Chapter 5
CONCLUSIONS

... When your mind is going hither and thither, discrimination will never be brought to a conclusion. With an intense, fresh and undelaying spirit, one will make his judgements within the space of seven breaths. It is a matter of being determined and having the spirit to break right through to the other side.

From 'The Book of the Samurai, Hagakure'

It has been demonstrated that Lie algebraic structures play an essential role in affine Toda models, which form a vast class of integrable quantum field theories. Coxeter geometry and its q-deformed extension accommodating the renormalization flow have been proved to provide a concise mathematical framework for the treatment of the quantum aspects of ATFT. The strategy to express various physical quantities like the mass spectrum, the fusing rules, and the S-matrices in generic and universal Lie algebraic formulas leads not only to a great simplification in the treatment of the on-shell structure of these models, but has also been shown to carry through to the level of the thermodynamic Bethe ansatz when investigating the underlying conformal field theories in the ultraviolet limit. Indeed, all relevant calculations have been performed in a complete generic Lie algebraic setting, treating all models at once.

In addition, the Lie algebraic structures have been explicitly exploited to formulate hitherto unknown exact S-matrices via the introduction of colour degrees of freedom, the so-called $\mathfrak{g}|\tilde{\mathfrak{g}}$-models. By means of a detailed TBA analysis the universal formula (4.201) giving the effective central charges of the associated conformal models in the high energy regime has been derived. For particular subclasses of $\mathfrak{g}|\tilde{\mathfrak{g}}$-scattering matrices this result could be directly linked to the central charges of WZNW coset models, see Table 5.1.

$\mathfrak{g}$	$\tilde{\mathfrak{g}}$	integrable model	WZNW coset
A_n	ADE	HSG models	$\mathfrak{g}_{n+1}/u(1)^{\times \operatorname{rank} \mathfrak{g}}$
ADE	A_1	minimal ATFT	$\mathfrak{g}_1 \otimes \mathfrak{g}_1/\mathfrak{g}_2$
ADE	A_n	"dual" HSG models	$\mathfrak{g}_1^{\otimes(n+1)}/\mathfrak{g}_{n+1}$

Table 5.1: $\mathfrak{g}|\tilde{\mathfrak{g}}$-theories related to WZNW cosets.

This demonstrates that in many cases the Lie algebraic structures occurring in S-matrices of affine Toda type can be traced back to the ones appearing in WZNW models. They are therefore particularly interesting candidates for further investigations relating massive and conformal spectra, since the Lie algebraic framework might serve as a guiding line in this task. The motivation to explore the interplay between conformal and integrable models in greater depth is manifold. For example, it might shed light on the origin of mass as already mentioned in the introduction. From a more technical point of view it will certainly help in the construction of correlation functions. The latter are to a much deeper extent explored in conformal theories than in massive ones. In fact, two and three-point functions can immediately be written down in presence of conformal symmetry. Moreover, in the case of WZNW models the higher n-point functions are accessible by differential equations, the so-called Knizhnik-Zamolodchikov equations. By means of conformal perturbation theory one might then check against outcomes of the form factor program. At the moment such relations are still a far posted goal, but the techniques of the bootstrap approach and the TBA analysis performed in this thesis are first steps in this direction.

As outlined in the introduction the development of efficient techniques to calculate correlation functions are of use in the area of quantum field theory as well as of two-dimensional statistical mechanics and condensed matter systems with second order phase transitions. In particular, it should be emphasized once more that the interpretation of integrable field theories as perturbed conformal models and the related TBA analysis are conceptually closely related to the ideas of the renormalization group originating in the study of critical phenomena. For example, the central object of interest in the TBA considerations, the scaling function, has been conjectured to be tightly linked to the β-function appearing in the renormalization group equations [58]. Exploring this connection in more detail one might learn from 1+1 dimensional theories about higher dimensional ones, since the concept of the renormalization group is applicable in any dimension.

These general remarks are now supplemented by a more detailed presentation of the results in order to pinpoint concrete starting points for further investigations.

5.1 The affine Toda S-matrix

A systematic treatment has been given for the application of geometrical arguments in context of ATFT. Starting with the theory of ordinary Coxeter elements their q-deformation has been motivated: first on an abstract mathematical level by generalizing numerous formulas and identities from the non-deformed case, second by exhibiting how q-deformation matches with the renormalization picture obtained from perturbative calculations, which have already been performed in the literature. Recall that the latter indicated that the renormalized particle masses flow between the classical values of two different affine Toda models belonging to a pair of algebras related by Langlands duality ($\alpha \leftrightarrow \alpha^{\vee}$). The crucial property of the q-deformed Coxeter element, twisted or untwisted, turned out to be the merging of the data of these two dual algebras in a consistent manner.

