Non-commutative Representation for Quantum Systems on Lie groups

PhD thesis

by

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List of Publications

This dissertation has led to the following publications:

1. Carlos Guedes, Daniele Oriti, Matti Raasakka,
   Quantization maps, algebra representation and non-commutative Fourier transform for Lie groups,
   Journal of Mathematical Physics **54**:083508 (2013).

2. Matti Raasakka,
   Non-commutative dual representation for quantum systems on Lie groups,
   Journal of Physics: Conference Series **360**:012052 (2012).

3. Daniele Oriti, Matti Raasakka,
   Quantum mechanics on SO(3) via non-commutative dual variables,
   Physical Review **D84**:025003 (2011).

4. Matti Raasakka,
   Group Fourier transform and the phase space path integral for finite dimensional Lie groups,
   preprint: arXiv:1111.6481 [math-ph].

5. Daniele Oriti, Matti Raasakka,
   Semi-classical analysis of Ponzano-Regge model with non-commutative metric boundary data,
   submitted for publication (preprint: arXiv:1401.5819 [gr-qc]).

Declaration

I declare that this thesis is the result of my own research and has not been accepted for another degree at any university or other institute of tertiary education. Information derived from the published and unpublished work of others has been acknowledged in the text and a list of references is given.

Contents

Chapter 1

Introduction

The topic of this thesis is a new representation for quantum systems on weakly exponential Lie groups in terms of a non-commutative algebra of functions, the associated non-commutative harmonic analysis, and some of its applications to specific physical systems.

Recall that in ordinary quantum mechanics of a point particle on a flat space $\mathbb{R}^d$, one can either choose to represent the wave functions in the position representation, that is, realizing the Hilbert space of the system as (square-integrable) L^2-functions on the configuration space $\mathbb{R}^d$, or in the momentum representation, given again by L^2-functions on the cotangent space $\mathcal{T}^*_x\mathbb{R}^d \cong \mathbb{R}^d$. These two realizations can be independently defined, once a quantization map of the classical Poisson algebra of observables has been chosen. On a Euclidean space the usual Fourier transform gives a map between these representations, i.e., between the two $L^2(\mathbb{R}^d)$ spaces, relating them self-dually.[1] Explicitly, for $\psi \in L^2(\mathbb{R}^d)$, the Fourier transform is given by

$$\tilde{\psi}(\vec{p}) = \int_{\mathbb{R}^d} \mathrm{d}^d x \, e^{-i\vec{p}\cdot\vec{x}} \, \psi(\vec{x}) \ \in L^2(\mathbb{R}^d) \,, \tag{1.1}$$

where $e^{-i\vec{p}\cdot\vec{x}}$ are unitary irreducible representations of the group of translations $\cong \mathbb{R}^d$ in $\mathbb{R}^d$, and $\vec{x}, \vec{p} \in \mathbb{R}^d$. Thus, in the flat case, points $\vec{p}$ of the cotangent space $\mathcal{T}^*_x\mathbb{R}^d$, which is the classical momentum space, are in one-to-one correspondence with unitary irreducible representations of the translational symmetry group of the configuration space.

For a generic curved manifold, a momentum representation in terms of L^2-functions on its cotangent space cannot be defined in the absence of symmetries, nor a notion of Fourier transform. On the other hand, for symmetric spaces and, in particular, for Lie groups

[1] Actually, there are certain technicalities related to the convergence of the Fourier transform, so that one needs to initially consider a smaller space than $L^2(\mathbb{R}^d)$ as the domain of the transform, and then complete the construction in the L^2-norm. In this introductory chapter we neglect these subtleties, to which we will come back later.

the notion of Fourier transform can be generalized as an expansion in terms of unitary irreducible representations of the same group, acting transitively on the configuration manifold. For locally compact abelian groups the transform is mediated by the Pontryagin duality, while for compact non-abelian groups the exact formulation is given by the Peter-Weyl theorem. In both cases, the Fourier transform is defined as a unitary map between $L^2(\mathcal{G})$ and $L^2(\widehat{\mathcal{G}})$, where $\widehat{\mathcal{G}}$ denotes the spectrum of a suitable set of differential operators on the group. Such harmonic analysis has proven a very useful tool in quantum mechanics, quantum field theory in curved spaces, and quantum gravity.

However, some of the nice features of the usual momentum representation and of the usual Euclidean Fourier transform are inevitably lost in this formalism. When considering a physical system, whose configuration space is a more general Lie group $\mathcal{G}$, the momentum space coincides with the dual of the Lie algebra $\mathfrak{g}^*$, which generically differs from $\widehat{\mathcal{G}}$. For example, for $U(1)$, $\widehat{U(1)} = \mathbb{Z}$, while $\mathfrak{u}(1)^* \cong \mathbb{R}$. That is, the dual space $\widehat{\mathcal{G}}$ is a very different object from the cotangent space of a configuration space, the classical momentum space, coinciding only in very special cases such as $\mathcal{G} = \mathbb{R}^d$ above. Therefore, the dual representation obtained from harmonic analysis is not in terms of functions of classical momenta, i.e., functions on the dual of the Lie algebra. This implies that one is bound to lose contact with the classical theory, at least at the formal level, when working with quantum observables that are functions of the momenta. Of course, the same physical information can be recovered in any representation of the quantum system, but in some cases it might be beneficial to maintain a closer formal resemblance with the classical quantities. For example, this may in turn help to have a clearer picture of the underlying physics. In particular, several quantum gravity approaches, most notably loop quantum gravity [66, 60, 61], spin foam models [56] and group field theories [52, 51, 9], work with an underlying classical phase space based on the cotangent bundle over a Lie group, specifically, a direct product of either $SU(2)$'s or $SL(2,\mathbb{C})$'s. While the group elements encode the degrees of freedom of the gravitational connection, the elements of the Lie algebra are related directly to the triad field, thus to the metric itself. A representation which makes directly use of functions of such Lie algebra elements then brings the geometric aspects of the theory to the forefront.

Another possible benefit of having a representation with classical-like continuous momentum variables at disposal in quantum theory, even in the case of a compact configuration space, is that (as we will show below) one then has a direct access to the classical limit of a model via the stationary phase approximation of the first order phase space path integral. To derive this approximation one needs to consider infinitesimal variations of the phase space variables, which is possible only if the variables are continuous.

Such a non-commutative representation has first been proposed in the context of Loop Quantum Gravity, where it also goes under the name of flux representation, and its development and application is now a growing area of research [25, 26, 36, 6, 7, 8, 5, 53, 57, 54, 20]. However, it had mainly been introduced as a derived product of the usual group representation, obtained from a non-commutative Fourier transform, whose mathematical basis had remained only partially explored, and which still had a certain flavour of arbitrariness in its defining details (e.g., plane waves and star-product).

In the first part of this thesis, Chapter 2, after introducing the necessary mathematical preliminaries, we show how the non-commutative representation can be defined independently of the group representation on the sole basis of the choice of a quantization map of the classical Poisson algebra, and we identify more clearly the conditions for its existence. Secondly, we clarify under which conditions a unitary map between such a non-commutative representation and the usual group representation can be constructed, and characterize the non-commutative Fourier transform together with the corresponding non-commutative plane waves. In looking to the above, we try to work with as general a Lie group $\mathcal{G}$ as possible. Thirdly, we consider specific and interesting choices of quantization maps and Lie groups, and exhibit the corresponding star-products, non-commutative representations and the plane waves. On the one hand, the examples presented prove the non-emptiness of the definitions provided together with the existence of their non-commutative representation and of their non-commutative Fourier transforms; on the other hand, the results of specific quantization maps can find direct applications, as we discuss in the following, to physics models. In particular, we identify the non-commutative plane waves and star-product for the Duflo map — a special case of the Kontsevich star-product —, which has been suggested to be useful in several quantum gravity contexts [1, 63, 62, 49].

The construction we present extends earlier work on the non-commutative Fourier transform by several authors. The concept arose originally in considerations of the phase space structure of 3-dimensional Euclidean quantum gravity models. The earliest notion (to our knowledge) of a non-commutative Fourier transform for the group SU(2) appeared in a paper by Schroers [64] (see also [43] by Schroers & Majid), where the construction is based on the duality structure of the quantum double DSU(2), which is introduced as a quantization of the classical phase space ISO(3). Later, more explicit notions of what became to be called 'group' Fourier transform were introduced, first for the group SO(3) by Freidel & Livine in [25], and later extended to SU(2) and related to the quantum group Fourier transform by Freidel & Majid [26], Joung, Mourad & Noui [36] and Dupuis, Girelli & Livine [20], each in their own different ways. See also [59, 28]. To a certain extent, our construction can be considered as yet another extension of the original concept in [25] to more general classes of non-commutative structures and Lie groups. However, it derives from the canonical structures of the classical phase space, the cotangent bundle of $\mathcal{G}$, of the quantization map applied to it, and of the corresponding quantum observable algebra. Thus, it also provides a better general understanding of the relation of the non-commutative Fourier transform to these fundamental underlying structures.[2]

Our first physics application of the above non-commutative methods is the formulation of a non-commutative momentum representation for quantum mechanics on a Lie group in Section 3.1. Quantum mechanics on a Lie group has been considered, following the seminal work of DeWitt [15] on quantum mechanics on general curved manifolds, first by Schulman [65] for $SU(2)$ and later by Dowker [19] for semi-simple Lie groups. This work has been later expanded upon, e.g., by Marinov and Terentyev [45]. In the case of group manifolds, the group structure, and thus ensuing homogeneity and representation theory, admit a considerable simplification compared to the general case considered by DeWitt. These formulations are considered largely satisfactory, apart from some disagreement about quantum correction terms in the path integral formulation [15, 47, 39]. Therefore, it is of a particular interest to apply the new non-commutative methods to a well-known physics model, quantum mechanics on $SO(3)$, to be able to compare the results, and to gain more insight into the interpretation of the non-commutative variables. We show that the phase space path integral obtained via the non-commutative approach yields the correct classical equations of motion in the classical limit, via the stationary phase approximation, and produces quantum corrections to the action consistent with those obtained originally by DeWitt [15], provided that one takes into account the non-commutative deformation of the phase space in the variational calculus. In Section 3.1, we review our results from [54], while generalizing to other Lie groups using the results of [29] where possible.

Another physics application we consider is in the context of spin foam models, which have in recent years arisen to prominence as a possible candidate formulation for the quantum theory of spacetime geometry. (See [56] for a recent review.) The formalism of spin foams derives mainly from topological quantum field theories [3] and Loop Quantum Gravity [60, 66], but it can also be seen as a generalization of matrix models for 2d quantum gravity via group field theory [52, 23]. For 3d quantum gravity, the relation between spin foam models and canonical quantum gravity is understood. In particular, it is known that the Turaev-Viro model is the covariant version of the canonical quantization (à la Witten) of 3d Riemannian gravity with a positive cosmological constant, while the Ponzano-Regge model is the limit of the former for a vanishing cosmological constant [2]. However, in 4d the situation is less clear. Several different spin foam models for 4d quantum gravity have been proposed in the literature, and there is thus far no solid consensus on the correct choice in the sense of defining a spacetime covariant formulation

[2]For other directions to Fourier analysis on Lie groups, let us in particular point to the extensive work on the Kirillov orbit method [38], subsequent (Fourier) analysis based on the decomposition of $\widehat{G}$ into orbits in $\mathfrak{g}^*$ [68], and the Helgason Fourier transform [34] for further reference.

of Loop Quantum Gravity. These 4d models differ specifically in their implementation of the necessary simplicity constraints on the underlying topological BF theory, which should impose geometricity of the two-complex corresponding to a discrete spacetime manifold and give rise to local degrees of freedom [55]. Therefore, further study of the geometric content of spin foam models is called for. In the 3d case, the Ponzano-Regge spin foam model is known to reproduce Regge gravity, the discretized version of general relativity, in the classical limit [18]. In 4d, Regge action was recovered first for a single 4-simplex [10] and later for a fixed spin labeling, when both boundary and bulk variables are scaled to the classical limit [14, 31, 32]. Recently, in [35], the first asymptotic analysis of the full 4d partition function was given in terms of wave front sets, which revealed some worrying accidental curvature constraints on the geometry of several widely studied 4d models.

Classically, spin foam models, as discretizations of continuum theories, are based on a phase space structure, which is a direct product of cotangent bundles over a Lie group that is the structure group of the corresponding continuum principal bundle (e.g., $SU(2)$ for 3d Riemannian gravity). The group part of the product of cotangent bundles thus corresponds to discrete connection variables on a triangulated spatial hypersurface, while the cotangent spaces correspond to discrete metric variables (e.g., edge vectors in 3d, or face bivectors in 4d). Accordingly, the geometric data of the classical discretized model is transparently encoded in the cotangent space variables. However, when one goes on to quantize the system to obtain the spin foam model, the cotangent space variables get quantized to differential operators on the group. Typically (for compact Lie groups), these geometric operators possess discrete spectra, and so the transparent continuous classical geometry gets replaced by somewhat more obscure quantum geometry, which is described by discrete spin labels. (Hence the name 'spin' foams.) This is bound to make the geometric content of the models less obvious.

Our aim in Section 3.2 is to initiate the application of the above non-commutative methods in analysing the geometric properties of spin foam models, in particular, in the classical limit. Indeed, in the context of spin foams, the non-commutative but continuous metric variables obtained through the non-commutative Fourier transform correspond to the classical metric variables in the sense of deformation quantization. Thus, it enables one to describe the quantum geometry of spin foam models and group field theory [6, 5] (and Loop Quantum Gravity [4, 16]) by classical-like continuous metric variables. We will restrict our considerations to the 3d Ponzano-Regge model to have a better control over the formalism in this simpler case. However, already for the Ponzano-Regge model we discover non-trivial properties of the metric representation related to the non-commutative structure, which elucidate aspects of the use of non-commutative Fourier transform in the context of spin foam models. In particular, the choice of quantization map for the algebra of geometric operators turns out to have unexpected consequences for the geometric interpretation of metric boundary data obtained through the transform. Again, in general, only if the deformation structure of the phase space is taken into account in the variational calculus, one finds the correct classical geometric constraints corresponding to simplicial geometry. In any case, the non-commutative Fourier transform is seen to facilitate a straightforward asymptotic analysis of the full partition function via stationary phase approximation.

In summary, the plan of the presentation is as follows: After introducing the necessary mathematical aparata in Section 2.1 that we will use in the rest of the thesis, we define the algebra that quantizes the canonical symplectic structure over a Lie group in Section 2.2. Subsequently, we define two complementary faithful representations of this quantum algebra as operators on spaces of functions. The first representation is the canonical one in terms of functions on the group itself, while the second is a representation defined on a non-commutative function space obtained via deformation quantization of the subalgebra corresponding to canonically conjugate cotangent space variables. Finally, we define the integral transform, which we call the non-commutative Fourier transform, that intertwines these two complementary representations.

In the second part of the thesis, Chapter 3, we consider the applications mentioned above of the just introduced formalism in the treatment of some specific models in physics. First, in Section 3.1, using the non-commutative Fourier transform, we write down the phase space path integral for quantum mechanics on an exponential Lie group. We show that the non-commutative conjugate variables allow for a convenient study of the semi-classical limit through variational methods even for compact groups, for which the spectra of invariant operators, the quantum mechanical momentum space, is discrete. Secondly, in Section 3.2, we apply the non-commutative methods to define a metric representation of the Ponzano-Regge model of 3d quantum gravity. In this case, the non-commutative variables correspond physically to discretized triad variables, which encode explicitly the discrete spacetime geometry of the model. Thus, the non-commutative representation allows for a more transparent view on the geometric properties of spin foam models. Last, as above for quantum mechanics, we show that one may conveniently study the geometric properties of the model in the semi-classical limit by utilizing the full power of path integral methods.

The research, whose results are to be exhibited hereon, has been conducted in collaborations with Carlos Guedes and Daniele Oriti, and derives from the publications [29, 58, 54, 50]. However, any errors in the following presentation are solely the responsibility of the current author.

Chapter 2

Non-commutative Representation for Lie Groups

2.1 Mathematical Preliminaries

2.1.1 Structure of Lie groups and algebras

In this subsection we recall some basic facts about the structure of Lie groups and their canonical fiber bundles. For more details, see for example [34, 33, 67].

A *group* is a set G equipped with a *multiplication map* $m : G \times G \to G$, $m(g,h) \equiv gh \in G$, which satisfies the following axioms:

1. The existence of a unit element e:
   There exists an element $e \in G$ such that $eg = ge = g$ for all $g \in G$.

2. The existence of inverse elements g^{-1}:
   For all $g \in G$ there exists an element $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$.

With these axioms, both the unit element and the inverse elements are unique. A group is called *abelian*, if the multiplication is *commutative*, i.e., $\forall\, g,h \in G\ gh = hg$ and *non-abelian* otherwise.

A *Lie group* $\mathcal{G}$ is a group, which is also a *differentiable manifold*, referred to as the group manifold of $\mathcal{G}$, or just as $\mathcal{G}$. In particular, there exists an atlas $\{(\mathcal{U}_i, \phi_i)\}_i$ of coordinate patches on $\mathcal{G}$, where $\{\mathcal{U}_i \subset \mathcal{G}\}_i$ is an open covering of $\mathcal{G}$, and the coordinate functions $\phi_i : \mathcal{U}_i \to R_i \subseteq \mathbb{R}^d$ are homeomorphisms to open subsets R_i of $\mathbb{R}^d$ for some constant $d \in \mathbb{N}$ called the dimension $\dim(\mathcal{G})$ of $\mathcal{G}$.[1] The transition functions $\phi_i \circ \phi_j^{-1} : \phi_j(\mathcal{U}_i \cap \mathcal{U}_j) \to \phi_i(\mathcal{U}_i \cap \mathcal{U}_j)$ are required to be smooth for all i, j. In addition, we require the multiplication map $(g,h) \mapsto gh$ to be differentiable.

[1] In this work we will be concerned only with finite dimensional Lie groups.

The group properties give a differential manifold important extra structure. In particular, the *left* and *right multiplications* $h \mapsto gh$ and $h \mapsto hg$, where $g, h \in \mathcal{G}$, respectively, induce automorphisms of the manifold. By the virtue of such automorphisms, we know everything local about the manifold just by knowing a neighborhood of the unit element.[2]

Let us denote the linear vector space of smooth functions on $\mathcal{G}$ by $C^\infty(\mathcal{G})$. A smooth curve in $\mathcal{G}$ passing through the point $g \in \mathcal{G}$ is a smooth map $\gamma : (-\epsilon, \epsilon) \subset \mathbb{R} \to \mathcal{G}$ such that $\gamma(0) = g$. We may define the *tangent space* $\mathcal{T}_g\mathcal{G}$ at the point $g \in \mathcal{G}$ as the linear vector space of directional derivative operators, or *tangent vectors*, $\hat{X}_g : C^\infty(\mathcal{G}) \to \mathbb{R}$ at $g \in \mathcal{G}$,

$$\hat{X}_g \phi := \frac{\mathrm{d}}{\mathrm{d}t} \phi(\gamma(t)) \Big|_{t=0} , \tag{2.1}$$

where the curve γ is defined as above. $\mathcal{T}_g\mathcal{G}$ is isomorphic to $\mathbb{R}^d$ for $d = \dim(\mathcal{G})$ and for all $g \in \mathcal{G}$. The union of the tangent spaces $\mathcal{T}\mathcal{G} := \cup_{g \in \mathcal{G}} \mathcal{T}_g\mathcal{G}$ of $\mathcal{G}$ is called the *tangent bundle* over $\mathcal{G}$. Sections of the tangent bundle are *vector fields* $\hat{X} : C^\infty(\mathcal{G}) \to C^\infty(\mathcal{G})$ defined via the point-wise restriction $(\hat{X}\phi)(g) \equiv \hat{X}_g \phi$.

For a Lie group the tangent space $\mathcal{T}_e\mathcal{G}$ at $e \in \mathcal{G}$ can be identified with the *Lie algebra* $\mathfrak{g}$ of the group $\mathcal{G}$: In a neighborhood $\mathcal{U}_e \subset \mathcal{G}$ of the unit element $e \in \mathcal{G}$, we may locally exponentiate the action of the directional derivative, so that $\exp(t\hat{X}_e)\phi \equiv \phi(e^{itX})$ for all $\phi \in C^\infty(\mathcal{G})$, $t \in \mathbb{R}$ small, for a unique group element denoted as $e^{itX} \in \mathcal{U}_e$. By definition, $X \in \mathfrak{g}$ is a unique Lie algebra element. The Baker-Campbell-Hausdorff (BCH) formula

$$\begin{aligned} B(X, Y) &:= -i \ln(e^{iX} e^{iY}) \\ &= X + Y + \frac{i}{2}[X, Y] - \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [X, Y]] + \ldots \in \mathfrak{g} , \end{aligned} \tag{2.2}$$

where $[X, Y] := XY - YX$ is the *Lie bracket*, pulls back the group multiplication onto the Lie algebra. It can be shown that all the higher order terms may also be expressed solely in terms of Lie brackets. From this relation we see that, importantly, the Lie algebra is closed under the Lie bracket, so that $-i[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$. Thus, we may write $[e_i, e_j] = i c_{ij}{}^k e_k$ for an orthonormal basis $e_i \in \mathfrak{g}$, where $c_{ij}{}^k \in \mathbb{R}$ are called the *structure constants* of $\mathfrak{g}$.[3]

A Lie group $\mathcal{G}$ is called *exponential*, if $\exp(i\mathfrak{g}) = \mathcal{G}$, i.e., if the exponential map $\exp : i\mathfrak{g} \to \mathcal{G}$ is surjective onto $\mathcal{G}$. If the image of exp is dense in $\mathcal{G}$, $\mathcal{G}$ is called weakly exponential. The logarithm map $\ln : \mathcal{G} \to i\mathfrak{g}$, the formal inverse of exp, is generally multivalued, but in the exponential case we can construct an injective map $-i \ln_R : \mathcal{G} \to \mathfrak{g}$ by restricting its values onto the principal branch. Then, we have $\exp \circ \ln_R = \mathrm{id}_\mathcal{G}$, the identity map $\mathrm{id}_\mathcal{G}(g) = g$ on $\mathcal{G}$, and $\ln_R \circ \exp =: R_\mathcal{G} : \mathfrak{g} \to \mathfrak{g}$ is the *canonical surjective restriction* onto

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[2]More generally, this is true about symmetric spaces, which have a transitive group action on them, but are not necessarily groups themselves.

[3]We will be using the physicists' convention of multiplying the Lie algebra by the imaginary unit, which gives Hermitean Lie algebra elements for unitary groups.

the principal branch that identifies all elements of $\mathfrak{g}$ that map to the same group element through the exponential map. We may also define a corresponding embedding map for functions[4] $i_{\mathfrak{g}} : C^\infty(\mathcal{G}) \to C^\infty(\mathfrak{g})$ as $i_{\mathfrak{g}}(f)(X) = f(e^{iX})$. $i_{\mathfrak{g}}(f) \in C^\infty(\mathfrak{g})$ is evidently $\mathcal{G}$-*periodic*, meaning that $i_{\mathfrak{g}}(f)(X) = i_{\mathfrak{g}}(f)(R_{\mathcal{G}}(X))$ for all $X \in \mathfrak{g}$. Let us denote the space of $\mathcal{G}$-periodic smooth functions on $\mathfrak{g}$ by

$$C^\infty_{\mathcal{G}}(\mathfrak{g}) := \{f \in C^\infty(\mathfrak{g}) : i_{\mathfrak{g}}(f)(X) = i_{\mathfrak{g}}(f)(R_{\mathcal{G}}(X)) \ \forall \ X \in \mathfrak{g}\} . \tag{2.3}$$

Any $\mathcal{G}$-periodic function on $\mathfrak{g}$ can be unambiguously mapped onto $\mathcal{G}$ via $i_{\mathfrak{g}}^{-1} : C^\infty_{\mathcal{G}}(\mathfrak{g}) \to C^\infty(\mathcal{G})$ given by $i_{\mathfrak{g}}^{-1}(f)(g) = f(-i \ln_R(g))$, because all the branches of the logarithm map to the same values.

A linear smooth map $\pi : \mathcal{G} \to \mathrm{Aut}(V)$, where $\mathrm{Aut}(V)$ is the algebra of automorphisms of a vector space V, is called a *representation* of $\mathcal{G}$, if $\pi(gh) = \pi(g)\pi(h)$ for all $g, h \in \mathcal{G}$ and $\pi(e) = \mathrm{id}_V$. A representation π is called *faithful* if $\ker \pi := \{g \in \mathcal{G} : \pi(g) = \mathrm{id}_V\} = \{e\}$. We call

$$\tilde{\pi} : \mathfrak{g} \to \mathrm{Aut}(V), \ \tilde{\pi}(X) = -i \left.\frac{\mathrm{d}}{\mathrm{d}t} \pi(e^{itX})\right|_{t=0} \tag{2.4}$$

the induced representation of the Lie algebra $\mathfrak{g}$ corresponding to the representation π of $\mathcal{G}$. Then, we may characterize the relation between the tangent space at unity and the Lie algebra more concretely: In a neighborhood of the identity we have $\pi(e^{iX}) = e^{i\tilde{\pi}(X)}$, and $\tilde{\pi}(X) = -i\hat{X}_e\pi$. Typically, we would have, for example, $V \cong \mathbb{C}^n$ and thus $\mathrm{Aut}(V) \cong GL(n, \mathbb{C})$, the algebra of $n \times n$ invertible complex-valued matrices, but other useful possibilities do exist, as we will see.

The right multiplication induces a corresponding translation on $C^\infty(\mathcal{G})$ via $(R_h\phi)(gh) = \phi(g)$. Now, with the push-forward of the right translation $R_{h*} : \mathcal{TG} \to \mathcal{TG}$, defined fiber-wise via

$$(R_{h*}\hat{X}_g)\phi \equiv \hat{X}_g(R_{h^{-1}}\phi) , \tag{2.5}$$

we may translate the tangent space $\mathcal{T}_e\mathcal{G} \cong \mathfrak{g}$ at the unit element $e \in \mathcal{G}$ to a tangent space $\mathcal{T}_g\mathcal{G}$ at any other element $g \in \mathcal{G}$ by the virtue of transitivity of the group multiplication. In particular, by applying R_{g*} to any vector $\hat{X}_e \in \mathcal{T}_e\mathcal{G}$ as

$$\hat{X}_g \equiv R_{g*}\hat{X}_e \in \mathcal{T}_g\mathcal{G} \tag{2.6}$$

we induce a *right-invariant* vector field $\hat{X}$ on $\mathcal{TG}$. (Similar remarks apply to the left multiplication.) Thus, any orthonormal basis of $\mathfrak{g} \cong \mathcal{T}_e\mathcal{G}$ induces an *orthonormal basis* $\hat{T}_i \in$

[4]These definitions apply obviously to other (generalized) function spaces on $\mathcal{G}$ just as well, but we restrict our formulation to smooth functions for the sake of concreteness.

$\mathcal{TG}$ *of right-invariant vector fields*, which satisfy the Lie algebra commutation relations[5]

$$[\hat{T}_i, \hat{T}_j] = c_{ij}{}^k \hat{T}_k \,. \tag{2.7}$$

Due to the existence of a global basis of vector fields, the tangent bundle is *parallelizable*, and we have $\mathcal{TG} \cong \mathcal{G} \times \mathfrak{g}$.

To each tangent space $\mathcal{T}_g\mathcal{G}$ we may relate a *cotangent space* $\mathcal{T}_g^*\mathcal{G}$, which is the vector space of linear maps $\alpha_g : \mathcal{T}_g\mathcal{G} \to \mathbb{R}$ called covectors. The union of the cotangent spaces at each $g \in \mathcal{G}$ is called the *cotangent bundle* $\mathcal{T}^*\mathcal{G} := \cup_{g\in\mathcal{G}} \mathcal{T}_g^*\mathcal{G}$ of $\mathcal{G}$. The *pull-back* of the right multiplication $R_{h^{-1}}^* : \mathcal{T}_{gh}^* \to \mathcal{T}_g^*$ in the cotangent bundle is defined via

$$(R_{h^{-1}}^* \alpha_{gh})(\hat{X}_g) \equiv \alpha_{gh}(R_{h*}\hat{X}_g) \tag{2.8}$$

for all $\hat{X}_g \in \mathcal{T}_g\mathcal{G}$, $\alpha_{hg} \in \mathcal{T}_{hg}^*\mathcal{G}$. An orthonormal basis e^i for $\mathcal{T}_e^*\mathcal{G}$ is defined through $e^i(e_j) = \delta_j^i$, where e_j is an orthonormal basis of $\mathcal{T}_e\mathcal{G} \cong \mathfrak{g}$. Thus, in fact, as a vector space $\mathcal{T}_e^*\mathcal{G} \cong \mathfrak{g} \cong \mathbb{R}^d$. As for the tangent bundle, we may then induce right-invariant covector fields, i.e., 1-forms $\alpha \in \mathcal{T}^*\mathcal{G}$ via the right multiplication as $\alpha_g = R_g^*\alpha_e$, and similarly obtain orthonormal bases of right-invariant 1-forms $\eta^i \in \mathcal{T}^*\mathcal{G}$ by applying R_g^* to $e^i \in \mathcal{T}_e^*\mathcal{G}$. Accordingly, also the cotangent bundle is parallelizable, and we have $\mathcal{T}^*\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}^*$. Obviously, $\eta^i(\hat{T}_j) = \delta_j^i$.

Two important invariant tensors may be defined in terms of a right-invariant basis of 1-forms: A right-invariant nowhere-vanishing *Haar integration measure* on $\mathcal{G}$ may be defined as given by the dim($\mathcal{G}$)-form

$$\mathrm{d}g := \eta^1 \wedge \eta^2 \wedge \cdots \wedge \eta^{\dim(\mathcal{G})} \,, \tag{2.9}$$

where $\wedge$ denotes the exterior product of 1-forms. The existence of dg makes $\mathcal{G}$ orientable. The *right-invariant metric tensor* on $\mathcal{G}$ may be written as $g_R = \sum_i \eta^i \otimes \eta^i$. One can show that right-invariant vector fields are Killing with respect to g_R, i.e., $\hat{X} g_R = 0$ for a right-invariant vector field $\hat{X}$.

$\mathcal{T}^*\mathcal{G}$ has the so-called canonical symplectic structure as a cotangent bundle: We may define a projective map $\tau : \mathcal{T}^*\mathcal{G} \to \mathcal{G}$, $(g, \alpha_g) \mapsto g$, from the cotangent bundle onto the base manifold. The pull-back of this map $\tau^* : \mathcal{T}^*\mathcal{G} \to \mathcal{T}^*(\mathcal{T}^*\mathcal{G}) \cong \mathcal{T}^*\mathcal{G} \times \mathcal{TG}$ defines the *canonical 1-form* θ fiber-wise as

$$\theta_{\pi_g} := \tau^*\pi_g = (\pi_g, \bar{0}_g) \in \mathcal{T}_{\pi_g}^*(\mathcal{T}^*\mathcal{G}) \tag{2.10}$$

on $\mathcal{T}^*\mathcal{G}$, where $\bar{0}_g \in \mathcal{T}_g\mathcal{G}$ is the zero vector. The *canonical symplectic structure* on $\mathcal{T}^*\mathcal{G}$ is

[5] Here, orthonormality is defined with respect to the right-invariant metric obtained via pull-back of the right multiplication applied to the Euclidean metric of $\mathfrak{g}$.

then given by the exterior derivative $\omega := -\mathrm{d}\theta$.

