmetric twice covariant tensors *on Σ by*

$$Sym_2(\Sigma) := \{v = \{v_{ij}\} \in \Gamma(T^*\Sigma \otimes T^*\Sigma) | v_{ij} = v_{ji}\}$$

and the space of smooth Riemannian metrics *on Σ by*

$$\mathcal{M}(\Sigma) := \{g = \{g_{ij}\} \in Sym_2(\Sigma) | g_{ij} > 0\} \ .$$

For a Riemannian metric g, we denote the Levi-Civita connection by ∇, the Christoffel symbols of some coordinate system by Γ and its curvature tensor by Rm which can be either the $(1,3)$ or the $(0,4)$ version, depending on the context. Its Ricci tensor is denoted by Rc, the scalar curvature by R, and the volume element by dV. If it is important to make clear to which metric these tensors belong, we write $Rm(g)$ and so on. Furthermore, the exterior derivative of a form is denoted by d. Recall that d is independent of the metric g. The Lie derivative of a tensor B with respect to $X \in \mathcal{X}(\Sigma)$ is denoted by $\mathcal{L}_X B$.

Given some coordinate system $\{x^1, \dots, x^n\}$, we abbreviate $\partial_i := \frac{\partial}{\partial x^i}$ for the partial derivatives with respect to the coordinates x, $\nabla_i := \nabla_{\frac{\partial}{\partial x^i}}$ for the covariant derivatives associated to g, and ∂_t for partial differentiation with respect to time. We denote the components of the Hessian ∇^2 of some function u by $\nabla_i \nabla_j u := \nabla^2_{ij} u$ and similar for higher derivatives. The components of the metric g itself are given by $\{g_{ij}\}$, and the metric g^{-1} on the cotangent bundle $T^*\Sigma$ is represented by $\{g^{ij}\}$. The Laplacian of a function u with respect to g is given by

$$\Delta^g u = g^{ij}\left(\partial_i \partial_j u - \Gamma^k_{ij} \partial_k u\right) \ . \tag{1.1}$$

The Riemannian metric g induces norms on the tensor bundles. In coordinates this norm is given for a tensor $B := \left\{B^{j_1 \dots j_l}_{i_1 \dots i_k}\right\}$ by

$$|B|^2 := g^{i_1 m_1} \cdot \dots \cdot g^{i_k m_k} \cdot g_{j_1 n_1} \cdot \dots \cdot g_{j_l n_l} \cdot B^{j_1 \dots j_l}_{i_1 \dots i_k} \cdot B^{n_1 \dots n_l}_{m_1 \dots m_k}, \tag{1.2}$$

where we use the Einstein summation convention, meaning that we sum over a repeated lower and upper index from 1 to n. When computing in normal coordinates, the summation can be over two lower indices. The convention is used throughout this paper, and deviations are marked explicitly. The Christoffel symbols of the Riemannian metric g are defined by

$$\Gamma_{ij}^k = \tfrac{1}{2} g^{kl} \big(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \big) \tag{1.3}$$

and we use the following formula for the construction of the components of the Riemann tensor:

$$R_{ijl}^k = \partial_i \Gamma_{jl}^k - \partial_j \Gamma_{il}^k + \Gamma_{is}^k \Gamma_{jl}^s - \Gamma_{js}^k \Gamma_{il}^s \ . \tag{1.4}$$

Furthermore, we contract the Riemann tensor as follows to get the Ricci curvature:

$$R_{ij} = g^{kl} R_{kilj} = R_{kij}^k \ . \tag{1.5}$$

For the interchange of covariant derivatives we have

$$\begin{aligned} \nabla_i \nabla_j \omega_k - \nabla_j \nabla_i \omega_k &= g^{lm} R_{ijkl} \omega_m \\ \nabla_i \nabla_j v^k - \nabla_j \nabla_i v^k &= R_{ijh}^k v^h \end{aligned} \tag{1.6}$$

for $\omega \in \Omega^1(\Sigma)$, $v \in \mathcal{X}(\Sigma)$ and similar for more complicated tensors. This convention is from [Ham82, §2]. Observe that the symmetry of the Christoffel symbols implies that the second derivatives of a function $u \in C^\infty(\Sigma)$ commute:

$$\nabla_i \nabla_j u = \nabla_j \nabla_i u \ . \tag{1.7}$$

We need several identities for the curvature tensor, in particular the second Bianchi identity together with two contracted versions:

$$\nabla_i R_{jkpq} + \nabla_j R_{kipq} + \nabla_k R_{ijpq} = 0 \tag{1.8}$$

$$\nabla_p R_{ijpq} + \nabla_j R_{iq} - \nabla_i R_{jq} = 0 \tag{1.9}$$

$$2 \nabla_p R_{pi} - \nabla_i R = 0 \ . \tag{1.10}$$

To simplify and shorten the forthcoming calculations, there is the following convenient notation: We write $A * B$ for a linear combination of contractions of components of the two tensors A and B when the precise form and number of these terms is irrelevant for the computation. In this notation, factors g and g^{-1} are suppressed. For example, we could write equation (1.6) in the form: $\nabla_i \nabla_j \omega - \nabla_j \nabla_i \omega = Rm * \omega$.

