Free boundary problems governed by mean curvature

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Contents

Abstract

In this thesis we consider the following three free boundary value problems for (hyper-)surfaces that are governed by the mean curvature of the (hyper-)surface:

1. *A monotonicity formula for free boundary surfaces with respect to the unit ball*

   We prove a monotonicity identity for compact surfaces with free boundaries inside the boundary of the unit ball in $\mathbb{R}^n$ that have square integrable mean curvature. As one consequence we obtain a Li-Yau type inequality in this setting, thereby generalizing results of Oliveira and Soret [RV95, Proposition 3], and Fraser and Schoen [FS11, Theorem 5.4]. Then we derive some sharp geometric inequalities for compact surfaces with free boundaries inside arbitrary orientable support surfaces of class C^2. Furthermore, we obtain a sharp lower bound for the L^1-tangent-point energy of closed curves in $\mathbb{R}^3$ thereby answering a question raised by Strzelecki, Szumańska, and von der Mosel [SSvdM13].

2. *Relative isoperimetric properties of asymptotically flat support surfaces*

   We define a notion of mass for asymptotically flat hypersurfaces S of euclidean space and prove a positive mass theorem in all dimensions. Then we establish a free boundary version of an obstruction discovered by Schoen and Yau in their proof of the positive mass theorem [SY79b], and refined by Eichmair and Metzger [EM12], and very recently by Carlotto [Car14]: positive mean curvature of $S \subset \mathbb{R}^3$ is not compatible with the existence of (certain) stable free boundary minimal surfaces. We then use this to prove that given a compact set K of $\mathbb{R}^3$, all volume-preserving stable free boundary constant mean curvature surfaces with respect to S of sufficiently large boundary length will avoid K, thereby obtaining a free boundary version of the main result in [EM12]. Finally, inspired by ideas of Eichmair and Metzger [EM13b] we prove the existence of arbitrarily large isoperimetric regions relative to S.

3. *Weak solutions of nonlinear mean curvature flow with Neumann boundary condition*

   We propose a new flow approach to obtain relative isoperimetric inequalities. As a first step in this program we develop a weak level set formulation for mean curvature flow and positive powers of mean curvature flow with Neumann boundary condition. We prove the existence of weak solution under natural conditions on the supporting surface and derive some properties for the evolving surfaces. The case of surfaces without boundary has been treated by Schulze [Sch08].

Introduction

Geometric calculus of variations deals with the question of the existence of optimal geometric objects and their properties. One of the oldest problems in geometric calculus of variations, dating back to Greek antiquity, is the so-called isoperimetric problem. The problem can be stated as follows: "How much n-dimensional area is needed to bound a given $(n+1)$-dimensional volume in a given $(n+1)$-dimensional ambient space?"

This interesting mathematical problem also serves as a model problem for other geometric variational problems (with constraints). Moreover, the isoperimetric problem has strong relevance in physics. Not only do optimizers of this problem serve as a model for the description of soap bubbles, but the isoperimetric behavior of asymptotically flat 3-manifolds is strongly related to the concept of mass in the theory of general relativity [Bra97,Hui06,Hui09,EM13b].

The critical points of the isoperimetric problem, i.e. critical points of the area functional under a volume constraint, satisfy the nice property that they have constant mean curvature, and are thus referred to as constant mean curvature (CMC) hypersurfaces. Critical points of the area functional without a volume constraint, which serve as a model for soap films spanned by a wire or as the black hole horizons of initial data sets in general relativity, are called minimal hypersurfaces since they locally minimize area. Understanding minimal and CMC hypersurfaces not only helps to find and understand optimal shapes for the isoperimetric problem but these surfaces can also be used to foliate asymptotically flat manifolds in a geometrically natural way leading to concepts such as the center of mass of an initial data set in general relativity [HY96,Ye96,Met07,QT07,Hua10,Ma11,EM13b,Ner14]. Moreover, minimal hypersurfaces have been successfully employed to solve other important mathematical problems in geometry such as the positive mass theorem [SY79b] and the Willmore conjecture [MN14].

The *relative* isoperimetric problem, also known as *partitioning problem*, or *Dido's problem*, after Dido, Queen of Carthage, is the problem of minimizing *relative* area subject to a volume constraint inside a fixed domain G (open, connected) with non-empty boundary (or more generally, a Riemannian manifold with boundary). I.e.

$$\text{minimize area}(\partial\Omega \cap G) \text{ in the class of sets } \Omega \subset G \text{ with } \mathrm{vol}(\Omega) = V.$$

Note that we do not account for the area of $\partial\Omega \cap \partial G$ as G is open. Optimizers are used to model liquid drops under negligible gravitation and adhesion effects. Identifying optimizers for the relative isoperimetric problem, also known as *relative isoperimetric domains*, remains mainly unsolved. So far, they have been explicitly characterized only for very few domains G. E.g. euclidean balls by Bokowski and Sperner [BS79], and Almgren [Alm87], slabs by Athanassenas [Ath87], Vogel [Vog87], and Pedrosa and Ritoré [PR99], solid cones by Lions and Pacella [LP90], and Ritoré and Rosales [RR04], and recently convex solid cylinders (under the condition that the prescribed volume is sufficiently large) by Ritoré and Vernadakis [RV14].

Existence of optimizers inside bounded domains easily follows from standard compactness results for functions of bounded variation. Local interior regularity of their relative boundary was first established by Giusti [Giu81], see also Gonzalez, Massari, and Tamanini [GMT83]. Local boundary regularity was proved by Grüter [Grü87]. Fall [Fal10] proved that relative isoperimetric domains inside bounded domains G concentrate along points of maximal mean curvature of ∂G as their volume tends to zero.

While existence of optimizers inside bounded domains is always ensured, relative isoperimetric regions inside non-compact domains need not exist in general as minimizing sequences can drift off to infinity. However, the *relative isoperimetric profile* I_G of a domain G is always well defined by

$$I_G(V) := \inf\{\mathrm{area}(\partial\Omega \cap G) : \Omega \subset G, \mathrm{vol}(\Omega) = V\}, \quad 0 < V < \mathrm{vol}(G)/2.$$

Sternberg and Zumbrun [SZ99] proved concavity of the function I_G for bounded convex domains $G \subset \mathbb{R}^{n+1}$ and concluded geometrical and topological consequences for relative isoperimetric domains. Kuwert [Kuw03] observed that in fact in this case the renormalized profile $I_G^{(n+1)/n}$ is concave. Bayle and Rosales [BR05] derived more general second order differential inequalities for I_G for convex domains G inside Riemannian manifolds and proved sharp comparison theorem for convex bodies.

A strongly related question, which is sometimes equivalent to Dido's problem, is the search for optimal inequalities that bound enclosed volume by relative boundary area of sets inside a given domain; so called *relative isoperimetric inequalities*. In other words, this problem consists of computing the number

$$\inf\left\{\frac{I_G(V)^{\frac{n+1}{n}}}{V} : 0 < V < \mathrm{vol}(G)/2\right\}.$$

Such inequalities (in cases optimizers are not explicitly known) have been obtained by Choe, Ghomi, and Ritoré [CGR07] (see also [CR07]) for sets in the complement of convex sets. A key ingredient in their proof is to find sharp lower bounds on the maximum mean curvature of the relative boundary of certain subsets in the complement of convex sets, as these in turn yield a lower bound on the first derivative of the relative isoperimetric profile.

Critical points of Dido's problem have relative boundaries with constant mean curvature, meeting the boundary of the domain orthogonally. These hypersurfaces are referred to as *free boundary constant mean curvature hypersurfaces*. Volume-preserving stable CMC hypersurfaces with free boundaries, i.e. critical points of Dido's problem with non-negative second variation, have been characterized in very few special cases only, see e.g. [Ath87, Vog87, RV95, PR99, RR04]. We also refer to [Ros05] for an overview of known results about volume-preserving stable free boundary CMC hypersurfaces inside convex domains.

Critical points of the area functional in the class of hypersurfaces inside Riemannian manifolds G the boundaries of which are confined to lie inside the boundary of G *without* a volume constraint are known as *free boundary minimal (hyper-)surfaces*, and have already been studied for a long time with first existence results going back to Courant [Cou40]. We refer to [DHS10, DHT10b, DHT10a] for an almost up to date historical account on the subject.

Very recently, there has been great interest in the study of free boundary minimal hypersurfaces inside compact Riemannian manifolds with boundary. Exemplarily, we mention the works of Fraser [Fra00, Fra02], Fraser and Schoen [FS11, FS13b, FS13a], Brendle [Bre12], Ambrozio [Amb13], Li [Li14b, Li14a], Fraser and Li [FL14], and Maximo, Nunes, and Smith [MNS13].

Of particular interest is the interplay between curvature of the boundary manifold and the free boundary minimal hypersurfaces via the second variation of area. For example, as can be seen by a simple calculation involving the second variation, there are no compact immersed stable free boundary minimal hypersurfaces inside a strictly convex domain. This fact stands in analogy to the fact that Riemannian manifolds of strictly positive Ricci curvature do not contain closed immersed stable minimal hypersurfaces. Moreover, it was observed by Ros [Ros08, Proposition 2] that every immersed stable free boundary minimal surface inside a bounded mean convex domain is a topological disk, which parallels the result of Schoen and Yau [SY79a, Theorem 5.1] that every immersed stable minimal surface inside a closed Riemannian manifold of positive scalar curvature is a topological sphere. This structural analogy between boundaryless Riemannian manifolds, positive Ricci curvature, positive scalar curvature, and minimal surfaces on the one hand, and domains in euclidean space, strict convexity, strict mean convexity, and free boundary minimal surfaces on the other hand, goes in fact much further and is also reflected in results about the relative isoperimetric problem or about free boundary CMC surfaces. As another example we mention the analogy between closed minimal hypersurfaces in the standard sphere, and free boundary minimal hypersurfaces in the unit ball: Almgren [Alm66] showed that the equator is the only immersed minimal surface in $\mathbb{S}^3$ of genus zero (up to congruences), whereas Nitsche [Nit85] could show that the flat unit disk is the only immersed free boundary minimal disk inside the unit ball B of $\mathbb{R}^3$ (up to congruences). The recent proof by Brendle [Bre13] of the Lawson conjecture that the only properly embedded minimal torus inside $\mathbb{S}^3$ is the Clifford torus (up to congruences) suggestes the conjecture by Fraser and Li [FL14] that the only properly embedded free boundary minimal annulus inside the unit ball B of $\mathbb{R}^3$ is the critical catenoid (up to congruences), which however is still unsolved.

The L^2-gradient flow of the area functional for free boundary hypersurfaces is the so-called *mean curvature flow with Neumann boundary condition* (sometimes also called mean curvature flow with Neumann *free* boundary condition).

Stahl [Sta96b] proved short-time existence and uniqueness of solutions on a maximal time interval, and showed that in case this interval is bounded the curvature of the evolving hypersurfaces blows up as time approaches the final existence time. Moreover, Stahl [Sta96a] could show that under mean curvature flow with Neumann boundary condition, convex free boundary hypersurfaces end up in a Type I singularity, and become asymptotically hemispherical after rescaling. Buckland [Buc05] derived a monotonicity formula for mean curvature flow with Neumann boundary condition and was able to classify the boundary singularities for mean convex evolving hypersurfaces. Koeller [Koe12] established a local monotonicity formula for mean curvature flow with Neumann boundary condition and proved estimates on the size of the singular set under certain regularity assumptions on the flow.

The non-parametric mean curvature flow of graphs with orthogonal contact angle on cylindrical domains had been studied earlier by Huisken [Hui89]. More recent results for the graphical mean curvature flow with Neumann boundary condition were obtained by Wheeler [Whe14b, Whe14a].

The L^2-gradient flow for Dido's problem is the *volume-preserving mean curvature flow with Neumann (free) boundary condition.* First results were obtained by Athanassenas [Ath97, Ath03] for rotationally symmetric surfaces with free boundaries inside two parallel planes, and by Athanassenas and Kandanaarachchi [AK12] for rotationally symmetric surfaces with free boundary inside a plane. Very recently, Mäder-Baumdicker [MB14] proved a monotonicity formula for the volume-preserving mean curvature flow with Neumann boundary condition. For the evolution of curves she could show longtime existence and convergence to a circular arc for certain initial configurations.

Outline of main results

A monotonicity formula for free boundary surfaces with respect to the unit ball

In Chapter 1 we consider compact *free boundary surfaces with respect to the unit ball B in* $\mathbb{R}^n$, i.e. compact surfaces $\Sigma \subset \mathbb{R}^n$, the boundaries $\partial\Sigma \neq \emptyset$ of which meet the boundary ∂B of the unit ball B orthogonally.