In order to obtain a more explicit form for the symplectic structure, let η^i and $\hat{T}_i$ constitute right-invariant bases of $\mathcal{T}^*\mathcal{G}$ and $\mathcal{T}\mathcal{G}$, respectively. Then the set of $(\eta^i, \bar{0}) =: \mathrm{d}g^i$ and $(\bar{0}, \hat{T}_i) =: \mathrm{d}X_i$ constitutes a right-invariant basis of $\mathcal{T}^*(\mathcal{T}^*\mathcal{G}) \cong \mathcal{T}^*\mathcal{G} \times \mathcal{T}\mathcal{G}$. Through an explicit calculation we find

$$\omega = \mathrm{d}g^i \wedge \mathrm{d}X_i + c_{ij}{}^k X_k \mathrm{d}g^i \wedge \mathrm{d}g^j \,, \tag{2.11}$$

where X_i are coordinate functions in $\mathcal{T}^*\mathcal{G}$ obtained by integrating $\mathrm{d}X_i$.[6] If we denote the dual basis of $\mathcal{T}(\mathcal{T}^*\mathcal{G}) \cong \mathcal{T}\mathcal{G} \times \mathcal{T}^*\mathcal{G}$ by $(\hat{T}_i, \bar{0}) =: \frac{\partial}{\partial g^i}, (\bar{0}, \eta^i) =: \frac{\partial}{\partial X_i}$, the matrix-inverse[7] Λ of ω is

$$\Lambda = \frac{\partial}{\partial g^i} \otimes \frac{\partial}{\partial X_i} - \frac{\partial}{\partial X_i} \otimes \frac{\partial}{\partial g^i} - c_{ij}{}^k X_k \frac{\partial}{\partial X_i} \otimes \frac{\partial}{\partial X_j} \,. \tag{2.12}$$

Now, the *Poisson bracket* of functions $f, f' \in C^\infty(\mathcal{T}^*\mathcal{G})$ can be defined through the action of Λ as

$$\{f, f'\} := -(m \circ \Lambda)(\mathrm{d}f \otimes \mathrm{d}f') = \frac{\partial f}{\partial X_i}\frac{\partial f'}{\partial g^i} - \frac{\partial f}{\partial g^i}\frac{\partial f'}{\partial X_i} + c_{ij}{}^k X_k \frac{\partial f}{\partial X_i}\frac{\partial f'}{\partial X_j} \,, \tag{2.13}$$

where m denotes the point-wise product of functions on $\mathcal{T}^*\mathcal{G}$.

Here, in fact, this notation is somewhat misleading, since in general there does not exist globally well-defined coordinates g^i on $\mathcal{G}$. Instead, $\frac{\partial f}{\partial g^i}$ denote here the Lie derivatives

$$\mathcal{L}_i f(g) \equiv \hat{T}_i f(g) := \left.\frac{\mathrm{d}}{\mathrm{d}t} f(e^{ite_i} g)\right|_{t=0} \tag{2.14}$$

with respect to the basis of right-invariant vector fields $\hat{T}_i \in \mathcal{T}\mathcal{G}$, which are not necessarily globally integrable. Having chosen a system of Euclidean coordinates X_i on the Lie algebra, the same Poisson bracket on $\mathcal{T}^*\mathcal{G}$ may be written more concretely as

$$\{f, f'\} \equiv \frac{\partial f}{\partial X_i}\mathcal{L}_i f' - \mathcal{L}_i f \frac{\partial f'}{\partial X_i} + c_{ij}{}^k \frac{\partial f}{\partial X_i}\frac{\partial f'}{\partial X_j} X_k \,, \tag{2.15}$$

where $\mathcal{L}_i$ act only on the first factor and $\frac{\partial}{\partial X_i}$ act only on the second factor of $\mathcal{T}^*\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}^*$. The *classical Poisson algebra* $\mathcal{P}(\mathcal{T}^*\mathcal{G})$ of $\mathcal{T}^*\mathcal{G}$ arising solely from the canonical symplectic structure is constituted by elements of $C^\infty(\mathcal{T}^*\mathcal{G})$ equipped with point-wise product and

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[6] $(\mathrm{d}g^i, \mathrm{d}X_i)$ is not a canonical basis, since ω is not of the form $\sum_i \mathrm{d}g^i \wedge \mathrm{d}X_i$ in the basis 1-forms. By Darboux theorem, a canonical basis always exists locally, but in general not globally. The basis we use respects the factorization of the cotangent bundle, and is globally defined in the second factor, which makes it a natural choice for a basis despite it not being canonical.

[7] Inverse in the sense $\Lambda \circ \omega = \mathrm{Id}\big(\mathcal{T}(\mathcal{T}^*\mathcal{G})\big)$, $\omega \circ \Lambda = \mathrm{Id}\big(\mathcal{T}^*(\mathcal{T}^*\mathcal{G})\big)$, where ω and Λ are considered as mappings $\omega : \mathcal{T}(\mathcal{T}^*\mathcal{G}) \to \mathcal{T}^*(\mathcal{T}^*\mathcal{G})$, $X \mapsto \omega(\cdot, X)$ and $\Lambda : \mathcal{T}^*(\mathcal{T}^*\mathcal{G}) \to \mathcal{T}(\mathcal{T}^*\mathcal{G})$, $\alpha \mapsto \Lambda(\cdot, \alpha)$, i.e., $\Lambda\big(\cdot, \omega(\cdot, X)\big) = X \ \forall\ X \in \mathcal{T}(\mathcal{T}^*\mathcal{G})$ and $\omega\big(\cdot, \Lambda(\cdot, \alpha)\big) = \alpha \ \forall\ \alpha \in \mathcal{T}^*(\mathcal{T}^*\mathcal{G})$, or in the component form $\omega_{ik}\Lambda^{kj} = \Lambda^{jk}\omega_{ki} = \delta_i^j$.

the Poisson bracket.

2.1.2 Universal enveloping algebras

As was mentioned above, to any Lie group $\mathcal{G}$ is associated a Lie algebra $\mathfrak{g}$, which can be identified with the linear vector space of right-invariant vector fields in $\mathcal{T}\mathcal{G}$. The Lie algebra is closed under the Lie bracket, $-i[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}$, and fully characterized by the structure constants ${c_{ij}}^k \in \mathbb{R}$, which determine the commutators $[e_i, e_j] = i{c_{ij}}^k e_k$ for an orthonormal basis $e_i \in \mathfrak{g}$. Note, however, that in general the Lie algebra does not fully characterize the Lie group $\mathcal{G}$, since as first order differential operators it only accounts for local properties of the group.

From the Lie algebra $\mathfrak{g}$ we may further construct the *universal enveloping algebra* $U(\mathfrak{g})$ of $\mathfrak{g}$. (See for example [17] for further details.) Consider first the tensor algebra $T(\mathfrak{g})$ over $\mathfrak{g}$

$$T(\mathfrak{g}) := \sum_{n\in\mathbb{N}} \mathfrak{g}^{\otimes n}\,, \tag{2.16}$$

where $\mathfrak{g}^{\otimes 0} \equiv \mathbb{C}\mathbb{1}$ and $\mathfrak{g}^{\otimes(n+1)} \equiv \mathfrak{g}^{\otimes n} \otimes \mathfrak{g}$. Addition and multiplication are defined as usual for $A, B \in T(\mathfrak{g})$

$$A + B \equiv \left(\sum_{n\in\mathbb{N}} A_n\right) + \left(\sum_{n\in\mathbb{N}} B_n\right) = \sum_{n\in\mathbb{N}} (A_n + B_n)\,, \tag{2.17}$$

$$A \otimes B \equiv \left(\sum_{n\in\mathbb{N}} A_n\right) \otimes \left(\sum_{n\in\mathbb{N}} B_n\right) = \sum_{n\in\mathbb{N}} \sum_{\substack{k,l\in\mathbb{N}\\ k+l=n}} A_k \otimes B_l\,, \tag{2.18}$$

where $A_n, B_n \in \mathfrak{g}^{\otimes n}$. Now, the set of elements

$$\{X \otimes Y - Y \otimes X - [X,Y] : X, Y \in \mathfrak{g}\}\,, \tag{2.19}$$

where $[X,Y] \in \mathfrak{g}$ is the Lie bracket of $X, Y \in \mathfrak{g}$, generates a two-sided ideal $I(\mathfrak{g})$ of $T(\mathfrak{g})$. The universal enveloping algebra is then defined as the quotient $U(\mathfrak{g}) := T(\mathfrak{g})/I(\mathfrak{g})$. This means that the elements

$$X \otimes Y - Y \otimes X - [X,Y] \in T(\mathfrak{g}) \tag{2.20}$$

are equated to zero, so that the tensor product factors of $U(\mathfrak{g})$ satisfy the Lie algebraic commutation relations. Accodingly, $U(\mathfrak{g})$ is naturally identified with the space of right-invariant differential operators on $\mathcal{G}$ of all orders. We call the subalgebra

$$Z(\mathfrak{g}) := \{A \in U(\mathfrak{g}) : [A, B] = 0\ \forall B \in U(\mathfrak{g})\} \tag{2.21}$$

the center of $U(\mathfrak{g})$.

As for the Lie algebra, also the universal enveloping algebra carries only local information about the group. In the following we will also consider completions $\overline{U(\mathfrak{g})}$ of the universal enveloping algebra, which contain infinite sums of elements of arbitrary high order. In practice, we will consider $\overline{U(\mathfrak{g})}$ to be the linear space of right-invariant differential operators, under which the space of smooth functions $C^\infty(\mathcal{G})$ on $\mathcal{G}$ is closed. It is important to realize that such a completion encodes also global information about $\mathcal{G}$. If we denote by $\pi_\mathcal{G}$ the representation of $U(\mathfrak{g})$ as right-invariant differential operators on $C^\infty(\mathcal{G})$, the exponentials

$$e^{itX} := \sum_{n=0}^{\infty} \frac{(it)^n}{n!} X^n \in \overline{U(\mathfrak{g})}\,, \tag{2.22}$$

in particular, induce finite translations

$$e^{t\pi_\mathcal{G}(X)}\phi(g) = \phi(e^{itX}g) \tag{2.23}$$

of $\phi \in C^\infty(\mathcal{G})$ along the integral curves of the right-invariant vector fields $\pi_\mathcal{G}(X)$. This corresponds exactly to the $\mathcal{G}$-periodicity of the embedding map $i_\mathfrak{g}$ discussed above. If $\pi_\mathcal{G}(X)$ generates a compact $U(1)$ subgroup of $\mathcal{G}$, then there exists $t \in \mathbb{R}$, $t \neq 0$, such that $e^{t\pi_\mathcal{G}(X)} = \mathrm{id}_{C^\infty(\mathcal{G})}$, and correspondingly $R_\mathfrak{g}(tX) = 0$, because the integral curve is periodic. Therefore, we will assume that the ideal $I'(\mathfrak{g})$ generated by the set of elements $\{e^{iX} - \mathbb{1} : X \in \ker(R_\mathfrak{g})\}$ is quotiented out in any such completion $\overline{U(\mathfrak{g})}$. Then, assuming that $\mathcal{G}$ is an exponential Lie group, the set of exponential elements

$$E(\mathfrak{g}) := \{e^{iX} \in \overline{U(\mathfrak{g})} : X \in \mathfrak{g}\} \tag{2.24}$$

of the completion constitute a faithful representation of $\mathcal{G}$, which is nothing but the group of left translations $L_h\phi(g) = \phi(hg)$ acting on the linear vector space of functions $\phi \in C^\infty(\mathcal{G})$. Accodingly, we have $E(\mathfrak{g}) \cong \mathcal{G}$.

2.1.3 Hopf structures of exponential Lie groups

In addition to universal enveloping algebras, we will be needing some basic notions of *Hopf algebras* (also known as *quantum groups*). In comparison to ordinary algebras, Hopf algebras carry some extra structure, which give them interesting duality properties. (For more details on Hopf algebras, see for example [37, 44].)

The Hopf algebra structure $(\mathcal{H}, e, m, \epsilon, \Delta, S)$ is given by a linear vector space $\mathcal{H}$ over a field $\mathbb{K}$ and five different maps:

- *Product:* $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ such that (associativity)

$$m \circ (\mathrm{id}_{\mathcal{H}} \otimes m) = m \circ (m \otimes \mathrm{id}_{\mathcal{H}}) \,, \tag{2.25}$$

  where $\mathrm{id}_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ is the identity map $A \overset{\mathrm{id}_{\mathcal{H}}}{\mapsto} A$.

- *Unit:* $e : \mathbb{K} \to \mathcal{H}$ such that

$$m \circ (e \otimes \mathrm{id}_{\mathcal{H}}) = \mathrm{id}_{\mathcal{H}} = m \circ (\mathrm{id}_{\mathcal{H}} \otimes e) \,. \tag{2.26}$$

- *Coproduct:* $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ such that (coassociativity)

$$(\mathrm{id}_{\mathcal{H}} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_{\mathcal{H}}) \circ \Delta \,. \tag{2.27}$$

- *Counit:* $\epsilon : \mathcal{H} \to \mathbb{K}$ such that

$$(\epsilon \otimes \mathrm{id}_{\mathcal{H}}) \circ \Delta = \mathrm{id}_{\mathcal{H}} = (\mathrm{id}_{\mathcal{H}} \otimes \epsilon) \circ \Delta \,. \tag{2.28}$$

- *Antipode:* $S : \mathcal{H} \to \mathcal{H}$ such that

$$m \circ (S \otimes \mathrm{id}_{\mathcal{H}}) \circ \Delta = e \circ \epsilon = m \circ (\mathrm{id}_{\mathcal{H}} \otimes S) \circ \Delta \,. \tag{2.29}$$

These maps are required to satisfy also the following consistency relations:

- $\Delta \circ m = (m \otimes m) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \Delta)$, where $\tau : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is the *transposition map* $A \otimes B \overset{\tau}{\mapsto} B \otimes A$.

- $\epsilon \circ m = \epsilon \otimes \epsilon$ and correspondingly $\Delta \circ e = e \otimes e$.

Let us then consider some concrete examples of Hopf structures related to Lie groups, which will turn out to be relevant for our later developments. First of all, the universal enveloping algebra $U(\mathfrak{g})$ of $\mathcal{G}$ can be endowed with a natural Hopf algebra structure with $\mathbb{K} = \mathbb{C}$ and the following definitions for the generators $\mathbb{1}$ and $X \in \mathfrak{g}$: We set for

- the coproduct $\Delta_{\mathfrak{g}}(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}$ and $\Delta_{\mathfrak{g}}(X) = X \otimes \mathbb{1} + \mathbb{1} \otimes X$,

- the counit $\epsilon_{\mathfrak{g}}(\mathbb{1}) = 1$ and $\epsilon_{\mathfrak{g}}(X) = 0$,

- the antipode $S_{\mathfrak{g}}(\mathbb{1}) = \mathbb{1}$ and $S_{\mathfrak{g}}(X) = -X$,

linearly extended to the whole of $U(\mathfrak{g})$. Notice that the Leibniz rule is encoded into the coproduct, when $U(\mathfrak{g})$ is represented as the space of right-invariant differential operators. We have a natural compatibility condition between the point-wise product

$$m_{\mathcal{G}} : C^\infty(\mathcal{G}) \otimes C^\infty(\mathcal{G}) \to C^\infty(\mathcal{G}), \quad m_{\mathcal{G}}(\phi_1 \otimes \phi_2) = \phi_1 \phi_2, \tag{2.30}$$

of smooth functions on $\mathcal{G}$ and the coproduct $\Delta_{\mathfrak{g}}$ on $U(\mathfrak{g})$, which reads

$$\pi_{\mathcal{G}}(A) \circ m_{\mathcal{G}} = m_{\mathcal{G}} \circ \pi_{\mathcal{G}}(\Delta_{\mathfrak{g}}(A)) \tag{2.31}$$

for all $A \in U(\mathfrak{g})$.

Importantly, this compatibility condition may be straightforwardly generalized to the case, where $\mathcal{H}$ is any Hopf algebra represented as operators acting on an algebra $\mathcal{A}$ (i.e., $\mathcal{A}$ is an $\mathcal{H}$-module) with product $m_{\mathcal{A}}$ as the commutative diagram

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} & \xrightarrow{m_{\mathcal{A}}} & \mathcal{A} \\ {\scriptstyle \pi(\Delta(h))}\downarrow & & \downarrow{\scriptstyle \pi(h)} \\ \mathcal{A} \otimes \mathcal{A} & \xrightarrow{m_{\mathcal{A}}} & \mathcal{A} \end{array} \tag{2.32}$$

for all $h \in \mathcal{H}$.

Another important example is given by the group algebra $\mathbb{C}[\mathcal{G}]$, which is the linear algebra over $\mathbb{C}$ generated by the elements of $\mathcal{G}$. For $\mathbb{C}[\mathcal{G}]$ we may set

- $\Delta_{\mathcal{G}}(g) = g \otimes g$,
- $\epsilon_{\mathcal{G}}(g) = 1$, and
- $S_{\mathcal{G}}(g) = g^{-1}$.

It is easy to see that, in the case of an exponential group, the Hopf algebra structure for $U(\mathfrak{g})$ extended to the exponential elements of a completion $\overline{U(\mathfrak{g})}$ corresponds exactly to the Hopf structure of the group algebra $\mathbb{C}[\mathcal{G}]$. Indeed, the Hopf algebra structure extends to representations $\pi_{\mathcal{G}} : \mathcal{G} \to \mathrm{Aut}(V)$ of $\mathcal{G}$ with the above definitions, since by definition $\pi_{\mathcal{G}}$ is linear. If we set $V = C^\infty(\mathcal{G})$ and $\pi_{\mathcal{G}}(g)\phi = L_g\phi$, this is exactly the action of the exponential elements of $\overline{U(\mathfrak{g})}$ on $C^\infty(\mathcal{G})$. One may again easily verify the compatibility condition (2.32) with the point-wise product for $C^\infty(\mathcal{G})$.

Now, let us consider the dual space $\mathcal{H}^*$ of linear forms $\phi : \mathcal{H} \to \mathbb{K}$. We may define a complementary Hopf algebra-like structure on $\mathcal{H}^*$ as dual to that of $\mathcal{H}$ by setting for

- the product $m_{\mathcal{H}^*}(Z \otimes W)(A) = (Z \otimes W)(\Delta_{\mathcal{H}}(A))$,
- the unit $u_{\mathcal{H}^*}(1)(A) = \epsilon_{\mathcal{H}}(A)$,
- the coproduct $\Delta_{\mathcal{H}^*}(Z)(A \otimes B) = Z(m_{\mathcal{H}}(A \otimes B))$,
- the counit $\epsilon_{\mathcal{H}^*}(Z) = Z(e_{\mathcal{H}})$,
- the antipode $S_{\mathcal{H}^*}(Z)(A) = Z(S_{\mathcal{H}}(A))$,

for all $Z, W \in \mathcal{H}^*$, $A, B \in \mathcal{H}$. For a finite dimensional Hopf algebra $\mathcal{H}$, the dual $\mathcal{H}^*$ really forms another Hopf algebra, whereas for the infinite dimensional case there are subtleties, which we will shortly encounter.[8] Notice that if $\mathcal{H}$ is cocommutative (i.e., $\tau \circ \Delta_{\mathcal{H}} = \Delta_{\mathcal{H}}$), then $\mathcal{H}^*$ is commutative (i.e., $m_{\mathcal{H}^*} \circ \tau = m_{\mathcal{H}^*}$), and vice versa.

In particular, for the smooth linear forms $Z \in \mathbb{C}[\mathcal{G}]^*$, $Z : \mathbb{C}[\mathcal{G}] \to \mathbb{C}$, we may write $Z(A) := (\pi_{\mathcal{G}}(A)\phi_Z)(e)$, where $\phi_Z \in C^\infty(\mathcal{G})$, and $\pi_{\mathcal{G}} : \mathbb{C}[\mathcal{G}] \to \mathrm{Aut}(C^\infty(\mathcal{G}))$ is the representation of $\mathbb{C}[\mathcal{G}]$ induced by the left translations. The duality with $\mathbb{C}[\mathcal{G}]$ makes $C^\infty(\mathcal{G})$, in a sense, *almost* a Hopf algebra. Namely, we obtain

- a counit $\epsilon'(\phi) = \phi(e)$,
- an antipode $S'(\phi)(g) = \phi(g^{-1})$, and
- a map analogous to a coproduct $\Delta' : C^\infty(\mathcal{G}) \to C^\infty(\mathcal{G} \times \mathcal{G})$ such that $\Delta'(\phi)(g_1, g_2) = \phi(g_1 g_2)$.

However, Δ' is not a proper coproduct (unlike in the case of group algebra of a finite group), since the image fails to be in $C^\infty(\mathcal{G}) \otimes C^\infty(\mathcal{G})$. As mentioned, this problem arises, because $\mathbb{C}[\mathcal{G}]$ is an infinite dimensional algebra. We note that the formal coproduct Δ' that arises for the algebra $C^\infty(\mathcal{G})$ this way is compatible, in the sense of the diagram (2.32), with the convolution product $(\varphi * \varphi')(g) := \int_{\mathcal{G}} \varphi(gh^{-1})\varphi'(h)$ on the space of smooth compactly supported functions $C_c^\infty(\mathcal{G})$, when we consider $\phi \in C^\infty(\mathcal{G})$ as operators acting on $C_c^\infty(\mathcal{G})$ via point-wise product.

We may also consider the dual $U(\mathfrak{g})^*$ of the universal enveloping algebra and its completion $\overline{U(\mathfrak{g})}$. Again, $U(\mathfrak{g})$ is infinite dimensional, so $U(\mathfrak{g})^*$ will not form exactly a proper Hopf algebra, but we may still derive some interesting and useful structure for it through the duality, which can be considered as a generalization to the usual Hopf algebra structure. In fact, by identifying the elements $A \in U(\mathfrak{g})$ with right-invariant differential operators on $\mathcal{G}$ through the representation $\pi_{\mathcal{G}}$, we may write again $Z(A) = (\pi_{\mathcal{G}}(A)\phi_Z)(e)$ for $\phi_Z \in C^\infty(\mathcal{G})$. The resulting dual structure on $C^\infty(\mathcal{G})$ agrees with that of the previous paragraph.

On the other hand, consider the linear operators $\partial^i : U(\mathfrak{g}) \to U(\mathfrak{g})$ given by

$$\partial^i := \sum_{n=1}^{\infty} \sum_{m=1}^{n} \mathrm{id}_{\mathfrak{g}}^{\otimes(m-1)} \otimes \eta^i \otimes \mathrm{id}_{\mathfrak{g}}^{\otimes(n-m)}, \tag{2.33}$$

where $\eta^i \in \mathfrak{g}^*$ constitute a right-invariant orthonormal basis of $\mathfrak{g}^*$. The idea is that ∂^i correspond to partial derivative operators on $U(\mathfrak{g})$, since we have $\partial^i(e_j^{\otimes n}) = n\delta^i_j e_i^{\otimes(n-1)}$ for the dual basis elements $e_i \in \mathfrak{g}$. Notice that $[\partial^i, \partial^j] = \partial^i\partial^j - \partial^j\partial^i = 0$ for all i, j. Now,

[8] Specifically, in the infinite dimensional case the duals may be too big: the dual of a dual is not the space itself. For example, the dual of the group algebra is the algebra of bounded functions on the group, but the dual of bounded functions on the group is a much bigger space than the group algebra. It contains integrations with arbitrary compactly supported weights on the group. Loosely speaking, these are continuous sums of point-wise evaluations, and therefore not in the group algebra, which contains only finite sums. The problem may in some cases be solved by restricting the dual to a smaller subalgebra.

consider the composite operators

$$\hat{Z} = \sum_{n=0}^{\infty} Z_{i_1 \cdots i_n} \partial^{i_1} \cdots \partial^{i_n} \,. \tag{2.34}$$

It is not difficult to verify that any element $Z \in U(\mathfrak{g})^*$ can be expressed formally as $Z(A) = (P_0 \circ \hat{Z})A$, where P_0 is the projection onto the zero degree part, i.e., $P_0(A) = A_0 \in \mathbb{C}$ for $A = \sum_n A_n$, $A_n \in \mathfrak{g}^{\otimes n}$.

We may extend the action of the operators ∂^i onto the completion $\overline{U(\mathfrak{g})}$. Remember that for an exponential Lie group we may identify the set of exponential elements $E(\mathfrak{g}) \subset \overline{U(\mathfrak{g})}$ with the group $\mathcal{G}$. Then, acting on the exponential elements gives $-i\partial^i e^{iX} = X^i e^{iX}$, where X^i are the coordinates of X in the dual basis η^i. This again shows that ∂^i may be considered as partial derivative operators on $\overline{U(\mathfrak{g})}$ that are dual to the basis elements $e_i \in \mathfrak{g}$. In fact, $X^i \equiv \eta^i(-i\ln(g))$ are exactly the canonical coordinates (of the first kind) on $\mathcal{G}$. In order to have consistency with the equivalence relations introduced by quotienting out the ideal $I'(\mathfrak{g})$, we will eventually restrict to consider only $\mathcal{G}$-periodic functions of ∂^i, i.e., pseudo-differential operators $(i_{\mathfrak{g}}\phi)(-i\vec{\partial})$, where $\phi \in C^\infty(\mathcal{G})$. Such functional operators may be considered to be defined through their action

$$(i_{\mathfrak{g}}\phi)(-i\vec{\partial})e^{iX} \equiv (i_{\mathfrak{g}}\phi)(X)e^{iX} = \phi(e^{iX})e^{iX} \tag{2.35}$$

on the exponential elements.

Now, we may define through duality some generalized Hopf structure for the derivative operators ∂^i on $\overline{U(\mathfrak{g})}$. Namely, let us define a coproduct for the operators $-i\partial^i$ through the duality by requiring $\Delta_\partial(-i\partial^i)(A \otimes B) = -i\partial^i(A \otimes B)$ for all $A, B \in \overline{U(\mathfrak{g})}$. For the exponential elements we obtain

$$\begin{aligned} \Delta_\partial(-i\partial^i)(e^{iX} \otimes e^{iY}) &= -i\partial^i e^{B(X,Y)} = B^i(X,Y)e^{iB(X,Y)} \\ &= B^i(-i\partial_1, -i\partial_2)(e^{iX} \otimes e^{iY}) \,, \end{aligned} \tag{2.36}$$

where the subindices for ∂_1, ∂_2 denote the first and second factor of the tensor product, on which ∂^i act, and $B^i(X,Y)$ are the components of the Baker-Campbell-Hausdorff (BCH) formula (2.2) in the basis e_i of $\mathfrak{g}$. Explicitly,

$$B^i(-i\partial_1, -i\partial_2) = \sum_{n=1}^{\infty} (-i)^n \sum_{\substack{k,l=1 \\ k+l=n}}^{\infty} B^i_{p_1 \cdots p_k q_1 \cdots q_l} \partial^{p_1} \cdots \partial^{p_k} \otimes \partial^{q_1} \cdots \partial^{q_l} \,, \tag{2.37}$$

where $B^i_{p_1 \cdots p_k q_1 \cdots q_l} \in \mathbb{R}$ are the expansion coefficients of the BCH formula. Consequently, we define $\Delta_\partial(-i\partial^i) = B^i(-i\partial_1, -i\partial_2)$. Notice that for $\mathcal{G}$-periodic functions of ∂^i this coproduct corresponds to the one defined for the dual of $\mathbb{C}[\mathcal{G}]$ above, namely,

$$\begin{aligned}(i_{\mathfrak{g}}\phi)(\Delta_{\partial}(-i\partial^i))(e^{iX}\otimes e^{iY}) &= \phi(e^{iX}e^{iY})(e^{iX}\otimes e^{iY}) \\ &= \Delta'(\phi)(e^{iX},e^{iY})(e^{iX}\otimes e^{iY})\,,\end{aligned} \tag{2.38}$$

and so we have

$$P_0\circ(i_{\mathfrak{g}}\phi)(\Delta_{\partial}(-i\partial^i))(e^{iX}\otimes e^{iY}) = \Delta'(\phi)(e^{iX},e^{iY}) \tag{2.39}$$

for all $\phi\in C^\infty(\mathcal{G})$.

We can also make an explicit identification to the dual $U(\mathfrak{g})^*$ above, as we have for $A=\sum_n a^{i_1\cdots i_n}e_{i_1}\otimes\cdots\otimes e_{i_n}\in U(\mathfrak{g})$ and $\phi\in C^\infty(\mathcal{G})$

$$\begin{aligned}(\pi_{\mathcal{G}}(A)\phi)(e) &= \left(\sum_{n\in\mathbb{N}}(-i)^n a^{i_1\cdots i_n}\frac{\partial}{\partial X^{i_1}}\cdots\frac{\partial}{\partial X^{i_n}}\right)_{X^{i_1}=\cdots=X^{i_n}=0} e^{\pi_{\mathcal{G}}(iX)}\phi(e) \\ &= \left(\sum_{n\in\mathbb{N}}(-i)^n a^{i_1\cdots i_n}\frac{\partial}{\partial X^{i_1}}\cdots\frac{\partial}{\partial X^{i_n}}\right)_{X^{i_1}=\cdots=X^{i_n}=0} P_0\circ(i_{\mathfrak{g}}\phi)(-i\partial^i)e^{iX} \\ &= P_0\circ(i_{\mathfrak{g}}\phi)(-i\partial^i)A\,,\end{aligned} \tag{2.40}$$

where we used the identity

$$e^{\pi_{\mathcal{G}}(iX)}\phi(e) = \phi(e^{iX}) = P_0\circ(i_{\mathfrak{g}}\phi)(-i\partial^i)e^{iX}\,. \tag{2.41}$$

Therefore, we have for all $Z\in U(\mathfrak{g})^*$ that $Z(A)\equiv(\pi_{\mathcal{G}}(A)\phi_Z)(e) = P_0\circ(i_{\mathfrak{g}}\phi_Z)(-i\partial^i)A$, and the two duals of $U(\mathfrak{g})$ defined above can be identified.

To conclude, let us emphasize the important structural observations above:

- We considered the Hopf algebras $\mathbb{C}[\mathcal{G}]$ and $U(\mathfrak{g})$, which were shown to coincide in the completion $\overline{U(\mathfrak{g})}$ for exponential Lie groups.
- The algebraic duals of linear forms of each were considered, for which dual generalized Hopf structures were derived. We noted that for $\mathbb{C}[\mathcal{G}]^*$ the dual coproduct Δ' : $\phi(g)\mapsto\phi(g_1g_2)$ is compatible in the sense of the commutative diagram (2.32) with the convolution product on $C_c^\infty(\mathcal{G})$, when $C^\infty(\mathcal{G})$ is considered as an algebra of operators acting by point-wise multiplication on $C_c^\infty(\mathcal{G})$.
- On the other hand, we identified the dual spaces $C^\infty(\mathcal{G})$ and $C^\infty_{\mathcal{G}}(-i\partial^i)$ of $U(\mathfrak{g})$ via the formula

$$(\pi_{\mathcal{G}}(A)\phi)(e) = P_0\circ(i_{\mathfrak{g}}\phi)(-i\partial^i)A\,, \tag{2.42}$$

and coproduct Δ' with the coproduct Δ_∂ of the derivative operators ∂^i on $\overline{U(\mathfrak{g})}$, which was derived from the compatibility with the product in $\overline{U(\mathfrak{g})}$.

