

Chapter 2

MATHEMATICAL PRELIMINARIES

11.15. *Restate my assumptions.*

1. *Mathematics is the language of nature.*
 2. *Everything around us can be represented and understood through numbers.*
 3. *If you graph these numbers patterns emerge.*
- Therefore: There are patterns everywhere in nature.*

Max Cohen in the film π .

This first chapter prepares the discussion of affine Toda field theory by introducing the mathematical framework used in the subsequent chapters to treat all affine Toda models at once in a complete and generic way. As pointed out in the introduction Toda and affine Toda field theories come naturally equipped with a Lie algebraic structure. Consider the classical Toda field equations (see e.g. [61]) which read

$$\partial_\mu \partial^\mu \varphi_i + \frac{m^2}{\beta} \sum_{j=1}^n A_{ij} e^{\beta \varphi_j} = 0, \quad i = 1, \dots, n \quad (2.1)$$

where the field $\varphi = (\varphi_1, \dots, \varphi_n)$ consists of n -components and transforms as a scalar under the Lorentz group. The constants m, β represent an overall mass scale and the coupling, respectively. Being classically unimportant they become relevant when the system is quantized. The field equations are characterized by the so-called **Cartan matrix** $A = (A_{ij})$ which has only integral entries and is non-singular, i.e. $\det A \neq 0$. Moreover it is subject to the following restrictions:

- (A1) Its diagonal elements take the value $A_{ii} = 2$.
- (A2) The off-diagonal elements are zero or negative, $A_{ij} \leq 0$.
- (A3) For $i \neq j$, $A_{ij} = 0$ implies $A_{ji} = 0$.

Every such matrix can be shown to code the structure of a finite dimensional simple Lie algebra as will be explained below. Similar, the field equations of affine Toda field theory [24], which can be viewed as perturbation of Toda theory, read

$$\partial_\mu \partial^\mu \varphi_i + \frac{m^2}{\beta} \sum_{j=0}^n \hat{A}_{ij} e^{\beta \varphi_j} = 0, \quad i = 0, 1, \dots, n, \quad (2.2)$$

where the set of fields has been extended by a component ($i = 0$) and $\hat{A} = (\hat{A}_{ij})$ is said to be the **extended** or **affine Cartan matrix**. The latter can be constructed from A by adding a row and a column in a specific way explained below. It satisfies the same properties (A1),(A2) and (A3) as the ordinary Cartan matrix but is singular*, i.e. $\det \hat{A} = 0$. Like in the non affine case, a Lie algebra can be linked to the matrix \hat{A} , but this time it is infinite dimensional. However, as it will turn out in the chapters to follow, the structure of the finite dimensional simple Lie algebra is sufficient to describe most of affine Toda field theory. Emphasis is therefore given to finite simple Lie algebras and their geometry, which will appear in its full complexity and elegance when we discuss the classical and quantum mass spectrum of affine Toda theory, its fusing processes of particles and the two-particle S-matrix.

In preparation to this discussion we recapitulate in the first section of this chapter how the Lie algebraic structure can be extracted from the Cartan matrix A appearing in the Toda equations. Starting point is the classification of all possible Cartan matrices which gives the complete set of possible Toda and affine Toda models with regard to (2.1) and (2.2). In the next step simple roots, Weyl groups and abstract root systems are constructed. They constitute the central objects in describing the structure of simple Lie algebras. The latter are introduced by means of the Chevalley-Serre relations and it is shown how the root systems naturally arise in the context of the adjoint representation.

In Section 2.2 the theory of Coxeter and twisted Coxeter elements of a Weyl group is reviewed. Their properties are stated and their action on the root system discussed. The formulas presented are preparatory for the subsequent section.

Section 2.3 contains the key results of the chapter. After introducing the concept of dual algebras q -deformed Coxeter and twisted Coxeter elements are defined. The q -deformation will play a crucial role in exhibiting the coupling dependence of affine Toda field theory as well as in the formulation of its fusing rules and its scattering matrix. Furthermore, it is shown how the information encoded in the root orbits of the q -deformed Coxeter elements is contained in so-called q -deformed Cartan matrices. The latter are of special importance for the integral representation of the ATFT S-matrix which will be exploited in Chapter 4 when we investigate the high energy limit of the associated integrable quantum field theories.

2.1 Simple Lie algebras

In this section we follow the Cartan-Killing classification of simple Lie algebras in reverse order. We start with the most compact and reduced object coding all the necessary information, the Cartan matrix A specified above. Step by step we then introduce more complex structures, fundamental systems, the Weyl group and root systems. The motivation for this procedure will become apparent when relating these objects to simple Lie algebras and their adjoint representation in the final step. The material presented can be found in more detail in several text books, e.g. [62, 25], however, the following summary ought to keep the discussion self-contained and is

*More precisely, it is degenerate positive definite. That is, there exists a diagonal matrix D such that DA is symmetric and positive semidefinite.

focused on relations and quantities which are relevant for our physical application.

2.1.1 Fundamental systems and Dynkin diagrams

Suppose we are given an $n \times n$ Cartan matrix A , then we can always assign to it an n -dimensional Euclidean vector space \mathfrak{E} on which it naturally acts. That is, we understand in the following \mathfrak{E} to be a real vector space with positive definite inner product $\langle \cdot, \cdot \rangle : \mathfrak{E} \times \mathfrak{E} \rightarrow \mathbb{R}$. The action of the Cartan matrix is then defined w.r.t. some special basis $\{\alpha_1, \dots, \alpha_n\}$ which spans \mathfrak{E} and whose scalar products give the Cartan matrix elements,

$$A_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle, \quad \alpha_i^\vee := \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}. \quad (2.3)$$

The elements α_i of such a basis are called **simple roots** and the vector α_i^\vee introduced above is named a **simple coroot**. The real vector space \mathfrak{E} with its inner product $\langle \cdot, \cdot \rangle$ and the basis $\{\alpha_1, \dots, \alpha_n\}$ together are said to form a **simple system**. Note that the set of simple roots is linear independent but in general not orthonormal. In fact, exploiting the relation (2.3) one immediately sees that

$$A_{ij}A_{ji} = 4 \langle \alpha_i, \alpha_j \rangle^2 / |\alpha_i|^2 |\alpha_j|^2 = 4 \cos^2 \delta_{ij}$$

where $0 \leq \delta_{ij} \leq \pi$ is the angle between the simple roots α_i, α_j . Since by its definition the Cartan matrix has only integral values, the possible values for the product $A_{ij}A_{ji}$ are restricted to 0, 1, 2, 3, 4. The latter is excluded by linear independence of the simple roots, because it implies $\delta_{ij} = 0$ or π . Furthermore, from the properties (A2) and (A3) we infer that the allowed off diagonal entries in A are 0, -1 , -2 , -3 corresponding to the angles $\delta_{ij} = \pi/2, 2\pi/3, 3\pi/4, 5\pi/6$. Thus, by virtue of introducing the Euclidean space \mathfrak{E} it follows from simple geometric arguments that the set of all possible Cartan matrices is fairly restricted. This observation turns out to be crucial for their classification, which we will now perform with the help of so-called Dynkin diagrams.

A **Dynkin diagram** Γ is a connected graph consisting of vertices and links encoding a given Cartan matrix and its assigned simple system in the following way. To each of the simple roots there corresponds a vertex and the vertices of two simple roots α_i, α_j are connected by $A_{ij}A_{ji} = 0, 1, 2$ or 3 lines. In particular, if α_i, α_j are orthogonal they are not connected at all. In case that $A_{ij}A_{ji} = 2$ or 3 we must have that $\alpha_i^2 < \alpha_j^2$ or $\alpha_i^2 > \alpha_j^2$ according to (2.3) and the fact that all matrix entries are integral. We then choose the convention to draw an arrow pointing towards the shorter root on the line connecting the corresponding vertices, i.e. $A_{ij} = -1$ and $A_{ji} = -2$ or -3 when $\alpha_i^2 < \alpha_j^2$. In this manner, to each simple system there exists a Dynkin diagram. Vice versa, we can now construct all such diagrams graphically where the number of links between vertices is at most three and interpret them in the above manner as Cartan matrices or simple systems. Then many combinations drop out by contradicting the positive definiteness of the inner product in \mathfrak{E} . The set of allowed diagrams is depicted in Figure 2.1 and Figure 2.2 together with its nomenclature. They are separated in two classes namely those where only simple links occur (the *ADE* series) and those which allow for multiple links (the *BCFG*

series). For obvious reasons, they are called **simply-laced** and **non simply-laced**. Clearly, in the former case all simple roots have the same length, $\alpha_i^2 = \alpha_j^2$, while in the latter they can be different as discussed above. The numeration of the vertices and their colour, black or white, will become important in due course. The dotted lines connecting two vertices in the Dynkin diagrams of the A , D series and E_6 indicate possible permutations of the vertices under which the inner product evaluated on the linear span of the simple roots stays invariant. They are called **Dynkin diagram automorphisms** and will become important later on when discussing dual algebras.

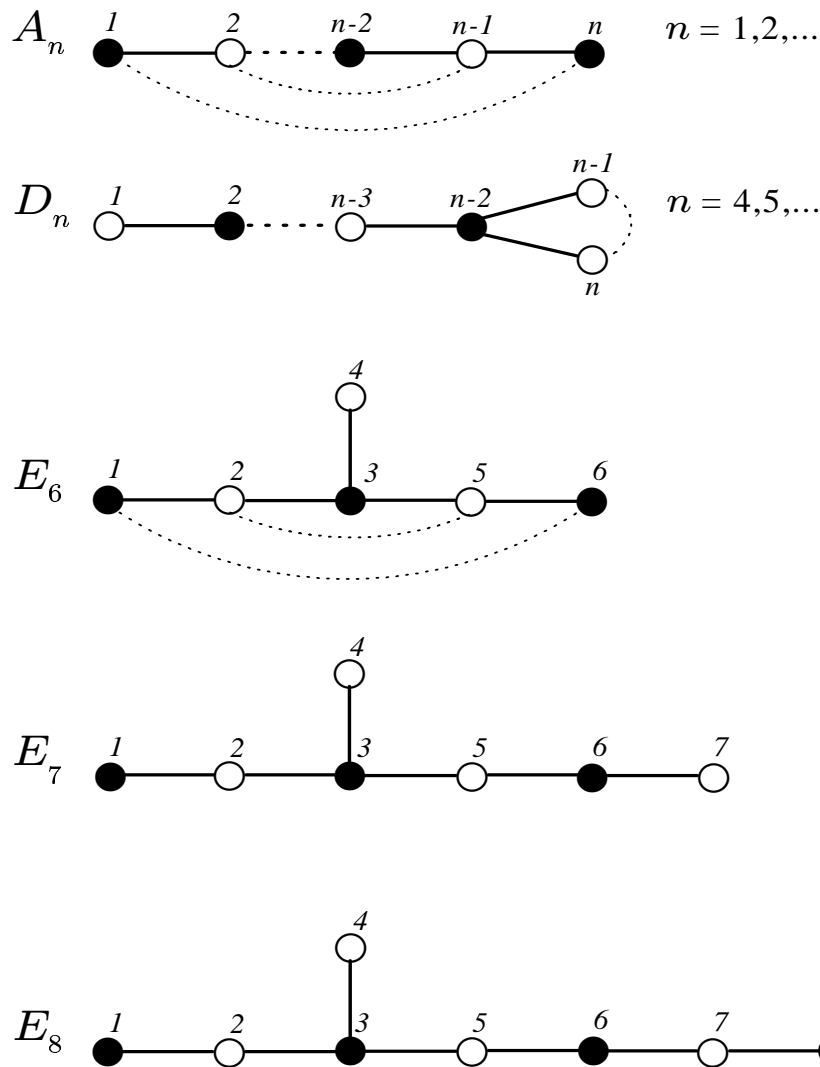


Figure 2.1: Simply-laced Dynkin diagrams

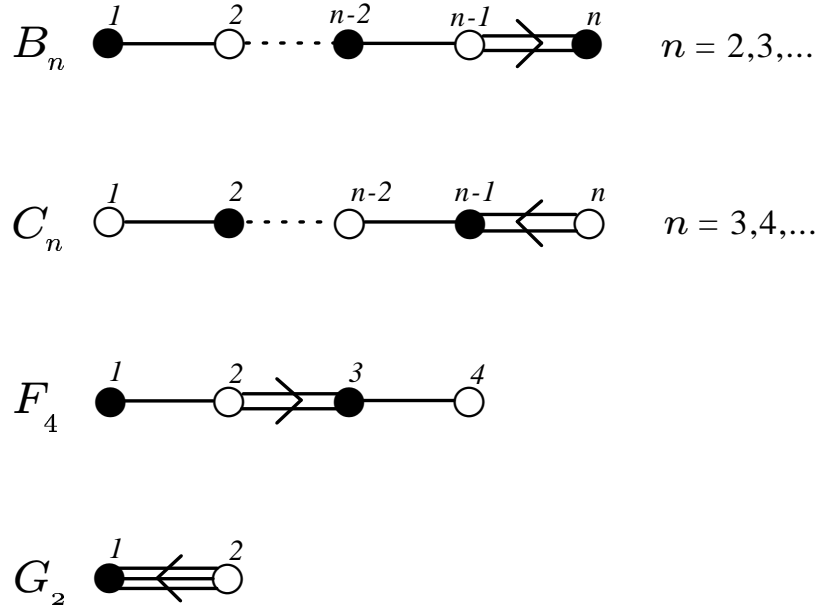


Figure 2.2: Non simply-laced Dynkin diagrams.

