Robust small area estimation under spatial non-stationarity for unit-level models

Theory and Empirical Results

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1 Introduction

By a small sample we may judge of the whole piece.

- Miguel de Cervantes (1605)

1.1 Motivation

The demand for reliable small area statistics has increased substantially in recent years. As an example, household characteristics such as income and unemployment status are not equally distributed across the geography of a country. Even within a city, the level of income can vary substantially between different neighborhoods. Thus, to develop effective policy measures policymakers are increasingly interested in highly disaggregated information on various crucial topics such as social and economic issues at a local level. Population surveys are generally designed to obtain precise estimates of the quantity of interest on an aggregated level, i.e., countries or regions. The target quantity can, for instance, be a mean, a count or a quantile of the variable of interest. Typically, the sample size for disaggregated levels from population surveys are very small or even zero. Therefore, these levels are called *small areas*. The term small area is not limited to geographic entities, but can also refer to cross-classifications of subgroups of a population, i.e., age $\times$ sex $\times$ education $\times$ income. In that case, these subgroups are usually called small domains. When political decisions are based on small area statistics, the bias and variability of the estimates for the target quantity should be sufficiently small. Direct estimates, which rely only on area-specific information, are unbiased but can produce results with high variability in case of small sample sizes. Due to budgetary constraints, increasing the sample size is usually not an option. Therefore, small area estimation (SAE) techniques have been developed to gain reliability compared to direct estimates by *borrowing strength* from similar domains and additional information. Reliability refers to the precision of an estimator and is usually measured by the mean squared error (MSE), which comprises the bias and the variability.
One well-known SAE method is the empirical best linear unbiased predictor

(EBLUP) that uses the linear mixed model approach to incorporate auxiliary variables. Mixed models typically include area-specific random effects that account for unexplained variation between areas. It is assumed here that population units in different small areas are uncorrelated. In practice, however, the boundaries of small areas can be set arbitrarily to define administrative subregions for a population area. In that case, units that are spatially close may be related even though they belong to different small areas. Therefore, it can be reasonable to borrow strength over space by integrating spatial effects into the model. As an example of spatial effects, Figure 1.1 shows a map of the average quoted apartment net rents per square meter (sqm) in urban planning areas of Berlin in 2015. The map shows clusters of very high values in the city center and the south-west, whereas there are low values at the eastern and western outskirts of Berlin. This pattern may be due to spatial correlation among neighboring areas or a result of spatial heterogeneity in the form of non-constant error variances or model coefficients.

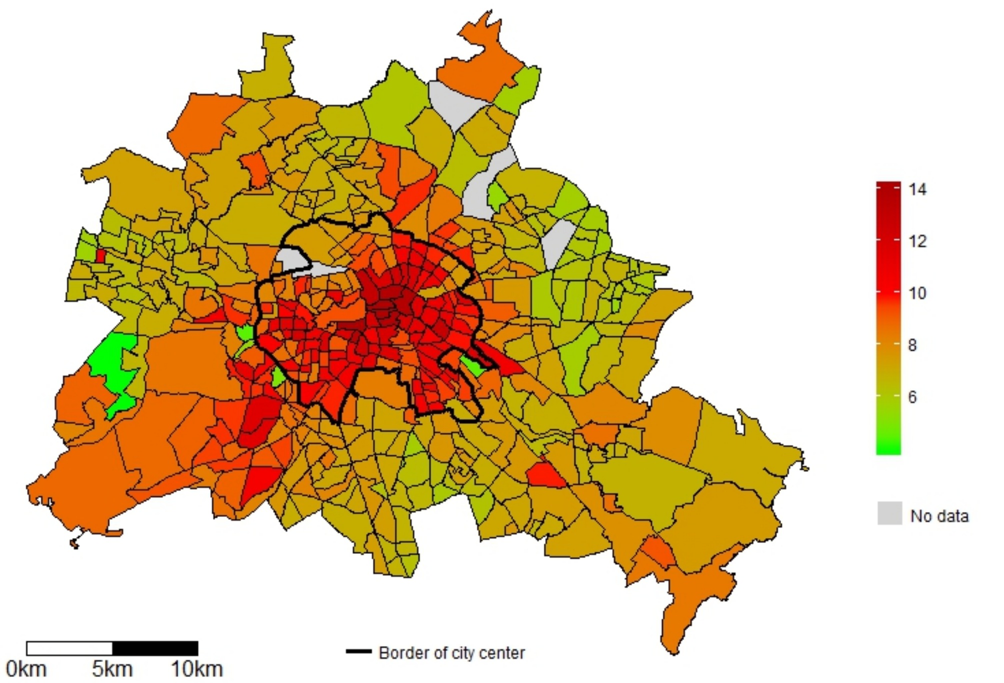


Figure 1.1: Average quoted apartment net rents per sqm in urban planning areas of Berlin. Data source: empirica-database (empirica-systeme.de).

Spatial correlation in the data can be accounted for by allowing spatially correlated random effects (cf. Singh et al., 2005; Pratesi and Salvati, 2008). One way to include the spatial heterogeneity allows the regression coefficients to vary across the study area, which is referred to as *spatial non-stationarity*

(Brunsdon et al., 1996; Fotheringham et al., 2002). In that case, the linear mixed model can be combined with a geographically weighted regression to produce reliable small area estimates (cf. Chandra et al., 2012).
These spatial methods for SAE extend the EBLUP approach and rely on the assumption of normally distributed error term components. This assumption can be violated in the presence *outliers*. Hence, it can be beneficial to reduce the influence of outliers and use robust small area methods (cf. Sinha and Rao, 2009; Chambers and Tzavidis, 2006). Outlier robust SAE under spatially correlated random effects have been discussed in the literature (cf. Schmid, 2012; Schmid and Münnich, 2014; Schmid et al., 2016). Robust SAE under spatial non-stationarity has been discussed in the context of M-quantile estimation (cf. Salvati et al., 2012). The aim in this thesis is to provide a robust EBLUP-based approach for SAE under spatial non-stationarity.
The remainder of this introduction is organized in the following order: Section 1.2 provides a literature overview and justifies in depth how this thesis contributes to the field of SAE. Section 1.3 outlines the structure of this thesis.

1.2 Related Literature

SAE methods are relevant when population parameters for areas or domains with small or even zero sample sizes are of interest. In recent decades, the demand for small area statistics has increased in public and private organizations. The results can be used for various purposes like targeted recourse allocation, health monitoring or poverty mapping. Methodological developments in SAE underline the increasing demand for reliable disaggregated population estimates. The purpose of this section is to link the contribution of this thesis to the current literature in the field of SAE. More comprehensive literature overviews for SAE are provided, for instance, in Pfeffermann (2002, 2013), Rao (2003) and Rao and Molina (2015).
SAE can be divided into design-based and model-based estimation methods. In general, design-based estimators rely on the sampling design and are evaluated under the randomization distribution resulting from the sampling process. In contrast, model-based methods are conditioned on the sample and inference is conducted with respect to the underlying model (Pfeffermann, 2013).
Design-based SAE methods broadly comprise direct estimators, indirect synthetic estimators, and composite estimators. For a comprehensive review of design-based methods in SAE see Lehtonen and Veijanen (2009). Direct estimators, such as the Horvitz-Thompson (cf. Horvitz and Thompson, 1952; Cochran, 1977) and the model-assisted generalized regression (GREG) estimator (cf. Särndal et al., 1992) only use the area-specific data from the sample. Therefore, the variance of these estimators can be unac-

ceptably large in the case of small sample sizes. Indirect synthetic estimators are based on implicit models and increase the effective sample size by borrowing strength from similar areas. They can produce estimates with higher precision but with a possibly large bias. Composite estimators are defined as the weighted sum of an indirect synthetic and a direct estimator. Therefore, they can be seen as a compromise between a large bias of indirect synthetic estimators and a large variance of direct estimators. The design-based indirect methods implicitly make use of statistical models that ignore unexplained between-area variations. In practice, however, unexplained heterogeneity between different areas may be present.
Model-based SAE methods on the other hand are typically based on mixed effects models. Mixed models include area-specific random effects that account for the between-area variation that cannot be explained by the fixed part of the model (cf. Jiang and Lahiri, 2006). In the context of SAE mixed models can be divided into unit-level and area-level mixed models. Area-level models are suitable in situations where the target variable can only be observed at small area-level (cf. Fay and Herriot, 1979) whereas for unit-level models the target variable needs to be available for all units (cf. Battese et al., 1988). Model-based methods for SAE rely on auxiliary variables with good predictive power that are available in the sample and (at least at area level) for the non-sampled population. Given these variables, model-based estimators strongly rely on the model assumptions (cf. Pfeffermann, 2013). Based on mixed models, small area estimates can be obtained using the empirical best unbiased predictor (EBLUP), empirical Bayes and hierarchical Bayes methods. A comprehensive review of these frameworks in the context of SAE is provided by Rao and Molina (2015). Unit-level EBLUP approaches for SAE are the main subject of this thesis, hence, Bayes methods are not discussed in further detail.
The production of reliable small area estimates is based on the availability of accurate auxiliary information. When additional spatial information such as coordinates and distances between small areas is available, it can be incorporated into the model to borrow strength over space. For instance, accounting for spatial effects can offer some gains in terms of precision. In SAE, this has been demonstrated by several authors (cf. Pratesi and Salvati, 2008; Longford, 2010; Chandra et al., 2012; Porter et al., 2014). Spatial effects can be divided into spatial dependence and spatial heterogeneity (Anselin, 1988). When spatial dependency is present, outcomes from different locations are related, i.e., they are spatially autocorrelated. Spatial heterogeneity can be interpreted as structural instability such as varying error variances or model coefficients.
One approach to incorporate the spatial correlation in the data is to allow for spatially correlated random effects, e.g. by using conditional (CAR) or simultaneous autoregressive (SAR) processes (cf. Cressie (1993) for SAR and CAR models). In fact, mixed models that include area-specific random

effects usually assume that population units within an area are correlated whereas units from different areas are uncorrelated. This assumption of independence can become implausible when similarities between population units can be explained by spatial proximity. In that case it is very likely that adjacent areas are more similar than distant areas. Singh et al. (2005) and Pratesi and Salvati (2008) proposed a spatial extension of the EBLUP (SEBLUP) for area-level SAE under spatial dependence. Following these ideas, Chandra et al. (2007) considered unit-level SAE under spatial dependence. An approach to include the spatial heterogeneity assumes that the regression coefficients vary spatially across the study area which can be referred to as *spatial non-stationarity* (Brunsdon et al., 1996). In particular, Chandra et al. (2012) proposed a geographically weighted EBLUP (GWEBLUP) for unit-level SAE under spatial non-stationarity. This approach combines the geographically weighted regression (cf. Fotheringham et al., 2002) and the mixed model approach. Chandra et al. (2015) presented a spatially non-stationary area-level model for SAE. Alternative approaches for incorporating spatial effects in SAE based on non-parametric extensions to the mixed model have been proposed by Opsomer et al. (2008) and Ugarte et al. (2009). Here, spatial proximities in the data are captured by adding the geographic locations (the coordinates) into the model via bivariate P-spline smoothing (cf. Ruppert et al., 2003). P-spline models can handle situations where the target variable is affected by a spatial trend of arbitrary form. Opsomer et al. (2008) proposed a non-parametric EBLUP (NPEBLUP) under the non-parametric spatial P-spline model for unit-level SAE. Giusti et al. (2012) extended this idea for area-level SAE.

Approaches based on mixed models have in common that they rely on strong distributional assumptions. In particular, the error term components are assumed to be normally distributed. This implies that estimators evolved under a mixed model can be strongly influenced by departures from normality. These departures can be the result of extreme observations in the sample, so-called *outliers*. Chambers (1986) distinguishes between two categories of outlying observations: (i) the term *representative* outliers is used to describe correctly recorded observations which are extremely different compared to the whole sample and cannot be assumed to be unique in the population; (ii) *non-representative* can be described as incorrectly recorded observations in the sample. Outliers of the second type should be detected and corrected in the data editing process. If these values are unique in the population, they can be excluded from the estimation.

Outlier robust estimation methods have been vividly discussed in the field of SAE. Sinha and Rao (2009) used influence functions to estimate robust parameters in a linear mixed model and propose a robust EBLUP (REBLUP) for SAE. Chambers and Tzavidis (2006) propose an alternative for outlier robust SAE based on M-quantile methods which is a robust and distribution free approach for modeling the relationship between a dependent

and a set of explanatory variables at different quantiles of the target variable. Both methods, the REBLUP and the M-Quantile approach, can lead to efficiency gains in the presence of outliers but can produce biased small area estimators. Both ideas are developed under the assumption of non-representative outliers, implying that all non-sampled values have to follow the assumed robust working model and can hence be biased in the presence of representative outliers. Chambers et al. (2014) denote such approaches as *robust projective*. Dongmo-Jiongo et al. (2013) and Chambers et al. (2014) correct for the bias of the robust small area estimators by an additional correction term. Such bias-corrected robust estimators are denoted as *robust predictive* since these predict the contribution of non-sampled outliers in the population on the target statistic (Chambers et al., 2014).
Recently, robust methods have been combined with spatial models in SAE. In particular, Schmid and Münnich (2014) and Schmid et al. (2016) discuss robust projective and predictive estimators under spatially correlated random effects. Rao et al. (2014) propose a REBLUP approach under semi-parametric mixed models. Robust SAE under spatial non-stationarity has been discussed in the context of M-Quantiles (cf. Salvati et al., 2012). Robust EBLUP-based methods for SAE under spatial non-stationarity have not been considered in the literature so far.
As a contribution to the current literature, this thesis provides a robust extension to the GWEBLUP of Chandra et al. (2012). In particular, a robust projective and a robust predictive version of the GWEBLUP is proposed together with two analytic MSE estimates as measures of precision. More precisely, conditional MSE estimation is developed based on the pseudo-linearization approach of Chambers et al. (2011) and under the full linearization approach proposed by Chambers et al. (2014). In addition, the proposed methods are implemented using the R language (R Core Team, 2016) and provided as supplementary material to this thesis.

1.3 Outline

This thesis is organized around two main topics and is correspondingly divided into two parts. Part I contains the theoretical foundations for the proposed methodology of this thesis. Part II presents results of empirical evaluations that assess the relative performance of the proposed estimators. Part I consists of three chapters: Chapters 2 and 3 review the methodological background motivating the proposed robust method that is presented in Chapter 4. The review in Chapter 2 includes the classical unit-level EBLUP method and its spatial extensions in SAE. These methods are based on the linear mixed model which is introduced in Section 2.2. The EBLUP approach for SAE is introduced in Section 2.3. The EBLUP ignores spatial effects such as spatial dependence or spatial heterogeneity. Therefore, spatial extensions of the EBLUP are introduced in Section 2.4.

The review in Chapter 3 introduces the robust EBLUP approach in Section 3.1 including bias correction and conditional MSE estimations. Robust extensions of spatial EBLUP approaches are reviewed in Section 3.2.
So far, robust EBLUP-based methods for spatial non-stationary data has not been considered in the literature. Therefore, robust extensions of the GWEBLUP are proposed in Chapter 4. In particular, a robust projective and a robust predictive version of the GWEBLUP for the area mean are presented in Section 4.1. As measures of precision, two analytic solutions for conditional MSE estimation are presented in Section 4.2. In addition to the theoretical presentation, the proposed estimators are implemented in the package `saeRGW` which is presented in Section 4.3.
Part II consists of two chapters: a model-based simulation in Chapter 5 and a design-based simulation in Chapter 6. The model-based simulation in Chapter 5 compares the statistical performance of the reviewed and the proposed methods under different outlier scenarios. This includes scenarios with and without spatial stationarity in the model coefficients. Even though this thesis concentrates on small area statistics, robust and non-robust estimates of model the parameters under the geographically weighted linear mixed model are examined in Section 5.2. The performance of the area means is analyzed in Section 5.3 and that of the conditional MSE in Section 5.3. The stability of the estimation and calculation times are discussed in Section 5.5. The design-based simulation in Chapter 6 compares selected estimators under the realistic data situation of the Berlin apartment rental market. Here, the target statistic is the average of the quoted apartment net rents in urban planning areas in Berlin. The chapter includes a case study in Section 6.2 where the application of robust SAE methods under spatial non-stationarity is justified for the underlying data situation. The results of the simulation study are presented in Section 6.3.2. This includes a discussion regarding the performance of the area means and the conditional MSE estimates.
The main results of this thesis are summarized in Chapter 7. In addition, open research questions are also presented here.

Part I

Theory

EBLUP Approaches in Small Area Estimation 2

This chapter reviews selected methods for SAE. These methods are based on the assumptions of the linear mixed model (LMM). Therefore, the LMM is introduced in Section 2.2. This section includes parameter estimation under the LMM and introduces the empirical best unbiased predictor (EBLUP) together with the mean squared error (MSE) estimation for quantities of interest. The remaining sections review EBLUP-based small area estimators that make use of the LMM. In particular, the EBLUP of the small area mean is presented in Section 2.3. Thereafter, spatial extensions of the EBLUP that make use of auxiliary geographical information are reviewed in Section 2.4. This review includes the spatial EBLUP (cf. Petrucci et al., 2005; Pratesi and Salvati, 2008, 2009), the non-parametric EBLUP (cf. Opsomer et al., 2008), and the geographically weighted EBLUP (cf. Chandra et al., 2012). The latter approach is the basis for the proposed outlier robust SAE under spatial non-stationarity. Since this method requires unit-level information, this review is confined to unit-level models for SAE. To begin with, the following section introduces the notation that is used throughout this thesis and defines the target quantity.

2.1 Notation and Target Quantity

Consider a finite target population U of size N which is divided into M disjoint small areas such that $U = U_1 \cup \ldots \cup U_M$. The population units are denoted by j and the small areas by i. Each small area i contains a known number of units N_i, with $N = \sum_{i=1}^{M} N_i$. It is further assumed that a sample $s = s_1 \cup \ldots \cup s_m$ of size n is drawn from the population by using a non-informative sampling design where m denotes the number of sampled areas and n_i denotes the area-specific sample size with $n = \sum_{i=1}^{m} n_i$. A sampling design is non-informative when population models are valid for the population as well as the sampled observations (Pfeffermann and Sverchkov, 2009). Let y be the characteristic of interest, where y_{ij} denotes the realization of y for individual j in area i. The target quantity for this thesis is the population mean $\bar{y}_i$ of y in area i, which is defined by

$$\bar{y}_i = \frac{1}{N_i} \sum_{j \in U_i} y_{ij}. \tag{2.1}$$

Note that the target quantity can be defined differently, i.e., as the population total of the response variable. Under simple random sampling without replacement (SRSWOR), the design-based direct Horvitz-Thompson estimator of the area mean, denoted as $\hat{\bar{y}}_i^d$, is given by (cf. Horvitz and Thompson, 1952; Cochran, 1977)

$$\hat{\bar{y}}_i^d = \frac{1}{n_i} \sum_{j \in s_i} y_{ij}. \tag{2.2}$$

The sampling variability of this estimator is given by $Var(\hat{\bar{y}}_i^d) = (1 - n_i N_i^{-1}) S_i^2 / n_i$, where $S_i^2 = \sum_{j \in U_i} (y_{ij} - \bar{y}_i)/(N_i - 1)$ denotes the variance of the target variable in area i. An unbiased estimate of S_i^2 is given by $\hat{S}_i^2 = \sum_{j \in s_i} (y_{ij} - \hat{\bar{y}}_i^d)/(n_i - 1)$ (cf. Lehtonen and Veijanen, 2009, p.227). Under SRSWOR, the area-specific sample sizes n_i can be very small or even zero. Small sample sizes can cause the variability of $\hat{\bar{y}}_i^d$ to be unacceptably large. To achieve more accurate area-specific estimates, model-based estimation techniques can be employed. Among those, the LMM proved to be useful for developing population estimates of continuous target variables in the context of SAE.

2.2 The Linear Mixed Model

The class of LMMs allows us to take advantage of correlated data. Dependencies between observations are usually induced by a hierarchical data structure, i.e., when individual observations can be grouped into higher level observation units. In the context of SAE, observations can be grouped to a certain area. Due to unobserved heterogeneity between areas, the outcomes of a target variable from individuals of the same area can be more alike than the outcomes from different areas, i.e., observation within one area might be correlated. LMMs can account for correlations within the data by including random effects for higher level observation units in the model. Detailed introductions to LMM theory and applications can be found, for instance, in Faraway (2016) or McCulloch and Searle (2004). Following Jiang and Lahiri (2006) a general LMM with area-level random effects can be expressed as

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Zv} + \boldsymbol{e}, \tag{2.3}$$

where

$$\boldsymbol{v} \sim N(0, \boldsymbol{\Sigma}_v),$$
$$\boldsymbol{e} \sim N(0, \boldsymbol{\Sigma}_e).$$

Here $\boldsymbol{y}$ denotes a $(n \times 1)$ vector of the target variable, $\boldsymbol{X}$ is a $(n \times p)$ design matrix, which contains p explanatory variables, and $\boldsymbol{\beta}$ is a p-vector of fixed effects, which describe the linear relation of the respective covariates in $\boldsymbol{X}$

with the target variable. The unexplained part in model (2.3) consists of two components, the random effect part $\boldsymbol{Zv}$ and the $(n \times 1)$ vector of individual errors $\boldsymbol{e}$, where $\boldsymbol{Z}$ is a $(n \times mq)$ design matrix for the $(mq \times 1)$ vector of random effects $\boldsymbol{v}$ and q denotes the number of random parameters within $\boldsymbol{v}$. The included random effects add structure to the error term that accounts for variations between areas, which cannot be explained by the fixed part of the model. Note that in this thesis $q = 1$, as only random intercept models are considered. Both, $\boldsymbol{v}$ and $\boldsymbol{e}$, are assumed to be independently normally distributed with a mean of zero and variance matrix $\boldsymbol{\Sigma}_e$ and $\boldsymbol{\Sigma}_v$, respectively. Based on model (2.3) the marginal distribution of $\boldsymbol{y}$ can be expressed as

$$\boldsymbol{y} \sim N(\boldsymbol{X\beta}, \boldsymbol{V}), \tag{2.4}$$

where the variance matrix is given by

$$\boldsymbol{V}(\boldsymbol{\theta}) = \boldsymbol{Z}\boldsymbol{\Sigma}_v\boldsymbol{Z}^T + \boldsymbol{\Sigma}_e.$$

The variance components, $\boldsymbol{\Sigma}_v$ and $\boldsymbol{\Sigma}_e$, depend on a vector $\boldsymbol{\theta}$ of unknown variance parameters.

2.2.1 Best Linear Unbiased Prediction

Under model (2.3), quantities of interest for the target variable, such as the population total or the mean for the target variable, can be expressed as a linear combination of the model parameters $\boldsymbol{\beta}$ and $\boldsymbol{v}$

$$\mu = \boldsymbol{l}^T\boldsymbol{\beta} + \boldsymbol{m}^T\boldsymbol{v}, \tag{2.5}$$

where $\boldsymbol{l}$ and $\boldsymbol{m}$ are specified vectors and μ denotes the quantity of interest. For known variance parameters $\boldsymbol{\theta}$, Henderson (1950) proposed the best linear unbiased predictor (BLUP) for μ, i.e., a predictor for μ, which minimizes the MSE in the class of linear unbiased estimators. The BLUP of μ can be expressed as

$$\tilde{\mu} = \tilde{\mu}(\boldsymbol{\theta}) = \boldsymbol{l}^T\tilde{\boldsymbol{\beta}} + \boldsymbol{m}^T\tilde{\boldsymbol{v}}, \tag{2.6}$$

where

$$\tilde{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}}(\boldsymbol{\theta}) = (\boldsymbol{X}^T\boldsymbol{V}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{V}^{-1}\boldsymbol{y} \tag{2.7}$$

is the best unbiased estimator (BLUE) of $\boldsymbol{\beta}$ and

$$\tilde{\boldsymbol{v}} = \tilde{\boldsymbol{v}}(\boldsymbol{\theta}) = \boldsymbol{\Sigma}_v\boldsymbol{Z}^T\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\tilde{\boldsymbol{\beta}}) \tag{2.8}$$

denotes the BLUP of $\boldsymbol{v}$. Proofs that (2.6) is the BLUP estimator for μ can be found in Henderson (1963) and Robinson (1991). Note, that $\tilde{\boldsymbol{\beta}}$ and $\tilde{\boldsymbol{v}}$ can be obtained by maximizing the joint density of $\boldsymbol{y}$ and $\boldsymbol{v}$ with respect to $\boldsymbol{\beta}$ and $\boldsymbol{v}$. This maximization is equivalent to maximizing the penalized likelihood function

$$l(\boldsymbol{\beta}, \boldsymbol{\theta}) = -\frac{1}{2}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{Z}\boldsymbol{v})^T \boldsymbol{\Sigma}_e^{-1}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{Z}\boldsymbol{v}) - \frac{1}{2}\boldsymbol{v}^T \boldsymbol{\Sigma}_v^{-1}\boldsymbol{v}. \quad (2.9)$$

Setting the first derivatives with respect to $\boldsymbol{\beta}$ and $\boldsymbol{v}$ equal to zero yields the ML estimating equations

$$\boldsymbol{X}^T \boldsymbol{\Sigma}_e^{-1}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{Z}\boldsymbol{v}) = \boldsymbol{0} \quad (2.10)$$

$$\boldsymbol{Z}^T \boldsymbol{\Sigma}_e^{-1}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{Z}\boldsymbol{v}) - \boldsymbol{\Sigma}_v^{-1}\boldsymbol{v} = \boldsymbol{0}. \quad (2.11)$$

For known $\boldsymbol{\theta}$, the solutions equations (2.10) and (2.11) are equivalent to $\tilde{\boldsymbol{\beta}}$ and $\tilde{\boldsymbol{v}}$ from (2.7) and (2.8), respectively. In Section 3.1, these estimating equations become relevant for developing robust estimators. In practice, the variance parameters $\boldsymbol{\theta}$ are usually unknown and need to be estimated. Substituting the unknown variance parameters with suitable estimators leads to the empirical best linear unbiased predictor (EBLUP) for μ

$$\hat{\mu} = \hat{\mu}(\hat{\boldsymbol{\theta}}) = \boldsymbol{l}^T \hat{\boldsymbol{\beta}} + \boldsymbol{m}^T \hat{\boldsymbol{v}}. \quad (2.12)$$

Here, $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{v}}$ are defined as in (2.7) and (2.8), respectively, but with estimated variance parameters $\hat{\boldsymbol{\theta}}$.

2.2.2 Estimation of the Variance Parameters

There exist various methods for estimating the variance parameters $\boldsymbol{\theta}$. Among those, the maximum likelihood (ML) and restricted maximum likelihood (REML) estimation are frequently applied in SAE. As the ML estimation of $\boldsymbol{\theta}$ becomes important in Section 3.1, where I introduce the robust EBLUP approach, some details are provided here. In what follows, I mainly refer to Rao and Molina (2015, Chapter 5.2.4).

Maximizing the density of $\boldsymbol{y}$ with respect to $\boldsymbol{\theta}$, which is equivalent to maximizing the log-likelihood function, would yield the ML estimates $\hat{\boldsymbol{\theta}}_{ML}$ of $\boldsymbol{\theta}$. Under the normality assumption stated in (2.4), the log-likelihood function of $\boldsymbol{y}$ is given by

$$l(\boldsymbol{\beta}, \boldsymbol{\theta}) = h - \frac{1}{2} log|\boldsymbol{V}| - \frac{1}{2}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})^T \boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}), \quad (2.13)$$

where h denotes an additive constant. Let θ_l be the lth element in $\boldsymbol{\theta}$. Then, the partial derivatives of $l(\boldsymbol{\beta}, \boldsymbol{\theta})$ with respect to θ_l is given by

$$s_l(\boldsymbol{\beta}, \boldsymbol{\theta}) = -\frac{1}{2} tr(\boldsymbol{V}^{-1}\frac{\partial \boldsymbol{V}}{\partial \theta_l}) + \frac{1}{2}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})^T \frac{\partial \boldsymbol{V}^{-1}}{\partial \theta_l}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}), \quad (2.14)$$

where $\partial \boldsymbol{V}^{-1}/\partial \theta_l = -\boldsymbol{V}^{-1}(\partial \boldsymbol{V}/\partial \theta_l)\boldsymbol{V}^{-1}$. Let $\mathcal{I}(\boldsymbol{\theta})$ be the matrix of expected second-order-derivatives of $-l(\boldsymbol{\beta}, \boldsymbol{\theta})$. Then the element (l, k) of $\mathcal{I}(\boldsymbol{\theta})$ is given by

$$\mathcal{I}_{l,k}(\boldsymbol{\beta}) = \frac{1}{2} tr(\boldsymbol{V}^{-1}\frac{\partial \boldsymbol{V}}{\partial \theta_l}\boldsymbol{V}^{-1}\frac{\partial \boldsymbol{V}}{\partial \theta_k}). \quad (2.15)$$

The ML estimator of $\boldsymbol{\theta}$ can be obtained iteratively using a Fisher-scoring algorithm, where the expression

$$\boldsymbol{\theta}^{(t+1)} = \boldsymbol{\theta}^{t} + \left[\mathcal{I}(\boldsymbol{\theta}^{(t)})\right]^{-1} s_l(\tilde{\boldsymbol{\beta}}(\boldsymbol{\theta}^{(t)}), \boldsymbol{\theta}^{(t)}), \quad t = 1, 2, \ldots \quad (2.16)$$

is updated until convergence is achieved. Note that $\tilde{\boldsymbol{\beta}}(\boldsymbol{\theta}^{(t)})$ is obtained by (2.7), with $\boldsymbol{\theta}^{(t)}$ treated as a known variance parameter, where t denotes the current iteration. At the convergence of equation (2.16), $\boldsymbol{\theta}^{(t)}$ is defined as the ML estimate $\hat{\boldsymbol{\theta}}_{ML}$ of $\boldsymbol{\theta}$. The ML estimator $\hat{\boldsymbol{\theta}}_{ML}$ does not account for the loss of degrees of freedom, due to the estimation of $\tilde{\boldsymbol{\beta}}(\boldsymbol{\theta}^{(t)})$, and is therefore biased. Patterson and Thompson (1971, 1974) introduced the unbiased REML method for the estimation of $\boldsymbol{\theta}$. The REML approach involves transforming $\boldsymbol{y}$, such that the likelihood of $\boldsymbol{y}^* = \boldsymbol{A}^T\boldsymbol{y}$ does not depend on the fixed effects any more. Here, $\boldsymbol{A}$ is a matrix of dimension $n \times (n-p)$ that is orthogonal to $\boldsymbol{X}$. Maximizing the log-likelihood of $\boldsymbol{y}^*$ with respect to $\boldsymbol{\theta}$ leads to the unbiased REML estimator $\hat{\boldsymbol{\theta}}_{RM}$ of $\boldsymbol{\theta}$. A comprehensive overview of further methods for estimating $\boldsymbol{\theta}$ can be found, e.g., in Jiang and Lahiri (2006). Technical details on estimation methods for the variance parameters can be found in Searle and Gruber (2016, Chapters 9 and 10).

2.2.3 Mean Squared Error Estimation

To judge the precision of an estimator it is necessary to assess its variability. A measurement frequently used in SAE is the MSE. Let a be an arbitrary estimator. Then, in general the MSE of a is given by

$$MSE(a) = Var(a) + Bias(a)^2, \quad (2.17)$$

where $Var(a)$ denotes the variance and $Bias(a)$ the bias of a, respectively. Assuming that model (2.3) holds, it can be shown that the EBLUP in (2.6) is asymptotically unbiased (cf. Kackar and Harville, 1981). Hence, under the model, the MSE in (2.17) reduces to the variance part since the bias is equal to zero. The MSE of the EBLUP can be derived in two steps: first, the MSE of the BLUP assuming known variance parameters $\boldsymbol{\theta}$; secondly, a term is added that takes the variability caused by the estimation of $\boldsymbol{\theta}$ into account. The MSE of the BLUP $\tilde{\mu}$ in equation (2.6) can be expressed as (cf. Rao and Molina, 2015, Chapter 5.2.2)

$$MSE(\tilde{\mu}) = E(\tilde{\mu} - \mu)^2 = g_1(\boldsymbol{\theta}) + g_2(\boldsymbol{\theta}), \quad (2.18)$$

where

$$g_1(\boldsymbol{\theta}) = \boldsymbol{m}^T \left(\boldsymbol{\Sigma}_v - \boldsymbol{\Sigma}_v \boldsymbol{Z}^T \boldsymbol{V}^{-1} \boldsymbol{\Sigma}_v\right) \boldsymbol{m},$$
$$g_2(\boldsymbol{\theta}) = \boldsymbol{d}^T \left(\boldsymbol{X}^T \boldsymbol{V}^{-1} \boldsymbol{X}\right)^{-1} \boldsymbol{d}.$$

Here, $\boldsymbol{d}^T = \boldsymbol{l}^T - \boldsymbol{b}\boldsymbol{X}$ and $\boldsymbol{b} = \boldsymbol{m}^T \boldsymbol{\Sigma}_v \boldsymbol{Z}^T \boldsymbol{V}^{-1}$. Note that $g_1(\boldsymbol{\theta})$ is the variance of $\tilde{\mu}$ for a known $\boldsymbol{\beta}$, and $g_2(\boldsymbol{\theta})$ is the variance induced by the estimation of $\boldsymbol{\beta}$.

To develop an MSE for the EBLUP in equation (2.12), the variation induced by the estimation of the variance parameters, needs to be taken into account. Kackar and Harville (1984) showed that the MSE of the EBLUP in (2.12) can be expressed as

$$MSE(\hat{\mu}) = MSE(\tilde{\mu}) + E\left[\tilde{\mu} - \hat{\mu}\right]^2, \tag{2.19}$$

where $MSE(\tilde{\mu})$ is given in (2.18). Thus, approximating $MSE(\hat{\mu})$ by $MSE(\tilde{\mu})$ can lead to an underestimation of $MSE(\hat{\mu})$, especially when the last term of (2.19) is large. In general, this term has no closed-form solution and needs to be approximated. Kackar and Harville (1984) provide a Taylor series approximation of that term. Based on this work, Prasad and Rao (1990) proposed an approximation, which can be expressed as

$$E\left[\tilde{\mu} - \hat{\mu}\right]^2 \approx tr\left[\Delta \boldsymbol{b}^T V (\Delta \boldsymbol{b}^T) V_{\hat{\boldsymbol{\theta}}}\right], \tag{2.20}$$

where $\Delta \boldsymbol{b}^T = \delta \boldsymbol{b}^T / \delta \boldsymbol{\theta}$, and $V_{\hat{\boldsymbol{\theta}}}$ is the asymptotic variance matrix of $\hat{\boldsymbol{\theta}}$. Thus, (2.19) can be expressed as

$$MSE(\hat{\mu}) \approx g_1(\boldsymbol{\theta}) + g_2(\boldsymbol{\theta}) + g_3(\boldsymbol{\theta}), \tag{2.21}$$

where $g_3(\boldsymbol{\theta})$ equals (2.20). In practical applications, the $MSE(\hat{\mu})$ needs to be estimated. According to (Rao and Molina, 2015, Chapter 5.2.6), this can be done by substituting $\hat{\boldsymbol{\theta}}$ for $\boldsymbol{\theta}$ in equation (2.21). Datta and Lahiri (2000) showed that this substitution will lead to approximately unbiased estimators for $g_2(\boldsymbol{\theta})$ and $g_3(\boldsymbol{\theta})$. However, the estimation of $g_1(\boldsymbol{\theta})$ is biased. The bias of $g_1(\hat{\boldsymbol{\theta}})$ can be approximated by $-g_3(\boldsymbol{\theta})$, when $\boldsymbol{\theta}$ is obtained by an unbiased estimator, such as the REML estimator. Thus, an unbiased estimator for $MSE(\hat{\mu})$ is given by

$$\widehat{MSE}(\hat{\mu}) = g_1(\hat{\boldsymbol{\theta}}) + g_2(\hat{\boldsymbol{\theta}}) + 2g_3(\hat{\boldsymbol{\theta}}). \tag{2.22}$$

When the estimation of $\boldsymbol{\theta}$ is biased, as in the ML approach, the bias of $\hat{\boldsymbol{\theta}}$ needs to be taken into account for the MSE estimation. In that case, an estimator of $MSE(\hat{\mu})$ can be approximated by (cf. Prasad and Rao, 1990)

$$\widehat{MSE}(\hat{\mu}) = g_1(\hat{\boldsymbol{\theta}}) + g_2(\hat{\boldsymbol{\theta}}) + 2g_3(\hat{\boldsymbol{\theta}}) - \boldsymbol{c}^T(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}) \nabla g_1(\hat{\boldsymbol{\theta}}), \tag{2.23}$$

where $\boldsymbol{c}^T(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta})$ is an approximation of the bias of $\hat{\boldsymbol{\theta}}$, and $\nabla g_1(\hat{\boldsymbol{\theta}})$ denotes the vector of first derivatives of $g_1(\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$, evaluated at $\hat{\boldsymbol{\theta}}$. Proofs for the unbiasedness of the MSE estimates in equations (2.22) and (2.23) are provided in Das et al. (2004).

2.2.4 Discussion

The LMM is useful for modeling continuous outcome variables. In some cases however, it might be necessary to model discrete outcomes, such as categorial or count data. In the class of generalized linear mixed models (GLMM) the distributional assumption for the target variable is relaxed. Here, the target variable is assumed to follow a distribution that belongs to the exponential family, which includes the normal, the binomial, the Poisson or the multinomial distribution. Detailed information on the theory and applications of GLMMs is provided, for instance, in Jiang (2006) and McCulloch and Searle (2004).
Under the LMM in equation (2.3), the variance matrices $\boldsymbol{\Sigma}_e$ and $\boldsymbol{\Sigma}_v$ for the error term components have a very general form which provides some flexibility in modeling the error term. This feature of the LMM becomes important in Section 2.4, where selected spatial extensions of the EBLUP approach are reviewed.
As the LMM assumes normality of the error term components, the parameter estimation under this model is sensitive to outliers. When it comes to small sample sizes as in SAE, the impact of outliers on the parameter estimation can be even more severe. In the past, estimation strategies in the presence of outliers have been developed for special cases of the LMM. A review of these developments is given in Chapter 3. The outlier sensitivity of the LMM is also the main motivation for the proposed method in this thesis, where an outlier robust parameter estimation for LMMs under spatial non-stationarity is developed.

2.3 The EBLUP of the Small Area Mean

The LMM, introduced in the last section, provides a useful framework for SAE to estimate small area statistics for continuous target variables. Using a special case of the LMM in (2.3), Battese et al. (1988) introduced an EBLUP for small area means based on unit-level data. Their aim was to predict the crop area at the county level in Iowa (USA), using survey data from the U.S. Department of Agriculture and additional satellite data.
This section provides details regarding this unit-level EBLUP approach for SAE. This includes the estimation of the small area mean in Section 2.3.1 and its MSE estimation in Section 2.3.2.

2.3.1 Estimation of the Area Mean

The Battese, Harter and Fuller (BHF) model, introduced in Battese et al. (1988), is a linear nested error regression model. This is a special case of the LMM from (2.3), where $\boldsymbol{v}$ contains area-specific random intercepts. Analogously to (2.3), the BHF model can be expressed as

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{v} + \boldsymbol{e}, \tag{2.24}$$

where

$$\boldsymbol{v} \sim N(0, \boldsymbol{\Sigma}_v),$$
$$\boldsymbol{e} \sim N(0, \boldsymbol{\Sigma}_e).$$

In this model the number of random parameters is $q = 1$ and thus the dimension of $\boldsymbol{Z}$ reduces to $(n \times m)$. Without loss of generality, it is assumed that the units are sorted by areas. In that case $\boldsymbol{y} = col_{1\leq i\leq m} col_{1\leq j\leq n_i}(y_{ij})$, $\boldsymbol{X} = col_{1\leq i\leq m} col_{1\leq j\leq n_i}(\boldsymbol{x}_{ij})$ and $\boldsymbol{Z} = col_{1\leq i\leq m} col_{1\leq j\leq n_i}(\boldsymbol{z}_{ij})$, where $\boldsymbol{x}_{ij}$ is the vector of explanatory variables for individual j in area i, and $\boldsymbol{z}_{ij}$ the respective vector in $\boldsymbol{Z}$. Then $\boldsymbol{Z} = diag_{1\leq i\leq m}(\mathbf{1}_{n_i})$, where $\mathbf{1}_{n_i}$ is a vector of ones with length n_i. The variance matrix $\boldsymbol{\Sigma}_e$ can be expressed as $\boldsymbol{\Sigma}_e = \sigma_e^2 \boldsymbol{I}_n$, where σ_e^2 denotes the error term variance and $\boldsymbol{I}_n$ is an identity matrix of dimension n. Under the assumption of independent random effects, the variance matrix $\boldsymbol{\Sigma}_v$ simplifies to $\boldsymbol{\Sigma}_v = \sigma_v^2 \boldsymbol{I}_m$, where σ_v^2 denotes the variance of the random effects and I_m is an identity matrix of dimension m. Under the BHF model in (2.24) the variance matrix $\boldsymbol{V} = diag_{1\leq i\leq m}(\boldsymbol{V}_i)$ is a block-diagonal matrix with $\boldsymbol{V}_i = \sigma_e^2 \boldsymbol{I}_{n_i} + \sigma_v^2 \mathbf{1}_{n_i} \mathbf{1}_{n_i}^T$, where $\boldsymbol{I}_{n_i}$ is an identity matrix of dimension n_i. Let $\boldsymbol{l}_{ij} = \boldsymbol{x}_{ij}$ and $\boldsymbol{m}_{ij} = \boldsymbol{z}_{ij}$. According to equation (2.12), the EBLUP of the target variable for an individual j in area i can then be expressed as

$$\begin{aligned} \hat{\mu}_{ij} = \hat{y}_{ij} &= \boldsymbol{l}_{ij}^T \hat{\boldsymbol{\beta}} + \boldsymbol{m}_{ij}^T \hat{\boldsymbol{v}} \\ &= \boldsymbol{x}_{ij}^T \hat{\boldsymbol{\beta}} + \hat{v}_i, \end{aligned} \tag{2.25}$$

where $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{v}}$ are defined as in (2.12), and $\hat{v}_i$ denotes the area-specific element from $\hat{\boldsymbol{v}}$ for area i. Assuming that model (2.24) also holds for the non-sampled population units, the EBLUP of the small area mean $\bar{y}_i$ in area i can be expressed as

$$\begin{aligned} \hat{\bar{y}}_i^{EBLUP} &= N_i^{-1}\Big\{\sum_{j\in s_i} y_{ij} + \sum_{j\in r_i} \hat{y}_{ij}\Big\} \\ &= N_i^{-1}\Big\{\sum_{j\in s_i} y_{ij} + \sum_{j\in r_i} (\boldsymbol{x}_{ij}^T \hat{\boldsymbol{\beta}} + \hat{v}_i)\Big\}, \end{aligned} \tag{2.26}$$

where s_i and r_i denote the sampled and the non-sampled part in area i, respectively. For unit-level small area models, it is generally assumed that the explanatory variables can be observed for the samples and non-sampled units. This assumption can be relaxed in some cases, such that only the population means $\bar{\boldsymbol{x}}_i$ need to be known. Note that $(N_i - n_i)^{-1} \sum_{j \in r_i} \boldsymbol{x}_{ij}^T \hat{\boldsymbol{\beta}} = \bar{\boldsymbol{x}}_{ir}^T \hat{\boldsymbol{\beta}}$, where $\bar{\boldsymbol{x}}_{ir}$ denotes the area mean of the explanatory variables from the non-sampled part of the population. Let $\bar{\boldsymbol{x}}_{is}$ denote the area mean of the explanatory variables from the sample, then $\bar{\boldsymbol{x}}_{ir} = (N_i \bar{\boldsymbol{x}}_i - n_i \bar{\boldsymbol{x}}_{is})/(N_i - n_i)$. It follows that for the EBLUP in (2.25) it is sufficient to know the area means of the explanatory variables (cf. Rao and Molina, 2015, p.179). For non-sampled areas the EBLUP of the area mean in (2.25) reduces to a synthetic estimator $\hat{\bar{y}}_i^{EBLUP-Syn} = \bar{\boldsymbol{x}}_i^T \hat{\boldsymbol{\beta}}$ with an area-specific random effect equal to zero.

2.3.2 Mean Squared Error Estimation

The MSE estimation introduced in Section 2.2.3 can be considered to be unconditional in the sense that it does not depend on the realizations of the random effects. Here, the expectation of the squared prediction error in equation (2.18) is averaged over the assumed distribution of the random effects. The unconditional MSE in equation (2.18) is valid under the assumed model in (2.3). However, when the assumptions of this model are violated, the unconditional MSE estimation can be biased. Hence, Chambers et al. (2011) proposed a conditional MSE, which is bias-robust with respect to model misspecifications regarding the variance parameters in the underlying model. Their MSE is conditional in the sense that it treats the random effects as fixed, but unknown. The unconditional MSE for the EBLUP of the small area mean in equation (2.26) is presented in Section 2.3.2.1 followed by the conditional MSE in Section 2.3.2.2.

2.3.2.1 Unconditional Mean Squared Error Estimation

Based on the MSE estimation in equation (2.22), Prasad and Rao (1990) proposed MSE estimates for special cases of the LMM, which are frequently used in SAE. Among those, they provided an MSE estimate for the EBLUP of the area mean in equation (2.26), which is based on the nested error regression model in (2.24). The prediction error of the EBLUP in (2.26) can be rearranged as

$$\begin{aligned} \hat{\bar{y}}_i^{EBLUP} - \bar{y}_i &= N_i^{-1} \Big\{ \sum_{j \in s_i} y_{ij} + \sum_{j \in r_i} \hat{y}_{ij} \Big\} - N_i^{-1} \sum_{j \in U_i} y_{ij} \\ &= N_i^{-1} \Big\{ \sum_{j \in r_i} \hat{y}_{ij} - \sum_{j \in r_i} y_{ij} \Big\} \\ &= (1 - f_i) \Big\{ \hat{\bar{y}}_{ir} - \mu_{ir} - \bar{e}_{ir} \Big\}, \end{aligned} \tag{2.27}$$

where $\mu_{ir} = \bar{\boldsymbol{x}}_{ir}^T\boldsymbol{\beta} + v_i$ is the true area mean, and $\bar{e}_{ir}$ the mean of the errors from the non-sampled units in area i. Furthermore, $\hat{\bar{y}}_{ir} = \hat{\mu}_{ir}$ denotes the EBLUP of μ_{ir}, with $\boldsymbol{l}_i = \bar{\boldsymbol{x}}_{ir}$ and $\boldsymbol{m}_i = \boldsymbol{z}_i$, where $\boldsymbol{z}_i$ is 1 at the ith position and 0 otherwise. Based on the prediction error from (2.27), the unconditional MSE of the EBLUP in (2.26) can then be expressed as

$$MSE(\hat{\bar{y}}_i^{EBLUP}) = (1-f_i)^2\Big\{MSE(\hat{\mu}_{ir}) + (N_i - n_i)^{-1}\sigma_e^2\Big\}, \tag{2.28}$$

where, analogously to equation (2.19), $MSE(\hat{\mu}_{ir})$ is defined by

$$MSE(\hat{\mu}_{ir}) = g_{1i}(\boldsymbol{\theta}) + g_{2i}(\boldsymbol{\theta}) + g_{3i}(\boldsymbol{\theta}). \tag{2.29}$$

Prasad and Rao (1990) showed that for the BHF case $g_{1i}(\boldsymbol{\theta})$, $g_{2i}(\boldsymbol{\theta})$ and $g_{3i}(\boldsymbol{\theta})$ simplify to the following terms

$$\begin{aligned} g_{1i}(\boldsymbol{\theta}) =& (1-\gamma_i)\sigma_v^2 \\ g_{2i}(\boldsymbol{\theta}) =& (\bar{\boldsymbol{x}}_{ir} - \gamma_i\bar{\boldsymbol{x}}_{is})^T \left(\boldsymbol{X}^T\boldsymbol{V}^{-1}\boldsymbol{X}\right)^{-1} (\bar{\boldsymbol{x}}_{ir} - \gamma_i\bar{\boldsymbol{x}}_{is}) \\ g_{3i}(\boldsymbol{\theta}) =& n_i^{-2}(\sigma_v^2 + \sigma_e^2/n_i)^{-3} \\ & \times \left[\sigma_e^4 Var(\hat{\sigma}_v^2) + \sigma_v^4 Var(\hat{\sigma}_e^2) - 2\sigma_e^2\sigma_v^2 Cov(\hat{\sigma}_e^2, \hat{\sigma}_v^2)\right], \end{aligned}$$

where $\gamma_i = \sigma_v^2(\sigma_v^2 + \sigma_e^2/n_i)^{-1}$. Expressions for the covariance $Cov(\hat{\sigma}_e^2, \hat{\sigma}_v^2)$ and the variances $Var(\hat{\sigma}_e^2)$ and $Var(\hat{\sigma}_v^2)$ of σ_e^2 and σ_v^2, respectively, are provided in Prasad and Rao (1990). For the estimation of the unconditional MSE in (2.29), the variance parameters σ_e^2 and σ_e^2 need to be substituted by suitable estimates $\hat{\sigma}_e^2$ and $\hat{\sigma}_e^2$, respectively. Analogously to equation (2.22) for unbiased estimations of σ_e^2 and σ_e^2, such as REML estimates, an unbiased estimator for (2.29) is given by

$$\widehat{MSE}(\hat{\mu}_{ir}) = g_{1i}(\hat{\boldsymbol{\theta}}) + g_{2i}(\hat{\boldsymbol{\theta}}) + 2g_{3i}(\hat{\boldsymbol{\theta}}). \tag{2.30}$$

For ML estimators of σ_e^2 and σ_e^2, an unbiased estimator for (2.29) can be obtained by

$$\widehat{MSE}(\hat{\mu}_{ir}) = g_{1i}(\hat{\boldsymbol{\theta}}) + g_{2i}(\hat{\boldsymbol{\theta}}) + 2g_{3i}(\hat{\boldsymbol{\theta}}) - \boldsymbol{c}^T(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta})\nabla g_1(\hat{\boldsymbol{\theta}}), \tag{2.31}$$

where an expression for $\boldsymbol{c}^T(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta})$ is given in Rao and Molina (2015, Chapter 7.2.2). It follows that an unbiased estimator for the unconditional MSE in (2.28) is given by

$$\widehat{MSE}(\hat{\bar{y}}_i^{EBLUP}) = (1-f_i)^2\Big\{\widehat{MSE}(\hat{\mu}_{ir}) + (N_i - n_i)^{-1}\hat{\sigma}_e^2\Big\}, \tag{2.32}$$

where $\widehat{MSE}(\hat{\mu}_{ir})$ can be obtained by equation (2.30) or (2.31) depending on the method used for the estimation of $\hat{\boldsymbol{\theta}}$.

2.3.2.2 Conditional Mean Squared Error Estimation

To develop a conditional MSE for the EBLUP of the area mean in equation (2.26), Chambers et al. (2011) proposed applying a pseudo-linearization approach, which is based on rewriting the EBLUP estimator as a weighted sum of the sampled values of the target variable $\boldsymbol{y}$. Recall that the EBLUP for the mean $\bar{y}_i$ in area i from equation (2.26) is defined as

$$\hat{\bar{y}}_i^{EBLUP} = N_i^{-1}\Big\{\sum_{j\in s_i} y_{ij} + \sum_{j\in r_i}(\boldsymbol{x}_{ij}^T\hat{\boldsymbol{\beta}} + \hat{v}_i)\Big\}.$$

Given the definition of $\hat{\boldsymbol{\beta}}$ and $\hat{v}_i$ from equation (2.25), the latter expression can be reformulated as

$$\begin{aligned}\hat{\bar{y}}_i^{EBLUP} &= N_i^{-1}\sum_{j\in s_i} y_{ij} \\ &\quad + N_i^{-1}\sum_{j\in r_i}\left[\boldsymbol{x}_{ij}^T\underbrace{(\boldsymbol{X}^T\hat{\boldsymbol{V}}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}^T\hat{\boldsymbol{V}}^{-1}}_{=\boldsymbol{A}}\boldsymbol{y}\right] \\ &\quad + N_i^{-1}\sum_{j\in r_i}\left[\boldsymbol{z}_i^T\underbrace{\hat{\boldsymbol{\Sigma}}_v\boldsymbol{Z}^T\hat{\boldsymbol{V}}^{-1}}_{=\boldsymbol{Q}}(\boldsymbol{I}_n - \boldsymbol{X}\boldsymbol{A})\boldsymbol{y}\right] \\ &= N_i^{-1}\left[\boldsymbol{\delta}_i^T + (N_i - n_i)\left\{\bar{\boldsymbol{x}}_{ri}^T\boldsymbol{A} + \boldsymbol{z}_i^T\boldsymbol{Q}(\boldsymbol{I}_n - \boldsymbol{X}\boldsymbol{A}\right\}\right]\boldsymbol{y} \\ &= \boldsymbol{d}_i^T\boldsymbol{y}. \end{aligned} \tag{2.33}$$

Here, $\boldsymbol{\delta}_i$ is an indicator vector of size n which is one for the sampled units in area i and zero otherwise, and $\boldsymbol{d}_i$ is a vector of weights specific to area i with $\sum_{j\in s} d_{ij} = 1$. The vector of weights $\boldsymbol{d}_i$ in (2.33) depends on the estimated variance parameters $\hat{\boldsymbol{\theta}}$. As estimates of $\boldsymbol{\theta}$ itself depend on the target variable $\boldsymbol{y}$, this approach is called pseudo-linearization. Chambers et al. (2011) used expression (2.33) to develop an approximation for the conditional MSE for the EBLUP in (2.26), which can be estimated by

$$\widehat{MSE}_{CCT}(\hat{\bar{y}}_i^{EBLUP}) = \widehat{Var}_{\boldsymbol{v}}(\hat{\bar{y}}_i^{EBLUP}) + \widehat{Bias}_{\boldsymbol{v}}(\hat{\bar{y}}_i)^2, \tag{2.34}$$

where the first term on the right-hand side denotes an estimate for the conditional prediction variance, and the second term denotes the squared prediction bias of (2.26). The subscript $\boldsymbol{v}$ indicates that the terms are conditioned on the realization of the random effects. According to the surnames of the authors who contributed to Chambers et al. (2011), this MSE estimator is referred to by adding the subscript CCT. Chambers et al. (2011) showed that the conditional prediction bias can be estimated by

$$\widehat{Bias}_{\boldsymbol{v}}(\hat{\bar{y}}_i^{EBLUP}) = \sum_{j\in s} d_{ij}\hat{\mu}_j - N_i^{-1}\sum_{j\in U_i}\hat{\mu}_j, \tag{2.35}$$

where $\hat{\mu}_j$ is an unbiased linear estimator of the conditional expected value $\mu_j = E(y_j|\boldsymbol{x}_j, \boldsymbol{v})$ under model (2.24), which is defined in equation (2.25). Following the same principal as in (2.33), $\hat{\mu}_j$ can be expressed as

$$\hat{\mu}_j = \sum_{k \in s} \gamma_{kj} y_k = \boldsymbol{\gamma}_j^T \boldsymbol{y}, \tag{2.36}$$

where

$$\boldsymbol{\gamma}_j^T = \boldsymbol{x}_j^T \boldsymbol{A} + \boldsymbol{z}_i^T \boldsymbol{Q}(\boldsymbol{I}_n - \boldsymbol{X}\boldsymbol{A})$$

is a vector of weights specific to unit j. Due to the shrinkage effect, which can lead to biased estimates of μ_j, Chambers et al. (2011) recommend computing $\hat{\mu}_j$ as the *unshrunken* version $\hat{\mu}_j^u$, which is defined as $\hat{\mu}_j$ in (2.36) but with matrix $\boldsymbol{Q}$ replaced by $\boldsymbol{Q}^u = (\boldsymbol{Z}\boldsymbol{Z}^T)^{-1}\boldsymbol{Z}^T$. Let $I(j \in i)$ be an indicator function which is equal to 1 whenever unit j is in area i. Then, an estimate of the conditional prediction variance is given by

$$\widehat{Var}_{\boldsymbol{v}}(\hat{\bar{y}}_i^{EBLUP}) = N_i^{-2} \sum_{j \in s} \left\{ a_{ij}^2 + (N_i - n_i)n^{-1} \right\} \hat{\lambda}_j^{-1} (y_j - \hat{\mu}_j)^2, \tag{2.37}$$

with $a_{ij} = N_i d_{ij} - I(j \in i)$, $\hat{\mu}_j$ is defined as in (2.35), and $\hat{\lambda}_j = 1 - 2\gamma_{jj} + \sum_{k \in s} \gamma_{kj}^2$ is a scaling constant, specific to unit j, where the weights γ_{kj} are defined by (2.36). Note that when $\hat{\mu}_j$ is replaced by its unshrunken version $\hat{\mu}_j^u$, $\hat{\lambda}_j$ is of order $1 + O(n^{-1})$ and can be set to $\hat{\lambda}_j = 1$.