Based on the above observations, one may then expect that there, in fact, should exist a relation between the convolution product on $C_c^\infty(\mathcal{G})$ and the operator product in $\overline{U(\mathfrak{g})}$ and, in general, the respective algebras. We will see that this link is provided by the non-commutative harmonic analysis introduced below.

2.1.4 $*$-algebras and quantization

An *abstract $*$-algebra* over $\mathbb{C}$ is defined as a linear vector space $\mathcal{A}$ over $\mathbb{C}$ equipped with

- a multiplication map $m_\mathcal{A} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, $a \otimes b \mapsto ab$, and
- an involutive anti-automorphism $* : \mathcal{A} \to \mathcal{A}$, $a \mapsto a^*$, such that $(\lambda a+\mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$ and $(ab)^* = b^* a^*$ for all $\lambda, \mu \in \mathbb{C}$, $a, b \in \mathcal{A}$,

where the overline denotes complex conjugation. a^* is called the adjoint of a. An element $a \in \mathcal{A}$ such that $a^* = a$ is called *self-adjoint*.

A *$*$-algebra homomorphism* is a $\mathbb{C}$-linear map $\xi : \mathcal{A} \to \mathcal{B}$ between $*$-algebras such that

- $\xi(ab) = \xi(a)\xi(b)$, and
- $\xi(a^*) = \xi(a)^*$

for all $a, b \in \mathcal{A}$. A *representation* of a $*$-algebra $\mathcal{A}$ is a $*$-homomorphism $\pi : \mathcal{A} \to \mathrm{Aut}(V)$ onto the algebra of automorphisms of a vector space V. Such an explicit realization of a $*$-algebra is called a *concrete* $*$-algebra.

Now, let $\mathcal{P}(\mathcal{S})$ be a Poisson algebra, which is constituted by smooth (complex-valued) functions $f \in C^\infty(\mathcal{S})$ on $\mathcal{S}$ equipped with the point-wise product and a Poisson bracket $\{\cdot,\cdot\} : C^\infty(\mathcal{S}) \times C^\infty(\mathcal{S}) \to C^\infty(\mathcal{S})$. We call a *quantization* of $\mathcal{P}(\mathcal{S})$ a map $\mathfrak{Q} : C^\infty(\mathcal{S}) \to \mathcal{A}$, where $\mathcal{A}$ is an abstract $*$-algebra. Usually it is required that $\mathfrak{Q}(1) = \mathbb{1}$, the unit of $\mathcal{A}$, and possibly $\mathfrak{Q} \circ \phi = \phi \circ \mathfrak{Q}$ for $\phi : \mathbb{R} \to \mathbb{R}$, when both sides are well-defined. Optimally, one would like to also require that $\mathfrak{Q}(\{f, f'\}) = [\mathfrak{Q}(f), \mathfrak{Q}(f')]$, where $[a, b] := ab - ba$ is the operator commutator, in general for all $f, f' \in \mathcal{P}(\mathcal{S})$, but very strict *quantization obstructions* have been proved that prevent the definition of such a quantization map [27].

Physically acceptable quantization maps can be, however, defined on a case-by-case basis, which reflect the idea of the transition from classical to quantum physics by imposing a non-zero minimal action proportional to the Planck constant $\hbar$. The strategy is to pick a preferred set of classical observables, usually the canonical phase space variables, in terms of which all other observables may be expressed, and define the quantum commutators to reproduce the Poisson brackets of these variables exactly. The commutators of other observables may then receive quantum corrections proportional to Planck constant $\hbar$, which vanish in the classical limit $\hbar \to 0$.

For example, if we consider quantization of a Poisson algebra on a Euclidean space, $\mathcal{S} = \mathcal{T}^*\mathbb{R}^d \cong \mathbb{R}^d \times \mathbb{R}^d$, the Poisson brackets of the canonical coordinate variables x^i, p_i read

$$\{x^i, x^j\} = 0 = \{p_i, p_j\}\,, \quad \{x^i, p_j\} = \delta^i_j\,. \tag{2.43}$$

Then, the corresponding *quantum algebra* $\mathfrak{Q}(\mathcal{P}(\mathcal{T}^*\mathbb{R}^d)) =: \mathfrak{A}$ may be defined as the tensor algebra generated by the unit $\mathbb{1} \in \mathfrak{A}$ and the self-adjoint generators $\mathfrak{Q}(x^i) =: X^i$, $\mathfrak{Q}(p_i) =: P_i$ modulo the *canonical commutation relations*

$$[X^i, X^j] = 0 = [P_i, P_j]\,, \quad [X^i, P_j] = i\hbar\delta^i_j\mathbb{1}\,, \tag{2.44}$$

for all i, j. In other words, we take the quotient of the tensor algebra by the two-sided ideal that is generated by the elements

$$X^i \otimes X^j - X^j \otimes X^i\,, \quad P_i \otimes P_j - P_j \otimes P_i\,, \quad X^i \otimes P_j - P_j \otimes X_i - i\hbar\delta^i_j\mathbb{1}\,. \tag{2.45}$$

The involution satisfies $(A \otimes B)^* = B^* \otimes A^*$. We may further consider, at least formally, the quantization map for the full Poisson algebra, $\mathfrak{Q} : \mathcal{P}(\mathcal{T}^*\mathbb{R}^d) \to \mathfrak{A}$, via some completion. The above strategy can be applied also in many other instances, in particular, for the Poisson algebra $\mathcal{P}(\mathcal{T}^*\mathcal{G})$ of a Lie group, as we will see below.

It is important to notice that the definition of a quantization map, associating a non-commutative algebra to a commutative one, is never unique. As the simplest example, there is no unique way to map the function $x^i p_i \in \mathcal{P}(\mathcal{T}^*\mathbb{R}^d)$ into $\mathfrak{A}$, since X^i and P_i do not commute in $\mathfrak{A}$. We could define, for example, $\mathfrak{Q}(x^i p_i) = X^i \otimes P_i$, $\mathfrak{Q}(x^i p_i) = P_i \otimes X^i$, or $\mathfrak{Q}(x^i p_i) = (X^i \otimes P_i + P_i \otimes X^i)/2$, which differ by elements proportional to $\hbar$. Thus, one is forced to make a choice of the map $\mathfrak{Q}$. Of course, there are further requirements that one may impose on $\mathfrak{Q}$, often motivated by physical considerations. For example, one could ask $\mathfrak{Q}(x^i p_i)$ to be a self-adjoint element of $\mathfrak{A}$, which would prefer the symmetric choice $\mathfrak{Q}(x^i p_i) = (X^i \otimes P_i + P_i \otimes X^i)/2$ over the two others, but this is still by no means a unique choice. There are certain standard choices of $\mathfrak{Q}$ that one can make, such as the symmetric quantization map $\mathfrak{Q}_S$ defined by symmetrization of the tensor product factors, or the Duflo quantization map. We will explore further the different choices of quantization maps in Section 2.4, where we consider some concrete examples.

2.1.5 Representations of $\mathfrak{Q}(\mathcal{T}^*\mathbb{R}^d)$ and Fourier transform

Let $\mathfrak{Q}(\mathcal{T}^*\mathbb{R}^d)$ now be the quantum algebra of Euclidean space defined as above.[9] Now, let us define the canonical representations of this quantum algebra. The set of operators X^i constitute a maximal set of commuting self-adjoint operators, and similarly for P_i.

[9] We set $\hbar = 1$ for the rest of this chapter.

Accordingly, we may simultaneously diagonalize either one of the two sets of operators. Moreover, by the spectral theorem, their eigenstates constitute a basis of a representation space, which is the Hilbert space of square-integrable functions on the joint spectrum of X^i or P_i, respectively. In fact, since X^i and P_i can all be shown to be unbounded operators, strictly speaking, their eigenfunctions are generalized functions, which do not belong to the Hilbert space. However, any element of the Hilbert space may be expressed, through the spectral decomposition, in terms of the eigenstates. Moreover, we must restrict the domains of X^i and P_i to a dense subset of the Hilbert space in order for their images to lie in the Hilbert space.[10]

With the above technicalities in mind, we may then consider the space of smooth compactly supported functions on the joint spectrum $\vec{x} \in \mathbb{R}^d$ of the operators X^i, $C_c^\infty(\mathbb{R}^d)$. On this space, the X^i operators are represented by scalar multiplication $(\pi_x(X^i)\phi)(\vec{x}) = x^i\phi(\vec{x})$. More generally, we could consider the action of functional operators corresponding to functions $f \in C^\infty(\mathbb{R}^d)$ with the point-wise multiplication as

$$(\pi_x(f(X^i))\phi)(\vec{x}) = f(\vec{x})\phi(\vec{x}) \in C_c^\infty(\mathbb{R}^d)\,. \tag{2.46}$$

One may easily show that by setting

$$(\pi_x(P_i)\phi)(\vec{x}) = -i\frac{\partial}{\partial x^i}\phi(\vec{x}) \tag{2.47}$$

we obtain the correct commutation relations for the canonical variables. Notice that the action of P_i operators may be exponentiated to induce finite translations $(\pi_x(e^{i\vec{y}\cdot\vec{P}})\phi)(\vec{x}) = \phi(\vec{x}+\vec{y})$, and that there is no ambiguity here due to the integrability of the coordinate vector fields $\frac{\partial}{\partial x^i}$ in Euclidean space.

One may then finally complete $C_c^\infty(\mathbb{R}^d)$ in the L^2-norm $|\phi|^2 = \int_{\mathbb{R}^d} \mathrm{d}^d x\ \overline{\phi(\vec{x})}\phi(\vec{x})$ to obtain the Hilbert space $L^2(\mathbb{R}^d)$.[11] A similar treatment applies in the case of the operators P_i. In this case we consider again functions $\varphi \in C_c^\infty(\mathbb{R}^d)$. The representation of the quantum algebra is given by

$$(\pi_p(P_i)\varphi)(\vec{p}) = p_i\varphi(\vec{p}) \quad \text{and} \quad (\pi_p(X^i)\varphi)(\vec{p}) = i\frac{\partial}{\partial p_i}\varphi(\vec{p})\,. \tag{2.48}$$

Now, we want to define an *intertwiner* between these two representations, namely, a linear invertible transformation $\mathcal{F} : C_c^\infty(\mathbb{R}^d) \to C_c^\infty(\mathbb{R}^d)$ such that $\mathcal{F}\circ\pi_x = \pi_p\circ\mathcal{F}$. It may be shown [29] that the unique unitary intertwiner is given by an integral transform

$$\mathcal{F}(\phi)(\vec{p}) = \int_{\mathbb{R}^d} \mathrm{d}^d x\ e^{-i\vec{p}\cdot\vec{x}}\phi(\vec{x})\,. \tag{2.49}$$

[10] Alternatively, we could generalize our considerations to the rigged Hilbert space formalism [12].

[11] Due to the Stone-von Neumann theorem, the above representation is unique up to unitary equivalence.

Thus, the usual Euclidean Fourier transform may be understood as the intertwiner between the canonical representations of the quantum algebra $\mathfrak{Q}(\mathcal{T}^*\mathbb{R}^d)$. The above line of reasoning will serve as a general strategy below to understand the non-commutative Fourier transform for exponential Lie groups. Indeed, since $\mathbb{R}^d$ itself is an exponential Lie group, our construction can be considered as a generalization of the usual Fourier transform. (See Subsection 2.4.1.)

2.1.6 Deformation quantization

In the previous subsections we defined an abstract $*$-algebra as the quantization of a Poisson algebra, and considered its canonical representations and the resulting harmonic analysis. However, it is often difficult to work with abstract or concrete operators. Moreover, physically the connection to the classical phase space structure is somewhat indirect and obscure. For these reasons, alternative ways to represent the algebra in terms of ordinary functions have been considered. (See, for example, [11, 30, 40, 22].)

Specifically, we may define a non-commutative product, denoted by $\star$, for functions on $\mathcal{T}^*\mathbb{R}$, which reflects the quantum operator product. Namely, we require

$$\mathfrak{Q}(f \star f') = \mathfrak{Q}(f)\mathfrak{Q}(f') \tag{2.50}$$

for f, f' functions on $\mathcal{T}^*\mathbb{R}$. This implies that $\mathfrak{Q} : F_\star(\mathcal{T}^*\mathbb{R}) \to \mathfrak{A}$ is an algebra homomorphism, where we denote an algebra of functions on $\mathcal{T}^*\mathbb{R}$ equipped with the $\star$-product by $F_\star(\mathcal{T}^*\mathbb{R})$. For polynomials the $\star$-product is rather easy to compute explicitly from the expression above, as soon as the quantization map is determined. Obviously, the form of the $\star$-product depends on the non-unique choice of the quantization map, but generically it is of the form

$$f \star f' = ff' + \sum_{k=1}^{\infty} \hbar^k B_k(f, f')\,, \tag{2.51}$$

where B_k are linear bidifferential operators of degree at most k.[12] We see that in the limit $\hbar \to 0$ the $\star$-product coincides with the point-wise product. From this point of view, quantization may be considered as a deformation of the commutative pointwise product with the deformation parameter $\hbar$.

Accordingly, we may consider the formal inverse map $\mathfrak{D} : \mathfrak{A} \to F_\star(\mathcal{T}^*\mathbb{R})/\ker\mathfrak{Q}$, $a \overset{\mathfrak{D}}{\mapsto} \pi_\star(a)$, to the quantization map $\mathfrak{Q}$ such that

$$\mathfrak{D} \circ \mathfrak{Q} = \mathrm{id}_{F_\star(\mathcal{T}^*\mathbb{R})} - P_{\ker\mathfrak{Q}} \quad \text{and} \quad \mathfrak{Q} \circ \mathfrak{D} = \mathrm{id}_{\mathfrak{A}}\,, \tag{2.52}$$

where $P_{\ker\mathfrak{Q}}$ is the projection onto the kernel of $\mathfrak{Q}$. $\mathfrak{D}$ defines a representation of the

[12] In general, this series diverges, and convergence has to be established for suitable subalgebras.

quantum algebra $\mathfrak{A}$, since we have by definition $\pi_\star(ab) = \pi_\star(a) \star \pi_\star(b)$ and $\pi_\star(\mathbb{1}) = 1$. The elements in $F_\star(\mathcal{T}^*\mathbb{R})/\ker\mathfrak{Q}$ act by $\star$-multiplication on $F_\star(\mathcal{T}^*\mathbb{R})/\ker\mathfrak{Q}$ itself, thus defining automorphisms of $F_\star(\mathcal{T}^*\mathbb{R})/\ker\mathfrak{Q}$. We call such $\mathfrak{D}$ the *deformation quantization* corresponding to $\mathfrak{Q}$.

One may also consider the deformation quantization associated to the universal enveloping algebra $U(\mathfrak{g})$. In this case the quantization $\mathfrak{Q}$ is a map from the symmetric tensor algebra to the universal enveloping algebra, where the non-commutativity $[e_i, e_j] = i{c_{ij}}^k e_k$ is controlled by the structure constants. Notice that if we identify the symmetric tensor algebra with the space of polynomial functions on $\mathfrak{g}^*$, its quantization to $U(\mathfrak{g})$ is exactly the quantization of the Poisson algebra $\mathcal{P}(\mathcal{T}^*\mathcal{G})$ associated to the corresponding Lie group $\mathcal{G}$, restricted to the second factor of $\mathcal{T}^*\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}^*$. A uniform scaling of the structure constants ${c_{ij}}^k \mapsto \lambda {c_{ij}}^k$ corresponds to a respective scaling of the metric on the group manifold. Introducing such a scaling, one may define a $\star$-product for functions on $\mathfrak{g}^*$ of the form

$$f \star f' = ff' + \sum_{k=1}^{\infty} \lambda^k B_k(f, f')\,, \tag{2.53}$$

which reflects the Lie algebra commutation relations.

2.2 Formulation of the Non-commutative Representation

In this section we present the mathematical formulation of the non-commutative representation for exponential Lie groups, and the non-commutative Fourier transform intertwining the non-commutative representation. We start by defining the quantum algebra associated to the canonical symplectic structure of the cotangent bundle of an exponential Lie group. Accordingly, we define two conjugate representations for the quantum algebra in terms of group and algebra variables. The latter representation is given in terms of a non-commutative deformation quantization star-product. We further determine the integral transform that intertwines the two representations, and consider its properties. Finally, we present elementary examples of the Lie groups $\mathbb{R}^d$, $U(1)$ and $SU(2)$, which illustrate different aspects of the general formalism.

2.2.1 Quantum algebra of $\mathcal{T}^*\mathcal{G}$

As we have seen in Subsection 2.1.1 Equation (2.15), the canonical symplectic structure of the cotangent bundle $\mathcal{T}^*\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}^*$ over a Lie group $\mathcal{G}$ is described by the Poisson algebra $\mathcal{P}(\mathcal{T}^*\mathcal{G})$, whose Poisson bracket is explicitly given by

$$\{f, f'\} \equiv \frac{\partial f}{\partial X_i}\mathcal{L}_i f' - \mathcal{L}_i f \frac{\partial f'}{\partial X_i} + {c_{ij}}^k \frac{\partial f}{\partial X_i}\frac{\partial f'}{\partial X_j} X_k\,. \tag{2.54}$$

Here, as before, X_i are coordinate functions on $\mathfrak{g}^*$, $\mathcal{L}_i$ are Lie derivatives on $\mathcal{G}$ with respect to an orthonormal basis of right-invariant vector fields, and ${c_{ij}}^k$ are the structure constants of $\mathcal{G}$. Notice that $C^\infty(\mathcal{G})$ and $C^\infty(\mathfrak{g}^*)$ are Poisson subalgebras of the algebra $C^\infty(\mathcal{G} \times \mathfrak{g}^*)$, since they are closed under point-wise multiplication and the Poisson brackets.

We then want to quantize this Poisson algebra, along the lines described in Subsection 2.1.4. Accordingly, we want to define a quantization map $\mathfrak{Q} : \mathcal{P}(\mathcal{T}^*\mathcal{G}) \to \mathfrak{A}$ from the Poisson algebra $\mathcal{P}(\mathcal{T}^*\mathcal{G})$ to an abstract $*$-algebra $\mathfrak{A}$. However, due to the quantization obstructions, we must first restrict to a subalgebra of $\mathcal{P}(\mathcal{T}^*\mathcal{G})$ by choosing a preferred set of elements in terms of which the rest can be expressed, for which the quantization is exact (by definition). We will therefore first consider only elements of the polynomial ring $\mathbb{C}[X_i]$ of the coordinate functions X_i on $\mathfrak{g}^*$ instead of the full subalgebra $C^\infty(\mathfrak{g}^*)$ of $C^\infty(\mathcal{G} \times \mathfrak{g}^*)$. For $C^\infty(\mathcal{G})$ we do not have such problems, since it is an abelian subalgebra of the Poisson algebra.

We then denote $\mathfrak{Q}(f) =: \hat{f} \in \mathfrak{A}$ for $f \in C^\infty(\mathcal{G})$ and $\mathfrak{Q}(X_i) =: \hat{X}_i \in \mathfrak{A}$. We require $\hat{f}^* = \hat{\overline{f}}$ for all $f \in C^\infty(\mathcal{G})$ and $\hat{X}_i^* = \hat{X}_i$. The algebra $\mathfrak{A}$ is considered to be a completion of the tensor algebra generated by the elements $\hat{f}, \hat{X}_i$ and quotiented by the ideal generated by the relations

$$\begin{aligned} \hat{f} \otimes \hat{f}' &= \widehat{ff'}\,, \\ \hat{X}_i \otimes \hat{f} - \hat{f} \otimes \hat{X}_i &= i\widehat{\mathcal{L}_i f}\,, \\ \hat{X}_i \otimes \hat{X}_j - \hat{X}_j \otimes \hat{X}_i &= i{c_{ij}}^k \hat{X}_k \end{aligned} \tag{2.55}$$

for all $f, f' \in C^\infty(\mathcal{G})$, which follow from the Poisson structure. The quantum algebra $\mathfrak{A}$ again contains the subalgebras $\mathfrak{Q}(C^\infty(\mathcal{G}))$ and $\mathfrak{Q}(\mathbb{C}[X_i])$ as they are closed under the algebra product and the commutation relations.

In particular, we have $\mathfrak{Q}(\mathbb{C}[X_i]) \cong U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$, as they have the same generators and commutation relations. Accordingly, we may identify $\mathfrak{Q}(\mathbb{C}[X_i])$ with the algebra of right-invariant differential operators on $\mathcal{G}$. Thus, the extension of $\mathfrak{Q}$ onto $C^\infty(\mathfrak{g}^*)$ amounts to a completion $\mathfrak{Q}(C^\infty(\mathfrak{g}^*)) \cong \overline{U(\mathfrak{g})}$ of the universal enveloping algebra. As discussed above, in such a completion we have $e^{iX} \sim \mathbb{1}$ for X such that $R_\mathfrak{g}(X) = 0$. Let $E \in C^\infty(\mathfrak{g})$ be such that $\mathfrak{Q}(E) = e^{i\hat{X}}$ and $R_\mathfrak{g}(X) = 0$. Then, for any such function E, $E - 1 \in C^\infty(\mathfrak{g})$ must be in the kernel of the quantization map $\mathfrak{Q}$.

Notice that, in general, we do not have a globally well-defined set of coordinates on $\mathcal{G}$, and accordingly there are no natural canonically conjugate variables as in the Euclidean case. However, for an exponential Lie group, we may always consider $f \in C^\infty(\mathcal{G})$ as $\mathcal{G}$-periodic functions of the canonical coordinates $k^i = \eta^i(-i\ln(g))$ on $\mathcal{G}$. Writing $\hat{f} \equiv f(\hat{k}^i)$, we may then formally write for the commutators of the coordinate operators $\mathfrak{Q}(k^i) =: \hat{k}^i$, such that $(\hat{k}^i)^* = \hat{k}^i$,

$$[\hat{k}^i, \hat{k}^j] = 0 \;, \quad [\hat{X}_i, \hat{k}^j] = i\widehat{\mathcal{L}_i k^j} \;, \quad [\hat{X}_i, \hat{X}_j] = i c_{ij}{}^k \hat{X}_k \,. \tag{2.56}$$

Moreover, we have

$$\widehat{\mathcal{L}_i k^j} = \sum_{n=1}^{\infty} B^j_{iq_1\cdots q_{n-1}} \hat{k}^{q_1} \cdots \hat{k}^{q_{n-1}} \,, \tag{2.57}$$

where $B^j_{iq_1\cdots q_{n-1}} \in \mathbb{R}$ are the coefficients of the BCH formula (2.2). We saw before that the coordinates k^i correspond also to the 'partial derivative' operators $-i\partial^i$ on the universal enveloping algebra $U(\mathfrak{g})$, which suggests to consider k^i and X_i as canonically conjugate variables, even though their commutation relations are not exactly canonical. We must of course restrict to consider only $\mathcal{G}$-periodic functions of k^i in the end.

2.2.2 Group representation

Let us then consider representations of the quantum algebra $\mathfrak{A}$ defined above. We may define a canonical representation $\pi_{\mathcal{G}}$ in terms of automorphisms of smooth functions on $\mathcal{G}$ as $\pi_{\mathcal{G}}(\hat{f})\phi = f\phi$ and $\pi_{\mathcal{G}}(\hat{X}_i)\phi = i\mathcal{L}_i\phi$ for $\phi \in C_c^\infty(\mathcal{G})$. It is easy to check that the commutation relations are correctly reproduced. Again, we may consider formally the coordinate operators $\hat{k}^i$ acting as $\pi_{\mathcal{G}}(\hat{k}^i)\phi = k^i\phi$, but eventually restrict to consider $\mathcal{G}$-periodic functions of $\hat{k}^i$. Furthermore, as in the Euclidean case, we may restrict to consider compactly supported functions in $C_c^\infty(\mathcal{G})$, and then complete $C_c^\infty(\mathcal{G})$ in the L^2-norm in order to obtain a representation of $\mathfrak{A}$ acting on a Hilbert space.

We note the important role the coproduct of $U(\mathfrak{g})$, following from the Leibniz rule for Lie derivatives, plays in the reproduction of the commutation relations by $\pi_{\mathcal{G}}$. In particular, we have

$$\begin{aligned}\pi_{\mathcal{G}}(\hat{X}_i)\pi_{\mathcal{G}}(\hat{f})\phi &= \pi_{\mathcal{G}}(\hat{X}_i)(f\phi) = m_{\mathcal{G}} \circ \Delta_{\mathfrak{g}}(\pi_{\mathcal{G}}(\hat{X}_i))(f \otimes \phi) \\ &= (\pi_{\mathcal{G}}(\hat{X}_i)f)\phi + f(\pi_{\mathcal{G}}(\hat{X}_i)\phi) \,,\end{aligned} \tag{2.58}$$

since $\Delta_{\mathfrak{g}}(\pi_{\mathcal{G}}(\hat{X}_i)) = \pi_{\mathcal{G}}(\hat{X}_i) \otimes \mathbb{1} + \mathbb{1} \otimes \pi_{\mathcal{G}}(\hat{X}_i)$. On the other hand

$$\pi_{\mathcal{G}}(\hat{f})\pi_{\mathcal{G}}(\hat{X}_i)\phi = f(\pi_{\mathcal{G}}(\hat{X}_i)\phi) \,, \tag{2.59}$$

so that we get for all $\phi \in C_c^\infty(\mathcal{G})$

$$[\pi_{\mathcal{G}}(\hat{X}_i), \pi_{\mathcal{G}}(\hat{f})]\phi = (\pi_{\mathcal{G}}(\hat{X}_i)f)\phi = (i\mathcal{L}_i f)\phi = \pi_{\mathcal{G}}(i\widehat{\mathcal{L}_i f})\phi \,, \tag{2.60}$$

as required.

2.2.3 Algebra representation

Another, in the sense of the factorization of the cotangent bundle, canonically conjugate representation of the quantum algebra $\mathfrak{A}$ can be defined in terms of the universal enveloping algebra. We set

$$\pi_{\mathfrak{g}}(\hat{X}_i)A = e_i \otimes A \quad \text{and} \quad \pi_{\mathfrak{g}}(\hat{f})A = (i_{\mathfrak{g}}f)(-i\vec{\partial})A \tag{2.61}$$

for all $A \in U(\mathfrak{g})$, where ∂^i are as in Equation (2.33). These definitions may be further extended to the completion $\overline{U(\mathfrak{g})} \cong \mathfrak{Q}(C^\infty(\mathfrak{g}^*))$.

The correct reproduction of the commutators among either $\hat{f}$ or $\hat{X}_i$ by themselves are trivial to show. The commutator $[\hat{X}_i, \hat{f}] = i\widehat{\mathcal{L}_i f}$ is slightly less trivial. We note that using the coproduct Δ_∂ we get

$$\begin{aligned} -i\partial^i(e_j \otimes A) &= \Delta_\partial(-i\partial^i)(e_j \otimes A) = B^i(-i\vec{\partial}_1, -i\vec{\partial}_2)(e_j \otimes A) \\ &= \left(\sum_{n=1}^{\infty}(-i)^n \sum_{k+l=n} B^i_{p_1\cdots p_k q_1\cdots q_l}\partial^{p_1}\cdots\partial^{p_k}\otimes\partial^{q_1}\cdots\partial^{q_l}\right)(e_j \otimes A) \\ &= e_j \otimes (-i\partial^i A) - i\sum_{l=1}^{\infty}(-i)^l B^i_{jq_1\cdots q_l}\partial^{q_1}\cdots\partial^{q_l}A\,, \end{aligned} \tag{2.62}$$

where we used $B^i_j = \delta^i_j$. We have $\pi_{\mathfrak{g}}(\hat{k}^i) = -i\partial^i$, and accordingly

$$\begin{aligned} \pi_{\mathfrak{g}}([\hat{X}_i, \hat{k}^j])A &= e_i \otimes (-i\partial^j A) - (-i\partial^j)(e_i \otimes A) \\ &= i\sum_{l=1}^{\infty}(-i)^l B^j_{iq_1\cdots q_l}\partial^{q_1}\cdots\partial^{q_l}A = \pi_{\mathfrak{g}}(i\widehat{\mathcal{L}_i k^j})A \end{aligned} \tag{2.63}$$

for all $A \in U(\mathfrak{g})$, which proves the correct reproduction of the commutation relations for the coordinate operators. This extends formally to the full algebra via

$$\Delta'((i_{\mathfrak{g}}f)(-i\vec{\partial}))(A \otimes B) = (i_{\mathfrak{g}}f)(\Delta_\partial(-i\vec{\partial}))(A \otimes B)\,, \tag{2.64}$$

for all $A, B \in \overline{U(\mathfrak{g})}$, which is equivalent for $\mathcal{G}$-periodic functions to the coproduct Δ' on $C^\infty(\mathcal{G})$ for the exponential elements in $\overline{U(\mathfrak{g})}$.

2.2.4 Non-commutative representation

Now, in order to have a more concrete connection to the original cotangent bundle variables, one may apply the formalism of deformation quantization to the above algebraic representation, as discussed in Subsection 2.1.6. In particular, we want to define a map $\mathfrak{D} : \mathfrak{Q}(C^\infty(\mathfrak{g}^*)) \cong \overline{U(\mathfrak{g})} \to F_\star(\mathfrak{g}^*)$ such that $\mathfrak{D}(AB) = \mathfrak{D}(A) \star \mathfrak{D}(B)$ for all $A, B \in \overline{U(\mathfrak{g})}$ and $\mathfrak{D}(\mathbb{1}) = 1$, corresponding to a quantization of the cotangent space $\mathfrak{Q} : C^\infty(\mathfrak{g}^*) \to$

$\mathfrak{Q}(C^\infty(\mathfrak{g}^*))$. As was mentioned above, the choice of a quantization map $\mathfrak{Q}$, and thus that of $\mathfrak{D}$, is not unique, and for each choice of $\mathfrak{Q}$ there is a corresponding choice of $\mathfrak{D}$.