2.1.2 Simple Weyl reflections, the Weyl group and root systems

In this subsection we show how to generate a group of isometries $\mathfrak{E} \rightarrow \mathfrak{E}$ from the simple system of a Cartan matrix A . Define the mapping $\sigma_i : \mathfrak{E} \rightarrow \mathfrak{E}$ to be the reflection w.r.t. to the hyperplane orthogonal to α_i ,

$$v \rightarrow \sigma_i v := v - \langle v, \alpha_i^\vee \rangle \alpha_i. \quad (2.4)$$

These reflections associated with each simple root are called **simple Weyl reflections**. They generate the so-called Weyl group W via composition. The latter belongs to the class of Coxeter groups, i.e. its structure is completely determined by a set of relations of the form

$$(\sigma_i \sigma_j)^{x_{ij}} = 1 \quad (2.5)$$

where x_{ij} is always a finite integer number depending on the angle between the two roots α_i, α_j . In particular, $x_{ii} = 1$. From the generating relations (2.5) it follows that the Weyl group is finite, since only a limited number of composite simple Weyl reflections are different from the identity. Thus, the action of the simple Weyl reflections on the set of simple roots yields a finite subset R of \mathfrak{E} which is called a **root system**,

$$R = W\{\alpha_1, \dots, \alpha_n\}. \quad (2.6)$$

The elements in R are called **roots** and will be denoted by Greek letters α, β , etc. in the following. As before for simple roots we define also so-called **coroots** by setting $\alpha^\vee := 2\alpha/|\alpha|^2$. Even though, we have introduced the root system as a set generated from the simple roots by the Weyl group, R can be defined independently of them by its properties:

Root systems. Let \mathfrak{E} be a finite dimensional Euclidean space of dimension n with inner product $\langle \cdot, \cdot \rangle$. Then a finite subset $R \subset \mathfrak{E}$ spanning \mathfrak{E} and not containing 0 is called an (abstract) root system if the following conditions are satisfied:

(R1) For any two elements $\alpha, \beta \in R$ the scalar product $\langle \alpha, \beta^\vee \rangle$ is an integer.

(R2) If $\alpha, \beta \in R$ the element $\beta - \langle \alpha, \beta^\vee \rangle \alpha$ also lies in R .

(R3) The only multiple of $\alpha \in R$ lying also in R is $-\alpha$.

It needs to be emphasized that any root system characterized by the properties (R1),(R2) and (R3) can be obtained from a simple system in the way described. Thus, Cartan matrices, simple systems and root systems constitute in a loose sense equivalent objects and all of them are classified by the Dynkin diagrams depicted in Figures 2.1 and 2.2. In particular, we might also define the Weyl group solely in terms of the root system by associating to each $\alpha \in R$ a Weyl reflection $\sigma_\alpha : \mathfrak{E} \rightarrow \mathfrak{E}$ completely analogous to (2.4). The latter then generate the same Weyl group W . Clearly, the condition (R2) just states the invariance of R under W .

From (2.6) we infer that every root $\alpha \in R$ can be written as an integral linear combination of simple roots by observing that the simple Weyl reflections act on simple roots via the Cartan matrix, $\sigma_i \alpha_j = \alpha_j - A_{ji} \alpha_i$. This naturally induces a \mathbb{Z} -gradation of R when associating to each root its height $\text{ht} : R \rightarrow \mathbb{Z}$,

$$\text{ht}(\alpha) := \sum_{i=1}^n k_i \quad \text{with} \quad \alpha = \sum_{i=1}^n k_i \alpha_i. \quad (2.7)$$

The height functional can be used to define a partial ordering \succ of the root system writing symbolically $\alpha \succ \beta$, whenever $\text{ht}(\alpha) > \text{ht}(\beta)$ for $\alpha, \beta \in R$. One then calls $\alpha \in R$ a **positive root**, in symbols $\alpha \succ 0$, if $\text{ht}(\alpha) > 0$. The set of all positive roots will be denoted by R_+ . In the obvious way, the set of negative roots R_- is defined and its elements are called **negative roots**, in symbols $\alpha \prec 0$. Clearly, $R_- = -R_+$ and more importantly $R = R_+ \cup R_-$. For the classified root systems the partial ordering singles out a unique **highest** or **maximal root** θ satisfying $\text{ht}(\theta) > \text{ht}(\alpha)$ for all roots α . Throughout this thesis the normalization of the inner product is chosen such that

$$\langle \theta, \theta \rangle = 2. \quad (2.8)$$

The linear coefficients $n_i, n_i^\vee \in \mathbb{N}$ of the highest root w.r.t. the basis of simple roots and coroots are called **Coxeter** and **dual Coxeter labels**, respectively. From the definition of the coroots we see that the Coxeter labels are related to their dual counterparts by $n_i = |\alpha_i|^2 n_i^\vee / 2$. Their sums define two important and characteristic constants of a simple Lie algebra, the so-called **Coxeter** and **dual Coxeter number**, both of which we will frequently encounter in the following,

$$h := 1 + \sum_{i=1}^n n_i \quad \text{and} \quad h^\vee := 1 + \sum_{i=1}^n n_i^\vee. \quad (2.9)$$

For later purposes it is important to note that the height functional can be implemented by an element of the vector space \mathfrak{E} . Let $\{\lambda_1, \dots, \lambda_n\}$ and $\{\lambda_1^\vee, \dots, \lambda_n^\vee\}$ denote the dual basis of the simple coroots and simple roots, respectively, i.e.

$$\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij} \quad \text{and} \quad \langle \lambda_i^\vee, \alpha_j \rangle = \delta_{ij}. \quad (2.10)$$

The elements λ_i are called **fundamental weights** and the elements λ_i^\vee **fundamental coweights**. Their sums define the so-called **Weyl** and **dual Weyl vector** which equivalently can also be expressed in terms of the positive roots and coroots, respectively,

$$\rho = \sum_{i=1}^n \lambda_i = \frac{1}{2} \sum_{\alpha \in R_+} \alpha \quad \text{and} \quad \rho^\vee = \sum_{i=1}^n \lambda_i^\vee = \frac{1}{2} \sum_{\alpha \in R_+} \alpha^\vee . \quad (2.11)$$

The Weyl vectors are uniquely determined by the properties $\langle \rho^\vee, \alpha_i \rangle = \langle \rho, \alpha_i^\vee \rangle = 1$ which is immediate by their definition in terms of fundamental (co)weights. The equivalent expressions in terms of (co)roots can be derived by use of the Weyl group. The height functional can now be expressed by the scalar product $\text{ht}(\alpha) = \langle \rho^\vee, \alpha \rangle$. In particular, Coxeter and dual Coxeter number read

$$h = 1 + \langle \rho^\vee, \theta \rangle \quad h^\vee = 1 + \langle \rho, \theta \rangle . \quad (2.12)$$

This closes the discussion of abstract root systems and we now move on from linear algebra to the theory of simple Lie algebras by associating to the elements of each root system a set of Lie algebra generators. However, most of the actual calculations will only use the framework of linear algebra introduced in this and the preceding subsection.

2.1.3 Lie algebras and the Chevalley-Serre relations

The concept of a Lie algebra naturally arises in physics when one considers the infinitesimal generators of symmetry operations mathematically linked to a Lie group G . For instance, if we consider the group $G = SU(2)$ their infinitesimal generators are given by the angular momentum operators $\{J_+, J_-, J_3\}$ which form the well known Lie algebra $su(2)$. The group structure is encoded in their commutation relations,

$$[J_+, J_-] = 2J_3 , \quad [J_3, J_\pm] = \pm J_\pm \quad (2.13)$$

together with the property that J_+ is the Hermitian conjugate of J_- and J_3 is self-conjugate. The Lie group is then regained by "exponentiation" of the generators in an appropriate way. However, from a mathematical point of view the concept of a Lie algebra can be treated separately from the one of a Lie group. We thus start with the abstract definition of a simple Lie algebra and then comment on its realizations in terms of Cartan matrices and simple systems.

Lie algebra. A vector space \mathfrak{g} (real or complex) is called a Lie algebra if it is equipped with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, the so-called **Lie product** or **bracket**, satisfying skew-symmetry and the Jacobi identity, i.e.,

$$[x, y] = -[y, x] \quad \text{and} \quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad (2.14)$$

for all $x, y, z \in \mathfrak{g}$.

The Lie algebras we are going to consider in this section are all finite dimensional, meaning that \mathfrak{g} viewed as a vector space has finite dimension. Furthermore,

we restrict ourselves to **simple Lie algebras**. The latter are characterized by the property that they are not abelian, i.e. $[\mathfrak{g}, \mathfrak{g}] \neq \{0\}$, and contain no proper ideal, i.e. there exists no subalgebra \mathfrak{i} which obeys the relation $[\mathfrak{g}, \mathfrak{i}] \subset \mathfrak{i}$. This particular class of Lie algebras is distinguished by the fact that all its elements are classified through the Cartan matrices or equivalently the root systems presented in the preceding subsections.

Suppose we are given a Cartan matrix A together with an assigned simple system. The most direct way to associate a Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ to them is to define to each simple root α_i a subalgebra $\{e_i, f_i, h_i/2\}$ isomorphic to $su(2)$ specified in (2.13). Here e_i, f_i are identified with the step operators J_+, J_- , respectively, and $h_i/2$ plays the role of J_3 . In order to determine the Lie algebra $\mathfrak{g}(A)$ generated by $\{e_i, f_i, h_i/2\}_{i=1}^n$ completely we need to define in addition the bracket relations between generators belonging to different simple roots. These are known as **Chevalley-Serre relations** and read,

$$\begin{aligned} [h_i, h_j] &= 0, & [h_i, e_j] &= A_{ij}e_j, & [h_i, f_j] &= -A_{ij}f_j, & [e_i, f_j] &= \delta_{ij}h_j, \\ (\text{ad}_{e_i})^{1-A_{ij}}e_j &= 0 & \text{and} & & (\text{ad}_{f_i})^{1-A_{ij}}f_j &= 0 \end{aligned} \quad (2.15)$$

where in the second set of relations the symbol ad stands for the **adjoint action**, i.e. $\text{ad}_x y := [x, y]$. Now, the important result is that the constructed Lie algebra $\mathfrak{g}(A)$ is simple and that any simple Lie algebra can be obtained in this way, i.e. there is a bijective mapping $A \rightarrow \mathfrak{g}(A)$ from the set of Cartan matrices into the set of simple Lie algebras. Hence, the classification scheme for Cartan matrices via Dynkin diagrams carries through to simple Lie algebras. To make the abstract definition of simple Lie algebras more explicit, some examples which can be realized in terms of matrix algebras are stated in the following table,

Lie algebra	matrix algebra
A_n	$su(n+1)$
B_n	$so(2n+1)$
C_n	$sp(2n)$
D_n	$so(2n)$

Table 2.1: Matrix representations of simple Lie algebras.