The synthetic estimator for non-sampled areas can also be expressed in the pseudo-linear form

$$\hat{\bar{y}}_i^{EBLUP-Syn} = \bar{\boldsymbol{x}}^T \hat{\boldsymbol{\beta}} = (\boldsymbol{d}_i^{Syn})^T \boldsymbol{y}, \tag{2.38}$$

with $\boldsymbol{d}_i^{Syn} = \boldsymbol{A}^T \bar{\boldsymbol{x}}_i$ and $\boldsymbol{A}$ defined as in (2.33). However, the conditional MSE estimator in (2.34) is not applicable for the synthetic estimator since equation (2.35) cannot be used for estimating the area-specific conditional bias. Chambers et al. (2011) suggest to use the expected bias under model (2.24) and estimate the conditional expectation of the square of this bias. The expected bias under model (2.24) is given by

$$E[\hat{\bar{y}}_i^{EBLUP-Syn} - \bar{y}_i] = \sum_{j \in s} d_{ij}^{Syn} (\boldsymbol{x}_j^T \boldsymbol{\beta} + v_i) - \bar{\boldsymbol{x}}_i^T \boldsymbol{\beta} - v_i. \tag{2.39}$$

The conditional expectation of the square of this expression is given by

$$E_{\boldsymbol{v}} \left\{ E^2[\hat{\bar{y}}_i^{EBLUP-Syn} - \bar{y}_i] \right\} = \left\{ \sum_{j \in s} d_{ij}^{Syn} (\boldsymbol{x}_j^T \boldsymbol{\beta} + v_i) - \bar{\boldsymbol{x}}_i^T \boldsymbol{\beta} \right\}^2 + \sigma_v^2. \tag{2.40}$$

Thus, for non-sampled areas the squared bias of the synthetic estimator in (2.38) can be estimated by

$$\widehat{Bias}_{\boldsymbol{v}}(\hat{\bar{y}}_i^{EBLUP-Syn})^2 = \left\{ \sum_{j \in s} d_{ij}^{Syn}(\boldsymbol{x}_j^T \hat{\boldsymbol{\beta}} + \hat{v}_i^u) - \bar{\boldsymbol{x}}_i^T \hat{\boldsymbol{\beta}} \right\}^2 + \hat{\sigma}_v^2, \qquad (2.41)$$

where $\hat{v}_i^u$ is the unshrunken estimated random effect. The conditional MSE of the synthetic EBLUP can be estimated using (2.34) where the conditional bias is replaced by (2.41).

The conditional MSE based on pseudo-linearization ignores the extra variability due to the estimation of $\boldsymbol{\theta}$, and can therefore be seen as a first-order approximation of the conditional MSE of the EBLUP. This approximation can lead to an underestimation when the EBLUP of the area mean (2.25) varies substantially with $\boldsymbol{\theta}$ and the variability of $\hat{\boldsymbol{\theta}}$ is large. However, the conditional MSE presented here can easily be developed for EBLUP estimators that can be written in a pseudo-linear form. Chambers et al. (2011) emphasize that this is one main advantage of the pseudo-linearization approach compared to the unconditional MSE estimation in Section 2.3.2.1. In Section 4.2 this advantage is used to develop a conditional MSE of the proposed estimator in this thesis.

2.3.2.3 Alternative Methods

For complex estimators, analytic expressions for the MSE estimation can be intractable. When analytic solutions for the MSE estimation are not available resampling methods can be applied as an alternative. A comprehensive overview of resampling methods is given, for instance, in Efron (1982). Several resampling methods have been proposed for the MSE estimation in the context of SAE. Among those are the parametric bootstrap approaches of Hall and Maiti (2006b) or Chatterjee et al. (2008), the non-parametric bootstrap of Hall and Maiti (2006a), the semi-parametric block bootstrap for hierarchically clustered data suggested by Chambers and Chandra (2013) and the jackknife method of Jiang et al. (2002).

Depending on the sample size, MSE estimations based on resampling methods can be computationally demanding. With increasing computing capacity, this argument becomes less important. However, since the proposed method in this thesis is computationally very demanding, even for common sample sizes, resampling methods are not feasible for the MSE estimation. Therefore, they will not be discussed any further detail in this thesis. References will be made, though, when necessary.

2.4 Spatial Extensions of the EBLUP Approach

This section provides an overview regarding spatial EBLUP-based estimators in SAE. The EBLUP of the area mean introduced in Section 2.3 is optimal in terms of efficiency under the assumptions of the LMM in (2.24). However, spatial effects are ignored in the underlying model assumptions. According to Anselin (1988, Chapter 2.2), spatial effects can be divided into spatial dependence and spatial heterogeneity. When spatial dependency is present, outcomes from different locations are related. Under spatial heterogeneity the functional forms and parameters of a model can vary with the geographic location.
A brief introduction to spatial data structures follows in Section 2.4.1 as well as a review of extensions to the EBLUP that account for spatial effects in the remaining subsections. In particular, the spatial EBLUP (Petrucci et al., 2005; Pratesi and Salvati, 2008, 2009, cf.) that accounts for spatial dependencies by allowing for spatial autocorrelation in the random effects is presented in Section 2.4.2. Section 2.4.3 describes the non-parametric EBLUP of Opsomer et al. (2008) which can account for a spatial trend of unknown functional form in the target variable. The geographically weighted EBLUP of Chandra et al. (2012) that can capture spatial heterogeneity by allowing the model parameters to vary over space is reviewed in Section 2.4.2. Note that the latter approach is the basis for the proposed robust method that is presented in Chapter 4 of this thesis.

2.4.1 Spatial Data Structures

According to Cressie (1993), spatial data can be classified into three categories: (i) geostatistical data; (ii) areal data; and (iii) point pattern data. Let D be the site where data is collected. For geostatistical data, quantities of interest can be measured continuously within the whole range of D. For example, air pollution can be recorded at arbitrary locations within the border of a city or a specified region. For areal data, quantities can only be measured at a fixed set of locations, which can be defined as points or polygons. For point data, D can be partitioned in a lattice or mosaic of polygons, where each polygon contains at least one measurement point. A mosaic is defined as an irregular arrangement of neighboring areas, whereas a lattice has a rectangular arrangement. Individual income data, for instance, is usually measured anonymously, such that observations cannot be associated with a certain location or address, but to subregions of a sampling area (D), e.g., municipalities. Both types, geostatistical and areal data, provide the possibility to explore neighborhood structures which are described by the arrangement of polygons or the distance between spatial points. In contrast, for point pattern data, the locations of points are of main interest. Here, D consists of locations in an area and the primary interest lies in the spatial distribution of these points. An example is a forest

(D), where each tree can be directly located. In that case the distribution pattern of certain species of trees can be of interest (cf. Plant, 2012, Chapter 1.2).
In survey sampling, data is collected at certain locations. In SAE, the interest lies in producing reliable estimates for subregions of a population, using the survey data and additional information. Therefore, geostatistical and areal data types are relevant in SAE as a source of such additional information. When the survey data can be connected to spatial information, the spatial data structure can be exploited to improve the estimates via spatial modeling. In the remainder of this section different strategies for spatial modeling in SAE are reviewed.

2.4.2 The Spatial EBLUP

The BHF model in (2.24) accounts for unobserved heterogeneity between the areas by including area-specific random intercepts. By assumption, these random effects are uncorrelated. However, when spatial correlation between neighboring areas is present this assumption becomes implausible. In theory, one way to take spatial dependencies into account is to include strong covariates into the model that explain the spatial correlation. In practice however, it can be difficult to find such covariates. In the absence of appropriate variables it is possible to redefine the model assumptions and include spatial dependencies into the model (cf. Molina et al., 2009). One possible way to account for the spatial correlation is to allow for spatially correlated random effects. Spatial correlation in the random effects can be modeled using a simultaneous autoregressive (SAR) or a conditional autoregressive (CAR) process. According to Shekhar and Xiong (2007, p.1104) the CAR model is appropriate when a first-order dependency is present, i.e., the spatial correlation is rather local. SAR models, on the other hand, are more suitable in situations with a spatial dependency of higher order or a more global spatial correlation. In the context of area-level SAE, Petrucci et al. (2005) and Pratesi and Salvati (2008, 2009) extended the random intercept model in (2.24) by assuming a SAR process for the area-specific random effects. Chandra et al. (2007) considered the BHF model in (2.24) for unit-level data and extended the EBLUP from equation (2.26) to account for spatial correlation between areas in the unobserved part of the model. The SAR process of order one for the random effects is defined by

$$\boldsymbol{v} = \rho \boldsymbol{W} \boldsymbol{v} + \boldsymbol{u}, \tag{2.42}$$

where $\rho \in (-1, 1)$ describes the strength of the spatial correlation between neighboring areas (cf. Plant, 2012, Chapter 13.3.1). Furthermore, $\boldsymbol{u}$ has length m and is a normally distributed random variable with a mean of zero and variance matrix $\boldsymbol{\Sigma}_u = \sigma_u I_m$. The quadratic matrix $\boldsymbol{W}$ of dimension m describes the spatial contiguity, which is based on the spatial arrangement of the areas. Pratesi and Salvati (2009) defined $\boldsymbol{W}$ as a 0-1 contiguity matrix that describes the neighborhood structure between the areas with zero-valued elements for non-adjacent pairs of areas, and non-zero elements for neighboring areas. According to them, this definition for $\boldsymbol{W}$ can be a good choice for applications in SAE. Equation (2.42) can be rearranged as

$$\boldsymbol{v} = (\boldsymbol{I}_m - \rho \boldsymbol{W})^{-1} \boldsymbol{u}. \tag{2.43}$$

It follows that the LMM with a SAR process for $\boldsymbol{v}$ can be expressed as

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{v} + \boldsymbol{e}, \tag{2.44}$$

where

$$\boldsymbol{v} \sim N(0, \boldsymbol{\Sigma}_v),$$
$$\boldsymbol{e} \sim N(0, \boldsymbol{\Sigma}_e).$$

Here, the variance matrix $\boldsymbol{\Sigma}_e$ is defined as in equation (2.24) and $\boldsymbol{\Sigma}_v$ is defined by

$$\boldsymbol{\Sigma}_v = \sigma_u^2 \left[\left(\boldsymbol{I}_m - \rho \boldsymbol{W}^T \right) (\boldsymbol{I}_m - \rho \boldsymbol{W})^{-1} \right]. \tag{2.45}$$

In contrast to the nested error regression model in (2.24) the variance matrix of the composed error term $\boldsymbol{V} = \boldsymbol{Z}\boldsymbol{\Sigma}_v\boldsymbol{Z}^T + \boldsymbol{\Sigma}_e$ in model (2.44) is more complex as it is no longer block-diagonal. Thus, the estimation of the model parameter can be computationally more demanding. Under model (2.44), the spatial EBLUP (SEBLUP) for the small area mean $\bar{y}_i$ can be expressed as

$$\hat{\bar{y}}_i^{sp} = N_i^{-1} \Big\{ \sum_{j \in s_i} y_{ij} + \sum_{j \in r_i} (\boldsymbol{x}_{ij}^T \hat{\boldsymbol{\beta}}^{sp} + \hat{v}_i^{sp}) \Big\}, \tag{2.46}$$

where the superscript sp indicates that $\boldsymbol{v}$ follows the previously defined SAR process. Analogously to the EBLUP for the small area mean in equation (2.26), the parameter estimates $\hat{\boldsymbol{\beta}}^{sp}$ and $\hat{\boldsymbol{v}}^{sp}$ are defined as in equation (2.12). The unknown variance parameters σ_e, σ_u and the spatial correlation parameter ρ can be estimated by ML or REML estimation as introduced in Section 2.3. See Salvati (2004) for details of the ML and REML estimation of these parameters. Petrucci and Salvati (2006) provide analytic MSE estimators for the SEBLUP, based on ML and REML estimates for the unknown variance parameters, which are equivalent to equations (2.22) and (2.23) from Section 2.2.3, respectively. In addition, parametric and non-parametric bootstrap procedures for MSE estimation were proposed by Molina et al. (2009).

2.4.3 The Non-Parametric EBLUP

In the absence of variables that can explain the spatial correlation between population units, the geographical information itself can be exploited to serve as a proxi-variable. By incorporating the geographical information into the fixed part of the model, spatial trends in the target can be accounted for. One approach, which can take into account a spatial trends of arbitrary functional form, is the non-parametric EBLUP (NPEBLUP) approach proposed by Opsomer et al. (2008), where the geographical coordinates can be incorporated into the model via penalized splines (P-splines). This method is not limited to a spatial context as it can be applied whenever the functional form of the relationship between the target variable and the covariates cannot be specified in a parametric form. In what follows, the non-parametric P-spline approach is introduced along the lines of Ruppert et al. (2003). Afterwards, the NPEBLUP approach is presented in the context of SAE. In general, a relationship of unknown functional form can be expressed as

$$y_i = m(x_i) + \epsilon_i, \ i = 1, \dots, n, \tag{2.47}$$

where $m(\cdot)$ denotes an unknown function, and ϵ_i is a random error term with a mean of zero and variance σ_ϵ. For a univariate vector $\boldsymbol{x}$ of the explanatory variable x, function $m(x_i)$ can be approximated, using a spline function with a truncated polynomial spline basis Ruppert et al. (cf. 2003, Chapter 3):

$$m(\boldsymbol{x}, \boldsymbol{\beta}, \boldsymbol{\gamma}) = \beta_0 + \beta_1 \boldsymbol{x} + \dots \beta_t \boldsymbol{x}^t + \sum_{k=1}^{K} \gamma_k (\boldsymbol{x} - \kappa_k)_+^t \tag{2.48}$$

with

$$(\boldsymbol{x} - \kappa_k)_+^t = \begin{cases} (\boldsymbol{x} - \kappa_k)^t & \text{for } \boldsymbol{x} \geq \kappa_k \\ 0 & \text{else.} \end{cases}$$

Here t is the degree of the spline, $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_K)^T$ is the coefficient vector for the spline portion of the model, and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_t)^T$ are the coefficients for the parametric part of the model. K denotes the number of fixed knots $\kappa_k < \dots < \kappa_K$. In general, the knots are equally spaced at quantiles of $\boldsymbol{x}$, where the number of knots should be sufficient to fit equation (2.48) to the data without overfitting. Some cross validation procedures are presented in Ruppert (2002) for an optimal choice of K. Besides that, he offers some guidelines for a simple default choice of K where he suggests a knot every four observations with a maximum number of 40 knots. In the P-spline approach, a potential overfitting is prevented by a penalty term for the magnitude of the spline parameters $\boldsymbol{\gamma}$. Here, the parameters $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are chosen such that the penalized quadratic loss function

$$(\boldsymbol{y} - m(\boldsymbol{x}, \boldsymbol{\beta}, \boldsymbol{\gamma}))^2 + \lambda \boldsymbol{\gamma}^T \boldsymbol{\gamma} \tag{2.49}$$

is minimized. In the latter expression λ denotes a fixed penalty parameter which defines the smoothness of the spline function. The larger λ, the more the parameters $\boldsymbol{\gamma}$ tend toward zero, and the more spline function (2.48) is reduced to its parametric part. Let $\boldsymbol{X} = col_{1\leq i\leq n}[(1, x_i, x_i^2, \ldots, x_i^t)]$, and $\boldsymbol{D} = col_{1\leq i\leq n}(\boldsymbol{d}_i)$ with $\boldsymbol{d}_i^T = col_{1\leq k\leq K}[(x_i - \kappa_k)_+^t]$. Then, solving (2.49) is equivalent to estimating the parameters of a linear mixed model where the parameters $\boldsymbol{\gamma}$ are treated as random effects, which can be expressed as (cf. Wand, 2003):

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{D\gamma} + \boldsymbol{e}, \tag{2.50}$$

where

$$\boldsymbol{\gamma} \sim N(0, \boldsymbol{\Sigma}_\gamma),$$
$$\boldsymbol{e} \sim N(0, \boldsymbol{\Sigma}_e).$$

In this model the variance matrix $\boldsymbol{\Sigma}_e$ is defined as in model (2.24) and $\boldsymbol{\Sigma}_\gamma$ is defined by $\boldsymbol{\Sigma}_\gamma = \sigma_\gamma^2 \boldsymbol{I}_K$, where σ_γ^2 denotes the variance of $\boldsymbol{\gamma}$ and $\boldsymbol{I}_K$ is an identity matrix of dimension K. Similar to the LMM introduced in Section 2.2, the model parameters $\boldsymbol{\beta}$ can be obtained as in equation (2.6), with a variance matrix $\boldsymbol{V} = \boldsymbol{\Sigma}_e + \boldsymbol{D}\boldsymbol{\Sigma}_\gamma \boldsymbol{D}^T$. The BLUP for $\boldsymbol{\gamma}$ is given by

$$\tilde{\boldsymbol{\gamma}} = \boldsymbol{\Sigma}_\gamma \boldsymbol{D}^T \boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\tilde{\boldsymbol{\beta}}). \tag{2.51}$$

The EBLUP $\hat{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$ can be obtained with substituting the unknown variance parameters σ_γ and σ_e in equation (2.51) by suitable estimates. In a spatial context $\boldsymbol{x}_i = (x_{i1}, x_{i2})$ has two dimensions: the longitude and latitude of a geographical coordinate. Thus, bivariate basis functions are required for bivariate smoothing. Ruppert et al. (2003, p.254) recommended using a spline function with a transformed radial basis function for two or more dimensional P-splines. In a two-dimensional case, $\boldsymbol{D}$ is defined as

$$\boldsymbol{D} = col_{1\leqslant i\leqslant n}[\underset{1\leqslant k\leqslant K}{C(\boldsymbol{x}_i - \boldsymbol{\kappa}_k)}][\underset{1\leqslant k,k'\leqslant K}{C(\boldsymbol{\kappa}_k - \boldsymbol{\kappa}_{k'})}]^{-1/2}, \tag{2.52}$$

where $C(\boldsymbol{r}) = ||\boldsymbol{r}||^2 \log ||\boldsymbol{r}||$ is the radial basis function, $\boldsymbol{x}_i = (x_{1i}, x_{2i})$ represents the spatial coordinates of unit i, and $\boldsymbol{\kappa}_k$ is a two-dimensional knot. For two dimensional spline functions it is no longer possible to place the knots at quantiles of $\boldsymbol{x}$. The knots should rather be a subset of the observations, which should be scattered to cover the domain. Ruppert et al. (2003, p.257) suggest choosing the number of knots by $K = max[20, min(n/4, 150)]$ and applying a space-filling algorithm for the position of the knots. See, e.g., Nychka et al. (1998) for details on space-filling algorithms.
In the context of SAE, model (2.50) can be combined with the random intercept model (2.24) to obtain the NPEBLUP proposed by Opsomer et al. (2008). The underlying model for the NPEBLUP can be expressed as

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{D\gamma} + \boldsymbol{Zv} + \boldsymbol{e}, \tag{2.53}$$

where

$$\boldsymbol{\gamma} \sim N(0, \boldsymbol{\Sigma}_\gamma),$$
$$\boldsymbol{v} \sim N(0, \boldsymbol{\Sigma}_v),$$
$$\boldsymbol{e} \sim N(0, \boldsymbol{\Sigma}_e).$$

Here, matrix $\boldsymbol{X}$ contains the parametric part from the spline function in (2.48) and additional variables that need to be included in the model. Matrix $\boldsymbol{D}$ is defined as in (2.50) for the univariate case or as in (2.52) for bivariate smoothing. The variance matrices $\boldsymbol{\Sigma}_e$ and $\boldsymbol{\Sigma}_e$ are defined as in model (2.24) and $\boldsymbol{\Sigma}_\gamma$ is defined as in model (2.50). In contrast to the nested error regression model in (2.24), the parameter vector $\boldsymbol{\theta} = (\sigma_\gamma, \sigma_v, \sigma_e)$ consists of three unknown variance parameters. These can be estimated by applying the ML or REML approach from Section 2.2.2. The unknown parameters $\boldsymbol{\beta}$, $\boldsymbol{v}$ and $\boldsymbol{\gamma}$ can be estimated by applying equations (2.7), (2.8) and (2.51), respectively, where the variance matrix $\boldsymbol{V}$ is defined by $\boldsymbol{V} = \boldsymbol{\Sigma}_e + \boldsymbol{D}\boldsymbol{\Sigma}_\gamma\boldsymbol{D}^T + \boldsymbol{Z}\boldsymbol{\Sigma}_v\boldsymbol{Z}^T$. This variance matrix is more complex compared to model (2.24) as it is not necessarily block-diagonal any more. Under model (2.53), the NPEBLUP for the mean $\bar{y}_i$ in area i is defined by

$$\hat{\bar{y}}_i^{np} = N_i^{-1}\Big\{\sum_{j\in s_i} y_{ij} + \sum_{j\in r_i}(\boldsymbol{x}_{ij}^T\hat{\boldsymbol{\beta}}^{np} + \boldsymbol{d}_{ij}^T\hat{\boldsymbol{\gamma}}^{np} + \hat{v}_i^{np})\Big\}. \tag{2.54}$$

where the superscript np indicates that nonlinear trends in y are taken into account using the non-parametric approach described above. Opsomer et al. (2008) provide an analytic MSE estimation, based on REML estimates for the variance components, which is an extension of the MSE estimation presented in (2.22). As an alternative to analytic MSE estimation, they also provide a non-parametric bootstrap approach. Salvati et al. (2010) develop a conditional MSE for the NPEBLUP based on the pseudo-linearization approach that was reviewed in Section 2.3.2.2.

2.4.4 The Geographically Weighted EBLUP

The EBLUP approaches introduced so far, the EBLUP in (2.26), the SEBLUP in (2.46) and the NPEBLUP in (2.54) assume that the regression coefficients are spatially stationary, i.e., the relationship between y_{ij} and $\mathbf{x}_{ij}$ is the same for the entire target area. If this assumption is violated, the coefficients are referred to as being *spatially non-stationary* and these estimators can be inefficient. Spatial non-stationarity can be present in the intercept and the slope coefficients, where a varying intercept can be interpreted as a spatial trend in the target variable. Thus, when spatial non-stationarity is only present in the intercept, the NPEBLUP approach from the last section can be applied. Chandra et al. (2012) propose a geographically weighted EBLUP (GWEBLUP) for SAE under spatial non-stationarity where all model coefficients are allowed to vary over space. The authors employ a geographically weighted regression (GWR), a method frequently used for modeling data under spatial non-stationarity (Brunsdon et al., 1996) and combine it with the LMM approach introduced in Section 2.2. It follows a short introduction to the GWR method where I mainly refer to Fotheringham et al. (2002). Thereafter, the GWEBLUP of the area mean is presented in the context of SAE.

The GWR approach is based on the classical linear regression model (LM) where all observations are assumed to be independent. The LM can be expressed as $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{e}$, where in contrast to the LMM defined in equation (2.3), there is no random effect part in the error term. By assumption, the LM has constant model coefficients $\boldsymbol{\beta}$ and can therefore be seen as a global model. In contrast, the GWR is a local approach where a separate LM is defined for each observed location. Here, the coefficients are estimated locally, rather than globally and can therefore vary over space with an arbitrary functional form. Let u_i be a location in the target area associated with individual i, which is defined by its geographical coordinates, the longitude and the latitude. The local LM with respect to location u_i is defined by

$$y_i = \boldsymbol{x}_i^T \boldsymbol{\beta}(u_i) + e_i, \quad i = 1, \dots, n \tag{2.55}$$

where $\boldsymbol{\beta}(u_i) = \boldsymbol{\beta}_i$ is the vector of p regression coefficients specific to location u_i, and e_i is the error term with a mean of zero and variance σ_e^2. Using a weighted least squares approach, as suggested by Brunsdon et al. (1996), the local coefficients $\boldsymbol{\beta}_i$ can be estimated by

$$\hat{\boldsymbol{\beta}}_i = (\boldsymbol{X}^T \boldsymbol{W}_i^{-1} \boldsymbol{X})^{-1} \boldsymbol{X}^T \boldsymbol{W}_i^{-1} \boldsymbol{y}. \tag{2.56}$$

Here $\boldsymbol{W}_i = diag_{k \in s}(w_i(u_k))$ is a fixed matrix diagonal matrix where $w_i(u_k)$ is a weighting function that decreases as the distance between the location of individual i and location u_k increases and u_k is the location associated to individual k. One possible choice to define the geographical weights $w_i(u_k)$ is the Euclidean weighting function which is defined by $w_i(u_k) = exp(-1/2(d_{i,k}/b))$, where b denotes a bandwidth parameter and $d_{i,k}$ is the spatial distance between the individual i and k. A comprehensive review of spatial weighting functions for GWR is provided in Fotheringham et al. (2002, Chapter 2.7.3). Most weighting functions have in common that they depend on a bandwidth parameter b which defines the speed at which the geographical weights decay. Note that when b goes to infinity, the weights become constant and model (2.55) is equivalent to the global LM. One possible choice is to define b such that it minimizes the leave-one-out cross validation criteria which is given by

$$CV(b) = \sum_{i \in s} (y_i - \hat{y}_{\neq i}(b))^2 \tag{2.57}$$

where $\hat{y}_{\neq i} = \boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}_{\neq i}$ denotes the estimated value of y_i where observation i is omitted from the estimation process. The estimate $\hat{\boldsymbol{\beta}}_{\neq i}$ is obtained by equation (2.56), where the weight for the ith data point is set to zero, such that it is omitted from the computation. Further calibration methods for b are described in detail in Fotheringham et al. (2002, Chapter 2.7.4). The local LM in (2.55) has the matrix representation

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta}_i + \boldsymbol{W}_i^{-1/2}\boldsymbol{e}. \tag{2.58}$$

where $\boldsymbol{e}$ is the error term with a mean of zero and variance matrix $\boldsymbol{\Sigma}_e = \sigma_e^2 \boldsymbol{I}_n$. The full model, unconditional to a specific location can be expressed as

$$\boldsymbol{y} = (\boldsymbol{X} \circ \boldsymbol{B})\boldsymbol{1}_p + \boldsymbol{e}, \tag{2.59}$$

where matrix $\boldsymbol{B} = col_{1 \leqslant i \leqslant n}(\boldsymbol{\beta}_i^T)$ consists of n sets of local coefficients $\boldsymbol{\beta}_i$, $\boldsymbol{1}_p$ is a p-vector of ones and $\circ$ denotes the element-wise product of matrices. In a global LM, the number of parameters for the fixed model part is defined by p. Fotheringham et al. (2002, Chapter 4.2.3) point out that in the GWR framework the concept of the *number of parameters* is meaningless. They suggest considering the *effective number of parameters* (p_e) which varies between p and the sample size n. When the bandwidth tends to infinity $p_e \approx p$ and when the bandwidth tends to zero $p_e \approx n$. They show that p_e is given by

$$p_e = 2tr(\boldsymbol{H}) - tr(\boldsymbol{H}^T\boldsymbol{H}) \tag{2.60}$$

where $\boldsymbol{H}$ is the $n \times n$ head matrix for mapping $\hat{\boldsymbol{y}} = \boldsymbol{H}\boldsymbol{y}$ with row vectors

$$\boldsymbol{h}_i = \boldsymbol{x}_i^T(\boldsymbol{X}^T\boldsymbol{W}_i^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{W}_i^{-1}.$$

To detect spatial non-stationarity, Fotheringham et al. (2002, p.92) suggest applying an approximate likelihood ratio test. In particular, they divide the residual sum of squares (RSS) for the global LM by that for the GWR model and carry out an F-test on this ratio with (d_1, d_2) degrees of freedom (DF). Here, d_1 refers to the DF for the global LM and d_2 to those of the GWR model. Alternative goodness-of-fit test based on the RSS can be found Leung et al. (2000).
In the context of SAE, Chandra et al. (2012), combined the GWR approach with the LMM approach to define the GWEBLUP. To motivate a local LMM let u_{ij} be a location in the target area, associated with individual j in area i. Then the local LMM with respect to location u_{ij} is given by

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta}_{ij} + \boldsymbol{Z}\boldsymbol{v} + \boldsymbol{W}_{ij}^{-1/2}\boldsymbol{e}, \tag{2.61}$$

where

$$\boldsymbol{v} \sim N(0, \boldsymbol{\Sigma}_v),$$
$$\boldsymbol{e} \sim N(0, \boldsymbol{\Sigma}_e).$$

In this model the variance matrices $\boldsymbol{\Sigma}_e$ and $\boldsymbol{\Sigma}_v$ are defined as in model (2.24), $\boldsymbol{\beta}_{ij}$ is the vector of regression coefficients specific to location u_{ij}, and $\boldsymbol{W}_{ij}$ the respective spatial weighting matrix. The parameter vector $\boldsymbol{\theta}$ for the variance components is assumed to be constant for all locations. Thus, for now, spatial non-stationarity is only allowed for the fixed regression coefficients. Equivalently to model (2.59), the full model, unconditional to a specific location, can be expressed as

$$\boldsymbol{y} = (\boldsymbol{X} \circ \boldsymbol{B})\boldsymbol{1}_p + \boldsymbol{Z}\boldsymbol{v} + \boldsymbol{e}, \tag{2.62}$$

where

$$\boldsymbol{v} \sim N(0, \boldsymbol{\Sigma}_v),$$
$$\boldsymbol{e} \sim N(0, \boldsymbol{\Sigma}_e).$$

Analogously to the equation (2.7), the BLUE of the local coefficients $\boldsymbol{\beta}_{ij}$ is given by

$$\tilde{\boldsymbol{\beta}}_{ij} = (\boldsymbol{X}^T\boldsymbol{V}_{ij}{}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{V}_{ij}{}^{-1}\boldsymbol{y}, \tag{2.63}$$

where $\boldsymbol{V}_{ij} = \boldsymbol{Z}\boldsymbol{\Sigma}_v\boldsymbol{Z}^T + \sigma_e^2\boldsymbol{W}_{ij}^{-1}$ is the local variance matrix of $\boldsymbol{y}$ under model (2.61). The number of effective parameters p_e for the fixed part of the full model in (2.62) can be obtained by equation (2.60) where the row vectors of head matrix $\boldsymbol{H}$ are replaced by $\boldsymbol{h}_{ij} = \boldsymbol{x}_{ij}^T(\boldsymbol{X}^T\boldsymbol{V}_{ij}{}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{V}_{ij}{}^{-1}$. Analogously to equation (2.8), the BLUP of the random effect $\boldsymbol{v}$ is given by

$$\tilde{\boldsymbol{v}} = \boldsymbol{\Sigma}_v\boldsymbol{Z}^T\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{\lambda}), \tag{2.64}$$

where the projection $\boldsymbol{\lambda}$ is given by $\boldsymbol{\lambda} = (\boldsymbol{X} \circ \tilde{\boldsymbol{B}}_s)\boldsymbol{1}_p$ and $\boldsymbol{V} = \boldsymbol{Z}\boldsymbol{\Sigma}_v\boldsymbol{Z}^T + \sigma_e^2\boldsymbol{I}_n$ is the unconditional variance matrix of $\boldsymbol{y}$ from model (2.62). Matrix $\tilde{\boldsymbol{B}}_s$ consists of n sets of local coefficient estimates $\tilde{\boldsymbol{\beta}}_{ij}$. Note that the subscript s is used here to emphasize that the residuals in equation (2.64), and therefore the estimation of the random effects, depends on the local in-sample coefficients.

Under model (2.61), the GWEBLUP for the mean $\bar{y}_i$ in area i is defined by

$$\hat{\bar{y}}_i^{gw} = N_i^{-1}\Big\{\sum_{j\in s_i} y_{ij} + \sum_{j\in r_i}(\boldsymbol{x}_{ij}^T\hat{\boldsymbol{\beta}}_{ij}^{gw} + \hat{v}_i^{gw})\Big\}, \tag{2.65}$$

where the superscript gw indicates that the parameters depend on geographical weighting. The estimates, $\hat{\boldsymbol{\beta}}_{ij}^{gw}$ and $\hat{v}_i^{gw}$ are obtained by the empirical counterparts of (2.63) and (2.64), respectively, where the unknown variance parameters σ_e and σ_u are replaced by estimates. These estimates can be obtained by applying the ML or the REML approach from Section 2.2.2. It is important to notice that for the GWEBLUP in (2.65) the local coefficients $\boldsymbol{\beta}_{ij}$ have to be estimated for each sampled and non-sampled population unit which makes the estimation computationally demanding. In practical applications, however, it can be unrealistic to have access to the geographical information for the non-sampled units. In that case it is possible to use the centroid information of the areas as an approximation of the unknown locations. Here, the coefficients are constant within an area and the GWEBLUP for the mean $\bar{y}_i$ in area i becomes

$$\hat{\bar{y}}_i^{gw} = N_i^{-1}\Big\{\sum_{j\in s_i} y_{ij} + \sum_{j\in r_i}(\boldsymbol{x}_{ij}^T\hat{\boldsymbol{\beta}}_i^{gw} + \hat{v}_i^{gw})\Big\}. \tag{2.66}$$

Details regarding the algorithms for estimating the model parameters are provided in Chandra et al. (2012). As this algorithm is the basis for the estimation of the proposed estimator in this thesis, it will be discussed in more detail in Section 4.1. The authors also provide a conditional MSE estimator for the GWEBLUP, which is based on the pseudo-linearization approach from Section 2.3.2.2. The underlying weights that are necessary to develop the conditional MSE are discussed in more detail in Section 4.2.1 where the same approach is applied to develop a conditional MSE for the proposed robust extension of the GWEBLUP in equation (2.65).

2.5 Summary and Outlook

When spatial data is available, it can be exploited to improve small area estimates via spatial modeling. Here, spatial effects are taken into account within the model specifications. The three spatial methods described in the last section -the SEBLUP, the NPEBLUP and the GWEBLUP- extend the random intercept model from Section 2.3. In the SEBLUP approach, spatial dependencies are modeled in the error term by allowing for spatially correlated random effects. In both, the NPEBLUP and the GWEBLUP approach, spatial effects are modeled in the fixed part of the model. For the NPEBLUP the geographical information is added to the model as an explanatory variable using a P-Spline approach whereas for the GWEBLUP approach the fixed coefficients are defined locally rather than globally.

The EBLUP approaches in this chapter, the basic EBLUP in equation (2.26) and the spatial extensions in Section 2.4, are asymptotically unbiased and efficient under the correct model specification and distributional assumption. The assumed normality of the error term components can be violated in the presence of outliers. Outlier robustified versions of the EBLUP in equation (2.26) have recently been investigated by several authors (cf. Sinha and Rao, 2009; Chambers et al., 2014; Dongmo-Jiongo et al., 2013). These developments are the main subject of the next chapter.

Robust Extensions to EBLUP Approaches 3

In the last chapter, selected EBLUP estimators which are based on the LMM were reviewed in the context of SAE. In general, the LMM is based on the normality assumption of the error term components which can be violated in the presence of outliers. Extreme observations can have a large influence on the parameter estimates, leading to inefficient predictions of the area statistic. This lack of outlier robustness becomes even more severe in small samples and hence in the context of SAE. Applying outlier robust estimation methods is expected to yield more reliable results.

This chapter reviews robustified versions of EBLUP estimators that have been proposed in the past. In particular, the robust EBLUP (REBLUP) of Sinha and Rao (2009) is introduced in Section 3.1 which includes the robust estimation process for the model parameters and a bias-corrected version of the REBLUP (Chambers et al., 2014, REBLUP-bc). In addition, conditional MSE estimation for the REBLUP is presented in this section. Thereafter, robust extensions of spatial small area estimators are reviewed in Section 3.2. This includes a robust version of the SEBLUP (Schmid and Münnich, 2014, SREBLUP) and a robust extension of the NPEBLUP (Rao et al., 2014, RNPEBLUP). A robust extension of the GWEBLUP has not been considered in the literature and is proposed in Chapter 4 as the contribution of this thesis. This chapter concludes with a discussion in Section 3.3 about alternative approaches for robust SAE.

3.1 The Robust EBLUP of the Small Area Mean

One approach for robustifying the EBLUP from equation (2.26) was introduced by Sinha and Rao (2009). The authors used formerly developed robust estimation methods for the LMM and introduced these into the field of SAE. It follows an outline of these underlying ideas. Thereafter, the robust EBLUP (REBLUP) of the area mean proposed by Sinha and Rao (2009) is introduced.

Fellner (1986) investigated robust estimations for $\boldsymbol{\beta}$ and $\boldsymbol{v}$ from model (2.3). He suggested restricting the influence of extreme observations by using robust versions of the ML estimation equations (2.10) and (2.11) from Section 2.2.1 for the estimation of $\boldsymbol{\beta}$ and $\boldsymbol{v}$. As a reminder, under the LMM in (2.3) these ML estimation equations are given by

$$\boldsymbol{X}^T \boldsymbol{\Sigma}_e^{-1}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{Z}\boldsymbol{v}) = \boldsymbol{0}$$

$$\boldsymbol{Z}^T\boldsymbol{\Sigma}_e^{-1}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta}-\boldsymbol{Z}\boldsymbol{v})-\boldsymbol{\Sigma}_v^{-1}\boldsymbol{v}=\boldsymbol{0}.$$

Fellner's robust ML equations are given by

$$\boldsymbol{X}^T\boldsymbol{\Sigma}_e^{-1/2}\psi(\boldsymbol{\Sigma}_e^{-1/2}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta}-\boldsymbol{Z}\boldsymbol{v}))=\boldsymbol{0} \quad (3.1)$$

$$\boldsymbol{Z}^T\boldsymbol{\Sigma}_e^{-1/2}\psi(\boldsymbol{\Sigma}_e^{-1/2}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta}-\boldsymbol{Z}\boldsymbol{v}))-\boldsymbol{\Sigma}_v^{-1/2}\psi(\boldsymbol{\Sigma}_v^{-1/2}\boldsymbol{v})=\boldsymbol{0}, \quad (3.2)$$

where $\psi(a)$ denotes an influence function. Following Fellner (1986), $\psi(a)$ should be defined such that it restricts $\psi(a)$ to be smaller in absolute values than a for very large values of a. Furthermore, it should approximate a when a is sufficiently small, i.e., when it is not an outlier. A popular choice for $\psi(a)$ is Huber's influence function, which is defined as

$$\psi(a)=a\min(1,c/|a|), \quad (3.3)$$

where c is a tuning constant that defines the strength of the restriction (Huber, 1964). For very small values of c, the estimator is very robust as the share of restricted observations is high. When c grows large, fewer observations are restricted and the estimator becomes less robust. A common choice is to set c equal to 1.345.
For known variance parameters $\boldsymbol{\theta}$, robust estimates for $\boldsymbol{\beta}$ and $\boldsymbol{v}$ are obtained by finding solutions for (3.1) and (3.2), respectively. Chambers et al. (2014) note that the usefulness of these estimating equations is limited, unless robust estimates of $\boldsymbol{\theta}$ can be defined.
Huggins (1993) defines a different set of robust ML estimates by maximizing the log-likelihood stated in equation (2.13) with respect to $\boldsymbol{\beta}$ and $\boldsymbol{\theta}$. Setting the first derivatives with respect to $\boldsymbol{\beta}$ and $\boldsymbol{\theta}$ equal to zero yields the ML estimating equations

$$\boldsymbol{X}^T\boldsymbol{V}^{-1}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})=\boldsymbol{0} \quad (3.4)$$

$$-\frac{1}{2}tr(\boldsymbol{V}^{-1}\frac{\partial\boldsymbol{V}}{\partial\theta_l})-\frac{1}{2}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})^T\frac{\partial\boldsymbol{V}^{-1}}{\partial\theta_l}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})=\boldsymbol{0}, \quad (3.5)$$

where θ_l denotes the lth element of $\boldsymbol{\theta}$. Huggin's robust versions of these estimation equations are defined as

$$\boldsymbol{X}^T\boldsymbol{V}^{-1/2}\psi(\boldsymbol{z})=\boldsymbol{0} \quad (3.6)$$

$$-\frac{1}{2}tr(\boldsymbol{K}_1\boldsymbol{V}^{-1}\frac{\partial\boldsymbol{V}}{\partial\theta_l})+\frac{1}{2}\psi(\boldsymbol{z})^T\boldsymbol{V}^{-1/2}\frac{\partial\boldsymbol{V}}{\partial\theta_l}\boldsymbol{V}^{-1/2}\boldsymbol{z}=\boldsymbol{0}, \quad (3.7)$$

where $\boldsymbol{z}=\boldsymbol{V}^{-1/2}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})$ is a vector of standardized residuals, and $\boldsymbol{K}_1=E(\boldsymbol{a}\psi(\boldsymbol{a})^T)$ with $a\sim N(0,\boldsymbol{I}_n)$ being standard normally distributed. Based on Huggin's estimating equations Richardson and Welsh (1995) propose several modifications for the robust ML estimating equation of the variance parameter $\boldsymbol{\theta}$ in (3.7). Among those, they suggest solving the robust ML estimating equation

$$-\frac{1}{2}tr(\boldsymbol{K}_2\boldsymbol{V}^{-1}\frac{\partial \boldsymbol{V}}{\partial \theta_l}) + \frac{1}{2}\psi(\boldsymbol{z})^T\boldsymbol{V}^{-1/2}\frac{\partial \boldsymbol{V}}{\partial \theta_l}\boldsymbol{V}^{-1/2}\psi(\boldsymbol{z}) = \boldsymbol{0}, \tag{3.8}$$

for finding a robust estimator of $\boldsymbol{\theta}$. Here $\boldsymbol{K}_2 = E(\psi(\boldsymbol{a})\psi(\boldsymbol{a})^T)$ with $a \sim N(0, \boldsymbol{I}_n)$. The authors called the solution of equation (3.8) robust ML Proposal II, as it can be seen as a generalization of Huber's Proposal 2 (cf. Huber, 1964).

For developing the REBLUP in the context of SAE, Sinha and Rao (2009) build on the results presented above. In their approach, they apply robust ML estimating equations in two steps: (1) obtain robust estimates for the model parameters $\boldsymbol{\beta}$ and $\boldsymbol{\theta}$ simultaneously by solving modified versions of equations (3.6) and (3.8) using an iterative algorithm; (2) insert the estimates from step one into Fellner's equation (3.2) to obtain robust estimates for $\boldsymbol{v}$ iteratively. For step one the robust ML estimation equations in (3.6) and (3.8) are replaced by the modified estimating equations

$$\boldsymbol{X}^T\boldsymbol{V}^{-1}\boldsymbol{U}^{1/2}\psi(\boldsymbol{r}) = \boldsymbol{0} \tag{3.9}$$

$$-\frac{1}{2}tr(\boldsymbol{K}\boldsymbol{V}^{-1}\frac{\partial \boldsymbol{V}}{\partial \theta_l}) + \frac{1}{2}\psi(\boldsymbol{r})^T\boldsymbol{U}^{1/2}\boldsymbol{V}^{-1}\frac{\partial \boldsymbol{V}}{\partial \theta_l}\boldsymbol{V}^{-1}\boldsymbol{U}^{1/2}\psi(\boldsymbol{r}) = \boldsymbol{0}. \tag{3.10}$$

Here, $\boldsymbol{r} = \boldsymbol{U}^{-1/2}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})$ is a vector of standardized residuals, $\boldsymbol{K}$ is a diagonal matrix with $\boldsymbol{K} = E(\psi^2(\boldsymbol{a}))\boldsymbol{I}_n$, where a follows a standard normal distribution and ψ is Huber's influence function, as defined in (3.3). Matrix $\boldsymbol{U}$ is a diagonal matrix with $\boldsymbol{U} = diag(\boldsymbol{V})$. In contrast to the robust estimating equations in (3.6) and (3.8) where matrix $\boldsymbol{V}^{-1/2}$ is used for normalizing the residuals, Sinha and Rao (2009) simplify the normalization by using matrix $\boldsymbol{U}^{-1/2}$ instead. This replacement can lead to a stabilization of the parameter estimation. Further details on the estimation process are presented below. Given robust estimates for $\boldsymbol{\beta}$ and $\boldsymbol{v}$, Sinha and Rao (2009) suggest plugging these into the EBLUP from equation (2.26) to obtain the REBLUP. Thus, under model (2.24) the REBLUP for the mean $\bar{y}_i$ in area i is defined by

$$\hat{\bar{y}}_i^{\psi} = N_i^{-1}\Big\{\sum_{j\in s_i} y_{ij} + \sum_{j\in r_i}(\boldsymbol{x}_{ij}^T\hat{\boldsymbol{\beta}}^{\psi} + \hat{v}_i^{\psi})\Big\}, \tag{3.11}$$

where the superscript ψ indicates that the estimates $\hat{\boldsymbol{\beta}}^{\psi}$ and $\hat{\boldsymbol{v}}^{\psi}$ depend on the influence function ψ. Analogously to the EBLUP of the area mean, the REBLUP in (3.11) reduces to a synthetic estimator $\hat{\bar{y}}_i^{\psi,syn} = \bar{\boldsymbol{x}}_i^T\hat{\boldsymbol{\beta}}^{\psi}$ with an area-specific random effect equal to zero for non-sampled areas.

3.1.1 Solving the Robust Estimation Equations

Sinha and Rao (2009) suggest using the Newton-Raphson algorithm to simultaneously solve equations (3.9) and (3.10) to obtain robust parameter estimates of $\boldsymbol{\beta}$ and $\boldsymbol{\theta}$, respectively. This algorithm turns out to be unstable when the starting values for $\boldsymbol{\beta}$ and $\boldsymbol{\theta}$ differ substantially from the true values (cf. Rao and Molina, 2015, Chapter 7.4.1). As an alternative to the Newton-Raphson method a fixed point algorithm has been developed by Chatrchi (2012) that solves equation (3.10) to obtain $\hat{\boldsymbol{\theta}}^{\psi}$. In addition, Schoch (2012) suggests an iteratively re-weighted least squares (IRWLS) algorithm for solving equation (3.9) to obtain $\hat{\boldsymbol{\beta}}^{\psi}$. To obtain robust estimates for $\boldsymbol{v}$, given $\hat{\boldsymbol{\beta}}^{\psi}$ and $\hat{\boldsymbol{v}}^{\psi}$, Sinha and Rao (2009) applied the Newton-Raphson algorithm which yields stable results (cf. Rao and Molina, 2015, p.196).
The fixed point algorithm, the IRWLS algorithm and the Newton-Raphson algorithm are described in the following as strategies to obtain robust estimates for $\boldsymbol{\theta}$, $\boldsymbol{\beta}$ and $\boldsymbol{v}$, respectively. These algorithms are also relevant for the proposed method of this thesis (Chapter 4) as they are applied for the parameter estimation in the robustified version of the GWEBLUP.

3.1.1.1 Solving for $\boldsymbol{\theta}$

To develop a fixed-point algorithm for the approximation of $\boldsymbol{\theta} = (\sigma_e^2, \sigma_v^2)^T$ it is necessary to find an expression of the form $\boldsymbol{\theta} = f(\boldsymbol{\theta})$, where $f(\cdot)$ is an arbitrary continuous function. The fixed point algorithm, developed by Chatrchi (2012) is based on reformulating equation (3.10) as a system of equations. Using the fact that $tr(\boldsymbol{K}\boldsymbol{V}^{-1}(\partial \boldsymbol{V}/\partial\theta_l)) = tr(\boldsymbol{K}\boldsymbol{V}^{-1}(\partial \boldsymbol{V}/\partial\theta_l)\boldsymbol{V}^{-1}\boldsymbol{V})$ with $(\partial \boldsymbol{V}/\partial\sigma_e^2) = \boldsymbol{I}_n$ and $(\partial \boldsymbol{V}/\partial\sigma_v^2) = \boldsymbol{Z}\boldsymbol{Z}^T$, the robust ML estimating equation in (3.10) can be expressed as

$$tr\left(\boldsymbol{K}\boldsymbol{V}^{-1}\boldsymbol{I}_n\boldsymbol{V}^{-1}(\boldsymbol{I}_n \;\; \boldsymbol{Z}\boldsymbol{Z}^T)\begin{pmatrix}\sigma_e^2\\ \sigma_v^2\end{pmatrix}\right) = a_1(\boldsymbol{\theta})$$
$$tr\left(\boldsymbol{K}\boldsymbol{V}^{-1}\boldsymbol{Z}\boldsymbol{Z}^T\boldsymbol{V}^{-1}(\boldsymbol{I}_n \;\; \boldsymbol{Z}\boldsymbol{Z}^T)\begin{pmatrix}\sigma_e^2\\ \sigma_v^2\end{pmatrix}\right) = a_2(\boldsymbol{\theta}),$$

with

$$a_1(\boldsymbol{\theta}) = \psi(\boldsymbol{r})^T\boldsymbol{U}^{1/2}\boldsymbol{V}^{-1}\boldsymbol{I}_n\boldsymbol{V}^{-1}\boldsymbol{U}^{1/2}\psi(\boldsymbol{r})$$
$$a_2(\boldsymbol{\theta}) = \psi(\boldsymbol{r})^T\boldsymbol{U}^{1/2}\boldsymbol{V}^{-1}\boldsymbol{Z}\boldsymbol{Z}^T\boldsymbol{V}^{-1}\boldsymbol{U}^{1/2}\psi(\boldsymbol{r}).$$

This equation system can be expressed as $\boldsymbol{\theta} = \boldsymbol{A}(\boldsymbol{\theta})^{-1}\boldsymbol{a}(\boldsymbol{\theta})$, where $\boldsymbol{a}(\boldsymbol{\theta}) = (a_1(\boldsymbol{\theta}), a_2(\boldsymbol{\theta}))^T$ and

$$\boldsymbol{A}(\boldsymbol{\theta}) = \begin{bmatrix} tr\left(\boldsymbol{K}\boldsymbol{V}^{-1}\boldsymbol{I}_n\boldsymbol{V}^{-1}\boldsymbol{I}_n\right) & tr\left(\boldsymbol{K}\boldsymbol{V}^{-1}\boldsymbol{I}_n\boldsymbol{V}^{-1}\boldsymbol{Z}\boldsymbol{Z}^T\right)\\ tr\left(\boldsymbol{K}\boldsymbol{V}^{-1}\boldsymbol{Z}\boldsymbol{Z}^T\boldsymbol{V}^{-1}\boldsymbol{I}_n\right) & tr\left(\boldsymbol{K}\boldsymbol{V}^{-1}\boldsymbol{Z}\boldsymbol{Z}^T\boldsymbol{V}^{-1}\boldsymbol{Z}\boldsymbol{Z}^T\right)\end{bmatrix}.$$

The fixed-point iterations for $\boldsymbol{\theta}$ are given by

$$\boldsymbol{\theta}^{(t+1)} = \boldsymbol{A}(\boldsymbol{\theta}^{(t)})^{-1}\boldsymbol{a}(\boldsymbol{\theta}^{(t)}), \quad t = 1, 2, \ldots \tag{3.12}$$

To obtain a robust estimate of $\boldsymbol{\theta}$, the latter expression is updated until convergence is achieved.

3.1.1.2 Solving for $\boldsymbol{\beta}$

The approximation of $\boldsymbol{\beta}$ is very similar to the process described above. It is based on rewriting the robust ML estimation equation in (3.9) as

$$\boldsymbol{X}^T\boldsymbol{V}^{-1}\boldsymbol{U}^{1/2}\boldsymbol{D}\boldsymbol{U}^{-1/2}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}) = \boldsymbol{0}. \tag{3.13}$$

Here, $\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{\beta})$ is an $n \times n$ diagonal matrix with the jth diagonal elements defined as $d_j = \psi(r_j)/r_j$, where r_j represents the jth element of the standardized residuals $\boldsymbol{r}$. It follows that $\boldsymbol{\beta}$ can be approximated by repeatedly evaluating the fixed-point expression

$$\boldsymbol{\beta}^{(t+1)} = (\boldsymbol{X}^T\boldsymbol{V}^{-1}\boldsymbol{D}(\boldsymbol{\beta}^{(t)})\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{V}^{-1}\boldsymbol{D}(\boldsymbol{\beta}^{(t)})\boldsymbol{y}, \quad t = 0, 1, 2, \ldots \tag{3.14}$$

until convergence is reached. The latter expression is equivalent to the IRWLS algorithm, suggested by Schoch (2012). Let $\tilde{\boldsymbol{X}} = \boldsymbol{D}^{1/2}\boldsymbol{U}^{1/2}\boldsymbol{V}^{-1}\boldsymbol{X}$ and $\tilde{\boldsymbol{y}} = \boldsymbol{D}^{1/2}\boldsymbol{U}^{1/2}\boldsymbol{V}^{-1}\boldsymbol{y}$ be transformations of $\boldsymbol{X}$ and $\boldsymbol{y}$, respectively. Then equation (3.14) can be expressed as an iteration of IRWLS algorithm

$$\boldsymbol{\beta}^{(t+1)} = \left[(\tilde{\boldsymbol{X}}^{(t)})^T\tilde{\boldsymbol{X}}^{(t)}\right]^{-1}(\tilde{\boldsymbol{X}}^{(t)})^T\tilde{\boldsymbol{y}}^{(t)}, \quad t = 0, 1, 2, \ldots \tag{3.15}$$

To obtain a robust estimate of $\boldsymbol{\beta}$, the latter expression is updated until convergence is achieved.