The main result of this chapter is a monotonicity identity for these surfaces (see Theorem 1.1), which is analogous to Simon's monotonicity identity [Sim93] for closed surfaces. Inspired by the interpretation of Simon's test vector field, a desingularized-cut-off version of the vector field $Y(x) = (x - x_0)|x - x_0|^{-2}$, as the gradient of the Newtonian potential of $\mathbb{R}^2$ evaluated in $\mathbb{R}^n$, the main idea of the proof is to test the first variations identity with a desingularized-cut-off version of the gradient of the Neumann Green's function of the Laplacian with respect to the unit disk in $\mathbb{R}^2$ evaluated in $\mathbb{R}^n$.

As a consequence we obtain area bounds, and as a limiting case of the monotonicity identity we obtain the following theorem.

Theorem 0.1. *For any $F : \Sigma \to \mathbb{R}^n$, immersed, compact free boundary surface with respect to the unit ball in $\mathbb{R}^n$, we have*

$$\theta_{max} \leq \frac{1}{8\pi} \int_\Sigma |\vec{H}|^2 \, d\mathcal{H}^2_{F^*\delta} + \frac{1}{2\pi} \int_{\partial\Sigma} x \cdot \nu_{\partial\Sigma} \, d\mathcal{H}^1_{F^*\delta} =^{def} \frac{1}{2\pi} \mathcal{W}(F),$$

where θ_{max} denotes the maximal multiplicity of $F(\Sigma)$. In particular

$$W(F) \geq 2\pi, \tag{0.1}$$

and if

$$W(F) < 4\pi,$$

then F is an embedding. Moreover, equality in (0.1) *implies that $F(\Sigma)$ is a round spherical cap or a flat unit disk.*

This theorem can be seen as a generalization of a sharp isoperimetric inequality for free boundary minimal surfaces with respect to the unit ball in $\mathbb{R}^n$ due to Fraser and Schoen [FS11, Theorem 5.4] (see also Brendle [Bre12]) to not necessarily *minimal* surfaces.

We also prove the following geometric inequalities:

Proposition 0.2. *Let $\Omega \subset \mathbb{R}^n$ be convex set such that $h^{\partial\Omega}_{ij} \leq k\,\delta_{ij}$. Then for every compact free boundary surface Σ that meets $\partial\Omega$ from the inside we have*

$$2\pi \leq \frac{1}{4} \int_\Sigma |\vec{H}|^2 \, d\mathcal{H}^2 + k\, \mathcal{H}^1(\partial\Sigma).$$

Let $\Omega \subset \mathbb{R}^n$ be convex set such that $h_{ij}^{\partial\Omega} \geq k\,\delta_{ij}$. Then for every compact free boundary surface Σ that meets $\partial\Omega$ from the outside we have

$$2\pi \leq \frac{1}{4}\int_\Sigma |\vec{H}|^2\,d\mathcal{H}^2 - k\,\mathcal{H}^1(\partial\Sigma).$$

Moreover, equality holds if and only if Σ is a spherical cap or a flat unit disk.

Using a new observation we are able to prove the following proposition.

Proposition 0.3. *Let Γ be a closed curve in $\mathbb{R}^3$ of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$. Then*

$$2\pi\mathrm{length}(\Gamma) \leq \int_\Gamma\int_\Gamma \frac{2\mathrm{dist}(x-y, T_x\Gamma)}{|x-y|^2}\,d\mathcal{H}^1(x)\,d\mathcal{H}^1(y),$$

with equality only if Γ is a planar, convex curve.

Proposition 0.3 confirms a conjecture by Strzelecki, Szumańska, and von der Mosel [SSvdM13].

Relative isoperimetric properties of asymptotically flat support surfaces

In Chapter 2 we investigate relative isoperimetric properties of a certain class of non-compact hypersurfaces. Inspired by an analogy with asymptotically flat Riemannian manifolds we define a class of hypersurfaces called *asymptotically flat hypersurfaces*, which outside a compact set can be written as a graph of a function u which has a controlled decay of its first and second derivatives at infinity.

Similar to the ADM mass of asymptotically flat Riemannian manifolds we can assign a number to these asymptotically flat hypersurfaces. We define the *extrinsic mass* $m(S)$ of an asymptotically flat hypersurface S in $\mathbb{R}^{n+1}$ by

$$m(S) := \lim_{r\to\infty}\frac{1}{\omega_{n-1}}\int_{\partial B_r^n(0)}\frac{\partial u}{\partial\nu}\,d\mathcal{H}^{n-1},$$

where ν denotes the euclidean outward unit normal to $B_r^n(0) \subset \mathbb{R}^n$, and where $\omega_{n-1} = \mathrm{area}(\partial B_1^n(0))$. The extrinsic mass is a well defined geometric quantity (see Proposition 2.6).

An interesting subclass of asymptotically flat hypersurfaces are *asymptotically catenoidal* hypersurfaces which have the defining property that

$$u(x) = a + \phi_M(r) + \mathcal{O}(r^{-n+1}) \quad \text{as } r = |x| \to \infty,$$

where

$$\phi_M(r) = \begin{cases} M\log(r) & ,n=2 \\ -\frac{M}{(n-2)r^{n-2}} & ,n\geq 3, \end{cases}$$

for some constants $M, a \in \mathbb{R}$, and where we require this expansion to hold up to and including second order derivatives.

We then define a class of hypersurfaces that we call *exterior hypersurfaces*, which are simply asymptotically flat hypersurfaces with some extra condition in case their boundary is non-empty. Using a maximum principle argument we obtain a positive mass theorem:

Theorem 0.4. (Positive Mass Theorem) *Let $S \subset \mathbb{R}^{n+1}$ be an asymptotically catenoidal exterior hypersurface of non-negative mean curvature. Then*

$$m(S) \geq 0,$$

with equality if and only if S is a hyperplane.

Guided by this analogy we adapt methods by Eichmair-Metzger [EM13b] to prove the partial solvability of Dido's problem in this context:

Theorem 0.5. *Let $S = \partial G$ be an asymptotically catenoidal exterior hypersurface of $\mathbb{R}^{n+1}$. There exists a sequence of relative isoperimetric regions $\Omega_i \subset G$ with $\mathcal{L}^n(\Omega_i) \to \infty$.*

Inspired by the minimal surface proof of Schoen-Yau's positive mass theorem [SY79b] and works of Eichmair-Metzger [EM12] we prove the following results for free boundary surfaces in dimension 3. See also the recent results of Carlotto [Car14].

For non-compact stable free boundary minimal surfaces we have the following rigidity theorem:

Theorem 0.6. *Let S be an asymptotically catenoidal exterior surface in $\mathbb{R}^3$ such that S has non-negative mean curvature. Let Σ be a complete, non-compact, properly embedded, stable free boundary minimal surface with respect to S. Then S is a plane and Σ is a half-plane meeting S orthogonally.*

Moreover, we prove that volume-preserving stable free boundary CMC surfaces of sufficiently large boundary length must be outlying, i.e. avoid a given compact region:

Theorem 0.7. *Let S be an asymptotically catenoidal exterior surface in $\mathbb{R}^3$ of non-negative mean curvature that is not a plane. For every compact set $K \subset \mathbb{R}^3$ and every $\Theta > 0$ there exists a constant $L = L(S, \Theta, K) > 0$ with the following property:*

Let Σ be a connected, compact volume-preserving stable free boundary constant mean curvature surface with respect to S with $\mathcal{H}^2(\Sigma \cap B_\sigma) \leq \Theta\,\sigma^2$ for all $\sigma \geq 1$ and with $\mathcal{H}^1(\partial\Sigma) \geq L$. Then $\Sigma \cap K = \emptyset$.

Nonlinear mean curvature flow with Neumann boundary condition

In Chapter 3 we consider a new geometric flow called *nonlinear mean curvature flow with Neumann boundary condition*. It deforms a given initial hypersurface $M \setminus \partial M \subset G$ of $\mathbb{R}^{n+1}$ with free boundary ∂M inside a given support surface $S = \partial G$ along its normal direction with speed given by a positive power $k > 0$ of its mean curvature H, while maintaining an orthogonal contact angle along the evolution. In technical terms, one tries to find a family of immersions $\{F_t : M \to \mathbb{R}^{n+1}\}_{t \in [0,t)}$ that satisfies the following system of equations

$$(\star)\begin{cases} \frac{d}{dt}F(p,t) = -H(p,t)^k\,\nu(p,t), & (p,t) \in M \times (0,T), \\ F(p,0) = \mathrm{id}, & p \in M \\ F(p,t) \in S, & (p,t) \in \partial M \times [0,T), \\ \langle \nu, \gamma \circ F\rangle(p,t) = 0, & (p,t) \in \partial M \times [0,T), \\ F(p,t) \in G, & (p,t) \in (M \setminus \partial M) \times [0,T). \end{cases}$$

Here $H(\cdot,t)$ denotes the mean curvature and $\nu(\cdot,t)$ denotes a unit normal field of the immersion F_t.

As mentioned in the introduction the case $k = 1$, i.e. mean curvature flow with Neumann boundary condition has been studied in the classical setting by Stahl [Sta96b, Sta96a], Buckland [Buc05] and Koeller [Koe10, Koe12], and by Huisken [Hui89] and Wheeler [Whe14b, Whe14a] in the graphical case.

Even though smooth solutions under special geometric assumptions exist until the enclosed volume goes to zero (cf. [Sta96a]), singularities in general may occur in the interior as well as on the supporting hypersurface (cf. [Koe10]) before the enclosed volume vanishes. In order to continue the flow past those singularities we replace $(\star)$ by the following level set formulation.

Here we assume that M is the closure of the relative boundary $\partial\Omega \cap G$ of some bounded domain $\Omega \subset G$, and that the evolving hypersurfaces are then given by the relative boundaries of the superlevel sets of a function $u : \overline{\Omega} \to \mathbb{R}_{\geq 0}$, $u = 0$ on $\partial^{\mathcal{D}}\Omega := \overline{\partial\Omega \cap G}$ via

$$M_t = \overline{\partial\{x \in \Omega : u(x) > t\} \cap G}.$$

The system $(\star)$ is then replaced by the following degenerate elliptic mixed boundary value problem.

$$(\star\star) \begin{cases} \operatorname{div}\left(\frac{Du}{|Du|}\right) = -|Du|^{-\frac{1}{k}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial^{\mathcal{D}}\Omega \text{ and} \\ \frac{\partial u}{\partial \gamma} := \gamma^i \, D_i u = 0 & \text{on } \partial^{\mathcal{N}}\Omega, \end{cases}$$

where $\partial^{\mathcal{N}}\Omega := \partial\Omega \setminus \partial^{\mathcal{D}}\Omega$ and where γ denotes the outward unit normal to S. This formulation is inspired by the work of Schulze [Sch08] for the H^k-flow, which in turn was inspired by the work of Evans-Spruck [ES91] and Chen-Giga-Goto [CGG89] on mean curvature flow and by work of Huisken-Ilmanen [HI01] on the inverse mean curvature flow. A level set formulation for inverse mean curvature flow with Neumann boundary condition was put forward independently by Marquardt [Mar12].

In this setting using the so called elliptic regularization we obtain the following result:

Theorem 0.8. *Let $G \subset \mathbb{R}^{n+1}$ be a smooth domain and let $\Omega \subset G$ be such that $\partial^{\mathcal{D}}\Omega =^{def} \overline{\partial\Omega \cap G}$ is a smooth strictly mean convex free boundary hypersurface with respect to $S := \partial G$. If $(\star\star)$ admits a supersolution, then $(\star\star)$ has a weak solution $u \in C^{0,1}(\overline{\Omega})$. Moreover, the superlevel sets $\{u > t\}$ are minimizing area from the outside relative to G.*

A sufficient condition to ensure the existence of a supersolution of $(\star\star)$ is that $\partial^{\mathcal{N}}\Omega$ is graphical over part of a sphere (see Lemma 3.5).

Notation

Let $n \geq 2$ be an integer. Unless otherwise stated submanifolds of euclidean space will always be smooth, complete, properly embedded, orientable, two-sided, and possibly have non-empty boundary. As usual 2-dimensional submanifolds are called surfaces. Manifolds of dimension n and hypersurfaces in $\mathbb{R}^{n+1}$ will usually be denoted by M. Manifolds of dimension 2 and surfaces in euclidian space (of higher codimension) will be denoted by Σ.

The *scalar second fundamental* form $A = \{h_{ij}\}$ of a hypersurface $M \subset (N^{n+1}, g)$ with outward unit normal ν is the symmetric 2-tensor that is given by

$$h_{ij} = g(D_{e_i}\nu, e_j),$$

where $\{e_i\}_{i=1,\dots,n}$ denotes a local orthonormal frame for TM, and where D denotes the Levi-Civita connection of (N, g). The *mean curvature* H of M is defined to be the trace of A, i.e. $H = \operatorname{div}_M(\nu)$. For a surface $\Sigma \subset \mathbb{R}^3$ with non-empty boundary $\partial\Sigma$ the *geodesic curvature* κ_g of $\partial\Sigma$ is defined to be the mean curvature of $\partial\Sigma$ as a submanifold of Σ.