Here $su(n)$ denotes the Lie algebra of the unitary $n \times n$ matrices with unit determinant, $so(n)$ the Lie algebra of the orthogonal $n \times n$ matrices with unit determinant and $sp(n)$ the n -dimensional symplectic algebra.

Having constructed all simple Lie algebras via the above identification scheme concludes the Cartan-Killing classification program. In the next subsection we briefly comment on how the simple Lie algebras give naturally rise to root systems by means of the adjoint representation. To simplify this discussion we henceforth assume the Lie algebra to be defined over the complex numbers.[†]

[†]So far we have constructed $\mathfrak{g}(A)$ as a real Lie algebra using an underlying real Euclidean vector

2.1.4 The adjoint representations of simple Lie algebras

Every simple Lie algebra \mathfrak{g} might be represented as an operator algebra acting on itself. Regard \mathfrak{g} as a vector space then each element in it can be interpreted as operator by means of the adjoint action defined above,

$$\mathfrak{g} \ni x \mapsto \text{ad}_x . \quad (2.16)$$

Then as an immediate consequence of the relations (2.14) the Lie bracket might be identified with the commutator, $[x, y] \sim \text{ad}_x \circ \text{ad}_y - \text{ad}_y \circ \text{ad}_x$. This realization of a simple Lie algebra is called its **adjoint representation** and suggests in a natural way the introduction of a metric on the algebra by setting

$$(x, y) \mapsto \kappa(x, y) := \frac{1}{2h^\vee} \text{Tr}(\text{ad}_x \circ \text{ad}_y) . \quad (2.17)$$

Here the normalization constant is the so-called Dynkin index of the adjoint representation (see e.g. [62]) which due to our convention (2.8) equals twice the dual Coxeter number. The metric (2.17) is known as **Killing form**. Clearly, κ is symmetric, bilinear and invariant w.r.t. the bracket in the sense that the following relation holds, $\kappa([x, y], z) = \kappa(x, [y, z])$ for all $x, y, z \in \mathfrak{g}$. Moreover, for \mathfrak{g} simple it is non-degenerate, i.e. $\kappa(x, y) = 0$ for all $x \in \mathfrak{g}$ implies $y = 0$. In the following we now outline how the abstract root system defined above naturally appears in the adjoint representation.

From the Serre relations (2.15) we infer that we can write \mathfrak{g} as the following direct sum in the sense of vector spaces,

$$\mathfrak{g}(A) = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$$

where \mathfrak{g}_+ , \mathfrak{g}_- are the subspaces generated by the step operators e_i and f_i , respectively, and \mathfrak{h} denotes the subalgebra spanned by the h_i 's. According to (2.15) it is maximal abelian and such a subalgebra \mathfrak{h} is said to be a **Cartan subalgebra**.[‡] Its dimension as a vector space is called the **rank** of \mathfrak{g} and by construction it coincides with the dimension of the Euclidean space \mathfrak{E} introduced in the context of fundamental and root systems,

$$\text{rank } \mathfrak{g} := \dim \mathfrak{h} = n . \quad (2.18)$$

Because \mathfrak{h} is maximal abelian the adjoint representation restricted to \mathfrak{h} must decompose into a direct sum of one-dimensional representations, i.e. all the elements in \mathfrak{h}

space \mathfrak{E} , however, the abstract definition of a Lie algebra allows also for vector spaces over the complex numbers. In fact, we might associate to $\mathfrak{g}(A)$ the complex Lie algebra $\mathfrak{g}(A) \otimes \mathbb{C} = \mathfrak{g}(A) \oplus \sqrt{-1}\mathfrak{g}(A)$. On the contrary, every complex Lie algebra can be reduced to a real Lie algebra by restriction to real linear combinations. Under this mutual relation the classification scheme presented above carries over to complex simple Lie algebras as well, that is, all complex simple Lie algebras are obtained in this manner.

[‡]In fact, for any complex simple Lie algebra we might speak of *the* Cartan subalgebra, since one can show that independent of the choice of the fundamental system all Cartan subalgebras of \mathfrak{g} are conjugate to each other with respect to the group of inner automorphisms. That is, ghg^{-1} yields another Cartan subalgebra where g is an element of the Lie group G associated with \mathfrak{g} and every Cartan subalgebra may be obtained this way.

can be diagonalized simultaneously. Thus, we may write \mathfrak{g} as a direct sum of \mathfrak{h} and one-dimensional subspaces \mathfrak{g}_α invariant under the action of $\text{ad } \mathfrak{h}$,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \quad \text{with} \quad [h, \mathfrak{g}_\alpha] = \alpha(h)\mathfrak{g}_\alpha, \quad h \in \mathfrak{h}. \quad (2.19)$$

Here the labels α are elements in the dual space \mathfrak{h}^* of the Cartan subalgebra defined by $\alpha : h \rightarrow \alpha(h)$, where the scalar factors $\alpha(h)$ are the eigenvalues of the operators $\text{ad}_h, h \in \mathfrak{h}$ when restricted to the eigenspace \mathfrak{g}_α . As the notation indicates the functionals $\alpha \in \mathfrak{h}^*$ constitute a root system R by identifying the Euclidean space \mathfrak{E} as the subspace of \mathfrak{h}^* obtained when restricting the linear combinations to real numbers. Conversely, given an abstract root system R in an Euclidean space \mathfrak{E} there always exists a simple Lie algebra $\mathfrak{g}(R)$ determined by the direct sum (2.19). The latter is then called **root space decomposition**. In fact, from the above decomposition we immediately derive that the number of roots is related to the dimension of the simple Lie algebra by

$$\dim \mathfrak{g} = |R| + \text{rank } \mathfrak{g} = |R| + n \quad (2.20)$$

The remaining structure which needs to be identified is the inner product $\langle \cdot, \cdot \rangle$ in \mathfrak{E} . It is related to the Killing form of the adjoint representation by setting

$$\kappa(h_\alpha, h_\beta) = \langle \alpha, \beta \rangle$$

where the element $h_\alpha \in \mathfrak{h}$ is uniquely defined by the relation $\alpha(h) = \kappa(h_\alpha, h)$. Similar, like in the case of simple roots one finds to each $\alpha \in R$ a triplet $\{e_\alpha, e_{-\alpha}, h_{\alpha^\vee}/2\}$ of generators which forms a subalgebra isomorphic to $su(2)$ with $e_\alpha, e_{-\alpha}$ playing the role of the ladder operators. The latter span the one-dimensional subspaces $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ in (2.19) and are obtained from the Chevalley generators $e_i = e_{\alpha_i}, f_i = e_{-\alpha_i}$ as multiple commutators via the Chevalley-Serre relations (2.15). These describe how the whole root system R can be constructed from simple roots in terms of Lie algebra generators. The latter represents the decomposition of the Lie algebra $\mathfrak{g}(A)$ into $su(2)$ subalgebras whose mutual commutation rules are reflected by the linear structure of R . In particular, one infers by means of (2.14) that the subspaces obey the relation

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta},$$

where it is understood that $\mathfrak{g}_\alpha = \{0\}$ if α is not a root.

2.2 Affine Lie Algebras

We now come to an extension of the finite simple Lie algebras studied so far. As mentioned in the introduction to this chapter affine Toda field theories are parametrized by an extended or affine Cartan matrix. The latter describes in a similar fashion a Lie algebra $\hat{\mathfrak{g}}$, which is infinite dimensional and closely related to a simple finite dimensional Lie algebra \mathfrak{g} . In this section we will first turn to the Lie algebraic structure and then see at the end how the affine Cartan matrix emerges from the affine root system. However, throughout the subsequent chapters we will only use the structure of the finite Lie algebras but as the name of the integrable theories we are going

to investigate indicates there is naturally associated an affine structure. The latter is particularly important in the context of classical affine Toda field theory as for example in the construction of solutions to the classical equations of motion (2.2), see e.g. [61, 64, 65]. For completeness we therefore present a short survey on the construction of affine Lie algebras, but this exposition will be less detailed than the discussion of simple Lie algebras. For a more profound treatise of affine Lie algebras and their important role in physics see e.g. [25, 63].

Let \mathfrak{g} be a simple finite Lie algebra and $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ its Killing form (2.17). We now assign to this pair an infinite-dimensional graded Lie algebra $\hat{\mathfrak{g}}$ which is called (untwisted) **affine Lie algebra**. At the heart of this construction lies the following geometrical picture in terms of the associated groups. Suppose that G is a compact Lie group associated with \mathfrak{g} . Then we might consider the set of diffeomorphisms from the unit circle into the group, $\mathbb{S}^1 \rightarrow G$. Defining the group multiplication on this set by pointwise multiplication one obtains an infinite-dimensional Lie group, the so-called **loop group** of G whose infinitesimal generators give rise to the **loop algebra**,

$$L(\mathfrak{g}) := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]. \quad (2.21)$$

The above definition as a tensor product is understood in the sense of vector spaces and $\mathbb{C}[t, t^{-1}]$ denotes the commutative algebra of all Laurent polynomials over the field \mathbb{C} , i.e. an element $f \in \mathbb{C}[t, t^{-1}]$ in this algebra is of the form

$$f = \sum_{n \in \mathbb{Z}} f_n t^n, \quad f_n \in \mathbb{C},$$

where only a finite number of the coefficients f_n is nonzero. While the first factor in (2.21) represents the generators of G , the second is associated with an infinitesimal smooth mapping defined on the unit circle, which always allows for an expansion in a Laurent polynomial when setting $t = e^{i\tau}$, $\tau \in \mathbb{R}$. To determine the loop algebra completely we need to specify the Lie bracket structure, which is given by

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n}, \quad (2.22)$$

where $x, y \in \mathfrak{g}$. In context of quantum theory central extensions of Lie algebras play an immanent role. Their occurrence can be understood in the sense of anomalies which arise when classical symmetries get quantized. In Chapter 4 this will be discussed in more detail in the context of conformal field theory. The loop algebra $L(\mathfrak{g})$ has a unique nontrivial central extension $\hat{\mathfrak{g}}$ defined by the exact sequence

$$0 \rightarrow \mathbb{C}k \rightarrow \hat{\mathfrak{g}} \rightarrow L(\mathfrak{g}) \rightarrow 0, \quad (2.23)$$

where $\mathbb{C}k$ is a one dimensional space spanned by the so-called **central element** k . The latter is characterized by the property that it commutes with all other generators. Explicitly, $\hat{\mathfrak{g}}$ as vector space is given by

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k \quad (2.24)$$

and the bracket structure $[\cdot, \cdot] : \hat{\mathfrak{g}} \times \hat{\mathfrak{g}} \rightarrow \hat{\mathfrak{g}}$ takes the following form

$$[\hat{\mathfrak{g}}, k] = 0 \quad \text{and} \quad [x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + \langle x, y \rangle m \delta_{m+n,0} k. \quad (2.25)$$