3.1.1.3 Solving for $\boldsymbol{v}$

To develop the Newton-Raphson algorithm for the approximation of $\boldsymbol{v}$ let $\Delta(\boldsymbol{v})$ be Fellner's robust ML estimation equation (3.2). Then, the iterations can be expressed as

$$\boldsymbol{v}^{(t+1)} = \boldsymbol{v}^{(t)} - \Delta'(\boldsymbol{v}^{(t)})^{-1}\Delta(\boldsymbol{v}^{(t)}), \quad t = 1, 2, \ldots \tag{3.16}$$

where $\Delta'(\boldsymbol{v}^{(t)})$ is the derivative of $\Delta(\boldsymbol{v})$ with respect to $\boldsymbol{v}$ evaluated at $\boldsymbol{v}^{(t)}$. The derivative $\Delta'(\boldsymbol{v})$ is given by

$$\begin{aligned}\Delta'(\boldsymbol{v}) &= \boldsymbol{Z}^T\boldsymbol{\Sigma}_e^{-1}\psi'(\underbrace{\boldsymbol{\Sigma}_e^{-1/2}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{Z}\boldsymbol{v})}_{=\boldsymbol{t}})\boldsymbol{Z} - \boldsymbol{\Sigma}_v^{-1}\psi'(\underbrace{\boldsymbol{\Sigma}_v^{-1/2}\boldsymbol{v}}_{=\boldsymbol{l}}) \\ &= \boldsymbol{Z}^T\boldsymbol{\Sigma}_e^{-1}\psi'(\boldsymbol{t})\boldsymbol{Z} - \boldsymbol{\Sigma}_u^{-1}\psi'(\boldsymbol{l}),\end{aligned} \tag{3.17}$$

where $\psi'(\boldsymbol{t}) = diag_n[I(-c < \boldsymbol{t} < c)]$ and $\psi'(\boldsymbol{l}) = diag_m[I(-c < \boldsymbol{l} < c)]$ are diagonal matrices of dimension n and m, respectively, and c denotes the tuning constant. When the proportion of outliers is large, the diagonals of $\psi'(\boldsymbol{t})$ and $\psi'(\boldsymbol{l})$ can be very sparse. This can cause the convergence of expression (3.16) to fail when using the Newton-Raphson algorithm. To enhance stability of the iterative process, it can be advisable to replace $\Delta'(\boldsymbol{v}^{(t)})$ with its expectation, which would lead to the Fisher scoring algorithm (cf. Jennrich and Sampson, 1976). Let $\mathcal{A}(\boldsymbol{v})$ be the expectation of $\Delta'(\boldsymbol{v})$, which is given by

$$\mathcal{A}(\boldsymbol{v}) = \boldsymbol{Z}^T \boldsymbol{\Sigma}_e^{-1} E\left[\psi'(\boldsymbol{t})\right] \boldsymbol{Z} - \boldsymbol{\Sigma}_u^{-1} E\left[\psi'(\boldsymbol{l})\right], \tag{3.18}$$

where $E\left[\psi'(\boldsymbol{t})\right]$ and $E\left[\psi'(\boldsymbol{l})\right]$ are diagonal matrices of dimension n and m, respectively, with a constant diagonal element. Under model (2.24), $\boldsymbol{t}$ and $\boldsymbol{l}$ are standard normally distributed. The elements are then given by $E[-c \leq t, l \leq c] = 2\Phi(c) - 1$, where Φ denotes the cumulative distribution function of the standard normal distribution. The iterations of the Fisher scoring algorithm are defined by

$$\boldsymbol{v}^{(t+1)} = \boldsymbol{v}^{(t)} - \mathcal{A}(\boldsymbol{v}^{(t)})^{-1} \Delta(\boldsymbol{v}^{(t)}), \quad t = 1, 2, \ldots \tag{3.19}$$

To obtain a robust estimate of $\boldsymbol{v}$, the latter expression is updated until convergence is achieved.

3.1.2 Bias Correction

Sinha and Rao (2009) conducted a simulation study to examine the properties of their proposed estimator from equation (3.11) in the presence of outliers. Here, they generated populations with different specifications for the outlier distribution, which all had in common that the outliers were generated symmetrically around zero. The REBLUP of the area mean appeared to perform well in terms of bias and efficiency under their specifications. According to Chambers et al. (2014) this estimator can be referred to as being *robust projective* because it replaces outlying sample values with estimates which are expected under the working model. This implies that departures from the model assumptions caused by outliers should only be present in the sample and not in the non-sampled part of the population. Thus, the non-sampled population units are assumed to follow model (2.24), unless departures from that model vary around zero, which is the case for symmetric outliers. However, in the presence of non-symmetric outliers with an expected value different from zero, the REBLUP of Sinha and Rao (2009) can be biased (Chambers et al., 2014; Dongmo-Jiongo et al., 2013).
To account for non-symmetric outliers in the population Chambers (1986) suggested adding a bias correction to robust projective estimators, which accounts for the contribution of population outliers to the population pa-

rameter of interest. These bias-corrected robust estimators are referred to as being *robust predictive*. Following this idea, Chambers et al. (2014) introduced a robust predictive extension of the REBLUP (REBLUP-bc) which is given by

$$\hat{\bar{y}}_i^{\psi-bc} = \hat{\bar{y}}_i^{\psi} + bc_i, \tag{3.20}$$

with

$$bc_i = \left(\frac{N_i - n_i}{N_i}\right) n_i^{-1} \sum_{j \in s_i} \omega_i \psi_b \Big\{ (y_{ij} - \mathbf{x}_{ij} \hat{\boldsymbol{\beta}}^{\psi} - \hat{v}_i{}^{\psi}) / \omega_i \Big\}.$$

Here ω_i is a robust scale estimator, given by the median absolute deviation of the residuals in area i. The function ψ_b is defined by Huber's influence function in (3.3) with a tuning constant $b > c$, implying that ψ_b is less restrictive than ψ in equation (3.11). The additional term bc_i is an area-specific estimate of the prediction bias and can be considered as a local bias correction. Thus, for non-sampled areas, a bias correction based on equation (3.20) is not possible. If n_i is very small the variability of this bias correction can be high (Rao and Molina, 2015, p.196).
Dongmo-Jiongo et al. (2013) mention that the estimator in (3.20) may still be biased. They argue that the bias correction depends only on the local information in area i, whereas the robust estimators $\hat{\boldsymbol{\beta}}^{\psi}$ and $\hat{\boldsymbol{v}}^{\psi}$ are influenced by all sampled units. Dongmo-Jiongo et al. (2013) propose two *fully bias-corrected* robust estimators for the area mean that make use of the residuals from all areas in the sample. In their first approach, they show that the EBLUP in equation (2.25) can be decomposed into a sum of the REBLUP in equation (3.11) and a correction term that quantifies the prediction bias from using the REBLUP instead of the unbiased EBLUP for the prediction of the area mean. Following the ideas of Chambers (1986), they define a robust predictive estimator for the area mean by robustifying this correction term using Huber's influence function. In their second approach the correction term is based on the *conditional* prediction bias for the estimation of the area mean. For the proposed method in Chapter 4, a fully bias-corrected estimation of the population parameters has not been considered yet. Therefore, a further description of this method is dispensed at this point. For detailed information on fully bias-corrected estimators I refer to Dongmo-Jiongo et al. (2013) and Rao and Molina (2015, Chapter 7.4.1) and the literature cited there.

3.1.3 Mean Squared Error Estimation

A robust estimation of the small area mean also requires MSE estimates as a measure of precision for the point estimates. For the MSE estimation of the REBLUP in (3.11), Sinha and Rao (2009) propose a parametric bootstrap method that is based on the robust estimates $\hat{\boldsymbol{\beta}}^{\psi}$ and $\hat{\boldsymbol{\theta}}^{\psi}$ for generating the bootstrap populations. Dongmo-Jiongo et al. (2013) note that using robust estimators for the bootstrap population that do not reflect all population

units can lead to poor results in the sense that the MSE can be negatively biased. They propose a parametric bootstrap method that is based on the robust coefficient estimates $\hat{\boldsymbol{\beta}}^{\psi}$, but non-robust estimates for the variance parameters. However, this can lead to a positive bias for the MSE estimation. Chambers et al. (2014) proposed two analytic methods of MSE estimation for robust predictors. First, they adopted the pseudo-linearization of Chambers et al. (2011) to obtain a conditional MSE estimator for the robust projective REBLUP in (3.11). They use the same approach to obtain a conditional MSE estimator for the robust predictive REBLUP-bc in (3.20) as well. As mentioned in Section 2.3.2.2, the conditional MSE based on pseudo-linearization ignores the extra variability due to the estimation of $\boldsymbol{\theta}$, and can therefore be seen as a first-order approximation for the conditional MSE of the REBLUP or the REBLUP-bc, respectively. Hence, Chambers et al. (2014) also proposed a second-order approximation of the conditional MSE for the REBLUP that is based on a linearization of the conditional prediction variance. In addition, they showed adjustments of this approach to obtain a conditional MSE for the robust predictive REBLUP-bc.
For the proposed method in this thesis, two analytic MSE estimation methods will be provided. As these are based on the two approaches proposed by Chambers et al. (2014), more details are provided for both methods. In particular, the conditional MSE estimation for the REBLUP and REBLUP-bc of the area mean based on the pseudo-linearization approach is presented in Section 3.1.3.1 followed by respective MSE estimators based on the linearization approach in Section 3.1.3.2.

3.1.3.1 Based on the Pseudo-Linearization Approach

To develop a conditional MSE estimation using the pseudo-linearization approach, Chambers et al. (2014) showed that the REBLUP of the area mean in equation (3.11) can be rewritten as a weighted sum of the sampled values of $\boldsymbol{y}$

$$\hat{\bar{y}}_i^{\psi} = \sum_{j \in s} d_{ij}^{\psi} y_j = \left(\boldsymbol{d}_i^{\psi}\right)^T \boldsymbol{y}, \tag{3.21}$$

where $\boldsymbol{d}_i^{\psi}$ is an area-specific vector of weights defined by

$$\left(\boldsymbol{d}_i^{\psi}\right)^T = N_i^{-1} \left[\boldsymbol{\delta}_i^T + (N_i - n_i) \left\{\bar{\boldsymbol{x}}_{ri}^T \boldsymbol{A}^{\psi} + \boldsymbol{z}_i^T \boldsymbol{Q}^{\psi} (\boldsymbol{I}_n - \boldsymbol{X}\boldsymbol{A}^{\psi})\right\}\right]. \tag{3.22}$$

Note that $\boldsymbol{d}_i^{\psi}$ has the same form as the weighting vector $\boldsymbol{d}_i$ in equation (2.33). Here matrix $\boldsymbol{A}^{\psi}$ is given by

$$\boldsymbol{A}^{\psi} = (\boldsymbol{X}^T \hat{\boldsymbol{V}}^{-1} \hat{\boldsymbol{U}}^{1/2} \boldsymbol{D}_1 \hat{\boldsymbol{U}}^{-1/2} \boldsymbol{X})^{-1} \boldsymbol{X}^T \hat{\boldsymbol{V}}^{-1} \hat{\boldsymbol{U}}^{1/2} \boldsymbol{D}_1 \hat{\boldsymbol{U}}^{-1/2}. \tag{3.23}$$

$\boldsymbol{D}_1$ is an $n \times n$ diagonal matrix which is identical to $\boldsymbol{D}$ in equation (3.13). Matrix $\boldsymbol{Q}^\psi$ is given by

$$\boldsymbol{Q}^\psi = (\boldsymbol{Z}^T \hat{\boldsymbol{\Sigma}}_e^{-1/2} \boldsymbol{D}_2 \hat{\boldsymbol{\Sigma}}_e^{-1/2} \boldsymbol{Z} + \hat{\boldsymbol{\Sigma}}_v^{-1/2} \boldsymbol{D}_3 \hat{\boldsymbol{\Sigma}}_v^{-1/2})^{-1} \boldsymbol{Z}^T \hat{\boldsymbol{\Sigma}}_e^{-1/2} \boldsymbol{D}_2 \hat{\boldsymbol{\Sigma}}_e^{-1/2}, \quad (3.24)$$

where $\boldsymbol{D}_2$ and $\boldsymbol{D}_3$ are $n \times n$ and $m \times m$ diagonal matrices, respectively. The diagonal elements $d_{2,ij}$ and $d_{3,i}$, $i = 1, \dots m$ are given by

$$d_{2,ij} = \psi\Big\{(\hat{\sigma}_e)^{-1}(y_{ij} - \mathbf{x}_{ij}\hat{\boldsymbol{\beta}} - v_i)\Big\} / \Big\{(\hat{\sigma}_e)^{-1}(y_{ij} - \mathbf{x}_{ij}\hat{\boldsymbol{\beta}} - \hat{v}_i)\Big\}$$

and

$$d_{3,i} = \psi\Big\{(\hat{\sigma}_v)^{-1}\hat{v}_i\Big\} / \Big\{(\hat{\sigma}_v)^{-1}\hat{v}_i\Big\}.$$

Given the weights in (3.22), developing the conditional MSE for the REBLUP in (3.11) is straightforward. Similar to (2.34), the conditional MSE for the REBLUP is given by

$$\widehat{MSE}_{CCT}(\hat{\bar{y}}_i^\psi) = \widehat{Var}_{\boldsymbol{v}}(\hat{\bar{y}}_i^\psi) + \widehat{Bias}_{\boldsymbol{v}}(\hat{\bar{y}}_i^\psi)^2. \quad (3.25)$$

The prediction bias and the variance can be estimated by (2.35) and (2.37), respectively, with $\boldsymbol{d}_i$ substituted by $\boldsymbol{d}_i^\psi$ from equation (3.22). Thus, the conditional prediction bias can be estimated by

$$\widehat{Bias}_{\boldsymbol{v}}(\hat{\bar{y}}_i^\psi) = \sum_{j \in s} d_{ij}^\psi \hat{\mu}_j - N_i^{-1} \sum_{j \in U_i} \hat{\mu}_j, \quad (3.26)$$

where $\hat{\mu}_j$ is an unbiased linear estimator of the conditional expected value $\mu_j = E(y_j | \boldsymbol{x}_j, \boldsymbol{v})$ under model (2.24). An unbiased estimate for μ_j can be expressed as

$$\hat{\mu}_j = \sum_{k \in s} \gamma_{kj}^\psi y_k = (\boldsymbol{\gamma}_j^\psi)^T \boldsymbol{y}, \quad (3.27)$$

where

$$(\boldsymbol{\gamma}_j^\psi)^T = \boldsymbol{x}_j^T \boldsymbol{A}^\psi + \boldsymbol{z}_i^T \boldsymbol{Q}^\psi (\boldsymbol{I}_n - \boldsymbol{X}\boldsymbol{A}^\psi)$$

is a vector of weights specific to unit j. Analogously to equation (2.35), $\boldsymbol{Q}^\psi$ needs to be replaced by $(\boldsymbol{Z}\boldsymbol{Z}^T)^{-1}\boldsymbol{Z}^T$ in (3.27) to obtain an unshrunken version of $\hat{\mu}_j$. An estimate of the conditional prediction variance is given by

$$\widehat{Var}_{\boldsymbol{v}}(\hat{\bar{y}}_i^\psi) = N_i^{-2} \sum_{j \in s} \Big\{(a_{ij}^\psi)^2 + (N_i - n_i)n^{-1}\Big\}(\hat{\lambda}_j^\psi)^{-1}(y_j - \hat{\mu}_j)^2, \quad (3.28)$$

where $a_{ij}^{\psi} = N_i d_{ij}^{\psi} - I(j \in i)$ and $\hat{\lambda}_j^{\psi} = 1 - 2\gamma_{jj} + \sum_{k \in s} \gamma_{kj}^2$ is a scaling constant, specific to unit j. The weights γ_{kj} are defined by equation (3.27). Analogously to the synthetic EBLUP of the area mean in (2.38), the synthetic REBLUP can be expressed in the pseudo-linear form

$$\hat{\bar{y}}_i^{\psi,Syn} = \bar{\boldsymbol{x}}_i^T \hat{\boldsymbol{\beta}}^{\psi} = (\boldsymbol{d}_i^{\psi,Syn})^T \boldsymbol{y}, \tag{3.29}$$

with $(\boldsymbol{d}_i^{\psi,Syn})^T = \bar{\boldsymbol{x}}_i^T \boldsymbol{A}^{\psi}$ and $\boldsymbol{A}^{\psi}$ defined as in (3.22). Accordingly, for non-sampled areas the squared bias of the synthetic REBLUP can be estimated by

$$\widehat{Bias}_{\boldsymbol{v}}(\hat{\bar{y}}_i^{\psi,Syn})^2 = \left\{ \sum_{j \in s} d_{ij}^{\psi,Syn} (\boldsymbol{x}_j^T \hat{\boldsymbol{\beta}}^{\psi} + \hat{v}_i^{\psi,u}) - \bar{\boldsymbol{x}}_i^T \hat{\boldsymbol{\beta}}^{\psi} \right\}^2 + (\hat{\sigma}_v^{\psi})^2. \tag{3.30}$$

where $\hat{v}_i^{\psi,u}$ is the unshrunken estimated random effect. The conditional MSE of the synthetic REBLUP can be estimated using (3.25) where the conditional bias is replaced by (3.30).
The conditional MSE estimator for the robust predictive REBLUP-bc of the area mean in (3.20) can be obtained by replacing the weights in $\boldsymbol{d}_i^{\psi}$ with the corresponding weights for the REBLUP-bc. Let $\boldsymbol{q}_i$ be a vector specific to area i with elements

$$q_{ij} = \begin{cases} \frac{\psi_b\left(\omega_i^{-1}(y_{ij} - \mathbf{x}_{ij}\hat{\boldsymbol{\beta}}^{\psi} - \hat{v}_i{}^{\psi})\right)}{\omega_i^{-1}(y_{ij} - \mathbf{x}_{ij}\hat{\boldsymbol{\beta}}^{\psi} - \hat{v}_i{}^{\psi})} & \text{for } j \in s_i \\ & , \\ 0 \;\; \text{else} & \end{cases} \tag{3.31}$$

where ω_i is defined as in equation (3.20). Then, the weights for the REBLUP-bc are defined by

$$\begin{aligned} \left(\boldsymbol{d}_i^{\psi,bc}\right)^T = N_i^{-1} \Bigg\{ & \left(\boldsymbol{\delta}_i + \frac{N_i - n_i}{n_i} \boldsymbol{q}_i\right)^T \\ & + \left(\sum_{j \in r_i} \boldsymbol{x}_{ij}^T - \frac{N_i - n_i}{n_i} \sum_{j \in s_i} \boldsymbol{x}_{ij}^T q_{ij} \right) \boldsymbol{A}^{\psi} \\ & + \left((N_i - n_i) - \frac{N_i - n_i}{n_i} \sum_{j \in s_i} q_{ij} \right) \boldsymbol{z}_i^T \boldsymbol{Q}^{\psi} (\boldsymbol{I}_n - \boldsymbol{X}\boldsymbol{A}^{\psi}) \Bigg\}. \end{aligned} \tag{3.32}$$

As the REBLUP-bc is an approximately unbiased estimator of the area mean, the conditional bias in (3.25) can be omitted for the MSE estimation. Thus, the conditional MSE for the REBLUP-bc of the area mean $\bar{y}_i$ is given by

$$\widehat{MSE}_{CCT}(\hat{\bar{y}}_i^{\psi,bc}) = \widehat{Var}_{\boldsymbol{v}}(\hat{\bar{y}}_i^{\psi,bc}), \tag{3.33}$$

and conditional variance can be expressed as

$$\widehat{Var}_{\boldsymbol{v}}(\hat{\bar{y}}_i^{\psi,bc}) = N_i^{-2} \sum_{j \in s} \left\{ (a_{ij}^{\psi,bc})^2 + (N_i - n_i)n^{-1} \right\} (\hat{\lambda}_j^{\psi,bc})^{-1} (y_j - \hat{\mu}_j)^2, \tag{3.34}$$

with $a_{ij}^{\psi,bc} = N_i d_{ij}^{\psi,bc} - I(j \in i)$. Analogously to equation (3.28), $\hat{\mu}_j$ can be expressed as $\hat{\mu}_j = \sum_{k \in s} \gamma_{k,j}^{\psi,bc} y_k = (\boldsymbol{\gamma}_j^{\psi,bc})^T \boldsymbol{y}$, where $\boldsymbol{\gamma}_j^{\psi,bc}$ is a vector of weights specific to unit j, given by

$$\begin{aligned}(\boldsymbol{\gamma}_j^{\psi,bc})^T = &\left(\boldsymbol{\delta}_i^T + n_i^{-1} \boldsymbol{q} \right)^T \\ &+ \left(\sum_{j \in r_i} \boldsymbol{x}_{ij}^T - n_i^{-1} \sum_{j \in s_i} \boldsymbol{x}_{ij}^T q_j \right) \boldsymbol{A}^{\psi} \\ &+ \left(1 - n_i^{-1} \sum_{j \in s_i} q_j \right) \boldsymbol{z}_i^T \boldsymbol{Q}^{\psi} (\boldsymbol{I}_n - \boldsymbol{X}\boldsymbol{A}^{\psi}),\end{aligned} \tag{3.35}$$

and $\hat{\lambda}_j^{\psi} = 1 - 2\gamma_{jj}^{\psi} + \sum_{k \in s} (\gamma_{kj}^{\psi})^2$ is the scaling constant, specific to unit j. As mentioned before, the conditional MSE based on pseudo-linearization ignores the extra variability due to the estimation of $\boldsymbol{\theta}$, and can therefore be seen as a first-order approximation for the conditional MSE of the REBLUP. A second-order approximation to the conditional MSE of the REBLUP is presented in the next section.

3.1.3.2 Based on the Linearization Approach

In their work Chambers et al. (2014) develop a second-order approximation of the conditional MSE estimation for the REBLUP based on a linearization of the conditional prediction variance where the linearization is conducted in two steps. First they develop a conditional MSE for the REBLUP, assuming the variance parameters $\boldsymbol{\theta}$ are known, i.e., for the robust BLUP (RBLUP). In a second step they derive an additional term that accounts for the variability caused by the estimation of the variance components. For the review of this method, I follow these steps. The conditional MSE for the RBLUP of the area mean can be expressed by

$$MSE_{\boldsymbol{v}}(\tilde{\bar{y}}_i^{\psi}) = Var_{\boldsymbol{v}}(\tilde{\bar{y}}_i^{\psi}) + Bias_{\boldsymbol{v}}(\tilde{\bar{y}}_i^{\psi})^2. \tag{3.36}$$

The conditional prediction bias can be estimated by (3.26) for sampled or by (3.30) for non-sampled areas. Thus, an expression for the conditional variance is needed for the estimation of the conditional MSE in (3.36). Let $\tilde{\boldsymbol{\beta}}^{\psi}$ and $\tilde{\boldsymbol{v}}^{\psi}$ denote robust estimates of $\boldsymbol{\beta}$ and $\boldsymbol{v}$, respectively, where the variance parameters $\boldsymbol{\theta}$ are known. Then, the prediction error of the RBLUP for the area mean is given by

$$\begin{aligned}\tilde{\bar{y}}_i^{\psi} - \bar{y}_i &= N_i^{-1}\Big\{\sum_{j\in s_i} y_{ij} + \sum_{j\in r_i}(\boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}^{\psi} + \tilde{v}_i^{\psi})\Big\} - N_i^{-1}\sum_{j\in U_i} y_{ij} \\ &= N_i^{-1}\Big\{\sum_{j\in r_i}(\boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}^{\psi} + \tilde{v}_i^{\psi}) - \sum_{j\in r_i} y_{ij}\Big\} \\ &= \left(\frac{N_i - n_i}{N_i}\right)\left\{\bar{\boldsymbol{x}}_{ri}^T\tilde{\boldsymbol{\beta}}^{\psi} + \boldsymbol{z}_i^T\tilde{\boldsymbol{v}}^{\psi} - \bar{y}_{ri}\right\}. \end{aligned} \tag{3.37}$$

Assuming independence between $\boldsymbol{\beta}$ and $\boldsymbol{v}$, the conditional prediction variance of the RBLUP for the area mean is given by

$$\begin{aligned}&Var_{\boldsymbol{v}}(\tilde{\bar{y}}_i^{\psi} - \bar{y}_i) = \\ &\quad\left(\frac{N_i - n_i}{N_i}\right)^2\left\{\bar{\boldsymbol{x}}_{ri}^T Var_{\boldsymbol{v}}(\tilde{\boldsymbol{\beta}}^{\psi})\bar{\boldsymbol{x}}_{ri} + \boldsymbol{z}_i^T Var_{\boldsymbol{v}}(\tilde{\boldsymbol{v}}^{\psi})\boldsymbol{z}_i + Var_{\boldsymbol{v}}(\bar{e}_{ri})\right\}. \end{aligned} \tag{3.38}$$

To estimate (3.38), estimates for $Var_{\boldsymbol{v}}(\tilde{\boldsymbol{\beta}}^{\psi})$, $Var_{\boldsymbol{v}}(\tilde{\boldsymbol{v}}^{\psi})$ and $Var_{\boldsymbol{v}}(\bar{e}_{ri})$ are needed. An estimate for the last term can either be estimated using only the area-specific sample residuals or by using the residuals from the entire sample. Chambers et al. (2014) recommend using the latter option since it can yield MSE estimates which are more stable when the area-specific sample sizes are very small. Following that suggestion, $Var_{\boldsymbol{v}}(\bar{e}_{ri})$ can be estimated by

$$\widehat{Var}_{\boldsymbol{v}}(\tilde{\bar{e}}_{ri}) = (N_i - n_i)^{-1}(n-1)^{-1}\sum_{i=1}^{m}\sum_{j\in s_i}(y_{ij} - \boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}^{\psi} - \tilde{v}_i^{\psi})^2. \tag{3.39}$$

To obtain first-order approximations for $Var_{\boldsymbol{v}}(\tilde{\boldsymbol{\beta}}^{\psi})$ and $Var_{\boldsymbol{v}}(\tilde{\boldsymbol{v}}^{\psi})$, Chambers et al. (2014) used the robust estimation equations (3.9) and (3.2), respectively. Let $\tilde{\boldsymbol{\alpha}} = col(\tilde{\boldsymbol{\beta}}^{\psi}, \tilde{\boldsymbol{v}}^{\psi})$ be the vector of estimated fixed and random effects, and $\boldsymbol{\alpha} = col(\boldsymbol{\beta}, \boldsymbol{v})$ the corresponding vector of true values under model (2.24). Then, $\boldsymbol{H}(\tilde{\boldsymbol{\alpha}}) = \boldsymbol{0}$ is the vector of the robust ML equations, where

$$\boldsymbol{H}(\boldsymbol{\alpha}) = \begin{pmatrix}\boldsymbol{H}_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) \\ \boldsymbol{H}_{\boldsymbol{v}}(\boldsymbol{\alpha})\end{pmatrix} = \begin{pmatrix}\boldsymbol{X}^T\boldsymbol{V}^{-1}\boldsymbol{U}^{1/2}\psi(\boldsymbol{r}) \\ \boldsymbol{Z}^T\boldsymbol{\Sigma}_e^{-1/2}\psi(\boldsymbol{t}) - \boldsymbol{\Sigma}_v^{-1/2}\psi(\boldsymbol{l})\end{pmatrix}. \tag{3.40}$$

The vectors $\boldsymbol{t}$ and $\boldsymbol{l}$ are defined as in equation (3.17), and vector $\boldsymbol{r}$ is defined as in equation (3.9). The first-order approximations for $Var_{\boldsymbol{v}}(\tilde{\boldsymbol{\beta}}^{\psi})$ and $Var_{\boldsymbol{v}}(\tilde{\boldsymbol{v}}^{\psi})$ are given by

$$Var_{\boldsymbol{v}}(\tilde{\boldsymbol{\beta}}^{\psi}) \approx \left[E_{\boldsymbol{v}}(\partial_{\boldsymbol{\beta}}\boldsymbol{H}_{\boldsymbol{\beta}})^{-1}\right] Var_{\boldsymbol{v}}\left(\boldsymbol{H}_{\boldsymbol{\beta}}\right)\left[E_{\boldsymbol{v}}(\partial_{\boldsymbol{\beta}}\boldsymbol{H}_{\boldsymbol{\beta}})^{-1}\right]^T \tag{3.41}$$

and

$$Var_{\boldsymbol{v}}(\tilde{\boldsymbol{v}}^{\psi}) \approx \left[E_{\boldsymbol{v}}(\partial_{\boldsymbol{v}}\boldsymbol{H}_{\boldsymbol{v}})^{-1}\right] Var_{\boldsymbol{v}}\left(\boldsymbol{H}_{\boldsymbol{v}}\right)\left[E_{\boldsymbol{v}}(\partial_{\boldsymbol{v}}\boldsymbol{H}_{\boldsymbol{v}})^{-1}\right]^{T}, \tag{3.42}$$

respectively, with

$$\begin{aligned}
E_{\boldsymbol{v}}(\partial_{\boldsymbol{\beta}}\boldsymbol{H}_{\boldsymbol{\beta}}) &= -\boldsymbol{X}\boldsymbol{V}^{-1}\boldsymbol{U}^{1/2}\boldsymbol{R}\boldsymbol{U}^{-1/2}\boldsymbol{X}\\
E_{\boldsymbol{v}}(\partial_{\boldsymbol{v}}\boldsymbol{H}_{\boldsymbol{v}}) &= -\boldsymbol{Z}^{T}\boldsymbol{\Sigma}_e^{-1/2}\boldsymbol{T}\boldsymbol{\Sigma}_e^{-1/2}\boldsymbol{Z} - \boldsymbol{\Sigma}_v^{-1/2}\boldsymbol{L}\boldsymbol{\Sigma}_v^{-1/2}\\
Var_{\boldsymbol{v}}\left(\boldsymbol{H}_{\boldsymbol{\beta}}\right) &= \nu(\psi(\boldsymbol{r}))\boldsymbol{X}\boldsymbol{V}^{-1}\boldsymbol{U}\boldsymbol{V}^{-1}\boldsymbol{X}\\
Var_{\boldsymbol{v}}\left(\boldsymbol{H}_{\boldsymbol{v}}\right) &= \nu(\psi(\boldsymbol{t}))\boldsymbol{Z}^{T}\boldsymbol{\Sigma}_e^{-1}\boldsymbol{Z}.
\end{aligned}$$

Here c denotes the tuning constant, $\boldsymbol{T} = diag_n E_{\boldsymbol{v}}\left[I(-c < \boldsymbol{t} < c)\right]$, $\boldsymbol{L} = diag_m E_{\boldsymbol{v}}\left[I(-c < \boldsymbol{l} < c)\right]$, and $\boldsymbol{R} = diag_n E_{\boldsymbol{v}}\left[I(-c < \boldsymbol{r} < c)\right]$. Expression $\nu(\psi(\boldsymbol{r}))$ denotes the conditional variance of $\psi(\boldsymbol{r})$, and $\nu(\psi(\boldsymbol{t}))$ the respective variance of $\psi(\boldsymbol{t})$. Estimates of (3.41) and (3.42) are obtained by replacing $\boldsymbol{\beta}$ and $\boldsymbol{v}$ with their estimates $\tilde{\boldsymbol{\beta}}^{\psi}$ and $\tilde{\boldsymbol{v}}^{\psi}$, respectively, leading to

$$\widehat{Var}_{\boldsymbol{v}}(\tilde{\boldsymbol{\beta}}^{\psi}) = \left[\hat{E}_{\boldsymbol{v}}(\partial_{\boldsymbol{\beta}}\boldsymbol{H}_{\boldsymbol{\beta}})^{-1}\right] \widehat{Var}_{\boldsymbol{v}}\left(\boldsymbol{H}_{\boldsymbol{\beta}}\right)\left[\hat{E}_{\boldsymbol{v}}(\partial_{\boldsymbol{\beta}}\boldsymbol{H}_{\boldsymbol{\beta}})^{-1}\right]^{T} \tag{3.43}$$

and

$$\widehat{Var}_{\boldsymbol{v}}(\tilde{\boldsymbol{v}}^{\psi}) = \left[\hat{E}_{\boldsymbol{v}}(\partial_{\boldsymbol{v}}\boldsymbol{H}_{\boldsymbol{v}})^{-1}\right] \widehat{Var}_{\boldsymbol{v}}\left(\boldsymbol{H}_{\boldsymbol{v}}\right)\left[\hat{E}_{\boldsymbol{v}}(\partial_{\boldsymbol{v}}\boldsymbol{H}_{\boldsymbol{v}})^{-1}\right]^{T}, \tag{3.44}$$

with

$$\begin{aligned}
\hat{E}_{\boldsymbol{v}}(\partial_{\boldsymbol{\beta}}\boldsymbol{H}_{\boldsymbol{\beta}}) &= -\boldsymbol{X}\boldsymbol{V}^{-1}\boldsymbol{U}^{1/2}\tilde{\boldsymbol{R}}\boldsymbol{U}^{-1/2}\boldsymbol{X}\\
\hat{E}_{\boldsymbol{v}}(\partial_{\boldsymbol{v}}\boldsymbol{H}_{\boldsymbol{v}}) &= -\boldsymbol{Z}^{T}\boldsymbol{\Sigma}_e^{-1/2}\tilde{\boldsymbol{T}}\boldsymbol{\Sigma}_e^{-1/2}\boldsymbol{Z} - \boldsymbol{\Sigma}_v^{-1/2}\tilde{\boldsymbol{L}}\boldsymbol{\Sigma}_v^{-1/2}\\
\widehat{Var}_{\boldsymbol{v}}\left(\boldsymbol{H}_{\boldsymbol{\beta}}\right) &= \hat{\nu}(\psi(\tilde{\boldsymbol{r}}))\boldsymbol{X}\boldsymbol{V}^{-1}\boldsymbol{U}\boldsymbol{V}^{-1}\boldsymbol{X}\\
\widehat{Var}_{\boldsymbol{v}}\left(\boldsymbol{H}_{\boldsymbol{v}}\right) &= \hat{\nu}(\psi(\tilde{\boldsymbol{t}}))\boldsymbol{Z}^{T}\boldsymbol{\Sigma}_e^{-1}\boldsymbol{Z}.
\end{aligned}$$

Similarly, $\tilde{\boldsymbol{T}} = diag_n\left[I(-c < \tilde{\boldsymbol{t}} < c)\right]$, $\tilde{\boldsymbol{L}} = diag_m\left[I(-c < \tilde{\boldsymbol{l}} < c)\right]$, and $\tilde{\boldsymbol{R}} = diag_n\left[I(-c < \tilde{\boldsymbol{r}} < c)\right]$. An estimator for the conditional variance $\nu(\psi(\tilde{\boldsymbol{r}}))$ is given by

$$\hat{\nu}(\psi(\tilde{\boldsymbol{r}})) = \left(\eta\sum_{j=1}^{n}\psi^2(\tilde{r}_i)\right)/(n-p) \tag{3.45}$$

with

$$\eta = \left(1 + p\widehat{Var}_{\boldsymbol{v}}(\psi'(\tilde{\boldsymbol{r}}))/n\hat{E}_{\boldsymbol{v}}[\psi'(\tilde{\boldsymbol{r}})]^2\right),$$

as a bias correction term, that was suggested by Huber (1964). An estimate for the conditional variance $\nu(\psi(\tilde{\boldsymbol{r}}))$ is obtained by replacing $\tilde{\boldsymbol{r}}$ with $\tilde{\boldsymbol{t}}$ in equation (3.45). Finally, an estimate for the conditional prediction variance in equation (3.38) is given by

$$\widehat{Var}_{\boldsymbol{v}}\left(\tilde{\bar{y}}_i^{\psi} - \bar{y}_i\right) = h_{1i}(\tilde{\boldsymbol{\alpha}}) + h_{2i}(\tilde{\boldsymbol{\alpha}}) + h_{3i}(\tilde{\boldsymbol{\alpha}}), \tag{3.46}$$

where the first component,h_{1i}, measures the variance, caused by the estimation of the fixed regression coefficients:

$$h_{1i}(\tilde{\boldsymbol{\alpha}}) = \left(\frac{N_i - n_i}{N_i^2}\right)^2 \bar{\boldsymbol{x}}_{ri}^T \widehat{Var}_{\boldsymbol{v}}\left(\tilde{\boldsymbol{\beta}}^{\psi}\right) \bar{\boldsymbol{x}}_{ri}.$$

The second component, h_{2i}, in (3.46) measures the variance caused by the estimation of the random effects, and is given by

$$h_{2i}(\tilde{\boldsymbol{\alpha}}) = \left(\frac{N_i - n_i}{N_i^2}\right)^2 \boldsymbol{z}_i^T \widehat{Var}_{\boldsymbol{v}}(\tilde{\boldsymbol{v}}^{\psi}) \boldsymbol{z}_i.$$

The last term, h_{3i}, in (3.46) measures the variance of $\tilde{\bar{y}}_i^{\psi}$ for known $\boldsymbol{\beta}$ and $\boldsymbol{v}$, and is given by

$$h_{3i}(\tilde{\boldsymbol{\alpha}}) = \left(\frac{N_i - n_i}{N_i}\right)^2 \widehat{Var}_{\boldsymbol{v}}(\tilde{\bar{e}}_{ri}).$$

Using equations (3.46) and (3.26), the conditional MSE of the RBLUP can be estimated by

$$\widehat{MSE}_{\boldsymbol{v}}(\tilde{\bar{y}}_i^{\psi}) = h_{1i}(\tilde{\boldsymbol{\alpha}}) + h_{2i}(\tilde{\boldsymbol{\alpha}}) + h_{3i}(\tilde{\boldsymbol{\alpha}}) + \widehat{Bias}_{\boldsymbol{v}}\left(\tilde{\bar{y}}_i^{\psi}\right)^2. \tag{3.47}$$

As noted above, equation (3.36) can be seen as a first-order approximation for the conditional MSE of the REBLUP in equation (3.11) as it ignores the variability induced by the estimation of the variance components $\boldsymbol{\theta}$. By using this approximation, the conditional MSE for the REBLUP can be underestimated. To obtain a second-order approximation for the conditional MSE of the REBLUP, Chambers et al. (2014) suggest adding an extra term to equation (3.36) that accounts for the variability caused by the estimation of the variance components $\boldsymbol{\theta}$. They show that the MSE of the REBLUP can be approximated by

$$MSE_{\boldsymbol{v}}(\hat{\bar{y}}_i^{\psi}) \approx MSE_{\boldsymbol{v}}(\tilde{\bar{y}}_i^{\psi}) + E_{\boldsymbol{v}}\left[(\hat{\bar{y}}_i^{\psi} - \tilde{\bar{y}}_i^{\psi})^2\right]. \tag{3.48}$$

The authors develop an approximation of the prediction error induced by the estimation of $\boldsymbol{\theta}$, which is given by

$$\hat{\bar{y}}_i^{\psi} - \tilde{\bar{y}}_i^{\psi} \approx \left(N_i^{-1} \sum_{j \in r_i} \boldsymbol{z}_{ij}^T\right) \sum_{k=1}^{2} (\partial_{\theta_k} \boldsymbol{Q})\,(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})\,(\hat{\theta}_k^{\psi} - \theta_k). \tag{3.49}$$

Here $\boldsymbol{Q}$ is defined as in equation (3.24) but for known variance parameters. Thus, $\boldsymbol{Q}$ can be expressed as

$$\begin{aligned}\boldsymbol{Q} &= (\underbrace{\boldsymbol{Z}^T\boldsymbol{\Sigma}_e^{-1/2}\boldsymbol{D}_2\boldsymbol{\Sigma}_e^{-1/2}\boldsymbol{Z} + \boldsymbol{\Sigma}_v^{-1/2}\boldsymbol{D}_3\boldsymbol{\Sigma}_v^{-1/2}}_{=\boldsymbol{B}_1})^{-1}\underbrace{\boldsymbol{Z}^T\boldsymbol{\Sigma}_e^{-1/2}\boldsymbol{D}_2\boldsymbol{\Sigma}_e^{-1/2}}_{=\boldsymbol{B}_2}\\ &= \boldsymbol{B}_1^{-1}\boldsymbol{B}_2. \end{aligned} \tag{3.50}$$

Then, the derivatives of $\boldsymbol{Q}$ with respect to σ_v is given by

$$\partial_{\sigma_v}\boldsymbol{Q} = -\boldsymbol{B}_1^{-1}\left(\partial_{\sigma_v}\boldsymbol{B}_1\right)\boldsymbol{B}_1^{-1}\boldsymbol{B}_2 \tag{3.51}$$

with

$$\partial_{\sigma_v}\boldsymbol{B}_1 = -\frac{1}{2}\boldsymbol{\Sigma}_v^{-1}\boldsymbol{\Sigma}_v^{-1/2}\boldsymbol{D}_3\boldsymbol{\Sigma}_v^{-1/2} - \frac{1}{2}\boldsymbol{\Sigma}_v^{-1}\boldsymbol{\Sigma}_v^{-1/2}\psi'(\boldsymbol{l})\boldsymbol{\Sigma}_v^{-1/2},$$

where $\psi'(\boldsymbol{l})$ is defined as in equation (3.16). The derivatives of $\boldsymbol{Q}$ with respect to σ_e are given by

$$\partial_{\sigma_e}\boldsymbol{Q} = -\boldsymbol{B}_1^{-1}\left(\partial_{\sigma_e}\boldsymbol{B}_1\right)\boldsymbol{B}_1^{-1}\boldsymbol{B}_2 + \boldsymbol{B}_1^{-1}\left(\partial_{\sigma_e}\boldsymbol{B}_2\right) \tag{3.52}$$

with

$$\begin{aligned}\partial_{\sigma_e}\boldsymbol{B}_1 &= -\boldsymbol{Z}^T\boldsymbol{\Sigma}_e^{-1}\boldsymbol{\Sigma}_e^{-1/2}\boldsymbol{D}_2\boldsymbol{\Sigma}_e^{-1/2}\boldsymbol{Z} - \boldsymbol{Z}^T\boldsymbol{\Sigma}_e^{-1}\boldsymbol{\Sigma}_e^{-1/2}\psi'(\boldsymbol{t})\boldsymbol{\Sigma}_e^{-1/2}\boldsymbol{Z}\\ \partial_{\sigma_e}\boldsymbol{B}_2 &= -\boldsymbol{Z}^T\boldsymbol{\Sigma}_e^{-1}\boldsymbol{\Sigma}_e^{-1/2}\boldsymbol{D}_2\boldsymbol{\Sigma}_e^{-1/2}\boldsymbol{Z}^T\boldsymbol{\Sigma}_e^{-1}\boldsymbol{\Sigma}_e^{-1/2}\psi'(\boldsymbol{t})\boldsymbol{\Sigma}_e^{-1/2}\end{aligned}$$

where $\psi'(\boldsymbol{t})$ is defined as in equation (3.16). The variance of the prediction error in (3.49) can be approximated by

$$\begin{aligned}&Var_{\boldsymbol{v}}(\hat{\bar{y}}_i^{\psi} - \tilde{\bar{y}}_i^{\psi}) \approx\\ &\left(N_i^{-1}\sum_{j\in r_i}\boldsymbol{z}_{ij}^T\right)Var_{\boldsymbol{v}}\left(\sum_{k=1}^{2}\left(\partial_{\theta_k}\boldsymbol{Q}\right)\left(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta}\right)\left(\hat{\theta}_k^{\psi}-\theta_k\right)\right)\left(N_i^{-1}\sum_{j\in r_i}\boldsymbol{z}_{ij}^T\right)^T,\end{aligned} \tag{3.53}$$

where

$$\begin{aligned}&E_{\boldsymbol{v}}\left((y_j - \boldsymbol{x}_j\boldsymbol{\beta})(\hat{\theta}_k^{\psi}-\theta_k), (y_l - \boldsymbol{x}_l\boldsymbol{\beta})(\hat{\theta}_g^{\psi}-\theta_g)\right)\\ &\approx \left\{(\boldsymbol{z}_j\boldsymbol{v})(\boldsymbol{z}_l\boldsymbol{v}) + \sigma_e^2 I(j=l)\right\} E_{\boldsymbol{v}}[(\hat{\theta}_k^{\psi}-\theta_k)(\hat{\theta}_g^{\psi}-\theta_g)].\end{aligned}$$

Using equation (3.53), an estimate for the conditional MSE of the REBLUP in equation (3.48) is given by

$$\widehat{MSE}_{\boldsymbol{v}}(\hat{\bar{y}}_i^{\psi}) = h_{1i}(\tilde{\boldsymbol{\alpha}}) + h_{2i}(\tilde{\boldsymbol{\alpha}}) + h_{3i}(\tilde{\boldsymbol{\alpha}}) + h_{4i}(\tilde{\boldsymbol{\alpha}}) + \widehat{Bias}\left(\tilde{\bar{y}}_i^{\psi}\right)^2, \tag{3.54}$$

where h_{4i} measures the variability caused by the estimation of the variance components and is given by

$$h_{4i}(\tilde{\boldsymbol{\alpha}}) = \left(N_i^{-1} \sum_{j \in r_i} \boldsymbol{z}_{ij}^T \right) \Upsilon \; Var_{\boldsymbol{v}}(\hat{\boldsymbol{\theta}}^{\psi}) \left(N_i^{-1} \sum_{j \in r_i} \boldsymbol{z}_{ij}^T \right)^T , \tag{3.55}$$

with

$$\Upsilon = \sum_{k=1}^{2} \sum_{g=1}^{2} \left\{ (\partial_{\theta_k} \boldsymbol{Q}) \left[\sum_{j=1}^{m} \sum_{l=1}^{m} (\boldsymbol{z}_{ij}^T \boldsymbol{v})(\boldsymbol{z}_{il}^T \boldsymbol{v}) + \sigma_e^2 I(j = l) \right] \left(\partial_{\theta_g} \boldsymbol{Q} \right)^T \right\} .$$

To estimate the variance matrix $Var(\hat{\boldsymbol{\theta}})$ of the variance components, Chambers et al. (2014) suggest using the robust estimation equations in (3.10) of Sinha and Rao (2009), and apply a first-order approximation. Explicit formulas are provided in the supplemental material to Chambers et al. (2014). To estimate the conditional MSE from equation (3.54), the parameters $\tilde{\boldsymbol{v}}$ and $\tilde{\boldsymbol{\beta}}$ and $\boldsymbol{\theta}$ need to be replaced with $\hat{\boldsymbol{v}}^{\psi}$ and $\hat{\boldsymbol{\beta}}^{\psi}$ and $\hat{\boldsymbol{\theta}}^{\psi}$, respectively, leading to

$$\widehat{MSE}_{CCST}(\hat{\bar{y}}_i^{\psi}) = h_{1i}(\hat{\boldsymbol{\alpha}}) + h_{2i}(\hat{\boldsymbol{\alpha}}) + h_{3i}(\hat{\boldsymbol{\alpha}}) + h_{4i}(\hat{\boldsymbol{\alpha}}) + \widehat{Bias}\left(\hat{\bar{y}}_i^{\psi} \right)^2 . \tag{3.56}$$

According to the surnames of the authors who contributed to Chambers et al. (2014), I refer to their MSE estimator in (3.56) by adding the subscript CCST.
The linearization approach can also be applied to develop a conditional MSE estimate for the REBLUP-bc in (3.20). Again, Chambers et al. (2014) first develop a conditional MSE for the REBLUP-bc assuming that the variance parameter $\boldsymbol{\theta}$ are known, i.e., for the RBLUP-bc. As the RBLUP-bc for the area mean $\bar{y}_i$ is approximately unbiased, the conditional MSE is given by

$$MSE_{\boldsymbol{v}}(\tilde{\bar{y}}_i^{\psi-bc}) = Var_{\boldsymbol{v}}(\tilde{\bar{y}}_i^{\psi-bc}). \tag{3.57}$$

Analogously to equation (3.37), the prediction error for RBLUP-bc of the area mean is given by

$$\tilde{\bar{y}}_i^{\psi} - \bar{y}_i = \left(\frac{N_i - n_i}{N_i} \right) \left\{ \bar{\boldsymbol{x}}_{ri}^T \tilde{\boldsymbol{\beta}}^{\psi} + \boldsymbol{z}_i^T \tilde{\boldsymbol{v}}^{\psi} + (1 - n_i N_i^{-1})^{-1} \widetilde{bc}_i - \bar{y}_{ri} \right\}. \tag{3.58}$$

The bias correction $\widetilde{bc}_i$ is defined as in equation (3.1.2), but for known variance parameters $\boldsymbol{\theta}$. Chambers et al. (2014) approximate the bias correction term by using a Taylor series approximation leading to

$$(1 - n_i N_i^{-1})^{-1} \widetilde{bc}_i \approx n_i^{-1} \sum_{j \in s_i} \omega_i \psi_b \Big\{ (y_{ij} - \boldsymbol{x}_{ij}^T \boldsymbol{\beta} - v_i)/\omega_i \Big\}$$

$$
\begin{aligned}
&- (\tilde{\boldsymbol{v}}^{\psi} - \boldsymbol{v})^T \left\{ n_i^{-1} \sum_{j \in s_i} \psi_b' \Big\{ \frac{y_{ij} - \boldsymbol{x}_{ij}^T \boldsymbol{\beta} - v_i}{\omega_i} \Big\} \boldsymbol{z}_{ij} \right\} \\
&- (\tilde{\boldsymbol{\beta}}^{\psi} - \boldsymbol{\beta})^T \left\{ n_i^{-1} \sum_{j \in s_i} \psi_b' \Big\{ \frac{y_{ij} - \boldsymbol{x}_{ij}^T \boldsymbol{\beta} - v_i}{\omega_i} \Big\} \boldsymbol{x}_{ij} \right\}.
\end{aligned} \tag{3.59}
$$

When the tuning constant b in ψ_b is sufficiently large the outer derivative becomes $\psi' \approx 1$. Then, the approximation in (3.59) reduces to

$$
\begin{aligned}
(1 - n_i N_i^{-1})^{-1} \widetilde{bc}_i \approx n_i^{-1} \sum_{j \in s_i} \omega_i \psi_b \Big\{ (y_{ij} - \boldsymbol{x}_{ij}^T \boldsymbol{\beta} - v_i)/\omega_i \Big\} \\
- (\tilde{\boldsymbol{v}}^{\psi} - \boldsymbol{v})^T \bar{\boldsymbol{z}}_{si} - (\tilde{\boldsymbol{\beta}}^{\psi} - \boldsymbol{\beta})^T \bar{\boldsymbol{x}}_{si}.
\end{aligned} \tag{3.60}
$$

The latter expression can be inserted into the prediction error from (3.58), leading to

$$
\begin{aligned}
\tilde{\bar{y}}_i^{\psi - bc} - \bar{y}_i \approx \left(\frac{N_i - n_i}{N_i} \right) \Bigg\{ & (\bar{\boldsymbol{x}}_{ri} - \bar{\boldsymbol{x}}_{si})^T (\tilde{\boldsymbol{\beta}}^{\psi} - \boldsymbol{\beta}) + (\bar{\boldsymbol{z}}_{ri} - \bar{\boldsymbol{z}}_{si})^T (\tilde{\boldsymbol{v}}^{\psi} - \boldsymbol{v}) \\
& + n_i^{-1} \sum_{j \in s_i} \omega_i \psi_b \Big\{ (y_{ij} - \boldsymbol{x}_{ij}^T \boldsymbol{\beta} - v_i)/\omega_i \Big\} - \bar{e}_{ri} \Bigg\}.
\end{aligned} \tag{3.61}
$$

Note that under the nested error regression model in (2.24) $\bar{\boldsymbol{z}}_{ri} - \bar{\boldsymbol{z}}_{si} = \mathbf{0}$. Thus, the conditional prediction variance of the RBLUP-bc is given by

$$
\begin{aligned}
Var_{\boldsymbol{v}} \left(\tilde{\bar{y}}_i^{\psi - bc} - \bar{y}_i \right) = \left(\frac{N_i - n_i}{N_i} \right)^2 \Bigg\{ & (\bar{\boldsymbol{x}}_{ri} - \bar{\boldsymbol{x}}_{si})^T Var_{\boldsymbol{v}}(\tilde{\boldsymbol{\beta}}^{\psi}) (\bar{\boldsymbol{x}}_{ri} - \bar{\boldsymbol{x}}_{si}) \\
& + Var_{\boldsymbol{v}} \left(n_i^{-1} \sum_{j \in s_i} \omega_i \psi_b \Big\{ (y_{ij} - \boldsymbol{x}_{ij}^T \boldsymbol{\beta} - v_i)/\omega_i \Big\} \right) \\
& + Var_{\boldsymbol{v}}(\bar{e}_{ri}) \Bigg\}.
\end{aligned} \tag{3.62}
$$

The conditional MSE in (3.57) can be estimated by

$$
\widehat{MSE}_{\boldsymbol{v}}(\tilde{\bar{y}}_i^{\psi - bc}) = h_{1i}^{bc}(\tilde{\boldsymbol{\alpha}}) + h_{2i}^{bc}(\tilde{\boldsymbol{\alpha}}) + h_{3i}(\tilde{\boldsymbol{\alpha}}), \tag{3.63}
$$

where the first component, h_{1i}^{bc}, is given by

$$
h_{1i}^{bc}(\tilde{\boldsymbol{\alpha}}) = \left(\frac{N_i - n_i}{N_i} \right)^2 (\bar{\boldsymbol{x}}_{ri} - \bar{\boldsymbol{x}}_{si})^T \widehat{Var}_{\boldsymbol{v}} \left(\tilde{\boldsymbol{\beta}}^{\psi} \right) (\bar{\boldsymbol{x}}_{ri} - \bar{\boldsymbol{x}}_{si}),
$$

the second component h_{2i}^{bc} arises due to the bias correction and is given by

$$\begin{aligned} h_{2i}^{bc}(\tilde{\boldsymbol{\alpha}}) &= \left(\frac{N_i - n_i}{N_i}\right)^2 \widehat{Var}_{\boldsymbol{v}} \left(n_i^{-1} \sum_{j \in s_i} \omega_i \psi_b \Big\{ (y_{ij} - \boldsymbol{x}_{ij}^T \boldsymbol{\beta} - v_i)/\omega_i \Big\} \right) \\ &= \left(\frac{N_i - n_i}{N_i}\right)^2 \left\{ n_i^{-2} \sum_{j \in s_i} \hat{E}_{\boldsymbol{v}} \left[\omega_i \psi_b \Big\{ (y_{ij} - \boldsymbol{x}_{ij}^T \boldsymbol{\beta} - v_i)/\omega_i \Big\} \right]^2 \right\} \\ &= \left(\frac{N_i - n_i}{N_i}\right)^2 \left\{ \frac{1}{n_i(n_i - p)} \sum_{j \in s_i} \left[\omega_i \psi_b \Big\{ (y_{ij} - \boldsymbol{x}_{ij}^T \tilde{\boldsymbol{\beta}}^{\psi} - \tilde{v}_i^{\psi})/\omega_i \Big\} \right]^2 \right\}, \end{aligned}$$

and the third component h_{3i} is identical to h_{3i} from equation (3.47). Analogously to equation (3.48), a second-order approximation for the conditional MSE of the REBLUP-bc is obtained by adding an extra term that accounts for the variability caused by the estimation of $\boldsymbol{\theta}$. Chambers et al. (2014) show that the prediction error induced by the estimation of $\boldsymbol{\theta}$ can be approximated by

$$\begin{aligned} \hat{\bar{y}}_i^{\psi-bc} - \tilde{\bar{y}}_i^{\psi-bc} \approx \sum_{k=1}^{2} \Bigg\{ & N_i^{-1} \sum_{j \in r_i} \boldsymbol{z}_{ij}^T \left(\partial_{\theta_k} \boldsymbol{Q} \right) \left(\boldsymbol{y} - \boldsymbol{X} \boldsymbol{\beta} \right) \\ & + \left(\frac{N_i - n_i}{N_i n_i} \right) \sum_{j \in s_i} \partial_{\theta_k} \left(\omega_i \psi_b \Big\{ (y_{ij} - \boldsymbol{x}_{ij}^T \boldsymbol{\beta} - v_i)/\omega_i \Big\} \right) \Bigg\} (\hat{\theta}_k - \theta_k). \end{aligned} \tag{3.64}$$

An approximation for the conditional variance of the latter expression is given by

$$Var_{\boldsymbol{v}} \left(\hat{\bar{y}}_i^{\psi-bc} - \tilde{\bar{y}}_i^{\psi-bc} \right) \approx \left(\frac{N_i - n_i}{N_i} \right)^2 \boldsymbol{\Omega}_i^T \Upsilon \; Var_{\boldsymbol{v}}(\hat{\boldsymbol{\theta}}) \boldsymbol{\Omega}_i, \tag{3.65}$$

where

$$\boldsymbol{\Omega}_i = \bar{\boldsymbol{z}}_{ri} - n_i^{-1} \sum_{j \in s_i} \psi_b' \Big\{ (y_{ij} - \boldsymbol{x}_{ij}^T \boldsymbol{\beta} - v_i)/\omega_i \Big\} \boldsymbol{z}_{ij},$$

and Υ is defined as in equation (3.55). When the tuning constant b is sufficiently large, then $\psi_b' \approx 1$. In that case $\boldsymbol{\Omega}_i = \boldsymbol{0}$ under model (2.24), and the conditional variance in (3.65) reduces to zero. Using these results, an estimate for the conditional MSE of the REBLUP-bc in equation (3.20) is given by

$$\widehat{MSE}_{\boldsymbol{u}}(\hat{\bar{y}}_i^{\psi-bc}) = h_{1i}^{bc}(\tilde{\boldsymbol{\alpha}}) + h_{2i}^{bc}(\tilde{\boldsymbol{\alpha}}) + h_{3i}(\tilde{\boldsymbol{\alpha}}) + h_{4i}^{bc}(\tilde{\boldsymbol{\alpha}}), \tag{3.66}$$

where h_{4i}^{bc} is given by (3.65). To estimate the conditional MSE in equation (3.66) the parameters $\tilde{\boldsymbol{v}}^{\psi}$ and $\tilde{\boldsymbol{\beta}}^{\psi}$ and $\boldsymbol{\theta}$ in (3.66) need to be replaced by $\hat{\boldsymbol{v}}^{\psi}$

and $\hat{\boldsymbol{\beta}}^{\psi}$ and $\hat{\boldsymbol{\theta}}^{\psi}$, respectively, leading to

$$\widehat{MSE}_{CCST}(\hat{\bar{y}}_i^{\psi-bc}) = h_{1i}^{bc}(\hat{\boldsymbol{\alpha}}) + h_{2i}^{bc}(\hat{\boldsymbol{\alpha}}) + h_{3i}(\hat{\boldsymbol{\alpha}}) + h_{4i}^{bc}(\hat{\boldsymbol{\alpha}}). \tag{3.67}$$

Chambers et al. (2014) examined the performance of their proposed conditional MSE estimators in a simulation study. The overall results showed that both approaches, the pseudo-linearization (CCT) and the linearization (CCST), perform comparably to parametric bootstrap methods with robust estimates for the variance components. In addition, both analytic methods were able to track the area-specific variability of the tested estimators. However, the CCST approach appeared to be more stable compared to the CCT approach as the variation between the iterations of the simulation study was considerably lower for the CCST estimator of the conditional MSE.