Let us first, as the simplest concrete example, consider the symmetric quantization map. As a formal power series, we may write it as

$$\mathfrak{Q}_S(f) = \sum_{n=0}^{\infty} e_{i_1} \otimes \cdots \otimes e_{i_n} \left(\frac{\partial}{\partial X_{i_1}}\right) \cdots \left(\frac{\partial}{\partial X_{i_n}}\right) f(\vec{X})\bigg|_{\vec{X}=0} \in \overline{U(\mathfrak{g})} \tag{2.65}$$

for all $f \in C^\infty(\mathfrak{g}^*)$. In particular, $\mathfrak{Q}_S(e^{i\vec{k}\cdot\vec{X}}) = e^{ik}$ for $k := k^i e_i \in \mathfrak{g}$, so that the plane waves $e^{i\vec{k}\cdot\vec{X}}$ are mapped to the exponential elements. Accordingly, $\mathfrak{Q}_S(e^{i\vec{k}\cdot\vec{X}}) = \mathbb{1}$ in $\overline{U(\mathfrak{g})}$ for $R_\mathfrak{g}(k) = 0$, where $R_\mathcal{G}$ is again the canonical restriction onto the principal branch of the logarithm map. Thus, we have for the linear span

$$\text{span}\,\{e^{i\vec{k}\cdot\vec{X}} - 1 \in C^\infty(\mathfrak{g}^*) : R_\mathcal{G}(k) = 0\} \subset \ker \mathfrak{Q}_S\,. \tag{2.66}$$

The corresponding deformation quantization $\mathfrak{D}_S$ is then given by

$$\mathfrak{D}_S(A)(\vec{X}) = P_0 \left(\sum_{n=0}^{\infty} X_{i_1} \cdots X_{i_n} \partial^{i_1} \cdots \partial^{i_n} A\right) \tag{2.67}$$

for all $A \in \overline{U(\mathfrak{g})}$. We have $\mathfrak{D}_S(e^{ik})(\vec{X}) = e^{iR_\mathcal{G}(\vec{k})\cdot\vec{X}}$, and thus

$$\begin{aligned} e^{iR_\mathcal{G}(\vec{k})\cdot\vec{X}} \star e^{iR_\mathcal{G}(\vec{k}')\cdot\vec{X}} &= \mathfrak{D}_S(e^{ik})(\vec{X}) \star \mathfrak{D}_S(e^{ik'})(\vec{X}) = \mathfrak{D}_S(e^{ik}e^{ik'})(\vec{X}) \\ &= \mathfrak{D}_S(e^{iB(k,k')})(\vec{X}) = e^{iR_\mathcal{G}(\vec{B}(k,k'))\cdot\vec{X}}\,. \end{aligned} \tag{2.68}$$

More generally, one may write

$$\mathfrak{Q}(f) = \sum_{n=0}^{\infty} e_{i_1} \otimes \cdots \otimes e_{i_n}\; Q^{i_1\cdots i_n}(\vec{\partial}_X) f(\vec{X})\bigg|_{\vec{X}=0} \in \overline{U(\mathfrak{g})}\,, \tag{2.69}$$

where $\vec{\partial}_X = (\partial/\partial X_i)_i$, and $Q^{i_1\cdots i_n}(\vec{\partial}_X)$ are (pseudo-)differential operators determining the quantization map. In order for this expression to make sense, one must of course impose appropriate convergence properties. The corresponding deformation quantization may be written as

$$\mathfrak{D}(A)(\vec{X}) = P_0 \left(\sum_{n=0}^{\infty} X_{i_1} \cdots X_{i_n} D^{i_1\cdots i_n}(\vec{\partial}) A\right)\,. \tag{2.70}$$

In the following, we will consider in particular quantization maps of the form

$$Q^{i_1\cdots i_n}(\vec{\partial}_X) = \eta(|\vec{\partial}_X|)^{-1} \prod_{k=1}^{n} \xi(|\vec{\partial}_X|) \frac{\partial}{\partial X_{i_k}}\,, \tag{2.71}$$

and the corresponding deformation quantization maps of the form

$$D^{i_1\cdots i_n}(\vec{\partial}) = \eta(|\vec{\partial}|)\prod_{k=1}^{n}\zeta(|\vec{\partial}|)\partial^{i_k}\,, \tag{2.72}$$

such that $\xi(\zeta(|\vec{k}|)|\vec{k}|)\zeta(|\vec{k}|) = 1$ for all $\vec{k} \in \mathbb{R}^d$, where $\eta(0) = 1$, and $\zeta(|\vec{k}|)\vec{k}$ define coordinates on $\mathcal{G}$ with $\zeta(0) = \frac{\mathrm{d}}{\mathrm{d}|k|}\zeta(0) = 1$. This gives

$$\mathfrak{D}(e^{ik}) = e_\star^{i\vec{k}\cdot\vec{X}} = \eta(|\vec{k}|)e^{i\zeta(|\vec{k}|)\vec{k}\cdot\vec{X}}\,, \tag{2.73}$$

where we introduced the star-exponential notation

$$e_\star^{f(X)} := \sum_{n=0}^{\infty}\frac{1}{n!}(\underbrace{f\star\cdots\star f}_{n\ \text{times}})(X)\,. \tag{2.74}$$

Then, one has

$$\begin{aligned}
&\eta(|\vec{k}|)e^{i\zeta(|\vec{k}|)\vec{k}\cdot\vec{X}}\star\eta(|\vec{k}'|)e^{i\zeta(|\vec{k}'|)\vec{k}'\cdot\vec{X}}\\
&= \mathfrak{D}(e^{ik})(\vec{X})\star\mathfrak{D}(e^{ik'})(\vec{X}) = \mathfrak{D}(e^{ik}e^{ik'})(\vec{X})\\
&= \mathfrak{D}(e^{iB(k,k')})(\vec{X}) = \eta(|R_{\mathcal{G}}\vec{B}(k,k')|)e^{i\zeta(|R_{\mathcal{G}}\vec{B}(k,k')|)R_{\mathcal{G}}\vec{B}(k,k')\cdot\vec{X}}\,.
\end{aligned} \tag{2.75}$$

In addition to the symmetric quantization map, we will consider two other special cases of this more general form, namely, the Duflo and the Freidel-Majid-Livine quantization maps specified below.

As we have discussed above, the deformation quantization map $\mathfrak{D}$ gives a representation $\pi_\star : \overline{U(\mathfrak{g})} \to F_\star(\mathfrak{g}^*)$ of the completion of the universal enveloping algebra, where the latter can be taken to act by $\star$-multiplication onto $F_\star(\mathfrak{g}^*)$. Importantly, this representation can be extended to the whole of $\mathfrak{A}$ for some cases of quantization maps. For example, for the symmetric quantization map we have

$$\mathfrak{D}_S(e_i\otimes A) = X_i\star\mathfrak{D}_S(A) \quad \text{and} \quad \mathfrak{D}_S(\partial^i A) = -i\frac{\partial}{\partial X_i}\mathfrak{D}_S(A)\,. \tag{2.76}$$

More generally, if $\mathfrak{D}$ is of the form (2.72), we have $\mathfrak{D}(\partial^i A) = -i\xi(|\vec{\partial}_X|)\frac{\partial}{\partial X_i}\mathfrak{D}(A)$. Accordingly, $\mathfrak{D}$ intertwines the algebraic representation with another given in terms of a deformation quantization $\star$-product, and thus setting

$$\pi_\star(\hat{X}_i)\varphi = X_i\star\varphi \quad \text{and} \quad \pi_\star(\hat{k}^i)\varphi = -i\xi(|\vec{\partial}_X|)\frac{\partial}{\partial X_i}\varphi \tag{2.77}$$

for all $\varphi \in F_\star(\mathfrak{g}^*)$ defines a representation of $\mathfrak{A}$ on $F_\star(\mathfrak{g}^*)$. We call this the non-commutative representation of $\mathfrak{A}$. Alternatively, one may write $\zeta(|\hat{k}|)\hat{k}^i =: \hat{\zeta}^i$ and

$$\pi_\star(\hat{\zeta}^i) := \pi_\star(\zeta(|\hat{k}|)\hat{k}^i)\varphi = -i\frac{\partial}{\partial X_i}\varphi \tag{2.78}$$

for the second operator, when $\zeta^i := \zeta(|\vec{k}|)k^i$ determine a set of coordinates on $\mathcal{G}$.

2.2.5 Non-commutative Fourier transform

We have defined above different representations of the quantum algebra $\mathfrak{A}$, the group representation $\pi_\mathcal{G}$, the algebra representation $\pi_\mathfrak{g}$ and the non-commutative representation $\pi_\star$. Now, we want to find the intertwiners between these representations. Let us first consider a map $\mathcal{F} : C_c^\infty(\mathcal{G}) \to \overline{U(\mathfrak{g})}$ of the form

$$\mathcal{F}(\phi) := \int_\mathcal{G} \mathrm{d}g\ e^{ik(g)}\phi(g) \in \overline{U(\mathfrak{g})} \tag{2.79}$$

for all $\phi \in C_c^\infty(\mathcal{G})$, where we identify $g \in \mathcal{G}$ with the exponential element $e^{ik(g)} \in \overline{U(\mathfrak{g})}$ for $k(g) = \ln_R(g) \in \mathfrak{g}$, and $\mathrm{d}g$ is the right-invariant Haar measure. $\mathcal{F}$ is an intertwiner between the representations $\pi_\mathcal{G}$ and $\pi_\mathfrak{g}$, since we have for the generators of $\mathfrak{A}$

$$\mathcal{F}(\pi_\mathcal{G}(\hat{f})\phi) = \int_\mathcal{G} \mathrm{d}g\ e^{ik(g)}f(g)\phi(g) = \int_\mathcal{G} \mathrm{d}g\ (i_\mathfrak{g}f)(-i\vec{\partial})e^{ik(g)}\phi(g) = \pi_\mathfrak{g}(\hat{f})\mathcal{F}(\phi)\,, \tag{2.80}$$

and

$$\begin{aligned}\mathcal{F}(\pi_\mathcal{G}(\hat{X}_i)\phi) &= \int_\mathcal{G} \mathrm{d}g\ e^{ik(g)}(i\mathcal{L}_i\phi)(g) = \int_\mathcal{G} \mathrm{d}g\ (-i\mathcal{L}_i e^{ik(g)})\phi(g)\\ &= \int_\mathcal{G} \mathrm{d}g\ (e_i \otimes e^{ik(g)})\phi(g) = \pi_\mathfrak{g}(\hat{X}_i)\mathcal{F}(\phi)\,,\end{aligned} \tag{2.81}$$

where we used integration by parts for the second equality. The possible boundary terms resulting from the integration are in $\ker \mathfrak{Q}$ and therefore vanish in $\overline{U(\mathfrak{g})}$. Moreover, we find by an explicit calculation

$$\mathcal{F}(\phi)\mathcal{F}(\phi') = \mathcal{F}(\phi * \phi')\,, \tag{2.82}$$

which provides the link between the operator (tensor) product in $\overline{U(\mathfrak{g})}$ and convolution product $(\phi * \phi')(g) := \int_\mathcal{G} \mathrm{d}g\ \phi(gh^{-1})\phi'(h)$ in $C_c^\infty(\mathcal{G})$ alluded to before in relation to the Hopf structures.

Now, consider a formal linear trace operation Tr on $\overline{U(\mathfrak{g})}$ defined through the relation $\mathrm{Tr}(e^{ik(g)}) = \delta(g)$, where δ is the Dirac distribution with respect to the (right-invariant) Haar measure on $\mathcal{G}$ peaked at the unit element. An inverse transform may be then defined as

$$\mathcal{F}^{-1}(A)(g) = \mathrm{Tr}(e^{-ik(g)} \otimes A)\,. \tag{2.83}$$

We have

$$\mathcal{F}^{-1}(\mathcal{F}(\phi))(g) = \int_{\mathcal{G}} \mathrm{d}h \; \mathrm{Tr}(e^{-ik(g)} \otimes e^{ik(h)})\phi(h) = \phi(g)\,, \tag{2.84}$$

where we used $\mathrm{Tr}(e^{-ik(g)} \otimes e^{ik(h)}) = \mathrm{Tr}(e^{ik(g^{-1}h)}) = \delta(g^{-1}h)$. Thus, we have $\mathcal{F}^{-1} \circ \mathcal{F} = \mathrm{id}_{C_c^\infty(\mathcal{G})}$ and, correspondingly, $\mathcal{F} \circ \mathcal{F}^{-1}$ gives the projection onto the image of $\mathcal{F}$.

As before, in order to make the expressions more concrete, we may translate the above algebraic construct onto a non-commutative space using a deformation quantization map, which gives

$$\mathcal{F}_\star(\phi)(X) := \mathfrak{D}(\mathcal{F}(\phi))(X) = \int_{\mathcal{G}} \mathrm{d}g \; \mathfrak{D}(e^{ik(g)})(X)\phi(g) = \int_{\mathcal{G}} \mathrm{d}g \; e_\star^{ik(g)\cdot X}\phi(g)\,. \tag{2.85}$$

By construction, $\mathfrak{D}$ being an algebra homomorphism, $\mathcal{F}_\star$ intertwines the representations $\pi_{\mathcal{G}}$ and $\pi_\star$ of $\mathfrak{A}$.

Let us list some important properties of the kernel $E_g(X) := \mathfrak{D}(e^{ik(g)}) = e_\star^{ik(g)\cdot X}$ of the transformation $\mathcal{F}_\star$, the non-commutative plane wave [29]:

$$E_e(X) = 1\,, \tag{2.86}$$

$$\mathfrak{Q}(E_g(X)) = e^{i\vec{k}(g)\cdot\hat{X}} \in \mathfrak{A}_{\mathfrak{g}^*}\,, \tag{2.87}$$

$$E_{g^{-1}}(X) = \overline{E_g(X)} = E_g(-X)\,, \tag{2.88}$$

$$E_{gh}(X) = E_g(X) \star E_h(X)\,, \tag{2.89}$$

$$E_g(\mathrm{Ad}_h^* X) = E_{hgh^{-1}}(X)\,, \tag{2.90}$$

where $(\mathrm{Ad}_h^* X)(Y) \equiv X(\mathrm{Ad}_h Y)$ for all $X \in \mathfrak{g}^*$, $Y \in \mathfrak{g}$. The non-commutative plane wave also acts as the generating function of $\star$-monomials, as we have

$$X_{i_1} \star \cdots \star X_{i_n} = (-i)^n \mathcal{L}_{i_1} \cdots \mathcal{L}_{i_n} E_g(X)\big|_{g=e}\,. \tag{2.91}$$

Now, assume that the deformation quantization map is such that we have

$$\int_{\mathfrak{g}^*} \frac{\mathrm{d}^d X}{(2\pi)^d}\; e_\star^{i\vec{k}(g)\cdot\vec{X}} = \delta^d(\vec{k}(g)) \equiv \delta(g)\,, \tag{2.92}$$

which is the Dirac delta distribution with respect to the Haar measure peaked at the unit element. In other words, we require the intertwining property to extend to the formal trace operation Tr, so that $\mathfrak{D}(\mathrm{Tr}(A)) = \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d X}{(2\pi)^d}\, \mathfrak{D}(A)(X)$ for all $A \in \overline{U(\mathfrak{g})}$, where $\mathrm{d}^d X$ is the Lebesgue measure on $\mathfrak{g}^*$.[13] Indeed, this is true for example for the symmetric deformation quantization map, for which $\mathfrak{D}(e^{ik})(X) = e^{i\vec{k}\cdot\vec{X}}$. This key property allows us to define an inverse transform $\mathcal{F}_\star^{-1}$ as

$$\mathcal{F}_\star^{-1}(\varphi)(g) = \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d X}{(2\pi)^d} e_\star^{-i\vec{k}(g)\cdot\vec{X}} \star \varphi(X)\,, \tag{2.93}$$

since then we have

$$\begin{aligned}
\mathcal{F}_\star^{-1}(\mathcal{F}_\star(\phi))(g) &= \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d X}{(2\pi)^d} e_\star^{-i\vec{k}(g)\cdot\vec{X}} \star \int_{\mathcal{G}} \mathrm{d}h\; e_\star^{i\vec{k}(h)\cdot\vec{X}} \phi(h) \\
&= \int_{\mathcal{G}} \mathrm{d}h\; \left(\int_{\mathfrak{g}^*} \frac{\mathrm{d}^d X}{(2\pi)^d} e_\star^{-i\vec{k}(g)\cdot\vec{X}} \star e_\star^{i\vec{k}(h)\cdot\vec{X}} \right) \phi(h) \\
&= \int_{\mathcal{G}} \mathrm{d}h\; \left(\int_{\mathfrak{g}^*} \frac{\mathrm{d}^d X}{(2\pi)^d} e_\star^{-i\vec{k}(gh)\cdot\vec{X}} \right) \phi(h) \\
&= \int_{\mathcal{G}} \mathrm{d}h\; \delta(gh)\phi(h) = \phi(g)\,,
\end{aligned} \tag{2.94}$$

and thus $\mathcal{F}_\star^{-1} \circ \mathcal{F}_\star = \mathrm{id}_{C_c^\infty(\mathcal{G})}$. Moreover, we have

$$\begin{aligned}
\mathcal{F}_\star(\mathcal{F}_\star^{-1}(\varphi))(X) &= \int_{\mathcal{G}} \mathrm{d}g\; e_\star^{i\vec{k}(g)\cdot\vec{X}} \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d Y}{(2\pi)^d} e_\star^{-i\vec{k}(g)\cdot\vec{Y}} \star \varphi(Y) \\
&= \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d Y}{(2\pi)^d} \left(\int_{\mathcal{G}} \mathrm{d}g\; e_\star^{i\vec{k}(g)\cdot\vec{X}} e_\star^{-i\vec{k}(g)\cdot\vec{Y}} \right) \star \varphi(Y)\,.
\end{aligned} \tag{2.95}$$

If $\varphi = \mathcal{F}_\star(\phi)$ for some $\phi \in C_c^\infty(\mathcal{G})$, then we must have $\mathcal{F}_\star(\mathcal{F}_\star^{-1}(\varphi)) = \varphi$ due to $\mathcal{F}_\star^{-1} \circ \mathcal{F}_\star = \mathrm{id}_{C_c^\infty(\mathcal{G})}$. Accordingly, $\mathcal{F}_\star \circ \mathcal{F}_\star^{-1} =: \mathcal{P}_\star$ is a projection, $\mathcal{P}_\star \circ \mathcal{P}_\star = \mathcal{P}_\star$, and

$$\delta_\star(X, Y) := \int_{\mathcal{G}} \mathrm{d}g\; e_\star^{i\vec{k}(g)\cdot\vec{X}} e_\star^{-i\vec{k}(g)\cdot\vec{Y}} \tag{2.96}$$

is the kernel of the projection onto $\mathcal{F}_\star(C_c^\infty(\mathcal{G})) =: F_\star^c(\mathfrak{g}^*)$, which acts as the delta distribution with respect to the $\star$-product on $F_\star^c(\mathfrak{g}^*)$.

Consequently, we have defined the non-commutative Fourier transform

$$\tilde{\psi} := \mathcal{F}_\star(\psi)(X) = \int_{\mathcal{G}} \mathrm{d}g\; e_\star^{ik(g)\cdot X} \psi(g)\,, \tag{2.97}$$

[13] Here it might be possible to generalize the construction to allow for a more general measure, so that $\mathfrak{D}(\mathrm{Tr}(A)) = \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d X}{(2\pi)^d}\, \omega(X)\mathfrak{D}(A)(X)$ for some weight function ω on $\mathfrak{g}^*$. Indeed, such a relation occurs for the spinorial construction of [20]. However, we will restrict to consider the translationally invariant case in what follows.

and its inverse

$$\psi = \mathcal{F}_\star^{-1}(\tilde{\psi})(g) = \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d X}{(2\pi)^d} e_\star^{-i\vec{k}(g)\cdot\vec{X}} \star \tilde{\psi}(X)\,, \tag{2.98}$$

that intertwine the two conjugate representations, the group and the non-commutative representation, of the quantum algebra $\mathfrak{A}$ in terms of automorphisms of $C_c^\infty(\mathcal{G})$ and $F_\star^c(\mathfrak{g}^*)$.

2.3 Properties of the Non-commutative Representation

Let us now consider some properties of the transform $\mathcal{F}_\star$ and the non-commutative function space $F_\star(\mathfrak{g}^*)$:

- Group multiplication from the right is dually represented on $\mathcal{F}_\star(\psi)(X)$ by $\star$-multiplication by $E_{g^{-1}}(X)$ from the right:

$$\begin{aligned}\mathcal{F}_\star(R_g\psi)(X) &= \int_{\mathcal{G}} \mathrm{d}h\, E_h(X)\, \psi(hg) \\ &= \int_{\mathcal{G}} \mathrm{d}h\, E_{hg^{-1}}(X)\, \psi(h) = \mathcal{F}_\star(\psi)(X) \star E_{g^{-1}}(X)\end{aligned} \tag{2.99}$$

  using the right-invariance of the Haar measure.

- If $\mathcal{G}$ is unimodular, i.e., the left- and right-invariant Haar measures coincide, we have

$$E_g(X) \star f(X) = f(\mathrm{Ad}_g^* X) \star E_g(X)\,, \tag{2.100}$$

  where $(\mathrm{Ad}_g^* X)(Y) \equiv X(\mathrm{Ad}_g(Y))$ for all $Y \in \mathfrak{g}$.

- Consider the $L_\star^2(\mathfrak{g}^*)$ inner product of two functions obtained through the transform

$$\begin{aligned}\langle \tilde{\psi}, \tilde{\psi}' \rangle_{\mathfrak{g}^*} &:= \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d X}{(2\pi)^d} \overline{\tilde{\psi}(X)} \star \tilde{\psi}'(X) \\ &= \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d X}{(2\pi)^d} \left[\int_{\mathcal{G}} \mathrm{d}g\, E_{g^{-1}}(X)\, \overline{\psi(g)}\right] \star \left[\int_{\mathcal{G}} \mathrm{d}h\, E_h(X)\, \psi'(h)\right] \\ &= \int_{\mathcal{G}} \mathrm{d}g \int_{\mathcal{G}} \mathrm{d}h\, \overline{\psi(g)}\, \psi'(h) \left[\int_{\mathfrak{g}^*} \frac{\mathrm{d}^d X}{(2\pi)^d} E_{g^{-1}h}(X)\right]\,.\end{aligned} \tag{2.101}$$

  Using (2.92), we find

$$\langle \tilde{\psi}, \tilde{\psi}' \rangle_{\mathfrak{g}^*} \equiv \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d X}{(2\pi)^d} \overline{\tilde{\psi}(X)} \star \tilde{\psi}'(X) = \int_{\mathcal{G}} \mathrm{d}g\, \overline{\psi(g)}\, \psi'(g) \equiv \langle \psi, \psi' \rangle_{\mathcal{G}}\,, \tag{2.102}$$

  so $\mathcal{F}_\star$ is, in fact, an isometry from $L^2(\mathcal{G})$ to $L_\star^2(\mathfrak{g}^*)$. Therefore, we may identify $L_\star^2(\mathfrak{g}^*) = \mathcal{F}_\star(L^2(\mathcal{G}))$.

- It is easy to check that the kernel of $\mathcal{P}_\star = \mathcal{F}_\star \circ \mathcal{F}_\star^{-1}$, $\ker(\mathcal{P}_\star) = \{\tilde{\psi} \in L^2_\star(\mathfrak{g}^*) : \mathcal{P}_\star(\tilde{\psi}) = 0\}$, contains all functions of the form $(e^{ik(e)\cdot X} - e^{ik'(e)\cdot X}) \star \tilde{\psi}(X)$, $\tilde{\psi} \in L^2_\star(\mathfrak{g}^*)$, where $k(e), k'(e) \in \mathfrak{g}$ are any two values of $-i\ln(e)$, and therefore $\mathcal{P}_\star$ implements the aforementioned $\ker\mathfrak{Q}$-equivalence classes in $L^2_\star(\mathfrak{g}^*)$.

- If $\mathfrak{D}$ is of the form (2.72), We have two equivalent expressions for the $\star$-product under integration in terms of a pseudo-differential operator σ, namely,

$$\begin{aligned}\int_{\mathfrak{g}^*} \mathrm{d}^d X\, \overline{\tilde{\psi}(X)} \star \tilde{\psi}'(X) &= \int_{\mathfrak{g}^*} \mathrm{d}^d X\, \left(\sigma(i\vec{\partial})\, \overline{\tilde{\psi}(X)}\right) \tilde{\psi}'(X) \\ &= \int_{\mathfrak{g}^*} \mathrm{d}^d X\, \overline{\tilde{\psi}(X)} \left(\sigma(-i\vec{\partial})\, \tilde{\psi}'(X)\right) \quad \forall\, \tilde{\psi}, \tilde{\psi}' \in L^2_\star(\mathfrak{g}^*)\,,\end{aligned} \tag{2.103}$$

  where $\sigma(\zeta) := \left(\omega(\zeta)|\eta(\zeta)|^2\right)^{-1}$ for $\zeta \in \mathfrak{g}$, $\mathrm{d}g \equiv \omega(\zeta(g))\,\mathrm{d}\zeta(g)$ for the right-invariant Haar measure, and $\eta(\zeta(g)) \equiv E(g, 0)$. For the proof of this identity we refer to [29], Appendix B.

- Due to (2.103), we may write the inverse transform $\mathcal{F}_\star^{-1} : L^2_\star(\mathfrak{g}^*) \to L^2(\mathcal{G})$ from (2.98) explicitly without a star-product as

$$\mathcal{F}_\star^{-1}(\tilde{\psi})(g) = \sigma(g) \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d X}{(2\pi)^d}\, \overline{E_g(X)}\, \tilde{\psi}(X)\,, \tag{2.104}$$

  where $\sigma(g) := \left(\omega(\zeta(g))|\eta(g)|^2\right)^{-1}$.

- Finally, due to $E_g \star E_h = E_{gh}$ the $\star$-product is dual to the convolution product under the non-commutative Fourier transform, that is,

$$\tilde{\psi} \star \tilde{\psi}' = \widetilde{\psi * \psi'}\,, \tag{2.105}$$

  where the convolution product is defined on the group as usual

$$\psi * \psi'(g) = \int_{\mathcal{G}} \mathrm{d}h\, \psi(gh^{-1})\psi'(h)\,. \tag{2.106}$$

2.4 Basic Examples

In this section, we consider some concrete examples of the non-commutative representation and the Fourier transform for specific elementary Lie groups, namely, $\mathbb{R}^d$, $U(1)$ and $SU(2)$. For each example we will determine the explicit form of the lowest order star-polynomials, the non-commutative plane wave, the corresponding Fourier transform and its inverse for some choices of deformation quantization map. On the one hand, the examples presented prove the non-emptiness of the definitions, together with the existence of their non-commutative representation and of their non-commutative Fourier transforms. On the other hand, the results on specific quantization maps find direct applications to physics models in Chapter 3.

2.4.1 $\mathbb{R}^d$

The Euclidean vector space $\mathbb{R}^d$ equipped with vector addition constitutes a Lie group, and provides the simplest possible example of our general construction above. In Section 2.1, we have already reviewed how the usual Fourier transform can be undestood as an intertwiner between the group/position and the algebra/momentum representations. Since the group $\mathbb{R}^d$ is abelian and has no compact subgroups, the deformation quantization $\star$-product for $\overline{U(\mathbb{R}^d)}$ coincides with the point-wise product, we have $\mathfrak{D}(e^{ik}) = e^{i\vec{k}\cdot\vec{X}}$ and $e^{i\vec{k}\cdot\vec{X}} \star e^{i\vec{k}'\cdot\vec{X}} = e^{i(\vec{k}+\vec{k}')\cdot\vec{X}}$. Accordingly, we obtain the familiar formalism of Fourier transform on Euclidean space. Therefore, the non-commutative Fourier transform formulated above presents a generalization to the Euclidean Fourier transform.

2.4.2 $U(1)$

U(1) is given by the set of complex numbers $z \in \mathbb{C}$ with modulus one $|z| = 1$. Accordingly, we can set $z = e^{i\theta}$, $\theta \in \mathbb{R}$. The canonical coordinates $k(g) = -i\ln(g) \equiv \theta$ are restricted to the principal branch of the logarithm as $\theta \in]-\pi, \pi]$. The dual of the Lie algebra $\mathfrak{u}(1)^*$ is simply given by the real numbers $X \in \mathbb{R}$.

In the abelian case, and in particular for $\mathfrak{u}(1)$, which has just one generator, no ordering ambiguity arises, so that there is no difference between quantization maps in this respect. Therefore, the natural choice of a star-product coincides with the pointwise product. However, first of all the group is compact, and this topological feature already makes things a little more interesting. Second, we have seen how the quantization map also affects the choice of coordinates appearing in the plane waves. It is then worth to consider this simple case in some detail.

For the symmetrization map $\mathfrak{Q}_S$ (and also for the Duflo map $\mathfrak{Q}_D$ which we will consider below, as they coincide for abelian groups) we indeed have $\mathfrak{Q}_S(X^n) = \hat{X}^n$ and, therefore,

$$\mathfrak{Q}_S(e^{i\theta X}) = e^{i\theta\hat{X}}\,, \tag{2.107}$$

that is, as expected, the plane waves are given by $e^{i\theta X}$, for $\theta \in]-\pi, \pi]$, $X \in \mathbb{R}$, and the corresponding $\star$-product on monomials is simply the pointwise product

$$\underbrace{X \star \cdots \star X}_{n \text{ times}} = X^n\,. \tag{2.108}$$

Nevertheless, the $\star$-product $e^{i\theta X} \star e^{i\theta' X} = e^{iR(\theta+\theta')X}$ of plane waves, where R is the restriction map onto the principal branch $]-\pi,\pi]$ of the logarithm, is still non-trivial due to the compactness of the group.

Furthermore, from (2.103) we have that

$$\int \mathrm{d}X\; f(X)\star f'(X) = \int \mathrm{d}X\; f(X)f'(X)\,, \tag{2.109}$$

since in this case $\mathrm{d}g = \mathrm{d}\theta \Rightarrow \omega(\theta) = 1$ and $E_g(X) = e^{i\theta X} \Rightarrow \eta(\theta) = 1$, so $\sigma = 1$. The non-commutative Fourier transform is thus given by

$$\tilde{\psi}(X) = \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi}\, e^{i\theta X}\, \psi(e^{i\theta})\,, \tag{2.110}$$

while its inverse is

$$\psi(e^{i\theta}) = \int_{\mathbb{R}} \mathrm{d}X\, e^{-i\theta X}\, \tilde{\psi}(X)\,. \tag{2.111}$$

Let us now point out the consequence of the existence of normal subgroups corresponding to the identity element in this simple case. The periodicity of the group is taken care of by the restriction map R, which translates it into the equivalence class of functions on the Lie algebra $\tilde{\psi}(X) = e^{i2\pi nX} \star \tilde{\psi}(X)$, $n \in \mathbb{Z}$, which are all mapped to the same function in $F_\star(\mathfrak{u}(1)^*)$ by the projection $\mathcal{P}_\star = \mathcal{F}_\star \circ \mathcal{F}_\star^{-1}$. This is the counterpart, in our setting, of the the restriction $X \in \mathbb{Z}$ for the usual Fourier transform on the circle, where the inverse transform is given by a sum over the integers.[14]

We have thus seen that the symmetric (and Duflo) map leads to plane waves equivalent to the usual ones. Still, we have also seen within the general formalism that the choice of quantization maps affects non-trivially also the coordinates appearing in the plane waves. Vice versa, by choosing non-linear coordinates on the group, one can end up with non-trivial star-products, despite the abelianess of the group. Let us say we have $\mathfrak{D}$ such that

$$\mathfrak{D}(e^{i\theta\hat{X}}) = e^{2i\sin\frac{\theta}{2}X}\,. \tag{2.112}$$

$\zeta(\theta) = 2\sin\frac{\theta}{2}$ can be seen as new coordinates on the group valid for $\theta \in]-\pi,\pi]$. According

[14] In fact, it was proved in [16] that this U(1) non-commutative Fourier transform defined for the full $\mathbb{R}$ can, in fact, be determined by its values on the integers; thus, even though the U(1) non-commutative Fourier transform is defined distinctively from the usual Fourier transform on the circle, they were shown to coincide due to this form of sampling.

to (2.91), we get for the $\star$-product on monomials already a diverting result at third order

$$X \star X = X^2 \,, \tag{2.113}$$

$$X \star X \star X = X^3 + \frac{1}{4} X \,, \tag{2.114}$$

$$\vdots$$

and, therefore, as remarked before, we see that the quantization map, choice of coordinates, and a star-product are related in a highly non-trivial way.