As a rule we will call $\hat{\mathfrak{g}}$ the **affine Lie algebra** associated with \mathfrak{g} . Analogous to the case of simple Lie algebras one might consider the adjoint representation in order to extract the root system as the eigenvalues of the generators in the Cartan subalgebra. However, the latter turn out to be infinitely degenerate, whence one usually introduces a grading of the affine Lie algebra via the derivation $d \equiv t \frac{d}{dt}$ acting on the polynomial part of the Lie algebra elements setting

$$[d, k] = 0 \quad \text{and} \quad [d, x \otimes f] = x \otimes df \quad (2.26)$$

with $f \in \mathbb{C}[t, t^{-1}]$. Thus, there is a natural gradation of $\hat{\mathfrak{g}}$ obtained by considering the eigenspaces of the degree operator d ,

$$\hat{\mathfrak{g}} = \bigoplus_{n \in \mathbb{Z}} \hat{\mathfrak{g}}_n, \quad \hat{\mathfrak{g}}_n := \{x \in \hat{\mathfrak{g}} : [d, x] = nx\}. \quad (2.27)$$

Clearly, these eigenspaces are spanned by the monomials $f(t) = t^n$. Adding this degree operator to the affine Lie algebra gives rise to what is called the **extended affine Lie algebra**,

$$\tilde{\mathfrak{g}} = \hat{\mathfrak{g}} \oplus \mathbb{C}d = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k \oplus \mathbb{C}d. \quad (2.28)$$

In order to obtain the same structures as in the finite-dimensional, non-affine case it remains to extend the Killing form $\langle \cdot, \cdot \rangle$ to $\tilde{\mathfrak{g}}$. This is done by setting

$$\langle x \otimes f, y \otimes g \rangle := \frac{\langle x, y \rangle}{2\pi i} \oint_{\mathbb{S}^1} \frac{dt}{t} f(t)g(t), \quad f, g \in \mathbb{C}[t, t^{-1}] \quad (2.29)$$

and

$$\langle k, k \rangle = \langle d, d \rangle = \langle x \otimes f, k \rangle = \langle x \otimes f, d \rangle = 0 \quad \langle k, d \rangle = 1. \quad (2.30)$$

One might verify that this definition makes $\langle \cdot, \cdot \rangle$ non-degenerate and invariant on the extended affine Lie algebra, while its restriction to the affine Lie algebra $\hat{\mathfrak{g}}$ is degenerate.

2.2.1 Affine roots and the generalized Cartan matrix

Similar to the non-affine case discussed in the previous section the root system can be recovered from the dual space to the Cartan subalgebra. In analogy to the finite-dimensional case one therefore defines the so-called **affine Cartan subalgebras**,

$$\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}k \subset \hat{\mathfrak{g}} \quad \text{and} \quad \tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}k \oplus \mathbb{C}d \subset \tilde{\mathfrak{g}} \quad (2.31)$$

with \mathfrak{h} being the Cartan subalgebra of the non-affine finite simple Lie algebra \mathfrak{g} . The dual spaces are then given by

$$\hat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\lambda_0 \quad \text{and} \quad \tilde{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\lambda_0 \oplus \mathbb{C}\delta, \quad (2.32)$$

respectively. The new extra elements λ_0, δ are the linear functionals corresponding to the central element and the degree operator and are defined by

$$\lambda_0(k) = 1, \quad \lambda_0(d) = \lambda_0(h) = 0 \quad \text{and} \quad \delta(d) = 1, \quad \delta(k) = \delta(h) = 0,$$

for all $h \in \mathfrak{h}$. As in the finite dimensional case (2.19) one has now the following affine root space decomposition,

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} \oplus \bigoplus_{\hat{\alpha} \in \hat{R}} \tilde{\mathfrak{g}}_{\hat{\alpha}}, \quad (2.33)$$

where the set of **affine roots** is obtained to

$$\hat{R} = \{\alpha + n\delta : \alpha \in R, n \in \mathbb{Z}\} \cup \{n\delta : n \in \mathbb{Z} \setminus \{0\}\} \quad (2.34)$$

and the subspaces $\tilde{\mathfrak{g}}_{\hat{\alpha}} := \{x \in \tilde{\mathfrak{g}} : [h, x] = \hat{\alpha}(h)x, h \in \tilde{\mathfrak{h}}\}$ invariant under the adjoint action of the affine Cartan subalgebras can be explicitly written down as

$$\tilde{\mathfrak{g}}_{\alpha+n\delta} = \{e_{\alpha} \otimes t^n\} \quad \text{and} \quad \tilde{\mathfrak{g}}_{n\delta} = \mathfrak{h} \otimes t^n. \quad (2.35)$$

Note that the set of affine roots contains also the roots of the non-affine Lie algebra \mathfrak{g} , i.e. $R \subset \hat{R}$. In particular, for $n = 0$ we can identify $\tilde{\mathfrak{g}}_{\alpha+n\delta}$ with the non-affine subspace \mathfrak{g}_{α} in (2.19).

The next step is to find a basis for set of affine roots. Looking at the structure of the affine root set (2.34) one immediately infers, that the set of simple roots $\{\alpha_1, \dots, \alpha_n\} \subset R$ of the non-affine Lie algebra must be supplemented by an additional element generating the dependence on the linear functional δ . This element is called the **affine root** and is defined as

$$\alpha_0 \equiv -\theta + \delta.$$

Recall from the non-affine case that there is to each of the simple roots a $su(2)$ subalgebra associated, the so-called Chevalley generators. From the latter the simple Lie algebra can be constructed by means of the Chevalley-Serre relations (2.15) encoded in the Cartan matrix A . The same is true for affine Lie algebras w.r.t. the generalized Cartan matrix \hat{A} which is defined as

$$\hat{A}_{ij} = \langle \alpha_i^{\vee}, \alpha_j \rangle, \quad i, j = 0, 1, \dots, n.$$

Defining the Chevalley generators for the affine root as

$$e_0 \equiv e_{-\theta} \otimes t, \quad f_0 \equiv e_{\theta} \otimes t^{-1}, \quad h_0 \equiv k + h_{-\theta}$$

the set $\{e_i, f_i, h_i\}_{i=0}^n$ generates the affine Lie algebra $\hat{\mathfrak{g}}$ upon invoking the affine version of the Chevalley-Serre relations,

$$\begin{aligned} [h_i, h_j] &= 0, & [h_i, e_j] &= \hat{A}_{ij} e_j, & [h_i, f_j] &= -\hat{A}_{ij} f_j, & [e_i, f_j] &= \delta_{ij} h_j, \\ (\text{ad}_{e_i})^{1-\hat{A}_{ij}} e_j &= 0 & \text{and} & & (\text{ad}_{f_i})^{1-\hat{A}_{ij}} f_j &= 0. \end{aligned} \quad (2.36)$$

The remarkable fact that the above algebraic relations lead to same Lie algebra $\hat{\mathfrak{g}}$ as the geometrical construction in terms of the loop algebra was proven in [25]. The interplay between these two aspects gives rise to a rich mathematical structure and is the deeper reason why affine Lie algebras play an important role in physics [25, 63]. After this short digression we now turn back to simple Lie algebras and their finite-dimensional root spaces.

2.3 Coxeter and twisted Coxeter elements

In the preceding sections we saw how to each simple Lie algebra a Euclidean space and a root system is naturally assigned. In this section we explore some of their intrinsic geometry which we already encountered when constructing the root system by means of simple Weyl reflections. Their collection forms the Weyl group which belongs to the class of Coxeter groups. These kind of groups are distinguished by the existence of special group elements, so-called Coxeter elements, whose properties we are going to exploit in Chapter 3 when discussing the Lie algebraic structure of affine Toda theory.

2.3.1 Bicolouration and Coxeter elements

Consider the Weyl group W of a simple Lie algebra \mathfrak{g} . All elements in the Weyl group are generated from simple Weyl reflections σ_i , i.e. for any $w \in W$ there is a decomposition $w = \sigma_{i_1} \cdots \sigma_{i_\ell}$ in simple Weyl reflections. There exists a longest element in the sense that it is built up from a maximal number of simple Weyl reflections. It is called **Coxeter element** or **transformation** and defined by the product over all simple Weyl reflections,

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_n .$$

Clearly, this definition depends on the particular choice of simple roots, which define the simple Weyl reflections. Also the ordering of the reflections is only a matter of choice. The Coxeter element is therefore only defined up to conjugacy, i.e., given a longest element σ we get another one by the adjoint action of the Weyl group, $\sigma \rightarrow w\sigma w^{-1}$. In fact, this adjoint action is exhaustive on the set of possible Coxeter elements. However, the geometric properties of the Coxeter element we are interested in are shared by all representatives of this conjugacy class [66]:

- (C1) *The Coxeter element fixes no non-zero vector.*
- (C2) *It is of finite order, $\sigma^h = 1$, where h is the Coxeter number defined in (2.9). This means that the Coxeter element permutes the roots in orbits of length h .*
- (C3) *The eigenvalues of σ are of the form*

$$\exp \frac{i\pi s_j}{h} , \quad j = 1, \dots, n$$

where the characteristic set of integers $1 = s_1 \leq s_2 \leq \dots \leq s_n = h - 1$ are called the exponents of the Lie algebra \mathfrak{g} satisfying the relation $s_{n+1-i} = h - s_i$.

Below we will calculate concrete physical quantities in terms of Coxeter elements, whence we need a unique prescription to determine which of the possible Coxeter elements we are going to use. This is achieved by introducing the concept of **bicolouration** for Dynkin diagrams. To every vertex in the Dynkin diagram $\Gamma(\mathfrak{g})$ we assign a colour, black or white, such that two vertices linked to each other are differently coloured. See Figure 2.1 and 2.2 for our conventions. This bicolouration polarizes the index set $\Delta = \{1, \dots, n\}$ into two subsets Δ_+ and Δ_- corresponding to

white and black coloured vertices, respectively. Grouping all simple Weyl reflections according to this polarization a Coxeter element is unambiguously specified by setting [30]

$$\sigma := \sigma_- \sigma_+, \quad \sigma_{\pm} := \prod_{i \in \Delta_{\pm}} \sigma_i. \quad (2.37)$$

This particular definition fixes the Coxeter element uniquely for the following reason. By the definition of the bicolouration the sub-elements σ_{\pm} only contain simple Weyl reflections corresponding to roots which are orthogonal to each other. Therefore, all of them commute in accordance with (2.5) and their relative order in σ_{\pm} does not matter. Moreover, as an immediate consequence of this construction one has the following identities between the sub-elements σ_{\pm} and the Coxeter element,

$$\sigma_{\pm}^2 = 1, \quad \sigma^{-1} = \sigma_+ \sigma_-, \quad \sigma_{\pm} \sigma^x = \sigma^{-x} \sigma_{\pm}. \quad (2.38)$$

Having these formulas at hand facilitates to calculate the action of the Coxeter element on the sets of simple roots and fundamental weights.

Coxeter orbits and coloured simple roots

In the course of our argumentation **Coxeter orbits** denoted by Ω_i will play an essential role. Following the conventions in [30, 29] they are generated by the successive action of the above Coxeter element (2.37) on a “**coloured**” **simple root** $\gamma_i = c_i \alpha_i$ with $c_i = +1$ or -1 when the i^{th} node in the Dynkin diagram $\Gamma(\mathfrak{g})$ is white or black, respectively. The associated orbit reads explicitly

$$\Omega_i := \{ \sigma^x \gamma_i : 1 \leq x \leq h \}. \quad (2.39)$$

Note that we have used the period (C2) of σ in this definition. The motivation to define the Coxeter orbits Ω_i via the coloured roots is that they do not intersect, i.e. $\Omega_i \cap \Omega_j = \emptyset$, and are exhaustive on the set of roots. Moreover, all γ_i 's lie in different orbits and all elements in one orbit are linear independent [30]. This let the coloured simple roots appear as natural entities in the context of the bicolouration of Dynkin diagrams and the definition (2.37), since they constitute a complete set of representatives for the Coxeter orbits Ω_i .