The first physical application where this property has been used was the formulation of the quantum fusing rules in ATFT indicating the vanishing of the three-point coupling, which in turn is associated with the fusing of two quantum particles to a third. While the fusing rules found by Chari and Pressley have to be formulated in terms of both the non-deformed Coxeter *and* the non-deformed twisted Coxeter element, q-deformation allows to state them either in terms of the orbits Ω^q *or* $\hat{\Omega}^q$ solely. The latter are generated by the q-deformed Coxeter and the q-deformed twisted Coxeter element, respectively. One of the central results presented in this thesis is the derivation of the precise relation between these different versions of the fusing rules (3.54) and the proof of their equivalence. This clarifies the interplay between the two dual Lie algebraic structures and shows their equivalence. In this context, it would be particularly interesting to investigate, whether for the q-deformed fusing rules a similar connection to representation theory exists as for the non-deformed ones [34]. This might give rise to new "quantum symmetries". However, the link to representation theory is so far only partially understood and the formalism presented here is by now the most restrictive one.

The q-deformed fusing rules were then shown to be consistent with the description of the mass spectrum and other conserved quantities as nullvector of the q-deformed Cartan matrix (2.67). Because of its compactness and mathematical beauty this characterization of the quantum masses is recalled here (compare (3.44)),

$$\sum_{j=1}^{n} [I_{ij}]_{\hat{q}}\, m_j = 2\cos\pi \left(\frac{2-B}{2h} + \frac{t_i B}{2H} \right) m_i \ , \quad \hat{q} = e^{i\pi \frac{sB}{2H}} \ .$$

From the above formula the renormalized mass flow of all affine Toda theories can be directly read off. It would be desirable to construct the corresponding "quantum" Lagrangian from which this mass spectrum can be deduced in terms of a mass matrix, as it was done for the ADE case w.r.t. the classical Lagrangian. It is most likely that q-deformation will also play a vital role in this construction.

Furthermore, the q-deformed fusing rules have been directly employed in the construction of the S-matrix by means of the bootstrap equations. Their generic formulation allowed to write down a universal formula for the two-particle scattering amplitude in terms of hyperbolic functions whose powers can be directly inferred from the q-deformed Coxeter orbits. These have been explicitly worked out for the first time in [37] (see also the appendix). Alternatively, one might use the M-matrix (2.66), which is a slightly modified version of the inverse q-deformed Cartan matrix. It was argued that the matrix elements of the latter always simplify to polynomials in the two deformation parameters involved. These polynomials reflect the structure of the building blocks of hyperbolic functions and encode the complete information about the S-matrix. Using the structure of the M-matrix, it has been systematically demonstrated that the two-particle scattering amplitude fulfills all required bootstrap properties. Equivalently, this also followed from the inner product identities for the q-deformed Coxeter elements derived in Section 2.4.4 and 2.4.7. Moreover, it is intriguing that the combined bootstrap equation (3.71), which is closely linked to the structure of the M-matrix, incorporates the information of *all* individual fusing processes. The discussion of the analytic properties of the S-matrix has been completed by providing a simple criterion which enables one to exclude unphysical poles from the participation in the bootstrap.

The matrix structure was also exploited in the rigorous derivation of the universal integral representation of the ATFT S-matrix (3.82)

$$S_{ij}(\theta) = \exp 8 \int_0^\infty \frac{dt}{t} \sinh(\vartheta_h t) \sinh(t_j \vartheta_H t) A(e^{t\vartheta_h}, e^{t\vartheta_H})^{-1} \sinh\left(\frac{\theta t}{i\pi}\right) .$$

This formula is not only a neat and compact expression for all ATFT S-matrices, but also of direct use in several applications. For instance, the two-particle form factor can be immediately extracted from the above identity. The discovery of similar powerful Lie algebraic structures might then yield a valuable advantage in the calculation of higher particle form factors. In this thesis the above integral representation has been explicitly applied when discussing the thermodynamic Bethe ansatz for ATFT, where it lead to additional universal formulas.