A *free boundary (hyper-)surface* $M \subset \mathbb{R}^{n+1}$ with respect to a *support hypersurface* $S \subset \mathbb{R}^{n+1}$, is a (hyper-)surface in $\mathbb{R}^{n+1}$ the boundary ∂M of which is a subset of S and such that M and S meet orthogonally. In case $S = \partial G$ for a domain $G \subset \mathbb{R}^{n+1}$ we sometimes say that M is a free boundary (hyper-)surface with respect to G. We will always denote by ν the *outward unit normal* of a free boundary hypersurface M, η will always denote its *outward unit conormal.* The *outward unit normal of a support surface* S will always be denoted by γ. Its mean curvature and second fundamental form will be denoted by H_S and A_S, respectively.

We remark here that in the different chapters and certain sections we will make certain additional assumptions on what we mean by a free boundary (hyper-)surface. E.g. that $\Sigma \subset \overline{G}$ such that $\Sigma \cap \overline{G} = \partial\Sigma$.

For a set $A \subset \mathbb{R}^{n+1}$ we denote by π_A the *nearest point projection* onto the set A wherever it is well defined. The *distance function* of A will be denoted by d_A, i.e. $d_A(x) := \inf\{|x - a| : a \in A\}$. For an open set A and a positive number $\delta > 0$ we let $A_\delta := \{x \in A : d_{\partial A}(x) > \delta\}$.

For sets $A_1, A_2 \subset \mathbb{R}^{n+1}$ we set $\partial_{A_2} A_1 := \partial A_1 \cap A_2$.

The letter $x = (x^1, ..., x^{n+1})$ will be used to denote a point in $\mathbb{R}^{n+1}$ and also to denote the position vector in $\mathbb{R}^{n+1}$, depending on the context.

We will often identify, without further mentioning, the hyperplane $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ with the space $\mathbb{R}^n$. For $\omega \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$ we set $\mathbb{R}^n_\omega := \{x \in \mathbb{R}^{n+1} : x \cdot \omega = 0\}$.

Numerical constants will be denoted by capital C. In case we want to emphasise their dependencies we write them in brackets. E.g. $C(\alpha, p)$ denotes a constant that depends on α and p.

Universal constants will be denoted by a small c. All these constants may vary from line to line. Reappearing specific constant will be numbered with positive integers in their subscript. These numberings are however only valid in their respective chapters.

In Chapter 2 we will frequently have to choose an initial radius large enough. We will denote radii of this kind by σ_0 and emphasise their dependencies in brackets. These radii may change from line to line. Reappearing initial radii will be denoted by a capital R and will be numbered with positive integers in their subscript.

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1 A monotonicity formula for free boundary surfaces with respect to the unit ball

The main goal of this chapter is to establish a monotonicity formula for compact free boundary surfaces (unless otherwise stated this means 2-dimensional, smooth, embedded) with respect to the unit ball in $\mathbb{R}^n$. The corresponding result for closed, i.e. compact and boundaryless, surfaces was proved by Simon [Sim93]. (See also Kuwert and Schätzle [KS04] for a generalization to integer rectifiable 2-varifolds with square integrable generalized mean curvature.) For a closed surface Σ, and radii $0 < \sigma < \rho < \infty$ Simon's monotonicity identity reads as follows.

$$g_{x_0}(\rho) - g_{x_0}(\sigma) = \frac{1}{\pi}\int_{\Sigma\cap B_\rho(x_0)\setminus B_\sigma(x_0)} \left|\frac{1}{4}\vec{H} + \frac{(x-x_0)^\perp}{|x-x_0|^2}\right|^2 d\mathcal{H}^2,$$

where

$$g_{x_0}(r) := \frac{\mathcal{H}^2(\Sigma\cap B_r(x_0))}{\pi r^2} + \frac{1}{16\pi}\int_{\Sigma\cap B_r(x_0)} |\vec{H}|^2\, d\mathcal{H}^2 + \frac{1}{2\pi r^2}\int_{\Sigma\cap B_r(x_0)} \vec{H}\cdot(x-x_0)\, d\mathcal{H}^2.$$

This monotonicity formula plays an important role in the existence proof of surfaces minimizing the Willmore functional [Sim93]. It also yields an alternative proof of the so called Li-Yau inequality [LY82]. Very recently, Lamm and Schätzle [LS14] used it to establish a quantitative version of Codazzi's theorem, thereby extending results of De Lellis and Müller [DLM05, DLM06] to arbitrary codimension.

In this chapter we prove a monotonicity identity for compact free boundary surfaces with respect to the unit ball in $\mathbb{R}^n$, i.e. compact surfaces with non-empty boundary meeting the boundary of the unit ball orthogonally. In fact, our results hold in the varifold context (see Section 1.1 for the precise assumptions).

As a consequence we obtain area bounds, and the existence of the density at *every* point on the surface. As a limiting case of the monotonicity identity we obtain the Li-Yau type inequality

$$2\pi\theta_{max} \leq \frac{1}{4}\int_\Sigma |\vec{H}|^2\, d\mathcal{H}^2 + \int_{\partial\Sigma} x\cdot\eta\, d\mathcal{H}^1, \tag{1.1}$$

where θ_{max} denotes the maximal multiplicity of the surface Σ (see Theorem 1.5).

A special case of (1.1) (for free boundary CMC surfaces inside the unit ball in $\mathbb{R}^3$) has appeared in a work of Ros and Vergasta [RV95, Proposition 3], attributing the result to Oliveira and Soret. The proof given in [RV95] seems to also work for any compact free boundary surface with respect to the unit ball in $\mathbb{R}^n$. Unaware of this result Fraser and Schoen independently established the inequality for free boundary minimal surfaces inside the unit ball in $\mathbb{R}^n$ (see [FS11, Theorem 5.4]). In this context we also mention the work of Brendle [Bre12] in which the author generalizes the inequality [FS11, Theorem 5.4] to higher-dimensional free boundary minimal surfaces inside the unit ball in $\mathbb{R}^n$.

The chapter is organized as follows. In Section 1.1 we introduce the notation and describe the setting we work in. In Section 1.2 we establish the monotonicity formula (Theorem 1.1) and prove the existence of the density (Theorem 1.4). In Section 1.3 we give some geometric applications that follow from the results of Section 1.2. Finally, in Section 1.4 we prove sharp geometric inequalities for compact free boundary surfaces with respect to arbitrary orientable support surfaces of class C^2. We also include a sharp lower bound for the L^1-tangent-point energy of closed curves in $\mathbb{R}^3$.

I would like to thank Dr. Simon Blatt for bringing the paper [SSvdM13] to my attention.

1.1 The setting

We use essentially the same notation as in [KS04]. Unless stated otherwise we assume that μ is an integer rectifiable 2-varifold in $\mathbb{R}^n$ of compact support $\Sigma := \operatorname{spt}(\mu)$, $\Sigma \cap \partial B \neq \emptyset$, with generalized mean curvature $\vec{H} \in L^2(\mu; \mathbb{R}^n)$ such that

$$\int \operatorname{div}_\Sigma X \, d\mu = -\int \vec{H} \cdot X \, d\mu \tag{1.2}$$

for all vector fields $X \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ with $X \cdot \gamma = 0$ on ∂B, where $\gamma(x) = x$ denotes the outward unit normal to B (the open unit ball in $\mathbb{R}^n$). Furthermore, we assume that $\mu(\partial B) = 0$.

It follows from the work of Grüter and Jost [GJ86b] that μ has bounded first variation $\delta\mu$. Hence, by Lebesgue's decomposition theorem there exists a Radon measure $\sigma = |\delta\mu| \llcorner Z$ ($Z = \{x \in \mathbb{R}^n : D_\mu |\delta\mu|(x) = +\infty\}$) and a vector field $\eta \in L^1(\sigma; \mathbb{R}^n)$ with $|\eta| = 1$ σ-a.e. such that

$$\delta\mu(X) =^{def} \int \operatorname{div}_\Sigma X \, d\mu = -\int \vec{H} \cdot X \, d\mu + \int X \cdot \eta \, d\sigma \tag{1.3}$$

for all $X \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$. It easily follows from (1.2) that

$$\operatorname{spt}(\sigma) \subset \partial B \quad \text{and} \quad \eta \in \{\pm\gamma\} \ \ \sigma\text{-a.e.}.$$

We shall henceforth refer to such varifolds μ as compact *free boundary varifolds* (with respect to the unit ball).

In case μ is given by a smooth embedded surface Σ (i.e. $\mu = \mathcal{H}^2 \llcorner \Sigma$) η is the outward unit conormal to Σ and $\sigma = \mathcal{H}^1 \llcorner \partial\Sigma$, and we say that Σ is a compact free boundary surface (with respect to the unit ball).

Note that since Σ is compact we may use the position vector field as a test function to obtain

$$2\mu(\mathbb{R}^n) = -\int \vec{H} \cdot x \, d\mu + \int x \cdot \eta \, d\sigma. \tag{1.4}$$

1.2 The monotonicity formula

The following monotonicity identity is the free boundary analogue of the monotonicity identity [Sim93, (1.2)], [KS04, (A.3)].

Theorem 1.1. (monotonicity identity) *For $x_0 \in \mathbb{R}^n$ consider the functions g_{x_0} and $\hat{g}_{x_0}$ given by*

$$g_{x_0}(r) := \frac{\mu(B_r(x_0))}{\pi r^2} + \frac{1}{16\pi}\int_{B_r(x_0)} |\vec{H}|^2\, d\mu + \frac{1}{2\pi r^2}\int_{B_r(x_0)} \vec{H}\cdot(x - x_0)\, d\mu$$

and

$$\begin{aligned}\hat{g}_{x_0}(r) &:= g_{\xi(x_0)}(r/|x_0|)\\ &\quad - \frac{1}{\pi(|x_0|^{-1}r)^2}\int_{\hat{B}_r(x_0)} (|x-\xi(x_0)|^2 + P_x(x-\xi(x_0))\cdot x)\, d\mu\\ &\quad - \frac{1}{2\pi(|x_0|^{-1}r)^2}\int_{\hat{B}_r(x_0)} \vec{H}\cdot(|x-\xi(x_0)|^2 x)\, d\mu\\ &\quad + \frac{1}{2\pi}\int_{\hat{B}_r(x_0)} \vec{H}\cdot x\, d\mu + \frac{\mu(\hat{B}_r(x_0))}{\pi},\end{aligned}$$

for $x_0 \neq 0$, and

$$\hat{g}_0(r) = -\frac{\min(r^{-2},1)}{2\pi}\int x\cdot\eta\, d\sigma.$$

Here $\xi(x) := \frac{x}{|x|^2}$ and $\hat{B}_r(x_0) = B_{r/|x_0|}(\xi(x_0))$. Then for any $0 < \sigma < \rho < \infty$ we have

$$\begin{aligned}&\frac{1}{\pi}\int_{B_\rho(x_0)\setminus B_\sigma(x_0)} \left|\frac{1}{4}\vec{H} + \frac{(x-x_0)^\perp}{|x-x_0|^2}\right|^2 d\mu\\ &\quad + \frac{1}{\pi}\int_{\hat{B}_\rho(x_0)\setminus\hat{B}_\sigma(x_0)} \left|\frac{1}{4}\vec{H} + \frac{(x-\xi(x_0))^\perp}{|x-\xi(x_0)|^2}\right|^2 d\mu\\ &\quad = (g_{x_0}(\rho) + \hat{g}_{x_0}(\rho)) - (g_{x_0}(\sigma) + \hat{g}_{x_0}(\sigma)),\end{aligned} \tag{1.5}$$

where the second integral in (1.5) is to be interpreted as 0 in case $x_0 = 0$. Here $(x-x_0)^\perp := (x-x_0) - P_x(x-x_0)$, where P_x denotes the orthogonal projection onto $T_x\mu$, the approximate tangent space of μ at x. In particular, $g + \hat{g}$ is non-decreasing.