From these facts we can now easily derive the number of roots of the Lie algebra \mathfrak{g} and its dimension. Every coloured simple root gives rise to an orbit with h elements when the Coxeter element acts on it. The set of all these orbits gives the total root system R whence we deduce $|R| = nh$. Together with formula (2.20) derived from the root space decomposition we obtain the dimension of the Lie algebra in terms of the Coxeter number

$$\dim \mathfrak{g} = n(h + 1). \quad (2.40)$$

Action on simple roots and fundamental weights

Exploiting that the simple Weyl reflections in σ_{\pm} all commute we obtain via the definition of simple Weyl reflections (2.4) the following simple relation for the action

of σ on coloured roots

$$\sigma_{c_i}\gamma_i = -\gamma_i \quad \text{and} \quad \sigma_{-c_i}\gamma_i = \gamma_i - \sum_{j \in \Delta_{-c_i}} I_{ij}\gamma_j. \quad (2.41)$$

Here the notation σ_{c_i} is understood in the obvious sense, that $\sigma_{c_i} = \sigma_{\pm}$ for $c_i = \pm 1$, and we have defined the so-called **incidence matrix**

$$I := 2 - A, \quad (2.42)$$

which will frequently be used in the course of our argumentation. Adding both identities (2.41) yields the relation

$$(\sigma_+ + \sigma_-)\gamma_i = - \sum_{j \in \Delta_{-c_i}} I_{ij}\gamma_j \quad (2.43)$$

which upon squaring is seen to relate the eigenvalues of the Cartan matrix with the eigenvalues of the Coxeter element defined in (C3). Indeed, the former can be determined to be $4 \sin^2 \pi s_k / h$, where s_k , $k = 1, \dots, n$ are the exponents.

The action of simple Weyl reflections can be extended to weights. Of particular interest are the fundamental weights (2.10) which form the dual basis of the simple coroots. The action of the special elements is then derived to

$$\sigma_{c_i}\lambda_i = \lambda_i - \alpha_i \quad \text{and} \quad \sigma_{-c_i}\lambda_i = \lambda_i. \quad (2.44)$$

Using the identities (2.41) and (2.44) the Coxeter element can be shown to relate simple roots and fundamental weights which is of practical importance,

$$\gamma_i = (\sigma_- - \sigma_+)\lambda_i = (1 - \sigma^{-1})\sigma^{\frac{1-c_i}{2}}\lambda_i.$$

Note that $(1 - \sigma^{-1})$ never vanishes because of (C1) and the fact that $\sigma^{-1} = \sigma_- \sigma \sigma_-$ is also an Coxeter element. Hence, it can be inverted and by exploiting (C2) we then obtain

$$\lambda_i = \sum_{x=1}^h \frac{x}{h} \sigma^{x+\frac{c_i-1}{2}} \gamma_i \quad (2.45)$$

which supplements the former equation giving γ_i in terms of λ_i . This last identity will play a central role for our further discussion since it allows to relate the information combined in a Coxeter orbit of a simple root to the Cartan matrix via the variant,

$$A_{ji}^{-1} \frac{|\alpha_i|^2}{2} = \langle \lambda_j, \lambda_i \rangle = \sum_{x=1}^h \frac{x}{h} \left\langle \lambda_j, \sigma^{x+\frac{c_i-1}{2}} \gamma_i \right\rangle, \quad (2.46)$$

where the first equation follows from the definition of the Cartan matrix (2.3) and the fundamental weights (2.10). Below q -deformed versions of the relations presented here will be introduced and turn out to be the hinge for revealing the Lie algebraic structure behind affine Toda theory.

2.3.2 Twisted Coxeter elements

In case the Dynkin diagram Γ of the simple Lie algebra \mathfrak{g} allows for symmetry operations leaving Γ invariant, there is a generalized notion of Coxeter elements [67, 25]. From Figure 2.1 we infer that such symmetries are present in the A and D series of simple algebras as well as in E_6 . That is, for any of these algebras there exists an automorphism $\mathfrak{g} \rightarrow \mathfrak{g}$ induced by a reordering of the labels in the Dynkin diagram, $\omega : \{1, \dots, n\} \rightarrow \{\omega(1), \dots, \omega(n)\}$, such that all inner products, i.e. the Killing form, are invariant. A natural action of ω on the root system is given by

$$\omega(\alpha_i) := \alpha_{\omega(i)}.$$

Note that we use the same symbol ω for the mapping defined on the index set and for the one defined on the simple roots. Necessarily, the automorphism ω has finite order, i.e. there is a finite integer ℓ such that $\omega^\ell = 1$, and from Figure 2.1 we infer that the latter is given by $\ell = 1, 2$ or 3 . While $\ell = 1$ represents the case when no symmetry is present, the last possibility $\ell = 3$ only occurs for $\mathfrak{g} = D_4$ which has an enhanced symmetry compared to the other Dynkin diagrams of the D series.

Exploiting the Dynkin diagram automorphism ω we might now define twisted Coxeter elements which share similar properties as the ordinary ones defined in the preceding subsection. Choose an index set $\Delta^\omega \subset \{1, \dots, n\}$ containing a representative of each orbit of ω such that for any pair $i, j \in \Delta^\omega$ the roots γ_i, γ_j lie in different orbits. Set $\Delta_\pm^\omega := \Delta^\omega \cap \Delta_\pm$ where Δ_\pm are the white and black vertices defined in the preceding subsection and note that the Dynkin diagram automorphism is colour preserving, i.e. $\gamma_{\omega(i)} \in \Delta_\pm$ if $\gamma_i \in \Delta_\pm$. Then the corresponding twisted Coxeter element is defined by

$$\hat{\sigma} := \omega^{-1} \hat{\sigma}_- \hat{\sigma}_+, \quad \hat{\sigma}_\pm := \prod_{i \in \Delta_\pm^\omega} \sigma_i. \quad (2.47)$$

The above definition of the twisted Coxeter element differs from the original one given in [67] by conjugation with the automorphism ω^{-1} . However, this does not alter any of its properties but rather amounts to a different choice of the set of representatives, $\Delta^\omega \rightarrow \omega^{-1}(\Delta^\omega)$. This is immediate to see by deriving the transformation properties of the simple Weyl reflections from which one then infers that

$$\omega \hat{\sigma}_\pm \omega^{-1} = \prod_{i \in \omega(\Delta_\pm^\omega)} \sigma_i.$$

Hence, the adjoint action of ω on $\hat{\sigma}$ gives another twisted Coxeter element associated with the index set $\omega(\Delta^\omega)$. Note that this action is not exhaustive on the set of possible twisted Coxeter elements, that is, there exist index sets which can not be obtained from Δ^ω by successive actions of ω . We list further properties of twisted Coxeter elements [67]:

- (TC1) *The twisted Coxeter element fixes no non-zero vector.*
- (TC2) *It has finite order, $\hat{\sigma}^H = 1$ with $H = \ell h$ being the so-called ℓ^{th} Coxeter number [25]. This means in particular that $\hat{\sigma}$ permutes the roots in orbits of length H .*

(TC3) *The eigenvalues of σ are of the form*

$$\varepsilon_j^{-1} \exp \frac{2\pi i s_j}{H}, \quad j = 1, \dots, n.$$

Here ε_i denote the eigenvalues of ω and s_i are the exponents defined in context of the ordinary Coxeter element, see (C3).

Analogous to the non-twisted Coxeter element we now investigate the action on simple roots. Since the twisted Coxeter element only contains the simple Weyl reflections associated with the choice of representatives Δ^ω it is convenient to define the integers

$$\hat{t}_i = \begin{cases} 1 & \text{for } \alpha_i \in \Delta^\omega \\ 0 & \text{for } \alpha_i \notin \Delta^\omega \end{cases}. \quad (2.48)$$

The action of the special elements $\hat{\sigma}_\pm$ on a simple root can now conveniently be written in the following form

$$\hat{\sigma}_{c_i} \gamma_i = (-1)^{\hat{t}_i} \gamma_i \quad \text{and} \quad \hat{\sigma}_{-c_i} \gamma_i = \gamma_i - \sum_{j \in \hat{\Delta}_{-c_i}^\omega} I_{ij} \gamma_j, \quad (2.49)$$

where the sum involving the incidence matrix $I = 2 - A$ is restricted to the representatives $\hat{\Delta}_\pm^\omega$. Analogously to the untwisted case we also define twisted Coxeter orbits by setting

$$\hat{\Omega}_i := \{\hat{\sigma}^x \gamma_i : 1 \leq x \leq H\}$$

where in the definition the finite period of the twisted Coxeter element has been employed, see (TC2).

2.4 Dual algebras and q -deformation

For the discussion of the ATFT S-matrix it will become necessary to introduce the concept of a pair of **dual affine Lie algebras** $(\mathfrak{g}, \mathfrak{g}^\vee)$ which are related to each other by exchanging short and long roots. In other words the identification of roots and coroots, $\alpha \rightarrow \alpha^\vee$, maps the root system of the one algebra onto the other. This is sometimes referred to as Langlands duality. Clearly, simply-laced Lie algebras are left invariant under this duality transformation since all roots are of the same length, i.e. $\mathfrak{g} = \mathfrak{g}^\vee$. The remaining pairs of algebras which are not self-dual must therefore involve non simply-laced algebras. It turns out that one factor, say \mathfrak{g} , is given by an untwisted affine Lie algebra denoted by $X_n^{(1)}$, while the second factor \mathfrak{g}^\vee involves a twisted affine Lie algebra denoted by $\hat{X}_{\hat{n}}^{(\ell)}$. Here we adopted the notation of [25]. The lower indices n, \hat{n} state the rank of the algebras while the upper index $\ell = 1, 2$ or 3 refers to the order of a Dynkin diagram automorphism ω used in the construction of the twisted algebra, see [25] for details. We list all non self-dual pairs in the following table,

untwisted	twisted
$B_n^{(1)}$	$A_{2n-1}^{(2)}$
$C_n^{(1)}$	$D_{n+1}^{(2)}$
$F_4^{(1)}$	$E_6^{(2)}$
$G_2^{(1)}$	$D_4^{(3)}$

Table 2.2: Pairs of non self-dual affine Lie algebras.

In terms of the associated affine Dynkin diagrams these pairs of Lie algebras are related by reversing the arrow between vertices associated with short and long roots. However, in the following we will not use the affine structure, since it will turn out that all the necessary information is already contained in the finite dimensional algebras X_n and $\hat{X}_{\hat{n}}$ whose relation to each other can be easily understood via folding.

2.4.1 Pairs of finite algebras and folding

From the above table we infer that $\hat{X}_{\hat{n}}$ is simply-laced and equipped with a Dynkin diagram automorphism ω of order $\ell = 2$ except for $\hat{X}_{\hat{n}} = D_4$, where $\ell = 3$. Folding the algebra $\hat{X}_{\hat{n}}$ w.r.t. to this automorphism, i.e. fixing the subalgebra invariant under ω , we obtain the Langlands dual of the finite non-simply laced algebra X_n . Following the procedure outlined in [68] this means that given a Chevalley-basis $\{\hat{h}_i, \hat{e}_i, \hat{f}_i\}$ of $\hat{X}_{\hat{n}}$ we choose an index set of representatives $\hat{\Delta}^\omega \subset \{1, \dots, \hat{n}\}$ (as explained in the context of twisted Coxeter elements in subsection 2.3.2) and define the generators of the “folded” non simply-laced Lie algebra by setting

$$h_i^\omega := \sum_{n=1}^{\ell_i} \hat{h}_{\omega(i)}, \quad e_i^\omega := \sum_{n=1}^{\ell_i} \hat{e}_{\omega(i)} \quad \text{and} \quad f_i^\omega := \sum_{n=1}^{\ell_i} \hat{f}_{\omega(i)}$$

where $i \in \hat{\Delta}^\omega$ runs over the representatives of the different orbits and ℓ_i denotes the length of the orbit of the i^{th} simple root, i.e. $\omega^{\ell_i} \hat{\alpha}_i = \hat{\alpha}_i$. It is straightforward to verify that these generators satisfy the Chevalley-Serre relations (2.15) w.r.t. the **“folded” Cartan matrix**,

$$A_{ij}^\omega = \sum_{n=1}^{\ell_i} \hat{A}_{\omega(i)j}, \quad i, j \in \hat{\Delta}^\omega. \quad (2.50)$$

By the Cartan-Killing classification scheme this determines the folded Lie algebra uniquely and one immediately verifies that the folded Cartan matrix A^ω just coincides with the Cartan matrix A of the non simply-laced algebra X_n given in Table 2.2. In particular, A^ω is a $n \times n$ matrix and the rank n of X_n equals the number of orbits $|\hat{\Delta}^\omega|$ of the automorphism ω .