We may give an expression for the corresponding $\star$-product under integral, from (2.103), as a (non-trivial) pseudo-diffential operator

$$\int \mathrm{d}X \; f(X) \star f'(X) = \int \mathrm{d}X \; f(X) \sqrt{1 + \tfrac{1}{4} \left(\tfrac{\mathrm{d}}{\mathrm{d}X}\right)^2} \, f'(X) \,, \tag{2.115}$$

(where $\frac{\mathrm{d}}{\mathrm{d}X}$ may act either left or right) as we now have, in contrast to the previous parametrization, a non-trivial relation between the Haar measure $\mathrm{d}\theta$ and the Lebesgue measure $\mathrm{d}\zeta$, namely, $\mathrm{d}\theta = (\sqrt{1-\zeta^2/4})^{-1}\mathrm{d}\zeta$, so $\sigma(\zeta) = \sqrt{1-\zeta^2/4}$.

The non-commutative Fourier transform is thus given by

$$\tilde{\psi}(X) = \int_{-\pi}^{\pi} \mathrm{d}\theta \; e^{2i \sin \frac{\theta}{2} X} \, \psi(e^{i\theta}) \,, \tag{2.116}$$

while its inverse is, from (2.104),

$$\psi(e^{i\theta}) = \cos(\tfrac{\theta}{2}) \int_{\mathbb{R}} \frac{\mathrm{d}X}{2\pi} \; e^{-2i \sin \frac{\theta}{2} X} \, \tilde{\psi}(X) \,. \tag{2.117}$$

2.4.3 $SU(2)$

We now consider a simple but very important non-abelian example, SU(2), which is particularly relevant also for quantum gravity applications.

The Lie algebra $\mathfrak{su}(2)$ has a basis given (in the defining representation) by a set of two-by-two traceless hermitian matrices $\{\sigma_j\}_{j=1,2,3}$, which read

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{2.118}$$

and satisfy $\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$, or rather, the relations $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$. Thus, a generic element $k \in \mathfrak{su}(2)$ can be written as $k = k^j \sigma_j$, $k^j \in \mathbb{R}$, while for any group element $g \in \mathrm{SU}(2)$ we may write $g = e^{ik^j\sigma_j}$ – SU(2) is an exponential Lie group. Another convenient parametrization of SU(2) can be written as

$$g = p^0 \mathbb{1} + ip^i \sigma_i \,, \quad (p^0)^2 + p^i p_i = 1 \,, \quad p^i \in \mathbb{R} \,. \tag{2.119}$$

Here, the p^i's are constrained by the $\mathbb{R}^3$ vector norm $|\vec{p}|^2 \leq 1$. Thus, this last parametrization naturally identifies SU(2) with the 3-sphere S^3. $p^0 \geq 0$ and $p^0 \leq 0$ correspond to the upper and lower hemispheres of S^3, respectively, in turn corresponding to two copies of SO(3). Parametrization of the group elements in terms of $\vec{p} \in \mathbb{R}^3$ is one-to-one only on either of the two hemispheres, whereas the canonical coordinates $\vec{k}$ parametrize the whole group except for $-\mathbb{1} \in \mathrm{SU}(2)$.

The relation between these two parametrizations is mediated by the following change of coordinates

$$\vec{p} = \frac{\sin|\vec{k}|}{|\vec{k}|}\vec{k}, \quad p_0 = \cos|\vec{k}|, \qquad k^i \in \mathbb{R}, \tag{2.120}$$

where $|\vec{k}| \in [0, \frac{\pi}{2}[$, or $|\vec{k}| \in [\frac{\pi}{2}, \pi[$ according to $p^0 \geq 0$, $p^0 \leq 0$ respectively, and $g \in \mathrm{SU}(2)$ assumes the form

$$g = \cos|\vec{k}|\mathbb{1} + i\frac{\sin|\vec{k}|}{|\vec{k}|}\vec{k}\cdot\vec{\sigma} = e^{i\vec{k}\cdot\vec{\sigma}}. \tag{2.121}$$

We call the coordinates introduced the $\vec{k}$-parametrization and the $\vec{p}$-parametrization, respectively. The Haar measure on the group takes then the form

$$\mathrm{d}g = \mathrm{d}^3\vec{k}\left(\frac{\sin|\vec{k}|}{|\vec{k}|}\right)^2, \quad \vec{k} \in \mathbb{R}^3, |\vec{k}| \in [0, \pi[, \tag{2.122}$$

$$\mathrm{d}g = \frac{\mathrm{d}^3\vec{p}}{\sqrt{1-|\vec{p}|^2}}, \quad \vec{p} \in \mathbb{R}^3, |\vec{p}|^2 < 1, \tag{2.123}$$

where the latter is again applicable only for one of the two hemispheres.

We now consider three choices of quantization maps, and derive the corresponding $\star$-product, non-commutative representation and plane waves.

Symmetrization map

Given a set of $\mathfrak{su}(2)$ coordinates $X_{i_1}, \ldots, X_{i_n}$, the symmetrization map $\mathfrak{Q}_S$ takes the symmetric ordering of the corresponding coordinate operators $\hat{X}_{i_1}, \ldots, \hat{X}_{i_n}$,

$$\mathfrak{Q}_S(X_{i_1}\cdots X_{i_n}) = \frac{1}{n!}\sum_{\sigma\in S_n}\hat{X}_{i_{\sigma_1}}\cdots\hat{X}_{i_{\sigma_n}}, \tag{2.124}$$

where S_n is the symmetric group of order n.

Thus, for instance, for an exponential of the form $e^{i\vec{k}\cdot\vec{X}}$, we have

$$\mathfrak{Q}_S(e^{i\vec{k}\cdot\vec{X}}) = e^{i\vec{k}\cdot\hat{X}}, \tag{2.125}$$

which implies that the function $e^{i\vec{k}\cdot\vec{X}}$ gives exactly the $\star$-exponential (plane wave) for symmetric quantization with the $\vec{k}$-parametrization.

The composition of coordinates can be inferred from

$$e^{i\vec{k}_1\cdot\vec{X}} \star_S e^{i\vec{k}_2\cdot\vec{X}} = \mathfrak{D}_S(\mathfrak{Q}_S(e^{i\vec{k}_1\cdot\vec{X}})\mathfrak{Q}_S(e^{i\vec{k}_2\cdot\vec{X}})) = e^{iRB(\vec{k}_1,\vec{k}_2)\cdot\vec{X}}\,, \tag{2.126}$$

where $RB(\vec{k}_1,\vec{k}_2)$ is the value of the BCH formula restricted onto the principal branch of the logarithm. This star-product is referred to as the Gutt (or 'standard') $\star$-product [30].

Under integration, using (2.103) and (2.122), the $\star_S$-product acquires the form

$$\int_{\mathfrak{g}^*} \mathrm{d}^3X\ f(\vec{X}) \star_S f'(\vec{X}) = \int_{\mathfrak{g}^*} \mathrm{d}^3X\ f(\vec{X}) \left(\frac{|\vec{\partial}_X|}{\sin|\vec{\partial}_X|}\right)^2 f'(\vec{X})\,. \tag{2.127}$$

Given the plane waves just computed, we may then write the explicit form for the non-commutative Fourier transform as

$$\tilde{\psi}(\vec{X}) = \int_{\mathbb{R}^3,|\vec{k}|\in[0,\pi[} \mathrm{d}^3k \left(\frac{\sin|\vec{k}|}{|\vec{k}|}\right)^2 e^{i\vec{k}\cdot\vec{X}}\,\psi(\vec{k})\,, \tag{2.128}$$

with the inverse, from (2.104), being

$$\psi(\vec{k}) = \left(\frac{|\vec{k}|}{\sin|\vec{k}|}\right)^2 \int_{\mathbb{R}^3} \frac{\mathrm{d}^3X}{(2\pi)^3}\ e^{-i\vec{k}\cdot\vec{X}}\,\tilde{\psi}(\vec{X})\,. \tag{2.129}$$

Duflo map

The defining property of the Duflo quantization map $\mathfrak{Q}_D$ is that it provides an algebra isomorphism between the $\mathcal{G}$-invariant (Casimir) operators of $\overline{U(\mathfrak{g})}$ and the subspace of functions in $F_\star(\mathfrak{g}^*)$ that are invariant under the adjoint action of $\mathcal{G}$ on $\mathfrak{g}^*$. In this respect, it is the most natural quantization map to consider, as it translates maximally the set of invariants from the classical level to the quantum level. Duflo map is given explicitly by

$$\mathfrak{Q}_D = \mathfrak{Q}_S \circ j^{\frac{1}{2}}(\partial)\,, \tag{2.130}$$

where j is the following function on $\mathfrak{g}$:

$$j(X) = \det\left(\frac{\sinh\frac{1}{2}\mathrm{ad}_X}{\frac{1}{2}\mathrm{ad}_X}\right)\,. \tag{2.131}$$

For $X \in \mathfrak{su}(2)$, j computes to

$$j(X) = \left(\frac{\sinh|X|}{|X|}\right)^2 . \tag{2.132}$$

The application of the Duflo quantization map to exponentials $e^{i\vec{k}\cdot\vec{X}}$ gives

$$\mathfrak{Q}_D(e^{i\vec{k}\cdot\vec{X}}) = \frac{\sin|\vec{k}|}{|\vec{k}|} e^{i\vec{k}\cdot\hat{X}} , \tag{2.133}$$

which can be inverted to yield

$$\mathfrak{D}_D(e^{i\vec{k}\cdot\hat{X}}) = \frac{|\vec{k}|}{\sin|\vec{k}|}\, e^{i\vec{k}\cdot\vec{X}} \equiv e_\star^{i\vec{k}\cdot\vec{X}} , \tag{2.134}$$

that is, we have found the plane wave $E_g(X)$ under the Duflo deformation quantization $\mathfrak{D}_D$ with the $\vec{k}$-parametrization.

Once again, we may now use (2.91) to compute the $\star_\mathcal{D}$-product on monomials:

$$X_i \star_\mathcal{D} X_j = X_i X_j + i\epsilon_{ij}{}^k X_k - \frac{1}{3}\delta_{ij} , \tag{2.135}$$

$$\begin{aligned} X_i \star_\mathcal{D} X_j \star_\mathcal{D} X_k &= X_i X_j X_k + i(\epsilon_{ij}{}^m X_k + \epsilon_{ik}{}^m X_j + \epsilon_{jk}{}^m X_i) X_m \\ &\quad + \frac{1}{3}\delta_{jk} X_i - \frac{2}{3}\delta_{ik} X_j + \frac{1}{3}\delta_{ij} X_k , \\ &\;\vdots \end{aligned} \tag{2.136}$$

This star-product coincides with the star-product introduced by Kontsevich in [40]. For the non-commutative plane wave we again have the corresponding projected star-product $\star_\mathcal{D}$, which satisfies

$$\frac{|\vec{k}_1|}{\sin|\vec{k}_1|}\, e^{i\vec{k}_1\cdot\vec{X}} \star_\mathcal{D} \frac{|\vec{k}_2|}{\sin|\vec{k}_2|}\, e^{i\vec{k}_2\cdot\vec{X}} = \frac{|RB(\vec{k}_1,\vec{k}_2)|}{\sin|RB(\vec{k}_1,\vec{k}_2)|}\, e^{iRB(\vec{k}_1,\vec{k}_2)\cdot\vec{X}} . \tag{2.137}$$

Again, an expression for the $\star_\mathcal{D}$-product under integration can be obtained from (2.103). However, for the Duflo map the factors ω and η^2 cancel out exactly, and we have $\sigma(\zeta)^{-1} \equiv \omega(\zeta)|\eta(\zeta)|^2 = 1$. Accordingly,

$$\int_{\mathfrak{g}^*} \mathrm{d}^3X\ f(\vec{X}) \star_\mathcal{D} f'(\vec{X}) = \int_{\mathfrak{g}^*} \mathrm{d}^3X\ f(\vec{X}) f'(\vec{X}) , \tag{2.138}$$

i.e., the Duflo star-product of two functions coincides with the pointwise product (only) under integration. In particular, this implies that the Duflo $L^2_\star$ inner product coincides with the usual L^2 inner product, and therefore $L^2_\star(\mathfrak{g}^*) \subseteq L^2(\mathfrak{g}^*)$ (as an L^2 norm-complete vector space) for the Duflo map.

The explicit form of the non-commutative Fourier transform is thus

$$\tilde{\psi}(\vec{X}) = \int_{\mathbb{R}^3, |\vec{k}| \in [0,\pi[} \mathrm{d}^3 k \left(\frac{\sin|\vec{k}|}{|\vec{k}|} \right) e^{i\vec{k}\cdot\vec{X}} \, \psi(\vec{k}) \,, \tag{2.139}$$

while the inverse is

$$\psi(\vec{k}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 X}{(2\pi)^3} \left(\frac{|\vec{k}|}{\sin|\vec{k}|} \right) e^{-i\vec{k}\cdot\vec{X}} \, \tilde{\psi}(\vec{X}) \,. \tag{2.140}$$

Freidel-Livine-Majid map

The Freidel-Livine-Majid ordering map $\mathcal{Q}_{\mathrm{FLM}}$ [26], which has found several applications in the quantum gravity literature (cited in the introduction), can be essentially seen as symmetrization map in conjunction with a change of parametrization for SU(2). In particular, for exponentials of the form $e^{i\vec{p}\cdot\vec{X}}$ it is defined as

$$\mathfrak{Q}_{\mathrm{FLM}}(e^{i\vec{p}\cdot\vec{X}}) := e^{i\frac{\sin^{-1}|\vec{p}|}{|\vec{p}|}\vec{p}\cdot\hat{X}} \,, \tag{2.141}$$

which implies

$$\mathfrak{Q}_{\mathrm{FLM}}(e^{i\frac{\sin|\vec{k}|}{|\vec{k}|}\vec{k}\cdot X}) = e^{i\vec{k}\cdot\hat{X}}, \tag{2.142}$$

that is, with the $\vec{k}$-parametrization, the plane wave is given by $e_\star^{i\vec{k}\cdot\vec{X}} = e^{i\frac{\sin|\vec{k}|}{|\vec{k}|}\vec{k}\cdot\vec{X}}$. Accordingly, we have

$$\mathfrak{D}_{\mathrm{FLM}}(e^{i\vec{k}\cdot\hat{X}}) = e^{i\frac{\sin|\vec{k}|}{|\vec{k}|}\vec{k}\cdot\vec{X}} \,. \tag{2.143}$$

Of course, the transformation $\frac{\sin|\vec{k}|}{|\vec{k}|}\vec{k}$ defines the $\vec{p}$-parametrization as of (2.120), and therefore we may simply write $e_\star^{i\vec{k}\cdot\vec{X}} = e^{i\vec{p}(\vec{k})\cdot\vec{X}}$.[15] However, the coordinates $\vec{p}$ only cover the upper (or lower) hemisphere $\mathrm{SU}(2)/\mathbb{Z}_2 \cong \mathrm{SO}(3)$, and the resulting group Fourier transform is applicable only for functions on SO(3).

Now, since the $\vec{p}$-parametrization is applicable only for the upper hemisphere of SU(2), that is SO(3), instead of restricting the parametrization of the non-commutative plane waves to the principal branch of the logarithm, we restrict to the upper hemisphere, and obtain by an explicit calculation [29]

$$e^{i\vec{p}_1\cdot\vec{X}} \star_{\mathrm{FLM}} e^{i\vec{p}_2\cdot\vec{X}} = e^{i(\vec{p}_1 \oplus \vec{p}_2)\cdot\vec{X}} \,, \tag{2.144}$$

[15] Notice the close resemblance of the $\vec{p}$-coordinates to the non-linear coordinates $2\sin(\theta/2)$ we defined for $U(1)$ above. Indeed, the two coincide for the $U(1)$ subgroups of $SO(3)$.

where

$$\vec{p}_1 \oplus \vec{p}_2 = \epsilon(\vec{p}_1, \vec{p}_2) \left(\sqrt{1 - |\vec{p}_2|^2}\, \vec{p}_1 + \sqrt{1 - |\vec{p}_1|^2}\, \vec{p}_2 - \vec{p}_1 \times \vec{p}_2 \right) . \tag{2.145}$$

The sign factor

$$\epsilon(\vec{k}_1, \vec{k}_2) := \mathrm{sgn}(\sqrt{1 - |\vec{p}_1|^2}\sqrt{1 - |\vec{p}_2|^2} - \vec{p}_1 \cdot \vec{p}_2) , \tag{2.146}$$

introduced by the restriction, is 1 if both $\vec{p}_1, \vec{p}_2$ are close to zero or one of them is infinitesimal, and -1 when the addition of two upper hemisphere vectors ends up in the lower hemisphere (thus projecting the result to its antipode on the upper hemisphere).

The $\star_{\mathrm{FLM}}$-monomials thus read

$$X_i \star_{\mathrm{FLM}} X_j = X_i X_j + i\epsilon_{ij}{}^k X_k , \tag{2.147}$$

$$\begin{aligned} X_i \star_{\mathrm{FLM}} X_j \star_{\mathrm{FLM}} X_k = {} & X_i X_j X_k + i(\epsilon_{ijm} X_k + \epsilon_{ikm} X_j + \epsilon_{jkm} X_i) X_m \\ & + \delta_{jk} X_i - \delta_{ik} X_j + \delta_{ij} X_k , \end{aligned} \tag{2.148}$$

$$\vdots$$

which coincide with $\star_{\mathcal{S}}$ to second order, but no further.

As was already shown in [24, 41], but rederivable from the general expression (2.103) and (2.123), for the Freidel-Livine-Majid star-product we have under integration

$$\int_{\mathfrak{g}^*} \mathrm{d}^3 X \; f(\vec{X}) \star_{\mathrm{FLM}} f'(\vec{X}) = \int_{\mathfrak{g}^*} \mathrm{d}^3 X \; f(\vec{X}) \sqrt{1 + \vec{\partial}_X^2}\, f'(\vec{X}) . \tag{2.149}$$

Now, given the plane waves just computed, we may write the explicit form of the non-commutative Fourier transform as

$$\tilde{\psi}(\vec{X}) = \int_{\mathbb{R}^3, |\vec{p}|^2 < 1} \frac{\mathrm{d}^3 p}{\sqrt{1 - |\vec{p}|^2}} \; e^{i\vec{p}\cdot\vec{X}} \, \psi(\vec{p}) , \tag{2.150}$$

as well as the inverse

$$\psi(\vec{p}) = \sqrt{1 - |\vec{p}|^2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 X}{(2\pi)^3} \; e^{-i\vec{p}\cdot\vec{X}} \, \tilde{\psi}(\vec{X}) . \tag{2.151}$$

Chapter 3

Applications to Physics

3.1 Phase Space Path Integral for Quantum Mechanics on a Lie Group

In this section, using the non-commutative representation for Lie groups reviewed in the previous chapter, we formulate the first order phase space path integral for quantum mechanics on a Lie group $\mathcal{G}$. We show that its classical limit yields the correct classical equations of motion, if the deformation of the phase space structure introduced by the non-commutativity is taken into account in the variational calculus. As a concrete example of the general formalism, we consider a free quantum particle on $SU(2)$, and show the agreement with previous results in the literature. We will follow closely the exposition in [54], except for generalizing, where possible, from the Freidel-Majid-Livine transform for $SO(3)$ considered in [54] by taking the full advantage of the more general formalism introduced above.

3.1.1 Classical mechanics on $\mathcal{G}$

The formulation of classical mechanics of a physical system is based on the canonical symplectic structure of its phase space. (For more details on the differential geometric formulation of classical mechanics, see [46, 48].) For a system with a Lie group configuration space $\mathcal{G}$, the phase space will be the cotangent bundle $\mathcal{T}^*\mathcal{G}$. As we have seen in Subsection 2.1.1, the canonical symplectic structure of $\mathcal{T}^*\mathcal{G}$ is given by the Poisson bracket

$$\{f, f'\} = \frac{\partial f}{\partial P_i}(\mathcal{L}_i f') - (\mathcal{L}_i f)\frac{\partial f'}{\partial P_i} + \lambda c_{ij}{}^k P_k \frac{\partial f}{\partial P_i}\frac{\partial f'}{\partial P_j} \tag{3.1}$$

for $f, f' \in \mathcal{P}(\mathcal{T}^*\mathcal{G})$, where we now denote the cotangent space variables by P_i, since they correspond to the classical momentum variables. Moreover, we have introduced a dimensionful quantity λ that controls the physical scale that is associated to the variable described by the group manifold. In particular, the dimensionful Lie derivatives satisfy

$[\mathcal{L}_i, \mathcal{L}_j] = \lambda {c_{ij}}^k \mathcal{L}_k$, where ${c_{ij}}^k$ are the dimensionless structure constants of $\mathcal{G}$, and so in the limit $\lambda \to 0$ the commutators vanish, and the Poisson algebra becomes that of (the one-point-compactification of) $\mathbb{R}^d$.

The classical Hamiltonian dynamics of a dynamical system, whose configuration space is $\mathcal{G}$, is determined by specifying the *Hamiltonian function*, $H \in \mathcal{P}(\mathcal{T}^*\mathcal{G})$, typically corresponding to the total energy of the system, which gives the time-evolution of the observables $f \in \mathcal{P}(\mathcal{T}^*\mathcal{G})$ via

$$\frac{\mathrm{d}f}{\mathrm{d}t} \equiv \{H, f\} \,. \tag{3.2}$$

In other words, we specify a vector field $\frac{\mathrm{d}}{\mathrm{d}t} \in \mathcal{T}(\mathcal{T}^*\mathcal{G})$ as

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \{H, \cdot\} = \frac{\partial H}{\partial P_i}\frac{\partial}{\partial g^i} - \frac{\partial H}{\partial g^i}\frac{\partial}{\partial P_i} + \lambda {c_{ij}}^k P_k \frac{\partial H}{\partial P_i}\frac{\partial}{\partial X_j} \,, \tag{3.3}$$

which generates the time-evolution of the system. In particular, we obtain the classical Hamiltonian equations of motion for the coordinate functions as

$$\begin{aligned} \frac{\mathrm{d}k^i}{\mathrm{d}t} &\equiv \{H, k^i\} = \frac{\partial H}{\partial P_j}\mathcal{L}_j k^i \,, \\ \frac{\mathrm{d}P_i}{\mathrm{d}t} &\equiv \{H, P_i\} = -\mathcal{L}_i H + \lambda {c_{ij}}^k P_k \frac{\partial H}{\partial P_j} \,. \end{aligned} \tag{3.4}$$

Whereas the Hamiltonian mechanics are formulated in terms of the cotangent bundle $\mathcal{T}^*\mathcal{G}$, the *Lagrangian formulation* of mechanics uses the tangent bundle $\mathcal{T}\mathcal{G}$ instead. However, contrary to the case of cotangent bundle, on the tangent bundle we do not have any canonical symplectic structure, and therefore we must utilize a differentiable function $L : \mathcal{T}\mathcal{G} \to \mathbb{R}$, the *Lagrangian function*, to begin with, which we use to pull back the canonical structure on $\mathcal{T}^*\mathcal{G}$ onto $\mathcal{T}\mathcal{G}$. In particular, we may define a map $\mathbb{F}L : \mathcal{T}\mathcal{G} \to \mathcal{T}^*\mathcal{G}$, s.t. $\mathbb{F}L(X_g) \in \mathcal{T}^*_g\mathcal{G} \;\forall\; X_g \in \mathcal{T}_g\mathcal{G}$, called the *Legendre transform*, via the relation

$$\mathbb{F}L(X_g) \cdot Y_g := \frac{\mathrm{d}}{\mathrm{d}s} L(X_g + sY_g)\bigg|_{s=0} \qquad \forall\, X, Y \in \mathcal{T}\mathcal{G} \,, \tag{3.5}$$

i.e., it gives the directional derivative of L to the direction Y_g in the fiber at X_g. By an explicit calculation we find

$$\mathbb{F}L_i = \frac{\partial L}{\partial \dot{g}^i} \tag{3.6}$$

in a right-invariant basis, where we introduced the notation $(\bar{0}, e_i) =: \frac{\partial}{\partial \dot{g}^i}$ for the basis vectors in the tangent spaces on $\mathcal{T}\mathcal{G}$. Since $\mathbb{F}L$ is fiber-preserving, i.e., $\mathbb{F}L : \mathcal{T}_g\mathcal{G} \to$

$\mathcal{T}_g^*\mathcal{G}\ \forall\ g \in \mathcal{G}$, we may further write

$$\mathbb{F}L(g, X_g) = \left(g, \frac{\partial L}{\partial \dot{g}^i}(g, X_g)e^i\right) \in \mathcal{T}_g^*\mathcal{G}\,. \tag{3.7}$$

The associated *energy function* $E : \mathcal{TG} \to \mathbb{R}$ is defined as

$$E_L(X_g) = \mathbb{F}L(X_g) \cdot X_g - L(X_g)\,. \tag{3.8}$$

The Hamiltonian H corresponding to the Lagrangian L is then obtained as $H = E_L \circ \mathbb{F}L^{-1}$, i.e, as the inverse Legendre transformation of the energy function.[1]

Now, using $\mathbb{F}L$ we may pull back the canonical 2-form ω on $\mathcal{T}^*\mathcal{G}$ onto $\mathcal{TG}$, and define the *Lagrangian 2-form* $\omega_L := \mathbb{F}L^*\omega$. Explicitly, it reads

$$\omega_L \equiv \mathbb{F}L^*\omega = \left(\frac{\partial^2 L}{\partial \dot{g}^i \partial g^j} + c_{ij}{}^k \frac{\partial L}{\partial \dot{g}^k}\right) \mathrm{d}g^i \wedge \mathrm{d}g^j + \frac{\partial^2 L}{\partial \dot{g}^i \partial \dot{g}^j} \mathrm{d}g^i \wedge \mathrm{d}\dot{g}^j\,. \tag{3.9}$$

Now, correspondingly to the Hamiltonian case, the physical trajectories of the system are the integral curves of *Lagrangian vector fields* $Z \equiv (\dot{g}, \ddot{g}) \in \mathcal{T}(\mathcal{TG})$, which satisfy the condition

$$\omega_L(Z, \cdot) = \mathrm{d}E\,. \tag{3.10}$$

By substituting the definition (3.8) of the energy function into the condition (3.10), we may find the Lagrangian equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{g}^i}\right) = \frac{\partial L}{\partial g^i} - 2c_{ij}{}^k \dot{g}^j \frac{\partial L}{\partial \dot{g}^k}\,. \tag{3.11}$$

Thus, we find an extra term $-2c_{ij}{}^k \dot{g}^j \frac{\partial L}{\partial \dot{g}^k}$ arising from the noncommutativity. However, interestingly this term vanishes for Lagrangians, which have the usual quadratic kinetic term and no velocity dependence in the potential term, if the structure constants are totally antisymmetric.

One may show that the above Lagrangian equations of motion are also obtained from the *principle of least action*, which states that the classical trajectories are those which extremise the action, defined as the integral of the Lagrangian over the trajectory. In other words, we require of the action

$$\delta S = \int_{t_0}^{t_1} \mathrm{d}t\ \delta L\big((g(t), \dot{g}(t)\big) = 0 \tag{3.12}$$

under any infinitesimal variation of the path $g(t)$. This is the form of the classical mechanics

[1]Strictly speaking this is only true if $\mathbb{F}L$ is a diffeomorphism. This is true if and only if det $\left(\frac{\partial^2 L}{\partial \dot{g}^i \partial \dot{g}^j}\right) \neq 0$ everywhere on $\mathcal{TG}$ [46]. In such a case the Lagrangian is called *hyperregular*.

that is most immediately connected to the corresponding quantum mechanics in the path integral approach. It is therefore important to make a connection to our subsequent semi-classical analysis of the non-commutative phase space path integral.

3.1.2 Dual bases of generalized states for quantum theory

In Subsection 2.2.1 we defined the quantum algebra $\mathfrak{A}$ corresponding to the cotangent bundle $\mathcal{T}^*\mathcal{G}$, whose commutators read with all the constants in place as

$$[\hat{f}, \hat{f}'] = 0\,,\ [\hat{P}_i, \hat{f}] = i\hbar\widehat{\mathcal{L}_i f}\,,\ [\hat{P}_i, \hat{P}_j] = i\hbar\lambda {c_{ij}}^k \hat{P}_k\,. \tag{3.13}$$

We also formulated two complementary representations of $\mathfrak{A}$ in terms of Hilbert spaces $L^2(\mathcal{G})$ and $L^2_\star(\mathfrak{g}^*)$, which are intertwined by the non-commutative Fourier transform.

Corresponding to the first representation, using Dirac notation, we may abstractly consider the complete set of orthonormal basis states $\{|g\rangle : g \in \mathcal{G}\}$, labelled by group elements, which we will call the group basis.[2] This basis is chosen such that it satisfies

$$\langle g|g'\rangle \equiv \lambda^d \delta(g^{-1}g') \quad , \quad \int_{\mathcal{G}} \frac{\mathrm{d}g}{\lambda^d}\, |g\rangle\langle g| \equiv \hat{\mathbb{1}} \quad , \text{and} \quad \hat{f}|g\rangle \equiv f(g)|g\rangle \tag{3.14}$$

for any function $f \in C^\infty(\mathcal{G})$. Thus, $|g\rangle$ simultaneously diagonalize the operators $\hat{f} \in \mathfrak{A}_{\mathcal{G}} = \mathfrak{Q}(C^\infty(\mathcal{G}))$. As usual, we then define the Hilbert space of states $\mathcal{H}$ to consist of those states $|\psi\rangle$, whose decomposion in the $|g\rangle$ basis can be expressed in the form

$$|\psi\rangle = \int_{\mathcal{G}} \frac{\mathrm{d}g}{\lambda^d}\, \psi(g)|g\rangle\,, \tag{3.15}$$

where $\psi \in L^2(\mathcal{G}, \mathrm{d}g/\lambda^d)$, and $\langle\psi|\psi\rangle = \int_{\mathcal{G}} \frac{\mathrm{d}g}{\lambda^d}|\psi|^2 = 1$.