Before we give a proof of the above theorem we note (cf. [DiB10]) that the Neumann Green's function of the disk of radius R in $\mathbb{R}^2$ is, up to a multiplicative and additive constant, given by

$$G(x,y) = \log(|x-y|) + \log\left(\frac{|x|}{R}|\xi(x)-y|\right) + \frac{1}{2R^2}|y|^2,$$

where $\xi(x) := R^2\frac{x}{|x|^2}$. We have, for $R = 1$,

$$(D_yG)(x,y) = -\frac{x-y}{|x-y|^2} - \frac{\xi(x)-y}{|\xi(x)-y|^2} - y.$$

Proof. (of the theorem) Let $x_0 \in \mathbb{R}^n$. We define

$$Y(x) := \begin{cases} \frac{x-x_0}{|x-x_0|^2} + \frac{x-\xi(x_0)}{|x-\xi(x_0)|^2} - x & ,x_0 \neq 0\\ \frac{x}{|x|^2} - x & ,x_0 = 0.\end{cases}$$

For $0 < \sigma < \rho < \infty$ we define the vector field X by

$$X(x) := X_1(x) + X_2(x), \tag{1.6}$$

where we set

$$X_1(x) := (|x-x_0|_\sigma^{-2} - \rho^{-2})^+(x-x_0)$$

and

$$X_2(x) := \begin{cases} (|x-\xi(x_0)|_{\sigma|x_0|^{-1}}^{-2} - |x_0|^2\rho^{-2})^+(x-\xi(x_0)) & \\ \quad -\sigma^{-2}\min(|x_0||x-\xi(x_0)|,\sigma)^2 x & \\ \quad +\rho^{-2}\min(|x_0||x-\xi(x_0)|,\rho)^2 x & , x_0 \neq 0 \\ -\sigma^{-2}\min(1,\sigma)^2 x + \rho^{-2}\min(1,\rho)^2 x & , x_0 = 0, \end{cases}$$

and where $|v|_\sigma := \max(|v|,\sigma)$.

First, assume that $x_0 \neq 0$. Then, we set for $r > 0$

$$\hat{B}_r(x_0) = B_{r/|x_0|}(\xi(x_0)).$$

To simplify notation, we shall write B_r and $\hat{B}_r$ instead of $B_r(x_0)$ and $\hat{B}_r(x_0)$, respectively. We may decompose $\mathbb{R}^n$ into a disjoint union over the elements of the family of sets $\mathcal{F}_1$ or $\mathcal{F}_2$ given by

$$\mathcal{F}_1 := \{B_\sigma,\, B_\rho \setminus B_\sigma,\, \mathbb{R}^n \setminus B_\rho\} \quad \text{and} \quad \mathcal{F}_2 := \{\hat{B}_\sigma,\, \hat{B}_\rho \setminus \hat{B}_\sigma,\, \mathbb{R}^n \setminus \hat{B}_\rho\},$$

respectively. For $x \in \partial B$ we have $|x-x_0| = |x_0||x-\xi(x_0)|$. Therefore, ∂B can be decomposed into a disjoint union over the elements of the family of sets $\mathcal{F}_{\partial B}$ given by

$$\mathcal{F}_{\partial B} := \{\partial B \cap (B_\sigma \cap \hat{B}_\sigma),\, \partial B \cap [(B_\rho \setminus B_\sigma) \cap (\hat{B}_\rho \setminus \hat{B}_\sigma)],\, \partial B \setminus (B_\rho \cup \hat{B}_\rho)\},$$

and so we have for $x \in \partial B$

$$X(x) = \begin{cases} (\sigma^{-2}-\rho^{-2})|x-x_0|^2 Y(x) & , 0 \le |x-x_0| \le \sigma \\ Y(x) - \rho^{-2}|x-x_0|^2 Y(x) & , \sigma < |x-x_0| < \rho \\ 0 & , \rho \le |x-x_0|. \end{cases} \tag{1.7}$$

This implies that X is a valid test vector field in (1.2) in case $\partial B_\sigma, \partial\hat{B}_\sigma, \partial B_\rho$ and $\partial\hat{B}_\rho$ have μ measure zero, i.e. for a.e. σ and ρ. We compute

$$\int_A \mathrm{div}_\Sigma X_i \, d\mu \quad \text{and} \quad \int_A \vec{H} \cdot X_i \, d\mu$$

for all sets $A \in \mathcal{F}_i$, $i = 1,2$, separately. We have

$$\begin{aligned} \int \mathrm{div}_\Sigma X_2 \, d\mu &= \sum_{A \in \mathcal{F}_2} \int_A \mathrm{div}_\Sigma X_2 \, d\mu \\ &= 2|x_0|^2\sigma^{-2}\mu(\hat{B}_\sigma) - 2|x_0|^2\rho^{-2}\mu(\hat{B}_\rho) \\ &\quad - 2|x_0|^2\sigma^{-2}\int_{\hat{B}_\sigma} |x-\xi(x_0)|^2 \, d\mu + 2|x_0|^2\rho^{-2}\int_{\hat{B}_\rho} |x-\xi(x_0)|^2 \, d\mu \\ &\quad - 2|x_0|^2\sigma^{-2}\int_{\hat{B}_\sigma} P_x(x-\xi(x_0)) \cdot x \, d\mu + 2|x_0|^2\rho^{-2}\int_{\hat{B}_\rho} P_x(x-\xi(x_0)) \cdot x \, d\mu \\ &\quad + 2\int_{\hat{B}_\rho \setminus \hat{B}_\sigma} \frac{|(x-\xi(x_0))^\perp|^2}{|x-\xi(x_0)|^4} \, d\mu \\ &\quad - 2\mu(\hat{B}_\rho \setminus \hat{B}_\sigma), \end{aligned}$$

and

$$
\begin{aligned}
\int \vec{H} \cdot X_2 \, d\mu &= \sum_{A \in \mathcal{F}_2} \int_A \vec{H} \cdot X_2 \, d\mu \\
&= |x_0|^2 \sigma^{-2} \int_{\hat{B}_\sigma} \vec{H} \cdot (x - \xi(x_0)) \, d\mu - |x_0|^2 \rho^{-2} \int_{\hat{B}_\rho} \vec{H} \cdot (x - \xi(x_0)) \, d\mu \\
&\quad - |x_0|^2 \sigma^{-2} \int_{\hat{B}_\sigma} \vec{H} \cdot (|x - \xi(x_0)|^2 x) \, d\mu + |x_0|^2 \rho^{-2} \int_{\hat{B}_\rho} \vec{H} \cdot (|x - \xi(x_0)|^2 x) \, d\mu \\
&\quad + \int_{\hat{B}_\rho \setminus \hat{B}_\sigma} \vec{H} \cdot \frac{x - \xi(x_0)}{|x - \xi(x_0)|^2} \, d\mu - \int_{\hat{B}_\rho \setminus \hat{B}_\sigma} \vec{H} \cdot x \, d\mu.
\end{aligned}
$$

Using the fact that for any vector $v \in \mathbb{R}^n$

$$
2 \left| \frac{1}{4} \vec{H} + v^\perp \right|^2 = \frac{1}{8} |\vec{H}|^2 + 2|v^\perp|^2 + \vec{H} \cdot v, \tag{1.8}
$$

where we used Brakke's orthogonality theorem (cf. [Bra78, Chapter 5]), we get that

$$
\begin{aligned}
&\int \operatorname{div}_\Sigma X_2 \, d\mu + \int \vec{H} \cdot X_2 \, d\mu \\
&= 2|x_0|^2 \sigma^{-2} \mu(\hat{B}_\sigma) - 2|x_0|^2 \rho^{-2} \mu(\hat{B}_\rho) - \frac{1}{8} \int_{\hat{B}_\rho \setminus \hat{B}_\sigma} |\vec{H}|^2 \, d\mu \\
&\quad + |x_0|^2 \sigma^{-2} \int_{\hat{B}_\sigma} \vec{H} \cdot (x - \xi(x_0)) \, d\mu - |x_0|^2 \rho^{-2} \int_{\hat{B}_\rho} \vec{H} \cdot (x - \xi(x_0)) \, d\mu \\
&\quad - 2|x_0|^2 \sigma^{-2} \int_{\hat{B}_\sigma} |x - \xi(x_0)|^2 \, d\mu + 2|x_0|^2 \rho^{-2} \int_{\hat{B}_\rho} |x - \xi(x_0)|^2 \, d\mu \\
&\quad - 2|x_0|^2 \sigma^{-2} \int_{\hat{B}_\sigma} P_x(x - \xi(x_0)) \cdot x \, d\mu + 2|x_0|^2 \rho^{-2} \int_{\hat{B}_\rho} P_x(x - \xi(x_0)) \cdot x \, d\mu \\
&\quad - |x_0|^2 \sigma^{-2} \int_{\hat{B}_\sigma} \vec{H} \cdot (|x - \xi(x_0)|^2 x) \, d\mu + |x_0|^2 \rho^{-2} \int_{\hat{B}_\rho} \vec{H} \cdot (|x - \xi(x_0)|^2 x) \, d\mu \\
&\quad - 2\mu(\hat{B}_\rho \setminus \hat{B}_\sigma) - \int_{\hat{B}_\rho \setminus \hat{B}_\sigma} \vec{H} \cdot x \, d\mu + 2 \int_{\hat{B}_\rho \setminus \hat{B}_\sigma} \left| \frac{1}{4} \vec{H} + \frac{(x - \xi(x_0))^\perp}{|x - \xi(x_0)|^2} \right|^2 \, d\mu.
\end{aligned}
$$

Similarly, (in fact exactly as in [KS04]) we get that

$$
\begin{aligned}
&\int \operatorname{div}_\Sigma X_1 \, d\mu + \int \vec{H} \cdot X_1 \, d\mu \\
&\qquad = 2\sigma^{-2} \mu(B_\sigma) - 2\rho^{-2} \mu(B_\rho) - \frac{1}{8} \int_{B_\rho \setminus B_\sigma} |\vec{H}|^2 \, d\mu \\
&\qquad\quad + \sigma^{-2} \int_{B_\sigma} \vec{H} \cdot (x - x_0) \, d\mu - \rho^{-2} \int_{B_\rho} \vec{H} \cdot (x - x_0) \, d\mu \\
&\qquad\quad + 2 \int_{B_\rho \setminus B_\sigma} \left| \frac{1}{4} \vec{H} + \frac{(x - x_0)^\perp}{|x - x_0|^2} \right|^2 \, d\mu.
\end{aligned}
$$

Since, as mentioned above $X = X_1 + X_2$ is an admissible vector field for (1.2), we get after rearranging that

$$
\begin{aligned}
&2\int_{B_\rho\setminus B_\sigma}\left|\frac{1}{4}\vec{H}+\frac{(x-x_0)^\perp}{|x-x_0|^2}\right|^2 d\mu + 2\int_{\hat{B}_\rho\setminus \hat{B}_\sigma}\left|\frac{1}{4}\vec{H}+\frac{(x-\xi(x_0))^\perp}{|x-\xi(x_0)|^2}\right|^2 d\mu \\
&= 2\rho^{-2}\mu(B_\rho) - 2\sigma^{-2}\mu(B_\sigma) + 2|x_0|^2\rho^{-2}\mu(\hat{B}_\rho) - 2|x_0|^2\sigma^{-2}\mu(\hat{B}_\sigma) \\
&\quad + \frac{1}{8}\int_{B_\rho\setminus B_\sigma}|\vec{H}|^2\,d\mu + \frac{1}{8}\int_{\hat{B}_\rho\setminus\hat{B}_\sigma}|\vec{H}|^2\,d\mu + 2\mu(\hat{B}_\rho\setminus\hat{B}_\sigma) \\
&\quad + \rho^{-2}\int_{B_\rho}\vec{H}\cdot(x-x_0)\,d\mu - \sigma^{-2}\int_{B_\sigma}\vec{H}\cdot(x-x_0)\,d\mu \\
&\quad + |x_0|^2\rho^{-2}\int_{\hat{B}_\rho}\vec{H}\cdot(x-\xi(x_0))\,d\mu - |x_0|^2\sigma^{-2}\int_{\hat{B}_\sigma}\vec{H}\cdot(x-\xi(x_0))\,d\mu \\
&\quad - |x_0|^2\rho^{-2}\int_{\hat{B}_\rho}\vec{H}\cdot(|x-\xi(x_0)|^2x)\,d\mu + |x_0|^2\sigma^{-2}\int_{\hat{B}_\sigma}\vec{H}\cdot(|x-\xi(x_0)|^2x)\,d\mu \\
&\quad - 2|x_0|^2\rho^{-2}\int_{\hat{B}_\rho}P_x(x-\xi(x_0))\cdot x\,d\mu + 2|x_0|^2\sigma^{-2}\int_{\hat{B}_\sigma}P_x(x-\xi(x_0))\cdot x\,d\mu \\
&\quad - 2|x_0|^2\rho^{-2}\int_{\hat{B}_\rho}|x-\xi(x_0)|^2\,d\mu + 2|x_0|^2\sigma^{-2}\int_{\hat{B}_\sigma}|x-\xi(x_0)|^2\,d\mu \\
&\quad + \int_{\hat{B}_\rho\setminus\hat{B}_\sigma}\vec{H}\cdot x\,d\mu.
\end{aligned}
$$