For convenience the numeration of the vertices in the Dynkin diagrams of the dual Lie algebras in Table 2.2 and the set of representatives $\hat{\Delta}^\omega$ will be chosen such that the orbits of the first n simple roots of $\hat{X}_{\hat{n}}$ are identified under folding with the simple roots of X_n , i.e.

$$\hat{\Delta}^\omega := \{1, \dots, n\} \subset \{1, \dots, \hat{n}\}.$$

Below this will enable us to identify particles in the affine Toda theories associated with the dual algebras without relabeling. Furthermore, the bicolouration of the Dynkin diagrams of the dual algebras will be fixed by requiring that in $\Gamma(X_n)$ the unique vertex of the short root which is connected to a long root is black. Upon the above identification between both dual algebras this determines also the bicolouration of $\Gamma(\hat{X}_{\hat{n}})$.

Another feature of the duality between both algebras $(X_n^{(1)}, \hat{X}_{\hat{n}}^{(\ell)})$ we will exploit in the following is the close relationship between their Coxeter numbers h and \hat{h} . The interchange of roots and coroots leads to an interchange of the Kac labels n_i and the dual Kac labels n_i^\vee , whence one must have that

$$h = \hat{h}^\vee \quad \text{and} \quad h^\vee = \hat{h}, \quad (2.51)$$

where h^\vee, \hat{h}^\vee are the dual Coxeter numbers of X_n and $\hat{X}_{\hat{n}}$, compare (2.9). Thus, we might express the Coxeter numbers of the one algebra completely in terms of the other and vice versa. This will become important in the following subsections where we are going to introduce q -deformed Coxeter and twisted Coxeter elements on the two dual algebras $(X_n^{(1)}, \hat{X}_{\hat{n}}^{(\ell)})$. As we already saw in the non-deformed case the Coxeter numbers determine the periods of the Coxeter elements and the length of the orbits. We will encounter this mutual dependence of quantities belonging to the two algebras X_n and $\hat{X}_{\hat{n}}$ more often in due course. In fact, the close relationship between the dual algebras builds the cornerstone for the definition of generalized Coxeter elements via q -deformation. The latter allows to combine the structures of both algebras in the orbits of one Coxeter transformation.

2.4.2 q -deformed Coxeter element of $X_n^{(1)}$

The idea of introducing q -deformed Coxeter elements was first put forward by Oota [36]. In his work many formulas were supported solely by a case-by-case study. The discussion of the following subsections will provide the necessary proofs in a rigorous, generic setting and includes also numerous entirely new relations [37].

Definitions

From the defining equation (2.3) of the Cartan matrix we know that all its entries are integers. This allows in a natural way to introduce q -deformation via so-called **q -deformed integers** notated in the standard fashion by

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad q \in \mathbb{C}.$$

In this chapter the q -deformation will be discussed as a purely mathematical operation only and q will have the status of a formal parameter. That is, for the time being we assume the deformation parameter q to be completely generic. However, in Chapter 3 when discussing the S-matrix of ATFT we will specify q to be a root of unity and also introduce a particular parameterization $q(\beta)$, where β is a coupling constant. In this situation the limit* $q \rightarrow 1$, which we are going to perform at several occasions in

*In the literature this limit is often referred to as "classical" in the sense that $\ln q \propto \hbar$ with $\hbar \rightarrow 0$ being the Planck constant.

this chapter as a consistency check, will correspond to the weak or strong coupling limit.

As we know from the preceding discussion the action of simple Weyl reflections σ_i on simple roots α_i can be expressed in terms of the Cartan matrix A . We now define a q -deformed **Weyl reflection** σ_i^q by deforming the off-diagonal entries of A ,

$$\sigma_i^q(\alpha_j) := \alpha_j - \left(2\delta_{ij} - [I_{ji}]_q\right) \alpha_i . \quad (2.52)$$

Here $I = 2 - A$ denotes the incidence matrix related to the non simply-laced Lie algebra X_n . As motivation for the above definition we observe that the q -deformed Weyl reflection shares two essential features of the non-deformed one, namely it is idempotent and reflections belonging to orthogonal roots commute, i.e.,

$$(\sigma_i^q)^2 = 1 \quad \text{and} \quad (\sigma_i^q \sigma_j^q)^2 = 1 \quad \text{for} \quad \langle \alpha_i, \alpha_j \rangle = 0 . \quad (2.53)$$

In particular, in the classical limit $q \rightarrow 1$ we obtain the usual Weyl reflection defined in (2.4). In general, however, the Weyl reflections (2.52) do not preserve the inner product.

We now proceed analogously as in the non-deformed case to define q -deformed Coxeter elements, i.e. we introduce the bicolouration of the Dynkin diagram with colour values $c_i = \pm 1$ as defined above and order the simple roots in two sets Δ_{\pm} whose elements are mutually orthogonal (see Subsection 2.3.1). This allows to define the two special elements

$$\sigma_{\pm}^q := \prod_{i \in \Delta_{\pm}} \sigma_i^q , \quad (2.54)$$

uniquely which analogously to (2.38) share the property $(\sigma_{\pm}^q)^2 = 1$. So far the definition has closely followed the one of ordinary Coxeter elements. As a new feature in the context of q -deformation we will now take into account the different length of the roots for the definition of the Coxeter element. By our convention (2.8) the long roots have squared length equal to two and we therefore define integers t_i by

$$t_i = \ell \frac{\langle \alpha_i, \alpha_i \rangle}{2} \quad (2.55)$$

The ratio is indeed an integer as follows directly from the definition of the Cartan matrix A . Here $\ell = 1, 2$ or 3 equals the order of the Dynkin diagram automorphism, since the dual of X_n is obtained from $\hat{X}_{\hat{n}}$ as folded algebra w.r.t. ω (see our discussion above). In particular, these integers symmetrize the Cartan matrix A when combined into a diagonal matrix D ,

$$AD = DA^t , \quad D_{ij} = \delta_{ij} t_i . \quad (2.56)$$

In fact, the last property fixes the integers t_i up to a normalization constant. In the present context we need the t_i 's for introducing the map

$$\alpha_i \rightarrow \tau(\alpha_i) := q^{t_i} \alpha_i \quad (2.57)$$

in terms of which the **q -deformed Coxeter element** is then defined by setting [36]

$$\sigma_q := \sigma_-^q \tau \sigma_+^q \tau . \quad (2.58)$$

Note that σ_q is defined unambiguously by our choice of the bicolouration of the Dynkin diagram of X_n . The introduction of the map τ and the particular ordering of the maps in (2.58) will be motivated in retrospect by their actions on simple roots and the identities to which they give rise. Before we start to investigate the latter, notice that the above definition (2.58) is consistent with the "classical" limit, i.e. sending the deformation parameter to one we recover the ordinary Coxeter element (2.37), $\lim_{q \rightarrow 1} \sigma_q = \sigma$.

Action of σ_q in the root space

From the definition of the simple Weyl reflections (2.52) it is straightforward to derive the action on coloured simple roots $\gamma_i = c_i \alpha_i$ yielding the q -deformed version of equation (2.41)

$$\sigma_{c_i}^q \gamma_i = -\gamma_i \quad \text{and} \quad \sigma_{-c_i}^q \gamma_i = \gamma_i - \sum_{j \in \Delta_{-c_i}} [I_{ij}]_q \gamma_j . \quad (2.59)$$

From this relation together with the definition of the q -deformed Coxeter element we see that σ_q generates polynomials in the variable q and simple roots[§]. Thus, denoting by Ω_i^q the orbit of the simple root γ_i a typical element will be of the form

$$(\sigma_q)^x \gamma_i = a_1 q^{y_1} \gamma_{i_1} + a_2 q^{y_2} \gamma_{i_2} + \dots + a_k q^{y_k} \gamma_{i_k}$$

where x and $a_1, \dots, a_k, y_1, \dots, y_k$ are some integers depending on i and i_1, \dots, i_k . Now, the most important feature which motivates the definition (2.58) is the remarkable identity

$$q^{-2H} (\sigma_q)^h = 1 . \quad (2.60)$$

A general proof of (2.60) is not known so far, but it is expected to be quite involved when considering the analogue of the non-deformed case. Therefore, we shall here be content with confirming it by means of a case-by-case analysis in the appendix. Clearly, the property (2.60) generalizes condition (C2) of the usual Coxeter element. It states, that the roots up to multiplication by the factor q^{2H} are permuted in orbits of length h rendering its action on the root space "finite". It needs to be stressed that the length of the orbit is determined by the Coxeter number h of the algebra X_n , while the quasi-periodicity in the deformation parameter is determined by the ℓ^{th} Coxeter number H of the dual algebra $\hat{X}_{\hat{n}}$. This indicates that by means of the chosen q -deformation the characteristics of *both* dual algebras are combined in the structure of the orbits Ω_i^q generated by σ_q .

In anticipation of the discussion of the ATFT S-matrix we mention that each of the orbits Ω_i^q will be associated with a particle species. The operation of charge

[§]Note that the possible nonzero entries in the incidence matrix are $I_{ij} = 1, 2, 3$ which upon q -deformation give rise to the polynomials $[1]_q = 1$, $[2]_q = q + q^{-1}$ and $[3]_q = 1 + q^2 + q^{-2}$.

conjugation yielding the antiparticle is then identified by the transition to another orbit, $C : \Omega_i^q \rightarrow \Omega_{\bar{i}}^q$, which contains the element [37]

$$-q^{-H + \frac{c_{\bar{i}} - c_i}{2} t_i} \sigma_q^{\frac{h}{2} + \frac{c_i - c_{\bar{i}}}{4}} \gamma_i \in \Omega_{\bar{i}}^q. \quad (2.61)$$

Note that despite the first impression the power of the Coxeter element in this relation is integral due to the property $c_i c_{\bar{i}} = (-1)^h$. Repeating the charge conjugation in (2.61) leads to (2.60), provided $t_i = t_{\bar{i}}$. Thus, the property $C^2 = 1$ required on physical grounds is satisfied for simply-laced algebras. For the non-simply laced algebras, all particles will turn out to be self-conjugate whence the relation (2.61) reduces to

$$\sigma_q^{\frac{h}{2}} \gamma_i = -q^H \gamma_{\bar{i}}. \quad (2.62)$$

Remark. *The motivation of the definition (2.61) is analogue to the one known from the simply laced case [30]. This means complex conjugating the field which creates the particle of type i in the classical theory corresponds to the creation of the anti-particle \bar{i} , suggesting to associate $-\gamma_i$ to the anti-particle. However, in context of the quantum theory it will turn out that the classical theory is only recovered in the extreme weak or extreme strong limit of the coupling constant. That is, for $q \rightarrow 1$ we recover the known identity [29] for the simply-laced case $\sigma^{\frac{h}{2} + \frac{c_i - c_{\bar{i}}}{4}} \gamma_i = \gamma_{\bar{i}}$ which relates particles and anti-particles.*

For the time being all further information about the action of the q -deformed Coxeter element we need to proceed is the relation

$$(q^{-c_i t_i} (\sigma_q)^{c_i} + q^{c_i t_i}) \gamma_i = \sum_{j \in \Delta_{-c_i}} q^{\frac{1+c_i}{2} t_i - \frac{1+c_j}{2} t_j} [I_{ij}]_q \gamma_j \quad (2.63)$$

which will turn out to be crucial subsequently. It is derived from (2.59) by acting with $\sigma_{-c_i}^q$ on the first identity and then using the second. Note that for $c_i = -1$ this amounts to the action of the inverse Coxeter element[¶] $(\sigma_q)^{-1} = \tau^{-1} \sigma_+^q \tau^{-1} \sigma_-^q$. We are now going to exploit (2.63) to relate the orbits of the Coxeter element σ_q to a q -deformed Cartan matrix.