3.2 Robust Extensions to Spatial EBLUP Approaches

The spatial extensions of the EBLUP introduced in Section 2.4 account for different kinds of spatial effects. In particular, they extend the LMM in (2.24) by incorporating spatial information into the model. Thus, estimated small area means obtained by these model extensions borrow strength over space. However, these estimators still rely on the normality assumption of the error term components which may be violated in the presence of outliers. Schmid and Münnich (2014) introduce a robust extension to the SEBLUP of the area mean from equation (2.46) based on the results of Sinha and Rao (2009). Rao et al. (2014) propose a robust extension to the NPEBLUP from equation (2.54) under the spline nested error regression model in (2.53) using the results of Fellner (1986) on robust mixed model equations. In this section both approaches are reviewed: the robust SEBLUP in Section 3.2.1 and the robust NPEBLUP in Section 3.2.2. So far, robust extensions for the GWEBLUP of Chandra et al. (2012) from Section 2.4.4 have not been considered in the literature. Hence, robust extensions for the GWEBLUP of the area mean are presented in the next chapter as the proposed methods of this thesis.

3.2.1 The Robust Spatial EBLUP

Following the same principals as in Sinha and Rao (2009), Schmid and Münnich (2014) propose a robust SEBLUP (SREBLUP), where they develop robust ML estimating equations for the parameters in the LMM model in (2.44) under spatial correlation. Under this model, maximizing the density of $\boldsymbol{y}$ with respect to $\boldsymbol{\beta}$, $\boldsymbol{\theta}$ and ρ leads to these ML estimation equations:

$$\boldsymbol{X}^T\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X\beta}) = \boldsymbol{0} \tag{3.68}$$

$$-\frac{1}{2}tr(\boldsymbol{V}^{-1}\frac{\partial \boldsymbol{V}}{\partial \theta_l}) - \frac{1}{2}(\boldsymbol{y} - \boldsymbol{X\beta})^T\frac{\partial \boldsymbol{V}^{-1}}{\partial \theta_l}(\boldsymbol{y} - \boldsymbol{X\beta}) = \boldsymbol{0} \tag{3.69}$$

$$-\frac{1}{2}tr(\boldsymbol{V}^{-1}\frac{\partial \boldsymbol{V}}{\partial \rho}) - \frac{1}{2}(\boldsymbol{y}-\boldsymbol{X\beta})^T\frac{\partial \boldsymbol{V}^{-1}}{\partial \rho}(\boldsymbol{y}-\boldsymbol{X\beta}) = \mathbf{0}. \tag{3.70}$$

To obtain robust estimates for $\boldsymbol{\beta}$, $\boldsymbol{\theta}$ and ρ, the equations from above were robustified analogously to the robust ML estimation equations in (3.9) and (3.10), leading to

$$\boldsymbol{X}^T\boldsymbol{V}^{-1}\boldsymbol{U}^{1/2}\psi(\boldsymbol{r}) = \mathbf{0} \tag{3.71}$$

$$-\frac{1}{2}tr(\boldsymbol{KV}^{-1}\frac{\partial \boldsymbol{V}}{\partial \theta_l}) + \frac{1}{2}\psi(\boldsymbol{r})^T\boldsymbol{U}^{1/2}\boldsymbol{V}^{-1}\frac{\partial \boldsymbol{V}}{\partial \theta_l}\boldsymbol{V}^{-1}\boldsymbol{U}^{1/2}\psi(\boldsymbol{r}) = \mathbf{0} \tag{3.72}$$

$$-\frac{1}{2}tr(\boldsymbol{KV}^{-1}\frac{\partial \boldsymbol{V}}{\partial \rho}) + \frac{1}{2}\psi(\boldsymbol{r})^T\boldsymbol{U}^{1/2}\boldsymbol{V}^{-1}\frac{\partial \boldsymbol{V}}{\partial \rho}\boldsymbol{V}^{-1}\boldsymbol{U}^{1/2}\psi(\boldsymbol{r}) = \mathbf{0}. \tag{3.73}$$

As in Sinha and Rao (2009), Schmid and Münnich (2014) applied robust ML equations in two steps, where they first obtain robust estimates for $\boldsymbol{\beta}$, $\boldsymbol{\theta}$ and ρ simultaneously by solving (3.71), (3.72) and (3.73), using a Newton-Raphson algorithm. Later, Schmid et al. (2016) suggest solving equation (3.72) for $\boldsymbol{\theta}$ using the fixed point algorithm, as presented in (3.12) instead of the Newton-Raphson algorithm. At convergence, they define the spatial robust estimators $\hat{\boldsymbol{\beta}}^{\psi,sp}$, $\hat{\boldsymbol{\theta}}^{\psi,sp}$ and $\hat{\rho}^{\psi}$. In a second step, $\hat{\boldsymbol{\beta}}^{\psi,sp}$, $\hat{\boldsymbol{\theta}}^{\psi,sp}$ and $\hat{\rho}^{\psi}$ are used to find robust estimates of $\boldsymbol{v}$ by plugging them into Fellner's equation (3.2), and solving it using a Newton-Raphson algorithm. Given the robust estimates of $\boldsymbol{\beta}$ and $\boldsymbol{v}$, the SREBLUP is obtained by plugging these estimates into the SEBLUP in equation (2.46). Thus, under model (2.44), the SREBLUP for the small area mean $\bar{y}_i$ is defined as

$$\hat{\bar{y}}_i^{\psi,sp} = N_i^{-1}\Big\{\sum_{j\in s_i} y_{ij} + \sum_{j\in r_i}(\boldsymbol{x}_{ij}^T\hat{\boldsymbol{\beta}}^{\psi,sp} + \hat{v}_i^{\psi,sp})\Big\}. \tag{3.74}$$

Schmid et al. (2016) also proposed robust predictive versions of the robust projective estimator above. They therefore develop bias corrections that account for spatial correlation between the area specific random effects. Analogously to the local bias correction of Chambers et al. (2014) in equation (3.20), they propose a robust predictive SREBLUP (SREBLUP-bc) for the mean $\bar{y}_i$ in area i, which is given by

$$\hat{\bar{y}}_i^{\psi,sp-bc} = \hat{\bar{y}}_i^{\psi} + bc_i^{sp} \tag{3.75}$$

with

$$bc_i^{sp} = \left(\frac{N_i - n_i}{N_i}\right) n_i^{-1}\sum_{j\in s_i}\omega_i^{sp}\psi_b\Big\{(y_{ij} - \mathbf{x}_{ij}\hat{\boldsymbol{\beta}}^{\psi,sp} - \hat{v}_i{}^{sp})/\omega_i^{sp}\Big\}, \tag{3.76}$$

where ω_i^{sp} is a robust scale estimator, given by the median absolute deviation of the residuals in area i that depend on the parameters $\hat{\boldsymbol{\beta}}^{\psi,sp}$ and $\hat{v}_i^{\psi,sp}$. Additionally, Schmid et al. (2016) develop fully bias-corrected, robust predictive estimators under spatially correlated random effects following the two approaches of Dongmo-Jiongo et al. (2013). To obtain an MSE estimation Schmid et al. (2016) propose a parametric bootstrap approach that accounts for the spatial structure in the data.

3.2.2 The Robust Non-Parametric EBLUP

Rao et al. (2014) note that the NPEBLUP of the area mean in equation (2.54) of Opsomer et al. (2008) can be sensitive to outliers and propose a robustified version of the NPEBLUP (RNPEBLUP). According to the authors, the robust method of Sinha and Rao (2009) runs into difficulties in the context of spline mixed models. This is due to the definition of the standardized residuals $\boldsymbol{r} = \boldsymbol{U}^{-1/2}(\boldsymbol{y}-\boldsymbol{X\beta})$ in the robust ML equations (3.9) and (3.10), where the spline term $\boldsymbol{D\gamma}$ from model (2.53) is not included. Thus, applying these estimation equations to the spline LMM in (2.53) can lead to large absolute values in $\boldsymbol{r}$ causing the restricted residuals $\psi(\boldsymbol{r})$ to be nearly constant. Rao et al. (2014) avoid this difficulty in a two-step procedure where they first define robust estimation equations for the estimation of σ_e^2 and σ_v^2. Based on the spline LMM in (2.53), the ML estimation equations for the estimation of $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ and $\boldsymbol{v}$ are given by

$$\boldsymbol{X}^T\boldsymbol{\Sigma}_e^{-1}(\boldsymbol{y}-\boldsymbol{X\beta}-\boldsymbol{D\gamma}-\boldsymbol{Zv}) = \boldsymbol{0} \tag{3.77}$$
$$\boldsymbol{Z}^T\boldsymbol{\Sigma}_e^{-1}(\boldsymbol{y}-\boldsymbol{X\beta}-\boldsymbol{D\gamma}-\boldsymbol{Zv}) - \boldsymbol{\Sigma}_v^{-1}\boldsymbol{v} = \boldsymbol{0} \tag{3.78}$$
$$\boldsymbol{D}^T\boldsymbol{\Sigma}_e^{-1}(\boldsymbol{y}-\boldsymbol{X\beta}-\boldsymbol{D\gamma}-\boldsymbol{Zv}) - \boldsymbol{\Sigma}_\gamma^{-1}\boldsymbol{\gamma} = \boldsymbol{0}. \tag{3.79}$$

Using this system of equation, Rao et al. (2014) develop fixed-point equations for σ_γ^2, σ_v^2 and σ_e^2, and robustify these for σ_v^2 and σ_e^2. Note that they do not robustify the estimation of σ_γ^2, as the spline parameters $\boldsymbol{\gamma}$ are not affected by outliers. Explicit expression of the fixed-point equations for σ_e^2, σ_v^2 and σ_γ^2 are provided in Rao et al. (2014), as their equations 21, 20 and 13, respectively. In a second step, they apply Fellner's robust ML estimation equations. Under model (2.53), these are given by

$$\boldsymbol{X}^T\boldsymbol{\Sigma}_e^{-1/2}\psi(\boldsymbol{\epsilon}) = \boldsymbol{0} \tag{3.80}$$
$$\boldsymbol{Z}^T\boldsymbol{\Sigma}_e^{-1/2}\psi(\boldsymbol{\epsilon}) - \boldsymbol{\Sigma}_u^{-1/2}\psi(\boldsymbol{\Sigma}_v^{-1/2}\boldsymbol{v}) = \boldsymbol{0} \tag{3.81}$$
$$\boldsymbol{D}^T\boldsymbol{\Sigma}_e^{-1/2}\psi(\boldsymbol{\epsilon}) - \boldsymbol{\Sigma}_\gamma^{-1/2}\psi(\boldsymbol{\Sigma}_\gamma^{-1/2}\boldsymbol{\gamma}) = \boldsymbol{0}, \tag{3.82}$$

where $\boldsymbol{\epsilon} = \boldsymbol{\Sigma}_e^{-1/2}(\boldsymbol{y}-\boldsymbol{X\beta}-\boldsymbol{D\gamma}-\boldsymbol{Zv})$. Using these robust ML estimation equations and the fixed-point equations from the first step, robust predictions for $\boldsymbol{\gamma}$ and $\boldsymbol{v}$ and robust estimators for $\boldsymbol{\beta}$,σ_e^2, σ_v^2 and σ_γ^2 are obtained simultaneously. Given robust estimates for $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ and $\boldsymbol{v}$, the RNPEBLUP is obtained by plugging these estimates into the NPEBLUP in equation (2.54). Thus, under model (2.53), the RNPEBLUP for the mean $\bar{y}_i$ in area

i is defined as

$$\hat{\bar{y}}_i^{\psi,np} = N_i^{-1}\Big\{\sum_{j\in s_i} y_{ij} + \sum_{j\in r_i}(\boldsymbol{x}_{ij}^T\hat{\boldsymbol{\beta}}^{\psi,np} + \boldsymbol{d}_{ij}^T\hat{\boldsymbol{\gamma}}^{\psi,np} + \hat{v}_i^{\psi,np})\Big\}. \tag{3.83}$$

Rao et al. (2014) proposed a parametric bootstrap approach for the MSE estimation of the RNPEBLUP in (3.83), following the procedure, that has been proposed by Sinha and Rao (2009) for the MSE estimation of the REBLUP in (3.11). A robust predictive extension to the robust projective NPEBLUP in equation (3.83) has not been developed yet, and remains an open research question (cf. Dumitrescu, 2017).

3.3 Discussion

It is important to emphasize that all EBLUP approaches presented in Chapter 2 are based on different specifications of the LMM which rely on normality assumptions for the error term components. The robust EBLUP approaches presented in this chapter attempt to enhance the reliability of these approaches in the presence of outliers, by restricting the influence of extreme observation on the parameter estimation.
As an alternative estimation method that avoids the strong distributional assumptions of the LMM, Chambers and Tzavidis (2006) introduce the M-quantile regression of Breckling and Chambers (1988) into the field of SAE. This approach is a generalization of the quantile regression (cf. Koenker and Bassett, 1978) as it is flexible regarding the choice of influence function underpinning the estimation procedure. The M-quantile approach is a robust and distribution free method for modeling the relationship between a dependent and a set of explanatory variables at different quantiles of the target variable. The basic idea is to estimate M-quantile coefficients for different quantiles of the target variable, i.e., for a fine grid of quantiles on the (0,1) interval. For the estimation of the area mean $\bar{y}_i$, predictions for the non-sampled population units are based on the average of these estimated M-quantile coefficients in area i (cf. Rao and Molina, 2015, Chapter 7.5). Predictions based on M-quantiles can be considered to be robust projective as these are biased in the presence of non-symmetric outliers. Hence, bias corrections were suggested by Tzavidis et al. (2010) and Chambers et al. (2014) to obtain robust predictive M-quantile estimators.
Chambers et al. (2014) compare the performance of small area estimators based on the robust projective REBLUP in (3.11) and the robust predictive REBLUP-bc in (3.20) with respective M-quantile estimators (MQ and MQ-bc). The main findings suggest that the robust predictive estimators (REBLUP-bc and MQ-bc) are less biased than robust projective estimators (REBLUP and MQ) in the presence of non-symmetric outliers. In addition, both robust EBLUP approaches, the REBLUP and the REBLUP-bc

performed better in terms of efficiency compared to their M-quantile counterparts.
The small area M-quantile approach of Chambers and Tzavidis (2006) has also been investigated in the context of spatial modeling. Salvati et al. (2009), for instance, obtain small area estimates using M-quantile coefficients based on a LMM specification with spatially correlated random effects. For non-linear relationships, Pratesi et al. (2009) develop an M-quantile version for the non-parametric approach of Opsomer et al. (2008). Salvati et al. (2012) specify a local M-quantile model for SAE where they explore the use of the geographically weighted regression based on M-quantile modeling. Similar to Chandra et al. (2012), they capture spatial non-stationarity by allowing the model parameters to vary over space using a distance matrix that defines the geographical weights.

The Robust Geographically Weighted EBLUP 4

The local LMM in (2.61) is defined separately for each population unit with respect to the geographical location. Here, the local model coefficients for a unit j in area i are estimated using spatial weights that depend on the distances to all sampled observations. Thus, observations that have a large distance to that unit have lower weights compared to observations that are closer. The implication of this weighting scheme regarding the presence of outliers is twofold: (i) extreme observations that are far away have a small influence on the parameter estimation for unit j in area i whereas (ii) the influence is large for extreme observations that are close. In the first case, restricting the influence of outliers may not be crucial as the spatial weights themselves can have a robustifying effect on the parameter estimation. In the second case, however, the restriction becomes more important as the spatial weights can aggravate the effect of outliers. In practice, it is difficult to identify or even locate outliers. Hence, it can be useful to combine the geographical weighting with an influence function that restricts the impact of extreme observations on the parameter estimation.
Following the ideas of Sinha and Rao (2009), a robust version for the GWEBLUP of the area mean in (2.65) is developed in this chapter. This extends the current literature for SAE as robust extensions for the GWEBLUP have not been considered in the literature, so far. In particular, the robust estimation of the area mean is presented in Section 4.1 where a robust projective GWEBLUP (RGWEBLUP) and a bias-corrected robust predictive GWEBLUP (RGWEBLUP-bc) is proposed. In addition, an algorithm for estimating the robust model parameters is provided. In Section 4.2, the MSE estimation for the proposed estimators is presented. Based on the pseudo-linearization and the linearization approach in Chambers et al. (2014), two analytic MSE estimators that account for the spatial weights in the point estimators are proposed. In addition to the theoretical derivations, the proposed methods are implemented in the package `saeRGWR` for the R-language (R Core Team, 2016). In Section 4.3, the main functions of this package are presented.

4.1 Estimation of the Small Area Mean

To develop an outlier robust parameter estimation for the GWEBLUP in equation (2.65), the two-step procedure of Sinha and Rao (2009) described in Section 3.1 is utilized for the local LMM in (2.61). Thus, robust ML estimating equations are applied in two steps: (1) robust estimates for $\boldsymbol{\theta}$ and the local coefficients $\boldsymbol{\beta}_{ij}$ are obtained simultaneously by solving modified versions of (3.10) and (3.9), respectively, with an iterative algorithm; (2) the estimates from step one are used to solve a modified version of Fellner's equation (3.2) iteratively to obtain a robust estimate for $\boldsymbol{v}$. Under the local LMM (2.61), maximizing the density of $\boldsymbol{y}$ with respect to $\boldsymbol{\beta}_{ij}$ and $\boldsymbol{\theta}$ leads to the ML estimation equations

$$\boldsymbol{X}^T\boldsymbol{V}_{ij}^{-1}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta}_{ij}) = \boldsymbol{0} \tag{4.1}$$

$$-\frac{1}{2}tr(\boldsymbol{V}^{-1}\frac{\partial\boldsymbol{V}}{\partial\theta_l}) - \frac{1}{2}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})^T\frac{\partial\boldsymbol{V}^{-1}}{\partial\theta_l}(\boldsymbol{y}-\boldsymbol{\lambda}) = \boldsymbol{0}. \tag{4.2}$$

To obtain robust estimates for $\boldsymbol{\beta}_{ij}$ and $\boldsymbol{\theta}$ in step (1), these equations are robustified analogously to the estimation equations of Sinha and Rao (2009) in (3.9) and (3.10), leading to

$$\boldsymbol{X}^T\boldsymbol{V}_{ij}^{-1}\boldsymbol{U}_{ij}^{1/2}\psi(\boldsymbol{r}_{ij}) = \boldsymbol{0} \tag{4.3}$$

$$-\mathrm{tr}(\boldsymbol{V}^{-1}\frac{\partial\boldsymbol{V}}{\partial\theta_l}\boldsymbol{K}) + \psi(\boldsymbol{\tau})^T\boldsymbol{U}^{1/2}\boldsymbol{V}^{-1}\frac{\partial\boldsymbol{V}}{\partial\theta_l}\boldsymbol{V}^{-1}\boldsymbol{U}^{1/2}\psi(\boldsymbol{\tau}) = \boldsymbol{0}. \tag{4.4}$$

Here, $\boldsymbol{r}_{ij} = \boldsymbol{U}_{ij}^{-1/2}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta}_{ij})$ is a vector of standardized local residuals with $\boldsymbol{U}_{ij} = diag(\boldsymbol{V}_{ij})$, and $\boldsymbol{\tau} = \boldsymbol{U}^{-1/2}(\boldsymbol{y}-\boldsymbol{\lambda})$ is a vector of standardized global residuals, where the projection $\boldsymbol{\lambda}$ is given by $\boldsymbol{\lambda} = (\boldsymbol{X}\circ\tilde{\boldsymbol{B}}_s)\boldsymbol{1}_p$. As there is no closed-form solution to equations (4.3) and (4.4), $\boldsymbol{\beta}_{ij}$ and $\boldsymbol{\theta}$ are approximated iteratively. At convergence, these approximations are defined as the robust geographically weighted estimators $\hat{\boldsymbol{\beta}}_{ij}^{\psi,gw}$ and $\hat{\boldsymbol{\theta}}^{\psi,gw}$ of $\boldsymbol{\beta}_{ij}$ and $\boldsymbol{\theta}$, respectively. In step (2), these estimates are plugged into a modified version of Fellner's robust ML estimating equation (3.2) given by

$$\boldsymbol{Z}^T\boldsymbol{\Sigma}_e^{-1/2}\psi(\boldsymbol{\Sigma}_e^{-1/2}(\boldsymbol{y}-\boldsymbol{\lambda}-\boldsymbol{Z}\boldsymbol{v})) - \boldsymbol{\Sigma}_v^{-1/2}\psi(\boldsymbol{\Sigma}_v^{-1/2}\boldsymbol{v}) = \boldsymbol{0}. \tag{4.5}$$

Again, there is no closed-form solution of this equation and $\boldsymbol{v}$ is approximated iteratively. Given the robust estimates for $\boldsymbol{\beta}$ and $\boldsymbol{v}$, the RGWEBLUP of the area mean is obtained by replacing the non-robust estimates $\hat{\boldsymbol{\beta}}_{ij}^{gw}$ and $\hat{\boldsymbol{v}}^{gw}$ in equation (2.65) with their robust versions $\hat{\boldsymbol{\beta}}_{ij}^{\psi,gw}$ and $\hat{\boldsymbol{v}}^{\psi,gw}$. Thus, under model (2.61), the RGWEBLUP for the mean $\bar{y}_i$ in area i is defined by

$$\hat{\bar{y}}_i^{\psi,gw} = N_i^{-1}\Big\{\sum_{j\in s_i} y_{ij} + \sum_{j\in r_i}(\boldsymbol{x}_{ij}^T\hat{\boldsymbol{\beta}}_{ij}^{\psi,gw} + \hat{v}_i^{\psi,gw})\Big\}. \tag{4.6}$$

In case of non-sampled areas this estimator reduces to a synthetic estimator $\hat{\bar{y}}_i^{\psi,gw-syn} = N^{-1}\sum_{j\in U_i}\boldsymbol{x}_{ij}^T\hat{\boldsymbol{\beta}}_{ij}^{\psi,gw}$ with an area-specific random effect equal to zero. When the geographical information for the non-sampled units is not available, the centroid information of the areas can be used as an approximation for the unknown locations, similar to equation (2.66). The coefficients here are constant within an area and the RGWEBLUP for the mean $\bar{y}_i$ in area i becomes

$$\hat{\bar{y}}_i^{\psi,gw} = N_i^{-1}\Big\{\sum_{j\in s_i} y_{ij} + \sum_{j\in r_i}(\boldsymbol{x}_{ij}^T\hat{\boldsymbol{\beta}}_i^{\psi,gw} + \hat{v}_i^{\psi,gw})\Big\}. \tag{4.7}$$

This estimator reduces to the synthetic estimator $\hat{\bar{y}}_i^{\psi,gw-syn} = \bar{\boldsymbol{x}}_i^T\hat{\boldsymbol{\beta}}_i^{\psi,gw}$ in case of non-samples areas.
The proposed RGWEBLUP of the small area mean is a robust projective estimator which can be biased in the presence of non-symmetric outliers as the non-sampled population units are assumed to follow the underlying model (2.61). In order to develop a robust predictive estimator, the local bias correction of Chambers et al. (2014) in equation (3.20) is extended to the case of spatial non-stationarity. A robust predictive GWEBLUP (RGWEBLUP-bc) for the mean $\bar{y}_i$ in area i, that accounts for the geographical weights, is given by

$$\hat{\bar{y}}_i^{\psi,gw-bc} = \hat{\bar{y}}_i^{\psi,gw} + bc_i^{gw} \tag{4.8}$$

with

$$bc_i^{gw} = \Big(\frac{N_i - n_i}{N_i}\Big)\frac{1}{\sum_{j\in s_i}\bar{w}_{ij}}\sum_{j\in s_i}\bar{w}_{ij}\omega_i^{gw}\psi_b\Big\{\frac{y_{ij} - \mathbf{x}_{ij}^T\hat{\boldsymbol{\beta}}_{ij}^{\psi,gw} - \hat{v}_i^{\psi,gw}}{\omega_i^{gw}}\Big\},$$

where $\bar{w}_{ij} = n_i^{-1}\sum_{l\in s_i} w_l(u_{ij})$ is the average geographical weight of the sampled observations in area i with respect to location u_{ij}. The robust scaling parameter ω_i^{gw} is given by the median absolute deviation of the estimated residuals in area i. Alternatively, $\bar{w}_{ij}$ in equation (4.8) can be replaced by the median geographical weight of the sampled observations from area i. Note that $\bar{w}_{ij} = 1$ for all sampled observations in area i when only the centroid coordinates are available. In that case, the proposed bias correction in equation (4.8) simplifies to the non-weighted bias correction of Chambers et al. (2014) in equation (3.20).
As previously mentioned, Dongmo-Jiongo et al. (2013) introduced an alternative robust predictive estimator that use a global bias correction which is based on information from all sampled units. They notify that the local bias correction only depends on the local information in area i, whereas the robust estimators $\hat{\boldsymbol{\beta}}^{\psi}$ and $\hat{\boldsymbol{v}}^{\psi}$ are influenced by all sampled units. However, for geographically weighted estimators as the RGWEBLUP the influence of sampled units on the parameter estimation in area i is not constant. The influence of sampled units outside area i rather decreases as their distance to units in area i increases. Therefore, a local bias correction appears to be a natural start for developing a robust predictive GWEBLUP of the area mean. An extension of the full bias correction proposed by Dongmo-Jiongo et al. (2013) to the case of spatial non-stationarity is not addressed here but remains an open research question.

For the estimation of the proposed estimators in (4.6) and (4.8), a modified version of the step-wise algorithm suggested by Chandra et al. (2012) is proposed. As mentioned above, there are no closed-form solutions for the robust estimates of $\boldsymbol{\beta}_{ij}$, $\boldsymbol{\theta}$ and $\boldsymbol{v}$. Thus, the iteration steps for approximating the robust parameters are integrated in the step-wise estimation procedure. First, it follows a discussion of the approximation algorithms for $\boldsymbol{\beta}_{ij}$, $\boldsymbol{\theta}$ and $\boldsymbol{v}$. Thereafter, the entire estimation procedure is presented.
For the approximation of the in-sample coefficients $\boldsymbol{B}_s$, a local version of the fixed-point algorithm described in Section 3.1.1.2 is applied. For that purpose the robust ML estimation equation in (4.3) can be rewritten as

$$\boldsymbol{X}^T\boldsymbol{V}_{ij}^{-1}\boldsymbol{U}_{ij}^{1/2}\boldsymbol{D}_{ij}\boldsymbol{U}_{ij}^{-1/2}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta}_{ij})=\boldsymbol{0},$$

where $\boldsymbol{D}_{ij}$ is a diagonal matrix with $diag(\boldsymbol{D}_{ij})=\psi(\boldsymbol{r}_{ij})/\boldsymbol{r}_{ij}$. It follows that $\boldsymbol{\beta}_{ij}$ can be approximated by repeatedly evaluating the fixed-point expression

$$\boldsymbol{\beta}_{ij}^{(t+1)}=(\boldsymbol{X}^T\boldsymbol{V}_{ij}{}^{-1}\boldsymbol{D}(\boldsymbol{\beta}_{ij}^{(t)})\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{V}_{ij}^{-1}\boldsymbol{D}(\boldsymbol{\beta}_{ij}^{(t)})\boldsymbol{y},\quad t=0,1,2,\ldots \tag{4.9}$$

until convergence is reached. To obtain a robust estimate of $\boldsymbol{\theta}$, equation (4.4) is solved using the fixed-point algorithm described in Section 3.1.1.1. As $\boldsymbol{\theta}$ is assumed to be spatially stationary, the tth iteration step is given by equation (3.12) where the vector of standardized residuals $\boldsymbol{r}$ is replaced by $\boldsymbol{\tau}$ from equation (4.4). To obtain a robust estimate for $\boldsymbol{v}$, equation (4.5) is solved using the Fisher scoring algorithm described in Section 3.1.1.3. Here, the tth iteration step is given by equation (3.19). Keeping these algorithms in mind, the step-wise procedure for estimating the RGWEBLUP in equation (4.6) is defined as follows.

1: Compute all in sample distances and obtain the weighting matrix W_{ij} for each in-sample location.

2: Define the optimal bandwidth b by cross validation using the criterion in equation (2.57).

3: Choose initial values $\boldsymbol{B}_s^{(0)}$ and $\boldsymbol{\theta}^{(0)}$, preferably $\hat{\boldsymbol{B}}_s$ and $\hat{\boldsymbol{\theta}}$ from the non-robust GWEBLUP in (2.65).

4: Update $\boldsymbol{B}_s^{(t+1)}$ using equation (4.9).

5: Estimate the residuals $\boldsymbol{\tau}$ from equation (4.4) and update $\boldsymbol{\theta}^{(t+1)}$ using the fixed-point expression in (3.12).

6: Return to *step 4* and repeat until the following stopping criteria are achieved:

$$\|\beta_{ij}^{(t+1)}-\beta_{ij}^{(t)}\|_1<\epsilon,\ \forall j\in s$$

$$|\sigma_e^{2,(t+1)} - \sigma_e^{2,(t)}|/\sigma_e^{2,(t)} + |\sigma_v^{2,(t+1)} - \sigma_v^{2,(t)}|/\sigma_v^{2,(t)} < \epsilon,$$

where ϵ is a small constant.

7: Define the estimates at convergence to be the robust ML estimators $\hat{\boldsymbol{B}}_s^{\psi,gw}$ and $\hat{\boldsymbol{\theta}}^{\psi,gw}$.

8: Plug $\hat{\boldsymbol{B}}_s^{\psi,gw}$ and $\hat{\boldsymbol{\theta}}^{\psi,gw}$ into equation (4.5) and estimate $\boldsymbol{v}$ by updating the iterations of the Fisher scoring algorithm in (3.19) until the following stopping criterion is achieved:

$$\|\boldsymbol{v}^{(t+1)} - \boldsymbol{v}^t\|_1 < \epsilon.$$

9: Define the estimates at convergence to be the robust ML estimator $\hat{\boldsymbol{v}}^{\psi,gw}$.

10: Compute the out-of-sample distances for each non-sampled unit $ij \in r_i$ and obtain the weighting matrix W_{ij} for each non-sampled location $ij \in r_i$, using the same weighting function as in *step 1*.

11: Compute $\hat{\boldsymbol{V}}_{ij} = \boldsymbol{Z}\hat{\boldsymbol{\Sigma}}_v\boldsymbol{Z}^T + \hat{\boldsymbol{\Sigma}}_e\boldsymbol{W}_{ij}^{-1}$ for each non-sampled unit.

12: Compute the pseudo-values $\boldsymbol{y^*}$ by

$$y_{ij}^* = y_{ij}\min(1, \frac{b\sqrt{\hat{\sigma}_v^2 + \hat{\sigma}_e^2}}{|y_{ij}|} + \boldsymbol{x}_{ij}^T\hat{\boldsymbol{\beta}}_{ij}). \tag{4.10}$$

13: Substitute $\boldsymbol{y}$ with $\boldsymbol{y}^*$ in equation (2.63) and get empirical coefficient estimates $\boldsymbol{\beta}_{ij}$ for all $ij \in r_i$ by

$$\hat{\boldsymbol{\beta}}_{ij}^{\psi,gw} = (\boldsymbol{X}^T\hat{\boldsymbol{V}}_{ij}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}^T\hat{\boldsymbol{V}}_{ij}^{-1}\boldsymbol{y}^*. \tag{4.11}$$

14: Compute the RGWEBLUP of the area mean using equation (4.6).

Note that in the last step, the RGEWBLUP of the area mean can be replaced by the RGEWBLUP-bc in (4.8) to obtain a robust predictive GWEBLUP. Chandra et al. (2012) suggested a similar algorithm to obtain the GWEBLUP of the area mean in (2.65), which differs as follows: the local in-sample coefficients in *step 4* are estimated explicitly using equation (2.63); the variance parameters in *step 5* are estimated by numerically maximizing the log-likelihood (treating the fixed coefficients as known) using the Nelder-Mead algorithm (cf. Nelder and Mead, 1965); the random effects in *step 8* are estimated explicitly using equation (2.64); *steps 9* and *12* are skipped.

As mentioned before, the estimation procedure is computationally demanding, even for common sample sizes n. Details regarding computation times are discussed in Part II of this Thesis, where the performance of the proposed estimator is examined.

4.2 Mean Squared Error Estimation

As a precision measure, two analytic methods for MSE estimation are proposed following the ideas in Chambers et al. (2014) presented in Section 3.1.3. In particular, conditional MSE estimates based on a pseudo-linearization of the RGWEBLUP from equation (4.6) and the RGWEBLUP-bc from equation (4.8) are presented in Section 4.2.1. Section 4.2.2 presents the respective linearization-based MSE estimates.
So far, developing a bootstrap MSE has not been considered, as the point estimation of the proposed method is very time-consuming even for common sample sizes. Since bootstrap methods are based on repeatedly re-estimating the parameter of interest, these are currently not feasible for MSE estimation with conventional computer capacities.

4.2.1 Based on the Pseudo-Linearization Approach

In this subsection, the pseudo-linearization approach is used to develop a conditional MSE for the robust projective RGWEBLUP. Thereafter, this MSE estimator is extended for the robust predictive RGWEBLUP-bc.

4.2.1.1 Robust Projective GWEBLUP

In order to develop an MSE estimator that extends the pseudo-linearization of Chambers et al. (2014) from Section 3.1.3.1 to the case of spatial non-stationarity, the RGEWBLUP of the area mean in (4.6) needs to be expressed as a weighted sum of the sample values of the target variable $\boldsymbol{y}$. Recall that the RGWEBLUP for the area mean is given by

$$\hat{\bar{y}}_i^{\psi,gw} = N_i^{-1}\Big\{\sum_{j\in s_i} y_{ij} + \sum_{j\in r_i}(\boldsymbol{x}_{ij}^T\hat{\boldsymbol{\beta}}_{ij}^{\psi,gw} + \hat{v}_i^{\psi,gw})\Big\}.$$

It can be shown that the latter expression can be reformulated as

$$\hat{\bar{y}}_i^{\psi,gw} = \sum_{j\in s} d_{ij}^{\psi,gw} y_j = (\boldsymbol{d}_i^{\psi,gw})^T\boldsymbol{y}. \tag{4.12}$$

The derivation is carried out in three steps: (i) find a pseudo-linear form for the in-sample estimates $\hat{\boldsymbol{\beta}}_{ij}^{\psi,gw}$, (ii) find a pseudo-linear form for $\hat{\boldsymbol{v}}^{\psi,gw}$, (iii) find a pseudo-linear form for the out-of-sample estimates $\hat{\boldsymbol{\beta}}_{ij}^{\psi,gw}$. For step (i), the robust ML estimating equations (4.3) for $\boldsymbol{\beta}_{ij}$ can be rewritten as

$$\begin{aligned}\mathbf{0} &= \boldsymbol{X}^T\boldsymbol{V}_{ij}^{-1}\boldsymbol{U}_{ij}^{1/2}\boldsymbol{D}_{1,ij}\boldsymbol{U}_{ij}^{-1/2}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta}_{ij})\\ &= \boldsymbol{X}^T\boldsymbol{V}_{ij}^{-1}\boldsymbol{D}_{1,ij}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta}_{ij}),\end{aligned}$$

where $\boldsymbol{D}_{1,ij}$ is an $n\times n$ diagonal matrix which is defined as $\boldsymbol{D}_{ij}$ in equation (4.9). Using this expression, the in-sample regression coefficients $\boldsymbol{\beta}_{ij}$ can be rewritten in a pseudo-linear form

$$\begin{aligned}\boldsymbol{\beta}_{ij} &= \underbrace{(\boldsymbol{X}^T\boldsymbol{V}_{ij}^{-1}\boldsymbol{D}_{1,ij}\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{V}_{ij}^{-1}\boldsymbol{D}_{1,ij}}_{=\boldsymbol{A}_{ij}}\boldsymbol{y}\\ &= \boldsymbol{A}_{ij}\boldsymbol{y}.\end{aligned} \tag{4.13}$$

The pseudo-linear form of the robust estimate $\hat{\boldsymbol{\beta}}_{ij}^{\psi,gw}$ can be obtained by replacing $\boldsymbol{\beta}_{ij}$, $\boldsymbol{\theta}$ and $\boldsymbol{v}$ with their robust, geographically weighted estimates, $\hat{\boldsymbol{\beta}}_{ij}^{\psi,gw}$, $\hat{\boldsymbol{\theta}}^{\psi,gw}$ and $\hat{\boldsymbol{v}}^{\psi,gw}$, respectively, leading to

$$\hat{\boldsymbol{\beta}}_{ij}^{\psi,gw} = \boldsymbol{A}_{ij}^{\psi,gw}\boldsymbol{y}. \tag{4.14}$$

For step (ii), the robust ML estimating equations of $\boldsymbol{v}$ in (4.5) can be rewritten as

$$\mathbf{0} = \boldsymbol{Z}^T\boldsymbol{\Sigma}_e^{-1/2}\boldsymbol{D}_2\boldsymbol{\Sigma}_e^{-1/2}(\boldsymbol{y}-\boldsymbol{\lambda}-\boldsymbol{Z}\boldsymbol{v}) - \boldsymbol{\Sigma}_v^{-1/2}\boldsymbol{D}_3\boldsymbol{\Sigma}_v^{-1/2}\boldsymbol{v}, \tag{4.15}$$

where $\boldsymbol{D}_2$ and $\boldsymbol{D}_3$ are $n\times n$ and $m\times m$ diagonal matrices, respectively. The diagonal elements $d_{2,ij}$ and $d_{3,i}$ are given by

$$d_{2,ij} = \psi\Big\{(\sigma_e)^{-1}(y_{ij}-\mathbf{x}_{ij}\boldsymbol{\beta}_{ij}-v_i)\Big\}/\Big\{(\sigma_e)^{-1}(y_{ij}-\mathbf{x}_{ij}\boldsymbol{\beta}_{ij}-v_i)\Big\}$$

and

$$d_{3,i} = \psi\Big\{(\sigma_v)^{-1}\hat{v}_i\Big\}/\Big\{(\sigma_v)^{-1}v_i\Big\}.$$

Using equation (4.15), the random effects $\boldsymbol{v}$ can be expressed as

$$\begin{aligned}\boldsymbol{v} &= \underbrace{(\boldsymbol{Z}^T\boldsymbol{\Sigma}_e^{-1/2}\boldsymbol{D}_2\boldsymbol{\Sigma}_e^{-1/2}\boldsymbol{Z}+\boldsymbol{\Sigma}_v^{-1/2}\boldsymbol{D}_3\boldsymbol{\Sigma}_v^{-1/2})^{-1}\boldsymbol{Z}^T\boldsymbol{\Sigma}_e^{-1/2}\boldsymbol{D}_2\boldsymbol{\Sigma}_e^{-1/2}}_{=\boldsymbol{Q}}(\boldsymbol{y}-\boldsymbol{\lambda})\\ &= \boldsymbol{Q}(\boldsymbol{y}-\boldsymbol{\lambda}).\end{aligned} \tag{4.16}$$

The projection $\boldsymbol{\lambda}$ can be rewritten as

$$\boldsymbol{\lambda} = (\boldsymbol{X}\circ\boldsymbol{B}_s)\mathbf{1}_p = \boldsymbol{H}\boldsymbol{y}, \tag{4.17}$$

where $\boldsymbol{H}$ is the $n \times n$ head matrix with row vectors $\boldsymbol{h}_{ij}$ given by $\boldsymbol{h}_{ij} = \boldsymbol{x}_{ij}^T \boldsymbol{A}_{ij}$. Using equations (4.16) and (4.17), the random effects $\boldsymbol{v}$ can be expressed in a pseudo-linear form, given by

$$\boldsymbol{v} = \boldsymbol{Q}(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{y}. \tag{4.18}$$

The pseudo-linear form of $\hat{\boldsymbol{v}}^{\psi,gw}$ can be obtained by replacing $\boldsymbol{\beta}_{ij}$, $\boldsymbol{\theta}$ and $\boldsymbol{v}$ with their robust, geographically weighted estimates $\hat{\boldsymbol{\beta}}_{ij}^{\psi,gw}$, $\hat{\boldsymbol{\theta}}^{\psi,gw}$ and $\hat{\boldsymbol{v}}^{\psi,gw}$, respectively. This leads to

$$\hat{\boldsymbol{v}}^{\psi,gw} = \boldsymbol{Q}^{\psi,gw}(\boldsymbol{I} - \boldsymbol{H}^{\psi,gw})\boldsymbol{y}. \tag{4.19}$$

For step (iii), the out-of-sample regression coefficients $\boldsymbol{\beta}_{ij}$ need to be expressed in pseudo-linear form. The explicit expression for the out-of-sample regression coefficients in equation (4.11) can be written as

$$\begin{aligned}\hat{\boldsymbol{\beta}}_{ij}^{\psi,gw} &= \underbrace{(\boldsymbol{X}^T\hat{\boldsymbol{V}}_{ij}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}^T\hat{\boldsymbol{V}}_{ij}^{-1}\boldsymbol{D}_4}_{=\boldsymbol{\Lambda}_{ij}^{\psi,gw}}\boldsymbol{y} \\ &= \boldsymbol{\Lambda}_{ij}^{\psi,gw}\boldsymbol{y},\end{aligned} \tag{4.20}$$

where $\boldsymbol{D}_4$ is an $n \times n$ diagonal matrix. Here the diagonal elements are defined by $d_{4,ij} = y_{ij}^*/y_{ij}$. The results from equations (4.19) and (4.20) can be plugged into the RGWEBLUP for the area mean in equation (4.6) leading to

$$\begin{aligned}\hat{\bar{y}}_i^{\psi,gw} &= N_i^{-1}\left\{\sum_{j\in s_i} y_{ij} + \sum_{j\in r_i} \boldsymbol{x}_{ij}^T\boldsymbol{\Lambda}_{ij}^{\psi,gw}\boldsymbol{y} + \boldsymbol{z}_{ij}^T\boldsymbol{Q}^{\psi,gw}(\boldsymbol{I}_n - \boldsymbol{H}^{\psi,gw})\boldsymbol{y}\right\} \\ &= N_i^{-1}\left\{\boldsymbol{\delta}_i^T\boldsymbol{y} + \sum_{j\in r_i}\boldsymbol{\eta}_{ij}^{\psi,gw}\boldsymbol{y}\right. \\ &\qquad \left. + (N_i - n_i)\boldsymbol{z}_i^T\boldsymbol{Q}^{\psi,gw}(\boldsymbol{I}_n - \boldsymbol{H}^{\psi,gw})\boldsymbol{y}\right\} \\ &= N_i^{-1}\left[\boldsymbol{\delta}_i^T + (N_i - n_i)\left\{\bar{\boldsymbol{\eta}}_i^{\psi,gw} + \boldsymbol{z}_i^T\boldsymbol{Q}^{\psi,gw}(\boldsymbol{I}_n - \boldsymbol{H}^{\psi,gw})\right\}\right]\boldsymbol{y} \\ &= (\boldsymbol{d}_i^{\psi,gw})^T\boldsymbol{y}.\end{aligned} \tag{4.21}$$

Here, $\boldsymbol{\eta}_{ij}^{\psi,gw} = \boldsymbol{x}_{ij}^T\boldsymbol{\Lambda}_{ij}^{\psi,gw}$, and $\bar{\boldsymbol{\eta}}_i^{\psi,gw} = (N_i - n_i)^{-1}\sum_{j\in r_i}\boldsymbol{\eta}_{ij}^{\psi,gw}$. Given the weights in equation (4.21), developing the conditional MSE for the RGWEBLUP is equivalent to the method described in Section 2.3.2.2. Similar to equation (2.34), the conditional MSE for the RGWEBLUP in (4.6) is given by

$$\widehat{MSE}_{CCT}(\hat{\bar{y}}_i^{\psi,gw}) = \widehat{Var}_{\boldsymbol{v}}(\hat{\bar{y}}_i^{\psi,gw}) + \widehat{Bias}_{\boldsymbol{v}}(\hat{\bar{y}}_i^{\psi,gw})^2, \tag{4.22}$$

where the prediction bias and the variance can be estimated by substituting $\boldsymbol{d}^T$ with $(\boldsymbol{d}_i^{\psi,gw})^T$ in equation (2.35) and (2.37), respectively. Thus, the prediction bias can be estimated by

$$\widehat{Bias}_{\boldsymbol{v}}(\hat{\bar{y}}_i^{\psi,gw}) = \sum_{j\in s} d_{ij}^{\psi,gw}\hat{\mu}_j - N_i^{-1}\sum_{j\in U_i}\hat{\mu}_j, \tag{4.23}$$

where $\hat{\mu}_j$ is an unbiased linear estimator of the conditional expected value $\mu_j = E(y_j|\boldsymbol{x}_j, \boldsymbol{v})$ under model (2.61) which can be expressed as $\hat{\mu}_j = \sum_{k\in s}\gamma_{k,j}^{\psi,gw} y_k = (\boldsymbol{\gamma}_j^{\psi,gw})^T\boldsymbol{y}$. Here, $(\boldsymbol{\gamma}_j^{\psi,gw})^T$ is a vector of weights specific to unit j in area i, given by

$$(\boldsymbol{\gamma}_j^{\psi,gw})^T = \boldsymbol{x}_j^T\boldsymbol{A}_j^{\psi,gw} + \boldsymbol{z}_i^T\boldsymbol{Q}^{\psi,gw}(\boldsymbol{I}_n - \boldsymbol{H}^{\psi,gw}). \tag{4.24}$$

To obtain the unshrunken version of $\hat{\mu}_j$, the matrix $\boldsymbol{Q}^{\psi,gw}$ in equation (4.24) is replaced by $(\boldsymbol{Z}\boldsymbol{Z}^T)^{-1}\boldsymbol{Z}^T$. An estimate of the conditional prediction variance is then given by

$$\widehat{Var}_{\boldsymbol{v}}(\hat{\bar{y}}_i^{\psi,gw}) = N_i^{-2}\sum_{j\in s}\left\{(a_{ij}^{\psi,gw})^2 + (N_i - n_i)n^{-1}\right\}(\hat{\lambda}_j^{\psi,gw})^{-1}(y_j - \hat{\mu}_j)^2, \tag{4.25}$$

where $a_{ij}^{\psi,gw} = N_i d_{ij}^{\psi,gw} - I(j \in i)$ and $\hat{\mu}_j$ is defined as in equation (4.23). Again, $\hat{\lambda}_j^{\psi,gw} = 1 - 2\gamma_{jj}^{\psi,gw} + \sum_{k\in s}(\gamma_{kj}^{\psi,gw})^2$ is a scaling constant, specific to unit j, where $\gamma_{kj}^{\psi,gw}$ is defined by equation (4.24). Note that by setting the tuning constant c to a very high value, i.e., $c = 100$, equation (4.22) can also be used to estimate the conditional MSE of the non-robust GWEBLUP for the area mean in equation (2.65). In that case, the conditional MSE is equivalent to that suggested by Chandra et al. (2012).
Similar to the synthetic EBLUP of the area mean in (2.38), the synthetic RGWEBLUP can also be expressed in the pseudo-linear form

$$\hat{\bar{y}}_i^{\psi,gw-Syn} = N^{-1}\sum_{j\in U_i}\boldsymbol{x}_{ij}^T\hat{\boldsymbol{\beta}}_{ij}^{\psi,gw} = (\boldsymbol{d}_i^{\psi,gw-Syn})^T\boldsymbol{y}, \tag{4.26}$$

with $(\boldsymbol{d}_i^{\psi,gw-Syn})^T = \bar{\boldsymbol{\eta}}_i^{\psi,gw}$, where $\bar{\boldsymbol{\eta}}_i^{\psi,gw}$ is defined as in (4.21). The expected bias of this synthetic estimator under model (2.62) is given by

$$E[\hat{\bar{y}}_i^{\psi,gw-Syn} - \bar{y}_i] = \sum_{j\in s} d_{ij}^{\psi,gw-Syn}(\boldsymbol{x}_j^T\boldsymbol{\beta}_j + v_i) - N^{-1}\sum_{j\in U_i}\boldsymbol{x}_j^T\boldsymbol{\beta}_j - v_i. \tag{4.27}$$

The conditional expectation of the square of this expression is given by

$$\begin{aligned} &E_{\boldsymbol{v}}\{E^2[\hat{\bar{y}}_i^{\psi,gw-Syn} - \bar{y}_i]\} \\ &\quad = \left\{\sum_{j\in s} d_{ij}^{\psi,gw-Syn}(\boldsymbol{x}_j^T\boldsymbol{\beta}_j + v_i) - N^{-1}\sum_{j\in U_i}\boldsymbol{x}_j^T\boldsymbol{\beta}_j\right\}^2 + \sigma_v^2. \end{aligned} \tag{4.28}$$

Accordingly, for non-sampled areas the squared bias of the synthetic RGWEBLUP can be estimated by

$$\widehat{Bias}_{\boldsymbol{v}}(\hat{\bar{y}}_i^{\psi,Syn})^2 = \left\{ \sum_{j \in s} d_{ij}^{\psi,Syn} (\boldsymbol{x}_j^T \hat{\boldsymbol{\beta}}_j^{\psi,gw} + \hat{v}_i^{\psi,gw-u}) - N^{-1} \sum_{j \in U_i} \boldsymbol{x}_j^T \hat{\boldsymbol{\beta}}_j^{\psi,gw} \right\}^2 + (\hat{\sigma}_v^{\psi,gw})^2, \tag{4.29}$$

where $\hat{v}_i^{\psi,gw-u}$ is the unshrunken estimated random effect. The conditional MSE of the synthetic RGWEBLUP can be estimated using (4.22) where the conditional bias is replaced by (4.29).