1.2 Static vacuum solutions of the Einstein equations

The aim of this section is to give a short introduction to static vacuum solutions of the Einstein equations. These equations play an important role in the motivation of the following work. For this section we refer to [EK62, chapter 2] and [Wal84, chapter 6].

From a geometrical point of view, a Lorentzian manifold (L^4, h) is said to be static, if there exists a 1-parameter group of isometries with timelike orbits and a hypersurface Σ which is orthogonal to these orbits and therefore spacelike. Physically this reflects the fact that the solutions are independent of time, therefore having a time translation and reflection symmetry. Equivalent is the existence of a timelike, hypersurface orthogonal Killing vector field ξ.

A vacuum solution to the Einstein equations satisfies

$$Rc(h) \equiv 0 \ .$$

These solutions describe the gravitational field in a region of spacetime that does not contain any matter.

Combining these two concepts we arrive at the notion of a static vacuum solution. The Schwarzschild solution is the most important example in this class. It describes the gravitational field in the exterior region around an isolated nonrotating spherically symmetric body which could be a star or a black hole.

We give a technical description of these solutions: Arbitrary coordinates on the hypersurface Σ can be extended to points in some spacetime neighborhood. For every p in such a neighborhood there is a unique point $q \in \Sigma$, such that p and q are connected by one of the orbit curves of ξ. To obtain coordinates for p, we attach the parameter of the orbit curve to the spatial coordinates of q. One can then conclude that the metric components in these coordinates are independent of the orbit parameter, and that all sets of spacetime points with the same "time" parameter are also spacelike hypersurfaces orthogonal to the orbits. Consequently, the metric has the following simple form:

$$h = -V^2(x^1, x^2, x^3)dt^2 + \sum_{i,j=1}^{3} g_{ij}(x^1, x^2, x^3)dx^i dx^j,$$

where $\{x^i\}$ are the coordinates on Σ, t denotes the coordinate along the orbits, $V := \sqrt{-|\xi|^2}$ is the square root of the negative norm of the (timelike) Killing field ξ, and g_{ij} are the components of a spatial Riemannian metric on Σ. Therefore the unknown Lorentz metric h is now given by a function V on Σ, the *Lapse* function of the static spacetime, and a Riemannian metric g, being the metric on the spatial slices. One can express the static Einstein vacuum equations in terms of V and g on the hypersurface and finds

$$\begin{aligned} Rc(g) &= V^{-1}\nabla^2 V \\ V^{-1}\Delta^g V &= 0 \ . \end{aligned} \tag{1.11}$$

This is done in [EK62, 3.4-3.5]. Vice versa, a pair (V, g) satisfying (1.11) gives rise to a uniquely determined static vacuum solution [EK62, Theorem 2-3.3]. In the following, we do not restrict ourselves to dimension $n = 4$ (where Σ has dimension 3), but work with a manifold Σ of arbitrary dimension $n \geq 3$. However, we still use the physical description of such a spacetime in a formal way to simplify the discussion.

We can apply these ideas to the above mentioned Schwarzschild solution. The Lorentz metric of the Schwarzschild spacetime is given in spherical spatial coordinates (t, r, θ, ϕ) on $\mathbb{R} \times \mathbb{R}^3$ by

$$h = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\big(d\theta^2 + \sin^2\theta d\phi^2\big), \tag{1.12}$$

where m is the mass parameter. One calculates that it satisfies (1.11) for $V = \left(1 - \frac{2m}{r}\right)^{1/2}$ and g the spatial part of h on the slices $t = const.$

Another example are solutions (g, u) on $\mathbb{R} \times M^n$ where M^n is an n-dimensional Riemannian manifold, g a Ricci-flat metric, and $u \equiv const.$

A very thorough and detailed discussion of static solutions is given in [BS99].