Through the non-commutative Fourier transform formulated in the previous chapter, we may also consider the dual momentum space representation of quantum mechanics on $\mathcal{G}$ in terms of the non-commutative representation of $\mathfrak{A}$ on $L^2_\star(\mathfrak{g}^*)$. (See, e.g., [42, 21] for earlier treatments of quantum mechanics on Lie algebraic non-commutative spaces.) Let us define a set of states $\{|P\rangle \mid P \in \mathfrak{g}^*\}$ via their inner product with the group basis

$$\langle g|P\rangle \equiv E_g(P)\,, \tag{3.16}$$

which satisfy, due to the properties of the non-commutative Fourier transform, the following identities

$$\langle P|P'\rangle = (2\pi\hbar)^d \delta_\star(P, P') \quad \text{and} \quad \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d P}{(2\pi\hbar)^d}\, |P\rangle \star \langle P| = \hat{\mathbb{1}}\,. \tag{3.17}$$

[2]Dirac notation can be given a fully rigorous definition in the framework of rigged Hilbert spaces [12]. Here, we apply this convenient notation in the usual physics fashion without striving for absolute rigor.

Accordingly, they form a basis with respect to the $\star$-product structure in the non-commutative momentum space, and any state may be expressed as

$$|\psi\rangle = \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d P}{(2\pi\hbar)^d} \, |P\rangle \star \tilde{\psi}(P) \,, \tag{3.18}$$

where $\tilde{\psi} := \mathcal{F}_\star(\psi)$ for $\psi \in L^2(\mathcal{G})$.

3.1.3 Non-commutative phase space path integral

Next we will give the first order path integral formulation of quantum mechanics on $\mathcal{G}$ using the non-commutative momentum space defined above, and in particular the momentum basis $\{|P\rangle \mid P \in \mathfrak{g}^*\}$. The derivation follows similar lines to the commutative Euclidean case, but some extra subtleties arise due to the non-commutative structure.

The quantum mechanical evolution operator is given by

$$\hat{U}(t'-t) \equiv e^{-\frac{i}{\hbar}(t'-t)\hat{H}} \,, \tag{3.19}$$

where $\hat{H}$ is the Hamiltonian operator, as usual. Accordingly, we have for the propagation amplitude from the group element g at time t to g' at time t'

$$\langle g', t' | g, t\rangle \equiv \langle g' | \hat{U}(t'-t) | g\rangle \,. \tag{3.20}$$

Now, we introduce the time-slicing via the decomposition

$$\hat{U}(t'-t) \equiv \prod_{k=0}^{N-1} \hat{U}(t_{k+1}-t_k) \,, \tag{3.21}$$

where $t_{k+1} > t_k \; \forall k$ and $t_0 = t, t_N = t'$. We set $t_{k+1} - t_k \equiv \epsilon \; \forall \; k = 0, \ldots, N-1$, so we have $t' - t \equiv N\epsilon$. By inserting the resolution of identity

$$\hat{\mathbb{1}} = \int_{\mathcal{G}} \frac{\mathrm{d}g}{\lambda^d} \, |g\rangle\langle g| \tag{3.22}$$

$N-1$ times in between the evolution operators we obtain

$$\langle g', t' | g, t\rangle = \lim_{N\to\infty} \left[\prod_{k=1}^{N-1} \int_{\mathcal{G}} \frac{\mathrm{d}g_k}{\lambda^d} \right] \left[\prod_{k=0}^{N-1} \langle g_{k+1} | \hat{U}(\epsilon) | g_k\rangle \right] \,. \tag{3.23}$$

Furthermore, for each of the factors we use the resolution of identity

$$\hat{\mathbb{1}} = \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d P}{(2\pi\hbar)^d} \, |P\rangle \star \langle P| \tag{3.24}$$

to express them as

$$\langle g_{k+1}|\hat{U}(\epsilon)|g_k\rangle = \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d P_k}{(2\pi\hbar)^d} \langle g_{k+1}|P_k\rangle \star \langle P_k|\hat{U}(\epsilon)|g_k\rangle\,. \tag{3.25}$$

At this point, we restrict to systems with Hamiltonian operators of the form $\hat{H} = \hat{H}_{\mathfrak{g}^*} + \hat{H}_{\mathcal{G}}$, where $\hat{H}_{\mathfrak{g}^*} \in \mathfrak{A}_{\mathfrak{g}^*}$ and $\hat{H}_{\mathcal{G}} \in \mathfrak{A}_{\mathcal{G}}$, to avoid additional operator ordering issues, on top of those stemming from the non-commutativity of momentum variables and encoded by the $\star$-product structure defined above.[3] Then we get $\langle P|\hat{H}|g\rangle = H_\star(P,g) \star \langle P|g\rangle$, where the function $H_\star(P,g)$ is now obtained from the Hamiltonian operator $\hat{H}$ by replacing the momentum operators $\hat{P}_i$ in $\hat{H}$ by the non-commutative momentum variables P_i and the operator product of the momentum operators is replaced by the $\star$-product, whereas the operators $\hat{f}$ are replaced by the corresponding functions $f \in C^\infty(\mathcal{G})$. We may take the linear approximation in ϵ as

$$\langle P|e^{-\frac{i}{\hbar}\epsilon\hat{H}}|g\rangle \approx \big(1 - \frac{i}{\hbar}\epsilon H_\star(P,g)\big) \star \langle P|g\rangle \approx e_\star^{-\frac{i}{\hbar}\epsilon H_\star(P,g)} \star \langle P|g\rangle\,, \tag{3.26}$$

since the linear order in ϵ for the time-slice propagators is sufficient in order to obtain the correct finite time propagator satisfying the Schrödinger equation [15]. Accordingly, we obtain

$$\begin{aligned}\langle g_{k+1}|\hat{U}(\epsilon)|g_k\rangle &\approx \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d P_k}{(2\pi\hbar)^d} E_{g_{k+1}}(P_k) \star e_\star^{-\frac{i}{\hbar}\epsilon H_\star(P_k,g_k)} \star E_{g_k^{-1}}(P_k) \\ &= \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d P_k}{(2\pi\hbar)^d} E_{g_{k+1}g_k^{-1}}(P_k) \star e_\star^{-\frac{i}{\hbar}\epsilon H_\star(\mathrm{Ad}^*_{g_k} P_k, g_k)}\,,\end{aligned} \tag{3.27}$$

where we used the properties $\overline{E_g(P)} = E_{g^{-1}}(P)$,

$$E_g(P) \star E_h(P) = E_{gh}(P) \quad \text{and} \quad E_g(P) \star f(P) = f(\mathrm{Ad}_g P) \star E_g(P) \tag{3.28}$$

of the $\star$-product. (The last equality holds for unimodular $\mathcal{G}$.) Furthermore, we have $E_{g_{k+1}g_k^{-1}}(\mathrm{Ad}^*_{g_k^{-1}} P_k) = E_{g_k^{-1}g_{k+1}}(P_k)$, so by making the change of variables $P_k \mapsto \mathrm{Ad}^*_{g_k^{-1}} P_k$

[3] To handle more general Hamiltonians with mixed terms in g and P variables, one should introduce an additional $\star$-product, encoding in the definition of the path integral the operator ordering between group and momentum operators [13]. This would then lead to more complicated forms for the discrete and continuum phase space path integrals. However, for the arguments we wish to present here, the generalization is not important, as we focus on how the $\star$-product between momentum variables encodes their non-commutativity in the same path integral representation of the dynamics, so we restrict to the simpler case.

the full propagator reads in the first order form

$$\langle g', t'|g, t\rangle = \lim_{N\to\infty} \left[\prod_{k=1}^{N-1} \int_{\mathcal{G}} \frac{\mathrm{d}g_k}{\lambda^d}\right] \left[\prod_{k=0}^{N-1} \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d P_k}{(2\pi\hbar)^d}\right] \left[\prod_{k=0}^{N-1} E_{g_k^{-1}g_{k+1}}(P_k) \star e_\star^{-\frac{i}{\hbar}\epsilon H_\star(P_k,g_k)}\right] . \tag{3.29}$$

We observe that each of the factors in the product over P_k's is exactly the group Fourier transform of the function $\exp_\star[-\frac{i}{\hbar}\epsilon H_\star(P_k, g_k)]$ from the momentum variable P_k to the group variable $g_k^{-1}g_{k+1}$. It is not difficult to verify that the time-slice propagator

$$\langle g_{k+1}|\hat{U}(\epsilon)|g_k\rangle = \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d P_k}{(2\pi\hbar)^d} E_{g_k^{-1}g_{k+1}}(P_k) \star e_\star^{-\frac{i}{\hbar}\epsilon H_\star(P_k,g_k)} \tag{3.30}$$

satisfies the Schrödinger equation exactly. However, we would like to express this as an integral over a single exponential. Let us assume that the non-commutative plane wave is of the form $E_g(P) = \eta(g)e^{i\zeta(g)\cdot P/\hbar}$, where η is a class function on $\mathcal{G}$. Using the expression (2.103) for the $\star$-product under integration, and taking again the linear approximation in ϵ, we obtain

$$\langle g_{k+1}|\hat{U}(\epsilon)|g_k\rangle = \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d P_k}{(2\pi\hbar)^d}\, \eta(g_k^{-1}g_{k+1}) \exp\left\{\frac{i}{\hbar}\epsilon \left[\frac{\zeta(g_k^{-1}g_{k+1})}{\epsilon} \cdot P_k - H_q(P_k, g_k)\right]\right\} , \tag{3.31}$$

where

$$H_q(P, g) := \sigma(i\vec{\partial}_P)H_\star(P, g) \tag{3.32}$$

is an effective Hamiltonian containing additional terms, which arise from the non-trivial phase space structure and ensure that the time-slice propagator satisfies the Schrödinger equation up to first order in ϵ. Accordingly, we may write

$$\langle g', t'|g, t\rangle = \lim_{N\to\infty} \left[\prod_{k=1}^{N-1} \int_{\mathcal{G}} \frac{\mathrm{d}g_k}{\lambda^d}\right] \left[\prod_{k=0}^{N-1} \int_{\mathfrak{g}^*} \frac{\mathrm{d}^d P_k}{(2\pi\hbar)^d}\right] \left[\prod_{k=0}^{N-1} \eta(g_k^{-1}g_{k+1})\right] \times \exp\left\{\frac{i}{\hbar}\sum_{k=0}^{N-1} \epsilon \left[\frac{\zeta(g_k^{-1}g_{k+1})}{\epsilon} \cdot P_k - H_q(P_k, g_k)\right]\right\} , \tag{3.33}$$

This is clearly analogous to the first order form of path integral in the usual Euclidean case. The second order path integral can, in principle, be obtained from this first order form by integrating out the momentum or group variables, but since these integrations can only be performed explicitly for certain special cases (quadratic Hamiltonians, in particular), for greater generality we will stay at the first order level. (See Subsection 3.1.5 for the case of a free particle on $SU(2)$.)

We may write (3.33) in the continuum limit as

$$\langle g', t'|g, t\rangle = \int_{\substack{g(t)=g \\ g(t')=g'}} \mathcal{D}g\, \mathcal{D}P\ \exp\left\{\frac{i}{\hbar}\int_t^{t'} \mathrm{d}s\ \Big[\dot{g}(s)\cdot P(s) - H_q(P(s), g(s))\Big]\right\}, \qquad (3.34)$$

where $\dot{g}(t) := -\frac{i}{\lambda} L_{g^{-1}(t)*}\frac{\mathrm{d}g}{\mathrm{d}t}(t) \in \mathfrak{g}$, since we have

$$\lim_{\epsilon\to 0}\frac{\zeta(g_k^{-1}g_{k+1})}{\epsilon} = -\frac{i}{\lambda}\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} g^{-1}(t_k)g(t_k+\epsilon) \equiv \dot{g}(t_k)\,. \qquad (3.35)$$

Moreover, since η is a class function, $\eta(g_k^{-1}g_{k+1}) = 1 + \mathcal{O}(\epsilon^2)$, so we may approximate $\eta(g_k^{-1}g_{k+1}) \approx 1$ for all k. In the expression (3.34) of the propagator, the action

$$\mathcal{S}_b[g, P] = \int_t^{t'} \mathrm{d}s\ \Big[\dot{g}(s)\cdot P(s) - H_q(P(s), g(s))\Big] \qquad (3.36)$$

appearing in the exponent is the classical action, i.e., the time integral over the classical Lagrangean function obtained through Legendre transformation, except that the classical Hamiltonian is replaced with the effective Hamiltonian H_q. The Hamiltonian H_q can be interpreted as introducing quantum corrections into the action, as it contains, in addition to the classical Hamiltonian function H in the zeroth order, higher order terms in $\hbar$. The presence of such quantum corrections to the classical action in the path integral formulation of the dynamics is necessary in order for the propagator to satisfy the Schrödinger equation, and a generic feature of path integrals on curved manifolds [15, 13]. Also, note that at the Euclidean (no curvature in configuration space, commutative in momentum space) limit $\lambda \to 0$ we have $H_q \to H$ and, as should be expected, in this limit the path integral (3.34) coincides with the path integral for a point particle in Euclidean space. The expression (3.34) can be taken to confirm the usefulness and interpretation of the non-commutative momentum basis.

The propagator $\langle P', t'|P, t\rangle$ in the non-commutative momentum basis is obtained by applying the group Fourier transform to both sides of the propagator $\langle g', t'|g, t\rangle$ in the group basis. This results in adding a boundary term into the action:

$$\begin{aligned}\langle P', t'|P, t\rangle &= \int_{\mathcal{G}}\frac{\mathrm{d}g'}{\lambda^d}\int_{\mathcal{G}}\frac{\mathrm{d}g}{\lambda^d}\ \langle P'|g'\rangle\langle g', t'|g, t\rangle\langle g|P\rangle \\ &= \int_{\mathcal{G}}\frac{\mathrm{d}g'}{\lambda^d}\int_{\mathcal{G}}\frac{\mathrm{d}g}{\lambda^d}\ e^{-i\zeta(g')\cdot P'/\hbar + i\zeta(g)\cdot P/\hbar} \\ &\quad\times \int_{\substack{g(t)=g \\ g(t')=g'}} \mathcal{D}g\, \mathcal{D}P\ \exp\left\{\frac{i}{\hbar}\int_t^{t'} \mathrm{d}s\ \Big(\dot{g}(s)\cdot P(s) - H_q(P(s), g(s))\Big)\right\}\end{aligned}$$

$$
\begin{aligned}
&= \int_{\substack{P(t)=P \\ P(t')=P'}} \mathcal{D}g \, \mathcal{D}P \, \exp\left\{ \frac{i}{\hbar} \int_t^{t'} \mathrm{d}s \, \Big(\dot{g}(s) \cdot P(s) - H_q(P(s), g(s)) \Big) \right. \\
&\qquad \left. - \frac{i}{\hbar} \Big(\zeta(g(t')) \cdot P(t') - \zeta(g(t)) \cdot P(t) \Big) \right\} \\
&\equiv \int_{\substack{P(t)=P \\ P(t')=P'}} \mathcal{D}g \, \mathcal{D}P \, e^{\frac{i}{\hbar}\mathcal{S}[g,P]} \,,
\end{aligned} \tag{3.37}
$$

where the action $\mathcal{S}[g,P]$ consists of bulk and boundary terms $\mathcal{S}[g,P] \equiv \mathcal{S}_b[g,P] + \mathcal{S}_{\partial b}[g,P]$, respectively,

$$
\begin{aligned}
\mathcal{S}_b[g,P] &= \int_t^{t'} \mathrm{d}s \, \Big(\dot{g}(s) \cdot P(s) - H_q(P(s), g(s)) \Big) \,, \\
\mathcal{S}_{\partial b}[g,P] &= -\zeta(g(t')) \cdot P(t') + \zeta(g(t)) \cdot P(t) \,.
\end{aligned} \tag{3.38}
$$

This boundary term is crucial for obtaining the correct semi-classical limit in our case, as we will observe in Subsection 3.1.4.

Let us summarize the results of this section. We have shown that one can derive a first order path integral for a quantum mechanical system with a (unimodular and exponential) Lie group configuration space $\mathcal{G}$ in terms of the non-commutative momentum space variables. For the propagator in the group basis we obtained the continuum limit expression (3.34),

$$
\langle g', t' | g, t \rangle = \int_{\substack{g(t)=g \\ g(t')=g'}} \mathcal{D}g \, \mathcal{D}P \, \exp\left\{ \frac{i}{\hbar} \int_t^{t'} \mathrm{d}s \, \Big(\dot{g}(s) \cdot P(s) - H_q(P(s), g(s)) \Big) \right\} \,, \tag{3.39}
$$

where the measure reads

$$
\mathcal{D}g \, \mathcal{D}P \; \equiv \lim_{N \to \infty} \left[\prod_{k=1}^{N-1} \frac{\mathrm{d}g_k}{\lambda^d} \right] \left[\prod_{k=0}^{N-1} \frac{\mathrm{d}^3 P_k}{(2\pi\hbar)^d} \right] \,. \tag{3.40}
$$

The $\star$-product structure gives naturally rise to quantum corrections into the action

$$
\mathcal{S}_b[g,P] = \int_t^{t'} \mathrm{d}s \, \Big(\dot{g}(s) \cdot P(s) - H_q(P(s), g(s)) \Big) \tag{3.41}
$$

via the form of the quantum corrected Hamiltonian $H_q(P,g) \equiv \sigma(i\vec{\partial}_P) H_\star(P,g)$. Crucially, these corrections ensure that the propagator obtained via path integral satisfies the Schrödinger equation. In the momentum basis we found that the action receives an additional boundary term

$$
\mathcal{S}_{\partial b}[g,P] = -\zeta(g(t')) \cdot P(t') + \zeta(g(t)) \cdot P(t) \,, \tag{3.42}
$$

and thus the path integral acquires the form (3.37),

$$\langle P',t'|P,t\rangle = \int_{\substack{P(t)=P\\ P(t')=P'}} \mathcal{D}g\,\mathcal{D}P\ \exp\Bigg\{\frac{i}{\hbar}\int_t^{t'}\mathrm{d}s\ \Big(\dot{g}(s)\cdot P(s) - H_q(P(s),g(s))\Big) - \frac{i}{\hbar}\Big(\zeta(g(t'))\cdot P(t') - \zeta(g(t))\cdot P(t)\Big)\Bigg\}. \tag{3.43}$$

The second order path integral, either in terms of the group or the momentum variables, can be obtained from the first order formalism by integrating out the momentum or the group variables, respectively.

3.1.4 Classical limit

We are now interested in the classical limit of the transition amplitudes we have derived above in general form. In [54] we performed the semi-classical analysis for $SO(3)$. Here we repeat the analysis, but in the more general case of a unimodular exponential Lie group.

Let us, first of all, study the variations to the action (3.38). We choose an arbitrary path $(\bar{g}(s),\bar{P}(s))$ in the phase space, and introduce a small variation of the path as $(\bar{g}(s)e^{i\eta Z(s)},\bar{P}(s)+\xi Q(s))$, where $Z(s),Q(s)\in\mathfrak{g}$ for $s\in[t,t']$, and we assume that the momentum variation vanishes at the boundary: $Q(t)=Q(t')=0$. We find for the first order variation of the tangent vector $\dot{g} = -\frac{i}{\lambda}L_{g^{-1}*}\frac{\mathrm{d}g}{\mathrm{d}s}$ the form

$$\delta\dot{g}(s) = \eta\left(\frac{\mathrm{d}Z}{\mathrm{d}s}(s) + i[\dot{\bar{g}},Z](s)\right) + \mathcal{O}(\eta^2)\in\mathfrak{g}\,. \tag{3.44}$$

Now, even though we seem to be dealing with a classical action in calculating the variations, we find that we cannot forget its quantum origin in the case of non-commutative phase space variables. What we are really dealing with are the underlying non-commutative quantum amplitudes in which the action appears. We find that, in order to obtain the correct classical equations of motion, we must take into account this non-commutative structure in calculating the variations by defining the variation of the action through the variation of the amplitude as

$$\begin{aligned}&\exp\Big\{\frac{i}{\hbar}\Big(\eta\frac{\delta\mathcal{S}}{\delta g}[\bar{g},\bar{P}]\delta g + \xi\frac{\delta\mathcal{S}}{\delta P}[\bar{g},\bar{P}]\delta P\Big) + \mathcal{O}(\eta^2,\xi^2,\eta\xi)\Big\}\\ &:= \exp\Big\{-\frac{i}{\hbar}S[\bar{g},\bar{P}]\Big\}\star\exp\Big\{\frac{i}{\hbar}S[\bar{g}e^{i\eta Z},\bar{P}+\xi Q]\Big\}\,,\end{aligned} \tag{3.45}$$

where the $\star$-product applies for momentum variables in the same time-slice. Although the mathematical reasons for considering such non-commutative variations are not fully clear at the moment, we may motivate this definition by noting that the path integral acts as an integral kernel with respect to the $\star$-product in calculating the transition amplitudes. We will comment further on the non-commutative variational calculus in Section 3.2.

In any case, due to the linear approximation in ϵ for the single time-slice actions in the bulk, we can happily neglect the deformation in the calculation of the variation of the bulk part of the action. Substituting the variations into the bulk action, we find for the first order variation in η and ξ

$$\begin{aligned}
\delta\mathcal{S}_b[\bar{g},\bar{P}] = \int_t^{t'} \mathrm{d}s \, \Big\{ & \eta\Big(\frac{\mathrm{d}Z}{\mathrm{d}s}(s) + i[\dot{\bar{g}},Z](s)\Big)\cdot\bar{P}(s) + \xi\big(\dot{\bar{g}}(s)\big)\cdot Q(s) \\
& - \eta Z^i(s)\mathcal{L}_i H_q(\bar{P}(s),\bar{g}(s)) - \xi Q_i(s)\frac{\partial H_q}{\partial P_i}(\bar{P}(s),\bar{g}(s)) \Big\} \\
= \eta Z^i(s)\bar{P}_i(s)\Big|_{s=t}^{s=t'} & \\
+ \int_t^{t'} \mathrm{d}s \, \Big\{ & \eta Z^i(s)\Big[-\frac{\mathrm{d}\bar{P}_i}{\mathrm{d}s}(s) + \lambda c_{ij}{}^k \dot{\bar{g}}^j(s)\bar{P}_k - \mathcal{L}_i H_q(\bar{P}(s),\bar{g}(s))\Big] \\
& + \xi Q_i(s)\Big[\dot{\bar{g}}^i(s) - \frac{\partial H_q}{\partial P_i}(\bar{P}(s),\bar{g}(s))\Big]\Big\}. \qquad (3.46)
\end{aligned}$$

Let us, at first, neglect the first term in (3.46) associated with the boundary. By requiring the variation given by the integral to vanish for arbitrary perturbations $Z^i(s), Q^i(s)$, we obtain the equations

$$\begin{aligned}
\dot{\bar{g}}^i(s) &= \frac{\partial H_q}{\partial P_i}(\bar{P}(s),\bar{g}(s)) \\
\frac{\mathrm{d}\bar{P}_i}{\mathrm{d}s}(s) &= \lambda c_{ij}{}^k \dot{\bar{g}}^j(s)\bar{P}_k - \mathcal{L}_i H_q(\bar{P}(s),\bar{g}(s))\,. \qquad (3.47)
\end{aligned}$$

Substituting the first equation into the second, and noting that $\dot{g}^i \equiv \frac{\mathrm{d}k_g^i}{\mathrm{d}t}\big|_g$ for $k_h(g) = -i\ln(h^{-1}g) \in \mathfrak{g}$, we arrive at the equations

$$\begin{aligned}
\frac{\mathrm{d}k^i}{\mathrm{d}t} &= \frac{\partial H_q}{\partial P_j}\mathcal{L}_j k^i\,, \\
\frac{\mathrm{d}P_i}{\mathrm{d}t} &= -\mathcal{L}_i H + \lambda c_{ij}{}^k P_k \frac{\partial H_q}{\partial P_j}\,. \qquad (3.48)
\end{aligned}$$

In the semi-classical limit $\hbar \to 0$ the dominating contribution to the path integral arises then from the paths satisfying the equations (3.48). Given that in this limit $H_q \to H$, the equations coincide with the classical equations of motion (3.4) we obtained from the canonical analysis in Subsection 3.1.1.

We still need to show that the boundary term in the first order variation of the action (3.46) is cancelled by the variation of the boundary term. Now, for the boundaries no approximation is available, such as the one in ϵ for the bulk, and therefore the deformation structure must be taken into account. On the other hand, we observe that it is exactly the non-commutative variation (3.45), which enables us to cancel the boundary term and arrive at the right classical equations of motion: From (3.45), we obtain for the first order variation in η of the boundary action

$$\delta\mathcal{S}_{\partial b}[\bar{g},\bar{P}] = -\zeta(e^{i\eta Z(s)})\cdot\bar{P}(s)\Big|_{s=t}^{s=t'} \approx -\eta Z(s)\cdot\bar{P}(s)\Big|_{s=t}^{s=t'}\,, \tag{3.49}$$

which exactly cancels the boundary term arising from the bulk action (3.46). This further confirms the correctness of the non-commutative Fourier transform in encoding the relation between g and P variables, needed to produce the boundary term in the action.

Accordingly, we obtain the correct semi-classical behavior from the path integral (3.37), but only by taking into account the non-commutative structure of the phase space. In particular, this means that in the semi-classical limit $\hbar \to 0$ we may approximate the full path integral by a sum over the amplitudes of solutions to the classical equations of motion,

$$\langle P',t'|P,t\rangle \approx \sum_{(g_{cl},P_{cl})} e^{\frac{i}{\hbar}\mathcal{S}[g_{cl},P_{cl}]}\,, \tag{3.50}$$

such that $P_{cl}(t) = P$, $P_{cl}(t') = P'$.

3.1.5 Free particle on $SU(2)$

In order to show explicitly the compatibility of our analysis and results with those obtained by more conventional methods, in particular, harmonic analysis on the group manifold, we first compared our path integral expression for the finite time propagator with the standard expression in [54], in the special case of a free particle on $SO(3)$. Here we consider its double-cover $SU(2)$. In this case the Hamiltonian operator reads $\hat{H} = \hat{P}^2/2m$, which is a multiple of the Casimir operator. The free particle is an important test case for the formalism we developed, since it is well-known from previous literature [15, 65, 19, 13].

We will calculate the second order form of the path integral for the free particle in terms of the group (configuration) variables. For the Hamiltonian $\hat{H} = \hat{P}^2/2m$ the corresponding quantum corrected Hamiltonian is found to be

$$H_q(P) = (P^2 + \hbar^2\lambda^2 c_q)/2m\,, \tag{3.51}$$

where we denote by $c_q \in \mathbb{R}$ a constant, which depends on the choice of the quantization map. By a direct calculation one finds for the symmetric quantization map $c_q = -2$, for the Duflo map $c_q = -1$, and for the FLM map $c_q = 3$. Then, each of the N integrals over

P_k in (3.29)

$$\langle g', t'|g, t\rangle = \lim_{N\to\infty}\left[\prod_{k=1}^{N-1}\int_{SU(2)}\frac{\mathrm{d}g_k}{\lambda^3}\right]\left[\prod_{k=0}^{N-1}\int_{\mathfrak{su}(2)}\frac{\mathrm{d}^3P_k}{(2\pi\hbar)^3}\right]$$
$$\times\exp\left\{\frac{i}{\hbar}\sum_{k=0}^{N-1}\epsilon\left[\frac{\zeta(g_k^{-1}g_{k+1})}{\epsilon}\cdot P_k - \frac{1}{2m}\left(P_k^2+\hbar^2\lambda^2 c_q\right)\right]\right\} \qquad (3.52)$$

becomes the usual Fourier transform of the function $\exp\left[-\frac{i\epsilon}{2m\hbar}\left(P_k^2+\hbar^2\lambda^2 c_q\right)\right]$ to the coordinate variables $\zeta(g_k^{-1}g_{k+1})$. Thus, by performing the Gaussian integrals we obtain

$$\langle g', t'|g, t\rangle = \lim_{N\to\infty}\left[\prod_{k=1}^{N-1}\int_{SU(2)}\frac{\mathrm{d}g_k}{\lambda^3}\right]$$
$$\times\left[\prod_{k=1}^{N}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{\frac{3}{2}}\exp\left\{\frac{i\epsilon}{\hbar}\frac{m}{2}\left(\frac{\zeta(g_k^{-1}g_{k+1})}{\epsilon}\right)^2\right\}\right]e^{-\frac{i(t'-t)\hbar\lambda^2 c_q}{2m}}$$
$$= \lim_{N\to\infty}\left[\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{\frac{3}{2}}\prod_{k=1}^{N-1}\int_{SU(2)}\frac{\mathrm{d}g_k}{(2\pi i\hbar\lambda^2\epsilon/m)^{\frac{3}{2}}}\right]$$
$$\times\exp\left\{\frac{i}{\hbar}\sum_{k=1}^{N}\epsilon\frac{m}{2}\left(\frac{\zeta(g_k^{-1}g_{k+1})}{\epsilon}\right)^2\right\}e^{-\frac{i(t'-t)\hbar\lambda^2 c_q}{2m}}. \qquad (3.53)$$

The product of integrals including the factors of $\left(\frac{m}{2\pi i\hbar\lambda^2\epsilon}\right)^{\frac{3}{2}}$ becomes the second order path integral measure in the continuum limit, as in the usual case of $\mathbb{R}^3$, and the function in the exponent becomes $i/\hbar$ times the classical action [65, 19], since by defining $\bar{V}_\epsilon(k\epsilon)$ by $g_{k+1}\equiv\exp[i\epsilon\bar{V}_\epsilon(k\epsilon)\cdot\bar{\sigma}]\,g_k$, where $|\epsilon\bar{V}_\epsilon(k\epsilon)|<\pi$, we have

$$\frac{m}{2}\left(\frac{\zeta(g_k^{-1}g_{k+1})}{\epsilon}\right)^2 = \frac{m}{2}\frac{\bar{V}_\epsilon^2(k\epsilon)}{\lambda^2}\overset{N\to\infty}{\to} -\frac{m}{2\lambda^2}\mathrm{tr}_{\frac{1}{2}}\left(g^{-1}(t)\frac{\mathrm{d}g}{\mathrm{d}t}(t)g^{-1}(t)\frac{\mathrm{d}g}{\mathrm{d}t}(t)\right), \qquad (3.54)$$

which is the classical Lagrangean of a free point particle on $SU(2)$, where $\mathrm{tr}_{\frac{1}{2}}$ is the normalized trace in the fundamental spin-$\frac{1}{2}$ representation. Finally, we can write for the continuum path integral

$$\langle g', t'|g, t\rangle = \int_{\substack{g(t)\equiv g\\ g(t')\equiv g'}}\mathcal{D}g(t)\ \exp\left[\frac{i}{\hbar}\int_t^{t'}\mathrm{d}t\ \frac{m}{2\lambda^2}\mathrm{tr}_{\frac{1}{2}}\left(\dot{g}^2(t)\right) - \frac{i(t'-t)\hbar\lambda^2 c_q}{2m}\right], \qquad (3.55)$$

where $\mathcal{D}g(t)$ is the continuum limit of the path integral measure given above, and $\dot{g} := -ig^{-1}\frac{\mathrm{d}g}{\mathrm{d}t}\in\mathfrak{su}(2)$ is the velocity of the particle. This agrees up to the choice of the quantization map dependent constant c_q with the path integral for free particle on $SU(2)$ obtained by other methods in [45, 13].

