Appendix A

Notation

$\emptyset$	empty set
$\mathbb{R}$	set of real numbers
$\mathbb{M}^n$	set of all real square matrices of order n
Ω	bounded open subset of $\mathbb{R}^n$
Γ	boundary of Ω
$L_2(\Omega)$	space of square integrable functions on Ω
$H^m(\Omega)$	Sobolev spaces of order $m = 1, 2, \ldots$
$\|\| \cdot \|\|$	norm in vector space
$\|\| \cdot \|\|_2$	Euclidean vector norm
$\|\| \cdot \|\|_{0,\Omega}$	norm in the space $L_2(\Omega)$
$\|\| \cdot \|\|_{m,\Omega}$	norm in a Sobolev space
$a_l b_l = \sum_i a_i b_i$	Einstein's sum notation
$\partial_i = \partial / \partial x_i$	partial derivative with respect to x_i

L	differential operator
$\boldsymbol{\Delta}$	Laplace-operator
$\boldsymbol{\nabla}$	Nabla-operator
$\boldsymbol{\nabla} \cdot \mathbf{u} = \mathbf{div}\, \mathbf{u} = \partial_i u_i$	divergence of a vector field
$\boldsymbol{\nabla}\phi = \mathbf{grad}\, \phi = (\partial_1\phi_1, \ldots, \partial_n\phi_n)^T$	gradient of a scalar function
$\boldsymbol{\nabla}\mathbf{u} = \partial_j u_i$	Jacobian of a vector field
$\mathbf{I}$	identity matrix
$\det(\mathbf{A})$	determinant of a matrix
$\text{tr}(\mathbf{A}) = \sum_i a_{ii}$	trace of a matrix
$\\|\mathbf{x}\\|_{\mathbf{A}} = \sqrt{\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x}}$	energy norm
$\lambda_{\text{min,max}}(\mathbf{A})$	min/max eigenvalue of a matrix
$\kappa(\mathbf{A}) = \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}}$	condition number of a matrix
$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$	basis vectors in $\mathbb{R}^n$
$\mathbf{div}\, \mathbf{T} = \sum_i \sum_j \partial_j T_{ij} \mathbf{e}_i$	divergence of a tensor field
$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^{\text{T}}\mathbf{b}$	vector inner product
$\langle \mathbf{A}, \mathbf{B} \rangle = \text{tr}(\mathbf{A}^{\text{T}}\mathbf{B})$	matrix inner product
$\mathbf{u}$	displacements
$\mathbf{t}$	tractions
$\mathbf{n}$	normal
$\boldsymbol{\sigma}$	second Piola-Kirchhoff stress tensor
$\boldsymbol{\varepsilon}$	strain tensor

$\lambda,\ \mu$	Lamé constants
E	Young modulus
ν	Poisson ratio
$\mathbf{C}$	tensor of elastic constants
$\mathbf{A}(\mathbf{u})$	operator of non-linear elasticity
$\mathbf{A}'(\mathbf{u})$	Fréchet derivative of $\mathbf{A}(\mathbf{u})$

Appendix B

Fundamental Solution of Linear Elasticity

A theoretical solution of the Lamé-Navier PDE (3.54) can usually be found for a very limited number of boundary value problems with the special intrinsic symmetry [38]. One of such important cases is the so-called *fundamental solution* that describes the deformation of an infi nitely extended linear elastic medium under the impact of the Dirac-delta $\delta(\mathbf{x})$ distributed force density applied at the point O (see Figure A.1)

$$\frac{E}{2(1+\nu)}\,(\mathbf{\Delta u} + \frac{1}{1-2\nu}\,\text{grad}\,\text{div}\,\mathbf{u}) = -\mathbf{f}\delta(\mathbf{x})\,, \tag{A.1}$$

where $\mathbf{f}$ is a constant vector (not a function of coordinates). The fundamental solution of (3.54), also known as the *Green's function* of linear elasticity is given by

$$\mathbf{u} = \frac{1+\nu}{8\pi E(1-\nu)}\frac{(3-4\nu)\mathbf{f} + f\,\text{cos}\alpha\,\mathbf{e}_r}{r}\,, \tag{A.2}$$

where $f = |\mathbf{f}|$, $r = |\mathbf{r}| = |OP|$ is the vector from the starting point O to the space point P, $\mathbf{e}_r = \mathbf{r}/r$ is the unit vector in the direction of $\mathbf{r}$ and α is the angle between $\mathbf{f}$ and $\mathbf{e}_r$, see Figure A.1. In elasticity theory, it can be shown that every solution of an arbitrary linear elastic boundary value problem can be represented as a linear combination of fundamental solutions. The singularity of the fundamental solution in the point O is the direct result of the ”unnatural” behavior of the Dirac-delta distribution

$$\lim_{r\to 0}\mathbf{u} = \infty\,. \tag{A.3}$$

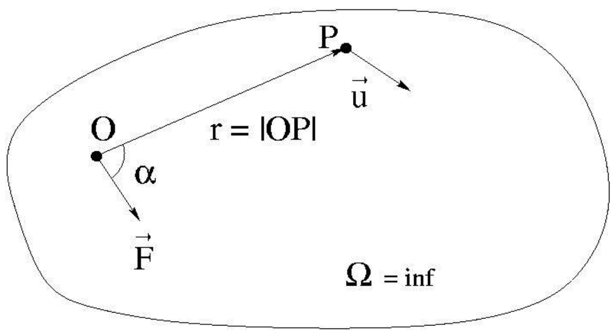

Figure A.1: Deformation of an infinite linear elastic medium under the impact of the Dirac-delta $\delta(\mathbf{x})$ distributed force density applied at the point O.

In [48], a singularity free solution of the Lamé-Navier PDE for the *Gauss*-distributed force density is derived

$$\frac{E}{2(1+\nu)}\,(\mathbf{\Delta u}+\frac{1}{1-2\nu}\,\mathrm{grad\,div}\,\mathbf{u})=-\frac{\mathbf{f}}{(\sqrt{2\pi}\sigma)^3}\,\exp(-\frac{r^2}{2\sigma^2})\,, \tag{A.4}$$

where σ denotes the characteristic width of the Gauss distribution. The solution of (A.4) is given by

$$\mathbf{u}=\frac{1+\nu}{8\pi E(1-\nu)}\{\Phi_f(\xi)\mathbf{f}+\Phi_r(\xi)f\cos\alpha\,\mathbf{e}_r\}\,, \tag{A.5}$$

where

$$\begin{aligned}\Phi_f(\xi)&=\frac{\sqrt{2}}{\sigma}\{\frac{(3-4\nu)\,(\xi)}{2\,\xi}+\frac{(\xi)}{4\,\xi^3}-\frac{\exp(-\xi^2)}{2\sqrt{\pi}\,\xi^2}\}\\ \Phi_r(\xi)&=\frac{\sqrt{2}}{\sigma}\{\frac{(\xi)}{2\,\xi}-\frac{3\,(\xi)}{4\,\xi^3}+\frac{3\exp(-\xi^2)}{2\sqrt{\pi}\,\xi^2}\}\end{aligned} \tag{A.6}$$

with $\xi=\frac{r}{\sqrt{2}\sigma}$. It can be easily shown that (A.5) converges to (A.2) for $r\gg\sigma$ and in the limit $r\to 0$ remains regular

$$\lim_{r\to 0}\mathbf{u}=\frac{(5-6\nu)(1+\nu)}{3(\sqrt{2\pi})^3(1-\nu)E\sigma}\,\mathbf{f}\,. \tag{A.7}$$