4.2.1.2 Robust Predictive RGWEBLUP-bc

As described in Section 3.1.3.1, a conditional MSE estimator for the robust predictive RGWEBLUP-bc in equation (4.8) can be obtained by replacing the weights $d_{ij}^{\psi,gw}$ in equation (4.22) with corresponding weights for the RGWEBLUP-bc. For this purpose, it can be shown that the RGWEBLUP-bc of the area mean $\bar{y}_i$ can be expressed as

$$\hat{\bar{y}}_i^{\psi,gw-bc} = \sum_{j \in s} d_{ij}^{\psi,gw-bc} y_j = (\boldsymbol{d}_i^{\psi,gw-bc})^T \boldsymbol{y}. \tag{4.30}$$

For the derivation of the latter expression let $\boldsymbol{q}_i$ be an n-vector specific to area i with elements

$$q_{ij} = \begin{cases} \bar{w}_{ij} \psi_b(t_{ij})/t_{ij} & \text{for } j \in s_i \\ 0 & \text{otherwise,} \end{cases} \tag{4.31}$$

where $t_{ij} = (y_{ij} - \mathbf{x}_{ij} \hat{\boldsymbol{\beta}}^{\psi,gw} - \hat{v}_i{}^{\psi,gw})/\omega_i^{gw}$, and ω_i^{gw} defined as in equation (4.8). Using the results from (4.14) and (4.19), the bias correction in equation (4.8) can be rewritten in a pseudo-linear form, given by

$$\begin{aligned}
bc_i^{gw} &= \left(\frac{N_i - n_i}{N_i}\right) \frac{1}{\sum_{j \in s_i} \bar{w}_{ij}} \sum_{j \in s_i} \bar{w}_{ij} \omega_i^{gw} \psi_b \left\{ \frac{y_{ij} - \mathbf{x}_{ij}^T \hat{\boldsymbol{\beta}}_{ij}^{\psi,gw} - \hat{v}_i{}^{\psi,gw}}{\omega_i^{gw}} \right\} \\
&\overset{(\text{eq. } 4.31)}{=} \left(\frac{N_i - n_i}{N_i}\right) \frac{1}{\sum_{j \in s_i} \bar{w}_{ij}} \sum_{j \in s_i} q_{ij} (y_{ij} - \mathbf{x}_{ij}^T \hat{\boldsymbol{\beta}}_{ij}^{\psi,gw} - \hat{v}_i{}^{\psi,gw}) \\
&= \left(\frac{N_i - n_i}{N_i}\right) \frac{1}{\sum_{j \in s_i} \bar{w}_{ij}} \\
&\quad \sum_{j \in s_i} q_{ij} \left(y_{ij} - \mathbf{x}_{ij}^T \boldsymbol{A}_{ij}^{\psi,gw} \boldsymbol{y} - \underbrace{\boldsymbol{z}_{ij} \boldsymbol{Q}^{\psi,gw} (\boldsymbol{I} - \boldsymbol{H}^{\psi,gw})}_{= \boldsymbol{\xi}_{ij}^{\psi,gw}} \boldsymbol{y} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{N_i - n_i}{N_i}\right) \frac{1}{\sum_{j\in s_i} \bar{w}_{ij}} \sum_{j\in s_i} q_{ij} \left(y_{ij} - \boldsymbol{h}_{ij}^{\psi,gw}\boldsymbol{y} - \boldsymbol{\xi}_{ij}^{\psi,gw}\boldsymbol{y}\right) \\
&= \left(\frac{N_i - n_i}{N_i}\right) \frac{1}{\sum_{j\in s_i} \bar{w}_{ij}} \left\{ \sum_{j\in s_i} q_{ij}y_{ij} - \left(\sum_{j\in s_i} q_{ij}\boldsymbol{h}_{ij}^{\psi,gw} + q_{ij}\boldsymbol{\xi}_{ij}^{\psi,gw}\right)\boldsymbol{y}\right\} \\
&= \left(\frac{N_i - n_i}{N_i}\right) \frac{1}{\sum_{j\in s_i} \bar{w}_{ij}} \left\{ \boldsymbol{q}_i^T\boldsymbol{y} - \left(\sum_{j\in s_i} q_{ij}\boldsymbol{h}_{ij}^{\psi,gw} + q_{ij}\boldsymbol{\xi}_{ij}^{\psi,gw}\right)\boldsymbol{y}\right\} \\
&= \left(\frac{N_i - n_i}{N_i}\right) \frac{1}{\sum_{j\in s_i} \bar{w}_{ij}} \left\{ \boldsymbol{q}_i^T - \sum_{j\in s_i} q_{ij}\boldsymbol{h}_{ij}^{\psi,gw} - \sum_{j\in s_i} q_{ij}\boldsymbol{\xi}_{ij}^{\psi,gw}\right\}\boldsymbol{y}.
\end{aligned}
$$

By plugging the last line of the upper expression into equation (4.8) it follows that

$$
\begin{aligned}
\hat{\bar{y}}_i^{\psi,gw-bc} &= \hat{\bar{y}}_i^{\psi,gw} + bc_i^{gw} \\
&= N_i^{-1}\left[\boldsymbol{\delta}_i^T + (N_i - n_i)\left\{\bar{\boldsymbol{\eta}}_i^{\psi,gw} + \underbrace{\boldsymbol{z}_i^T\boldsymbol{Q}^{\psi,gw}(\boldsymbol{I}_n - \boldsymbol{H}^{\psi,gw})}_{=\bar{\boldsymbol{\xi}}_i^{\psi,gw}}\right\}\right]\boldsymbol{y} \\
&\quad + \left(\frac{N_i - n_i}{N_i}\right) \frac{1}{\sum_{j\in s_i} \bar{w}_{ij}} \left\{ \boldsymbol{q}_i^T - \sum_{j\in s_i} q_{ij}\boldsymbol{h}_{ij}^{\psi,gw} - \sum_{j\in s_i} q_{ij}\boldsymbol{\xi}_{ij}\right\}\boldsymbol{y} \\
&= N_1^{-1}\left\{\left(\boldsymbol{\delta}_i + \frac{N_i - n_i}{\sum_{j\in s_i} \bar{w}_{ij}}\boldsymbol{q}_i\right)^T\right. \\
&\quad + (N_i - n_i)\left(\bar{\boldsymbol{\eta}}_i^{\psi,gw} - \frac{1}{\sum_{j\in s_i} \bar{w}_{ij}} \sum_{j\in s_i} q_{ij}\boldsymbol{h}_{ij}^{\psi,gw}\right) \\
&\quad \left. + (N_i - n_i)\left(\bar{\boldsymbol{\xi}}_i^{\psi,gw} - \frac{1}{\sum_{j\in s_i} \bar{w}_{ij}} \sum_{j\in s_i} q_{ij}\boldsymbol{\xi}_{ij}^{\psi,gw}\right)\right\}\boldsymbol{y} \\
&= (\boldsymbol{d}_i^{\psi,gw-bc})^T\boldsymbol{y}.
\end{aligned} \tag{4.32}
$$

As the RGWEBLUP-bc is an approximately unbiased estimator of the area mean, the conditional bias in (4.22) can be omitted for the MSE estimation. Thus, the conditional MSE for the RGWEBLUP-bc of the area mean $\bar{y}_i$ is given by

$$
\widehat{MSE}_{CCT}(\hat{\bar{y}}_i^{\psi,gw-bc}) = \widehat{Var}_{\boldsymbol{v}}(\hat{\bar{y}}_i^{\psi,gw-bc}). \tag{4.33}
$$

Here, the conditional variance $\widehat{Var}_{\boldsymbol{v}}(\hat{\bar{y}}_i^{\psi,gw-bc})$ is defined as

$$
\begin{aligned}
&\widehat{Var}_{\boldsymbol{v}}(\hat{\bar{y}}_i^{\psi,gw-bc}) = \\
&\qquad N_i^{-2}\sum_{j\in s}\left\{(a_{ij}^{\psi,gw-bc})^2 + (N_i - n_i)n^{-1}\right\}(\hat{\lambda}_j^{\psi,gw-bc})^{-1}(y_j - \hat{\mu}_j)^2,
\end{aligned} \tag{4.34}
$$

with $a_{ij}^{\psi,gw-bc} = N_i d_{ij}^{\psi,gw-bc} - I(j \in i)$. Like in equation (4.25), $\hat{\mu}_j$ can be expressed as $\hat{\mu}_j = \sum_{k\in s} \gamma_{k,j}^{\psi,gw-bc} y_k = (\boldsymbol{\gamma}_j^{\psi,gw-bc})^T \boldsymbol{y}$ where $\hat{\lambda}_j^{\psi,gw} = 1 - 2\gamma_{jj}^{\psi,gw} + \sum_{k\in s}(\gamma_{kj}^{\psi,gw})^2$ is the scaling constant, specific to unit j, and $\boldsymbol{\gamma}_j^{\psi,gw-bc}$ is a unit-specific vector of weights given by

$$(\boldsymbol{\gamma}_j^{\psi,gw-bc})^T = \left(\frac{1}{\sum_{j\in s_i} \bar{w}_{ij}} \boldsymbol{q}_i\right)^T + \left(\boldsymbol{\eta}_{ij}^{\psi,gw} - \frac{1}{\sum_{j\in s_i} \bar{w}_{ij}} \sum_{j\in s_i} q_{ij} \boldsymbol{h}_{ij}^{\psi,gw}\right) + \left(\boldsymbol{\xi}_{ij}^{\psi,gw} - \frac{1}{\sum_{j\in s_i} \bar{w}_{ij}} \sum_{j\in s_i} q_{ij} \boldsymbol{\xi}_{ij}^{\psi,gw}\right). \tag{4.35}$$

The proposed MSE estimators in (4.22) and (4.33) for the robust projective RGWEBLUP and the robust predictive RGWEBLUP-bc, respectively, are generalizations of the pseudo-linearization approach of Chambers et al. (2014). For constant spatial weights in model (2.62), equations (4.22) and (4.33) reduce to the conditional MSE estimates from equations (3.56) and (3.67) in Section 3.1.3.1, respectively. As previously mentioned, the conditional MSE based on pseudo-linearization ignores the extra variability due to the estimation of $\boldsymbol{\theta}$. Therefore, it can be seen as a first-order approximation for the conditional MSE. Thus, a second-order approximation is proposed in the next section. This extends the linearization approach of Chambers et al. (2014) from Section 3.1.3.2 to the case of spatial non-stationarity.

4.2.2 Based on the Linearization Approach

As in the last section, the linearization-based, conditional MSE estimation is first developed for the robust projective RGWEBLUP in equation (4.6). Thereafter, this conditional MSE estimation is extended for the robust predictive RGWEBLUP-bc in equation (4.8).

4.2.2.1 Robust Projective RGWEBLUP

Similar to Chambers et al. (2014), the linearization-based MSE estimation for the RGWEBLUP of the small area mean in equation (4.6) is developed in two steps. First, the conditional MSE is developed for the best unbiased predictor RGWBLUP where the variance parameters $\boldsymbol{\theta}$ are assumed to be known. Subsequently, an extra term is added that accounts for the variability caused by the estimation of the variance parameters. The conditional MSE for the RGWBLUP of the area mean $\bar{y}_i$, denoted by $\tilde{\bar{y}}_i^{\psi,gw}$, can be expressed as

$$MSE_{\boldsymbol{v}}(\tilde{\bar{y}}_i^{\psi,gw}) = Var_{\boldsymbol{v}}(\tilde{\bar{y}}_i^{\psi,gw}) + Bias_{\boldsymbol{v}}(\tilde{\bar{y}}_i^{\psi,gw})^2. \tag{4.36}$$

Here the conditional prediction bias can be obtained by (4.23) for sampled or by (4.29) for non-sampled areas. Let $\tilde{\boldsymbol{\beta}}^{\psi,gw}$ and $\tilde{\boldsymbol{v}}^{\psi,gw}$ denote the robust estimate of $\boldsymbol{\beta}$ and $\boldsymbol{v}$, respectively, for known variance parameters $\boldsymbol{\theta}$. Then the prediction error for the RGWBLUP of the area mean $\bar{y}_i$ is given by

$$\begin{aligned}\tilde{\bar{y}}_i^{\psi,gw} - \bar{y}_i &= N_i^{-1}\Big\{\sum_{j\in s_i} y_{ij} + \sum_{j\in r_i}(\boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw} + \tilde{v}_i^{\psi,gw})\Big\} - N_i^{-1}\sum_{j\in U_i} y_{ij} \\ &= N_i^{-1}\Big\{\sum_{j\in r_i}(\boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw} + \tilde{v}_i^{\psi,gw}) - \sum_{j\in r_i} y_{ij}\Big\} \\ &= N^{-1}\Big\{\sum_{j\in r_i}\boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\Big\} + \left(\frac{N_i - n_i}{N_i}\right)\Big\{\boldsymbol{z}_i^T\tilde{\boldsymbol{v}}^{\psi,gw} - \bar{y}_{ri}\Big\}. \qquad (4.37)\end{aligned}$$

Assuming independence between $\boldsymbol{\beta}$ and $\boldsymbol{v}$, the conditional variance of this prediction error is given by

$$\begin{aligned}Var_{\boldsymbol{v}}(\hat{\tilde{y}}_i^{\psi,gw} - \bar{y}_i) &= \frac{1}{N_i^2}Var_{\boldsymbol{v}}\Big(\sum_{j\in r_i}\boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\Big) \\ &\quad + \left(\frac{N_i - n_i}{N_i}\right)^2\Big\{\boldsymbol{z}_i^T Var_{\boldsymbol{v}}(\tilde{\boldsymbol{v}}^{\psi,gw})\boldsymbol{z}_i - Var_{\boldsymbol{v}}(\bar{e}_{ri})\Big\}. \qquad (4.38)\end{aligned}$$

To estimate this expression, estimates for $Var_{\boldsymbol{v}}(\tilde{\boldsymbol{v}}^{\psi,gw})$, $Var_{\boldsymbol{v}}(\bar{e}_{ri})$ and the variation caused by the estimation of the out-of-sample coefficients are needed. The latter component is defined in the first component on the right-hand side of equation (4.38).
For the proposed RGWEBLUP in equation (4.6) the variance parameter $\boldsymbol{\theta}$ is assumed to be spatial stationary. Then the results of Chambers et al. (2014) in Section 3.1.3.2 can be directly applied for estimating $Var_{\boldsymbol{v}}(\tilde{\boldsymbol{v}}^{\psi,gw})$ and $Var_{\boldsymbol{v}}(\bar{e}_{ri})$. In particular, $Var_{\boldsymbol{v}}(\tilde{\boldsymbol{v}}^{\psi,gw})$ can be estimated using equation (3.44). Here, the robust ML estimating equation in $\boldsymbol{H}_v$ is replaced by (4.5) leading to

$$\widehat{Var}_{\boldsymbol{v}}(\tilde{\boldsymbol{v}}^{\psi,gw}) = \left[\hat{E}_{\boldsymbol{v}}(\partial_{\boldsymbol{v}}\boldsymbol{H}_{\boldsymbol{v}})^{-1}\right]\widehat{Var}_{\boldsymbol{v}}\left(\boldsymbol{H}_{\boldsymbol{v}}\right)\left[\hat{E}_{\boldsymbol{v}}(\partial_{\boldsymbol{v}}\boldsymbol{H}_{\boldsymbol{v}})^{-1}\right]^T, \qquad (4.39)$$

where

$$\begin{aligned}\hat{E}_{\boldsymbol{v}}(\partial_{\boldsymbol{v}}\boldsymbol{H}_{\boldsymbol{v}}) &= -\boldsymbol{Z}^T\boldsymbol{\Sigma}_e^{-1/2}\tilde{\boldsymbol{T}}\boldsymbol{\Sigma}_e^{-1/2}\boldsymbol{Z} - \boldsymbol{\Sigma}_v^{-1/2}\tilde{\boldsymbol{L}}\boldsymbol{\Sigma}_v^{-1/2} \\ \widehat{Var}_{\boldsymbol{v}}\left(\boldsymbol{H}_{\boldsymbol{v}}\right) &= \hat{\nu}(\psi(\tilde{\boldsymbol{t}}))\boldsymbol{Z}^T\boldsymbol{\Sigma}_e^{-1}\boldsymbol{Z}.\end{aligned}$$

The elements of $\tilde{\boldsymbol{t}}$ are given by $\tilde{t}_{ij} = \sigma_e^{-1}(y_{ij} - \boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw} + \tilde{v}_i^{\psi,gw})$ and the matrices $\tilde{\boldsymbol{T}}$ and $\tilde{\boldsymbol{L}}$ are defined as in equation (3.44). An estimate of the conditional variance $\hat{\nu}(\psi(\tilde{\boldsymbol{t}}))$ can be obtained by equation (3.45). Similarly, $Var_{\boldsymbol{v}}(\bar{e}_{ri})$ in equation (4.38) can be estimated by equation (3.39), where $\tilde{\boldsymbol{\beta}}^{\psi}$

and $\tilde{\boldsymbol{v}}^{\psi}$ are substituted by $\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}$ and $\tilde{\boldsymbol{v}}^{\psi,gw}$, respectively, leading to

$$\widehat{Var}_{\boldsymbol{v}}(\tilde{\bar{e}}_{ri}) = (N_i - n_i)^{-1}(n-1)^{-1}\sum_{i=1}^{m}\sum_{j\in s_i}(y_{ij} - \boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw} - \tilde{v}_i^{\psi,gw})^2. \quad (4.40)$$

To estimate the conditional variance in equation (4.38), an estimate for the variation caused by the estimation of the local out-of-sample coefficients is required. Assuming independence between the local coefficients from different locations, i.e., $Cov\left(\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}, \tilde{\boldsymbol{\beta}}_{ik}^{\psi,gw}\right) = 0$ for $j \neq k$, this variance part can be expressed as

$$\begin{aligned}
\frac{1}{N_i^2}Var\left(\sum_{j\in r_i}\boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\right) &= \frac{1}{N_i^2}\sum_{j\in r_i}\sum_{k\in r_i}\boldsymbol{x}_{ik}^T Cov\left(\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}, \tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\right)\boldsymbol{x}_{ij} \\
&= \frac{1}{N_i^2}\sum_{k\in r_i}\sum_{j\in r_i}\boldsymbol{x}_{ik}^T Var\left(\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\right)\boldsymbol{x}_{ij} \\
&= \frac{1}{N_i^2}\sum_{k\in r_i}\boldsymbol{x}_{ik}^T\sum_{j\in r_i}Var\left(\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\right)\boldsymbol{x}_{ij} \\
&= \left(\frac{N_i - n_i}{N_i^2}\right)\bar{\boldsymbol{x}}_{ri}^T\sum_{j\in r_i}Var\left(\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\right)\boldsymbol{x}_{ij}. \quad (4.41)
\end{aligned}$$

It is important to note that the spatial weighting matrix $\boldsymbol{W}_{ij}$ from equation (4.11) is deterministic and only depends on the locations of the sampled observations but independent from the local coefficients from other locations. Therefore, the independence assumption for the local coefficients from different locations can be justified as their estimation is independent. The variance for the out-of-sample coefficients can be derived using their explicit form in equation (4.11), leading to

$$\widehat{Var}_{\boldsymbol{v}}\left(\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\right) = (\boldsymbol{X}^T\boldsymbol{V}_{ij}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{V}_{ij}^{-1}Var(\boldsymbol{y}^*)\boldsymbol{V}_{ij}^{-1}\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{V}_{ij}^{-1}\boldsymbol{X})^{-1}. \quad (4.42)$$

From equation (4.10) the pseudo observations $\boldsymbol{y}^*$ can be interpreted as a robust version of the original target variable $\boldsymbol{y}$ which would have been expected under model (2.62). Thus, the variance matrix $Var(\boldsymbol{y}^*)$ can be approximated by the global variance matrix $\boldsymbol{V}$ under that model. Using the results from equations (4.39), (4.40) and (4.42), an estimate of the conditional prediction variance in equation (4.38) is given by

$$\widehat{Var}_{\boldsymbol{v}}\left(\hat{\tilde{y}}_i^{\psi,gw} - \bar{y}_i\right) = h_{1i}(\tilde{\Theta}) + h_{2i}(\tilde{\Theta}) + h_{3i}(\tilde{\Theta}), \quad (4.43)$$

where $\tilde{\Theta} = \{\tilde{\boldsymbol{B}}^{\psi,gw}, \tilde{\boldsymbol{v}}^{\psi,gw}\}$ denotes the quantity of estimated random effects and local coefficients for known variance parameters. The first component, h_{1i}, in equation (4.43) reflects the variation due to the estimation of the local out-of-sample coefficients and is given by

$$h_{1i}(\tilde{\Theta}) = \left(\frac{N_i - n_i}{N_i^2}\right) \bar{\boldsymbol{x}}_{ri}^T \sum_{j \in r_i} \widehat{Var}\left(\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\right) \boldsymbol{x}_{ij}.$$

The second component, h_{2i}, in equation (4.43) measures the variance caused by the estimation of the random effects and is given by

$$h_{2i}(\tilde{\Theta}) = \left(\frac{N_i - n_i}{N_i^2}\right)^2 \boldsymbol{z}_i^T \widehat{Var}_{\boldsymbol{v}}(\tilde{\boldsymbol{v}}^{\psi,gw}) \boldsymbol{z}_i.$$

The third component, h_{3i}, in equation (4.43) is given by

$$h_{3i}(\tilde{\Theta}) = \left(\frac{N_i - n_i}{N_i}\right)^2 \widehat{Var}(\tilde{\bar{e}}_{ri}).$$

Using equations (4.43) and (4.23), the conditional MSE in equation (4.36) for the RGWBLUP of the area mean can be estimated by

$$\widehat{MSE}_{\boldsymbol{v}}(\tilde{\bar{y}}_i^{\psi,gw}) = h_{1i}(\tilde{\Theta}) + h_{2i}(\tilde{\Theta}) + h_{3i}(\tilde{\Theta}) + \widehat{Bias}_{\boldsymbol{v}}(\tilde{\bar{y}}_i^{\psi,gw})^2. \tag{4.44}$$

To account for the extra variability induced by the estimation of the variance parameters $\boldsymbol{\theta}$, a term has to be added to equation (4.44). Analogously to equation (3.48), the conditional MSE of the RGWEBLUP for the area mean can be approximated by

$$MSE_{\boldsymbol{v}}(\hat{\bar{y}}_i^{\psi,gw}) \approx MSE_{\boldsymbol{v}}(\tilde{\bar{y}}_i^{\psi,gw}) + E_{\boldsymbol{v}}\left[(\hat{\bar{y}}_i^{\psi,gw} - \tilde{\bar{y}}_i^{\psi,gw})^2\right]. \tag{4.45}$$

As $\boldsymbol{\theta}$ is assumed to be stationary, the results of Chambers et al. (2014) from Section 3.1.3.2 can be directly applied to the approximation of the additional term in equation (4.45). Now the global coefficients $\boldsymbol{\beta}$ in equation (3.49) need to be replaced by the local coefficients $\boldsymbol{\beta}_{ij}$ leading to

$$\hat{\bar{y}}_i^{\psi,gw} - \tilde{\bar{y}}_i^{\psi,gw} \approx \left(N_i^{-1} \sum_{j \in r_i} \boldsymbol{z}_{ij}^T\right) \sum_{k=1}^{2} (\partial_{\theta_k} \boldsymbol{Q})\,(\boldsymbol{y} - \boldsymbol{\lambda})\,(\hat{\theta}_k^{\psi,gw} - \theta_k), \tag{4.46}$$

where matrix $\boldsymbol{Q}$ is defined as in equation (4.16). The derivatives of $\boldsymbol{Q}$ with respect to σ_v and σ_e are equivalent to equations (3.51) and (3.52), respectively. The variance of the prediction error in equation (4.62) can be approximated by

$$Var_{\boldsymbol{v}}(\hat{\bar{y}}_i^{\psi,gw} - \tilde{\bar{y}}_i^{\psi,gw}) \approx$$

$$\left(N_i^{-1}\sum_{j\in r_i}\boldsymbol{z}_{ij}^T\right)Var_{\boldsymbol{v}}\left(\sum_{k=1}^{2}\left(\partial_{\theta_k}\boldsymbol{Q}\right)(\boldsymbol{y}-\boldsymbol{\lambda})\left(\hat{\theta}_k^{\psi,gw}-\theta_k\right)\right)\left(N_i^{-1}\sum_{j\in r_i}\boldsymbol{z}_{ij}^T\right)^T, \tag{4.47}$$

where, as in equation (3.53), the conditional covariance is given by

$$\begin{aligned}&E_{\boldsymbol{v}}\left((y_j-\boldsymbol{x}_j\boldsymbol{\beta}_j)(\hat{\theta}_k^{\psi,gw}-\theta_k),(y_l-\boldsymbol{x}_l\boldsymbol{\beta}_l)(\hat{\theta}_g^{\psi,gw}-\theta_g)\right)\\ &\approx\left\{(\boldsymbol{z}_j\boldsymbol{v})(\boldsymbol{z}_l\boldsymbol{v})+\sigma_e^2I(j=l)\right\}E_{\boldsymbol{v}}[(\hat{\theta}_k^{\psi,gw}-\theta_k)(\hat{\theta}_g^{\psi,gw}-\theta_g)].\end{aligned}$$

Replacing the global coefficients $\boldsymbol{\beta}$ with the local coefficients does not affect the conditional covariance in equation (4.47), which has exactly the same form as in equation (3.53). Thus, an estimate of the conditional MSE of the RGWEBLUP in (4.6) is given by

$$\widehat{MSE}_{\boldsymbol{u}}(\hat{\bar{y}}_i^{\psi,gw})=h_{1i}(\tilde{\Theta})+h_{2i}(\tilde{\Theta})+h_{3i}(\tilde{\Theta})+h_{4i}(\tilde{\Theta})+\widehat{Bias}_{\boldsymbol{v}}(\tilde{\bar{y}}_i^{\psi,gw})^2. \tag{4.48}$$

As in equation (3.54), h_{4i} measures the variability caused by the estimation of the variance components and is given by

$$h_{4_i}(\tilde{\Theta})=\left(N_i^{-1}\sum_{j\in r_i}\boldsymbol{z}_{ij}^T\right)\Upsilon\ Var_{\boldsymbol{v}}(\hat{\boldsymbol{\theta}}^{\psi,gw})\left(N_i^{-1}\sum_{j\in r_i}\boldsymbol{z}_{ij}^T\right)^T, \tag{4.49}$$

where

$$\Upsilon=\sum_{k=1}^{2}\sum_{g=1}^{2}\left\{(\partial_{\theta_k}\boldsymbol{Q})\left[\sum_{j=1}^{m}\sum_{l=1}^{m}(\boldsymbol{z}_{ij}^T\boldsymbol{v})(\boldsymbol{z}_{il}^T\boldsymbol{v})+\sigma_e^2I(j=l)\right]\left(\partial_{\theta_g}\boldsymbol{Q}\right)^T\right\}.$$

To finally obtain an estimate of the conditional MSE for the RGWEBLUP, the parameters $\tilde{\boldsymbol{B}}^{\psi,gw}$,$\tilde{\boldsymbol{v}}^{\psi,gw}$ and $\boldsymbol{\theta}$ in equation (4.48) need to be replaced with the estimates $\hat{\boldsymbol{B}}^{\psi,gw}$,$\hat{\boldsymbol{v}}^{\psi,gw}$ and $\hat{\boldsymbol{\theta}}^{\psi,gw}$ leading to

$$\widehat{MSE}_{CCST}(\hat{\bar{y}}_i^{\psi,gw})=h_{1i}(\hat{\Theta})+h_{2i}(\hat{\Theta})+h_{3i}(\hat{\Theta})+h_{4i}(\hat{\Theta})+\widehat{Bias}_{\boldsymbol{v}}(\hat{\bar{y}}_i^{\psi,gw})^2. \tag{4.50}$$

By setting the tuning constant c to a very high value, i.e., $c=100$, equation (4.50) can also be used to estimate the conditional MSE for the non-robust GWEBLUP of the area mean in equation (2.65).

4.2.2.2 Robust Predictive RGWEBLUP-bc

To develop the conditional MSE for the robust predictive RGWEBLUP-bc in equation (4.8), the same steps as before are followed by first developing the conditional MSE for the RGWBLUP-BC, where the variance parameters are assumed to be known, followed by adding an extra term which accounts for the variability caused by the estimation of $\boldsymbol{\theta}$. Because the RGWBLUP-BC for the area mean is approximately unbiased, the conditional MSE is given by

$$MSE_{\boldsymbol{v}}(\tilde{\bar{y}}_i^{\psi,gw-bc}) = Var_{\boldsymbol{v}}(\tilde{\bar{y}}_i^{\psi,gw-bc}). \tag{4.51}$$

The prediction error for the RGWBLUP-BC in (4.8) is given by

$$\begin{aligned}\tilde{\bar{y}}_i^{\psi,gw-bc} - \bar{y}_i = N_i^{-1}\Big\{ &\sum_{j\in r_i} \boldsymbol{x}_{ij}^T \tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\Big\} \\ &+ \left(\frac{N_i - n_i}{N_i}\right)\Big\{\boldsymbol{z}_i^T \tilde{\boldsymbol{v}}^{\psi,gw} + (1 - n_i N_i^{-1})^{-1}\widetilde{bc}_i^{gw} - \bar{y}_{ri}\Big\}.\end{aligned} \tag{4.52}$$

Here, the bias correction $\widetilde{bc}_i^{gw}$ is defined as in equation (4.8) where $\hat{\boldsymbol{\beta}}_{ij}^{\psi,gw}$ and $\hat{\boldsymbol{v}}^{\psi,gw}$ are replaced by $\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}$ and $\tilde{\boldsymbol{v}}^{\psi,gw}$, respectively. Thus, $\widetilde{bc}_i^{gw}$ is given by

$$\widetilde{bc}_i^{gw} = \left(\frac{N_i - n_i}{N_i}\right)\frac{1}{\sum_{j\in s_i}\bar{w}_{ij}}\sum_{j\in s_i}\bar{w}_{ij}\omega_i^{gw}\psi_b\Big\{\frac{y_{ij} - \mathbf{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw} - \boldsymbol{z}_{ij}^T\tilde{\boldsymbol{v}}^{\psi,gw}}{\omega_i^{gw}}\Big\}. \tag{4.53}$$

Similar to Chambers et al. (2014), this bias correction term can be approximated using a Taylor series approximation. Defining $g_{1i} = \sum_{j\in s_i}\bar{w}_{ij}$ and $\tilde{a}_{ij} = (y_{ij} - \boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij} - \boldsymbol{z}_{ij}^T\tilde{\boldsymbol{v}})/\omega_i^{gw}$, this approximation leads to

$$\begin{aligned}&(1-n_i N_i^{-1})^{-1}\widetilde{bc}_i \\ &\quad\approx g_{1i}^{-1}\sum_{j\in s_i}\bar{w}_{ij}\omega_i^{gw}\psi_b(\tilde{a}_{ij}) \\ &\qquad + (\tilde{\boldsymbol{v}}^{\psi,gw} - \boldsymbol{v})^T\partial_{\boldsymbol{v}}\left\{g_{1i}^{-1}\sum_{j\in s_i}\bar{w}_{ij}\omega_i^{gw}\psi_b(\tilde{a}_{ij})\right\} \\ &\qquad + \sum_{j\in s_i}(\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw} - \boldsymbol{\beta}_{ij})^T\partial_{\boldsymbol{\beta}_{ij}}\left\{g_{1i}^{-1}\sum_{j\in s_i}\bar{w}_{ij}\omega_i^{gw}\psi_b(\tilde{a}_{ij})\right\}.\end{aligned} \tag{4.54}$$

The partial derivatives are given by

$$\partial_{\boldsymbol{v}}\left\{g_{1i}^{-1}\sum_{j\in s_i}\bar{w}_{ij}\omega_i^{gw}\psi_b(a_{ij})\right\} = -g_{1i}^{-1}\sum_{j\in s_i}\bar{w}_{ij}\psi_b'(a_{ij})\boldsymbol{z}_{ij}$$

and

$$\partial_{\boldsymbol{\beta}_{ij}}\left\{g_{1i}^{-1}\sum_{j\in s_i}\bar{w}_{ij}\omega_i^{gw}\psi_b(a_{ij})\right\} = -g_{1i}^{-1}\bar{w}_{ij}\psi_b'(a_{ij})\boldsymbol{x}_{ij},$$

where

$$\psi_b'(a_{ij}) = \begin{cases} 1 \text{ for } |(y_{ij} - \boldsymbol{x}_{ij}^T\boldsymbol{\beta}_{ij} - \boldsymbol{z}_{ij}^T\boldsymbol{v})/\omega_i^{gw}| \leq b \\ 0 \text{ else} \end{cases}.$$

Chambers et al. (2014) note that if the tuning constant b in ψ_b is sufficiently large then $\psi_b' \approx 1$. In this case the approximation in equation (4.54) simplifies to

$$\begin{aligned}(1 - n_i N_i^{-1})^{-1}\widetilde{bc}_i \approx\ & g_{1i}^{-1} \sum_{j\in s_i} \bar{w}_{ij}\omega_i^{gw}\psi_b(a_{ij}) \\ & - g_{1i}^{-1} \sum_{j\in s_i} \bar{w}_{ij}\boldsymbol{z}_{ij}^T(\tilde{\boldsymbol{v}}^{\psi,gw} - \boldsymbol{v}) - g_{1i}^{-1} \sum_{j\in s_i} \bar{w}_{ij}\boldsymbol{x}_{ij}^T(\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw} - \boldsymbol{\beta}_{ij}) \\ \approx\ & g_{1i}^{-1} \sum_{j\in s_i} \bar{w}_{ij}\omega_i^{gw}\psi_b(a_{ij}) + \boldsymbol{z}_i^T\boldsymbol{v} - \boldsymbol{z}_i^T\tilde{\boldsymbol{v}}^{\psi,gw} \\ & + g_{1i}^{-1} \sum_{j\in s_i} \bar{w}_{ij}\boldsymbol{x}_{ij}^T\boldsymbol{\beta}_{ij} - g_{1i}^{-1} \sum_{j\in s_i} \bar{w}_{ij}\boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}. \end{aligned} \tag{4.55}$$

This approximation can be inserted into the prediction error from equation (4.52) leading to

$$\begin{aligned} & \tilde{\bar{y}}_i^{\psi,gw-bc} - \bar{y}_i \\ & = N_i^{-1}\Big\{\sum_{j\in r_i} \boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\Big\} + \left(\frac{N_i - n_i}{N_i}\right)\Big\{\boldsymbol{z}_i^T\tilde{\boldsymbol{v}}^{\psi,gw} + (1 - n_i N_i^{-1})^{-1}\widetilde{bc}_i^{gw} - \bar{y}_{ri}\Big\} \\ & \approx N^{-1}\Big\{\sum_{j\in r_i} \boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\Big\} + \left(\frac{N_i - n_i}{N_i}\right)\Big\{\boldsymbol{z}_i^T\tilde{\boldsymbol{v}}^{\psi,gw} + g_{1i}^{-1}\sum_{j\in s_i} \bar{w}_{ij}\omega_i^{gw}\psi_b(a_{ij}) \\ & \quad + \boldsymbol{z}_i^T\boldsymbol{v} - \boldsymbol{z}_i^T\tilde{\boldsymbol{v}}^{\psi,gw} + g_{1i}^{-1}\sum_{j\in s_i} \bar{w}_{ij}\boldsymbol{x}_{ij}^T\boldsymbol{\beta}_{ij} - g_{1i}^{-1}\sum_{j\in s_i} \bar{w}_{ij}\boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw} - \bar{y}_{ri}\Big\} \\ & \approx N^{-1}\Big\{\sum_{j\in r_i} \boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw} - \frac{(N_i - n_i)}{g_{1i}}\sum_{j\in s_i} \bar{w}_{ij}\boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\Big\} \\ & \quad + \left(\frac{N_i - n_i}{N_i}\right)\Big\{g_{1i}^{-1}\sum_{j\in s_i} \bar{w}_{ij}\omega_i^{gw}\psi_b(a_{ij}) + g_{1i}^{-1}\sum_{j\in s_i} \bar{w}_{ij}\boldsymbol{x}_{ij}^T\boldsymbol{\beta}_{ij} + \boldsymbol{z}_i^T\boldsymbol{v} - \bar{y}_{ri}\Big\} \\ & \approx \left(\frac{N_i - n_i}{N_i}\right)\Big\{\sum_{j\in U_i} (\boldsymbol{x}_{ij}^*)^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\Big\} \\ & \quad + \left(\frac{N_i - n_i}{N_i}\right)\Big\{g_{1i}^{-1}\sum_{j\in s_i} \bar{w}_{ij}\omega_i^{gw}\psi_b(a_{ij}) + g_{1i}^{-1}\sum_{j\in s_i} \bar{w}_{ij}\boldsymbol{x}_{ij}^T\boldsymbol{\beta}_{ij} + \boldsymbol{z}_i^T\boldsymbol{v} - \bar{y}_{ri}\Big\}, \end{aligned} \tag{4.56}$$

where

$$\boldsymbol{x}_{ij}^{*} = \begin{cases} \frac{1}{N_i - n_i}\boldsymbol{x}_{ij}, & j \in r_i \\ -\frac{1}{g_{1i}}\bar{w}_{ij}\boldsymbol{x}_{ij}, & j \in s_i. \end{cases}$$

It is important to notice that $\sum_{j\in U_i} \boldsymbol{x}_{ij}^{*} = \bar{\boldsymbol{x}}_{ri} - \bar{\boldsymbol{x}}_{si}^{gw}$, where $\bar{\boldsymbol{x}}_{si}^{gw}$ denotes the geographically weighted area mean $\bar{\boldsymbol{x}}_{si}^{gw} = g_{1i}^{-1}\sum_{j\in s_i}\bar{w}_{ij}\boldsymbol{x}_{ij}$. Using equation (4.56), the conditional variance of the prediction error in equation (4.56) can be approximated by

$$\begin{aligned} Var_{\boldsymbol{v}}(\tilde{\bar{y}}_i^{\psi,gw-bc} - \bar{y}_i) \approx & \left(\frac{N_i - n_i}{N_i}\right)^2 Var_{\boldsymbol{v}}\Big\{\sum_{j\in U_i}(\boldsymbol{x}_{ij}^{*})^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\Big\} \\ & + \left(\frac{N_i - n_i}{N_i}\right)^2 \left\{Var_{\boldsymbol{v}}\left(g_{1i}^{-1}\sum_{j\in s_i}\bar{w}_{ij}\omega_i^{gw}\psi_b(a_{ij})\right)\right. \\ & \left. + Var_{\boldsymbol{v}}(\bar{e}_{ri})\right\}. \end{aligned} \tag{4.57}$$

The first term of this expression can be estimated by

$$\begin{aligned} \widehat{Var}_{\boldsymbol{v}}\Big\{\sum_{j\in U_i}(\boldsymbol{x}_{ij}^{*})^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\Big\} &= \sum_{k\in U_i}\sum_{j\in U_i}(\boldsymbol{x}_{ik}^{*})^T\widehat{Var}\left(\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\right)\boldsymbol{x}_{ij}^{*} \\ &= \sum_{k\in U_i}(\boldsymbol{x}_{ik}^{*})^T\sum_{j\in U_i}\widehat{Var}\left(\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\right)\boldsymbol{x}_{ij}^{*} \\ &= (\bar{\boldsymbol{x}}_{ri} - \bar{\boldsymbol{x}}_{si}^{gw})^T\sum_{j\in U_i}\widehat{Var}\left(\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\right)\boldsymbol{x}_{ij}^{*} \\ &= (\bar{\boldsymbol{x}}_{ri} - \bar{\boldsymbol{x}}_{si}^{gw})^T(\boldsymbol{\zeta}_{ri} - \boldsymbol{\zeta}_{si}), \end{aligned} \tag{4.58}$$

where

$$\boldsymbol{\zeta}_{ri} = (N_i - n_i)^{-1}\sum_{j\in r_i}\boldsymbol{x}_{ij}\widehat{Var}_{\boldsymbol{v}}(\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw})$$

and

$$\boldsymbol{\zeta}_{si} = g_{1i}^{-1}\sum_{j\in s_i}\bar{w}_{ij}\boldsymbol{x}_{ij}\widehat{Var}_{\boldsymbol{v}}(\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}).$$

The variance of the out-of-sample coefficients in $\boldsymbol{\zeta}_{ri}$ can be estimated using equation (4.41). The variance of the in-sample coefficients in $\boldsymbol{\zeta}_{si}$ can be obtained with a first-order approximation that is comparable to the results in equation (3.41). To estimate $Var_{\boldsymbol{v}}(\tilde{\boldsymbol{\beta}}_{ij})$ for the in-sample coefficients, equation (3.43) can be directly applied, where $\boldsymbol{H}_{\boldsymbol{\beta}}$ is replace by $\boldsymbol{H}_{\boldsymbol{\beta}_{ij}}$, the local robust ML equation from (4.3), leading to

$$\widehat{Var}_{\boldsymbol{v}}(\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}) = \left[\hat{E}_{\boldsymbol{v}}(\partial_{\boldsymbol{\beta}_{ij}}\boldsymbol{H}_{\boldsymbol{\beta}_{ij}})^{-1}\right]\widehat{Var}_{\boldsymbol{v}}\left(\boldsymbol{H}_{\boldsymbol{\beta}_{ij}}\right)\left[\hat{E}_{\boldsymbol{v}}(\partial_{\boldsymbol{\beta}_{ij}}\boldsymbol{H}_{\boldsymbol{\beta}_{ij}})^{-1}\right]^T, \tag{4.59}$$

where

$$\hat{E}_{\boldsymbol{v}}(\partial_{\boldsymbol{\beta}_{ij}}\boldsymbol{H}_{\boldsymbol{\beta}_{ij}}) = -\boldsymbol{X}\boldsymbol{V}_{ij}^{-1}\boldsymbol{U}_{ij}^{1/2}\tilde{\boldsymbol{R}}_{ij}\boldsymbol{U}_{ij}^{-1/2}\boldsymbol{X}$$
$$\widehat{Var}_{\boldsymbol{v}}\left(\boldsymbol{H}_{\boldsymbol{\beta}_{ij}}\right) = \hat{\nu}(\psi(\tilde{\boldsymbol{r}}_{ij}))\boldsymbol{X}\boldsymbol{V}_{ij}^{-1}\boldsymbol{U}_{ij}\boldsymbol{V}_{ij}^{-1}\boldsymbol{X}.$$

Here, $\tilde{\boldsymbol{r}}_{ij} = \boldsymbol{U}_{ij}^{-\frac{1}{2}}(\boldsymbol{y} - \boldsymbol{X}\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw})$ and $\tilde{\boldsymbol{R}}_{ij} = diag_n\left[I(-c < \tilde{\boldsymbol{r}}_{ij} < c)\right]$. The conditional variance $\nu(\psi(\tilde{\boldsymbol{r}}_{ij}))$ can be estimated by applying equation (3.45). Using the conditional variance in equation (4.57), the conditional MSE for the RGWBLUP-BC can be estimated by

$$\widehat{MSE}_{\boldsymbol{u}}(\hat{\bar{y}}_i^{\psi,gw-bc}) = h_{1i}^{bc}(\tilde{\boldsymbol{\Theta}}) + h_{2i}^{bc}(\tilde{\boldsymbol{\Theta}}) + h_{3i}(\tilde{\boldsymbol{\Theta}}). \tag{4.60}$$

The first component, h_{1i}^{bc}, is given by

$$h_{1i}^{bc}(\tilde{\boldsymbol{\Theta}}) = \left(\frac{N_i - n_i}{N_i}\right)^2 (\bar{\boldsymbol{x}}_{ri} - \bar{\boldsymbol{x}}_{si}^{gw})^T(\boldsymbol{\zeta}_{ri} - \boldsymbol{\zeta}_{si}),$$

and the third component h_{3i} is identical to h_{3i} in equation (4.44). The second component, h_{2i}^{bc}, arises due to the bias correction and is given by

$$\begin{aligned}
h_{2i}^{bc}(\tilde{\boldsymbol{\Theta}}) &= \left(\frac{N_i - n_i}{N_i}\right)^2 \widehat{Var}_{\boldsymbol{v}}\left(g_{1i}^{-1}\sum_{j\in s_i}\bar{w}_{ij}\omega_i^{gw}\psi_b(a_{ij})\right)\\
&= \left(\frac{N_i - n_i}{N_i}\right)^2 g_{1i}^{-2}\sum_{j\in s_i}\bar{w}_{ij}^2\hat{E}_{\boldsymbol{v}}\left[\omega_i^{gw}\psi_b(a_{ij})\right]^2\\
&= \left(\frac{N_i - n_i}{N_i}\right)^2 \frac{g_{2i}}{g_{1i}^2(n_i - p_{i,e})}\sum_{j\in s_i}\left[\omega_i^{gw}\psi_b\left\{(y_{ij} - \boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw} - \tilde{v}_i^{\psi})/\omega_i^{gw}\right\}\right]^2,
\end{aligned}$$

where $g_{2i} = \sum_{j\in s_i}\bar{w}_{ij}^2$ and $p_{i,e}$ denotes the area-specific effective number of parameters which varies between p and n_i. If the bandwidth is sufficiently large or the sampled units in area i are spatially close, then $p_{i,e} \approx p$.
A second-order approximation for the conditional MSE of the RGWEBLUP-bc can be obtained by adding an extra term that accounts for the variability caused by the estimation of $\boldsymbol{\theta}$. Similar to equation (4.45), the MSE of the RGWEBLUP-bc can be approximated by

$$MSE_{\boldsymbol{v}}(\hat{\bar{y}}_i^{\psi,gw-bc}) \approx MSE_{\boldsymbol{v}}(\tilde{\bar{y}}_i^{\psi,gw-bc}) + E_{\boldsymbol{v}}\left[(\hat{\bar{y}}_i^{\psi,gw-bc} - \tilde{\bar{y}}_i^{\psi,gw-bc})^2\right]. \tag{4.61}$$

Using a first-order Taylor approximation, the prediction error induced by the estimation of $\boldsymbol{\theta}$ can be approximated by

$$\hat{\bar{y}}_i^{\psi,gw-bc} - \tilde{\bar{y}}_i^{\psi,gw-bc} \approx \frac{\partial\tilde{\bar{y}}_i^{\psi,gw-bc}}{\partial\boldsymbol{\theta}}\{\hat{\boldsymbol{\theta}}^{\psi,gw} - \boldsymbol{\theta}\}, \tag{4.62}$$

where

$$\begin{aligned}
\tilde{\bar{y}}_i^{\psi,gw-bc} &= N_i^{-1}\Big\{\sum_{j\in s_i} y_{ij} + \sum_{j\in r_i} \boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw} + \boldsymbol{z}_i^T\tilde{\boldsymbol{v}}^{\psi,gw} + (1-n_iN_i^{-1})^{-1}\widetilde{bc}_i^{gw}\Big\} \\
&\overset{\text{(eq. 4.55)}}{\approx} N_i^{-1}\Bigg\{\sum_{j\in s_i} y_{ij} + \sum_{j\in r_i} \boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw} + \boldsymbol{z}_i^T\tilde{\boldsymbol{v}}^{\psi,gw} + g_{1i}^{-1}\sum_{j\in s_i}\bar{w}_{ij}\omega_i^{gw}\psi_b(a_{ij}) \\
&\qquad + g_{1i}^{-1}\sum_{j\in s_i}\bar{w}_{ij}\boldsymbol{x}_{ij}^T\boldsymbol{\beta}_{ij} + \boldsymbol{z}_i^T\boldsymbol{v} - \boldsymbol{z}_i^T\tilde{\boldsymbol{v}}^{\psi,gw} - g_{1i}^{-1}\sum_{j\in s_i}\bar{w}_{ij}\boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\Bigg\} \\
&\approx N_i^{-1}\Bigg\{\sum_{j\in s_i} y_{ij} + \sum_{j\in r_i} \boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw} + g_{1i}^{-1}\sum_{j\in s_i}\bar{w}_{ij}\omega_i^{gw}\psi_b(a_{ij}) \\
&\qquad + g_{1i}^{-1}\sum_{j\in s_i}\bar{w}_{ij}\boldsymbol{x}_{ij}^T\boldsymbol{\beta}_{ij} + \boldsymbol{z}_i^T\boldsymbol{v} - g_{1i}^{-1}\sum_{j\in s_i}\bar{w}_{ij}\boldsymbol{x}_{ij}^T\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}\Bigg\}.
\end{aligned}$$

Following Chambers et al. (2014), it is assumed that the derivatives of $\tilde{\boldsymbol{\beta}}_{ij}^{\psi,gw}$ with respect to $\boldsymbol{\theta}$ are of a lower order. Then the derivative of $\tilde{\bar{y}}_i^{\psi,gw-bc}$ with respect to $\boldsymbol{\theta}$ can be approximated by

$$\begin{aligned}
&\frac{\partial\tilde{\bar{y}}_i^{\psi,gw-bc}}{\partial\boldsymbol{\theta}} \\
&\approx \partial_{\boldsymbol{\theta}}\Bigg\{N_i^{-1}\Big\{\sum_{j\in r_i}\boldsymbol{z}_i^T\boldsymbol{v} + g_{1i}^{-1}\sum_{j\in s_i}\bar{w}_{ij}\omega_i^{gw}\psi_b(a_{ij})\Big\}\Bigg\} \\
&\approx \partial_{\boldsymbol{\theta}}\Bigg\{\left(\frac{N_i-n_i}{N_i}\right)\Bigg\{\bar{\boldsymbol{z}}_i^T\boldsymbol{Q}(\boldsymbol{y}-\boldsymbol{\lambda}) \\
&\quad + g_{1i}^{-1}\sum_{j\in s_i}\bar{w}_{ij}\omega_i^{gw}\psi_b\left(\frac{(y_{ij}-\boldsymbol{x}_{ij}^T\boldsymbol{\beta}_{ij}-\boldsymbol{z}_{ij}^T\boldsymbol{Q}(\boldsymbol{y}-\boldsymbol{\lambda}))}{\omega_i^{gw}}\right)\Bigg\}\Bigg\} \\
&\approx \left(\frac{N_i-n_i}{N_i}\right)\Bigg\{\bar{\boldsymbol{z}}_{ri}^T\left(\partial_{\boldsymbol{\theta}}\boldsymbol{Q}\right)(\boldsymbol{y}-\boldsymbol{\lambda}) \\
&\quad - g_{1i}^{-1}\sum_{j\in s_i}\bar{w}_{ij}\omega_i^{gw}\psi_b'\left(\frac{(y_{ij}-\boldsymbol{x}_{ij}^T\boldsymbol{\beta}_{ij}-\boldsymbol{z}_{ij}^T\boldsymbol{Q}(\boldsymbol{y}-\boldsymbol{\lambda}))}{\omega_i^{gw}}\right)\boldsymbol{z}_{ij}^T\left(\partial_{\boldsymbol{\theta}}\boldsymbol{Q}\right)(\boldsymbol{y}-\boldsymbol{\lambda})\Bigg\} \\
&\approx \left(\frac{N_i-n_i}{N_i}\right)\boldsymbol{\Omega}_i^T\sum_{k=2}^{2}\left(\partial_{\theta_k}\boldsymbol{Q}\right)(\boldsymbol{y}-\boldsymbol{\lambda}). \qquad (4.63)
\end{aligned}$$

Analogously to equation (3.65), $\boldsymbol{\Omega}_i$ is defined by

$$\boldsymbol{\Omega}_i = \bar{\boldsymbol{z}}_{ri}^T - g_{1i}^{-1}\sum_{j\in s_i}\bar{w}_{ij}\omega_i^{gw}\psi_b'\left(\frac{(y_{ij}-\boldsymbol{x}_{ij}^T\boldsymbol{\beta}_{ij}-\boldsymbol{z}_{ij}^T\boldsymbol{Q}(\boldsymbol{y}-\boldsymbol{\lambda}))}{\omega_i^{gw}}\right)\boldsymbol{z}_{ij},$$

and ψ_b' is defined as in equation (4.54). Using equation (4.63), the variance of the prediction error in (4.62) can be approximated by

$$\begin{aligned} & Var_{\boldsymbol{v}}\left(\hat{\bar{y}}_i^{\psi-bc} - \tilde{\bar{y}}_i^{\psi-bc}\right) \\ & \approx \left(\frac{N_i - n_i}{N_i}\right)^2 \boldsymbol{\Omega}_i^T Var_{\boldsymbol{v}}\left(\sum_{k=2}^{2}(\partial_{\theta_k}\boldsymbol{Q})\left(\boldsymbol{y}-\boldsymbol{\lambda}\right)(\hat{\theta}_k^{\psi,gw} - \theta_k)\right)\boldsymbol{\Omega}_i \\ & \approx \left(\frac{N_i - n_i}{N_i}\right)^2 \boldsymbol{\Omega}_i^T \Upsilon \ Var_{\boldsymbol{v}}(\hat{\boldsymbol{\theta}}^{\psi,gw})\boldsymbol{\Omega}_i, \end{aligned} \tag{4.64}$$

where Υ is defined as in equation (4.49). Note that when the tuning constant b is sufficiently large then $\psi_b' \approx 1$. In that case $\boldsymbol{\Omega}_i = \mathbf{0}$ under the random intercept working model in (2.62), and the conditional variance in (4.64) is reduced to zero. Using the results from above, an estimate for the conditional MSE for the RGWEBLUP-bc is given by

$$\widehat{MSE}_{\boldsymbol{u}}(\hat{\bar{y}}_i^{\psi,gw-bc}) = h_{1i}^{bc}(\tilde{\boldsymbol{\Theta}}) + h_{2i}^{bc}(\tilde{\boldsymbol{\Theta}}) + h_{3i}(\tilde{\boldsymbol{\Theta}}) + h_{4i}^{bc}(\tilde{\boldsymbol{\Theta}}), \tag{4.65}$$

where h_{4i}^{bc} is given by (4.64). To finally estimate the conditional MSE for the RGWEBLUP-bc of the area mean, $\tilde{\boldsymbol{v}}^{\psi,gw}$ and $\tilde{\boldsymbol{\beta}}^{\psi,gw}$ and $\boldsymbol{\theta}$ need to be replaced by the estimates $\hat{\boldsymbol{v}}^{\psi,gw}$ and $\hat{\boldsymbol{\beta}}^{\psi,gw}$ and $\hat{\boldsymbol{\theta}}^{\psi,gw}$, respectively, leading to

$$\widehat{MSE}_{CCST}(\hat{\bar{y}}_i^{\psi,gw-bc}) = h_{1i}^{bc}(\hat{\boldsymbol{\Theta}}) + h_{2i}^{bc}(\hat{\boldsymbol{\Theta}}) + h_{3i}(\hat{\boldsymbol{\Theta}}) + h_{4i}^{bc}(\hat{\boldsymbol{\Theta}}). \tag{4.66}$$

The proposed MSE estimators in (4.50) and (4.66) for the robust projective RGWEBLUP and the robust predictive RGWEBLUP-bc, respectively, are generalizations of the linearization approach of Chambers et al. (2014). By setting the spatial weights in model (2.62) to be constant for all population units, equations (4.50) and (4.66) become identical to the conditional MSE estimates from equations (3.56) and (3.67), respectively. However, assessing the asymptotic properties of the conditional MSE estimates developed in this section remains an open research question. So far, these estimates can be seen as a rather heuristic adaptation of the linearization approach in Chambers et al. (2014) to the case of spatial non-stationarity.