3.2 Non-commutative Geometry of Ponzano-Regge Model

In this section we consider the application of the non-commutative representation to the Ponzano-Regge spin foam model for 3-dimensional quantum gravity. We will first introduce the Ponzano-Regge model, seen as a discretization of the continuum 3d BF theory. We then apply the non-commutative Fourier transform to the Ponzano-Regge model to obtain a representation of the model in terms of non-commutative metric variables, and write down an explicit expression for the quantum amplitude for fixed metric boundary data in the case of trivial topology. We further study the classical limit of the Ponzano-Regge amplitudes for fixed metric boundary data, and find again that the correct classical geometric constraints are obtained, in general, only by considering non-commutative variations of the action. We also compute the classical constraints for the usual commutative variations, and find that the results differ for different choices of non-commutative structures. Finally, we offer some comments on the obtained results.[4]

3.2.1 3d BF theory and Ponzano-Regge model

The Ponzano-Regge model can be understood as a discretization of 3-dimensional Riemannian BF theory. In this section, we will briefly review how it can be derived from the continuum BF theory, while keeping track of the dimensionful physical constants, which determine the various asymptotic limits of the theory.

Let $\mathcal{M}$ be a 3-dimensional base manifold to a frame bundle with the structure group $SU(2)$. Then the partition function of 3d BF theory on $\mathcal{M}$ is given by

$$\mathcal{Z}_{BF}^{\mathcal{M}} = \int \mathcal{D}E\ \mathcal{D}\omega\ \exp\left(\frac{i}{2\hbar\kappa}\int_{\mathcal{M}} \mathrm{tr}\big(E \wedge F(\omega)\big)\right), \tag{3.56}$$

where E is an $\mathfrak{su}(2)^*$-valued triad 1-form on $\mathcal{M}$, $F(\omega)$ is the $\mathfrak{su}(2)$-valued curvature 2-form associated to the connection 1-form ω, and the trace is taken in the fundamental spin-$\frac{1}{2}$ representation of $SU(2)$. The wedge $\wedge$ denotes the contraction of tensor indices with the Levi-Civita tensor ϵ^{ijk}. $\hbar$ is the reduced Planck constant and κ is a constant with dimensions of inverse momentum. The connection with Riemannian gravity in three spacetime dimensions gives $\kappa := 8\pi G$, where G is the gravitational constant [60]. Since the triad 1-form E has dimensions of length and the curvature 2-form F is dimensionless, the exponential is rendered dimensionless by dividing with $\hbar\kappa \equiv 8\pi l_p$, $l_p \equiv \hbar G$ being the Planck length in three dimensions. Integrating over the triad field in (3.56), we get heuristically

$$\mathcal{Z}_{BF}^{\mathcal{M}} \propto \int \mathcal{D}\omega\ \delta\big(F(\omega)\big), \tag{3.57}$$

so we see that the BF partition function is nothing but the volume of the moduli space of

[4]The findings exhibited in this section will be the subject of the forthcoming publication [50].

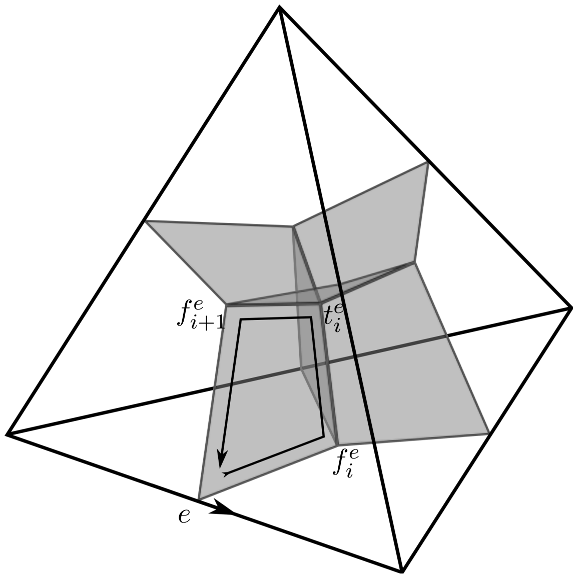

Figure 3.1: The subdivision of tetrahedra in Δ into a finer cellular complex Γ.

flat connections on $\mathcal{M}$. Generically, this is of course divergent, which (among other things) motivates us to consider discretizations of the theory. However, since BF theory is purely topological, that is, it does not depend on the metric structure of the base manifold, such a discretization should not affect its essential properties.

Now, to discretize the continuum BF theory, we first choose a triangulation Δ of the manifold $\mathcal{M}$, that is, a (homogeneous) simplicial complex homotopic to $\mathcal{M}$. The dual complex Δ^* of Δ is obtained by replacing each d-simplex in Δ by a $(3-d)$-simplex and retaining the connective relations between simplices. Then, the homotopy between Δ and $\mathcal{M}$ allows us to think of Δ, and thus Δ^*, as embedded in $\mathcal{M}$. We further form a finer cellular complex Γ by diving the tetrahedra in Δ along the faces of Δ^*. In particular, Γ then consists of tetrahedra $t \in \Delta$, with vertices $t^* \in \Delta^*$ at their centers, each subdivided into four cubic cells. Moreover, for each tetrahedron $t \in \Delta$, there are edges $tf \in \Gamma$, which correspond to half-edges of $f^* \in \Delta^*$, going from the centers of the triangles $f \in \Delta$ bounding the tetrahedron to the center of the tetrahedron t. Also, for each triangle $f \in \Delta$, there are edges $ef \in \Gamma$, which go from the center of the triangle $f \in \Delta$ to the centers of the edges $e \in \Delta$ bounding the triangle f. See Fig. 3.1 for an illustration of the subdivision of a single tetrahedron in Δ.

To obtain the discretized connection variables associated to the triangulation Δ, we integrate the connection along the edges $tf \in \Gamma$ and $ef \in \Gamma$ as

$$g_{tf} := \mathcal{P}e^{i\int_{tf}\omega} \in SU(2) \quad \text{and} \quad g_{ef} := \mathcal{P}e^{i\int_{ef}\omega} \in SU(2)\,, \tag{3.58}$$

where $\mathcal{P}$ denotes the path-ordered exponential. Thus, they are the Wilson line variables of the connection ω associated to the edges or, equivalently, the parallel transports from the source to the sink of the edges with respect to ω. We assume the triangulation Δ to be piece-wise flat, and associate frames to all simplices of Δ. We then interpret g_{tf} as the group element relating the frame of $t \in \Delta$ to the frame of $f \in \Delta$, and similarly g_{ef} as the group element relating the frame of $f \in \Delta$ to the frame of $e \in \Delta$. Furthermore, we integrate the triad field along the edges $e \in \Delta$ as

$$X_e := \int_e E \in \mathfrak{su}(2)^* \,, \tag{3.59}$$

where an orientation for the edge e may be chosen arbitrarily. X_e is interpreted as the vector giving the magnitude and the direction of the edge e in the frame associated to the edge e itself.

In the case that Δ has no boundary, a discrete version of the BF partition function (3.57), the Ponzano-Regge partition function, can then be written as

$$\mathcal{Z}_{PR}^{\Delta} = \int \Big[\prod_{tf} \mathrm{d}g_{tf}\Big] \prod_{e\in\Delta} \delta(H_{e^*}(g_{tf}))\,, \tag{3.60}$$

where $H_{e^*}(g_{tf}) \in SU(2)$ are holonomies around the dual faces $e^* \in \Delta^*$ obtained as products of g_{tf}, $f^* \in \partial e^*$, and $\mathrm{d}g_{tf}$ is again the Haar measure on $SU(2)$. Mimicking the continuum partition function of BF theory, the Ponzano-Regge partition function is thus an integral over the flat discrete connections, the delta functions $\delta(H_{e^*}(g_{tf}))$ constraining holonomies around all dual faces to be trivial.

Now, we can apply the non-commutative Fourier transform to expand the delta functions in terms of the non-commutative plane waves. We again assume that the non-commutative plane wave is of the form $E_g(P) = \eta(g)e^{i\zeta(g)\cdot X/\hbar}$. This yields

$$\begin{aligned}\mathcal{Z}_{PR}^{\Delta} = \int \Big[\prod_{tf} \mathrm{d}g_{tf}\Big]\Big[\prod_{e} \frac{\mathrm{d}X_e}{(2\pi\hbar\kappa)^3}\Big] \left[\prod_{e\in\Delta} \eta(H_{e^*}(g_{tf}))\right] \\ \times \exp\left\{\frac{i}{\hbar}\sum_{e\in\Delta} X_e \cdot \zeta(H_{e^*}(g_{tf}))\right\}\,.\end{aligned} \tag{3.61}$$

Comparing with (3.56), this expression has a straightforward interpretation as a discretization of the first order path integral of the continuum BF theory. We can clearly identify the discretized triad variables X_e in (3.59) with the non-commutative metric variables defined via non-commutative Fourier transform. We also see that, from the point of view of discretization, the form of the plane waves and thus the choice for the quantization map is directly related to the choice of the precise form for the discretized action and the path integral measure. In particular, the coordinate function $\zeta : SU(2) \to \mathfrak{su}(2)$ and the prefactor $\eta : SU(2) \to \mathbb{C}$ of the non-commutative plane wave are dictated by the choice of the quantization map, and the coordinates specify the discretization prescription for the curvature 2-form $F(\omega)$. Similar interplay between $\star$-product quantization and discretization is well-known in the case of the first order phase space path integral formulation of ordinary quantum mechanics [13].

3.2.2 Non-commutative metric representation

If the triangulated manifold Δ has a non-trivial boundary, we may assign connection data on the boundary by fixing the group elements g_{ef} associated to the boundary triangles $f \in \partial\Delta$. Then, the (non-normalized) Ponzano-Regge amplitude for the boundary can be written as

$$\mathcal{A}_{PR}(g_{ef}|f \in \partial\Delta) = \int \Big[\prod_{tf} \mathrm{d}g_{tf}\Big]\Big[\prod_{\substack{ef \\ f\notin\partial\Delta}} \mathrm{d}g_{ef}\Big] \prod_{e\in\Delta}\prod_{i=0}^{n_e-1} \delta\big(g_{ef^e_{i+1}} g^{-1}_{t^e_i f^e_{i+1}} g_{t^e_i f^e_i} g^{-1}_{ef^e_i}\big) . \quad (3.62)$$

The delta functions are over the holonomies around the wedges of the triangulation pictured in grey in Fig. 3.1. For this purpose, the tetrahedra t^e_i and the triangles f^e_i sharing the edge e are labelled by an index $i = 0, \ldots, n_e - 1$ in a right-handed fashion with respect to the orientation of the edge e and with the identification $f_{n_e} \equiv f_0$, as in Fig. 3.1. The expression (3.60) for the Ponzano-Regge partition function can be obtained in the absence of a boundary by integrating over all g_{ef}. However, in considering the amplitude for a boundary configuration, we need to fix the g_{ef} for $f \in \partial\Delta$, and therefore it is convenient to write the amplitude in terms of the wedges.

Let us introduce some simplifying notation. We will choose an arbitrary spanning tree of the dual graph to the boundary triangulation, pick an arbitrary root vertex for the tree, and label the boundary triangles $f_i \in \partial\Delta$ by $i \in \mathbb{N}_0$ in a compatible way with respect to the partial ordering induced by the tree, so that the root has the label 0. (See Fig. 3.2.) Moreover, we denote the set of ordered pairs of labels associated to neighboring boundary triangles by $\mathcal{N}$, and label the group elements associated to the pair of neighboring boundary triangles $(i, j) \in \mathcal{N}$ as illustrated in Fig. 3.2. The group elements g_{tf} for $f \notin \partial\Delta$ we will denote by a collective label h_l. As we integrate over g_{ef} for $f \notin \partial\Delta$ in (3.62), we obtain

$$\mathcal{A}_{PR}(g_{ij}) = \int \Big[\prod_{l} \mathrm{d}h_l\Big]\Big[\prod_{e\notin\partial\Delta} \delta(H_{e^*}(h_l))\Big]\Big[\prod_{\substack{(i,j)\in\mathcal{N} \\ i<j}} \delta(g_{ij} h_j^{-1} K_{ji}(h_l) h_i g_{ji}^{-1})\Big] . \quad (3.63)$$

Here h_i is the group element associated to the edge going from the boundary triangle i to the center of the bulk tetrahedron with triangle i on its boundary, and $K_{ij}(h_l)$ is the holonomy along the bulk dual edges from the center of the tetrahedron with triangle j to the center of the tetrahedron with triangle i. (See Fig. 3.2 for illustration.) There is a one-to-one correspondence between the pairs (i, j) of neighbouring boundary triangles and faces of the dual 2-complex touching the boundary. Notice that we have chosen here

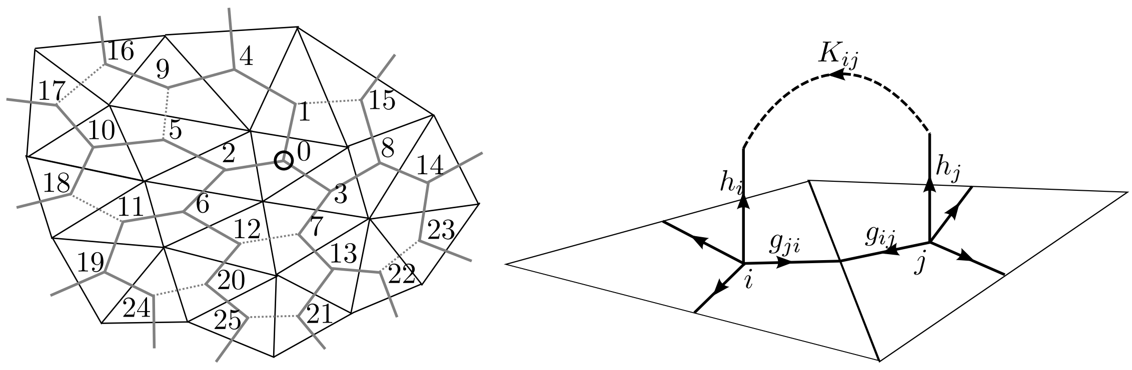

Figure 3.2: On the left: A portion of a rooted labelled spanning tree of the dual graph of a boundary triangulation (solid grey edges). On the right: Boundary triangles $f_i, f_j \in \partial\Delta$ and the associated group elements.

the base points for these holonomies as the boundary dual vertex with a smaller label. By expanding the delta distributions in (3.63) with boundary group variables into non-commutative plane waves, we get

$$\mathcal{A}_{PR}(g_{ij}) = \int \Big[\prod_{l} \mathrm{d}h_l\Big] \Bigg[\prod_{e \notin \partial\Delta} \delta(H_{e^*}(h_l))\Bigg] \\ \times \Bigg[\prod_{\substack{(i,j)\in\mathcal{N} \\ i<j}} \int \frac{\mathrm{d}Y_{ji}}{(2\pi\hbar\kappa)^3} \; E(g_{ij} h_j^{-1} K_{ji}(h_l) h_i g_{ji}^{-1}, Y_{ji})\Bigg] \,, \tag{3.64}$$

where we use the notation $E_g(X) \equiv E(g, X)$ for clarity.

To obtain the expression for metric boundary data, we employ the non-commutative Fourier transform,

$$\tilde{\mathcal{A}}_{PR}(X_{ij}) = \int \Big[\prod_{(i,j)\in\mathcal{N}} \frac{\mathrm{d}g_{ij}}{\kappa^3}\Big] \, \mathcal{A}_{PR}(g_{ij}) \prod_{(i,j)\in\mathcal{N}} E(g_{ij}^{-1}, X_{ij}) \,. \tag{3.65}$$

Here the variable X_{ij} is understood geometrically as the edge vector shared by the triangles i, j as seen from the frame of reference of the triangle j. From (3.64) and (3.65) the amplitude for metric boundary data is obtained by expanding the delta functions as

$$\tilde{\mathcal{A}}_{PR}(X_{ij}) = \int \Big[\prod_{(i,j)\in\mathcal{N}} \frac{\mathrm{d}g_{ij}}{\kappa^3}\Big]\Big[\frac{\mathrm{d}Y_{ji}}{(2\pi\hbar\kappa)^3}\Big]\Big[\prod_{l} \mathrm{d}h_l\Big]\Big[\frac{\mathrm{d}Y_e}{(2\pi\hbar\kappa)^3}\Big] \Bigg[\prod_{e\notin\partial\Delta} E(H_{e^*}(h_l), Y_e)\Bigg] \\ \times \Bigg[\prod_{\substack{(i,j)\in\mathcal{N} \\ i<j}} E(g_{ij} h_j^{-1} K_{ji}(h_l) h_i g_{ji}^{-1}, Y_{ji})\Bigg] \Bigg[\prod_{(i,j)\in\mathcal{N}} E(g_{ij}^{-1}, X_{ij})\Bigg] \,. \tag{3.66}$$

Exact amplitudes for metric boundary data with trivial topology

By integrating over all g_{ij}, Y_{ji}, Y_e in (3.66), we get

$$\tilde{\mathcal{A}}_{PR}(X_{ij}) \propto \int \Big[\prod_l \mathrm{d}h_l\Big] \Big[\prod_{e\notin\partial\Delta} \delta(H_{e^*}(h_l))\Big] \left[\prod_{\substack{(i,j)\in\mathcal{N}\\ i<j}} E(h_i^{-1}K_{ij}(h_l)h_j, X_{ji})\right] \star \left[\prod_{\substack{(i,j)\in\mathcal{N}\\ i<j}} \delta_\star(X_{ij}, -X_{ji})\right], \tag{3.67}$$

where we have dropped the immaterial finite propotionality constant cancelled by normalization. We see that the edge vectors X_{ij}, X_{ji} corresponding to the same edge in different frames of reference are simply identified by the non-commutative delta distributions $\delta_\star(X_{ij}, -X_{ji})$ with this choice of base points for the holonomies. We wish to integrate over the variables h_i. For every vertex i there is a unique path via the edges $(j_{n-1}, j_n)_{n=1,\dots,l}$, s.t. $j_0 = 0$, $j_l = i$, from the root to the vertex i along the spanning tree. Now, by making the changes of variables

$$h_i \mapsto \Big[\overleftarrow{\prod}_{n=0}^{l} K^{-1}_{j_{n-1}j_n}(h_l)\Big]\, h_i\,, \tag{3.68}$$

where by $\overleftarrow{\prod}$ we denote an ordered product for which the product index increases from right to left, we obtain

$$\begin{aligned}
\tilde{\mathcal{A}}_{PR}(X_{ij}) \propto &\int \Big[\prod_l \mathrm{d}h_l\Big] \Big[\prod_{e\notin\partial\Delta} \delta(H_{e^*}(h_l))\Big] \left[\prod_{\substack{(i,j)\in\text{tree}\\ i<j}} E(h_i^{-1}h_j, X_{ji})\right] \\
&\times \left[\prod_{\substack{(i,j)\notin\text{tree}\\ i<j}} E(h_i^{-1}L_{ij}(h_l)h_j, X_{ji})\right] \star \left[\prod_{\substack{(i,j)\in\mathcal{N}\\ i<j}} \delta_\star(X_{ij}, -X_{ji})\right] \\
= &\int \Big[\prod_l \mathrm{d}h_l\Big] \Big[\prod_{e\notin\partial\Delta} \delta(H_{e^*}(h_l))\Big] \\
&\times \left[\overrightarrow{\prod_i}^{\star} \left(E(h_i, \sum_j \epsilon_{ij}X_{ji}) \star \prod_{\substack{j\\ (i,j)\notin\text{tree}}} E(L_{ij}(h_l), X_{ji}) \right)\right] \\
&\star \left[\prod_{\substack{(i,j)\in\mathcal{N}\\ i<j}} \delta_\star(X_{ij}, -X_{ji})\right].
\end{aligned} \tag{3.69}$$

Here, $\epsilon_{ij} := \mathrm{sgn}(i-j)A_{ij}$, where A_{ij} is the adjacency matrix of the dual graph of the boundary triangulation. Moreover, $L_{ij}(h_l) \equiv G_{ij}^{-1}(h_l)H_{ij}(h_l)G_{ij}(h_l)$, where $H_{ij}(h_l)$ is the product of $K_{kl}(h_l)$'s around the unique cycle of the boundary dual graph formed by adding the edge (i,j) to the spanning tree, and $G_{ij}(h_l)$ is the product of $K_{kl}(h_l)$'s along the unique path from the root of the spanning tree to the cycle. The cycles formed from the spanning tree of a graph by adding single edges span the loop space of the graph. On the other hand, the product of $K_{kl}(h_l)$'s around a boundary vertex is constrained to be trivial by the flatness constraints for the bulk holonomies only if the neighbourhood of the vertex is a half-ball, since only in this case is the loop around the vertex contractible along the faces of the 2-complex. Thus, if and only if the neighborhoods of all boundary vertices have trivial topology, the flatness constraints impose $L_{ij}(h_l)$ to be trivial. In this case, we have

$$\tilde{\mathcal{A}}_{PR}(X_{ij}) \propto \int \Big[\prod_l \mathrm{d}h_l\Big] \Big[\prod_{e\notin\partial\Delta} \delta(H_{e^*}(h_l))\Big] \times \Big[\overrightarrow{\prod_i}_{\star} E(h_i, \epsilon_{ij} X_{ji})\Big] \star \Big[\prod_{\substack{(i,j)\in\mathcal{N} \\ i<j}} \delta_\star(X_{ij}, -X_{ji})\Big] . \tag{3.70}$$

Integrating over h_i then yields the closure constraints for the boundary triangles, and we end up with

$$\tilde{\mathcal{A}}_{PR}(X_{ij}) \propto [\delta(0)]^d \Big[\overrightarrow{\prod_i}_{\star} \delta_\star(\sum_j \epsilon_{ij} X_{ji})\Big] \star \Big[\prod_{\substack{(i,j)\in\mathcal{N} \\ i<j}} \delta_\star(X_{ij}, -X_{ji})\Big] , \tag{3.71}$$

where the sum is over vertices j connected to the vertex i, and d is the degree of divergence arising from the redundant delta distributions over the dual faces $e^* \in \Delta^*$, $e \notin \partial\Delta$.

It is clear that in the Euclidean limit $\kappa \to 0$, where the $\star$-product coincides with the point-wise product and $\delta_\star \to \delta$, the above amplitude imposes closure and identification of the edge vectors. However, the case of the classical limit $\hbar \to 0$ is more subtle: The whole notion of a non-commutative Fourier transform breaks down in this limit, since the non-commutative plane wave becomes ill-defined, having no well-defined limit. We will see in the following that these extra complications result into a discretization ambiguity in the classical limit, unless one applies non-commutative variational calculus to the phase space path integral, as in the case of quantum mechanics. From the above expression we see that the unambiguous result we will obtain from the non-commutative stationary phase analysis agrees with identifying the non-commutative dual space in the classical limit with the corresponding commutative space that would result from rederiving the dual representation as in Section 2.2 directly in the classical limit (even though in that case it would not result in a faithful representation of the full algebra). This provides an important consistency check for the non-commutative variational calculus.

3.2.3 Classical limit

The general treatment

Let us first consider the classical limit of the phase space path integral for Ponzano-Regge model, as given by the usual commutative variational calculus. We may use the expression for the non-commutative plane wave to express (3.66) as

$$\begin{aligned}\tilde{\mathcal{A}}_{PR}(X_{ij}) = \int \Big[\prod_{(i,j)\in\mathcal{N}} \frac{\mathrm{d}g_{ij}}{\kappa^3}\Big]\Big[\frac{\mathrm{d}Y_{ji}}{(2\pi\hbar\kappa)^3}\Big]\Big[\prod_l \mathrm{d}h_l\Big]\Big[\frac{\mathrm{d}Y_e}{(2\pi\hbar\kappa)^3}\Big] \\ \times \left[\prod_{e\notin\partial\Delta} \eta(H_{e^*}(h_l))e^{\frac{i}{\hbar}Y_e\cdot\zeta(H_{e^*}(h_l))}\right] \\ \times \left[\prod_{\substack{(i,j)\in\mathcal{N}\\ i<j}} \eta(g_{ij}h_j^{-1}K_{ji}(h_l)h_i g_{ji}^{-1})e^{\frac{i}{\hbar}Y_{ji}\cdot\zeta(g_{ij}h_j^{-1}K_{ji}(h_l)h_i g_{ji}^{-1})}\right] \\ \times \left[\prod_{(i,j)\in\mathcal{N}} \eta(g_{ij}^{-1})e^{\frac{i}{\hbar}X_{ij}\cdot\zeta(g_{ij}^{-1})}\right],\end{aligned} \tag{3.72}$$

and further by combining the exponentials we obtain

$$\begin{aligned}\tilde{\mathcal{A}}_{PR}(X_{ij}) = \int \Big[\prod_{(i,j)\in\mathcal{N}} \frac{\mathrm{d}g_{ij}}{\kappa^3}\eta(g_{ij}^{-1})\Big]\Big[\frac{\mathrm{d}Y_{ji}}{(2\pi\hbar\kappa)^3}\Big]\Big[\prod_l \mathrm{d}h_l\Big]\Big[\frac{\mathrm{d}Y_e}{(2\pi\hbar\kappa)^3}\Big] \\ \times \left[\prod_{e\notin\partial\Delta} \eta(H_{e^*}(h_l))\right]\left[\prod_{\substack{(i,j)\in\mathcal{N}\\ i<j}} \eta(g_{ij}h_j^{-1}K_{ji}(h_l)h_i g_{ji}^{-1})\right] \\ \times \exp\left\{\frac{i}{\hbar}\left[\sum_{e\notin\partial\Delta} Y_e\cdot\zeta(H_{e^*}(h_l)) + \sum_{\substack{(i,j)\in\mathcal{N}\\ i<j}} Y_{ji}\cdot\zeta(g_{ij}h_j^{-1}K_{ji}(h_l)h_i g_{ji}^{-1})\right.\right. \\ \left.\left. + \sum_{(i,j)\in\mathcal{N}} X_{ij}\cdot\zeta(g_{ij}^{-1})\right]\right\}.\end{aligned} \tag{3.73}$$

In this form the amplitude is amenable to stationary phase analysis through the study of the extrema of the exponential

$$\begin{aligned}\mathcal{S}_{PR} := \sum_{e\notin\partial\Delta} Y_e\cdot\zeta(H_{e^*}(h_l)) \quad + \sum_{\substack{(i,j)\in\mathcal{N}\\ i<j}} Y_{ji}\cdot\zeta(g_{ij}h_j^{-1}K_{ji}(h_l)h_i g_{ji}^{-1}) \\ + \sum_{(i,j)\in\mathcal{N}} X_{ij}\cdot\zeta(g_{ij}^{-1}),\end{aligned}$$

to which we now proceed.

There are five different kinds of integration variables: Y_e for $e \notin \partial\Delta$, Y_{ji}, h_l in the bulk, h_i touching the boundary and g_{ij}, whose variations we will consider in the following.

Variation of Y_e: Requiring the variation of the exponential to vanish simply gives

$$\zeta(H_{e^*}(h_l)) = 0 \Leftrightarrow H_{e^*}(h_l) = \mathbb{1} \tag{3.74}$$

for all $e \notin \partial\Delta$, i.e., the flatness of the connection around the dual faces e^* in the bulk. Thus, in particular, we have $\eta(H_{e^*}(h_l)) = 1$.

Variation of Y_{ji}: Similarly, this gives

$$\zeta(g_{ij}h_j^{-1}K_{ji}(h_l)h_i g_{ji}^{-1}) = 0 \Leftrightarrow g_{ij}h_j^{-1}K_{ji}(h_l)h_i g_{ji}^{-1} = \mathbb{1} \tag{3.75}$$

for all $(i,j) \in \mathcal{N}, i < j$, i.e., the triviality of the connection around the dual faces e^* to $e \in \partial\Delta$. We have $\eta(g_{ij}h_j^{-1}K_{ji}(h_l)h_i g_{ji}^{-1}) = 1$.

Variation of h_l in the bulk: The variations for the group elements are slightly more non-trivial. Taking right-invariant Lie derivatives of the exponential with respect to a group element $h_{l'} \equiv g_{tf}$ in the bulk, we obtain

$$\sum_{e\notin\partial\Delta} Y_e \cdot \mathcal{L}_k^{h_{l'}} \zeta(H_{e^*}(h_l)) + \sum_{\substack{(i,j)\in\mathcal{N}\\ i<j}} Y_{ji} \cdot \mathcal{L}_k^{h_{l'}} \zeta(g_{ij}h_j^{-1}K_{ji}(h_l)h_i g_{ji}^{-1}) = 0 \ \forall k\,.$$

Here, only the three terms in the sums depending on the holonomies around the boundaries of the three dual faces, which contain $l' := tf$ are non-zero. (Each dual edge f^* belongs to exactly three dual faces e^* of Δ^*, since Δ^* is dual to a 3-dimensional triangulation.) Now, using the fact uncovered through the previous variations that the holonomies around the dual faces are trivial for the stationary phase configurations, and the property $\zeta(\mathrm{ad}_g h) = \mathrm{Ad}_g\zeta(h)$ of the coordinates, we obtain

$$\sum_{\substack{e\in\Delta\\ e^*\ni f^*}} \epsilon_{fe}(\mathrm{Ad}_{G_{fe}} Y_e) = 0\,, \tag{3.76}$$

where $\mathrm{Ad}_{G_{fe}}$ implements the parallel transport from the frame of Y_e to the frame of f, and $\epsilon_{fe} = \pm 1$ accounts for the orientation of h_l with respect to the holonomy $H_{e^*}(h_l)$ and thus the relative orientations of the edge vectors. Clearly, this imposes the metric closure constraint for the three edge vectors of each bulk triangle $f \notin \partial\Delta$ in the frame of f.

Variation of h_i: Varying a h_i we get

$$\begin{aligned}
&\sum_{\substack{(i,j)\in\mathcal{N}\\ i<j}} Y_{ji}\cdot\mathcal{L}_k^{h_i}\zeta(g_{ij}h_j^{-1}K_{ji}(h_l)h_i g_{ji}^{-1})\\
&+\sum_{\substack{(j,i)\in\mathcal{N}\\ j<i}} Y_{ij}\cdot\mathcal{L}_k^{h_i}\zeta(g_{ji}h_i^{-1}K_{ij}(h_l)h_j g_{ij}^{-1}) = 0\ \forall k\,.
\end{aligned}$$

Again there are three non-zero terms in this expression, which correspond to the boundary triangles $f_j \in \partial\Delta$ neighboring f_i, i.e., such that $(i,j)\in\mathcal{N}$. We obtain the closure of the boundary integration variables Y_{ji} as

$$\sum_{\substack{f_j\in\partial\Delta\\ (i,j)\in\mathcal{N}}} \epsilon_{ji}(\mathrm{Ad}_{g_{ji}}Y_{ji}) = 0\,, \tag{3.77}$$

where $\mathrm{Ad}_{g_{ji}}$ parallel transports the edge vectors Y_{ji} to the frame of the boundary triangle f_i, and $\epsilon_{ji}=\pm1$ again accounts for the relative orientation.