5.2 The TBA analysis of ATFT

In Section 4.3 it has been demonstrated that it is possible to extract the leading order behaviour of the scaling function for *all* ATFT by simple analytical approximations schemes in the large and small density regime. By matching the approximate solutions of the TBA equations in the two different regimes at the point in which the particle density and the density of available states coincide, it is possible to fix the constant of integration, which originated in the approximation scheme of [58, 101, 43] and was left undetermined therein. Since the leading order behaviour derived for the scaling function is in agreement with the semi-classical results found in [42] and also [45, 46], one has an alternative method to fix the constant and to compare directly the different approaches. Thus, it is not necessary for this to proceed to higher order differential equations as was claimed in [99]. Since the solutions to the higher order differential equations may only be obtained approximately one does not gain any further structural insight this way and, moreover, one has lost the virtue of the leading order approximation, namely its simplicity.

With regard to future investigations it would be desirable to extend the analysis presented here and to find analytical expressions for the constant of integration totally within the TBA approach. This would allow to verify the semi-classical results in the literature on the basis of the “pure quantum” S-matrix. For this purpose further exact insight on the solutions of the TBA equations is required. This motivated the derivation of the Y-systems, which were also presented in a unique formula for all ATFT. As demonstrated they can be utilized to improve the large density approximations and can also be applied to put constraints on the constant of integration. Future considerations about possible periodicities and their analytic features might lead to additional requirements on the constant allowing to extract it analytically. There also exist interesting links between the Y-systems and spectral functions in quantum mechanics [116], such that one can expect more exact and universal results to follow.

The TBA results are also of interest in comparison to alternative methods which allow to calculate the ultraviolet central charge, such as the **c-theorem** [117].

The latter allows to compute the difference of the central charges recovered in the ultraviolet and infrared limit of the massive theory from the Schwinger function,

$$\Delta c = c_{\mathrm{UV}} - c_{\mathrm{IR}} = -\frac{3}{2}\int_0^\infty dr\, r^3 \left\langle 0|T^\mu_{\ \mu}(r)T^\mu_{\ \mu}(0)|0\right\rangle \ . \tag{5.1}$$

The correlation function might be calculated via the form factor approach upon inserting a complete set of states as explained in the introduction. In fact, in most cases the two-particle form factor already yields an excellent approximation. Since $c_{\mathrm{IR}} = 0$ for purely massive theories one might then compare with the outcome from the TBA. Closely related to this observation is the problem to compute the vacuum expectation value of the energy momentum tensor. Unlike in the situation of conformal invariance, in which the trace vanishes, this tensor is not unique and acquires some scaling behaviour for massive theories. One therefore needs an external input for the recursive form factor equations removing this ambiguity. The latter is provided by the TBA, where the vacuum expectation value can be directly obtained from the scaling function [23],

$$\left\langle 0|T^\mu_{\ \mu}(r)|0\right\rangle = -\frac{\pi^2}{3r}\frac{d}{dr}c(r) \ .$$

However, in case of ATFT one first needs a deeper physical understanding of the logarithmic corrections appearing in the leading order behaviour of the scaling function before one may proceed this way. Notice that in the context of fixing the vacuum expectation value by means of the TBA also the investigation of existence and uniqueness of the solution in Section 4.2.3 plays an important role. Restricting the number of possible solutions to one, this rules out any ambiguity inside this approach, except for different possibilities in choosing the statistics. The impact of statistics is another issue which needs to be clarified in more detail, especially in its relevance for relating conformal and massive spectra.

5.3 Colour valued S-matrices, HSG models and WZNW cosets

Suggesting a general construction principle a new class of exact scattering matrices exhibiting Lie algebraic structures very alike to those in affine Toda models has been generated. Extending the techniques originally developed in context of ATFT to these integrable systems, it has been shown that the proposed S-matrices for the $\mathfrak{g}|\tilde{\mathfrak{g}}$-theories provide consistent solutions of the bootstrap equations (3.17). The special feature of parity violation has been shown to enter this construction as a consequence of taking the square root of the CDD-factor occurring in ATFT. The motivation for this particular choice was to recover S-matrices already known in the literature, especially the ones proposed in context of the Homogeneous Sine-Gordon models.