2.4.3 q -deformed Cartan matrix of $X_n^{(1)}$

In this subsection we derive the analogue of identity (2.46) for the q -deformed Coxeter element. This will allow us to relate the action of σ_q on X_n to the action of the q -deformed twisted Coxeter element on $\hat{X}_{\hat{n}}$ defined below. In particular, the close interplay between the dual algebras will become manifest.

We start by defining a $n \times n$ matrix M depending on *two* deformation parameters $q, \hat{q} \in \mathbb{C}$, one of which will be related to X_n and the other to $\hat{X}_{\hat{n}}$,

$$M_{ij}(q, \hat{q}) := -\frac{[t_j]_{\hat{q}}}{2} \hat{q}^{\frac{1-c_j}{2} t_j - \frac{1+c_i}{2} t_i} \sum_{x=1}^h \left\langle \lambda_j^\vee, (\sigma_{\hat{q}})^{x + \frac{c_i - 1}{2}} \gamma_i \right\rangle q^{2x + \frac{c_i - c_j}{2} - 1}. \quad (2.64)$$

Here the λ_j^\vee 's denote the coweights (2.10) of X_n , the t_i 's are the integers defined in (2.55) and $\langle \cdot, \cdot \rangle$ is the inner product induced by the Killing form. The reason

[¶]We differ here from the definition of the inverse in [36].

for introducing the matrix M will become apparent momentarily. Multiplying now equation (2.63) by $q^{2x+\frac{c_i-c_j}{2}-1}\sigma_{\hat{q}}^x$ and performing the sum over the powers in the range $1 \leq x \leq h$ we obtain by taking the inner product with λ_j^\vee and exploiting the periodicity property (2.60) the determining equation

$$(q^{-1}\hat{q}^{-t_i} + q\hat{q}^{t_i})M_{ij}(q, \hat{q}) - \sum_{k=1}^n [I_{ik}]_{\hat{q}} M_{kj}(q, \hat{q}) = \frac{1-q^{2h}\hat{q}^{2H}}{2} [t_i]_{\hat{q}} \delta_{ij}. \quad (2.65)$$

Note that the dependence on the colour values of the roots has vanished and that the last equation only involves elementary data of the algebra X_n such as the incidence matrix $I = 2 - A$, the entries of the diagonal matrix D defined in (2.56) and the Coxeter numbers h, H . More importantly, (2.65) can be solved for the matrix M to give the alternative expression

$$M(q, \hat{q}) = \frac{1 - q^{2h}\hat{q}^{2H}}{2} A(q, \hat{q})^{-1} [D]_{\hat{q}}. \quad (2.66)$$

Here we have defined two new objects, the q -deformed symmetrizer $([D]_q)_{ij} := [D_{ij}]_q$ and the **doubly q -deformed Cartan matrix**^{||}

$$A(q, \hat{q}) := q^{-1}\hat{q}^{-D} + q\hat{q}^D - [I]_{\hat{q}}, \quad (2.67)$$

where $([I]_q)_{ij} := [I_{ij}]_q$ and $q^D \equiv \exp(\ln q \cdot D)$ is diagonal with entries $(q^D)_{ij} = \delta_{ij} q^{t_i}$. Note that the classical limit $q, \hat{q} \rightarrow 1$ yields again the non-deformed objects A and D . In particular, after dividing by $(1 - q^{2h}\hat{q}^{2H})$ in (2.65) we recover the identity (2.46) in this limit.

Some comments are due to fully appreciate the equivalence of the expressions (2.64) and (2.66). Looking at the defining relation (2.64) we know from the action of the q -deformed Coxeter element described in the previous subsection that the matrix elements $M_{ij}(q, \hat{q})$ consist of polynomials in the variables q, \hat{q} ,

$$M(q, \hat{q}) = \sum_{x=1}^{2h} \sum_{y=1}^{2H} \mu(x, y) q^x \hat{q}^y. \quad (2.68)$$

Looking at (2.66) this is far from obvious, since the expression contains the inverse of the deformed Cartan matrix (2.67). The latter is only well-defined for q, \hat{q} away from roots of unity, since then $A(q, \hat{q})$ might become singular. The singular values are given by the zeroes of the determinant of $A(q, \hat{q})$, which on a case-by-case basis can be established to

$$\det A(q, \hat{q}) = \prod_{k=1}^n (q\hat{q}^{H/h} + q^{-1}\hat{q}^{-H/h} - 2 \cos \frac{\pi s_k}{h}). \quad (2.69)$$

Here $\{s_1, \dots, s_n\}$ is the set of exponents of the Lie algebra X_n defined by (C3) in Section 2.3.1. However, the matrix $M(q, \hat{q})$ is defined for arbitrary complex values

^{||}In a different context not involving Coxeter geometry a similar expression was obtained in [69].

of the deformation parameters, since the determinant $\det A(q, \hat{q})$ is always contained as a factor in $(1 - q^{2h} \hat{q}^{2H})$ standing in front of the Cartan matrix in (2.66). Hence, possible poles of the determinant are cancelled. Later on we will see that the column vectors of the M -matrix evaluated at the zeroes of the determinant (2.69) can be directly linked to conserved quantities in ATFT. In particular, they will yield the mass spectrum.

2.4.4 Inner product identities

Having established the equivalence between the expressions (2.64) and (2.66) we might now turn the picture around and ask what we can learn from the matrix expression about the action of the q -deformed Coxeter element. In fact, we might plug in (2.66) into the l.h.s. of (2.64) and then solve for the inner product $\langle \lambda_j^\vee, \sigma_q^x \gamma_i \rangle$ by discrete Fourier transformation in the variable q ,

$$-\frac{[t_j]_{\hat{q}}}{2} \hat{q}^{\frac{1-c_j}{2}t_j - \frac{1+c_i}{2}t_i} \langle \lambda_j^\vee, (\sigma_{\hat{q}})^x \gamma_i \rangle = \frac{1}{2h} \sum_{n=1}^{2h} M_{ij}(\tau^n, \hat{q}) \tau^{n(2x - \frac{c_i+c_j}{2})} \quad (2.70)$$

where τ is any root of unity of order $2h$. Here we have used that the powers of the variable q run over the range $1, 2, \dots, 2h$ when the powers of the Coxeter element are restricted to $1, 2, \dots, h$. For practical purposes in calculating the orbits of the Coxeter element σ_q formula (2.70) might not yield an advantage. However, we can use it to prove some inner product identities which are immediate consequences of the matrix properties of (2.66). For example, noting that equation (2.56) also holds for the q -deformed quantities,

$$A(q, \hat{q})[D]_{\hat{q}} = [D]_{\hat{q}}A(q, \hat{q})^t, \quad (2.71)$$

we infer that the M -matrix is symmetric. Upon using the formula (2.70) we end up with the non-trivial identity [37]

$$(q^{2t_j} - 1) \langle \lambda_j^\vee, (\sigma_q)^x \gamma_i \rangle = (q^{2t_i} - 1) \langle \lambda_i^\vee, (\sigma_q)^x \gamma_j \rangle. \quad (2.72)$$

This particular equality has special significance, since it will be used in Chapter 3 to show parity invariance of the ATFT scattering matrix. Another inner product equality we are going to exploit in the discussion of the S-matrix is [37]

$$q^{\frac{(1-c_j)t_j - (1+c_i)t_i}{2}} \langle \lambda_j^\vee, \sigma_q^x \gamma_i \rangle + q^{2H + \frac{(c_j-1)t_j + (1+c_i)t_i}{2}} \langle \lambda_j^\vee, \sigma_{q^{-1}}^{h-x + \frac{c_i+c_j}{2}} \gamma_i \rangle = 0, \quad (2.73)$$

which will be linked to analyticity in the physical sheet. Similar like before, it follows by means of (2.66) and (2.70) from the relation

$$q^{2h} \hat{q}^{2H} M(q^{-1}, \hat{q}^{-1}) = -M(q, \hat{q}) \quad (2.74)$$

which is immediate to verify upon noting that $A(q^{-1}, \hat{q}^{-1}) = A(q, \hat{q})$ and $[D]_{q^{-1}} = [D]_q$. The simple derivation of both formulas (2.72) and (2.73) shows the usefulness of formula (2.66). It should, however, be noted that both inner product identities can also be proven directly [37].

2.4.5 q -deformed twisted Coxeter element of $\hat{X}_n^{(\ell)}$

In this subsection we perform a discussion similar to the previous one but now focussing on the dual algebra \hat{X}_n . We introduce the q -deformed twisted Coxeter element and afterwards show a matrix identity leading to the q -deformed folded Cartan matrix associated with (2.50). As is the non-deformed case the folded Cartan matrix will prove to be identical to the Cartan matrix (2.67) of X_n and, thus, relate the twisted and non-twisted q -deformed Coxeter elements of the dual algebras.

Definitions

We shortly recall the definitions from the beginning of this section. The dual algebra \hat{X}_n is always simply-laced and equipped with a Dynkin diagram automorphism ω of order ℓ . The simple roots $\hat{\alpha}_i$ fall in different orbits w.r.t. ω whose length is given by the integers ℓ_i such that $\omega^{\ell_i} \hat{\alpha}_i = \hat{\alpha}_i$. The largest value of ℓ_i corresponds to ℓ . Furthermore, we choose an index set of representatives $\hat{\Delta}^\omega \subset \{1, \dots, \hat{n}\}$ and separate it in two subsets $\hat{\Delta}_\pm^\omega$ according to the bicolouration of the Dynkin diagram associated to \hat{X}_n . Now, a q -deformation of the simple Weyl reflections

$$\hat{\sigma}_i(\hat{\alpha}_j) = \hat{\alpha}_j - \hat{A}_{ji} \hat{\alpha}_i \quad ,$$

analogously to (2.52) would not lead to a new structure since the incidence matrix $\hat{I} = 2 - \hat{A}$ only contains the entries 0 or 1. One possible way to introduce a non-trivial deformation is by defining the analogue of the map τ in (2.57) but this time using the integers \hat{t}_i defined in (2.48) which single out the representatives,

$$\hat{\tau}(\hat{\alpha}_i) := q^{2\hat{t}_i} \hat{\alpha}_i \quad . \quad (2.75)$$

Using the non-deformed elements $\hat{\sigma}_\pm = \prod_{i \in \hat{\Delta}_\pm^\omega} \hat{\sigma}_i$ of Subsection 2.3.2 a **q -deformed twisted Coxeter element** is defined as [36],

$$\hat{\sigma}_q := \omega^{-1} \hat{\sigma}_- \hat{\tau} \hat{\sigma}_+ \quad . \quad (2.76)$$

Once again the bicolouration ensures that $\hat{\sigma}_q$ is uniquely defined. The particular ordering of the maps will prove important for the characteristics of $\hat{\sigma}_q$. For $q \rightarrow 1$ we obtain the standard twisted Coxeter element (2.47).

Action of $\hat{\sigma}_q$ in the root space

Due to the occurrence of the automorphism ω in the definition (2.76) it turns out to be convenient to regard the objects $\hat{\gamma}_i^\omega := \omega^{\frac{c_i-1}{2}} \hat{\gamma}_i$ instead of the usual coloured roots. From (2.49) and (2.76) the action of $\hat{\sigma}_q$ is then obtained to

$$(-q^{-2c_i})^{\hat{t}_i} \hat{\sigma}_q^{c_i} \hat{\gamma}_i^\omega = \omega^{-c_i} \hat{\gamma}_i^\omega - \sum_{j \in \hat{\Delta}_{-c_i}^\omega} I_{ij} \hat{\gamma}_j^\omega \quad . \quad (2.77)$$

This tells us that the orbits $\hat{\Omega}_i^q$ generated by $\hat{\sigma}_q$ consist of polynomials in the deformation parameter q and simple roots analogous to the untwisted case. The structure

of $\hat{\Omega}_i^q$ is periodic because of the crucial relation

$$q^{-2h} \hat{\sigma}_q^H = 1 . \quad (2.78)$$

This on the one hand reflects the property (TC2) of the usual twisted Coxeter element (2.47) and on the other hand it shows the duality relation between the algebras X_n and $\hat{X}_{\hat{n}}$ by comparison with (2.60). We note that the roles of h and H are just interchanged. Like before we do not give a generic proof of this periodicity property, but verify it case-by-case in the appendix.