In view of the definition of g and $\hat{g}$ we may rewrite this as

$$
\begin{aligned}
&\frac{1}{\pi}\int_{B_\rho(x_0)\setminus B_\sigma(x_0)}\left|\frac{1}{4}\vec{H}+\frac{(x-x_0)^\perp}{|x-x_0|^2}\right|^2 d\mu + \frac{1}{\pi}\int_{\hat{B}_\rho(x_0)\setminus\hat{B}_\sigma(x_0)}\left|\frac{1}{4}\vec{H}+\frac{(x-\xi(x_0))^\perp}{|x-\xi(x_0)|^2}\right|^2 d\mu \\
&= (g_{x_0}(\rho)+\hat{g}_{x_0}(\rho)) - (g_{x_0}(\sigma)+\hat{g}_{x_0}(\sigma)).
\end{aligned}
$$

Now, assume that $x_0 = 0$. Then (1.7) still holds, and we may again test (1.2) with X. (Again first for a.e. σ and ρ.) We write B_r instead of $B_r(0)$, and may decompose $\mathbb{R}^n$ into a disjoint union over the elements of the family of sets $\mathcal{F}$ given by

$$\mathcal{F} := \{B_\sigma,\, B_\rho\setminus B_\sigma,\, \mathbb{R}^n\setminus B_\rho\}.$$

Recalling that

$$X_1(x) := (|x|_\sigma^{-2} - \rho^{-2})^+ x$$

and

$$X_2(x) := (\min(\rho^{-2},1) - \min(\sigma^{-2},1))x,$$

we compute

$$\int_A \mathrm{div}_\Sigma X_1\,d\mu \quad \text{and} \quad \int_A \vec{H}\cdot X_1\,d\mu$$

for all sets $A \in \mathcal{F}$. We have

$$\begin{aligned}\int \mathrm{div}_\Sigma X \, d\mu &= \int \mathrm{div}_\Sigma X_1 \, d\mu + \int \mathrm{div}_\Sigma X_2 \, d\mu \\ &= 2\sigma^{-2}\mu(B_\sigma) - 2\rho^{-2}\mu(B_\rho) \\ &\quad + 2\int_{B_\rho \setminus B_\sigma} \frac{|x^\perp|^2}{|x|^4} \, d\mu \\ &\quad + 2(\min(\rho^{-2}, 1) - \min(\sigma^{-2}, 1))\mu(\mathbb{R}^n)\end{aligned}$$

and

$$\begin{aligned}-\int \vec{H} \cdot X \, d\mu &= -\int \vec{H} \cdot X_1 \, d\mu - \int \vec{H} \cdot X_2 \, d\mu \\ &= -\sigma^{-2}\int_{B_\sigma} \vec{H} \cdot x \, d\mu + \rho^{-2}\int_{B_\rho} \vec{H} \cdot x \, d\mu \\ &\quad - \int_{B_\rho \setminus B_\sigma} \vec{H} \cdot (|x|^{-2}x) \, d\mu \\ &\quad - (\min(\rho^{-2}, 1) - \min(\sigma^{-2}, 1)) \int \vec{H} \cdot x \, d\mu.\end{aligned}$$

Using again (1.8) we get

$$\begin{aligned}2\int_{B_\rho \setminus B_\sigma} \left| \frac{1}{4}\vec{H} + \frac{x^\perp}{|x|^2} \right|^2 d\mu &= 2\rho^{-2}\mu(B_\rho) - 2\sigma^{-2}\mu(B_\sigma) + \frac{1}{8}\int_{B_\rho \setminus B_\sigma} |\vec{H}|^2 \, d\mu \\ &\quad - 2(\min(\rho^{-2}, 1) - \min(\sigma^{-2}, 1))\mu(\mathbb{R}^n) \\ &\quad + \rho^{-2}\int_{B_\rho} \vec{H} \cdot x \, d\mu - \sigma^{-2}\int_{B_\sigma} \vec{H} \cdot x \, d\mu \\ &\quad - (\min(\rho^{-2}, 1) - \min(\sigma^{-2}, 1)) \int \vec{H} \cdot x \, d\mu.\end{aligned}$$

In view of the definition of g_0 and $\hat{g}_0$, and equation (1.4) we may rewrite this as

$$\frac{1}{\pi}\int_{B_\rho(0) \setminus B_\sigma(0)} \left| \frac{1}{4}\vec{H} + \frac{x^\perp}{|x|^2} \right|^2 d\mu = (g_0(\rho) + \hat{g}_0(\rho)) - (g_0(\sigma) + \hat{g}_0(\sigma)).$$

This equality which was proved for a.e. σ and ρ is obviously also true for every σ and ρ by an approximation argument. $\square$

Proposition 1.2. *For every $x_0 \in \mathbb{R}^n$ the tilde-density*

$$\widetilde{\theta}^2(\mu, x_0) := \begin{cases} \lim_{r \downarrow 0} \left(\frac{\mu(B_r(x_0))}{\pi r^2} + \frac{\mu(\hat{B}_r(x_0))}{\pi(|x_0|^{-1}r)^2} \right) & , x_0 \neq 0, \\ \lim_{r \downarrow 0} \frac{\mu(B_r(0))}{\pi r^2} \end{cases}$$

exists. Moreover, the function $x \mapsto \widetilde{\theta}^2(\mu, x)$ is upper semicontinuous in $\mathbb{R}^n$.

Remark 1.3. Since $\hat{B}_r(x_0) = B_r(x_0)$ for $x_0 \in \partial B$ we have that $\widetilde{\theta}^2(\mu, \cdot) = 2\theta^2(\mu, \cdot)$ on ∂B.

Proof. Set, in case $x_0 \neq 0$,

$$R(r) := \frac{1}{2\pi r^2} \int_{B_r} \vec{H} \cdot (x - x_0)\, d\mu + \frac{1}{2\pi(|x_0|^{-1} r)^2} \int_{\hat{B}_r} \vec{H} \cdot (x - \xi(x_0))\, d\mu$$
$$- \frac{1}{\pi(|x_0|^{-1} r)^2} \int_{\hat{B}_r} (|x - \xi(x_0)|^2 + P_x(x - \xi(x_0)) \cdot x)\, d\mu$$
$$- \frac{1}{2\pi(|x_0|^{-1} r)^2} \int_{\hat{B}_r} \vec{H} \cdot (|x - \xi(x_0)|^2 x)\, d\mu.$$

We estimate with Hölder's inequality

$$|R(r)| \leq \left(\frac{\mu(B_r)}{\pi r^2}\right)^{\frac{1}{2}} \left(\frac{1}{4\pi} \int_{B_r} |\vec{H}|^2\, d\mu\right)^{\frac{1}{2}} + \left(\frac{\mu(\hat{B}_r)}{\pi(|x_0|^{-1} r)^2}\right)^{\frac{1}{2}} \left(\frac{1}{4\pi} \int_{\hat{B}_r} |\vec{H}|^2\, d\mu\right)^{\frac{1}{2}}$$
$$+ \frac{\mu(\hat{B}_r)}{\pi} + d \left(\frac{\mu(\hat{B}_r)}{\pi(|x_0|^{-1} r)^2}\right)^{\frac{1}{2}} \left(\frac{\mu(\hat{B}_r)}{\pi}\right)^{\frac{1}{2}}$$
$$+ d \left(\frac{\mu(\hat{B}_r)}{\pi}\right)^{\frac{1}{2}} \left(\frac{1}{4\pi} \int_{\hat{B}_r} |\vec{H}|^2\, d\mu\right)^{\frac{1}{2}}, \tag{1.9}$$

where $d := \sup\{|x| : x \in \Sigma\}$. Moreover, for $\varepsilon > 0$

$$|R(r)| \leq \varepsilon \frac{\mu(B_r)}{\pi r^2} + \frac{1}{16\pi\varepsilon} \int_{B_r} |\vec{H}|^2\, d\mu + \varepsilon \frac{\mu(\hat{B}_r)}{\pi(|x_0|^{-1} r)^2} + \frac{1}{16\pi\varepsilon} \int_{\hat{B}_r} |\vec{H}|^2\, d\mu$$
$$+ \frac{\mu(\hat{B}_r)}{\pi} + \varepsilon \frac{\mu(\hat{B}_r)}{\pi(|x_0|^{-1} r)^2} + \frac{1}{4\varepsilon} d^2 \frac{\mu(\hat{B}_r)}{\pi} + \frac{1}{4\pi} \int_{\hat{B}_r} |\vec{H}|^2\, d\mu + d^2 \frac{\mu(\hat{B}_r)}{4\pi}.$$

On the other hand, we have

$$\frac{\mu(B_\sigma)}{\pi\sigma^2} + \frac{\mu(\hat{B}_\sigma)}{\pi(|x_0|^{-1}\sigma)^2} \leq \frac{\mu(B_\rho)}{\pi\rho^2} + \frac{\mu(\hat{B}_\rho)}{\pi(|x_0|^{-1}\rho)^2} + \frac{1}{16\pi} \int_{(B_\rho \cup \hat{B}_\rho) \setminus (B_\sigma \cup \hat{B}_\sigma)} |\vec{H}|^2\, d\mu$$
$$+ \frac{1}{2\pi} \int_{\hat{B}_\rho \setminus \hat{B}_\sigma} \vec{H} \cdot x\, d\mu + \frac{\mu(\hat{B}_\rho \setminus \hat{B}_\sigma)}{\pi} + R(\rho) - R(\sigma).$$

Using (1.9) and

$$\int_{\hat{B}_\rho \setminus \hat{B}_\sigma} \vec{H} \cdot x\, d\mu \leq \frac{1}{4} \int_{\hat{B}_\rho} |\vec{H}|^2\, d\mu + d^2 \mu(\hat{B}_\rho),$$

we infer, upon redefining $0 < \varepsilon < 1$, that

$$\frac{\mu(B_\sigma)}{\pi\sigma^2} + \frac{\mu(\hat{B}_\sigma)}{\pi(|x_0|^{-1}\sigma)^2} \leq (1 + \varepsilon) \left(\frac{\mu(B_\rho)}{\pi\rho^2} + \frac{\mu(\hat{B}_\rho)}{\pi(|x_0|^{-1}\rho)^2}\right)$$
$$+ C(\varepsilon) \int_{B_\rho} |\vec{H}|^2\, d\mu + C(\varepsilon) \int_{\hat{B}_\rho} |\vec{H}|^2\, d\mu$$
$$+ C(\varepsilon) \left(1 + d^2\right) \mu(\hat{B}_\rho). \tag{1.10}$$

We infer that

$$\limsup_{\sigma \downarrow 0} \left(\frac{\mu(B_\sigma)}{\pi\sigma^2} + \frac{\mu(\hat{B}_\sigma)}{\pi(|x_0|^{-1}\sigma)^2}\right) < \infty,$$

and in view of (1.9) that

$$\lim_{r \downarrow 0} |R(r)| = 0.$$

Theorem 1.1 implies that the tilde-density $\widetilde{\theta}^2(\mu, x_0)$ exists, and that

$$\widetilde{\theta}^2(\mu, x_0) = \lim_{\sigma \downarrow 0} (g_{x_0}(\sigma) + \hat{g}_{x_0}(\sigma)).$$

Hence also

$$\begin{aligned}\widetilde{\theta}^2(\mu, x_0) \le (1+\varepsilon) & \left(\frac{\mu(B_\rho)}{\pi \rho^2} + \frac{\mu(\hat{B}_\rho)}{\pi (|x_0|^{-1} \rho)^2} \right) \\ & + C(\varepsilon) \int_{B_\rho} |\vec{H}|^2 \, d\mu + C(\varepsilon) \int_{\hat{B}_\rho} |\vec{H}|^2 \, d\mu \\ & + C(\varepsilon) \left(1 + d^2 \right) \mu(\hat{B}_\rho). \end{aligned} \tag{1.11}$$

Now, assume $x_0 = 0$, then set

$$R(r) := \frac{1}{2\pi r^2} \int_{B_r} \vec{H} \cdot x \, d\mu,$$

and we have that

$$|R(r)| \le \left(\frac{\mu(B_r)}{\pi r^2} \right)^{\frac{1}{2}} \left(\frac{1}{4\pi} \int_{B_r} |\vec{H}|^2 \, d\mu \right)^{\frac{1}{2}} \tag{1.12}$$

and for $\varepsilon > 0$

$$|R(r)| \le \varepsilon \frac{\mu(B_r)}{\pi r^2} + \frac{1}{16 \pi \varepsilon} \int_{B_r} |\vec{H}|^2 \, d\mu.$$