4.3 Implementation

Recent developments in the field of SAE are mainly driven by the increasing demand in public institutions for reliable information on a disaggregated level. To enable the application of new methods it is necessary to supply user-friendly software that can handle realistic data situations.

Several software packages for the R-language are currently available for SAE. Molina and Marhuenda (2015) introduced the package `sae` which provides a wide range of functions for area-level and unit-level SAE includ-

ing the EBLUP estimator of the area mean in equation (2.26) and MSE estimates based on a parametric bootstrap approach. Analytic MSE estimators, such as the conditional and unconditional estimators presented in Section 2.3.2 are not implemented, though. Some estimators implemented in this package are also available in the `emdi` package of Kreutzmann et al. (2017) whereas the focus here is to support the user in visualizing the results and exporting them to `MS-Excel` for further processing. Schoch (2014) developed the package `rsae` for robust SAE where the REBLUP of the area mean from equation (3.11) is implemented with a parametric bootstrap approach for MSE estimation. Yet, robust predictive estimators such as the REBLUP-bc in equation (3.20) are not available. The package `saeRobust` of Warnholz (2016) provides functions for robust area-level SAE with spatial and temporal extensions. So far, packages for robust and non-robust spatial unit-level SAE that were discussed in Sections 2.4 and 3.2 have not been published. However, an R-script implementing these methods is usually available from the authors.
As supplemental material to this thesis, the package `saeRGW` was developed which provides an implementation of the GWEBLUP in (2.65) of Chandra et al. (2012) and the proposed robust extensions, the RGWEBLUP in (4.6) and the RGWEBLUP-bc in (4.8), together with conditional MSE estimates (CCT and CCST). Section 4.3.1 gives guidance for handling the functions provided in this package and Section 4.3.2 outlines next steps for further development.

4.3.1 saeRGW: A Package for (Robust) Small Area Estimation under Spatial Non-Stationarity

The package `saeRGW` provides two categories of important functions: *fit* and *predict*. The two fit functions `gwlmm` and `rgwlmm` estimate the model parameters under the geographically weighted linear mixed model (GWLMM) in (2.62) using a population sample. The function `gwlmm` assumes normality for the error term components whereas the function `rgwlmm` uses the robust ML estimating equations (4.3) and (4.4) to estimate the local coefficients and the variance parameters, respectively. Both functions execute the first seven steps of the step-wise procedure from Section 4.1.
I decided to split this procedure as users do not necessarily have to be interested in SAE. The package can also be used when fitting a nested error regression model which takes into account spatial non-stationarity in the fixed effects coefficients is appropriate. Hence, it is applicable in situations where the coefficients vary over space and unobserved heterogeneity between sampled domains or clusters is present in the data.
The `predict` generic functions `predict.gwlmm` and `predict.rgwlmm` use the fitted model and estimate small area means together with the conditional MSE estimators from Section 4.2 when population data is provided. The predict function can handle aggregated population information with

centroid coordinates and unit-level population data with coordinates for each non-sampled unit. If no population data is provided these functions provide the in-sample predictions (random effects, residuals, etc.) based on the model fits. For users of this package who are interested in producing small area estimates, I recommend the following work flow:

1: Prepare the data to fit the model.

2: Fit a GWLMM to a population sample using the function `gwlmm`.

3: Check the results using model diagnostics provided by the `summary` and `plot` generic functions and decide whether a robust parameter estimation is appropriate.

No: Go to *step 4* and skip the rest.

Yes: Go to *step 5*.

4: Predict the small area means for aggregated or unit-level population data using the function `predict.gwlmm`.

5: Fit a GWLMM to a population sample with the function `rgwlmm` using the non-robust results from *step 2* as starting values for the local coefficients and the variance parameters.

6: Predict the robust and non-robust small area means for aggregated or unit-level population data using the functions `predict.gwlmm` and `predict.rgwlmm` and compare the results.

In what follows, these steps are demonstrated using the three simulated data sets attached to the package. When the package is installed and loaded, these data sets can be accessed by typing their names: `sampleData`, `popaggData`, `popData`. The data set `sampleData` is a sample of size $n = 200$ from a generated population that consists of $m = 40$ regions. From each region $n_i = 5$ units were selected by simple random sampling without replacement. The population is generated under the GLMM in (2.62) with a 5% outlier contamination. Details on the data-generating process are given in Chapter 5 as the provided data set `sampleData` represents one sample from the model-based simulation presented there. The two data sets `popaggData` and `popData` contain the population data and are described later in this section.

```
>library(saeRGW)
>head(sampleData)

#   long      lat clusterid        x        y
# 1  0.5 3.877223         1 5.407193 129.0080
# 2  3.0 2.763340         1 4.693331 136.0443
```

```
# 3  2.0 1.008894          1 2.518922 115.7804
# 4  0.5 5.877223          1 3.319369 121.6476
# 5  2.0 3.008894          1 2.718502 116.1293
# 6  6.5 1.903904          2 3.063594 125.2182
```

To use the fit functions `gwlmm` or `rgwlmm`, a data set must consist of at least two variables for the geographic coordinates (here: `long` and `lat`), an identifier for the areas (here: `clusterid`), an explanatory variable (here: `x`) and a dependent variable (here: `y`). In this example the data is already prepared and we can start directly at *step 2*. To fit the GWLMM in (2.62) to a sample using the function `gwlmm`, at least two arguments need to be defined:

`formula`	a two-sided formula object describing the fixed effects, the random effects and the geographical information of the model. The response is on the left of the ~ operator. The explanatory variables (numeric or factor) on the right side are separated by a + operator, a vertical bar (|) separates the identifier for the random intercept. After the second vertical bar, the two coordinates -longitude and latitude- are separated by a + operator.
`data`	a data frame containing the variables used in `formula`.

Additional arguments can be specified:

`band`	a numeric value defining an *a priori* defined bandwidth (default: `band = NULL`).
`maxit`	an integer value defining the maximum number of iterations for estimating the model parameters (default: `maxit = 100`).
`tol`	a numeric value defining the tolerance of the convergence (default `tol = 1e-04`).
`centroid`	a logical value defining weather coordinates in `formula` are centroids (default `centroid = FALSE`).

The function `gwlmm` returns an object of class 'gwlmm' that can be used for the `summary` generic function.

```
>formula <- y ~ x | clusterid | long + lat
>fitGWLMM <- gwlmm(formula, data = sampleData)
>summary(fitGWLMM)

# Call:
# gwlmm(formula = formula, data = sampleData)
#
# Geographic coordinates were available for each individual
#
# Bandwidth:  3.662756
#
```

```
# Number of iterations for ML estimates: 17
#
# Estimates of variance parameters:
# Random Effects       Residuals
#          2.691           9.561
#
# Shapiro-Wilk Test for the error term components:
#                    W          p-value
# Random effects:   9.568e-01  1.302e-01
# Residuals:        9.487e-01  1.430e-06
#
# Summary statistics for local coefficients:
#                Min. 1st Qu. Median   Mean 3rd Qu.   Max.
# (Intercept) 95.790 101.600 103.60 103.30  104.90 107.90
# x            6.938   9.844  10.91  11.32   12.86  17.69
#
# LR-Test for non-stationarity of model coefficients:
# Likelihood          LR          DF     P-Value
# -5.271e+02   3.309e+02   4.260e+01   2.897e-46
#
# Effective number of parameters 2*tr(H) - tr(H'H): 44.60187
```

The summary output contains basic information about the estimated model parameters. Here, we can see the bandwidth that is used for the spatial weights, the summary statistics for the local coefficients and the variance components. In addition, the summary output shows the results of a Shapiro-Wilk test (cf. Shapiro and Wilk, 1965) for normality for the error term components. Here normality can be rejected (p-value ≤ 0.05) for the unit-level residuals.
The bandwidth is estimated using the cross-validation criteria (2.57) implemented in the function `gwr.sel` from the package `spgwr` (Bivand and Yu, 2015). From the summary statistics for the local coefficients it appears that the local coefficients vary substantially, indicating that fitting a GWLMM rather than the global LMM in (2.24) is justified. In addition, `gwlmm` conducts a likelihood ratio test that compares the likelihood of the GWLMM and a global LMM fit, where the LMM is fitted using the function `lmer` from the package `lme4` (Bates et al., 2015). When the test rejects the Null, the model fit under the GWLMM is significantly better than for the LMM which indicates that spatial non-stationarity is present (cf. Fotheringham et al., 2002, p.92). In this case the Null is rejected and spatial non-stationarity in the model coefficients can be assumed.
The normality assumption can also be assessed using the `plot` generic function that provides quantile-quantile-plots for the residual components.

```
> plot(fitGWLMM)
```

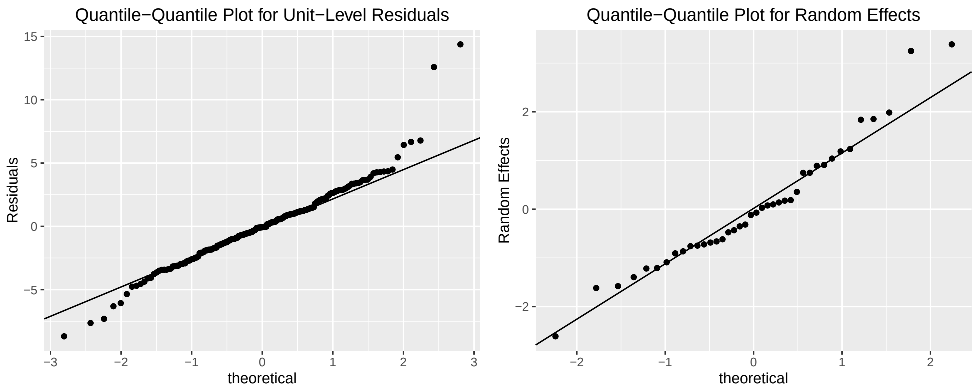


Figure 4.1: Output of plot method for objects of class ‘gwlmm’

Figure 4.1 shows substantial departures from the normal distribution of both error term components: the random effects and the individual error. This is expected as the data was generated with a 5% symmetric outlier contamination in the area-level and unit-level error term. When departures from the normality assumption are present, we can go to *step 5* and estimate the model parameters robustly using the function `rgwlmm`. This function has the same structure as `gwlmm`. In addition, two optional arguments can be specified:

- `Start` a list object defining starting values for the robust parameter estimation containing three objects: `betas` - Matrix with local coefficients; `sigma.v` - numeric value for the variance of the random effects; `sigma.e` - numeric value for the error term variance. Default = `NULL`.
- `k` a numeric value defining the tuning constant for Huber’s influence function (default `k = 1.345`).

If good starting values are available I recommend using the argument `Start` as this can decrease the number of iterations needed for the robust parameter estimation. Since the bandwidth was already estimated in the previous step, we can use it here by specifying the argument `band`. The function `rgwlmm` returns an object of class ‘rgwlmm’ that can be used for the `summary` generic function.

```
>Start <- list(betas = fitGWLMM$Coefficients,
               sigma.v = fitGWLMM$Variance$raneff,
               sigma.e = fitGWLMM$Variance$residual)

>fitRGWLMM <- rgwlmm(formula,
                     data = sampleData,
                     Start = Start,
```

```
                    band = fitGWLMM$bandwidth)
>summary(fitRGWLMM)

# Call:
# rgwlmm(formula = formula, data = sampleData,
#        band = fitGWLMM$bandwidth, Start = Start)
#
# Geographic coordinates were available for each individual
#
# Bandwidth:  3.662756
#
# Number of iterations for robust estimates: 29
#
# Robust estimates of variance parameters:
# Random Effects       Residuals
#          1.642           4.899
#
# Summary statistics for the robust local coefficients:
#               Min. 1st Qu. Median   Mean 3rd Qu.   Max.
# (Intercept) 96.560 101.500 103.10 102.80   104.5 107.50
# x            7.686   9.833  10.79  11.45    13.0  16.39
#
# Tuning constant: 1.345
```

The summary output shows the bandwidth that was used for the spatial weights, the robust variance parameters, the robust local coefficients and the tuning constant that was used to restrict the influence of extreme observations. It is noticeable that the robust variance estimates are substantially smaller than the non-robust, whereas this is not the case for the local coefficients. Here, the range of the coefficients appears to be slightly smaller compared to the non-robust estimates. To predict the robust and non-robust small area means in *step 6* a user can call the `predict` generic function. Two arguments have to be specified:

`object`	an object of class ‘gwlmm’ or ‘rgwlmm’.
`popdata`	a data frame with population data. Two options are possible: (1) containing the variables on the right of the ~ operator in the `formula` object for all non-sampled population units, (2) containing these variables aggregated on area-level and the area-specific population size. Default = NULL.

If the `object` argument is of class ‘rgwlmm’, three optional arguments are available:

bcConst	a numeric value defining the tuning constant for Huber's influence function in the bias correction (default bcConst = 3).
maxit	an integer value defining the maximum number of iterations for the random effects (default maxit=100).
tol	a numeric value defining the tolerance of the convergence (default tol = 1e-04).

When the argument popdata is unspecified, only in-sample predictions are produced.

```
>summary(predict(fitGWLMM))

# Call:
# predict.gwlmm(object = fitGWLMM)
#
# Summary statistics for estimates error term components:
#                    Min.  1st Qu. Median Mean 3rd Qu. Max.
# Random effects: -2.611 -0.751 -0.095 0.042 0.784  3.384
# Residuals:      -8.689 -1.718 -0.070 0.030 1.408 14.380
#

>summary(predict(fitRGWLMM))

# Call:
# predict.rgwlmm(object = fitRGWLMM)
#
# Iterations for robust random effects:  14
#
# Summary statistics for estimates error term components:
#                    Min.  1st Qu. Median  Mean  3rd Qu.  Max.
# Random effects:  -1.825 -0.734 -0.063  0.121  0.748  4.908
# Residuals:      -15.130 -1.289 -0.116  0.061  1.347 19.500
```

When popdata is defined, the attached data set popData is suitable for area mean predictions in case (1) and popaggData in case (2).

```
>head(popData)

#   long lat clusterid        x
# 1    0 0.0         1 3.451709
# 2    0 0.5         1 3.483688
# 3    0 1.0         1 1.768635
# 4    0 1.5         1 1.795835
# 5    0 2.0         1 2.314546
# 6    0 2.5         1 1.416395
```

```
>head(popaggData)

#  clusterid  long      lat        x Size
# 1         1  1.75 3.070282 3.067707  100
# 2         2  5.75 3.088069 2.823851  100
# 3         3  9.75 3.105856 3.354610  100
# 4         4 13.75 3.123643 3.289381  100
# 5         5 17.75 3.141431 3.363055  100
# 6         6 21.75 3.159218 3.315946  100
```

From a practical point of view it seems most realistic that a user has access to a population sample with unit-level information and aggregated population data. Therefore, I demonstrate an example using the data set `popaggData`. One additional argument has to be specified here:

`size`	a character value naming the variable in `popdata` containing the area-specific population sizes.

The `predict` generic function returns an object of class 'gwpred' that can be used inside the `summary` generic function.

```
>predGWLMM <- predict(fitGWLMM, popdata = popaggData,
                  size = "Size")
>summary(predGWLMM)

# Call:
# predict.gwlmm(object = fitGWLMM, popdata = popaggData,
#                size = "Size")
#
# ##################In-sample Predictions##################
#
# Summary statistics for estimates error term components:
#
#                 Min.  1st Qu. Median Mean 3rd Qu. Max.
# Random effects:-2.611 -0.751 -0.095 0.042 0.784  3.384
# Residuals:     -8.689 -1.718 -0.070 0.030 1.408 14.380
#
# ##################Area Mean Predictions##################
#
#          Min.   1st Qu. Median Mean   3rd Qu. Max.
# GWEBLUP  119.90 133.20  138.50 138.50 144.10  157.80
# MSE_CCST   0.62   0.75    1.01   1.30   1.47    5.76
# MSE_CCT    0.30   0.69    1.01   1.50   1.76    6.69

>predRGWLMM <- predict(fitRGWLMM, popdata = popaggData,
                   size = "Size")
```

```
>summary(rpredPop)

# Call:
# predict.rgwlmm(object = fitRGWLMM, popdata = popaggData,
#                size = "Size")
#
# #################In-sample Predictions#################
#
# Iterations for robust random effects:  14
#
# Summary statistics for estimates error term components:
#
#                    Min.  1st Qu. Median  Mean  3rd Qu. Max.
# Random effects: -1.825 -0.734 -0.063  0.121  0.748  4.908
# Residuals:     -15.130 -1.289 -0.116  0.061  1.347 19.500
#
# #################Area Mean Predictions#################
#
#             Min.   1st Qu. Median Mean   3rd Qu. Max.
# RGWEBLUP    120.50 133.10  138.60 138.40 144.00  158.30
# RGWEBLUP-bc 119.70 133.10  139.20 138.40 144.00  159.40
# MSE_CCST      0.34   0.46    0.89   1.34   1.43   10.69
# MSE_CCST-bc   0.34   0.94    1.26   1.61   2.26    4.66
# MSE_CCT       0.37   0.63    1.24   1.75   1.98   14.16
# MSE_CCT-bc    0.28   1.02    1.35   2.05   2.58   10.76
```

The `predict` generic function can also handle non-sampled areas. For those areas which are not in the sample but belong to the population, the returned point estimates are pure synthetic estimates. For robust predictions, the bias correction becomes zero. Hence, the robust projective and robust predictive estimators are equal for non-sampled areas. In addition, the package can handle three combinations of geographic information in the sampled and population data: (1) only centroid information in sampled and population data; (2) unit-level geographic information in sample and population data; (3) unit-level geographic information in the sample and centroid information in population data. A situation where unit-level geographic information is available for the population but not for the sample is not considered for this implementation as it seems unrealistic.

4.3.2 Further Developments

The package `saeRGW` is currently published on `GitHub` (repository: baldermann/saeRGW). Before submitting the package to a `CRAN` repository some work has to be done to guarantee a stable version for potential users. One important issue is to implement useful error messages that inform the user

how to continue after a problem occurs. Another topic concerns the computation time needed to fit a model in the GLMM framework. Even for common sample size, e.g., $n = 2000$ the fitting process can be very time-consuming. Some internal functions are already translated into the C++ language (ISO, 2012) to speed up the calculations. Further work can be done to optimize the linear algebra operations behind the fitting process.
As mentioned in the introduction of this section, there are already several package in existence for applications in SAE. With an increasing variety of packages, however, it may become difficult for users to choose between packages in each application. So far the package `saeRGW` covers SAE under spatial non-stationarity which is but one specific branch within the whole field. Hence, in a next step the package can be extended to cover unit-level (robust and non-robust) SAE where all the estimators from Chapters 2 and 3 are implemented with their point and MSE estimates. Developing such a general package would be beneficial for users as different estimators can be tested and compared for real applications using the interface as presented in the last section.
In a further step, this general package can be integrated in the package `saeRobust` of Warnholz (2016) to combine unit-level and area-level SAE in one package. Most software components in the package `saeRobust` can also be used for unit-level models as these are programmed in a very general form. In addition, some features of the `saeRGW` package, such as the summary output and the plot method, are already in line with this package. Thus, an integration in the package `saeRobust` seems natural for further developments.

4.4 Summary and Outlook

Outlier robust projective and predictive extensions to the currently available GWEBLUP of Chandra et al. (2012) were proposed in this chapter. The main findings can be summarized as follows:

- The influence of outliers on the parameter estimation has been reduced using the results of Sinha and Rao (2009). Their robust ML estimation equations were modified to account for spatial non-stationarity in the fixed effects coefficients.

- The robust parameter estimation has been integrated into the algorithm of Chandra et al. (2012) to estimate the robust projective GWEBLUP (RGWEBLUP) of the area mean. For that purpose a local fixed point algorithm has been developed for the approximation of the local in-sample coefficients $\boldsymbol{B}_s$.

- The local bias correction of Chambers et al. (2014) has been combined with geographical weighting to develop a robust predictive GWEBLUP (RGWEBLUP-bc) of the area mean.

- Two conditional MSE estimators have been developed for the RGWEBLUP and the RGWEBLUP-bc using the pseudo-linearization and the linearization approach in Chambers et al. (2014).

- The GWEBLUP of Chandra et al. (2012) and the proposed estimators, the RGWEBLUP and the RGWEBLUP-bc, have been implemented for the R-language in the package `saeRGW`.

Model-based and design-based simulation studies are conducted to assess the performance of the proposed estimators. These are presented in Part II of this thesis. Chapter 5 presents the results of a model-based simulation where the performance of the reviewed and the proposed methods is analyzed under different outlier scenarios. This includes scenarios with and without spatial stationarity in the model coefficients. Chapter 6 presents the results of a design-based simulation where selected estimators are compared for SAE under the realistic data situation of the Berlin apartment rental market.

Part II

Empirical Results

Model-based Simulation 5

This chapter presents the results from a model-based simulation study assessing the performance of the estimators that were described in Part I of this thesis. The aim of this study is twofold: to examine the performance of these estimators under spatial non-stationarity; and analyze their performance in the presence of outliers. Hence, the scenarios for the model-based simulations are a combination of settings under spatial stationarity and non-stationarity with different outlier contamination mechanisms. Details regarding the simulation setup are presented in Section 5.1. Thereafter, it follows a discussion regarding the estimated variance components and local coefficients under the GWLMM in Section 5.2. Section 5.3 presents the simulation results of the small area means and in Section 5.4 the performance of the proposed MSE estimators is discussed. The calculation times and the number of converged Monte Carlo replications are presented in Section 5.5. This is followed by a short discussion of the results in Section 5.6.

5.1 Simulation Setup

Following the simulation setup in Chambers et al. (2014), the population data is generated for $m = 40$ small areas. The population and sample sizes are fixed for all areas with $N_i = 100$ and $n_i = 5$, respectively. The samples are selected from the population by simple random sampling without replacement (SRSWOR) within each area. The target variable y_{ij} is generated using the nested error regression model:

$$y_{ij} = \beta_{0,ij} + \beta_{1,ij} x_{ij} + v_i + e_{ij}, \tag{5.1}$$

with

$$\beta_{0,ij} = 100 + a_0(longitude_{ij} + latitude_{ij})$$
$$\beta_{1,ij} = 5 + a_1(longitude_{ij} + latitude_{ij}).$$

The explanatory variable x_{ij} is drawn from a log-normal distribution with mean 1 and a standard deviation of 0.5. The population coordinates are generated as a rectangular grid of points covering the region $[0, \sqrt{N}/2] \times [0, \sqrt{N}/2]$. Figure 5.1 shows the resulting lattice of neighboring rectangular areas and the distribution of the coordinates (longitude and latitude) in the synthetic population. Here, the red target marks show the area centroid. The parameters a_0 and a_1 in the population model (5.1) define the spatial variation of the model coefficients. Following Chandra et al. (2012) these are set to zero for scenarios under spatial stationarity and to $a_0 = 0.1$ and $a_1 = 0.2$ for scenarios under spatial non-stationarity. Hence, under spatial stationarity the model coefficients are constant with $\beta_0 = 100$ and $\beta_1 = 5$. Under spatial non-stationarity the coefficients linearly depend on the coordinates where $\beta_{0,ij}$ lies within the range [100, 106.32] and $\beta_{1,ij}$ within the range [5, 17.65].

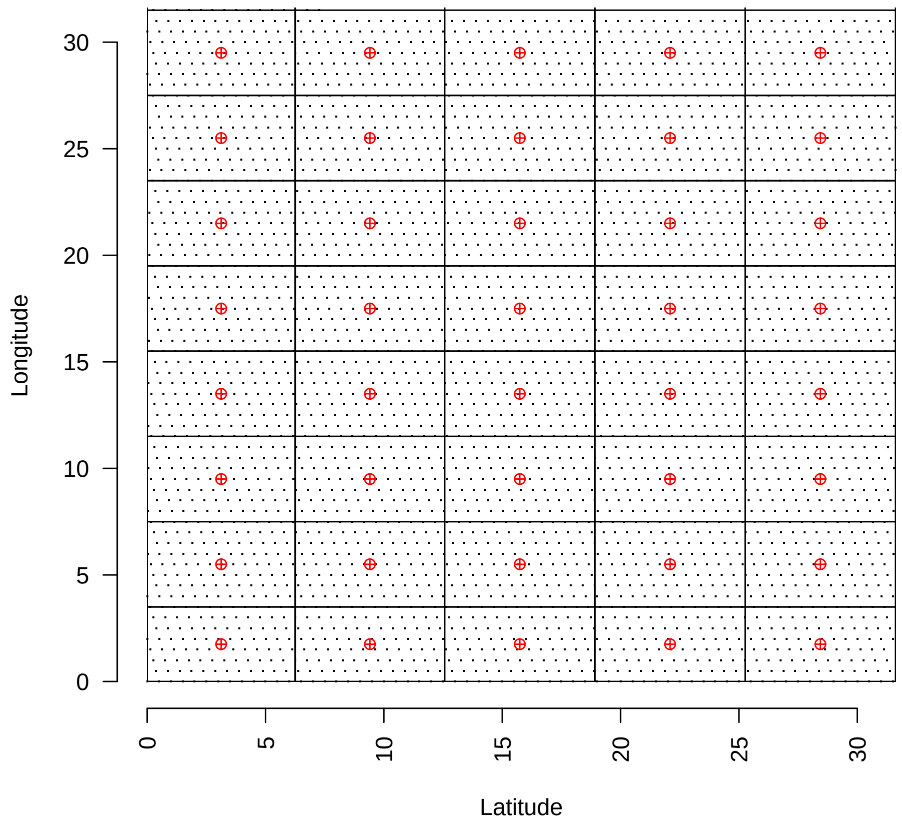


Figure 5.1: Coordinates of the synthetic population for the model-based simulation.

The random effects v_i and the individual error term e_{ij} in equation (5.1) are generated independently under different outlier contamination mechanisms:

- $(0,0)$ - no outliers with $e_{ij} \sim N(0,6)$ and $v_i \sim N(0,3)$;
- $(v,e)_s$ - symmetric outlier in the area and unit-level error term: $v_i \sim N(0,3)$ for the areas 1-36 and $v_i \sim N(0,20)$ for the areas 37-40, $e_{ij} \sim \delta N(0,6)+(1-\delta)N(0,150)$ where δ is Bernoulli distributed with $P(\delta = 1) = 0.95$;

- $(v, e)_{ns}$ - non-symmetric outliers the area and unit-level error term: $v_i \sim N(0, 3)$ for the areas 1-36 and $v_i \sim N(9, 20)$ for the areas 37-40, $e_{ij} \sim \delta N(0, 6) + (1 - \delta)N(20, 150)$ where δ is Bernoulli distributed with $P(\delta = 1) = 0.95$.

Under spatial stationarity, the scenario with no outliers, $(0, 0)$, serves as a reference when the assumptions of the LMM in (2.24) are valid whereas under spatial non-stationarity this scenario refers to the GWLMM in (2.62). The setting of the variance parameters induces a moderate intraclass correlation of $\sigma_v/(\sigma_v + \sigma_e) = 0.33$.
Following Sinha and Rao (2009), the focus in the second scenario $(v, e)_s$ is on symmetric outlier contamination in the area and unit-level error term. It is expected that the robust projective estimators (REBLUP, SREBLUP, RNPEBLUP and RGWEBLUP) are superior under these settings. However, the assumption of symmetric outlier contamination may be violated in real applications. Thus, following Chambers et al. (2014), non-symmetric outlier contamination is investigated in the third scenario $(v, e)_{ns}$. Here, outlier robust projective estimators are expected to suffer from a potential bias. The outlier robust predictive estimators (REBLUP-bc, SREBLUP-bc and RGWEBLUP-bc) should reduce this bias under this scenario.
These outlier scenarios and the general simulation setup, such as the number of areas and the sample and population sizes, are comparable to simulations conducted by Sinha and Rao (2009) and Chambers et al. (2014). Thus, this specific choice of the simulation setup provides the possibility to investigate the proposed SAE methods under extreme situations and produce comparable results to previous studies. Each scenario was repeated independently $T = 500$ times. For each Monte Carlo replication the population was generated according to the underlying setting and a sample was drawn. The sampled data was used to estimate the small area means and the corresponding MSE estimates. Before the results for the small area means are presented, the robust and non-robust parameter estimation under the GWLMM area discussed in brief.

5.2 Parameter Estimation under the GWLMM

In this thesis the main interest lies in producing reliable estimates of the small area mean rather than the model parameters itself. However, as mentioned while describing the package `saeRGW` in Section 4.3.1, users do not necessarily have to be interested in SAE when fitting the GWLMM.
First, the discussion focuses on the robust and non-robust estimation of the local coefficients $\beta_{0,ij}$ and $\beta_{1,ij}$. Finding an overall quality measure is not obvious since there exist as many local coefficients as there are units in the population. Therefore, I concentrate on the boundaries of the parameter space. Remember that in the population under spatial non-stationarity $\beta_{0,ij}$ lies within the range [100, 106.32] and $\beta_{1,ij}$ within the range [5, 17.65]. Con-

sequently, the estimated local coefficients should also lie within these ranges. Under spatial stationarity the estimated coefficients should be nearly constant with the expected values $\beta_0 = 100$ and $\beta_1 = 5$. The minimum (Min) and the maximum (Max) of the estimated local coefficients in each Monte Carlo replication serve as a measure for the observed boundaries. The median of these values over all replications are reported in 5.1 for all outlier scenarios under spatial stationarity and non-stationarity. In addition, the table reports the relative deviation of the Min and Max values from the respective lower and upper bound.

Scenario	*Coeff.*	*Non-robust*				*Robust*			
		Min	Δ%[1]	Max	Δ%[2]	Min	Δ%[1]	Max	Δ%[2]
Spatial non-stationary: $\beta_{0,ij} \in [100, 106.32]$, $\beta_{1,ij} \in [5, 17.65]$									
$(0,0)$	$\hat{\beta}_0$	96.20	-3.80	109.81	3.28	96.16	-3.84	109.85	3.32
	$\hat{\beta}_1$	6.01	20.27	16.59	-6.02	5.94	18.84	16.59	-6.01
$(v,e)_s$	$\hat{\beta}_0$	96.05	-3.95	110.65	4.07	96.45	-3.55	110.24	3.68
	$\hat{\beta}_1$	6.08	21.58	16.53	-6.35	6.18	23.64	16.52	-6.40
$(v,e)_{ns}$	$\hat{\beta}_0$	97.01	-2.99	118.26	11.23	97.07	-2.93	117.64	10.65
	$\hat{\beta}_1$	5.91	18.15	16.61	-5.90	6.26	25.10	16.69	-5.47
Spatial stationary: $\beta_0 = 100$, $\beta_1 = 5$									
$(0,0)$	$\hat{\beta}_0$	98.78	-1.22	101.03	1.03	98.85	-1.15	101.07	1.07
	$\hat{\beta}_1$	4.74	-5.13	5.31	6.27	4.76	-4.88	5.30	6.08
$(v,e)_s$	$\hat{\beta}_0$	99.01	-0.99	101.14	1.14	99.21	-0.79	100.85	0.85
	$\hat{\beta}_1$	4.77	-4.67	5.26	5.13	4.81	-3.74	5.18	3.67
$(v,e)_{ns}$	$\hat{\beta}_0$	97.71	-2.29	109.36	9.36	97.83	-2.17	107.68	7.68
	$\hat{\beta}_1$	3.85	-23.03	6.14	22.86	4.34	-13.17	5.70	14.10

[1] Relative deviation of Min from the lower boundary of the parameter space
[2] Relative deviation of Max from the upper boundary of the parameter space

Table 5.1: Median values for the Min and Max of the estimated local coefficients and under spatial stationarity and non-stationarity.

Comparing the robust and non-robust results of the Min and Max values, only very small differences in the range of the estimated coefficients can be observed in scenarios with no outliers, $(0,0)$. In scenarios with symmetric or asymmetric outliers,$(v,e)_s$ and $(v,e)_{ns}$, the range of the robust compared to the non-robust coefficients is smaller, especially for the estimated intercepts ($\hat{\beta}_0$). Since the robust parameter estimates are less affected by outliers, a smaller range can be expected.
Under spatial non-stationarity, in scenarios with no or symmetric outlier

contamination, $(0,0)$ and $(v,e)_s$, the range of the estimated robust and non-robust intercepts ($\hat{\beta}_0$) is larger than the expected range of [100, 106.32]. This effect is more severe with asymmetric outlier contamination $(v,e)_{ns}$. Here, the Max value of the intercept is about 10% higher than the expected upper boundary of 106.32. At the same time, in all outlier scenarios, $(0,0)$, $(v,e)_s$ and $(v,e)_{ns}$, the range of the estimated robust and non-robust slope coefficients ($\hat{\beta}_1$) is smaller than the expected range of [5, 17.65].
Turning to the spatial stationary scenarios, the range of the estimated local coefficients always includes the true population parameter for the intercept and the slope. In the scenario with asymmetric outliers, $(v,e)_{ns}$, the range becomes wider for both coefficients.

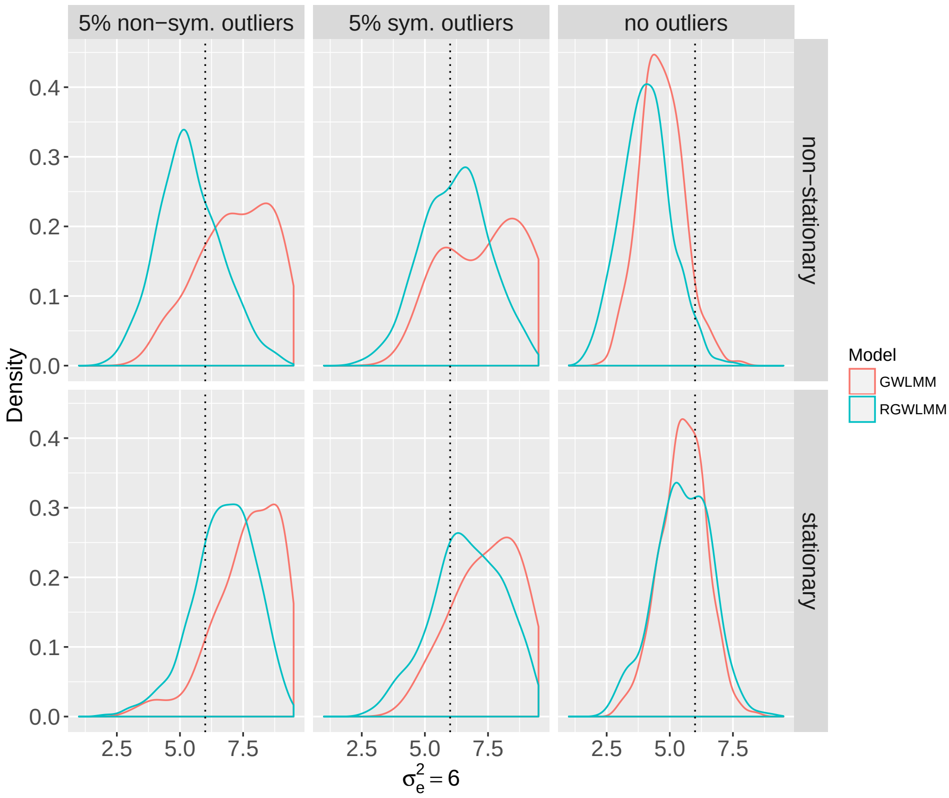


Figure 5.2: Density plots of σ_e^2 for all outlier scenarios under spatial stationarity and non-stationarity.

The focus now turns toward the estimation of the variance parameters. Figure 5.2 shows density plots for the estimated variance of the individual error term component σ_e^2. In general, it can be observed that the estimation of σ_e^2 is not very satisfying under the robust (blue) and non-robust (red) approach as in most cases the densities are not centered around the true value. One exception is the scenario with symmetric outlier contamination where the proposed robust estimation seems to estimate σ_e^2 quite well under spatial stationarity and non-stationarity. Looking at the area-level variance component in Figure 5.3 both approaches (robust and non-robust) seem to underestimate σ_v^2 in most of the scenarios. Hence, under the GWLMM the estimation of the variance parameters, either robust or non-robust, seems quite unreliable. For future simulations, it would be interesting to examine these results at different values of intraclass correlations (e.g. 10%, 40% and 60%).

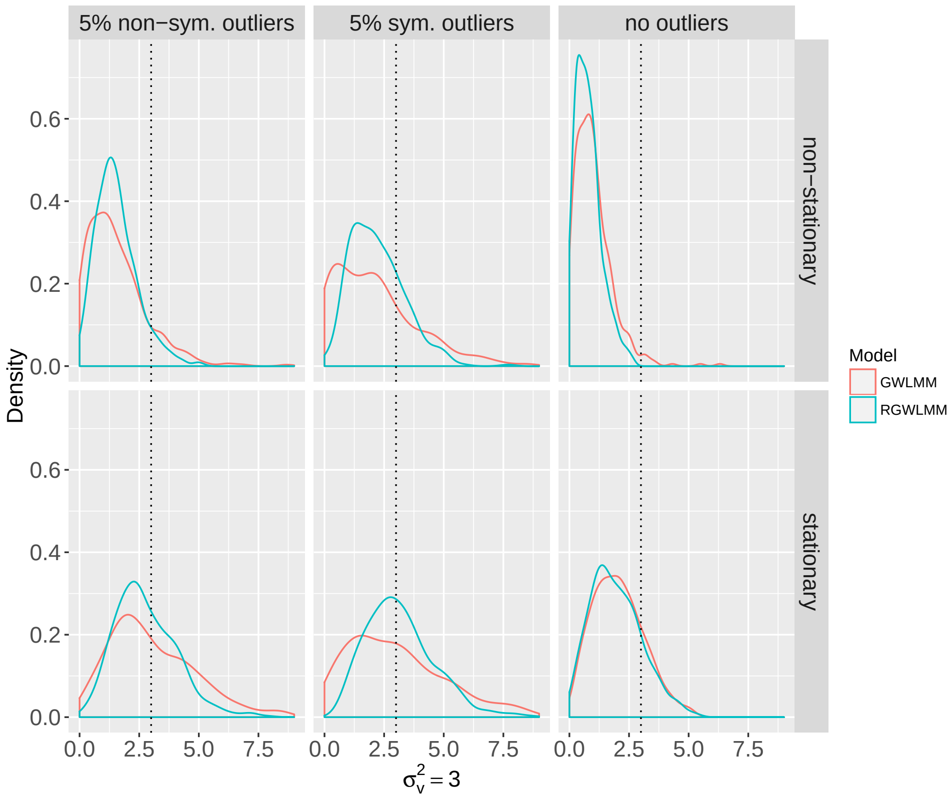


Figure 5.3: Density plots of σ_v^2 for all outlier scenarios under spatial stationarity and non-stationarity.

One explanation for the underestimation of the variance parameters can be found when recalling the simulation results for the local model coefficients in Table 6.3. In all scenarios the range of the estimated local intercepts was wider than in the synthetic population. This implies an overfitting of the model to the sampled data. More precisely, the estimated local intercepts partly capture the random variation of the error term components in the fixed part of the model. Consequently, the fraction of unexplained variation in the model becomes smaller, which leaves less variation for the two error term components.

To sum up this section, it is important to emphasize that the robust and non-robust parameter estimates under the GWLMM need to be treated carefully in applications. Based on the simulation results, the local coefficients seem to give an overall picture of the true parameter surface but should not be interpreted individually. In addition, the estimated variance parameters can be misleading. To find sound explanations and satisfying guidelines for parameter interpretations, further research in this direction is needed. The findings in this section can be seen as preliminary results.

5.3 Performance of the Small Area Means

Eleven estimators of the small area means are evaluated in the simulations. These are the EBLUP in (2.26), the robust projective REBLUP in (3.11) and the robust predictive REBLUP-bc in (3.20). In addition, the corresponding spatial estimators are evaluated –the SEBLUP in (2.46), SREBLUP in (2.46) and the SREBLUP-bc in (3.75)– which account for spatial dependencies between the area-specific random effects. The contiguity matrix $\boldsymbol{W}$ for the underlying model (2.44) is constructed using a binary adjacency matrix based on the neighborhood structure visible from Figure 5.1. Here, areas which have at least one point in common on the boundary are defined as neighbors. Based on this definition, the number of neighbors per area varies between three and eight with a median of five. The contiguity matrix is row standardized such that the rows sum up to one (cf. Pratesi and Salvati, 2009). Both the EBLUP and the spatial EBLUP estimators are suitable under spatial stationarity because they assume a global model. In the following these are referred to as global estimators. Furthermore, the simulation study includes the GWEBLUP in (2.65), the robust projective RGWEBLUP in (4.6) and the robust predictive RGWEBLUP-bc in (4.8), which are suitable under spatial non-stationarity. In addition, the non-parametric NPEBLUP in (2.54) and the robust projective non-parametric RNPEBLUP in (3.83) are evaluated. Both can cope with a spatial trend of an arbitrary form in the target variable.
Following Dongmo-Jiongo et al. (2013) and Chambers et al. (2014), the robust projective estimators are estimated setting the tuning constant $c = 1.345$. Furthermore, the bias correction for the robust predictive estimators is estimated using a larger tuning constant, $b = 3$. The simulation is conducted using the programming language R. The R-scripts are attached to the supplemental material, including a manual to reproduce the results. The performance measures for comparing the small area means are introduced in the next subsection. The results are presented thereafter.

5.3.1 Performance Measures

The performance of the examined estimators is evaluated using two quality measures: the relative bias (RB) and the relative root mean squared error

(RRMSE). Both measures are computed for each Monte Carlo replication. The RB of an estimator measures the relative difference from the true population value. Thus, the area-specific Monte Carlo RB of an estimator is defined as

$$\text{RB}_i = \frac{1}{T}\sum_{t=1}^{T}\frac{\hat{m}_{ti} - m_{ti}}{m_{ti}}, \tag{5.2}$$

where $\hat{m}_{it}$ is a generic notation to denote a small area estimator of the mean in area i for Monte Carlo replication t and m_{it} denotes the corresponding true population value. T denotes the number of Monte Carlo replications. The RRMSE is an indicator of the efficiency of an estimator as it measures the squared relative difference from the true population value. The Monte Carlo RRMSE of an estimator is defined as

$$\text{RRMSE}_i = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\hat{m}_{ti} - m_{ti}}{m_{ti}}\right)^2}. \tag{5.3}$$

Judging the quality of an estimator, the RB should be close to zero and the RRMSE should be small compared to alternative estimators.

5.3.2 Simulation Results

In what follows, the results for scenarios under spacial stationarity are presented first. This is followed by the results under spatial non-stationarity. Results for the scenarios under spatial stationarity are presented in Figure 5.4. Table 5.2 reports the median values of the RB and the RRMSE separated for areas with and without outliers in the area level error term.
In the scenarios with no or symmetric outliers, $(0, 0)$ and $(v, e)_s$, all estimators are nearly unbiased and the global estimators (EBLUP, REBLUP, REBLUP-bc and SEBLUP, SREBLUP, SREBLUP-bc) are more efficient compared to the geographically weighted and non-parametric estimators (GWEBLUP, RGWEBLUP, RGWEBLUP-bc and NPEBLUP). Here, the GWEBLUP tends to overfit the model since estimating local coefficients is not necessary. This effect is less severe for the NPEBLUP which estimates an unneeded spatial trend.
Within the global estimators, using the spatial EBLUP approach causes an efficiency loss compared to respective EBLUP estimators. As the data is driven by a LMM without spatial dependencies, this can be expected since estimating the additional spatial correlation parameter ρ is dispensable. In case of symmetric outliers $(v, e)_s$, the robust projective estimators (REBLUP, SREBLUP and RGWEBLUP) are superior in terms of efficiency compared to their non-robust counterparts (EBLUP, SEBLUP and GWEBLUP).
For non-symmetric outliers $(v, e)_{ns}$, the robust projective estimators (RE-

BLUP, SREBLUP and RGWEBLUP) suffer from bias as a consequence of their assumption that the outliers have zero expectations (Chambers et al., 2014). The robust predictive estimators (REBLUP-bc, SREBLUP-bc and RGWEBLUP-bc) can reduce the bias induced by the robustification at the cost of a higher RRMSE compared to the robust projective counterparts. This is especially true in the extreme areas 37−40 where outliers are present in the area and unit-level error term components.
Note that the RNPEBLUP is not reported for scenarios under spatial stationarity since the algorithm was unstable such that less than 50% of the Monte Carlo replications converged to a solution. One explanation could be that the three variance components in model (2.53) cannot be identified under the robust approach when the overall variation of the error term is limited. Details regarding the stability of all estimators under investigation are presented in Section 5.5.

Predictor	*Results (%) for the following scenarios and areas*						
	$(0,0)$ *1-40*	$(v,e)_s$ *1-40*	$(v,e)_s$ *1-36*	$(v,e)_s$ *37-40*	$(v,e)_{ns}$ *1-40*	$(v,e)_{ns}$ *1-36*	$(v,e)_{ns}$ *37-40*
Median values of RB							
GWEBLUP	0.01	0.01	0.01	0.07	0.08	0.11	-1.78
RGWEBLUP	0.00	0.02	0.01	0.07	-0.55	-0.54	-1.59
RGWEBLUP-bc	-0.01	0.03	0.02	0.04	-0.59	-0.58	-1.03
NPEBLUP	0.00	0.02	0.01	0.06	0.11	0.17	-1.72
SEBLUP	0.00	0.01	0.01	0.06	0.22	0.23	-1.97
SREBLUP	-0.00	0.03	0.02	0.05	-0.54	-0.53	-1.19
SREBLUP-bc	-0.02	0.02	0.02	0.03	-0.42	-0.41	-0.46
EBLUP	0.00	0.02	0.02	0.05	0.26	0.27	-2.08
REBLUP	-0.00	0.02	0.02	0.04	-0.50	-0.50	-1.32
REBLUP-bc	-0.01	0.02	0.02	0.03	-0.57	-0.57	-0.62
Median values of RRMSE							
GWEBLUP	0.84	1.10	1.10	1.80	1.49	1.48	2.82
RGWEBLUP	0.86	0.93	0.92	1.35	1.20	1.19	2.10
RGWEBLUP-bc	0.90	1.04	1.04	1.13	1.31	1.31	1.58
NPEBLUP	0.80	1.08	1.07	1.69	1.54	1.51	2.61
SEBLUP	0.80	1.08	1.07	1.69	1.55	1.54	2.76
SREBLUP	0.82	0.91	0.91	1.18	1.14	1.14	1.62
SREBLUP-bc	0.91	1.15	1.14	1.25	1.48	1.48	1.33
EBLUP	0.80	1.07	1.06	1.67	1.56	1.55	2.80
REBLUP	0.81	0.90	0.90	1.24	1.12	1.11	1.69
REBLUP-bc	0.90	1.04	1.04	1.08	1.28	1.28	1.24

Table 5.2: Median values for RB and RRMSE of estimated small area means under spatial stationarity.

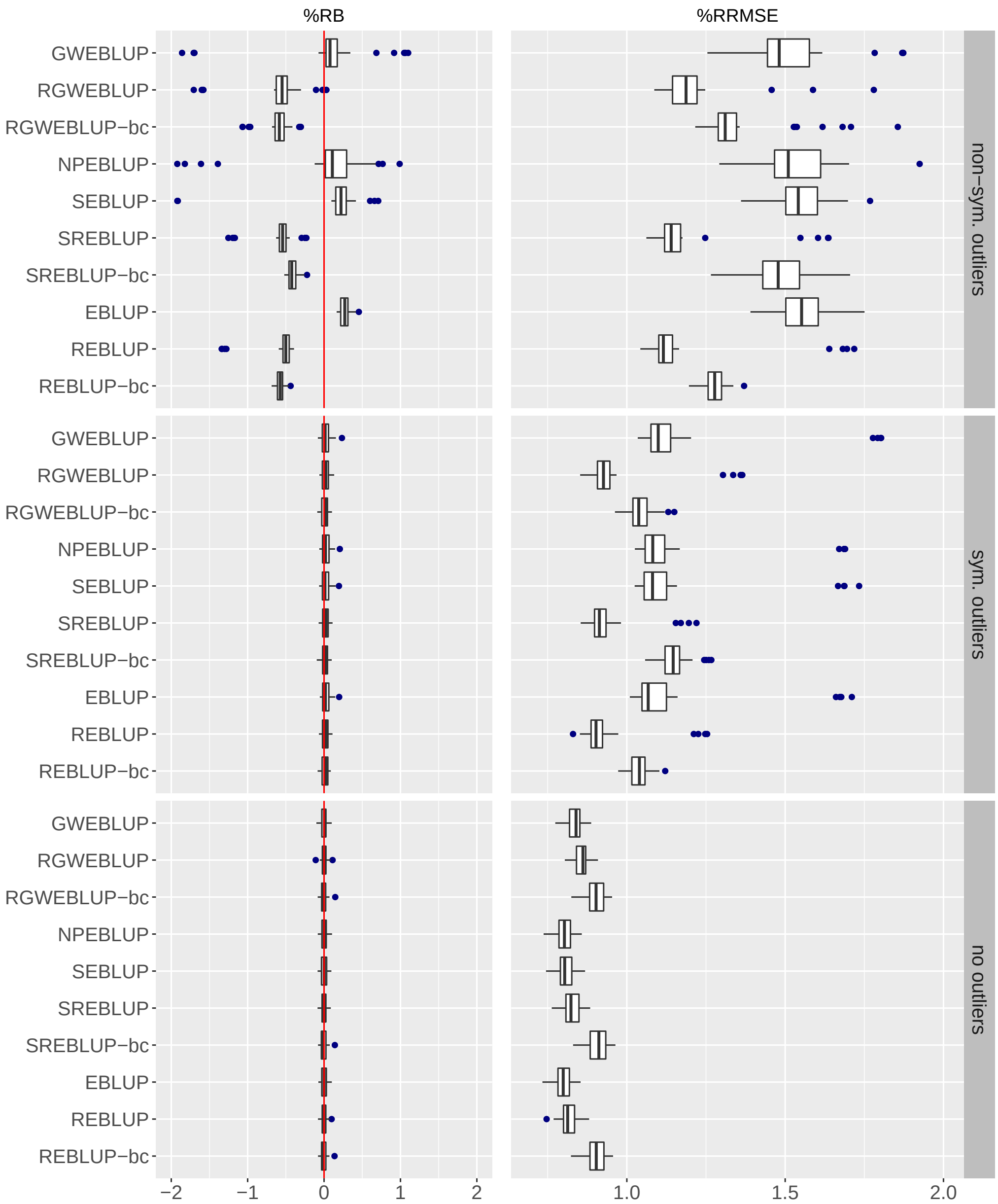


Figure 5.4: Boxplots of the RB (%) and the RRMSE (%) in the spatial stationary scenarios with different outlier contamination mechanisms.

Turning to the scenarios under spatial non-stationarity, the median values of the RB and the RRMSE are reported in Table 5.3 and Figure 5.5 shows the corresponding boxplots of these values over all areas. It can be observed that within the global approaches (EBLUP, REBLUP, REBLUP-bc and SEBLUP, SREBLUP, SREBLUP-bc) the non-spatial estimators are more efficient compared to their spatial counterparts. Thus, allowing for spatially correlated random effects does not seem to cause any efficiency gain when the spatial structure in the data is actually driven by spatial non-stationarity. To find an explanation it is important to bear in mind that under spatial non-stationarity both the SEBLUP and the EBLUP approaches are misspecified. The non-spatial EBLUP approaches ignore spatially varying coefficients in the fixed part of the model by assuming global coefficients; the error term components (area and unit level) on the other hand are correctly specified. In the SEBLUP approaches the fixed model part plus the area level random error term component are misspecified. Thus, the EBLUP is faced with one and the SEBLUP with two misspecifications under spatial non-stationarity. This may cause the efficiency loss of the SEBLUP approaches compared to the non-spatial EBLUP approaches.

Predictor	*Results (%) for the following scenarios and areas*						
	$(0,0)$ *1-40*	$(v,e)_s$ *1-40*	$(v,e)_s$ *1-36*	$(v,e)_s$ *37-40*	$(v,e)_{ns}$ *1-40*	$(v,e)_{ns}$ *1-36*	$(v,e)_{ns}$ *37-40*
Median values of RB							
GWEBLUP	0.00	0.02	0.05	-0.30	0.14	0.16	-1.68
RGWEBLUP	-0.00	0.02	0.04	-0.27	-0.41	-0.38	-1.66
RGWEBLUP-bc	0.01	0.00	0.02	-0.16	-0.45	-0.43	-1.16
NPEBLUP	0.01	0.05	0.05	-0.03	0.08	0.09	-1.19
RNPEBLUP	0.05	0.05	0.05	-0.43	-0.37	-0.34	-1.59
SEBLUP	0.02	-0.03	0.17	-2.22	0.08	0.24	-2.97
SREBLUP	0.01	-0.04	0.08	-1.99	-0.24	-0.13	-2.94
SREBLUP-bc	0.01	0.03	0.03	-0.10	-0.19	-0.17	-0.23
EBLUP	0.02	0.01	0.05	-0.63	0.04	0.13	-1.03
REBLUP	0.02	-0.01	0.04	-0.90	-0.40	-0.35	-1.23
REBLUP-bc	0.01	0.03	0.03	-0.47	-0.40	-0.35	-0.73
Median values of RRMSE							
GWEBLUP	0.81	1.06	1.05	1.53	1.40	1.34	2.41
RGWEBLUP	0.86	0.93	0.91	1.34	1.09	1.08	2.12
RGWEBLUP-bc	0.80	0.93	0.92	1.04	1.16	1.15	1.67
NPEBLUP	1.02	1.13	1.11	2.02	1.46	1.42	2.37
RNPEBLUP	0.98	1.02	1.01	1.93	1.17	1.14	2.39
SEBLUP	2.29	2.32	1.95	3.74	2.34	2.24	4.57
SREBLUP	1.30	1.97	1.72	3.40	2.13	2.00	4.10
SREBLUP-bc	1.28	1.45	1.42	2.24	1.88	1.79	2.34
EBLUP	1.29	1.53	1.46	2.27	2.02	1.99	2.52
REBLUP	1.25	1.32	1.27	2.24	1.53	1.47	2.42
REBLUP-bc	1.25	1.36	1.30	2.13	1.55	1.52	2.22

Table 5.3: Median values for RB and RRMSE of estimated small area means under spatial non-stationarity.