For later purposes we define the charge conjugation operation as done in Subsection 2.4.2 for the untwisted Coxeter element. Assign the anti-particle to the orbit $\hat{\Omega}_i^q$ in which we find the element [37]

$$-q^{-h + \frac{c_{\bar{i}} - c_i}{2} \hat{t}_i} \hat{\sigma}_q^{\frac{H}{2} + \frac{c_i - c_{\bar{i}}}{4} (2 - \ell_i)} \hat{\gamma}_i^\omega \in \hat{\Omega}_i^q , \quad i \in \hat{\Delta}^\omega . \quad (2.79)$$

Here i is assumed to belong to the set of representatives $\hat{\Delta}^\omega$ since only these will correspond to particle species later on. Repeating the conjugation yields again $\hat{\Omega}_i^q$ by use of (2.78) provided that $\ell_i = \ell_{\bar{i}}$. Assuming self-conjugation for the non-simply laced algebras as in (2.61), the latter relation reduces to

$$\hat{\sigma}_q^{\frac{H}{2}} \hat{\gamma}_i^\omega = -q^h \hat{\gamma}_{\bar{i}}^\omega . \quad (2.80)$$

In the limit $q \rightarrow 1$ we obtain $\hat{\sigma}^{\frac{H}{2} + \frac{c_i - c_{\bar{i}}}{4} (2 - \ell_i)} \hat{\gamma}_i^\omega = \hat{\gamma}_{\bar{i}}^\omega$, which relates particles and anti-particles in twisted algebras.

2.4.6 q -deformed folded Cartan matrix of $\hat{X}_{\hat{n}}^{(\ell)}$

In order to relate the action of $\hat{\sigma}_q$ to a matrix expression we now perform an analogous calculation as in Subsection 2.4.3 for the untwisted algebra. However, due to the appearance of the Dynkin diagram automorphism ω the computation turns out to be a bit more involved, whence we first state the result and then present the main steps of the derivation. Let λ_i^ω denote the fundamental weight which is dual to all elements inside one ω -orbit, i.e.

$$\lambda_i^\omega := \sum_{k=1}^{\ell_i} \hat{\lambda}_{\omega^k(i)} , \quad i \in \hat{\Delta}^\omega . \quad (2.81)$$

Then we will show below that the $n \times n$ matrix depending on two deformation parameters q, \hat{q} and defined by

$$N_{ij}(q, \hat{q}) := -\frac{q^{-\frac{c_i + c_j}{2}}}{2} \sum_{y=1}^H \left\langle \lambda_j^\omega, (\hat{\sigma}_q)^{y + \frac{c_i - 1}{2}} \hat{\gamma}_i^\omega \right\rangle \hat{q}^{2y + \frac{c_i - 1}{2} \ell_i - \frac{c_j - 1}{2} \ell_j - 1} , \quad i, j \in \hat{\Delta}^\omega \quad (2.82)$$

can be equivalently expressed in terms of the following q -deformed matrices,

$$N(q, \hat{q}) = \frac{1 - q^{2h} \hat{q}^{2H}}{2} A^\omega(q, \hat{q})^{-1} [\hat{D}]_{\hat{q}} . \quad (2.83)$$

The objects appearing in the last equation will be explained step by step. Consider the diagonal matrix $D_{ij} := \ell_i \delta_{ij}$ and the folded incidence matrix $I^\omega = 2 - A^\omega$ associated with (2.50), then we define

$$([\hat{D}]_q)_{ij} := [\ell_i]_q \delta_{ij}, \quad ([I^\omega]_q)_{ij} := \left[\sum_{n=1}^{\ell_i} \hat{I}_{\omega^n(i)j} \right]_q, \quad i, j \in \hat{\Delta}^\omega \quad (2.84)$$

to be their q -deformed counterparts. Setting now analogously to (2.67)

$$A^\omega(q, \hat{q}) = q^{-1} \hat{q}^{-\hat{D}} + q \hat{q}^{\hat{D}} - [I^\omega]_{\hat{q}} \quad (2.85)$$

we obtain a **q -deformed version of the folded Cartan matrix** (2.50) [37]. The restriction of the indices in (2.82) to the subset of representatives, which was chosen to be $\hat{\Delta}^\omega = \{1, \dots, n\}$, appears naturally due to the folding procedure explained at the beginning of this section. In fact, studying the latter we saw that $A^\omega = A$. Furthermore, one easily deduces that the length of the roots in X_n is related to the length of the ω -orbits in $\hat{X}_{\hat{n}}$, namely $\ell_i = t_i$, $i = 1, \dots, n$. Thus, we conclude that $A^\omega(q, \hat{q}) = A(q, \hat{q})$ holds for generic values of the deformation parameters, whence we have proven [37]

$$N(q, \hat{q}) = M(q, \hat{q}), \quad q, \hat{q} \in \mathbb{C}. \quad (2.86)$$

This establishes the previously mentioned equivalence of the roles played by the q -deformed Coxeter and twisted Coxeter elements on the dual algebras. It should be emphasized that the identities (2.64), (2.66) and (2.82), (2.83) together with the equality of the two q -deformed Cartan matrices (2.67) and (2.85) are the key results of this chapter. They will allow for a universal treatment of ATFT upon choosing the deformation parameters in a special coupling dependent way to be specified later. Moreover, they provide an important mathematical tool in which the duality relation between the two dual algebras becomes manifest.

Before we now turn to the proof of (2.83) we state another neat formula for the determinant of the q -deformed folded Cartan matrix [37],

$$\det A^\omega(q, \hat{q}) = \prod_{i=1}^{\hat{s}} (q^{h/H} \hat{q} + q^{-h/H} \hat{q}^{-1} - 2 \cos \frac{\pi \hat{s}_i}{H}) \quad (2.87)$$

where the product runs over the exponents of the algebra $\hat{X}_{\hat{n}}$. In view of (2.69) this is a different specification of the singular values of the Cartan matrix $A(q, \hat{q})$. However, since the exponents of the dual algebras only differ by multiples of the Coxeter number h [25] both formulas are consistent. Note that (2.87) is established on a case-by-case study.

We now turn to the proof of the identity (2.83). Let us first define the following auxiliary $\hat{n} \times \hat{n}$ matrix,

$$\hat{N}_{ij}(q, \hat{q}) := -\frac{1}{2} q^{-\frac{c_i+c_j}{2}} \sum_{y=1}^H \left\langle \hat{\lambda}_j, (\hat{\sigma}_q)^{y+\frac{c_i-1}{2}} \hat{\gamma}_i^\omega \right\rangle \hat{q}^{2y+\frac{c_i-1}{2} \ell_i - \frac{c_j-1}{2} \ell_j - 1}. \quad (2.88)$$

Here $\hat{\lambda}_j$ denotes a fundamental weight of the algebra $\hat{X}_{\hat{n}}$ and the indices run now over the full index set $\hat{\Delta} = \{1, \dots, \hat{n}\}$. Multiplying equation (2.77) from the left with $\hat{q}^{2y + \frac{c_i-1}{2}\ell_i - \frac{c_j-1}{2}\ell_j - 1} \hat{\sigma}_q^y$ and performing the sum over the powers y in the appropriate range yields upon using the periodicity (2.78) the determining equation

$$(-1)^{\hat{t}_i+1} (\hat{q}^{\hat{t}_i} \hat{q})^{-2c_i} \hat{N}_{ij} + \hat{N}_{\omega^{-c_i}(i)j} = \sum_{l \in \hat{\Delta}_{-c_i}^{\omega}} q^{-c_i} \hat{q}^{-2c_i + \frac{c_i-1}{2}\ell_i + \frac{c_i+1}{2}\ell_i} I_{il} \hat{N}_{lj} + (q\hat{q})^{-c_i} \frac{(1 - q^{2h} \hat{q}^{2H})}{2} \delta_{i\omega \frac{1+c_i}{2}(j)}.$$

In contrast to (2.65) it can not be directly solved for \hat{N} because of the indices transformed by ω appearing in the equation. Therefore, we successively replace $i \rightarrow \omega^{-c_i}(i)$ until the order ℓ of the automorphism is reached. The resulting set of iterated equations then allows to set up an equation for the restricted $n \times n$ matrix (2.82) obtained from \hat{N} by the prescription $N_{ij} := \sum_{n=1}^{\ell_j} \hat{N}_{i\omega^n(j)}$ with $i, j \in \hat{\Delta}^{\omega} = \{1, \dots, n\}$. The determining equation for N then reads

$$q^{-c_i} \hat{q}^{-\ell_i c_i} N_{ij} + q^{c_i} \hat{q}^{\ell_i c_i} (-1)^{\ell - \ell_i} (q\hat{q})^{2c_i(\ell - \ell_i)} N_{ij} = \sum_{n=0}^{\ell-1} I_{\omega^n(i)k} (-q^{2c_i})^{n\delta_{1,\ell_i}} \hat{q}^{2nc_i} \hat{q}^{\frac{c_i+1}{2}(\ell_k - \ell_i)} N_{kj} + \hat{q}^{c_i(1-\ell_i)} \sum_{n=0}^{\ell-1} (-q^{2c_i})^{n\delta_{1,\ell_i}} \hat{q}^{2nc_i} \frac{1 - q^{2h} \hat{q}^{2H}}{2} \delta_{ij}.$$

This expression can be simplified by discussing the cases $\ell_i = 1$ and $\ell_i = \ell$ separately and upon noting that $I_{\omega(i)j} = 0$ for $\ell_i = \ell$ and $I_{\omega(i)j} = I_{ij}$ for $\ell_i = 1$. In addition, using the identity

$$(q\hat{q})^{\ell} + (-1)^{\ell-1} (q\hat{q})^{\ell} = (q\hat{q} + q^{-1}\hat{q}^{-1})(q\hat{q})^{\ell-1} \sum_{n=0}^{\ell-1} (-1)^n (q\hat{q})^{2n}$$

the sums in the last expression can be simplified and we obtain the stringent formula (2.83).

2.4.7 Inner Product Identities

We close the section by stating the equivalent relations for the inner product identities (2.72) and (2.73) for the dual algebra $\hat{X}_{\hat{n}}$. They can be proven by the same procedure as in the non-twisted case. The identity linked to parity invariance is given by [37]

$$\langle \lambda_j^{\omega}, \hat{\sigma}_q^x \hat{\gamma}_i^{\omega} \rangle = \langle \lambda_i^{\omega}, \hat{\sigma}_q^{x + \frac{c_j - c_i}{2} + \frac{c_i - 1}{2}\ell_i + \frac{1 - c_j}{2}\ell_j} \hat{\gamma}_j^{\omega} \rangle \quad (2.89)$$

and the one related to analyticity of the S-matrix by [37]

$$\langle \lambda_j^{\omega}, \hat{\sigma}_q^x \hat{\gamma}_i^{\omega} \rangle = -q^{2h+c_i+c_j} \langle \lambda_j^{\omega}, \hat{\sigma}_{q^{-1}}^{H-x+c_i + \frac{1-c_i}{2}\ell_i + \frac{c_j-1}{2}\ell_j} \hat{\gamma}_i^{\omega} \rangle \quad (2.90)$$

It should be mentioned that the proof of both relations in terms of the Coxeter elements turns out to be more complicated as in the non-twisted case. However, it would be desirable to formulate these, since this might provide more profound insight of the Lie algebraic duality between the algebra X_n and $\hat{X}_{\hat{n}}$ and the action of the twisted Coxeter element.