Hence,

$$\frac{\mu(B_\sigma)}{\pi \sigma^2} \le (1+\varepsilon) \frac{\mu(B_\rho)}{\pi \rho^2} + C(\varepsilon) \int_{B_\rho} |\vec{H}|^2 \, d\mu + C(\varepsilon)(1 - \min(\rho^{-2}, 1)) \, \sigma(\partial B),$$

where we used that $\mathrm{spt}(\sigma) \subset \partial B$. We infer that

$$\limsup_{\sigma \downarrow 0} \frac{\mu(B_\sigma)}{\pi \sigma^2} < \infty,$$

and in view of (1.12) that

$$\lim_{r \downarrow 0} |R(r)| = 0.$$

Theorem 1.1 implies that the density $\theta^2(\mu, 0)$ exists, and that

$$\theta^2(\mu, 0) = \lim_{\sigma \downarrow 0} g_0(\sigma),$$

where we used that $\hat{g}_0(r) \equiv -\frac{1}{2\pi} \int x \cdot \eta \, d\sigma$ for all $0 < r \le 1$. Hence also

$$\begin{aligned}\widetilde{\theta}^2(\mu, 0) = \theta^2(\mu, 0) \le (1+\varepsilon) \frac{\mu(B_\rho)}{\pi \rho^2} & + C(\varepsilon) \int_{B_\rho} |\vec{H}|^2 \, d\mu \\ & + C(\varepsilon)(1 - \min(\rho^{-2}, 1)) \, \sigma(\partial B). \end{aligned} \tag{1.13}$$

Now, let x_j be a sequence in $\mathbb{R}^n$ such that $x_j \to x_0$. Then (1.11) and (1.13) with x_0 replaced by x_j implies

$$
\begin{aligned}
&\frac{\mu(\overline{B}_\rho)}{\pi\rho^2} + \frac{\mu(\overline{\hat{B}}_\rho)}{\pi(|x_0|^{-1}\rho)^2} \geq \limsup_{j\to\infty}\left(\frac{\mu(B_\rho(x_j))}{\pi\rho^2} + \frac{\mu(\hat{B}_\rho(x_j))}{\pi(|x_j|^{-1}\rho)^2}\right)\\
&\geq \frac{1}{1+\varepsilon}\limsup_{j\to\infty}\bigg(\widetilde{\theta}^2(\mu,x_j) - C(\varepsilon)\int_{B_\rho(x_j)\cup\hat{B}_\rho(x_j)}|\vec{H}|^2\,d\mu\\
&\qquad - C(\varepsilon)(1+d^2)\mu(\hat{B}_\rho(x_j)) - C(\varepsilon)(1-\min(\rho^{-2},1))\,\sigma(\partial B).\bigg)\\
&\geq \frac{1}{1+\varepsilon}\bigg(\limsup_{j\to\infty}\widetilde{\theta}^2(\mu,x_j) - C(\varepsilon)\int_{B_{2\rho}(x_0)\cup\hat{B}_{2\rho}(x_0)}|\vec{H}|^2\,d\mu\\
&\qquad - C(\varepsilon)\left(1+d^2\right)\mu(\hat{B}_{2\rho}(x_0)) - C(\varepsilon)(1-\min(\rho^{-2},1))\,\sigma(\partial B).\bigg),
\end{aligned}
$$

where we interpret $\hat{B}_r(0) = \emptyset$ and $\frac{\mu(\overline{\hat{B}}_\rho(0))}{\pi(|0|^{-1}\rho)^2} = 0$. Letting $\rho \downarrow 0$ and then $\varepsilon \downarrow 0$ implies the upper semicontinuity. $\square$

Since Σ is compact we may estimate

$$
\begin{aligned}
|R(r)| \leq{}& \frac{1}{2\pi r}\mu(B_r)^{\frac{1}{2}}\left(\int_{B_r}|\vec{H}|^2\,d\mu\right)^{\frac{1}{2}} + \frac{C(d,|x_0|)}{r^2}\mu(\hat{B}_r)\\
&+ \frac{C(d,|x_0|)}{r^2}\mu(\hat{B}_r)^{\frac{1}{2}}\left(\int_{\hat{B}_r}|\vec{H}|^2\,d\mu\right)^{\frac{1}{2}}.
\end{aligned}
$$

Hence,

$$\lim_{r\to\infty}|R(r)| = 0.$$

Also, by (1.3) and (1.4),

$$
\begin{aligned}
\lim_{r\to\infty}(g_{x_0}(r) + \hat{g}_{x_0}(r)) &= \frac{1}{8\pi}\int|\vec{H}|^2\,d\mu + \frac{1}{2\pi}\int\vec{H}\cdot x\,d\mu + \frac{\mu(\mathbb{R}^n)}{\pi}\\
&= \frac{1}{8\pi}\int|\vec{H}|^2\,d\mu + \frac{1}{2\pi}\int x\cdot\eta\,d\sigma
\end{aligned}
$$

for $x_0 \neq 0$, and

$$\lim_{r\to\infty}(g_0(r) + \hat{g}_0(r)) = \frac{1}{16\pi}\int|\vec{H}|^2\,d\mu.$$

Summarizing, we have proved the following theorem:

Theorem 1.4. *For every $x_0 \in \mathbb{R}^n$ the tilde-density*

$$
\widetilde{\theta}^2(\mu,x_0) := \begin{cases}\lim_{r\downarrow 0}\left(\frac{\mu(B_r(x_0))}{\pi r^2} + \frac{\mu(\hat{B}_r(x_0))}{\pi(|x_0|^{-1}r)^2}\right) & ,x_0 \neq 0,\\ \lim_{r\downarrow 0}\frac{\mu(B_r(0))}{\pi r^2}\end{cases}
$$

exists. The function $x \mapsto \widetilde{\theta}^2(\mu,x)$ is upper semicontinuous. Moreover, we have for all $0 < \sigma < \rho < \infty$

1. **(area bound)**

$$\begin{cases} \sigma^{-2}\mu(B_\sigma(x_0)) + (\sigma/|x_0|)^{-2}\mu(\hat{B}_\sigma(x_0)) \le C & , x_0 \neq 0, \\ \sigma^{-2}\mu(B_\sigma(0)) \le C & , \end{cases}$$

for $C = C(d, \mu(\mathbb{R}^n), \|\vec{H}\|_{L^2})$,

2. **(density bound)**

$$\begin{aligned} \widetilde{\theta}^2(\mu, x_0) \le (1+\varepsilon)\frac{\mu(B_\rho(x_0))}{\pi\rho^2} + (1+\varepsilon)\frac{\mu(\hat{B}_\rho(x_0))}{\pi(|x_0|^{-1}\rho)^2} \\ + C(\varepsilon)\int_{B_\rho(x_0)} |\vec{H}|^2\, d\mu + C(\varepsilon)\int_{\hat{B}_\rho(x_0)} |\vec{H}|^2\, d\mu \\ + C(\varepsilon)\left(1 + d^2\right)\mu(\hat{B}_\rho(x_0)) \end{aligned}$$

and

$$\theta^2(\mu, 0) \le (1+\varepsilon)\frac{\mu(B_\rho)}{\pi\rho^2} + C(\varepsilon)\int_{B_\rho} |\vec{H}|^2\, d\mu + C(\varepsilon)(1 - \min(\rho^{-2}, 1))\, \sigma(\partial B),$$

and

3. **(integral identity)**

$$\begin{aligned} \frac{1}{\pi}\int \left|\frac{1}{4}\vec{H} + \frac{(x-x_0)^\perp}{|x-x_0|^2}\right|^2 d\mu + \frac{1}{\pi}\int \left|\frac{1}{4}\vec{H} + \frac{(x-\xi(x_0))^\perp}{|x-\xi(x_0)|^2}\right|^2 d\mu \\ = \frac{1}{8\pi}\int |\vec{H}|^2\, d\mu + \frac{1}{2\pi}\int x\cdot\eta\, d\sigma - \widetilde{\theta}^2(\mu, x_0) \qquad \textit{for } x_0 \neq 0, \end{aligned}$$

and

$$\frac{1}{\pi}\int \left|\frac{1}{4}\vec{H} + \frac{x^\perp}{|x|^2}\right|^2 d\mu = \frac{1}{16\pi}\int |\vec{H}|^2\, d\mu + \frac{1}{2\pi}\int x\cdot\eta\, d\sigma - \theta^2(\mu, 0).$$

1.3 Applications

The Willmore energy $\mathcal{W}(F)$ of a smooth immersed compact orientable surface $F : \Sigma \to \mathbb{R}^n$ with boundary $\partial\Sigma$ is given by

$$\mathcal{W}(F) := \frac{1}{4}\int_\Sigma H^2\, d\mathcal{H}^2_{F^*\delta} + \int_{\partial\Sigma} \kappa_g\, d\mathcal{H}^1_{F^*\delta},$$

where κ_g denotes the geodesic curvature of $\partial\Sigma$ as a submanifold of Σ (cf. [Sch10]). By the Gauss equations and the Gauss-Bonnet theorem we have that

$$\mathcal{W}(F) = \frac{1}{2}\int_\Sigma |A^\circ|^2\, d\mathcal{H}^2_{F^*\delta} + 2\pi\chi(\Sigma),$$

where A° denotes the tracefree part of the second fundamental form, and $\chi(\Sigma)$ denotes the Euler characteristic of Σ. Since $\chi(\Sigma) = 2 - 2g(\Sigma) - r(\Sigma)$, $g(\Sigma) =$ genus of Σ, $r(\Sigma) =$ number of boundary components of Σ, we have that

$$\mathcal{W}(F) \geq 2\pi$$

for topological disks. For free boundary surfaces with respect to the unit ball we have that

$$\kappa_g = D_\tau \eta \cdot \tau = D_\tau(\eta \cdot x\, x) \cdot \tau = x \cdot \eta, \quad (\tau \in T(\partial\Sigma), |\tau| = 1)$$

hence the Willmore energy may be rewritten as

$$\mathcal{W}(F) = \frac{1}{4} \int_\Sigma |\vec{H}|^2 \, d\mathcal{H}^2_{F^*\delta} + \int_{\partial\Sigma} x \cdot \eta \, d\mathcal{H}^1_{F^*\delta}.$$

Motivated by the smooth case we may define the Willmore energy $\mathcal{W}(\mu)$ of a free boundary varifold μ with respect to the unit ball by

$$\mathcal{W}(\mu) = \frac{1}{4} \int |\vec{H}|^2 \, d\mu + \int x \cdot \eta \, d\sigma.$$

Theorem 1.5. *For any immersion $F : \Sigma \to \mathbb{R}^n$ of a compact free boundary surface with respect to the unit ball in $\mathbb{R}^n$ and the image varifold $\mu = \theta\mathcal{H}^2 \llcorner F(\Sigma)$, where $\theta(x) = \mathcal{H}^0(F^{-1}(\{x\}))$, we have*

$$\mathcal{H}^0(F^{-1}(\{x, \xi(x)\})) = \widetilde{\theta}^2(\mu, x) \leq \frac{1}{2\pi} \mathcal{W}(F),$$

in particular

$$W(F) \geq 2\pi, \tag{1.14}$$

and if

$$W(F) < 4\pi,$$

then F is an embedding. Moreover, equality in (1.14) *implies that F parametrizes a round spherical cap or a flat unit disk.*

Proof. The inequalities follow from Theorem 1.4. Assume now equality in (1.14) holds. In particular, we have that F is an embedding, and we may identify Σ with $F(\Sigma)$. The proof now follows from Proposition 1.7 below. □

Remark 1.6. The estimate is sharp, as can be seen by taking the union of two distinct free boundary flat disks.
It is also interesting to note that in case $0 \in \Sigma$ we have the stronger inequality

$$2\pi\theta^2(\mu, 0) + \frac{1}{8} \int |\vec{H}|^2 \, d\mu \leq \mathcal{W}(\mu).$$

Proposition 1.7. *Let $\mu \neq 0$ be a compact integer rectifiable free boundary 2-varifold with respect to ∂B such that*

$$\mathcal{W}(\mu) = 2\pi.$$

Then $\mu = \mathcal{H}^2 \llcorner \Sigma$, where Σ is a round spherical cap or a flat unit disk.