For these integrable theories strong evidence has been presented that the proposal originally made in [40] matches with the semi-classical picture obtained when perturbing the WZNW cosets $\tilde{\mathfrak{g}}/u(1)^{\mathrm{rank}\,\tilde{\mathfrak{g}}}$, see Table 5.1. In particular, the detailed analytical and numerical TBA analysis of the $su(3)$ model supported several assumptions made about unstable particles in the spectrum. They led to a direct physical interpretation of the staircase pattern in the scaling function resulting from the resonance poles in the S-matrix. The staircase pattern has then been interpreted as the sign for a more sophisticated renormalization group flow of these theories which dependent on the choice of external parameters might end in different fixed points, i.e. different conformal field theories. In particular, massless subflows might occur which have a non-trivial UV as well as a non-trivial IR fixed point. As one particular example the flow between the tricritical and the critical Ising model was provided. Similar findings are to be expected when generalizing to other HSG models.

However, the subject is still far from being closed. The exact nature of the unstable particles has to be explored in more detail, particularly in hindsight to a complete explanation of the resonance poles of the S-matrix. Also their relevance for recovering the correct conformal spectrum is not settled, since the expected coset central charge is reached irrespective whether or not the resonance parameter is chosen to be zero. In particular, on the level of the TBA equations the latter decides about the violation of parity. Thus, so far a definite statement concerning the loss of parity invariance can only be made w.r.t. the S-matrix which due to the phase factors breaks parity also when the resonances are removed. Further investigations of the quantum theory of these integrable models are necessary in order to settle the issues mentioned.

A first step towards this direction was performed in [14] where the complete set of form factors for the $su(3)_2$-HSG theory has been calculated exhibiting powerful determinant structures similar to those found in the context of the Yang-Lee [88] and the Sinh-Gordon model (see Fring et al. in [13]). The knowledge about the form factors of the energy-momentum tensor allowed to compare the UV central charge obtained in the TBA approach of Section 4.4.4 against the one obtained from the c-theorem mentioned above. Taking the form factors up to the six particle contribution into account one ends up with [14],

$$\text{TBA:} \quad c_{su(3)_2} = 1.2 \qquad\qquad \text{c-theorem:} \quad c^{(6)}_{su(3)_2} = 1.199\ldots\,.$$

This demonstrates perfect agreement between both methods and supports the findings presented here. Again it would be very interesting to extend this analysis to higher levels $k > 2$ and other algebras.

A further open question is to identify the corresponding Lagrangian for general $\mathfrak{g}|\tilde{\mathfrak{g}}$-theories. The knowledge of the ultraviolet central charge (4.201)

$$c^{\mathfrak{g}|\tilde{\mathfrak{g}}}_{\text{eff}} = \frac{\tilde{h}}{h+\tilde{h}}\, n\, \tilde{n}$$

obtained by extending the TBA analysis of the HSG models to the general case will certainly be helpful in this search. It provides the renormalization group fixed point and the perturbing operator might then be identified among the spinless relevant fields in the conformal spectrum. For the minimal affine Toda or scaling models this has already been achieved in the literature. Exploiting periodicities present in the Y-systems a series expansion of the scaling function of the following form has been derived from which the dimension d_Φ of the perturbing operator can be directly read off [23],

$$c(r) = c_{\text{eff}} + f_0\, r^2 + \sum_n c_n r^{2n(1-d_\Phi)} \,.$$

Here f_0 is a constant related to the bulk free energy of the massive model. One might now proceed similar for the more complex $\mathfrak{g}|\tilde{\mathfrak{g}}$-theories paying attention to the parity violation. For the HSG models it has already been demonstrated for the $su(3)$ model that periodicities in the Y-systems occur, which are consistent with the expected dimension of the perturbing field.

The general Lie algebraic classification of these new integrable models might also lead to the discovery of new "duality" relations, an intensively discussed issue in string theory. Analyzing the structure of the constant TBA equations (4.198) in Section 4.5, it became apparent that theories linked to each other by an exchange of the kind $\mathfrak{g} \leftrightarrow \tilde{\mathfrak{g}}$ share the same set of constant TBA solutions. Since structures analogous to the constant TBA equations have also been found in Virasoro characters associated to conformal models [112] one might conjecture on new Lie algebraic identities connecting different conformal field theories. For instance, the "dual" relation between the $A_n|\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}|A_n$-theory would relate the WZNW cosets $\tilde{\mathfrak{g}}/u(1)^{\times \operatorname{rank} \tilde{\mathfrak{g}}}$ and $\tilde{\mathfrak{g}}_1^{\otimes(n+1)}/\tilde{\mathfrak{g}}_{n+1}$ with each other. However, this issue has to be understood in more detail before definite conclusions can be drawn.

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