1.3 Conformal transformations

In the following, we consider conformally related metrics g and $\tilde{g}$, meaning that there exists a smooth function $\phi \in C^\infty(\Sigma)$ such that $\tilde{g} = \phi^2 \cdot g$. We compute the relation between the curvature of the two metrics:

Lemma 1.2 *If $\tilde{g} := \phi^2 \cdot g$ is conformally related to g, then the following equations hold:*

$$\begin{aligned}
\tilde{g}^{ij} &= \phi^{-2} \cdot g^{ij} \\
\tilde{\Gamma}^k_{ij} &= \Gamma^k_{ij} + \phi^{-1} g^{kl} \left\{\partial_i \phi \cdot g_{jl} + \partial_j \phi \cdot g_{il} - \partial_l \phi \cdot g_{ij}\right\} \\
\tilde{R}_{ij} &= R_{ij} - \phi^{-1} \Delta^g \phi \cdot g_{ij} - (n-2)\phi^{-1} \nabla_i \nabla_j \phi + 2(n-2)\phi^{-2} \partial_i \phi \partial_j \phi - (n-3)\phi^{-2} |d\phi|^2_g \cdot g_{ij} \\
\tilde{R} &= \phi^{-2} R - (n-1)(n-4)\phi^{-4} |d\phi|^2_g - 2(n-1)\phi^{-3} \Delta^g \phi \\
\Delta^{\tilde{g}} f &= \phi^{-2} \Delta^g f + (n-2)\phi^{-3} \langle df, d\phi \rangle_g \\
d\tilde{V} &= \phi^n dV ,
\end{aligned}$$

where all objects with a tilde correspond to $\tilde{g}$ and f is an arbitrary function.

□

Setting $\phi := e^{2\alpha\psi}$, the Ricci tensor is given in terms of ψ as follows:

Lemma 1.3 *If $\tilde{g} := e^{2\alpha\psi} \cdot g$ holds for a function $\psi \in C^\infty(\Sigma)$ and a constant $\alpha \in \mathbb{R}$, then the Ricci curvatures of $\tilde{g}$ and g are related in the following way:*

$$\tilde{R}_{ij} = R_{ij} - (n-2)\alpha \nabla_i \nabla_j \psi + (n-2)\alpha^2 \partial_i \psi \partial_j \psi - \alpha \Delta^g \psi \cdot g_{ij} - (n-2)\alpha^2 |d\psi|^2_g \cdot g_{ij} \ .$$

□

Let h be a solution to the static Einstein vacuum equations on a (n+1)-dimensional Lorentz manifold L which is given by a pair (g, V) satisfying (1.11). This system can be simplified considerably by removing the second derivatives of V on the right hand side via a conformal transformation. Defining $u := \ln V$, we calculate

$$\begin{aligned}
R_{ij} &= e^{-u} \nabla_i \nabla_j e^u = e^{-u} \nabla_i (\partial_j u e^u) = \nabla_i \nabla_j u + \partial_i u \partial_j u \\
0 &= e^{-u} \Delta^g e^u = e^{-u} \nabla_i \nabla_i e^u = \Delta^g u + |du|^2_g \ .
\end{aligned} \tag{1.13}$$

Using the conformal transformation $\tilde{g} := e^{\frac{2}{n-2}u} \cdot g$, we get from Lemma 1.3 and (1.13)

$$\begin{aligned}
\tilde{R}_{ij} &= R_{ij} - \frac{n-2}{n-2}\nabla_i\nabla_j u + \frac{n-2}{(n-2)^2}\partial_i u \partial_j u - \frac{1}{n-2}\Delta^g u\, g_{ij} - \frac{n-2}{(n-2)^2}|du|_g^2\, g_{ij} \\
&= \big(\nabla_i\nabla_j u + \partial_i u \partial_j u\big) - \nabla_i\nabla_j u + \frac{1}{n-2}\partial_i u\partial_j u - \frac{1}{n-2}\underbrace{\big(\Delta^g u + |du|_g^2\big)}_{=0} g_{ij} \\
&= \left(1 + \frac{1}{n-2}\right)\partial_i u \partial_j u = \frac{n-1}{n-2}\partial_i u \partial_j u \ .
\end{aligned} \tag{1.14}$$

Similarly we calculate

$$\Delta^{\tilde{g}} u = e^{\frac{-2}{n-2}u}\Delta^g u + (n-2)e^{\frac{-3}{n-2}u} g^{ij}\big(\partial_i e^{\frac{1}{n-2}u}\big)\partial_j u = e^{\frac{-2}{n-2}u}\left(\Delta^g u + \frac{n-2}{n-2}|du|_g^2\right) = 0 \ .$$

Together we obtain the conformal system for $(\tilde{g}, u)$:

$$\begin{aligned}
\tilde{R}_{ij} &= \tfrac{n-1}{n-2}\partial_i u \partial_j u \\
\tilde{\Delta} u &= 0
\end{aligned} \tag{1.15}$$

which is equivalent to $Rc(h) = 0$ for the Lorentz metric

$$h = -e^{\frac{2}{n-2}u}dt^2 + e^{-\frac{2}{n-2}u} g_{ij}\, dx^i dx^j \tag{1.16}$$

on $\mathbb{R} \otimes \mathbb{R}^n$. It is proven in [EK62, Theorem 2-3.4] for $n = 3$ that every solution of the static Einstein vacuum equations can be given that form. Note that after the conformal change of the metric, we have $\tilde{R} = \frac{n-1}{n-2}|du|^2$ instead of $R = 0$.