Proof. It follows from Theorem 1.4 that the tilde-density $\widetilde{\theta}^2(\mu, x)$ exists and is ≥ 1 for every $x \in \Sigma$. The assumption together with Theorem 1.4 then yield that $\widetilde{\theta}^2(\mu, x) = 1$ for every $x \in \Sigma$. In particular, we conclude that $\theta^2(\mu, x) = 1$ for every $x \in \Sigma \setminus \partial B$ and $\theta^2(\mu, x) = 1/2$ for every $x \in \Sigma \cap \partial B$. Since $\mu \neq 0$ and Σ is compact the area estimate in Theorem 1.4 implies that there exists a radius $R > 0$ such that $\Sigma \setminus B_R(x) \neq \emptyset$ for all $x \in \Sigma$. Pick any point $x_0 \in \Sigma$, then

$$1 + \frac{1}{\pi} \int \left| \frac{1}{4} \vec{H} + \frac{(x - x_0)^\perp}{|x - x_0|^2} \right|^2 d\mu + \frac{1}{\pi} \int \left| \frac{1}{4} \vec{H} + \frac{(x - \xi(x_0))^\perp}{|x - \xi(x_0)|^2} \right|^2 d\mu = \frac{1}{2\pi} \mathcal{W}(\mu) = 1.$$

We conclude that

$$\frac{1}{4} \vec{H}(x) + \frac{(x - x_0)^\perp}{|x - x_0|^2} = 0 \quad \text{for } \mu\text{-a.e. } x \in \Sigma. \tag{1.15}$$

In particular,

$$|\vec{H}(x)| = 4 \left| \frac{(x - x_0)^\perp}{|x - x_0|^2} \right| \leq \frac{8}{R} \quad \text{for } \mu\text{-a.e. } x \in \Sigma \setminus B_{\frac{R}{2}}(x_0).$$

And similarly, picking a second point $x_1 \in \Sigma \setminus B_R(x_0)$ we conclude that $|\vec{H}(x)| \leq \frac{8}{R}$ for μ-a.e. $x \in \Sigma \setminus B_{\frac{R}{2}}(x_1)$. Since $B_{\frac{R}{2}}(x_0) \cap B_{\frac{R}{2}}(x_1) = \emptyset$ we have that $|\vec{H}(x)| \leq \frac{8}{R}$ for μ-a.e. $x \in \Sigma$. In particular, $|\vec{H}| \in L^\infty(\mu)$. By Allard's regularity theorem [All72], Grüter-Jost's free boundary version [GJ86b], and Theorem 1.4 we conclude that Σ is a $C^{1,\alpha}$ manifold with boundary. We consider two cases:

First suppose that Σ is a free boundary minimal surface (cf. [Bre12]). Then writing Σ locally as the graph of a $C^{1,\alpha}$ function elliptic regularity theory (see for example [LU68]) implies that Σ is smooth. For any given point $y \in \Sigma$ we have that

$$\frac{(x - y)^{\perp_x}}{|x - y|^2} = 0 \quad \text{for } x \in \Sigma \setminus \{y\},$$

where ${}^{\perp_x}$ stands for the orthogonal projection onto the normal space of Σ at x. In particular, $y - x \in T_x\Sigma$ for all $y \in \partial\Sigma$ and all points $x \in \Sigma \setminus \partial\Sigma$. Hence, $\partial\Sigma$ is contained in a 2-dimensional plane. The maximum principle implies that Σ is itself contained in this plane. Since Σ is compact and $\partial\Sigma \subset \partial B$, Σ must be equal to a flat unit disk.

Now assume that Σ is not minimal. Then there exists a point $x_0 \in \Sigma \setminus \partial\Sigma$ such that $\vec{H}(x_0) \neq 0$ and equality holds in (1.15). After possibly rotating Σ we may assume that $T_{x_0}\Sigma = \operatorname{span}\{e_1, e_2\}$ and that $\vec{H}(x_0) = \frac{2}{r} e_3$ for some $r \neq 0$. This implies that for $j = 4, ..., n$

$$0 = \vec{H}(x_0) \cdot e_j = 4 \frac{(x - x_0)^{\perp_{x_0}}}{|x - x_0|^2} \cdot e_j = 4 \frac{(x - x_0)_j}{|x - x_0|^2} \tag{1.16}$$

for all $x \in \Sigma \setminus \{x_0\}$. (First for μ-almost all points, and by continuity in x of the right hand side of equation (1.16) for all points.) This implies that $\Sigma \subset x_0 + \mathbb{R}^3 \times \{0\}$. On the other hand,

$$\frac{2}{r} = \vec{H}(x_0) \cdot e_3 = 4 \frac{(x - x_0)_3}{|x - x_0|^2},$$

i.e. $\frac{1}{r} |x - x_0|^2 = 2(x - x_0)_3$, or equivalently

$$r^2 = (x - x_0)_1^2 + (x - x_0)_2^2 + ((x - x_0)_3 - r)^2 = |x - (x_0 + re_3)|^2$$

for all $x \in \Sigma \setminus \{x_0\}$, and $\Sigma \subset \partial B_r(x_0 + re_3) \cap \mathbb{R}^3 \times \{0\}$. Since $\partial\Sigma \subset \partial B$ we must have that either $\Sigma = (\partial B_r(x_0 + re_3) \cap \mathbb{R}^3 \times \{0\}) \cap \overline{B}$ or $\Sigma = (\partial B_r(x_0 + re_3) \cap \mathbb{R}^3 \times \{0\}) \setminus B$. $\square$

An immediate corollary of Theorem 1.5 is the following very special case of a Theorem due to Ekholm, White, and Wienholtz [EWW02].

Corollary 1.8. *Any immersed compact free boundary minimal surface with respect to the unit ball of boundary length strictly less that 4π (or equivalently of area strictly less that 2π) must be embedded.*

Remark 1.9. Bourni and Tinaglia [BT12] have extended the result of Ekholm, White, and Wienholtz to surfaces with small L^p-norm of the mean curvature with $p \geq 2$.

1.4 Geometric inequalites for free boundary surfaces

In this section we consider free boundary surfaces with respect to an orientable C^2-hypersurface S with outward unit normal γ that meet S from the inside. More precisely, we make the following assumptions.

We assume that μ is an integer rectifiable 2-varifold in $\mathbb{R}^n$ of compact support $\Sigma := \mathrm{spt}(\mu)$, $\Sigma \cap S \neq \emptyset$, with generalized mean curvature $\vec{H} \in L^p(\mu; \mathbb{R}^n)$, $p > 2$, such that

$$\int \mathrm{div}_\Sigma X \, d\mu = -\int \vec{H} \cdot X \, d\mu + \int X \cdot \gamma \, d\sigma \tag{1.17}$$

for all $X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$, and where $\sigma = |\delta\mu| \llcorner Z$ ($Z = \{x \in \mathbb{R}^n : D_\mu|\delta\mu|(x) = +\infty\}$). By [GJ86b, Corollary 3.2] we have that the density

$$\theta^2(\mu, x_0) = \lim_{r \downarrow 0} \frac{\mu(B_r(x_0))}{\pi r^2}$$

exists at every point $x_0 \in \mathrm{spt}(\mu)$, and that $\theta^2(\mu, x_0) \geq 1/2$ for every point $x_0 \in \mathrm{spt}(\sigma)$.

Lemma 1.10. *For every $x_0 \in \mathbb{R}^n$ we have*

$$\lim_{r \downarrow 0} \sigma(B_r(x_0)) = 0.$$

Proof. Let $x_0 \in \mathrm{spt}(\sigma) \subset S$. For $r > 0$ small enough so that the oriented distance function d_S of S is of class C^2. Let $\varphi \in C^1_c(\mathbb{R}^n)$, $0 \leq \varphi \leq 1$, be such that $\varphi = 1$ on $B_r(x_0)$, $\varphi = 0$ outside $B_{2r}(x_0)$, and $|D\varphi| \leq c$ for some constant c independent of r. Testing (1.17) with $X = -\varphi \, D d_S$ we obtain

$$\begin{aligned} \sigma(B_r(x_0)) &\leq \int \varphi \, d\sigma \leq \int \varphi |D^2 d_S| + |D\varphi| \, d\mu + \int_{B_{2r}(x_0)} |\vec{H}| \, d\mu \\ &\leq \left(C(S) + \frac{c}{r} \right) \mu(B_{2r}(x_0)) + \int_{B_{2r}(x_0)} |\vec{H}| \, d\mu, \end{aligned}$$

which by [GJ86b, Theorem 3.4] goes to zero as $r \downarrow 0$. □

We need the following definition.

Definition 1.11 (cf. [ALM13]). **(interior and exterior ball curvatures)** *The* interior (exterior) ball curvature $\overline{\kappa}(x)$ ($\underline{\kappa}(x)$) *of (S, γ) at $x \in S$ is defined by*

$$\overline{\kappa}(x) := \sup_{y \in S \setminus \{x\}} Z(x, y) \quad \left(\underline{\kappa}(x) := \inf_{y \in S \setminus \{x\}} Z(x, y) \right),$$

where
$$Z(x,y) := \frac{2(x-y)\cdot\gamma(x)}{|x-y|^2}.$$
The ball curvature $\kappa(x)$ *of* S *at* $x \in S$ *is defined by* $\kappa(x) := \max\{\overline{\kappa}(x), -\underline{\kappa}(x)\} \geq 0$. *For a subset* A *of* S *we set*
$$\overline{\kappa}_A(x) := \sup_{y\in A\setminus\{x\}} Z(x,y) \qquad \left(\underline{\kappa}_A(x) := \inf_{y\in A\setminus\{x\}} Z(x,y)\right),$$
and $\kappa(x) := \max\{\overline{\kappa}_A(x), -\underline{\kappa}_A(x)\} \geq 0$.

Remark 1.12. In case $S = \partial\Omega$ for a bounded and convex set Ω the interior (exterior) ball curvature is the curvature of the largest (smallest) ball enclosed by (enclosing) Ω and touching $\partial\Omega$ at x.

Writing S locally as a graph over its tangent plane one easily verifies the following lemma.

Lemma 1.13. *For any compact sets* $K_1, K_2 \subset S$ *we have*
$$\sup_{K_2} \kappa_{K_1} < \infty.$$

We test equation (1.17) with $X = \varphi|x-x_0|^{-2}(x-x_0)$, where $\varphi(x) = (|x-x_0|_\sigma^{-2} - \rho^{-2})^+|x-x_0|^2 \geq 0$, and where $x_0 \in S$. We have
$$\int X\cdot\eta\,d\sigma = \sigma^{-2}\int_{B_\sigma}(x-x_0)\cdot\gamma\,d\sigma - \rho^{-2}\int_{B_\rho}(x-x_0)\cdot\gamma\,d\sigma + \int_{B_\rho\setminus B_\sigma}\frac{x-x_0}{|x-x_0|^2}\cdot\gamma\,d\sigma,$$
where the double usage of the symbol σ should not lead to confusion. Then for a.e. $0 < \sigma < \rho < \infty$ we have
$$\begin{aligned}&\frac{1}{\pi}\int_{B_\rho(x_0)\setminus B_\sigma(x_0)}\left|\frac{1}{4}\vec{H} + \frac{(x-x_0)^\perp}{|x-x_0|^2}\right|^2 d\mu - \frac{1}{4\pi}\int_{B_\rho(x_0)\setminus B_\sigma(x_0)}\frac{2(x-x_0)}{|x-x_0|^2}\cdot\gamma\,d\sigma\\ &\quad = (g_{x_0}(\rho) + b_{x_0}(\rho)) - (g_{x_0}(\sigma) + b_{x_0}(\sigma)),\end{aligned}$$
where
$$b_{x_0}(r) = -\frac{1}{2\pi r^2}\int_{B_r}(x-x_0)\cdot\gamma\,d\sigma.$$
We note that this identity was originally derived in [Sim93] for smooth surfaces. Using Lemma 1.13 and the fact that (by Lemma 1.10)
$$|b_{x_0}(r)| \leq \frac{\sigma(B_r)}{4\pi}\sup_{B_r}\kappa_{\mathrm{spt}(\sigma)} \to 0 \quad \text{as } r \to 0$$
one easily concludes that one can let $\rho \to \infty$ and $\sigma \to 0$ to obtain
$$\begin{aligned}2\theta^2(\mu, x_0) &+ \frac{2}{\pi}\int\left|\frac{1}{4}\vec{H} + \frac{(x-x_0)^\perp}{|x-x_0|^2}\right|^2 d\mu\\ &= \frac{1}{8\pi}\int|\vec{H}|^2\,d\mu + \frac{1}{2\pi}\int\frac{2(x-x_0)\cdot\gamma}{|x-x_0|^2}\,d\sigma. \qquad (1.18)\end{aligned}$$

Even though the identity (1.18) is well known [Sim93], the geometric interpretation of the boundary term does not seem to have been exploited thus far. The quantity

$$\frac{2(x-x_0)\cdot\gamma(x)}{|x-x_0|^2}$$

is the curvature of the tangent ball, plane, or ball complement of S at x passing through x_0.

Proposition 1.14. *We have*

$$2\pi \leq \frac{1}{4}\int |\vec{H}|^2\,d\mu + \int \overline{\kappa}_{\mathrm{spt}(\sigma)}\,d\sigma.$$

Moreover, equality holds if and only if Σ is a round spherical cap or a flat unit disk.