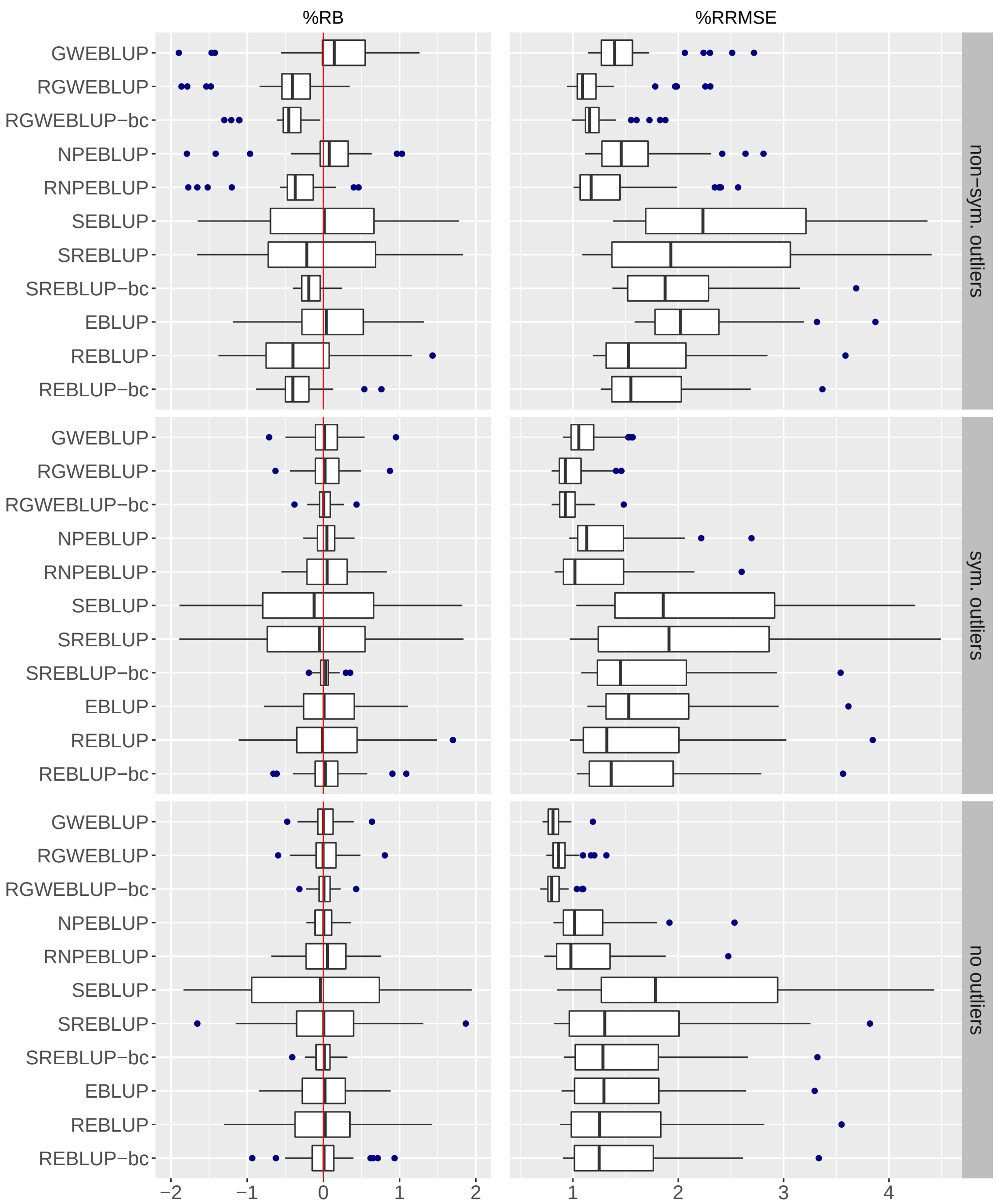


Figure 5.5: Boxplots of the RB (%) and the RRMSE (%) in the spatial non-stationary scenarios with different outlier contamination mechanisms.

In the scenario without outliers, $(0, 0)$, all estimators are nearly unbiased. Furthermore, the geographically weighted approaches (GWEBLUP, RGWEBLUP and RGWEBLUP-bc) and the non-parametric estimators (NPEBLUP, RNPEBLUP) are more efficient compared to the global approaches. This is expected as the working models for these estimators are more suitable under spatial non-stationarity. Within the global approaches, the robust estimators (REBLUP, REBLUP-bc and SREBLUP and SREBLUP-

bc) are more efficient compared to their non-robust counterparts (EBLUP and SEBLUP). Thus, robustifying against outliers seems to be beneficial when assumptions concerning the spatial stationarity are violated.
Interestingly, in the scenario without outliers $(0, 0)$, the robust predictive estimators (REBLUP-bc, SREBLUP-bc and RGWEBLUP-bc) are slightly more efficient compared to their robust projective counterparts (REBLUP, SREBLUP and RGWEBLUP). Actually, it is expected that the variation induced by the bias correction would cause a loss in efficiency. As this phenomenon does not occur in the scenarios under spatial stationarity, it is presumably caused by the spatial non-stationarity induced by the data generating process. One reason can be the local nature of the bias correction that may enhance the precision of the prediction. Further investigations are needed to find sound explanations, however.
The results for the scenarios with symmetric outlier contamination $(v, e)_s$ confirm the expectations regarding the behavior of the robust estimators. First, the robust projective estimators (REBLUP, SREBLUP, RNPEBLUP and RGWEBLUP) lead to more efficient results compared to the respective robust predictive estimators (REBLUP-bc, SREBLUP-bc and RGWEBLUP-bc) and compared to the non-robust estimators (EBLUP, SEBLUP, NPEBLUP and GWEBLUP). This can be observed for all areas 1-40, with and without outliers in the area-level error term. Second, the bias is almost negligible under symmetric outlier contamination, especially in the areas 1-36, where outliers are only present in the unit-level error term. Third, the geographically weighted methods offer some gains in RRMSE when compared to the global methods and the NPEBLUP.
As in the spatial stationary scenario, the robust projective estimators (REBLUP, SREBLUP, RNPEBLUP and RGWEBLUP) suffer from bias in the case of asymmetric outlier contamination $(v, e)_{ns}$. The robust predictive approaches (REBLUP-bc, SREBLUP-bc and RGWEBLUP-bc) correct for the bias at the cost of lower efficiency compared to their robust projective counterparts, especially in the extreme areas 37-40. Regarding the RRMSE results, the robust geographically weighted methods (RGWEBLUP and RGWEBLUP-bc) offer better efficiency compared to the non-robust GWEBLUP.
The overall results for the area means indicate that combining geographically weighted regression methods with robust protection –either robust predictive RGWEBLUP or projective RGWEBLUP-bc– can lead to gains in efficiency.

5.4 Performance of the MSE Estimates

The pseudo-linearization (CCT) and the linearization-based (CCST) approaches for MSE estimation are evaluated for six estimators of the small area mean. These are the CCT in (3.25) and the CCST in (3.48) for the robust projective REBLUP; the CCT in (3.33) and the CCST in (3.67) for the robust predictive REBLUP-bc; and the CCT in (2.34) and the CCST for the EBLUP. The CCST for the EBLUP can be obtained by setting the tuning constant in (3.48) to a very high value, $c = 100$. Furthermore, these MSE estimators are evaluated for the GWEBLUP and the proposed extensions, the robust projective RGWEBLUP and the robust predictive RGWEBLUP-bc. In particular, results are presented for the CCT in (4.22) and the CCST in (4.50) for the RGWEBLUP and the CCT in (4.33) and the CCST in (4.66) for the RGWEBLUP-bc. The CCT and the CCST for the GWEBLUP can be obtained by choosing a large tuning constant ($c = 100$) in (4.22) and (4.50), respectively. The performance measures for comparing the MSE estimators are introduced in the next subsection. The results are presented thereafter.

5.4.1 Performance Measures

Similar to the point estimates in Section 5.3, the quality of the MSE estimates is judged by means of the RB and the RRMSE of the estimated root MSE (RMSE). The true population value of the RMSE is given by the empirical RMSE of an estimator observed over the Monte Carlo replications:

$$\text{RMSE}_i = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(\hat{m}_{ti} - m_{ti})^2}, \tag{5.4}$$

where, as in equation (5.2), $\hat{m}_{it}$ is a generic notation to denote an estimator of the mean in area i for the Monte Carlo replication t and m_{it} denotes the corresponding true population value. Using this definition, the area-specific Monte Carlo RB for the RMSE estimator is given by

$$\text{RB}_i^L = \frac{1}{T}\sum_{t=1}^{T}\frac{\widehat{RMSE}_{ti}^L - RMSE_i}{RMSE_i}, \tag{5.5}$$

where $\widehat{RMSE}_i^L$ denotes the estimated RMSE of an estimator of the mean in area i using method L. Here, L can be either the CCT or the CCST approach. The area-specific Monte Carlo RRMSE for the RMSE estimator is given by

$$\text{RRMSE}_i^L = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\widehat{RMSE}_{ti}^L - RMSE_i}{RMSE_i}\right)^2}. \tag{5.6}$$

When comparing different methods for MSE estimation, stable estimators in terms of a low RRMSE with a RB close to zero are preferred. In addition, the MSE estimates are judged by the achieved coverage rate (CR) of a nominal 95% confidence interval. The area-specific Monte Carlo CR is given by

$$\mathrm{CR}_i^L = \frac{1}{T}\sum_{t=1}^{T} I\left(|\hat{m}_{ti} - m_{ti}| \leq 1.96\,\widehat{RMSE}_{ti}^{L}\right). \tag{5.7}$$

MSE estimators with a CR lower than the requested 95% are less trustworthy and those with a higher CR reveal low efficiency as the confidence intervals are unnecessarily wide. Thus, the CR of an MSE estimator should ideally be close to 95%.

5.4.2 Simulation Results

The performance results of the MSE estimators for spatial non-stationary scenarios are reported in Table 5.4. In particular, the median values of the RB, the RRMSE and the CR are presented separately for areas with and without outliers in the area-level error term. Results for scenarios under spatial stationarity are reported in Table A.1 in Appendix A. These are very similar and, hence, only the results under spatial non-stationarity are discussed here.
In general, it can be observed that the CCST method for the MSE estimation constantly offers better stability in terms of a lower RRMSE compared to the CCT method for the geographically weighted estimators (GWEBLUP, RGWEBLUP and RGWEBLUP-bc) and the global estimators (EBLUP, REBLUP and REBLUP-bc) for all scenarios.
Turning to the bias and coverage rate, the picture is not as clear. I start discussing the results of the global estimators (EBLUP, REBLUP and REBLUP-bc) and the non-robust GWEBLUP as these are quite similar. It can be observed here that the CCT method underestimates the MSE in all outlier scenarios $(0,0)$, $(v,e)_s$ and $(v,e)_{ns}$. The magnitude of the underestimation becomes more severe for scenarios $(v,e)_s$ and $(v,e)_{ns}$ in the extreme areas 37-40 with outliers in the area-specific error term. Since the CCT does not take into account the variation caused by the estimation of the variance parameters, an underestimation seems plausible. This underestimation is also reflected in the CRs of the CCT which are relatively low with values between 0.82 an 0.92. The CCST MSE estimates, on the other hand, tend to overestimate the MSE in scenario $(0,0)$ and for the areas 1-36 in scenarios $(v,e)_s$ and $(v,e)_{ns}$. In the extreme areas 37-40 an underestimation is revealed for both outlier scenarios. However, the magnitude of the bias is generally smaller compared to the CCT MSE estimates. The CRs of the CCST method vary between 0.94 and 0.97 in scenario $(0,0)$ and in scenarios $(v,e)_s$ and $(v,e)_{ns}$ for areas 1-36. Hence, the CCST provides 95% confidence intervals with a satisfying coverage performance at least in the areas 1-36.

Predictor	*MSE*	*Results for the following scenarios and areas*						
		$(0,0)$ *1-40*	$(v,e)_s$ *1-40*	$(v,e)_s$ *1-36*	$(v,e)_s$ *37-40*	$(v,e)_{ns}$ *1-40*	$(v,e)_{ns}$ *1-36*	$(v,e)_{ns}$ *37-40*
Median values of RB (%)								
GWEBLUP	CCST	3.73	5.00	6.29	-16.93	13.60	14.54	-19.75
	CCT	-23.05	-17.64	-17.30	-29.21	-7.22	-7.10	-27.58
RGWEBLUP	CCST	-5.81	4.34	4.70	-22.18	9.16	10.64	-37.37
	CCT	-25.57	-11.06	-10.38	-30.35	1.50	4.40	-38.40
RGWEBLUP-bc	CCST	-4.06	-5.06	-4.10	-17.84	-11.01	-10.33	-43.03
	CCT	1.65	5.63	6.43	-9.53	6.30	7.51	-31.32
EBLUP	CCST	17.12	9.43	13.57	-28.12	-0.96	2.35	-19.20
	CCT	-13.12	-17.42	-17.34	-20.12	-22.54	-22.66	-17.06
REBLUP	CCST	16.16	9.65	13.28	-23.70	1.11	7.19	-28.62
	CCT	-19.29	-16.97	-15.51	-34.02	-14.13	-13.75	-34.72
REBLUP-bc	CCST	9.32	7.00	7.74	-15.61	2.15	3.40	-19.84
	CCT	-14.78	-15.39	-14.22	-28.03	-14.83	-14.48	-29.49
Median values of RRMSE (%)								
GWEBLUP	CCST	30.24	37.46	36.04	38.78	46.96	47.63	44.30
	CCT	39.71	49.68	49.41	49.98	58.62	59.58	53.30
RGWEBLUP	CCST	32.80	42.20	42.68	39.82	56.43	56.79	50.88
	CCT	44.89	55.21	56.21	51.55	72.76	73.39	57.42
RGWEBLUP-bc	CCST	31.97	34.84	34.89	33.31	36.63	36.53	47.24
	CCT	39.34	51.63	52.37	45.01	64.87	66.00	49.75
EBLUP	CCST	32.54	25.16	24.24	29.69	17.82	16.70	23.23
	CCT	34.74	40.61	40.88	37.84	51.03	52.76	38.14
REBLUP	CCST	34.60	30.87	28.96	35.17	24.93	24.43	37.06
	CCT	30.64	33.91	33.16	41.20	41.06	40.78	43.31
REBLUP-bc	CCST	34.57	36.10	37.00	28.62	37.33	39.09	29.99
	CCT	30.89	34.82	34.76	38.20	41.41	41.41	41.28
Median values of CR								
GWEBLUP	CCST	0.95	0.97	0.97	0.93	0.95	0.96	0.91
	CCT	0.83	0.88	0.88	0.82	0.86	0.86	0.85
RGWEBLUP	CCST	0.93	0.94	0.94	0.73	0.94	0.94	0.88
	CCT	0.82	0.86	0.87	0.67	0.86	0.86	0.82
RGWEBLUP-bc	CCST	0.90	0.87	0.87	0.70	0.90	0.91	0.87
	CCT	0.91	0.89	0.89	0.73	0.91	0.92	0.88
EBLUP	CCST	0.97	0.95	0.95	0.88	0.96	0.97	0.84
	CCT	0.87	0.88	0.89	0.81	0.87	0.88	0.83
REBLUP	CCST	0.97	0.96	0.97	0.79	0.97	0.97	0.82
	CCT	0.85	0.85	0.87	0.73	0.87	0.88	0.75
REBLUP-bc	CCST	0.94	0.92	0.93	0.85	0.94	0.94	0.87
	CCT	0.87	0.86	0.86	0.78	0.88	0.88	0.80

Table 5.4: Median values of the performance measures of the RMSE estimators under spatial non-stationarity.

Based on these results it seems that for the global estimator (EBLUP, REBLUP and REBLUP-bc) and the non-robust GWEBLUP, the CCST MSE estimator constantly performs better (in terms of bias, stability and coverage) compared to the CCT MSE estimator over all outlier scenarios. Exceptions of these findings can be observed for the robust projective RGWEBLUP and the robust predictive RGWEBLUP-bc.
The performance results of the MSE estimators for the RGWEBLUP are discussed first. As already mentioned, the CCST MSE estimator is superior in terms of stability compared to the CCT MSE estimator as the RRMSE is constantly lower. In terms of bias, both MSE estimators underestimate the MSE in scenario $(0, 0)$ whereas the absolute value of the bias is substantially smaller for the CCST. For symmetric outlier contamination, $(v, e)_s$, the pattern is as follows: the CCST method overestimates the MSE except for the areas 37-40 where it underestimates; the CCT method underestimated the MSE in all areas; the magnitude of the bias is smaller for the CCST compared to the CCT. For asymmetric outlier contamination $(v, e)_{ns}$ both MSE estimates overestimate the MSE in areas 1-36 where the bias is smaller for the CCT, and in areas 37-40 both estimators underestimate with the same order of magnitude. Thus, in this scenario, the CCT seems to perform better in terms of bias. However, the CR of the CCT method is constantly lower compared to the CCST method. Here, the CCST method reveals a CR of 0.94 in scenario $(0, 0)$ and 0.94 in scenarios $(v, e)_s$ and $(v, e)_{ns}$ for areas 1-37 whereas the CCT does not reach an acceptable CR in any scenario.
Turning to the MSE estimators for the robust predictive RGWEBLUP-bc, again the CCST MSE estimator offers more stability compared to the CCT method. In addition, the CCT MSE estimator tends to overestimate and the CCST to underestimates the MSE, but the magnitude of the bias is in the same range. In all scenarios the CR of the CCT is slightly higher than for the CCST but does not in any case reach an acceptable level. This results is counter intuitive for both methods. An overestimation of the CCT method is unexpected as it ignores the variability caused by the estimation of the variance parameters. As the CCST method accounts for this source of variation, the MSE estimates should be larger than the CCT estimates. Further investigations are needed here to fins sound explanations.
The overall results in this section indicate that using the CCST method for the MSE estimation of the global estimators and the GWEBLUP appears to have appealing properties regarding stability, bias and coverage in all scenarios. Besides some limitations regarding the coverage in the case of the robust predictive RGWEBLUP-bc, the CCST method for MSE estimation also performs satisfyingly for the proposed robust methods.

5.5 Stability and Calculation Times

This section presents results regarding the stability of the estimation procedures that were investigated in this simulation study. This also includes a discussion regarding the calculation times needed for the estimation. Table 5.5 reports the number of successfully converged Monte Carlo replications separately for each estimator and scenario. Valid replications are defined as estimations where the iterative approximation algorithms converged with a tolerance of 10^{-4}. In addition, estimations with negative estimated variance parameters are treated as invalid. When estimations do not converge or produce implausible values, these replications are excluded from the analysis for all estimators. This assures that all estimators are compared based on the same number of replications.

	Successfully converged replications					
Predictor	*Non-stationarity*			*Stationarity*		
	$(0,0)$	$(v,e)_s$	$(v,e)_{ns}$	$(0,0)$	$(v,e)_s$	$(v,e)_{ns}$
GWEBLUP	500	488	490	500	499	497
RGWEBLUP	472	473	485	497	495	495
NPEBLUP	500	500	500	500	500	500
RNPEBLUP	500	500	500	233	263	467
SEBLUP	500	500	500	500	500	500
SREBLUP	500	500	500	500	500	500
EBLUP	500	500	500	500	500	500
REBLUP	500	500	500	500	500	498

Table 5.5: Number of converged Monte Carlo replications for scenarios under spatial stationarity and non-stationarity.

From Table 5.5 it can be noted that the estimators EBLUP, SEBLUP, SREBLUP and NPEBLUP do not face any stability problems as all the Monte Carlo replications converged successfully for these estimators. In contrast, the REBLUP, the RNPEBLUP, and the geographically weighted methods GWEBLUP and RGWEBLUP suffer from some stability problems. As has already been noted, the RNPEBLUP shows severe instability under spatial stationarity for scenarios without $(0,0)$ or symmetric outliers $(v,e)_s$ whereas it runs stable under spatial non-stationarity. The REBLUP also has minor stability problems under spatial stationarity but with non-symmetric outliers $(v,e)_{ns}$. The stability of the GWEBLUP and the RGWEBLUP mainly exhibits problems under non-stationarity caused by convergence issues for the variance parameters. The high variation of the fixed effects coefficients in these scenarios seems to disturb the convergence. The problem is more severe for the proposed RGWEBLUP than for the non-robust GWEBLUP. Table 5.6 reports the average calculation times for all estimators and scenarios based on the valid Monte Carlo replications.

	Mean calculation time in seconds per replication					
Predictor	*Non-stationarity*			*Stationarity*		
	$(0,0)$	$(v,e)_s$	$(v,e)_{ns}$	$(0,0)$	$(v,e)_s$	$(v,e)_{ns}$
GWEBLUP	30.48	28.59	28.08	18.63	20.28	20.08
RGWEBLUP	26.98	27.37	34.75	20.76	25.30	28.10
SEBLUP	6.24	5.98	5.67	0.90	1.01	1.09
SREBLUP	3.04	4.49	6.09	0.96	0.97	0.94
EBLUP	0.66	0.69	0.74	0.76	0.19	0.71
REBLUP	3.29	3.37	3.60	2.62	0.85	3.28
NPEBLUP	0.68	0.57	0.65	0.83	0.82	0.59
RNPEBLUP	0.80	0.68	0.77	0.80	0.83	0.68

Table 5.6: Calculation time for scenarios under spatial stationarity and non-stationarity, $n = 200$.

As previously mentioned, the geographically weighted methods (GWEBLUP and RGWEBLUP) are computationally demanding as the model coefficients have to be estimated for all locations in the sample. This is also reflected in the calculations times. On average, estimating the GWEBLUP or the RGWEBLUP was about 50 times slower compared to the EBLUP. This factor is expected to increase for larger samples as the number of operations needed for the calculation increases exponentially with the sample size n.

5.6 Discussion

The results from the model-based simulation indicate a potential benefit of combining geographically weighted regression with robust estimation. In the presence of spatial non-stationarity and extreme observations, applying the proposed robust projective RGWEBLUP and robust predictive RGWEBLUP-bc may lead to efficiency gains compared to the non-robust GWEBLUP. In addition, the proposed CCST MSE estimator for the RGWEBLUP and RGWEBLUP-bc appears to have appealing properties in terms of bias and stability in the investigated scenarios. In practice, the proposed robust SAE methods can be expensive in terms of calculation time compared to other methods. The potential efficiency gain, however, may be an acceptable trade-off.
The settings in this study are similar to those in Sinha and Rao (2009) and Chambers et al. (2014) which allows us to make some comparisons. With respect to the point estimates of the EBLUP and the REBLUP approach, the

spatial stationary results are very close to those in Sinha and Rao (2009) in that both are unbiased and the REBLUP yields efficiency gains under symmetric outlier contamination. The simulation in Chambers et al. (2014) also confirms this finding. In addition, they conclude that the CCST method for the MSE estimation consistently offers a better stability compared to the CCT method, which was also the finding here. As an avenue for future work it is possible to conduct model-based simulations similar to Schmid et al. (2016) by focusing on robust SAE under spatial correlation. It would be interesting here to assess the behavior of the geographically weighted estimators (GWEBLUP, RGWEBLUP and RGWEBLUP-bc) when the data is actually driven by spatially correlated random effects. First results under this additional setting can be found in Appendix B.
Model-based simulations allow us to assess the estimators in a controlled setup. However, in realistic data situations spatial stationarity in the model coefficients may be present but can have an arbitrary functional form and does not have to be a linear combination of the coordinates. In addition, the outlier contamination is difficult to identify in a real population sample. From a practical perspective, design-based simulations can be informative as these are usually based on realistic populations with an unknown data-generating process for the target variable. Therefore, the proposed method is also assessed in a design-based simulation study using data from the Berlin apartment rental market. The results of this study are presented in the next chapter.

Design-based Simulation 6

In SAE, design-based simulations complement model-based simulations. In applications the underlying model for small area estimates attempts to mimic the real data-generating process as precisely as possible to produce reliable results. However, in practice the actual process is unknown and all models, regardless of how advanced, can be misspecified to some extent. Hence, design-based simulation where samples are taken from a realistic finite population can be interesting regarding practical considerations. The design-based simulation in this chapter assesses the relative performance of geographically weighted EBLUP methods under the realistic data situation of the Berlin apartment rental market. The target statistic is the average quoted apartment net rent in urban subregions of Berlin.
The database for this simulation is introduced in Section 6.1 together with a description of the geographic system used in Berlin to divide the urban area into subregions. This is followed by a case study in Section 6.2 that justifies the use of geographically weighted SAE methods and robust parameter estimation in the underlying data situation. The simulation study is presented in Section 6.3.

6.1 The Data

The underlying data for this chapter is taken from a German real estate market database provided by Empirica-Systeme GmbH (www.empirica-systeme.de). The database includes 51,907 apartments for rent in Berlin in 2015 from various online real estate platforms and specialized print media sources. The data is collected continuously and all duplicate entries are dropped. In total, 120 characteristics for the apartments are available including information on the size, different kinds of costs, the quality of the facilities and attributes concerning the condition of an apartment and the building. In addition, the addresses for all units are geocoded at building level. This database is used in the following sections to estimate the average quoted apartment rent on different aggregation levels (cf. Empirca AG, 2014).
For planning and analyzing policy measures, the Berlin Senate uses a hierarchical system to divide the urban area into subregions. This system is called LOR (in German: Lebensweltlich orientierte Räume/common living spaces) and consists of three nested levels (Berlin Senate, 2006):

- 60 Forecasting areas (PRG, in German: **Prog**nose**r**äume),
- 138 District regions (BZR, in German: **Bez**irks**r**egionen),
- 447 Planning areas (PLR, in German: **Pl**anung**r**äume).

On average, the PRGs are inhabited by about 60,000 people, the BZRs by about 25,000 and the PLRs by about 7,500 people (Bömermann et al., 2006). Figure 6.1 shows the boundaries of the three levels. The black-rimmed areas identify the PGRs, the grey in conjunction with the black lines identify the BZRs and the combination of all three lines (black, gray and red) identify the PLRs.

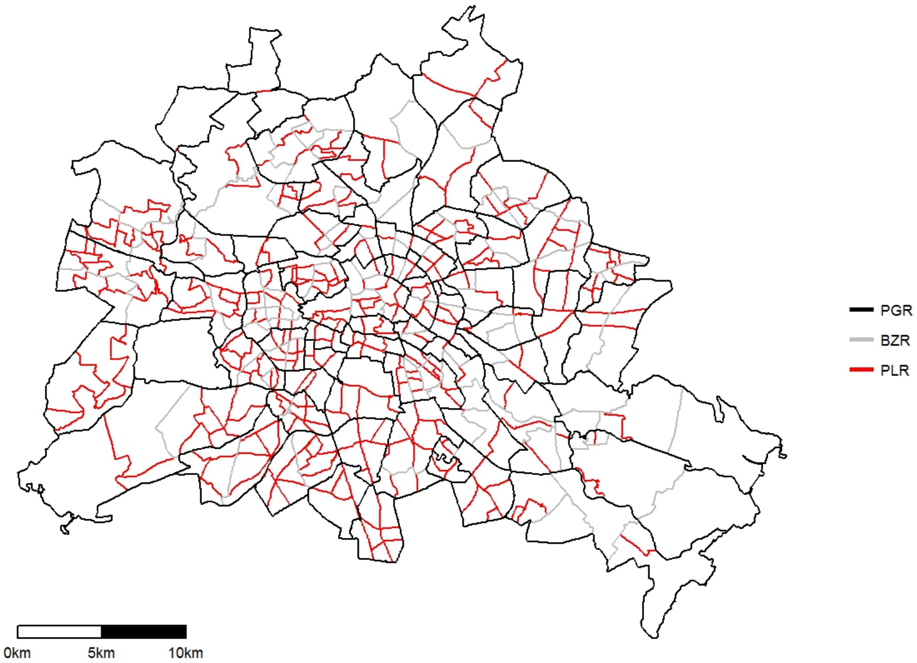


Figure 6.1: Boundaries within the hierarchical LOR system of the Berlin Senate. Data source: Berlin Senate - Department for Urban Development and Housing.

The PRGs are designed to monitor and forecast economic and demographic developments in order to provide sufficient resources in particular regions, such as residential homes and employment agencies. The BZRs are administrative subregions of Berlin. The PLR are mainly designed for monitoring social developments at neighborhood level and for planning targeted resource allocation (Bömermann, 2014).

6.2 Case Study

The aim in this case study is to justify the consideration of *robust SAE under spatial non-stationarity* in the underlying database. After introducing the model specifications, model diagnostics are presented in Section 6.2.2. The small area means are presented in Section 6.2.3.

6.2.1 Model Specifications

The variable of interest is the *quoted net rent* (QNR) per square meter (sq) and the target quantity is the mean value of QNR at the PLR level ($M = 447$), i.e., the lowest level of the LOR subregions in Berlin (cf. Figure 6.1). To enable purposeful and local policy measures, the Berlin Senate is often interested in information on this very disaggregated level (Berlin Senate, 2006). Hence, the choice of this target level has practical relevance for policymakers in Berlin.
The starting point in this case study is a situation which can be considered realistic in the field of survey sampling. Typically, researchers have access to a survey with unit-level information. In addition, aggregated information for the explanatory variables is available from publications supplied by official statistics. To simulate this situation, the Berlin real estate database introduced above is used to draw a sample of size $n = 2006$ which serves as a survey with unit-level information. In particular, the 138 BZRs (cf. Figure 6.1) of Berlin are employed as planned domains for the sampling design. The sample sizes are chosen to be proportional to the size of the BZRs, where the size is defined by the number of apartments listed in the database. The employed sampling design results in a sample with 86 out-of-sample areas and a medium sample size of three in the unplanned PLR domains. In a second step, the database is used to produce aggregates, i.e., median values on PLR level, which are used as the source of additional information for the explanatory variables.

Variable	Min	Mean	Max
Quoted net rent	3.97	8.96	21.54
Size	20.00	75.55	462.00
Facilities			
Simple standard	0.00	0.03	1.00
Normal standard	0.00	0.41	1.00
Good standard	0.00	0.42	1.00
High standard	0.00	0.14	1.00
Condition			
Bad	0.00	0.03	1.00
Normal	0.00	0.45	1.00
Good	0.00	0.52	1.00

Table 6.1: Summary statistics for the variables in the model.

In the field of housing and real estate, hedonic price models (cf. Rosen, 1974) are frequently applied to model the target variable (cf. Malpezzi, 2008). Considering geographically weighted models is also common in this field (cf. Bitter et al., 2007; Farber and Yeates, 2006; Fotheringham et al., 2002). In this case study QNR is modeled using an additive hedonic pricing model with three explanatory variables: (1) *size* in sq which is defined as the living space, (2) *facilities* as an ordered categorical variable (where 1 = simple, 2 = normal, 3 = good and 4 = high standard) which describes the quality of the apartment facilities, (3) *condition* as an ordered categorical variable (where 1 = bad, 2 = normal and 3 = good) which describes the state of the building. The variables *facilities* and *condition* are constructed as indexes which sum up several indicator variables. The variable *facilities* includes information about the type of heating and the standard of the bathroom and the kitchen. The variable *condition* includes characteristics concerning the renovation demand of the apartment and the building. The model is given by

$$QNR_{ij} = \beta_{0,ij} + \beta_{1,ij}S_{ij} + \boldsymbol{\beta}_{2,ij}F_{ij} + \boldsymbol{\beta}_{3,ij}C_{ij} + v_i + e_{ij}, \tag{6.1}$$

where S denotes the *size* variable. F comprises the categories of the variable *facilities* and C the categories of the variable *condition*. Level 1 serves as the reference category for both categorical variables, *facilities* and *condition*. Table 6.1 reports the summary statistics of the variables used in the model.

6.2.2 Model Diagnostics

In a global linear model (global LM) the explanatory variables capture around 25% of the variability of the target variable. Due to unobserved heterogeneity between PLRs, the error term is expected to be correlated. Employing a global linear mixed model (global LMM) with a random intercept at level 2 (level 1 = apartment, level 2 = PLR) accounting for the correlation within PLRs, yields an intraclass correlation coefficient (ICC) of 0.5.

Model	n	p_e	-2 log likelihood	p-value	AIC
Global LM	2006	8.00	8943.69	-	8959.69
Global LMM	2006	9.00	8218.72	-	8238.72
GWLM	2006	326.31	7542.53	2.20e-16	8195.15
GWLMM	2006	315.54	7456.97	8.34e-35	8088.06

p_e denotes the effective number of parameters

Table 6.2: Model comparison of different hedonic price models for the target variable QNR.

Evidence of spatial non-stationarity is also present in the data. Spatial non-stationarity of the model coefficients can be detected by a likelihood ratio (LR) test, as suggested by Fotheringham et al. (2002), where the likelihood of a global LM is compared to the likelihood of a geographically weighted LM (GWLM). Following that suggestion the likelihoods of the global LMM and the geographically weighted LMM (GWLMM) can be compared as well to test for spatial non-stationarity in the mixed model approaches. The distribution of that ratio can be approximated by a χ^2-distribution with $(p_e - p)$ degrees of freedom where p_e denotes the effective number of parameters in the fixed part of the GWLMM (Fotheringham et al., 2002, p.92). The local models are estimated using a Gaussian kernel with a bandwidth of 1.53 km. The bandwidth is determined by cross-validating the geographically weighted LM (GWLM) using the criteria in (2.57). According to the p-values of the LR-tests reported in Table 6.2, spatial stationarity of the regression coefficients can be rejected for both the GWLM and the GWLMM.

Variable	Min	Q1	Median	Q3	Max
Intercept	2.36	5.49	6.27	6.93	11.15
Size in sq	-0.05	-0.01	-0.00	0.00	0.02
Facilities					
Normal standard	-3.44	0.04	0.89	1.49	4.80
Good standard	-3.18	0.18	1.05	1.72	5.47
High standard	-2.20	1.63	2.54	3.59	6.26
Condition					
Normal	-1.51	0.42	0.83	1.31	4.08
Good	-0.87	1.14	2.04	2.62	6.22

Table 6.3: Summary statistics for the GWLMM coefficients.

By using a geographically weighted approach the share of explained variability in the dependent variable increases up to 60%, thus it can be doubled compared to the global approach. While the GWLM can capture parts of the correlation caused by unobserved heterogeneity between PLRs, the GWLMM still yields an ICC of 0.14. The necessity for random effects under spatial non-stationarity can also be assessed. Opsomer et al. (2008), Chandra et al. (2015) and Chandra et al. (2017) suggest a bootstrap procedure to test the area-level variance component against zero. Here, two models are fitted, first a model without random effects (null hypothesis H_0), and second a model with these effects (alternative hypothesis H_1). The test compares the restricted log-likelihoods under each hypothesis and the level of significance is calculated via a parametric bootstrap. In case of geographically weighted models, this procedure dramatically increases the computation time. As an alternative to this formal test, it is possible to compare the AIC values since these include a penalty for the additional variance parameter of the random effects. The GWLMM exhibits the lowest AIC (cf. Table 6.2), thus it seems superior compared to the other models. The summary statistics of the estimated local model coefficients from the GWLMM reported in Table 6.3 reveal much spatial variation in the parameter values. One exception is the *size* coefficient where, due to a very small scale, not much variation can be seen. The spatial surface of the estimated local intercepts is presented in Figure 6.2. The intercept is evaluated at an average-sized apartment (75.55 sqm) with a normal standard of facilities and normal a building condition. The spatial distribution of the local intercepts reflects the current situation on the Berlin apartment rental market with higher rents in the city center (marked by the inner black line) and the south-west, and with lower rents in the eastern and western outskirts of Berlin (IBB, 2015). The surface plots of the local coefficients from the other explanatory variables also show substantial variation and are reported in Figure A.1 in Appendix A.

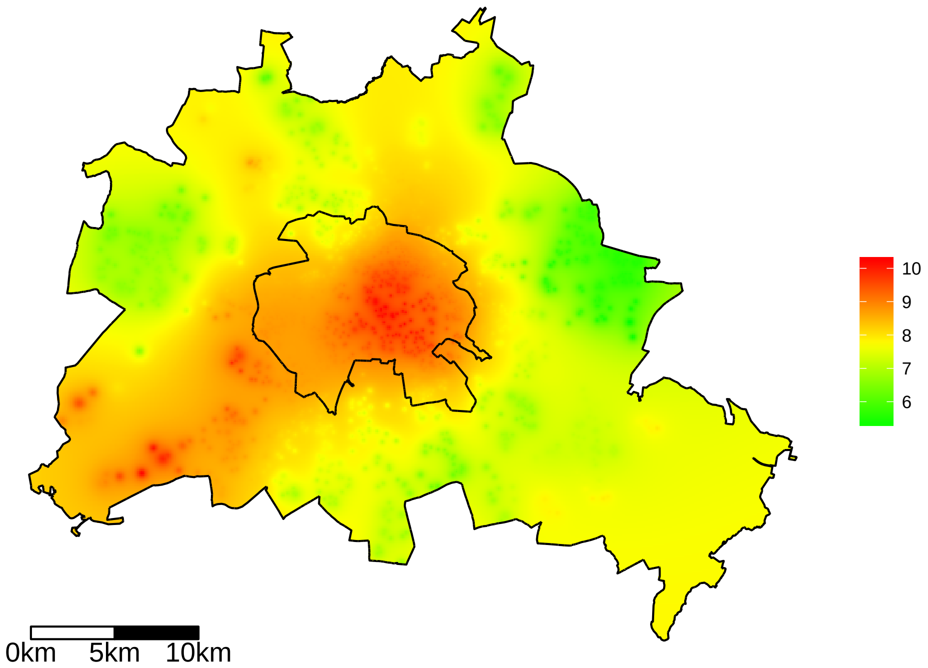


Figure 6.2: Spatial surface of the intercept from the GWLMM at an average apartment size of 75.55 sqm, normal standard of facilities and normal building condition.

The spatial variation of the model coefficients can be further assessed. Following Charlton et al. (2003) the variability of the estimated coefficients from the GWLMM can be compared to the estimated standard errors of the coefficients from the global LMM. They suggest calculating the interquartile range for each coefficient from the GWLMM and dividing it by two times the standard error of the regression coefficient for the same variable from the global LMM. This ratio is motivated by the fact that about 68% of a standard normally distributed variable lies within the range of ± one standard deviation and 50% within the quartiles. According to Charlton et al. (2003), this ratio provides an approximate measure of stationarity where values greater than one indicate spatial non-stationarity. Figure 6.3 shows this index plotted against different values for the bandwidth. At the estimated optimal bandwidth (vertical dashed line) the index of spatial stationarity is ≥ 1.5 for all coefficients, indicating substantial non-stationarity of the relationship between the quoted net rent and the explanatory variables. The graph also shows that the coefficients become stationary at different spatial scales, i.e., size becomes stationary at a bandwidth of 6 whereas the intercept is more stable since it already achieves stationarity at a bandwidth of 2.5.

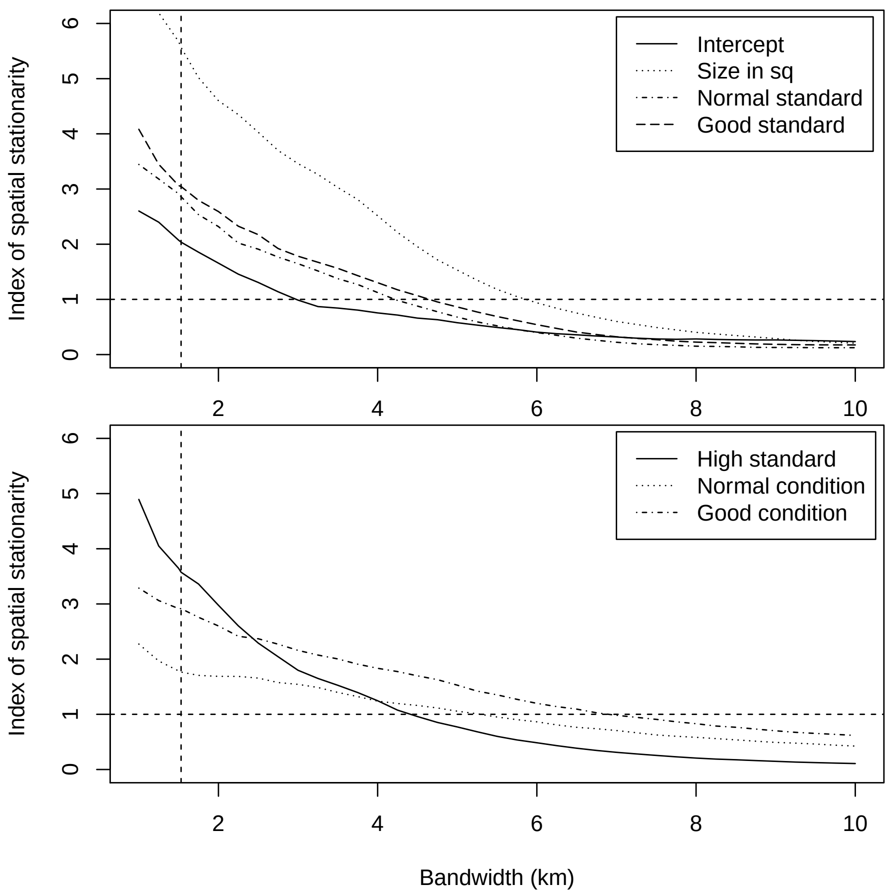


Figure 6.3: Index of spatial stationarity for the estimated coefficients of the GWLMM.

The GWLMM is based on the normality assumption for the error term components. To assess departures from this assumption, Figure 6.4 shows normal probability plots of level 1 and level 2 residuals. In both levels, severe departures from the Gaussian assumptions, which may be caused by outliers in the sample, can be observed.

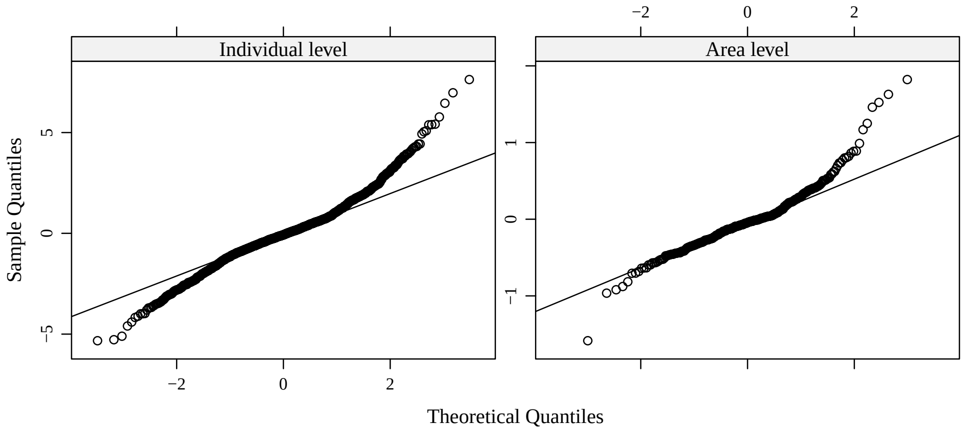


Figure 6.4: Normal probability plots of level 1 (left) and level 2 (right) residuals.

Thus, applying a robust estimation method to control for the influence of the extreme observations can be beneficial in this application. In addition, based on the test for spatial non-stationarity and the lowest AIC value, fitting a GWLMM to the data appears to be most suitable for the given data.

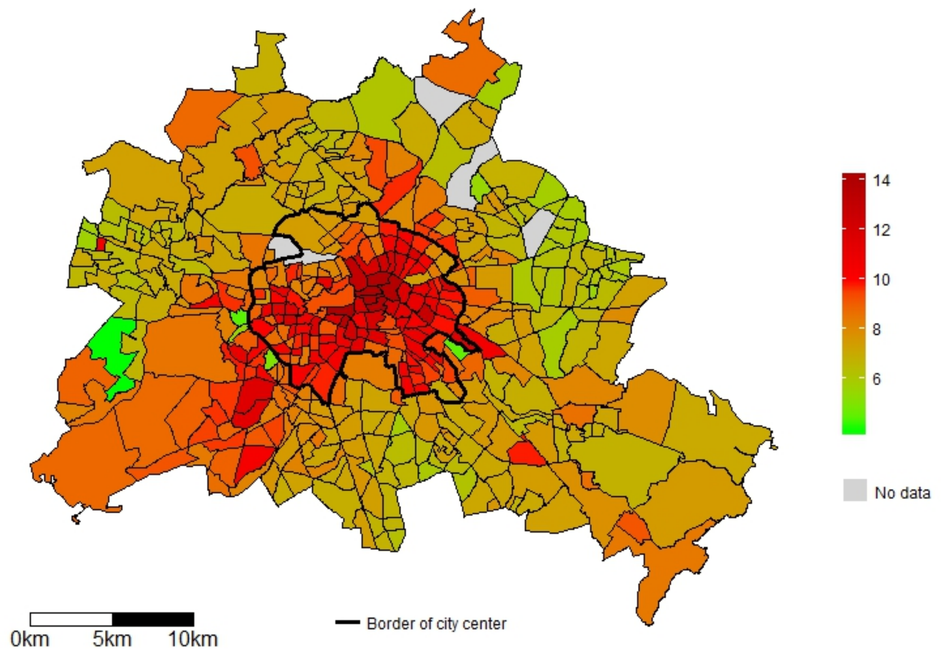
6.2.3 Small Area Estimates

The summary statistics of the estimated area means reported in Table 6.4 indicate that the robust projective RGWEBLUP and the robust predictive RGWEBLUP-bc estimates produce results of a very similar range.

Predictor	*Distribution across PLRs*					
	Min	Q1	Median	Mean	Q3	Max
RGWEBLUP	3.99	7.02	7.97	8.32	9.51	14.15
RGWEBLUP-bc	3.99	6.90	7.85	8.28	9.46	15.13
Bias correction	-2.05	-0.29	-0.05	-0.04	0.20	2.04

Table 6.4: Summary statistics for the estimated average quoted net rent across PLRs.

(a)



(b)

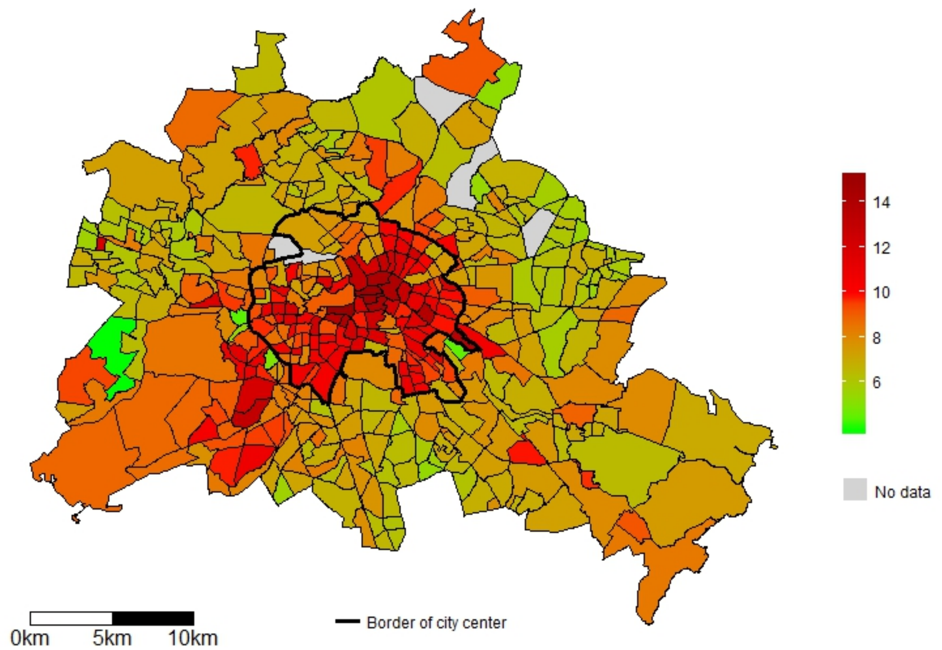


Figure 6.5: Map of estimated average quoted net rent per sqm in Berlin: (a) RGWEBLUP estimates; (b) RGWEBLUP-bc estimates.

Note that for the 86 out-of-sample PLRs the robust projective and robust predictive estimates are equal since the local bias correction is zero in these areas. It can further be observed that the bias correction seems to be symmetric around zero, indicating a symmetric contamination mechanism of

the data.
Figure 6.5 shows maps for the estimated average QNR for the PLRs in Berlin. At first glance both maps seem very similar. On closer inspection the RGWEBLUP estimates in the upper figure appear smoother compared to the RGWEBLUP-bc estimates especially in the city center. However, the spatial pattern in both maps confirm publications of the *Investitionsbank Berlin* (IBB), the public funding institute of the Berlin Senate (IBB, 2015). These patterns show clusters of very high values in the city center and the south-west, whereas there are low values at the eastern and western outskirts of Berlin.
In order to judge the quality of the model-based estimators, one possibility is to assess the proximity to the direct estimates (Chambers et al., 2016). Following Brown et al. (2001), this proximity can be investigated by computing a goodness-of-fit test. The basic idea of this test is that model-based estimators should not differ significantly from unbiased direct estimators. The test is computed as the value of a Wald test statistic

$$W\left(\hat{\bar{y}}_i^{model}\right) = \sum_{i=1}^{m} \frac{(\hat{\bar{y}}_i^{direct} - \hat{\bar{y}}_i^{model})^2}{\widehat{Var}(\hat{\bar{y}}_i^{direct}) + \widehat{MSE}(\hat{\bar{y}}_i^{model})},$$

where $\hat{\bar{y}}_i^{model}$ is a generic notation used to designate a model-based estimator of the small area mean. The MSE of $\hat{\bar{y}}_i^{model}$ is calculated by using the proposed CCT and CCST approaches. Note that in case of the robust predictive RGWEBLUP-bc, the CCST method for MSE estimation only provides results for certain areas. The reason can be found when recalling equation (4.66) for estimating the CCST MSE of the RGWEBLUP-bc. The second component h_{2i}^{bc} that arises due to the bias correction depends on the denominator $n_i - p$. Therefore, the CCST method for the MSE estimation can only be estimated for areas with sample sizes $n_i > p$, hence for $n_i > 7$ in case of the underlying model (6.1). Under the null hypothesis, the test statistic W is χ^2-distributed with m degrees of freedom (DF) where m denotes the number of sampled areas. As a direct estimate the Horvitz-Thompson estimator in (2.2) is used for PLRs with a sample size > 1. The test results are reported in Table 6.5.

MSE estimator	*CCT*			*CCST*		
Estimator	*W*	*DF*	*p-value*	*W*	*DF*	*p-value*
RGWEBLUP	293.43	314	0.79	308.65	314	0.57
RGWEBLUP-bc	293.65	314	0.79	39.67	94[1]	1.00

[1]DF equals the number of PLRs with sample size $n_i > 7$.

Table 6.5: Results of the goodness-of-fit test.

For the model-based robust estimators (RGWEBLUP and RGWEBLUP-bc) no significant difference to the direct estimates can be detected. In addition, the RGWEBLUP and the RGWEBLUP-bc seem to be consistent with the direct estimates with correlations of 0.86 and 0.92, respectively. The stronger correlation between the RGWEBLUP-bc estimates and the direct estimates provides evidence that using a robust predictive estimator appears to be appropriate in this application.
According to the German quality standards for official statistics the coefficient of variation (CV) is used as a precision measure for published estimates (DESTATIS, 2006). Even though these quality standards do not determine an acceptable CV, a value of at most 15% is frequently used as a benchmark for publishing statistics (FDZ, 2009). Thus, in order to assess the precision of the model-based robust estimators, the summary statistics for the CVs across the PLRs are reported in Table 6.6.

Predictor	*MSE*	*CVs across PLRs*					
		Min	Q1	Median	Mean	Q3	Max
For sampled PLRs							
RGWEBLUP	CCST	1.83	3.40	4.73	6.63	7.99	64.82
	CCT	1.87	3.76	5.18	7.05	8.38	64.81
RGWEBLUP-bc	CCST[1]	2.47	5.34	7.33	7.94	9.39	20.25
	CCT	1.63	3.88	5.06	5.52	6.63	14.76
For non-sampled PLRs							
RGWEBLUP	CCST	4.19	6.92	8.29	9.08	9.90	29.40
	CCT	4.92	7.53	8.77	9.78	10.57	32.79

[1]Only estimated for 94 PLRs with sample size $n_i > 7$.

Table 6.6: Summary statistics across the PLRs for the CVs obtained by different MSE estimators.

Apart from the maximum values, the CVs are in an acceptable range for the RGWEBLUP and the RGWEBLUP-bc estimates of the average QNR in the sampled and the non-sampled PLRs. It is noticeable that for the robust predictive RGWEBLUP-bc the CCT MSE estimates produce CVs smaller than 15% in all sampled PLRs. Since the median sample size of the unplanned PLR domain is three, there are many areas with a sample size $n_i < 7$. Hence, the CCST MSE estimator for the RGWEBLUP-bc can only be estimated for the 94 PLRs with $n_i > 7$ whereas the CCT MSE estimator delivers results for all sampled PLRs. Thus, the CCT method for the MSE estimation seems more appropriate in this case, even though it tends to underestimate the MSE as it ignores the estimation of the variance parameters. For the robust projective RGWEBLUP predictor of the area mean not much difference can be observed between CVs based on the CCT

and CCST MSE estimation.
Furthermore, it can be observed that the CVs are lower for estimates in the sampled PLRs compared to the non-sampled PLRs. This can be expected since the latter estimates are based on area-specific sampled information whereas the former are purely synthetic estimates. This also confirms the finding from the design-based simulation (Figure 6.7) where the RMSE lies on a higher level for the out-of-sample areas.
To compare the precision of the model-based and unbiased direct estimators it is possible to investigate the ratios between their estimated CVs where values greater than one indicate that the model-based estimator has a higher precision. These ratios are reported in Table 6.7 separately for sampled areas with $n_i > p$ and $1 < n_i \leq p$. The last column reports the percentage of areas where the ratio is greater than one. For larger areas with $n_i > 7$, in at least 70% of the PLRs the precision is higher for the model-based estimators. One exception are the CCST MSE estimates of the robust predictive RGWEBLUP-bc where only 20% of the PLRs have a higher precision. In PLRs with a sample size of $1 < n_i \leq p$, in 60% of the cases the precision is higher for the model-based estimators. This lower percentage compared to the larger areas is somehow counter-intuitive since model-based estimators which borrow strength from additional information are expected to be more efficient, especially in very small areas. Returning to the outcome variable in the sample, within some of these very small PLRs the observations are very homogeneous with only marginal within-area variation. Here, even direct estimators that are based only on few observations can be very precise.

Predictor	*MSE*	*CV ratios across PLRs*						
		Min	Q1	Median	Mean	Q3	Max	% > 1
For 94 sampled PLRs with $n_i > 7$								
RGWEBLUP	CCST	0.54	1.06	1.47	1.56	1.82	3.78	79
	CCT	0.51	0.98	1.29	1.33	1.56	2.67	70
RGWEBLUP-bc	CCST	0.27	0.58	0.74	0.77	0.94	1.52	20
	CCT	0.53	0.98	1.17	1.20	1.33	2.20	70
For 220 sampled PLRs with $1 < n_i \leq p$								
RGWEBLUP	CCST	0.03	0.71	1.24	1.51	1.95	6.86	58
	CCT	0.03	0.70	1.14	1.33	1.63	8.64	57
RGWEBLUP-bc	CCT	0.03	0.86	1.17	1.28	1.57	8.46	62

Table 6.7: Summary statistics across sampled PLRs for the ratio between estimated CVs of direct estimates and robust model-based estimators.