One solution to (1.15) is the conformal equivalent of the Schwarzschild metric (1.12). The spatial metric on the slices $t = const$ is given by

$$\tilde{g} = dr^2 + r(r - 2m)\big(d\theta^2 + \sin^2\theta d\phi^2\big),$$

while

$$u := \ln V = \tfrac{1}{2}\big(\ln(r - 2m) - \ln r\big)$$

is the logarithm of the Lapse function.

1.4 Evolution of a Riemannian metric

We consider time dependent metrics $g(t)$ satisfying an evolution equation like

$$\partial_t g_{ij}(t) = v_{ij}(t) \tag{1.17}$$

for some tensor $\{v_{ij}(t)\} \in \mathrm{Sym}_2(\Sigma)$ for all suitable t. This equation induces evolution equations for all the curvature expressions related to $g(t)$. Since these are well known, we just state them for reference reasons:

Lemma 1.4 *Suppose $g(t)$ is a solution of (1.17). Then the following evolution equations hold:*

$$\begin{aligned}
\partial_t g^{ij} &= -g^{ip} g^{jq} v_{pq} \\
\partial_t \Gamma_{ij}^k &= \tfrac{1}{2} g^{kl} (\nabla_i v_{jl} + \nabla_j v_{il} - \nabla_l v_{ij}) \\
\partial_t R_{ijl}^k &= \tfrac{1}{2} g^{ks} \big(\nabla_i \nabla_l v_{js} + \nabla_j \nabla_s v_{il} - \nabla_i \nabla_s v_{jl} - \nabla_j \nabla_l v_{is} + g^{pq} (R_{ijlp} v_{qs} + R_{ijsp} v_{lq}) \big) \\
\partial_t R_{ijkl} &= \tfrac{1}{2} \big(\nabla_i \nabla_k v_{js} + \nabla_j \nabla_s v_{ik} - \nabla_i \nabla_s v_{jk} - \nabla_j \nabla_k v_{is} + g^{pq} (R_{ijkp} v_{qs} + R_{ijsp} v_{kq}) \big) + v_{pk} R_{ijl}^p \\
\partial_t R_{ij} &= -\tfrac{1}{2} \Delta^g v_{ij} - \tfrac{1}{2} \nabla_i \nabla_j v + \tfrac{1}{2} g^{pq} (\nabla_p \nabla_j v_{iq} + \nabla_p \nabla_i v_{jq}) \\
\partial_t R &= -\Delta v + g^{pq} g^{rs} (\nabla_p \nabla_r v_{qs} - R_{pr} v_{qs}) \\
\partial_t dV &= \tfrac{1}{2} v \, dV
\end{aligned}$$

where $v = g^{ij} v_{ij}$ denotes the trace of v_{ij} computed with g.

Proof:
A proof is given in [CK04, §3.1].

□

An important technical tool to derive estimates for the solutions of parabolic partial differential equations is the maximum principle. We state one form here for reference reasons.

Theorem 1.5 *Let $g(t)$ and $X(t)$ be 1-parameter families of Riemannian metrics and vector fields respectively on a closed manifold M for $t \in [0, T)$. Let $f : [0, T) \times M \to \mathbb{R}$ be a C^2 function. If $f(0) \geq \alpha$ and on $[0, T) \times M$ we have the inequality*

$$\big(\partial_t - \Delta\big) f \geq \langle X, \nabla f \rangle,$$

then $f(t) \geq \alpha$ holds for all $t \in [0, T)$. If $f(0) \leq \alpha$ and on $[0, T) \times M$ we have

$$\big(\partial_t - \Delta\big) f \leq \langle X, \nabla f \rangle,$$

then $f(t) \leq \alpha$ holds for all $t \in [0, T)$.

Proof:
The first part of the theorem is [CK04, Theorem 4.2] and the second is analogous.

□

One can also prove maximum principles for heat equations with linear and nonlinear reaction terms on the right hand side. See for example [CK04, §4]. In the following, whenever we invoke the *maximum principle* on closed manifolds, we will apply Theorem 1.5. A version on complete manifolds is given by Theorem 6.10.