Variation of g_{ij}: Taking Lie derivatives with respect to a g_{ij} of the exponential, we obtain

$$\begin{aligned}
&Y_{ji}\cdot\mathcal{L}_k^{g_{ij}}\zeta(g_{ij}h_j^{-1}K_{ji}(h_l)h_i g_{ji}^{-1}) + X_{ij}\cdot\mathcal{L}_k^{g_{ij}}\zeta(g_{ij}^{-1}) = 0\ \forall k\\
\Leftrightarrow\quad &\mathrm{Ad}_{g_{ij}}Y_{ji} \mp D^{\zeta}(g_{ij})X_{ij} = 0\,,
\end{aligned} \tag{3.78}$$

where we denote $(D^{\zeta}(g))_{kl} := \tilde{\mathcal{L}}_k\zeta_l(g)$, and the signs $-$ and $+$ corresponds to the cases $i<j$ and $i>j$, respectively. We see that this equation identifies the boundary metric variables X_{ij} with the integration variables Y_{ji} up to a sign and a parallel transport between the frames of each vector, plus *a non-geometric deformation* given by the matrix $D^{\zeta}(g_{ij})$. (In varying g_{ij} we must assume that the measure $\eta(g)\mathrm{d}g$ on the group is continuous, which should be true for any reasonable choice of a quantization map, as it indeed is for all the cases we consider below.)

Thus, we have obtained the constraint equations corresponding to variations of all the integration variables. In particular, by combining the equations (3.78) with the boundary closure constraint (3.77), we obtain

$$\sum_{\substack{f_j\in\partial\Delta\\ (i,j)\in\mathcal{N}}} D^{\zeta}(g_{ij})X_{ij} = 0\ \forall i\,, \tag{3.79}$$

which gives, in general, a *deformed* closure constraint for the boundary metric edge variables X_{ij}. In addition, from (3.78) alone we obtain a *deformed* identification

$$\mathrm{Ad}_{g_{ij}}^{-1}(D^{\zeta}(g_{ij})X_{ij}) = -\mathrm{Ad}_{g_{ji}}^{-1}(D^{\zeta}(g_{ji})X_{ji})\,, \tag{3.80}$$

naturally up to a parallel transport, of the boundary edge variables X_{ij} and X_{ji}. Accordingly, we obtain for the amplitude

$$\begin{aligned}\tilde{\mathcal{A}}_{PR}(X_{ij}) \propto \int \Big[\prod_{(i,j)\in\mathcal{N}} \frac{\mathrm{d}g_{ij}}{\kappa^3}\eta(g_{ij}^{-1})\Big] & \left[\prod_{f_i\in\partial\Delta} \delta_\star\Big(\sum_{\substack{f_j\in\partial\Delta \\ (i,j)\in\mathcal{N}}} D^\zeta(g_{ij})X_{ij}\Big)\right] \\ & \star \left[\prod_{\substack{(i,j)\in\mathcal{N} \\ i<j}} \delta_\star\Big(\mathrm{Ad}^{-1}_{g_{ij}}(D^\zeta(g_{ij})X_{ij}) + \mathrm{Ad}^{-1}_{g_{ji}}(D^\zeta(g_{ji})X_{ji})\Big)\right] \\ & \star \exp\left\{ \frac{i}{\hbar} \sum_{(i,j)\in\mathcal{N}} X_{ij}\cdot\zeta(g_{ij}^{-1})\right\} \big(1+\mathcal{O}(\hbar)\big)\,, \end{aligned} \tag{3.81}$$

where the proportionality constant is given by the configuration space volume for the geometric configurations of the Y_e metric variables in the bulk, which is generically infinite but is cancelled by normalization, and the delta functions impose the boundary constraints from above. (Note that one must write the integrand in terms of $\star$-products and $\star$-delta functions in order for the constraints to be correctly imposed, since the amplitude acts on wave functions through $\star$-multiplication.) The exact form of the deformation matrix $D^\zeta_{kl}(g) \equiv \{X_k, \zeta_l\}(g) = \delta_{kl} + \mathcal{O}(\kappa, |\ln(g)|)$, and accordingly the geometric content of these constraints, depends on the coordinates ζ, and therefore on the discretization of the continuum BF action or, equivalently, the initial choice of the quantization map. We see that only in the Euclidean limit $\kappa \to 0$, $|\zeta| = \text{const.}$, do the different choices agree, in general, producing the undeformed discrete geometric constraints

$$\sum_{\substack{f_j\in\partial\Delta \\ (i,j)\in\mathcal{N}}} X_{ij} = 0 \ \forall f_i \in \partial\Delta \quad \text{and} \quad \mathrm{Ad}^{-1}_{g_{ij}} X_{ij} = \mathrm{Ad}^{-1}_{g_{ji}} X_{ji} \ \forall (i,j) \in \mathcal{N} \tag{3.82}$$

for the discretized boundary metric variables $X_{ij} \in \mathfrak{su}(2)^*$.

We emphasize that in the above variation of the amplitude we did not take into account the deformation of phase space structure, which appeared crucial for obtaining the correct classical equations of motion in Sec. 3.1 in the case of quantum mechanics. Indeed, we may define the non-commutative variation $\delta_\star S$ of the action S in the amplitude via $e_\star^{i\delta_\star S+\mathcal{O}(\delta^2)} \equiv e_\star^{iS^\delta} \star e_\star^{-iS}$ as in (3.45), where $\mathcal{O}(\delta^2)$ refers to terms higher than first order in the variations. Then, all the above results for variations remain the same by requiring the non-commutative variation $\delta_\star S$ of the action to vanish except for Eq. (3.78), which becomes undeformed, i.e., we obtain the geometric relation $\mathrm{Ad}_{g_{ij}} Y_{ji} \mp X_{ij} = 0$. Thus, the non-geometric deformation of the constraints does not appear, and we recover exactly the simplicial geometry relations for the boundary metric variables, regardless of the choice of a quantization map. In this case the leading order semi-classical contribution to the

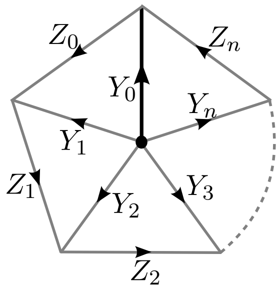

Figure 3.3: Labeling of boundary edge vectors around a vertex for solving the closure constraints.

Ponzano-Regge amplitude (3.81) reads

$$\tilde{\mathcal{A}}_{PR}(Y_{ji}) \propto \int \left[\prod_{\substack{(i,j)\in\mathcal{N} \\ i<j}} \frac{\mathrm{d}g_{ij}\mathrm{d}g_{ji}}{\kappa^3} \eta(g_{ji}^{-1} g_{ij}) \right] \left[\prod_{f_i \in \partial\Delta} \delta_\star \Big(\sum_{\substack{(i,j)\in\mathcal{N} \\ i<j}} \mathrm{Ad}_{g_{ji}} Y_{ji} - \sum_{\substack{(i,j)\in\mathcal{N} \\ j<i}} \mathrm{Ad}_{g_{ji}} Y_{ij} \Big) \right]$$
$$\star \ \exp \left\{ \frac{i}{\hbar} \sum_{\substack{(i,j)\in\mathcal{N} \\ i<j}} Y_{ji} \cdot \zeta(g_{ji}^{-1} g_{ij}) \right\} \big(1 + \mathcal{O}(\hbar)\big) \,, \tag{3.83}$$

where we have identified $Y_{ji} := \mathrm{Ad}_{g_{ji}}^{-1} X_{ji} = -\mathrm{Ad}_{g_{ij}}^{-1} X_{ij}$ for all $(i,j) \in \mathcal{N}$ such that $i < j$. In fact, the integrand is invariant under the change of variables $g_{ji} \mapsto g_{ji}k_{ji}$ and $g_{ij} \mapsto g_{ij}k_{ji}$ for $k_{ji} \in U(1)_{Y_{ji}} \subset SU(2)$, $i < j$, i.e., k_{ji} belong to the stabilizing $U(1)$ subgroup of elements such that $\mathrm{Ad}_{k_{ji}} Y_{ji} = Y_{ji}$. Therefore, we may further integrate over the subgroups $U(1)_{Y_{ji}}$, and we get exactly the same expression (3.83) as above (modulo the immaterial propotionality constant), but where now $g_{ji}, g_{ij} \in SU(2)/U(1)_{Y_{ji}} \cong SO(2)$ for all $(i,j) \in \mathcal{N}$, $i < j$.

The $\star$-delta functions in (3.83) impose closure of boundary triangles up to parallel transports. In order to connect this expression for the path integral to the exact treatment we gave before, let us now show how the closure constraints imply the flatness of the boundary connection for trivial boundary topology. Choose an arbitrary boundary vertex and label the boundary edge vectors associated to the edges around the vertex as in Fig. 3.3, where we indicate the orientations of the vectors by arrows. The closure constraints for the boundary triangles formed by these edges now read

$$Y_k = \mathrm{Ad}_{G_{k,k-1}} Y_{k-1} + \mathrm{Ad}_{H_{k,k-1}} Z_{k-1} \qquad \forall\, k = 1, \dots, n \,, \tag{3.84}$$

where $G_{k,k-1}$ and $H_{k,k-1}$ are the parallel transports from the frames of Y_{k-1} and Z_{k-1}, respectively, to the frame of Y_k (along the edges of the dual graph to the triangulation).

By combining these equation for all $k = 1, \ldots, n$, we obtain the equality

$$Y_0 = \mathrm{Ad}_{G_{0,0}} Y_0 + \sum_{k=1}^{n} \mathrm{Ad}_{H_{0,k}} Z_k \,, \tag{3.85}$$

where $G_{0,0} := G_{0,n} \overleftarrow{\prod}_{k=1}^{n} G_{k,k-1}$ is the holonomy around the boundary vertex, and $H_{0,k}$ are parallel transports from the frame of Z_k to the frame of Y_0. Clearly, this can be satisfied only if the holonomy $G_{0,0} \in U(1)_{Y_0} \backslash SU(2)/U(1)_{Y_0}$ around the boundary vertex is trivial, and the boundary edges Z_k satisfy closure $\sum_{k=1}^{n} \mathrm{Ad}_{H_{0,k}} Z_k = 0$ up to parallel transports. If the boundary has genus 0 (i.e., trivial topology), then the triviality conditions for holonomies around vertices impose triviality for all holonomies around cycles on the dual graph to the triangulation, since any cycle is contractible in that case.

Despite leading to the correct geometric constraints that agree with the exact analysis, it is not clear to us at the moment if there exists a rigorous mathematical argument for considering the non-commutative variational method. If one considers $\tilde{\mathcal{A}}_{PR}(X_{ij})$ just as an ordinary function, then the dominant contribution to it is definitely given by the normal variational method. However, in calculating transition amplitudes for boundary states, $\tilde{\mathcal{A}}_{PR}(X_{ij})$ acts as an integral kernel with respect to the $\star$-product, which may justify the use of such a non-commutative analysis. Unfortunately, we have not yet found any general argument for using the non-commutative variations, except for the fact that it leads to the correct classical equations of motion, and agrees with the intuitive classical limit of the exact expression (3.71). Therefore, it is still of interest to consider the normal commutative variations of the amplitudes for different explicit choices of quantization maps to see how these differ from each other. We will return to the question of non-commutative versus commutative variational calculus and to a further analysis of the non-commutative semi-classical limit of the Ponzano-Regge model in our forthcoming publication [50].

Before we go on to consider for the ordinary commutative variational calculus the stationary phase boundary configurations for some specific choices of the coordinates ζ, let us make a few general remarks on the dependence of the limit on this choice. As we have already emphasized above, the exact functional form of the non-commutative plane waves, and thus the coordinate choice, is determined ultimately by the choice of the quantization map and the $\star$-product that we thus obtain. We have found the general expression for the plane wave as a $\star$-exponential

$$E(g, X) = e_\star^{\frac{i}{\hbar\kappa} k(g)\cdot X} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{\hbar\kappa}\right)^n k^{i_1}(g) \cdots k^{i_n}(g) X_{i_1} \star \cdots \star X_{i_n} \,. \tag{3.86}$$

From this expression we may observe that the way the Planck constant $\hbar$ enters into the plane wave is very subtle. There are negative powers of $(\hbar\kappa)$ coming from the prefactor in the exponential, while from the $\star$-monomials arise positive powers of $(\hbar\kappa)$. The way these different contributions go together determines the explicit functional form of the non-commutative plane wave, and accordingly the behavior in the classical limit $\hbar \to 0$. Therefore, it is not too surprising that we may eventually find different classical limits for different choices of plane waves. In particular, it is important to realize that the non-commutative plane wave itself is purely a quantum object with an ill-defined classical limit, and therefore has no duty to coincide with anything in this limit. (On the contrary, in the Euclidean limit $\kappa \to 0$ we also scale the coordinates k^i on the group, so that k^i/κ stays constant, since κ determines the scale associated to the group manifold. Therefore, the non-commutative plane wave agrees with the usual Euclidean plane wave in this limit.) For this reason, the stationary phase solutions corresponding to different $\star$-products may also differ from each other, even though the $\star$-product itself coincides with the pointwise product in this limit. Only in the Euclidean limit, for which the class angles of the group elements are scaled simultaneously, do the different choices coincide for the commutative variational calculus.

Some choices of quantization maps

Symmetric & Duflo quantization maps

We may calculate for the deformation matrix as a function of the canonical coordinates $k(g) = -i \ln_R(g)$ the expression

$$D^{\mathrm{s}}_{ab}(k) = \left(\frac{\kappa|k|}{\sin(\kappa|k|)}\right)\left[\cos(\kappa|k|)\delta_{ab} + \left(\frac{\sin(\kappa|k|)}{\kappa|k|} - \cos(\kappa|k|)\right)\frac{k_a k_b}{|k|^2} - \kappa\epsilon_{abc}k^c\right]. \quad (3.87)$$

This deformation matrix has the following nice property: $D^{\mathrm{s}}_{ab}(k)k^b = k_a$. This implies, in particular, that when the edge vectors are stable under the dual connection variables, $\mathrm{Ad}_{g_{ij}}X_{ij} = X_{ij} \;\Leftrightarrow\; k(g_{ij}) \propto X_{ij}$, we have $D^{\mathrm{s}}(g_{ij})X_{ij} = X_{ij}$, and therefore recover the undeformed closure constraints from (3.79). Accordingly, classical geometric boundary data with $\mathrm{Ad}_{g_{ij}}X_{ij} = X_{ij}$, $X_{ij} = -X_{ji}$ and $\sum_j X_{ij} = 0$ satisfies the constraint equations for the symmetric quantization map. Therefore, in the case of geometric boundary data, we effectively recover the Regge action (modulo the ambiguous path integral measure) in the classical limit of the first order Ponzano-Regge amplitude for the Duflo (or symmetric) quantization.

Except for the stability ansatz $\mathrm{Ad}_{g_{ij}}X_{ij} = X_{ij}$, however, there are undoubtedly other solutions to the constraint equations in addition to the classical geometries, but we have not explored the possibilities in this general case. It is nevertheless clear that these additional solutions do not correspond to classical geometries, in general, since for them the closure constraint is again deformed. Still, at least, we obtain the classical boundary geometries as a subset of the space of solutions.

Freidel-Livine-Majid quantization map

For the coordinates $p^i = -\frac{i}{2\kappa}\mathrm{tr}_{\frac{1}{2}}(g\sigma^i)$ associated to the Freidel-Livine-Majid quantization map, it is straightforward to calculate the deformation matrix

$$D^{\mathrm{FLM}}_{kl}(g) = \frac{1}{2}\mathrm{tr}_{\frac{1}{2}}(g)\delta_{kl} + \frac{i}{2}\mathrm{tr}_{\frac{1}{2}}(g\sigma^j)\epsilon_{jkl} \equiv \sqrt{1-\kappa^2|p(g)|^2}\,\delta_{kl} - \kappa p^j(g)\epsilon_{jkl}\,. \tag{3.88}$$

Thus, according to our general description above, the classical discrete geometricity constraints are satisfied by the deformed boundary metric variables

$$D^{\mathrm{FLM}}(g_{ij})X_{ij} = \sqrt{1-\kappa^2|p(g_{ij})|^2}\,X_{ij} - \kappa(p(g_{ij}) \wedge X_{ij})\,. \tag{3.89}$$

We have not solved these constraints explicitly, which would generically impose relations between the stationary phase boundary connection g_{ij} and the given boundary metric data X_{ij}. However, one can easily confirm that data corresponding to generic classical discrete geometries does not satisfy the constraints, and therefore the geometry resulting from the constraints does not, in general, describe discrete geometries. In particular, for a classical discrete geometry we would have that $\mathrm{Ad}_{g_{ij}}X_{ij} = X_{ij}$, and that the usual closure constraint $\sum_j X_{ij} = 0$ holds for all i. But data satisfying these requirements does not generically solve the deformed constraints for the FLM quantization map. In fact, the deformed and the undeformed closure constraints are compatible only for $g_{ij} \equiv \mathbb{1}$, or equivalently, in the Euclidean limit. Therefore, we conclude that the non-commutative metric boundary variables do not have a classical geometric interpretation in the case of FLM quantization map outside the Euclidean approximation, unless one studies the non-commutative variation of the action instead.

Chapter 4

Conclusions and Outlook

Let us conclude with a summary of the results, and some further research directions, to which they point.

In the first part of this thesis, we have studied the representations of the quantum algebra $\mathfrak{A}$ obtained by canonically quantizing the Poisson algebra $\mathcal{P}(\mathcal{T}^*\mathcal{G})$ associated to the cotangent bundle of a Lie group $\mathcal{G}$. In addition to the usual representation of $\mathfrak{A}$ on the Hilbert space of square-integrable functions $L^2(\mathcal{G})$ on $\mathcal{G}$, we have seen that a dual *non-commutative representation* of $\mathfrak{A}$ in terms of a function space, which we denote as $L^2_\star(\mathfrak{g}^*)$, on the Lie algebra dual $\mathfrak{g}^*$ can be defined by introducing a suitable deformation quantization $\star$-product, depending only on the chosen quantization map between $\mathcal{P}(\mathcal{T}^*\mathcal{G})$ and $\mathfrak{A}$. We further identified the conditions for the existence of the representation. The *non-commutative Fourier transform* is then derived as the intertwining map between these two representations, generalizing in a natural way the usual Fourier transform on Euclidean space. We have seen that the explicit form of the *non-commutative plane wave*, which acts as the integral kernel of the transform, depends again only on the choice of a quantization map or, equivalently, a deformation quantization $\star$-product. In terms of the canonical coordinates $k(g) = -i \ln_R(g) \in \mathfrak{g}$ on $\mathcal{G}$ obtained through the logarithm map restricted on the principal branch, the plane wave was shown to be given by the star-exponential

$$E_g(X) = e_\star^{ik(g)\cdot X} ,$$

where $X \in \mathfrak{g}^*$. This can then, at least in the cases we have considered, be equivalently expressed as a standard exponential for some (*a priori* different) choice of coordinates on the group multiplied by a class function, also following canonically from the choice of quantization map.

These results show that the possibility of a non-commutative representation does not require the existence of the group representation, but only a choice of quantization map. The non-commutative representation for the quantum system, in other words, can stand on its own feet. Of course, which representation is more convenient to use depends on the specific question being tackled, as different representations have different advantages. The results also offer a new perspective on the non-commutative Fourier transform and some more insights into the various elements entering in its definition (e.g., the choice of coordinates), and lead to a prescription for how to define plane waves for generic quantization maps. This also clarifies the relation with the so-called quantum group Fourier transform of Majid, extending the work of Freidel & Majid [26].

In general, for an arbitrary quantization map and corresponding $\star$-product, the necessary conditions for the existence of the non-commutative representation would not be satisfied. However, we have provided some explicit and non-trivial examples satisfying the necessary conditions of the above construction for the Lie groups $\mathbb{R}$, $U(1)$ and $SU(2)$. In particular, for $SU(2)$ we considered examples corresponding to three different choices of quantization maps: the symmetric map, the Duflo map, and the so-called Freidel-Livine-Majid map used in the quantum gravity literature. For all examples, we have provided the corresponding $\star$-product, non-commutative representation and plane waves explicitly. On the one hand, these examples prove the non-emptiness of the definitions provided together with the existence of their non-commutative representation and of their non-commutative Fourier transforms. On the other hand, the results of specific quantization maps find direct applications to physics, as shown in the second part of the thesis.

Besides clarifying some aspects and the underlying logic of the construction of the non-commutative representation and of the non-commutative Fourier transform, we expect these results to have further interesting applications in the study of specific quantum systems arising from the quantization of the phase space we started from. In particular, we hope to have provided new tools to the development of quantum gravity models in the context of loop quantum gravity and group field theory. For example, a possible application of our construction would be to study the flux representation of loop quantum gravity and the corresponding coherent states for the Duflo map, extending the work of [16, 53]. In the same direction, the construction of a new 4d gravity model along the same lines as [8] can now be performed for the non-commutative representation corresponding, again, to the Duflo map, and it would be very interesting to identify clearly the consequences for the resulting model of the nice mathematical properties of such a quantization map. On the other hand, a generalization of the non-commutative representation and the non-commutative Fourier transform to homogeneous spaces that are endowed with a transitive action of a weakly exponential Lie group should also be possible.

In the second part of the thesis we considered applications of the non-commutative representation to some models of physics. We first considered quantum mechanics on a Lie group, and showed that, starting from the canonical formulation of quantum mechanics on a Lie group $\mathcal{G}$, a first order path integral can be derived using non-commutative momentum variables obtained via the non-commutative Fourier transform. This was shown to produce the correct equations of motion in the classical limit $\hbar \to 0$, but only by taking into account the quantum deformation of phase space structure in the variational calculus. Moreover, in the case $\mathcal{G} = SU(2)$ we verified the quantum corrections to the classical Hamiltonian, which in this approach arise from the non-commutative structure, to be consistent with earlier results in the literature.

The advantages of the approach we have studied are the following: On the one hand, it provides an alternative to the use of representation theory, and a more intuitive picture of the quantum dynamics by representing the system with continuous non-commutative momentum variables, which correspond to the classical momentum variables in the classical limit. On the other hand, this approach makes the semi-classical analysis much more straightforward, because one can easily apply the stationary phase approximation to the phase space path integral.

We have learned from the analysis the following:

- Given the classical phase space of the theory, the use of the non-commutative Fourier transform and of the $\star$-product described above is very natural, and seems to provide a correct conjugate representation for quantum states.

- Similarly, it is welcome that the dynamics of the theory in terms of the dual non-commutative variables takes the form of the expected first order path integral.

- In the case considered, the non-trivial phase space structure gives naturally rise to quantum corrections into the classical action.

- The use of non-commutative dual variables is advantageous in the study of the semi-classical approximation, because it brings the quantum dynamics in the form of a path integral.

- Finally, the additional parameter determining the physical scale of the group manifold (denoted λ above) could play an important role in studying the commutative Euclidean approximation, which is independent from the classical one.

Altogether, these results further indicate that the new non-commutative variables make sense, both mathematically and physically, and that the non-commutative methods can be applied successfully, where found advantageous.

As a second application to physics, we considered the non-commutative metric representation of the Ponzano-Regge spin foam model for 3-dimensional quantum gravity obtained via the non-commutative Fourier transform. In particular, we applied the non-commutative Fourier transform to express the Ponzano-Regge spin foam amplitude as a first order phase space path integral in terms of non-commutative metric boundary data. This reformulation then allowed us to study conveniently the classical limit of the full amplitude. We discovered that depending on the choice of non-commutative structure arising from the deformation quantization applied to the geometric operators, different limiting behaviors appear for the boundary data. For the normal commutative variational method, the constraints that arise as the classical equations of motion do not always correspond to discrete geometries, since the edge vectors in the constraint equations are deformed due to the non-linearity of the group manifold. In general, only by taking into account the deformation of phase space structure in studying the variations, as in the case of quantum mechanics above, we find the undeformed geometrical constraints.

For the examples we considered, we found that in the cases of non-commutative structures arising from symmetric and Duflo quantization maps, we recover the geometric closure constraints when each boundary edge vector is stable under the dual parallel transport, as for classical discrete geometries. *Vice versa*, if we impose closure on the boundary data, then the constraints are satisfied by bulk connections corresponding to classical discrete geometries. However, other non-geometric solutions also exist. For the Freidel-Livine-Majid non-commutative metric representation, popular in the literature, we found that the classical limit does not give rise to discrete geometries in the boundary metric variables, for classical metric boundary data does not solve the constraints in this case.

There are several conclusions to be drawn from these results. First, we have seen that the non-commutative Fourier transform facilitates the full asymptotic analysis of spin foam models. As the formalism of non-commutative Fourier transform has been extended to all exponential Lie groups in the first part of this thesis, and in particular to the case of the double cover $SL(2,\mathbb{C})$ of the Lorentz group, we hope also to extend our asymptotic analysis to the 4-dimensional spin foam models in future work. However, for the ordinary commutative variations, different choices for the explicit form of the non-commutative Fourier transform seem to lead to different properties of the metric boundary variables, some of which do not allow a discrete geometrical interpretation in the classical limit. Only by studying the non-commutative variations we recover the classical geometric constraints for all cases of non-commutative structures. This curious feature asks for further analysis, and must be taken into account in any future application of the non-commutative methods to spin foam models. We intend to address this issue further in a forthcoming publication [50].

In summary, we have provided a mathematical basis for the non-commutative representation of quantum algebras associated to exponential Lie groups, and demonstrated its usefulness in the analysis of different physics models. Naturally, the scope of further mathematical and physical applications of such a dual representation may be much vaster than what we have been able to present within this thesis. It is left for the future research to show its full extent.

Summary

The topic of this thesis is a new representation for quantum systems on weakly exponential Lie groups in terms of a non-commutative algebra of functions, the associated non-commutative harmonic analysis, and some of its applications to specific physical systems.

In the first part of the thesis, after a review of the necessary mathematical background, we introduce a $*$-algebra that is interpreted as the quantization of the canonical Poisson structure of the cotangent bundle over a Lie group. From the physics point of view, this represents the algebra of quantum observables of a physical system, whose configuration space is a Lie group. We then show that this quantum algebra can be represented either as operators acting on functions on the group, the usual group representation, or (under suitable conditions) as elements of a completion of the universal enveloping algebra of the Lie group, *the algebra representation*. We further apply the methods of deformation quantization to obtain a representation of the same algebra in terms of a non-commutative algebra of functions on a Euclidean space, which we call *the non-commutative representation* of the original quantum algebra. The non-commutative space that arises from the construction may be interpreted as the quantum momentum space of the physical system. We derive the transform between the group representation and the non-commutative representation that generalizes in a natural way the usual Fourier transform, and discuss key properties of this new non-commutative harmonic analysis. Finally, we exhibit the explicit forms of the non-commutative Fourier transform for three elementary Lie groups: $\mathbb{R}^d$, $U(1)$ and $SU(2)$.

In the second part of the thesis, we consider application of the non-commutative representation and harmonic analysis to physics. First, we apply the formalism to quantum mechanics of a point particle on a Lie group. We define the dual non-commutative momentum representation, and derive the phase space path integral with the help of the non-commutative dual variables. In studying the classical limit of the path integral, we show that we recover the correct classical equations of motion for the particle, if we account for the deformation of the phase space in the variational calculus. The non-commutative variables correspond in this limit to the classical momentum variables, further verifying their physical interpretation. We conclude that the non-commutative harmonic analysis facilitates a convenient study of the classical limit of quantum dynamics on a Lie group even if the group is compact, in which case variational calculus cannot easily be applied. As the second physics application, we repeat our above considerations for the case of Ponzano-Regge spin foam model for 3-dimensional quantum gravity. The non-commutative dual variables correspond in this case to discrete metric variables, thus illuminating the geometrical interpretation of the model. Again, we find that a convenient study of the classical limit is made possible through the non-commutative phase space path integral.

Zusammenfassung

Das Thema dieser Arbeit ist eine neue Darstellung für Quantensysteme auf schwach exponentiellen Lie-Gruppen im Sinne einer nichtkommutativen Algebra von Funktionen, die zugehörige nichtkommutative harmonische Analyse, und einige ihrer Anwendungen auf bestimmte physikalische Systeme.

Im ersten Teil der Arbeit, nach einem überblick über den notwendigen mathematischen Hintergrund, führen wir eine $*$-Algebra ein, die als Quantisierung der kanonischen Poisson-Struktur des Kotangentialbündels über eine Lie-Gruppe interpretiert wird. Aus physikalischer Sicht stellt diese die Algebra der Quantenobservablen eines physikalischen Systems dar, dessen Konfigurationsraum eine Lie-Gruppe ist. Wir zeigen dann, dass diese Quantenalgebra eine Darstellung entweder als Operatoren hat, welche auf Funktionen auf der Gruppe wirken, die übliche Gruppen-Darstellung, oder (unter geeigneten Bedingungen) als Elemente einer Vervollständigung der universellen einhüllenden Algebra der Lie-Gruppe, *die Algebra-Darstellung*. Weiterhin wenden wir die Methode der Deformierungs-Quantisierung an, um eine Darstellung derselben Algebra als nichtkommutative Algebra von Funktionen auf einem euklidischen Raum zu erhalten, die wir als *nichtkommutative Darstellung* der ursprünglichen Quantenalgebra bezeichnen. Der aus dieser Konstruktion resultierende nichtkommutative Raum kann als Quanten-Impulsraum des physikalischen Systems interpretiert werden. Wir leiten die Transformation zwischen der Gruppen-Darstellung und der nichtkommutativen Darstellung her, die auf natürliche Weise die übliche Fourier-Transformation verallgemeinert, und besprechen wichtige Eigenschaften dieser neuen nichtkommutativen harmonischen Analyse. Schlie'ilich präsentieren wir die explizite Form der nichtkommutativen Fourier-Transformation am Beispiel dreier elementarer Lie-Gruppen: $\mathbb{R}^d$, $U(1)$ und $SU(2)$.

Im zweiten Teil der Arbeit betrachten wir die Anwendung der nichtkommutativen Darstellung und der harmonischen Analyse auf die Physik. Zuerst wenden wir den Formalismus auf die Quantenmechanik eines Punktteilchens auf einer Lie-Gruppe an. Wir definieren die duale nichtkommutative Impulsdarstellung und leiten das Phasenraum-Pfadintegral mit Hilfe der nichtkommutativen dualen Variablen her. Bei der Untersuchung des klassischen Limes des Pfadintegrals zeigen wir, dass wir die richtigen klassischen Bewegungsgleichungen für das Teilchen erhalten, wenn wir in der Variationsrechnung die Deformierung des Phasenraums berücksichtigen. Die nichtkommutativen Variablen entsprechen in diesem Limes den klassischen Impuls-Variablen, womit ihre physikalischen Interpretation weiterhin bestätigt wird. Wir schlie'ien daraus, dass die nichtkommutative harmonische Analyse ein zweckmäßiges Studium des klassischen Limes der Quantendynamik auf einer Lie-Gruppe ermöglicht, selbst wenn die Gruppe kompakt ist, in welchem Fall die Variationsrechnung nicht leicht angewendet werden kann. Als zweite physikalische Anwendung wiederholen wir unsere obigen Betrachtungen für den Fall des Ponzano-Regge-Spinfoam-Modells für dreidimensionale Quantengravitation. Die dualen nichtkommutativen Variablen entsprechen in diesem Fall diskreten Metrik-Variablen und erhellen damit die geometrische Interpretation des Modells. Auch hier finden wir, dass das Studium des klassischen Limes in geeigneter Weise durch das nichtkommutative Phasenraum-Pfadintegral ermöglicht wird.

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