Based on this case study, it seems that considering robust SAE methods that account for spatial non-stationarity in the data can be a good choice for producing small area estimates of the average QNR on PLR level in Berlin. With a correlation of 0.92, the robust predictive estimator RGWEBLUP-bc (4.8) is consistent with the direct estimates and the CCT MSE estimator shows good properties with CVs $\leq 15\%$ in the sampled PLRs.

6.3 Simulation Study

The design-based simulation in this section assesses the relative performance of geographically weighted EBLUP methods using the database introduced in Section 6.1. The simulation setup is presented in Section 6.3.1 and the results are discussed Section 6.3.2. This includes a discussion concerning the performance of the area means and the conditional MSE estimates.

6.3.1 Simulation Setup

For the design-based simulation, the real estate database of Berlin introduced in Section 6.1 is used as a fixed population with $N = 51.907$. As in the case study, QNR is the variable of interest which is modeled using the hedonic price model in (6.1). For the design-based simulation, the sample size is reduced to $n = 505$ and the target statistic is the average QNR on PGR level ($M = 60$). Given the sample size of $n = 2006$ from the case study, fitting this model under the GWLMM approach lasts about 2.5 hours for the non-robust and 5.5 hours for the robust parameters. Based on the reduced sample size, estimating the GWEBLUP and RGWEBLUP of the area means took about 8.5 and 12.5 minutes, respectively. To guarantee area-specific sample sizes similar to those in the case study, the target level is shifted from PLR to PGR level (see Figure 6.1).
More precisely, in each of the $T = 500$ Monte Carlo replications, an independent random sample of size $n = 505$ is drawn from the fixed population by independently sampling from each region proportional to the size of the PGR. The PGR size is defined by the number of apartments listed in the database. For 11 out of the 60 PGRs the population sizes N_i are very small compared to the other regions such that the sample sizes of these areas were set to zero. This results in Monte Carlo samples with 49 in-sample and 11 out-of-sample PGRs. The median area-specific sample size for the sampled PGRs is 8 with a minimum of 4 and a maximum of 33 units.

6.3.2 Simulation Results

Six estimators of the small area mean are evaluated in this simulation study. These are the EBLUP in (2.26), the robust projective REBLUP in (3.11) and the robust predictive REBLUP-bc in (3.20) which are evaluated as

global estimators. Furthermore, the GWEBLUP in (2.65), the robust projective RGWEBLUP in (4.6) and the robust predictive RGWEBLUP-bc in (4.8) are evaluated. These are suitable under spatial non-stationarity. For the out-of-sample PGRs, the synthetic versions of the non-robust and robust projective estimators are estimated. Since the robust predictive estimators (REBLUP-bc and RGWEBLUP-bc) rely on area-specific bias corrections, these cannot be estimated for non-sampled areas. Note that a full bias correction as in Dongmo-Jiongo et al. (2013) could be a solution here to produce bias corrected small area estimates for the non-sampled areas.
In addition, the pseudo-linearization (CCT) and the linearization-based (CCST) approach for MSE estimation are analyzed for all six estimators. During the simulation study the global estimators did not face any stability problems whereas the parameter estimation for the proposed RGWEBLUP only converged in 495 out of the 500 Monte Carlo replications. Aborted replications are excluded from the analysis such that all estimators are compared based on 495 replications.
In what follows, the focus is first on results for the small area means using the quality measures presented in Section 5.3.1 from the model-based simulation. Thereafter, the performance of the MSE estimators is presented using the quality measures from Section 5.4.1. Since in the design-based simulation the population is fixed, $m_{it} = m_i$ for all quality measures.
The results for the area mean predictions are presented in Figure 6.6 separately for the sampled and non-sampled PGRs. In addition, the median values of the RB and the RRMSE are given in Table 6.8.

Predictor	*49 Sampled PGRs*		*11 Non-sampled PGRs*	
	RB	RRMSE	RB	RRMSE
GWEBLUP	1.62	4.92	3.62	13.74
RGWEBLUP	0.97	4.69	4.68	13.95
RGWEBLUP-bc	-0.11	6.01	-	-
EBLUP	1.00	6.00	9.60	10.35
REBLUP	-0.50	5.47	7.56	10.21
REBLUP-bc	-0.65	6.33	-	-

Table 6.8: Median values of RB (in %) and RRMSE (in %) for point estimates in the designs-based simulation.

For the in-sample PGRs the global (EBLUP, REBLUP and REBLUP-bc) and the geographically weighted methods (GWEBLUP, RGWEBLUP and RGWEBLUP-bc) perform reasonably well in terms of RB with absolute median values between 0.11% and 1.62%. However, the RGEWBLUP-bc has the lowest median RB for the in-sample PGRs. Interestingly, the bias-corrected global estimator REBLUP-bc increases the bias compared to the

robust projective REBLUP while it should be reducing it.
In addition, the geographically weighted predictors (GWEBLUP, RGWEBLUP and RGWEBLUP-bc) are more efficient in terms of lower RRMSE compared to their global counterparts (EBLUP, REBLUP and REBLUP-bc). Here, the proposed robust projective RGWEBLUP reveals the lowest RRMSE which is about one percentage point lower compared to the global REBLUP. The bias reduction for RGWEBLUP-bc comes at the price of lower efficiency compared to the non-robust GWEBLUP. However, it is still more efficient compared to the global REBLUP-bc.

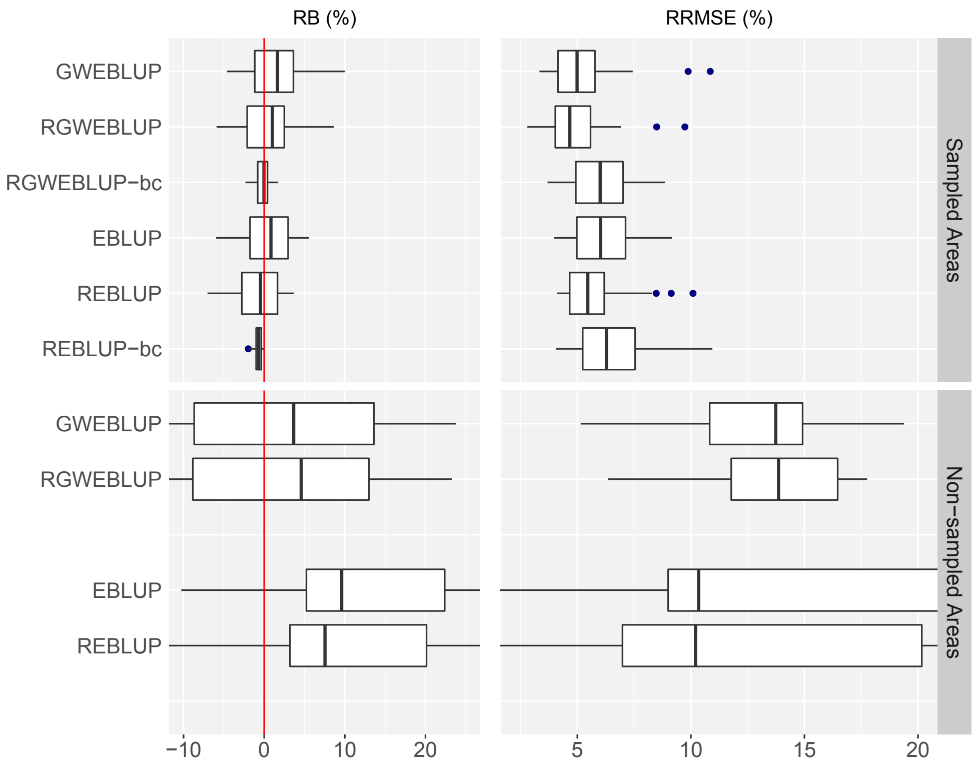


Figure 6.6: Boxplots of the RB (%) and the RRMSE (%) across PGRs in the design-based simulation.

For the out-of-sample PGRs, the synthetic global estimators based on the EBLUP and REBLUP suffer from a large bias. This is not as severe for the synthetic geographically weighted estimators based on the GWEBLUP and RGWEBLUP. Thus, using geographically weighted methods can improve the prediction especially in the non-sampled PGRs which can be an important advantage in SAE. This bias reduction comes at the price of lower efficiency in terms of RRMSE. Here, the median RRMSE is approximately three percentage points higher compared to the global approaches (cf. Table 6.8). However, the lower part of Figure 6.6 shows that the RRMSE of the

synthetic GWEBLUP and RGWEBLUP is considerably more stable across all 11 non-sampled PGRs which can also be an important feature for the reliability of an estimator. It can moreover be observed that the non-robust and robust geographically weighted estimators (GWEBLUP and RGWEBLUP) perform very similarly. Thus, combining the geographical weights with an influence function that gives lower weights to extreme observations does not lead to efficiency gains for the 11 out-of-sample PGRs. This seems plausible, recalling that the spatial weights for the synthetic estimators are based entirely on the distance between the sampled units of the 49 in-sample PGRs and the population units from the out of sample PGRs. Thus, the influence of all sampled units (including outliers) on the small area means of the non-sampled PGRs is already reduced by lower spatial weights. Furthermore, using the global robust method (REBLUP) to estimate the small area mean can lead to a bias reduction and efficiency gains compared to the global EBLUP for the out-of-sample PGRs.
Turning toward the performance of the MSE estimators in Table 6.9, the results for the in-sample PGRs (left part of Table 6.9) are presented first. The table shows the median values of the performance measures. The distributions of these measures across all 49 sampled PGRs are reported in Table A.2 in Appendix A.

Predictor	*MSE*	*49 Sampled PGRs*			*11 Non-sampled PGRs*		
		RB	RRMSE	CR	RB	RRMSE	CR
GWEBLUP	CCST	25.01	49.62	0.98	-14.80	37.08	0.93
	CCT	-2.51	37.88	0.90	-3.79	33.51	0.98
RGWEBLUP	CCST	25.36	52.05	0.96	-23.75	66.07	0.90
	CCT	4.30	49.74	0.89	-18.70	50.22	0.95
RGWEBLUP-bc	CCST	23.47	36.38	0.97	-	-	-
	CCT	-4.91	28.95	0.92	-	-	-
EBLUP	CCST	14.88	20.43	0.96	76.09	76.67	1.00
	CCT	-5.14	27.56	0.92	115.00	115.50	1.00
REBLUP	CCST	6.97	22.99	0.96	99.21	99.83	1.00
	CCT	-8.17	24.49	0.91	143.00	143.50	1.00
REBLUP-bc	CCST	40.66	51.63	0.98	-	-	-
	CCT	-14.31	25.90	0.88	-	-	-

Table 6.9: Median values of RB (in %), RRMSE (in %) and CR for the conditional MSE estimates in the design-based simulation.

For the geographically weighted methods (GWEBLUP, RGWEBLUP and RGWEBLUP-bc), the CCT method for MSE estimation shows better stability in terms of a lower RRMSE. Interestingly, this is in contrast to the findings in the model-based simulation study from the previous chapter where the CCST method is consistently more stable over all scenarios. With median values between 0.96 and 0.98, the CCST provides 95% confidence intervals with a reasonably good coverage performance with a tendency to over-coverage. The CCT method reveals under-coverage with values between 0.88 and 0.92.

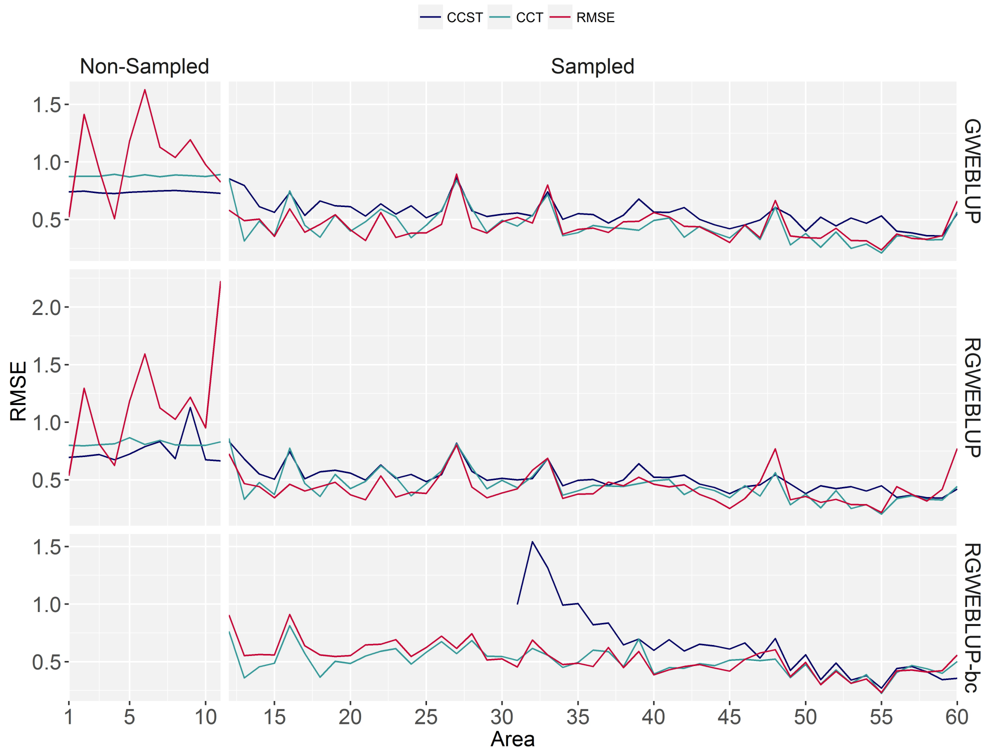


Figure 6.7: PGR-specific values of true RMSE (red line) for geographically weighted EBLUP estimators and average estimated RMSE based on CCT (light blue line) and CCST (dark blue line) method for MSE estimation. Areas are sorted by sample size.

For all predictors, it can be observed that the CCST method for MSE estimation tends to overestimate the MSE whereas the CCT tends toward an underestimation. However, in absolute value the bias of the CCT method is considerably lower for the geographically weighted methods (GWEBLUP, RGWEBLUP and RGWEBLUP-bc) and the global EBLUP and REBLUP-bc methods (cf. Table 6.9). The RMSE estimates for the in-sample PGRs shown in Figures 6.7 and 6.8 lead to a similar conclusion since for these predictors (GWEBLUP, RGWEBLUP, RGWEBLUP-bc, EBLUP and REBLUP-bc) the CCT MSE estimators seem to be more accurate in tracking the actual RMSE especially with a growing area-specific sample size.
Given that $n_i > 7$, the RMSE based on the CCST method shows a large

overestimation but advances toward the actual RMSE with a growing area-specific sample size. At the same time, the estimated RMSE based on the CCT method for the robust predictive REBLUP-bc and RGWEBLUP-bc can track the actual RMSE reasonably well even for small sample sizes.

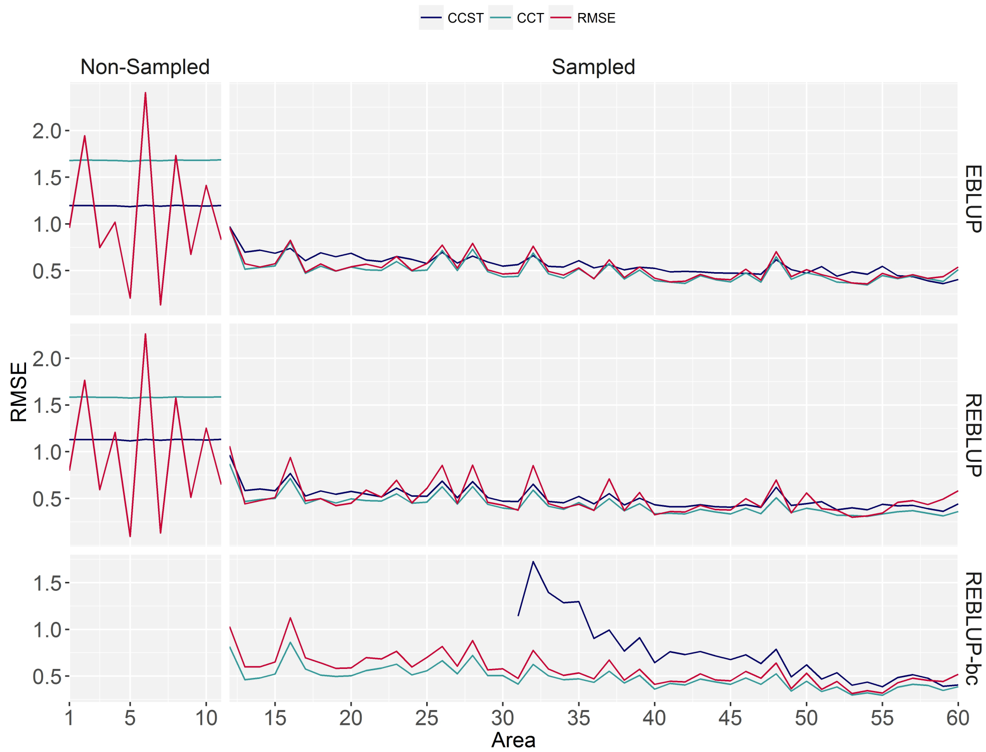


Figure 6.8: PGR-specific values of true RMSE (red line) for global EBLUP estimators and average estimated RMSE based on CCT (light blue line) and CCST (dark blue line) method for MSE estimation. Areas are sorted by sample size.

The right part of Table 6.9 reports the performance of the MSE estimators for the synthetic estimators in PGRs with zero sample size. The distributions of the performance measures across all 11 non-sampled PGRs are reported in Table A.3 in Appendix A. For the synthetic geographically weighted methods (GWEBLUP and RGWEBLUP) both MSE estimators underestimate the actual MSE whereas for the global estimators reveal an enormous overestimation. The results for the out-of-sample PGRs, shown on the left-hand side of Figures 6.7 and 6.8 confirm this finding. In addition, these figures reveal that both types of MSE estimation are unable to track the true MSE behavior of the synthetic predictors (GWEBLUP, RGWEBLUP and EBLUP, REBLUP).
One explanation for the extreme overestimation of the RMSE for the global

estimators can be found when recalling the conditional MSE estimation for the synthetic EBLUP estimators. The squared conditional bias is of the EBLUP and the REBLUP is estimated using (2.41) and (3.30), respectively. Here, it can be seen that these terms additively depend on the estimated variance of the area-specific random effect. The estimation results for the variance component from the design-based simulation are presented in Table 6.10. First, it can be observed that the estimates of σ_v based on the global model (LMM) are substantially larger compared to the geographically weighted model (GWLMM). This can be explained by the superior model fit of the GWLMM, which was discovered in the case study in Section 6.2 (cf. Table 6.2). Secondly, the mean estimated variance parameter from the LMM is 1.18 for the non-robust and 1.11 for the robust estimation. Compared to the scale of the RMSEs of the EBLUP and the REBLUP in the left part of Figure 6.8 this variance parameter is quite large. Thus, the estimated RMSEs for the global synthetic estimators are mainly driven by the large estimated variance parameter of the random effect, which may have caused the enormous overestimation in the non-sampled PGRs.

Model	*Method*	*Results for σ_v across replications*					
		Min	Q1	Median	Mean	Q3	Max
GWLMM	non-robust	0.23	0.37	0.44	0.45	0.52	0.80
	robust	0.23	0.33	0.40	0.41	0.47	0.70
LMM	non-robust	0.95	1.12	1.18	1.18	1.24	1.51
	robust	0.87	1.05	1.11	1.11	1.17	1.42

Table 6.10: Design-based simulation results: estimated variance parameter σ_v of the area specific random effects.

The overall results in this simulation study indicate that in the case of the Berlin real estate database it can be beneficial to apply geographically weighted rather than global methods for estimating small area means. In addition, combining the spatial weights with robust protection can lead to bias reduction and efficiency gains for the in-sample areas. For the out-of-sample areas, the robust protection cannot improve the prediction but neither does it deteriorate it. The CCST method for MSE estimation performs well in terms of bias and coverage rate for the GWEBLUP and the RGWEBLUP in the sampled areas. However, the CCT MSE estimates can track the actual RMSE more accurately and is capable of producing MSE estimates for the RGWEBLUP-bc even for very small area-specific sample sizes.

6.4 Discussion

The results of this chapter indicate that combining the local LMM in (2.62) with down weighting the influence of extreme observations can be beneficial in the field of SAE. Under the realistic data situation of the Berlin apartment rental market, applying the proposed robust projective RGWEBLUP and robust predictive RGWEBLUP-bc to estimate the average quoted net rent in urban subregions can lead to bias reduction and efficiency gains compared to global methods.

In the simulation study I compared geographically weighted methods with their global counterparts that do not account for spatial effects at all. The likelihood ratio test showed evidence of spatial non-stationarity in the model coefficients. Thus, using a geographically weighted model is indicated in this case. However, other spatial effects such as spatial autocorrelation may also be present in the data. In empirical studies it can be difficult to distinguish between the two spatial effects of dependence and heterogeneity. Hence, in further design-based simulation studies it would be interesting to assess the performance of other spatial EBLUP estimators such as the SEBLUP and NPEBLUP together with their robust extensions.

According to de Graaff et al. (2001) spatial dependency and spatial heterogeneity should be considered jointly. One reason is that there may be no difference between these two effects in an observational sense as a spatial clustering of similar values may be due to dependence among neighboring units or a result of changing coefficients. In a further step it may be useful to extend the GWLMM in (2.62) by allowing for spatially correlated random effects in order to account for spatial dependence and heterogeneity simultaneously.

7 Conclusion

This final chapter gives a summary of the key results of this thesis and outlines some avenues for future research. The main findings are summarized in Section 7.1 which includes the theoretical contributions and the empirical results of this thesis. Section 7.2 addresses some potential fields of further research.

7.1 Main Findings

The main objective of this thesis was to provide a robust EBLUP-based approach for SAE under spatial non-stationarity. The theoretical foundations underpinning the proposed methodology were set out in Part I and empirical evaluations in Part II.
In the theoretical part, different EBLUP-based methods for SAE were reviewed. This review includes the EBLUP of the area mean and selected spatial extensions in Chapter 2. These spatial extension are: (i) the spatial EBLUP (SEBLUP) that accounts for spatial dependencies by allowing for spatial autocorrelation in the random effects; (ii) the non-parametric EBLUP (NPEBLUP) which can account for spatial trends of unknown functional form in the target variable; (iii) and the geographically weighted EBLUP (GWEBLUP) that can capture spatial non-stationarity in the model coefficients. The reviewed EBLUP methods have in common that they are based on the linear mixed model. Accordingly, these methods rely on the assumption of normally distributed error term components which can be violated in the presence of outliers. Therefore, outlier robust extension to EBLUP approaches were reviewed in Chapter 3. This review includes the robust EBLUP (REBLUP) of the small area mean and robust extension to the SEBLUP and the NPEBLUP.
So far, robust EBLUP-based methods under spatial non-stationarity have not been considered in the literature. Hence, robust extensions of the GWEBLUP were proposed in Chapter 4. The main contributions to the current literature can be summarized as follows:

- The influence of outliers on the parameter estimation has been reduced using the results of Sinha and Rao (2009). Their robust ML estimation equations were modified to account for spatial non-stationarity in the fixed effects coefficients.

- The robust parameter estimation has been integrated in the algorithm of Chandra et al. (2012) to estimate the robust projective GWEBLUP (RGWEBLUP) of the area mean. For that purpose a local fixed point algorithm has been developed for the approximation of the local in-sample coefficients.

- The local bias correction of Chambers et al. (2014) has been combined with geographical weighting to develop a robust predictive GWE-BLUP (RGWEBLUP-bc) of the area mean.

- Two conditional MSE estimators have been developed for the RGWE-BLUP and the RGWEBLUP-bc using the pseudo-linearization and the linearization approach following the ideas set out in Chambers et al. (2014).

- The GWEBLUP of Chandra et al. (2012) and the proposed estimators, the RGWEBLUP and the RGWEBLUP-bc, have been implemented for the R-language in the package `saeRGW`.

The performance of the proposed methods was assessed in the empirical part of this thesis which includes a model-based simulation in Chapter 5 and a design-based simulation in Chapter 6. The aim of the model-based simulation was to examine the relative performance of the proposed estimators under spatial non-stationarity and in the presence of outliers. Hence, the scenarios for the model-based simulations were chosen to be a combination of settings under spatial stationarity and non-stationarity with different outlier contamination mechanisms. The overall results from the model-based simulation indicate a potential benefit from combining geographically weighted regression with robust parameter estimation. In the presence of spatial non-stationarity and outliers, applying the proposed robust projective RGWEBLUP and robust predictive RGWEBLUP-bc led to efficiency gains compared to the non-robust GWEBLUP of the small area mean. In addition, the proposed CCST MSE estimator for the RGWE-BLUP and RGWEBLUP-bc appeared to have good properties in terms of bias and stability in the investigated scenarios. Even though the main interest in this thesis was to provide reliable estimates for small area means, the estimated model parameters were studied as well. Based on the simulation results, the estimated local coefficient seemed to give an overall picture of the true parameter surface but should not be interpreted individually. In addition, the estimated variance parameters mainly suffered from underestimation. These results indicate that the robust and non-robust parameter estimates under the GWLMM should be treated carefully in applications.
The practical implications from a model-based simulation are limited since the data-generating process in the population is known, which is generally not the case in applications. Design-based simulations can be more informative as these are usually based on realistic populations with an unknown data-generating process for the target variable. Therefore, in Chapter 6 the

relative performance of the proposed method was assessed in a design-based simulation study using data from the Berlin apartment rental market. The target variable in that study is the quoted net apartment rent per square meter and the target statistic is the average of that variable on the level of the 60 forecasting areas in Berlin. Applying robust SAE methods that account for spatial non-stationarity in the data was justified beforehand in a case study in Section 6.2. The overall results of the simulation indicated that in the case of the Berlin real estate database it can be beneficial to apply geographically weighted rather than global methods for estimating the small area means of the quoted net rent. In addition, combining the spatial weights with robust protection led to bias reduction and efficiency gains for in-sample areas. For the out-of-sample areas, the robust protection had no effect on the efficiency of the estimated small area means. The proposed MSE estimators have also been investigated. Here, the CCST method for the MSE estimation performed well in terms of bias and coverage rate for the GWEBLUP and the RGWEBLUP of the area mean in the sampled areas. However, the CCT MSE estimates were able to track the actual RMSE more accurately and were capable of producing MSE estimates for the RGWEBLUP-bc even for very small area-specific sample sizes.

7.2 Further Research

The proposed robust extension of the GWEBLUP extends the current literature in the field of SAE. In addition, it showed promising results for estimating small area means in model and design-based simulation studies and is implemented in the R-language with a user-friendly interface for practitioners. However, there are several outstanding research issues that should be addressed in the future. These issues are of theoretical and empirical concern.
Starting with potential theoretical improvements, the choice of the tuning constant can be further investigated. The proposed robust GWEBLUP restricts the influence of outliers on the parameter estimation to produce reliable small area estimates. The strength of this restriction is defined by the tuning constant which is defined a priori. As the choice of the tuning constant is a challenging problem in robust SAE, a data-driven alternative can be considered as suggested by Wang et al. (2007). A further theoretical issue concerns the bias correction. In this thesis a local bias correction is considered following the ideas of Chambers et al. (2014). Dongmo-Jiongo et al. (2013) argue that the local bias correction may still lead to biased small area estimates as it only depends on area-specific information, whereas the robust parameter estimates are influenced by all sampled units. This remaining bias may be less severe in the geographically weighted framework since the influence of distant observations on the parameter estimation is small. However, an extension of the global bias correction of Dongmo-Jiongo

et al. (2013) to the case of spatial non-stationarity is left open for further research. Another path for further research concerns the spatial modeling. Under the GWLMM the random effects are assumed to be independently distributed. It may be useful to extend the GWLMM by allowing for spatially correlated random effects to account for spatial dependence and non-stationarity simultaneously. Furthermore, the results from the model-based simulation in Section 5.2 show that the robust and non-robust parameter estimates under the GWLMM need to be treated carefully in applications. To find sound explanations and satisfying guidelines for parameter interpretations, further research in this direction is needed. The findings in this thesis can only be seen as preliminary results. Moreover, further investigation is required for the MSE estimation. In addition to the conditional MSE estimators proposed in this thesis, bootstrap methods for MSE estimation can be developed. This could improve area-specific MSE estimation especially for non-sampled areas. With growing computing capacity the issue of calculation time may become less relevant in future. The last theoretical issue raised here is related to the sampling design which is normally assumed to be non-informative in model-based SAE. This assumption is violated when the sample selection probabilities are related to the target variable. In that case design weights can be incorporated into model-based SAE as suggested by Pfeffermann and Sverchkov (2007) and Verret et al. (2015). Based on these ideas, it would be interesting to include design effects into robust SAE methods such as the proposed RGWEBLUP and RGWEBLUP-bc.

Turning to outstanding empirical evaluations, further simulation studies can be conducted to assess the relative performance of the proposed method in a broader context. For instance, a comparison with alternative robust methods such as the M-quantile approach would be interesting. In this context, a natural competitor for the proposed method would be the geographically weighted M-quantile estimator of Salvati et al. (2012). Furthermore, in the model-based simulation in Chapter 5 the local coefficients linearly depend on the coordinates, which might be unrealistic. Thus, different settings for spatial non-stationarity could be analyzed in the future. Another avenue for future work would be to conduct model-based simulations similar to Schmid et al. (2016) and focus on robust SAE under spatial correlation. Here, it would be interesting to assess the behavior of the geographical estimators (GWEBLUP, RGWEBLUP and RGWEBLUP-bc) when the data is actually driven by spatially correlated random effects. Finally, in a design-based framework it would be interesting to analyze the effect of different sampling designs on the performance of the estimators.

To raise new questions, new possibilities, to regard old problems
from a new angle, requires creative imagination
and marks real advance in science.
- Albert Einstein (1938)

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Appendix A

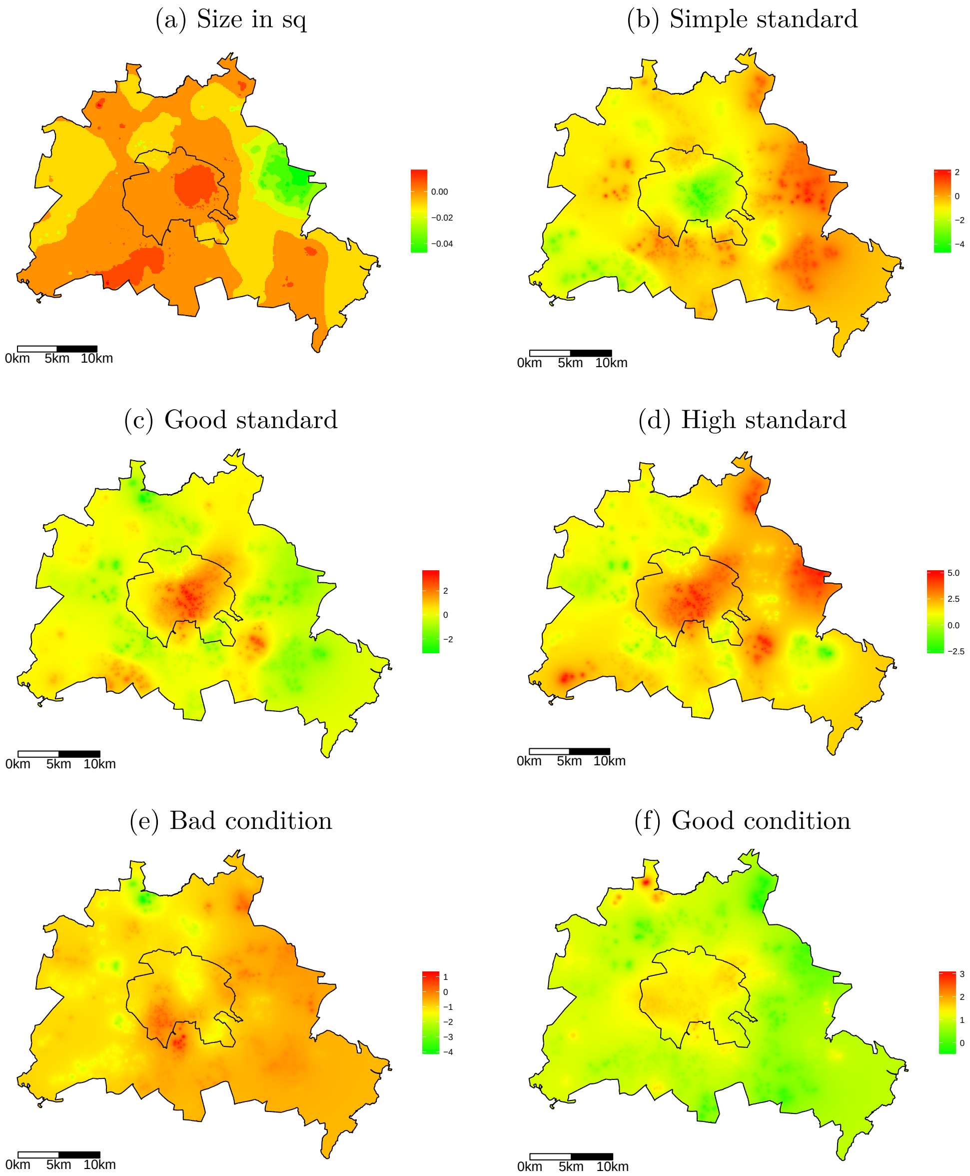

Figure A.1: Spatial surface of the model coefficients across Berlin from the hedonic model for QNR based on the GWLMM.

Predictor	*MSE*	*Results for the following scenarios and areas*						
		$(0,0)$ *1-40*	$(v,e)_s$ *1-40*	$(v,e)_s$ *1-36*	$(v,e)_s$ *37-40*	$(v,e)_{ns}$ *1-40*	$(v,e)_{ns}$ *1-36*	$(v,e)_{ns}$ *37-40*
Median values of RB (%)								
GWEBLUP	CCST	4.85	6.70	7.35	-12.23	17.96	18.63	-14.30
	CCT	-11.92	-10.74	-10.56	-21.29	-3.80	-3.16	-21.78
RGWEBLUP	CCST	0.59	7.39	7.79	-14.96	7.50	7.89	-26.47
	CCT	-15.70	-4.04	-3.32	-21.26	1.76	2.38	-27.24
RGWEBLUP-bc	CCST	-3.17	-4.15	-3.79	-9.94	-13.38	-12.74	-29.21
	CCT	-9.24	-5.75	-5.46	-11.81	-1.73	-0.46	-17.74
EBLUP	CCST	0.13	3.00	3.23	-10.93	4.13	4.70	-9.59
	CCT	-7.30	-10.33	-9.45	-17.14	-14.21	-14.09	-14.85
REBLUP	CCST	-2.15	1.74	1.92	-9.53	-8.25	-8.11	-26.76
	CCT	-13.58	-5.90	-5.25	-18.94	-7.27	-7.18	-29.23
REBLUP-bc	CCST	8.71	8.06	8.33	4.47	-2.37	-1.85	-7.21
	CCT	-15.13	-12.37	-11.87	-15.04	-12.06	-11.44	-14.65
Median values of RRMSE (%)								
GWEBLUP	CCST	27.34	39.15	38.85	43.25	52.68	53.37	46.67
	CCT	34.66	52.14	52.50	50.88	63.58	64.24	55.54
RGWEBLUP	CCST	29.30	38.01	38.85	32.42	49.39	49.65	43.49
	CCT	36.58	47.67	48.62	41.41	68.06	68.63	53.26
RGWEBLUP-bc	CCST	30.35	33.72	33.90	33.01	35.73	35.51	38.26
	CCT	31.77	41.25	41.29	40.28	58.76	59.73	51.21
EBLUP	CCST	23.85	35.63	35.48	42.89	32.30	32.15	40.48
	CCT	32.18	51.66	51.84	49.87	58.85	59.16	49.13
REBLUP	CCST	29.57	34.21	34.21	33.85	31.18	30.95	33.63
	CCT	31.45	41.03	41.61	36.29	52.81	53.20	47.13
REBLUP-bc	CCST	35.00	38.86	39.22	37.36	37.69	37.90	36.53
	CCT	30.19	37.89	37.78	37.98	53.92	53.56	54.31
Median values of CR								
GWEBLUP	CCST	0.95	0.96	0.97	0.93	0.95	0.96	0.93
	CCT	0.87	0.89	0.89	0.84	0.89	0.89	0.86
RGWEBLUP	CCST	0.94	0.93	0.93	0.80	0.95	0.95	0.91
	CCT	0.86	0.86	0.86	0.74	0.90	0.90	0.86
RGWEBLUP-bc	CCST	0.90	0.86	0.86	0.78	0.91	0.91	0.90
	CCT	0.88	0.87	0.87	0.79	0.90	0.90	0.89
EBLUP	CCST	0.94	0.96	0.96	0.96	0.95	0.95	0.92
	CCT	0.89	0.92	0.92	0.88	0.90	0.90	0.89
REBLUP	CCST	0.93	0.91	0.91	0.83	0.93	0.93	0.92
	CCT	0.88	0.86	0.86	0.73	0.89	0.89	0.87
REBLUP-bc	CCST	0.93	0.89	0.89	0.88	0.93	0.94	0.93
	CCT	0.87	0.82	0.83	0.81	0.88	0.88	0.88

Table A.1: Model-based simulation results: Performance of RMSE estimators under spatial stationarity.

Predictor	*MSE*	*Performance across PGRs*					
		Min	Q1	Median	Mean	Q3	Max
Distribution of RB (%)							
GWEBLUP	CCST	-13.90	12.80	25.01	29.89	45.28	132.60
	CCT	-36.57	-9.53	-2.51	1.04	7.70	51.37
RGWEBLUP	CCST	-44.65	9.03	25.36	24.11	43.83	113.70
	CCT	-41.72	-9.18	4.30	6.01	18.33	67.51
RGWEBLUP-bc	CCST	-34.47	8.97	23.47	37.38	52.06	135.60
	CCT	-36.11	-10.56	-4.91	-3.96	1.40	29.13
EBLUP	CCST	-23.98	-1.00	14.88	10.03	20.60	33.00
	CCT	-12.24	-6.74	-5.14	-5.05	-2.59	1.50
REBLUP	CCST	-26.52	-11.02	6.97	4.33	18.61	31.58
	CCT	-37.23	-21.56	-8.17	-11.38	-1.69	8.57
REBLUP-bc	CCST	-21.28	21.75	40.66	51.91	66.41	153.20
	CCT	-24.87	-18.66	-14.31	-14.40	-10.81	-4.68
Distribution of RRMSE (%)							
GWEBLUP	CCST	13.79	34.87	49.62	50.58	60.12	135.00
	CCT	19.38	33.42	37.88	43.93	48.88	98.07
RGWEBLUP	CCST	23.54	42.59	52.05	55.11	63.96	116.10
	CCT	28.66	40.32	49.74	53.95	61.21	117.50
RGWEBLUP-bc	CCST	13.67	23.72	36.38	53.69	66.57	146.70
	CCT	13.90	24.50	28.95	31.78	37.41	83.68
EBLUP	CCST	7.33	16.13	20.43	21.20	27.49	40.07
	CCT	12.74	23.14	27.56	26.13	29.88	40.91
REBLUP	CCST	8.88	18.54	22.99	23.51	28.75	39.75
	CCT	12.33	19.61	24.49	24.99	29.35	37.80
REBLUP-bc	CCST	14.26	31.43	51.63	64.73	76.90	167.10
	CCT	16.78	22.88	25.90	27.17	30.38	50.35
Distribution of CR							
GWEBLUP	CCST	0.78	0.93	0.98	0.95	0.99	1.00
	CCT	0.77	0.86	0.90	0.89	0.93	1.00
RGWEBLUP	CCST	0.71	0.92	0.96	0.94	0.99	1.00
	CCT	0.74	0.86	0.89	0.89	0.94	0.99
RGWEBLUP-bc	CCST	0.81	0.97	0.97	0.97	0.99	1.00
	CCT	0.76	0.89	0.92	0.91	0.94	0.97
EBLUP	CCST	0.82	0.93	0.96	0.94	0.97	1.00
	CCT	0.81	0.89	0.92	0.91	0.95	0.98
REBLUP	CCST	0.77	0.90	0.96	0.93	0.98	1.00
	CCT	0.74	0.83	0.91	0.88	0.94	0.97
REBLUP-bc	CCST	0.88	0.97	0.98	0.98	0.99	1.00
	CCT	0.83	0.86	0.88	0.88	0.91	0.92

Table A.2: Design-based simulation results: Performance of RMSE estimators for sampled areas.

Predictor	*MSE*	*Performance across PGRs*					
		Min	Q1	Median	Mean	Q3	Max
Distribution of RB (%)							
GWEBLUP	CCST	-45.99	-26.02	-14.80	-2.59	1.28	74.99
	CCT	-37.95	-15.91	-3.79	10.32	14.48	97.87
RGWEBLUP	CCST	-65.02	-32.20	-23.75	-13.80	4.98	53.19
	CCT	-60.68	-24.67	-18.70	-6.57	5.46	70.38
EBLUP	CCST	-30.11	8.26	76.09	213.70	137.40	1153.00
	CCT	-14.46	32.44	115.00	284.90	189.90	1437.00
REBLUP	CCST	-29.99	13.69	99.21	301.50	187.50	1566.00
	CCT	-14.57	38.89	143.00	393.60	250.50	1962.00
Distribution of RRMSE (%)							
GWEBLUP	CCST	29.46	32.86	37.08	46.95	45.11	98.12
	CCT	25.75	26.94	33.51	45.98	40.06	115.10
RGWEBLUP	CCST	30.00	45.68	66.07	114.40	138.40	426.20
	CCT	24.36	31.20	50.22	52.56	67.12	107.60
EBLUP	CCST	6.00	25.35	76.67	223.10	138.00	1155.00
	CCT	7.84	33.04	115.50	288.20	190.30	1439.00
REBLUP	CCST	5.83	28.92	99.83	310.20	188.20	1568.00
	CCT	11.31	39.46	143.50	397.00	251.00	1964.00
Distribution of CR							
GWEBLUP	CCST	0.55	0.87	0.93	0.89	0.97	1.00
	CCT	0.75	0.92	0.98	0.94	0.99	1.00
RGWEBLUP	CCST	0.44	0.82	0.90	0.86	0.97	1.00
	CCT	0.66	0.90	0.95	0.92	0.99	1.00
EBLUP	CCST	1.00	1.00	1.00	1.00	1.00	1.00
	CCT	1.00	1.00	1.00	1.00	1.00	1.00
REBLUP	CCST	1.00	1.00	1.00	1.00	1.00	1.00
	CCT	1.00	1.00	1.00	1.00	1.00	1.00

Table A.3: Design-based simulation results: Performance of RMSE estimators for non-sampled areas.

Appendix B

Model-based simulation under spatially correlated random effects

The synthetic population ($m = 40$ and $N_i = 100$) was generated using a spatial nested error regression model with correlated random area effects:

$$y_{ij} = 100 + 5x_{ij} + v_i + e_{ij}.$$

The covariate x_{ij} was drawn from a log-normal distribution with mean 1 and a standard deviation of 0.5. The random effects v_i and the individual error e_{ij} were generated as

$$\begin{aligned} v_i &\sim N(0, \boldsymbol{G}) \\ e_{ij} &\sim N(0, 6) \end{aligned}$$

with $\boldsymbol{G} = \sigma_u^2[(\boldsymbol{I} - \rho\boldsymbol{W})(\boldsymbol{I} - \rho\boldsymbol{W}^T)]^{-1}$, $\rho = 0.8$ and $\sigma_u = 3$. As contiguity matrix, a binary adjacency matrix was constructed based on the neighborhood structure visible from Figure 5.1. Areas which have at least one point on the boundary in common are defined as neighbors (queens structure). Based on this definition, the number of neighbors per area varies between 3 and 8 with median 5. To obtain the neighborhood matrix $\boldsymbol{W}$ the contiguity matrix is row standardized such that the rows sum up to one (cf. Pratesi and Salvati, 2009). Three outlier contamination mechanisms are investigated:

1. $(0, 0)$ - no outliers;

2. $(v, e)_s$ - symmetric outliers in area-level and individual error, $v_i \sim N(0, \boldsymbol{G})$ for the areas 1-36 and $v_i \sim N(0, 20)$ for the areas 37-40; $e \sim \delta N(0, 6) + (1 - \delta)N(0, 150)$ where δ is Bernoulli distributed with $P(\delta = 1) = 0.95$;

3. $(v, e)_{ns}$ - non-symmetric outliers in area-level and individual error, $v_i \sim N(0, \boldsymbol{G})$ for the areas 1-36 and $v_i \sim N(9, 20)$ for the areas 37-40; $e \sim \delta N(0, 6) + (1 - \delta)N(20, 150)$ where δ is Bernoulli distributed with $P(\delta = 1) = 0.95$.

Predictor	*Results (%) for the following scenarios and areas*						
	$(0,0)$ *1-40*	$(v,e)_s$ *1-40*	$(v,e)_s$ *1-36*	$(v,e)_s$ *37-40*	$(v,e)_{ns}$ *1-40*	$(v,e)_{ns}$ *1-36*	$(v,e)_{ns}$ *37-40*
Median values of RB							
GWEBLUP	0.02	0.02	0.02	0.15	0.01	0.02	-1.81
RGWEBLUP	0.02	0.02	0.01	0.16	-0.58	-0.58	-1.72
RGWEBLUP-bc	0.02	0.03	0.02	0.07	-0.59	-0.58	-1.00
NPEBLUP	0.02	0.02	0.02	0.11	0.05	0.07	-1.61
RNPEBLUP	0.02	0.02	0.02	0.07	-0.59	-0.58	-1.23
SEBLUP	0.02	0.03	0.02	0.10	0.13	0.14	-1.77
SREBLUP	0.02	0.02	0.00	0.06	-0.59	-0.58	-1.16
SREBLUP-bc	0.02	0.02	0.02	0.03	-0.41	-0.41	-0.38
EBLUP	0.02	0.02	0.02	0.09	0.23	0.24	-1.95
REBLUP	0.02	0.02	0.01	0.07	-0.50	-0.50	-1.24
REBLUP-bc	0.02	0.02	0.02	0.04	-0.58	-0.58	-0.56
Median values of RRMSE							
GWEBLUP	0.81	1.05	1.05	1.93	1.40	1.38	3.00
RGWEBLUP	0.82	0.89	0.88	1.80	1.14	1.13	2.33
RGWEBLUP-bc	0.90	1.04	1.04	1.44	1.29	1.28	1.72
NPEBLUP	0.78	1.04	1.03	1.61	1.45	1.43	2.58
RNPEBLUP	0.80	0.88	0.87	1.17	1.14	1.13	1.67
SEBLUP	0.76	1.03	1.02	1.66	1.47	1.46	2.64
SREBLUP	0.78	0.86	0.86	1.18	1.11	1.11	1.58
SREBLUP-bc	0.91	1.13	1.13	1.18	1.45	1.46	1.36
EBLUP	0.80	1.07	1.07	1.59	1.54	1.53	2.70
REBLUP	0.82	0.90	0.90	1.17	1.12	1.11	1.67
REBLUP-bc	0.91	1.03	1.03	1.02	1.27	1.27	1.24

Table B.1: Model-based simulation results: Performance of estimated small area means under spatially correlated random effects with $\rho = 0.8$.

The main results can be summarized as follows:

- Without outlier contamination, $(0,0)$, all estimators are nearly unbiased and the SEBLUP is the most efficient estimator. This is expected as the population model is generated under the assumptions of this estimator.

- With symmetric outlier contamination, $(v,e)_s$, all estimators are nearly unbiased what can be expected as the outliers sum up to zero in the out-of-sample population and can be seen as noise. Further, the robust projective estimators (REBLUP, SREBLUP, RNPEBLUP and

RGWEBLUP) are more efficient compared to their non-robust counterparts (EBLUP, REBLUP, NPEBLUP and GWEBLUP). As expected, the most efficient estimator in this scenario is the SREBLUP.

- With asymmetric outlier contamination, $(v, e)_{ns}$, the robust projective estimators (REBLUP, SREBLUP, RNPEBLUP and RGWEBLUP) are biased and the robust predictive estimators (REBLUP-bc, SREBLUP - bc and RGWEBLUP-bc) can reduce this bias at least in the extreme areas 37-40. This bias reduction leads to an efficiency loss in the areas 1-36 compared to the respective robust projective estimators, but improves efficiency in the areas 37-40.

- Comparing the global EBLUP approaches (EBLUP, REBLUP, REBLUP - bc) and the local counterparts (GWEBLUP, RGWEBLUP, RGWEBLUP - bc) the picture is not as clear: Without outlier contamination, $(0, 0)$, there is almost no difference in efficiency; with symmetric outlier contamination, $(v, e)_s$, the local approaches seem slightly more efficient; with asymmetric outlier contamination, $(v, e)_{ns}$, the global approaches are more efficient.

Abstract

The demand for reliable small area statistics has been growing in public and private organizations. Sample surveys are designed to produce reliable estimates for quantities of interest on higher geographic levels, but can have very small or even zero sample sizes for lower geographic levels. Direct estimates, which only rely on area-specific information, are usually unbiased but can produce results with high variability in the case of small sample sizes. *Small area estimation* (SAE) techniques have been developed to gain reliability compared to direct estimates by *borrowing strength* from additional information. One well known SAE method is the empirical best linear unbiased predictor (EBLUP) of the small area mean that incorporates auxiliary variables using the linear mixed model approach. In addition, spatial information can be used to borrow strength over space. One approach to account for geographical information is to extend the linear mixed model and allow for spatially correlated random area effects (cf. Pratesi and Salvati, 2008, SEBLUP). An alternative is to include the spatial information by non-parametric mixed models (cf. Opsomer et al., 2008, NPEBLUP). Another option is the geographic weighted regression where the model coefficients vary across the study area (cf. Chandra et al., 2012, GWEBLUP). Under the assumption of normally distributed error terms, these approaches are useful for estimating small area means efficiently. The normality assumption can be violated in the presence of outliers and, hence, it can be beneficial to reduce the influence of outliers and use robust methods for SAE (cf. Sinha and Rao, 2009).

This thesis extends the current literature by providing robust extensions for the GWEBLUP of the area mean. In particular, a robust projective and a robust predictive version of the GWEBLUP is proposed. In addition, two analytic MSE estimates are developed based on the pseudo-linearization approach of Chambers et al. (2011) and under the full linearization approach of Chambers et al. (2014). The proposed methods have been implemented for the R-language (R Core Team, 2016) in the package `saeRGW`. The performance of the proposed methods is assessed in model and design-based simulation studies. The model-based simulation indicates that in the presence of spatial non-stationarity and outliers, applying the proposed robust methods can lead to efficiency gains compared to the non-robust GWEBLUP of the small area mean. In addition, the proposed MSE estimators show good properties in terms of bias and stability in the investigated scenarios. The design-based simulation also indicates that in case of the Berlin real estate database it can be beneficial to combine the GWEBLUP with robust protection for estimating small area means of the quoted net rent.

Zusammenfassung

Small-Area-Verfahren gewinnen zunehmend an Bedeutung in der amtlichen Statistik und privaten Institutionen. Bevölkerungsstichproben liefern in der Regel verlässliche Schätzungen für aggregierte Populationsgrößen, können jedoch sehr kleine Stichprobenumfänge in tieferen regionalen Ebenen aufweisen. Direkte Schätzverfahren, welche ausschließlich auf regionalen Beobachtungen basieren, liefern bei kleinem Stichprobenumfang zwar unverzerrte, jedoch meist sehr unpräzise Schätzungen. Methoden der Small-Area-Schätzung (SAE - Small Area Estimation) hingegen können bei kleinen Stichprobenumfängen durch Zuhilfenahme zusätzlicher Informationen verlässlichere Regionalstatistiken liefern. Eine gängige Methode für SAE ist der Empirical Best Linear Predictor (EBLUP). Hier werden für die Schätzung von kleinräumigen Mittelwerten Hilfsvariablen mithilfe gemischter linearer Modelle einbezogen. Darüber hinaus können auch räumliche Informationen verwendet werden, um die Schätzgenauigkeit weiter zu erhöhen. Geographische Information können zum Beispiel berücksichtigt werden, indem eine räumliche Korrelation zwischen den zufälligen Effekten im gemischten Modell zugelassen wird (vgl. Pratesi and Salvati, 2008, SEBLUP). Außerdem können räumliche Informationen auch nichtparametrisch in das gemischte Modell aufgenommen werden (vgl. Opsomer et al., 2008, NPEBLUP). Eine weitere Option ist die geographisch gewichtete Erweiterung des gemischten Modells, in der die Koeffizienten räumlich variieren (cf. Chandra et al., 2012, GWEBLUP). Unter der Annahme normalverteilter Fehlerterme sind diese Ansätze effizient um detaillierte Regionalstatistiken zu Schätzen. Diese Normalverteilungsannahme kann jedoch durch einzelne extreme Beobachtungen (Ausreißer) verletzt werden. Der Einfluss von Ausreißern kann durch die Verwendung von robusten SAE- Methoden reduziert werden (vgl. Sinha and Rao, 2009).

In dieser Arbeit wird eine Erweiterung für den GWEBLUP zur robusten Schätzung von kleinräumigen Mittelwerten entwickelt. Hierbei werden eine robuste Variante nach Sinha and Rao (2009) und eine korrigierte nach Chambers et al. (2014) eingeführt. Darüber hinaus werden zwei analytische Methoden für die MSE-Schätzungen entwickelt: (i) nach dem Pseudo-Linearisierungsansatz von Chambers et al. (2011); (ii) und nach dem vollständigen Linearisierungsansatz von Chambers et al. (2014). Die entwickelten Methoden wurden zusätzlich im R-Paket `saeRGW` implementiert. Die statistischen Eigenschaften der entwickelten Methoden wurden in modell- und designbasierten Simulationsstudien untersucht. Unter räumlicher nicht-stationarität mit Ausreißern führen die entwickelten robusten Methoden in der modellbasierten Simulation zu Effizienzgewinnen. Darüber hinaus zeigt die MSE Schätzung in den untersuchten Szenarien gute Eigenschaften in Bezug auf Verzerrung und Stabilität. Die Ergebnisse der designbasierten Simulation deuten ebenfalls auf einen Effizienzgewinn durch die Verwendung der entwickelten Methoden bei der Schätzung der mittleren Nettokaltmieten in Planungsräumen der Stadt Berlin hin.

Publications

Some ideas and figures have appeared previously in the following peer reviewed article:

Baldermann, C., N. Salvati and T. Schmid (2017). Robust small area estimation under spatial non-stationarity. International Statistical Review. forthcoming.

The results of this article are partly the subject matter of the Chapters 5 and 6 in this thesis.

Declaration

I certify that this work contains no material which has been accepted for the award of any other degree or diploma in my name, in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.

Berlin, July 2017

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