Proof. The inequality follows immediately from (1.18), the definition of $\overline{\kappa}_{\mathrm{spt}(\sigma)}$, and the fact that the density at a boundary point is at least $1/2$. Now assume that equality holds. Then for σ-a.e. $x \in \mathrm{spt}(\sigma)$ we have that

$$\overline{\kappa}_{\mathrm{spt}(\sigma)}(x) = Z(x,y) \quad \text{for all } y \in \mathrm{spt}(\sigma)\setminus\{x\}. \tag{1.19}$$

Moreover, by (1.19) we see that $\mathrm{spt}(\sigma)$ must lie on the tangent sphere of S at x. Since this is true for σ-a.e. point $x \in \mathrm{spt}(\sigma)$ there exists a *single* sphere that is the tangent sphere of S at every point $x \in \mathrm{spt}(\sigma)$. After rescaling and translating we are in the situation of Proposition 1.7, which completes the proof. □

Remark 1.15. A weaker, but also sharp, inequality that can be obtained from (1.18) was observed by Rivière [Riv13, Lemma 1.2].

Lemma 1.16. *Let Ω be a convex domain of class C^2. Then*

$$\sup_{x\in\partial\Omega}\overline{\kappa} = \sup_{v\in T(\partial\Omega),|v|=1} A^{\partial\Omega}(v,v) \quad \text{and} \quad \inf_{x\in\partial\Omega}\underline{\kappa} = \inf_{v\in T(\partial\Omega),|v|=1} A^{\partial\Omega}(v,v),$$

where $A^{\partial\Omega}$ denotes the second fundamental form of $\partial\Omega$ with outward unit normal γ.

Proof. We have

$$\overline{\kappa}(x) \geq \limsup_{y\to x}\frac{2(x-y)\cdot\gamma(x)}{|x-y|^2} = \sup_{v\in T_x\partial\Omega,|v|=1} A^{\partial\Omega}(x)(v,v),$$

which establishes one inequality. Now assume by contradiction that the inequality is strict, i.e.

$$\sup_{\partial\Omega}\overline{\kappa} > \sup_{v\in T_x\partial\Omega,|v|=1} A^{\partial\Omega}(v,v). \tag{1.20}$$

By (1.20) we can find two distinct points $\overline{x},\overline{y}\in\partial\Omega$ such that

$$Z(\overline{x},\overline{y}) = \sup_{\partial\Omega}\overline{\kappa} =: R^{-1}.$$

By definition of $\overline{\kappa}$ we have that for every $x \in \partial\Omega$

$$B_R(x - R\gamma(x)) \subset \Omega,$$

and since $Z(\overline{x},\overline{y}) = R^{-1}$ we also have that

$$\overline{y} \in \partial B_R(\overline{x} - R\gamma(\overline{x})). \tag{1.21}$$

W.l.o.g. we assume that $\overline{x} - R\gamma(\overline{x}) = 0$. Since Ω is convex we have that

$$\Omega \subset \{\overline{x} + x : x \cdot \overline{x} < 0\} \cap \{\overline{y} + x : x \cdot \overline{y} < 0\} =: W.$$

That is, Ω is contained inside the slab or the wedge bounded by its affine tangent spaces at $\overline{x}$ and $\overline{y}$. We consider two cases. First assume that W is a wedge, i.e.

$$P := \operatorname{span}\{\overline{x},\overline{y}\}$$

is a 2-dimensional subspace of $\mathbb{R}^n$. Then $\Omega \cap P$ is contained inside the cone $W \cap P$. By convexity and by definition of $\sup_{\partial\Omega} \overline{\kappa} = R^{-1}$ we must have that the segment

$$\partial B_R(0) \cap \{x : x \cdot (\gamma(\overline{x}) + \gamma(\overline{y})) \geq 0\} \cap P$$

is completely contained inside $\partial\Omega$, which however contradicts (1.20). Now, assume that W is a slab, i.e. $\overline{x}$ and $\overline{y}$ are co-linear. Choose a point $z \in \partial\Omega \cap W$. (If no such point existed, we would have $\Omega = W$, contradicting (1.20).) Now let

$$P := \operatorname{span}\{\overline{x}, z\}.$$

Arguing similarly to the first case we see that $\partial\Omega$ must contain a circular segment of radius R inside P connecting $\overline{x}$ and z, which again contradicts (1.20). This establishes the first claim. The proof of the second claim is similar. □

Corollary 1.17. *Suppose $S = \partial\Omega$ for a convex set $\Omega \subset \mathbb{R}^n$ such that $h_{ij}^{\partial\Omega} \leq k\,\delta_{ij}$. Then*

$$2\pi \leq \frac{1}{4}\int |\vec{H}|^2 \, d\mu + k\,\sigma(\mathbb{R}^n).$$

Suppose $S = \partial(\mathbb{R}^n \setminus \Omega)$ for a convex set $\Omega \subset \mathbb{R}^n$ such that $h_{ij}^{\partial\Omega} \geq k\,\delta_{ij}$. Then

$$2\pi \leq \frac{1}{4}\int |\vec{H}|^2 \, d\mu - k\,\sigma(\mathbb{R}^n).$$

Moreover, equality holds if and only if Σ is a round spherical cap or a flat unit disk.

Remark 1.18. The assumption that $\vec{H} \in L^p(\mu;\mathbb{R}^n)$ with $p > 2$ was only needed to ensure that the singular part σ of the total variation measure $|\delta\mu|$ has no point masses which ensures that the integral

$$\int \frac{2(x - x_0)\cdot\gamma}{|x - x_0|^2} \, d\sigma$$

exists, and to ensure that the density at every boundary point is at least $1/2$. Alternatively, we could have supposed that $p = 2$ and that μ is the image varifold of a C^1-immersion.

Some observations concerning the L^1-tangent-point energy

Integration of (1.18) yields

$$2\pi \le \frac{1}{4}\int |\vec{H}|^2\, d\mu + \fint \int \frac{2\mathrm{dist}(x-y, T_x\partial\Omega)}{|x-y|^2}\, d\sigma(x)\, d\sigma(y).$$

We note that in case σ is 1-rectifiable the double integral can be estimated in terms of the so called (cf. [SvdM12]) L^1*-tangent-point energy* $\mathcal{E}_1(\sigma)$. By definition we have

$$\mathcal{E}_p(\sigma) := \int\int \frac{1}{R_{tp}(x,y)^p}\, d\sigma(x)\, d\sigma(y),$$

where $R_{tp}(x,y)$ denotes the so called (cf. [SvdM12]) *tangent-point radius* of σ at (x,y) given by

$$R_{tp}(x,y) = \frac{|x-y|^2}{2\mathrm{dist}(x-y, T_x\sigma)}.$$

This leads to the following.

Proposition 1.19. *Let Γ be a closed curve in $\mathbb{R}^3$ of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$. Then*

$$2\pi\mathcal{H}^1(\Gamma) \le \mathcal{E}_1(\Gamma), \tag{1.22}$$

with equality only if Γ is a planar, convex curve.

Proof. Let Σ be a compact orientable minimal surface with boundary $\partial\Sigma = \Gamma$. Such a surface may be obtained by solving the Plateau problem. See for example [HS79] and the references therein. The identity (1.18) in this context still holds with γ replaced by η, the outward unit conormal of Σ. Integrating the identity (1.18) over $\partial\Sigma = \Gamma$ yields

$$\begin{aligned} 2\pi\mathcal{H}^1(\Gamma) + 4\int_\Gamma\int_\Sigma & \frac{\left|(x-y)^{\perp_x}\right|^2}{|x-y|^4}\, d\mathcal{H}^2(x)\mathcal{H}^1(y) \\ &= \int_\Gamma\int_\Gamma \frac{2(x-y)\cdot\eta(x)}{|x-y|^2}\, d\mathcal{H}^1(x)\, d\mathcal{H}^1(y), \end{aligned}$$

which is no greater than

$$\int_\Gamma\int_\Gamma \frac{2\mathrm{dist}(x-y, T_x\Gamma)}{|x-y|^2}\, d\mathcal{H}^1(x)\, d\mathcal{H}^1(y) = \mathcal{E}_1(\Gamma).$$

This establishes the inequality (1.22). Now assume that equality holds in (1.22). Then for any given point $y \in \Gamma$

$$\frac{(x-y)^{\perp_x}}{|x-y|^2} = 0 \quad \text{for } x \in \Sigma \setminus \{y\}.$$

Arguing as in the proof of Proposition 1.7 we see that Σ is contained in a 2-dimensional plane. Since in the equality case we have equalities everywhere in our estimates we also conclude that

$$(x-y)\cdot\eta(x) = \mathrm{dist}(x-y, T_x\Gamma) \ge 0 \quad \text{for all } x, y \in \Gamma.$$

That is, Γ is convex. In particular, Γ must be connected. $\square$

Remark 1.20. After informing Simon Blatt about our inequality (1.22) he communicated to us the following alternative proof of Proposition 1.19 that works for closed C^1-curves in $\mathbb{R}^n$.

Proof. ([Bla14]) Let $y \in \Gamma$. Choose an arc length parametrization starting at y, i.e. let $c : [0, L] \to \mathbb{R}^n$ be a curve with $c(0) = c(L) = y$, $|c'(s)| \equiv 1$, and $\operatorname{trace}(c) = \Gamma$. We define the curve w by

$$w(s) := \frac{c(s) - c(0)}{|c(s) - c(0)|}.$$

The curve w is of class C^1 on the open interval $(0, L)$, has limits $\lim_{s\downarrow 0} w(s) = c'(0)$ and $\lim_{s\uparrow L} w(s) = -c'(0)$, and maps into the unit sphere $\mathbb{S}^{n-1}$. Thus we have

$$\pi = \lim_{\varepsilon\downarrow 0} \operatorname{dist}(w(\varepsilon), w(L-\varepsilon)) \leq \liminf_{\varepsilon\downarrow 0} \int_{\varepsilon}^{L-\varepsilon} |w'(s)|\, ds = \int_0^L |w'(s)|\, ds.$$

A straightforward calculation shows that

$$|w'(s)| = \frac{1}{2} \frac{1}{R_{tp}(c(s), c(0))},$$

and therefore

$$2\pi \leq \int_\Gamma \frac{1}{R_{tp}(x, y)}\, d\mathcal{H}^1(x).$$

Integrating over y yields the desired inequality. Note that we have equality if and only if the curve w is a geodesic in $\mathbb{S}^{n-1}$, that is if *and only* if c is planar and convex. □

Applying Hölder's inequality twice we immediately obtain the following.

Corollary 1.21. *Let Γ be a closed curve in $\mathbb{R}^n$ of class C^1. Then for any $p > 1$ we have*

$$2\pi \leq \mathcal{E}_p(\Gamma)^{\frac{1}{p}} \mathcal{H}^1(\Gamma)^{1-\frac{2}{p}}$$

with equality if and only if Γ is a round circle.

Remark 1.22. Corollary 1.21 answers a question raised by Strzelecki, Szumańska, and von der Mosel [SSvdM13].

2 Relative isoperimetric properties of asymptotically flat support surfaces

The main goal of this chapter is to derive relative isoperimetric properties of a certain class of non-compact support hypersurfaces in euclidean space which we call *asymptotically flat* and *asymptotically catenoidal* hypersurfaces. Our results turn out to almost perfectly parallel results for asymptotically flat and asymptotically Schwarzschildian Riemannian manifolds cf. [SY79b, Lam11, EM12, EM13a, Car14].

The chapter is organized as follows. In Section 2.1 we introduce the classes of support hypersurfaces that we are interested in, define a geometric invariant associated to these hypersurfaces which we call *extrinsic mass*, and prove a positive mass theorem (Theorem 2.8). In the graphical case we also obtain a Penrose type inequality (Proposition 2.9).

In Section 2.2 we study non-compact stable free boundary minimal surfaces and compact stable free boundary constant mean curvature surfaces with respect to asymptotically catenoidal support surfaces. Our two main results (Theorem 2.13 and Theorem 2.16) are free boundary analogues of [Car14, Theorem 1] and Theorem [EM12, Theorem 1.5], respectively.

2.1 Extrinsic mass for asymptotically flat hypersurfaces in euclidean space

In this section we define the classes of support (hyper-)surfaces that we will be studying in the sequel. The study of these (hyper-)surfaces can be motivated by a certain structural analogy to asymptotically flat Riemannian manifolds that manifests itself in the validity of a positive mass theorem.

Definition 2.1. (asymptotically flat hypersurface) *We say that a connected hypersurface S in $\mathbb{R}^{n+1}$ with outward unit normal γ, and possibly with compact boundary, is* asymptotically flat *if it satisfies the following conditions:*

1. *There exists a vector $\omega \in \mathbb{S}^n$, a domain $E \subset \mathbb{R}^n_\omega := \{x \in \mathbb{R}^{n+1} : x \cdot \omega = 0\}$ that is diffeomorphic to $\mathbb{R}^n \setminus \overline{B}_1(0)$, and a function $u \in C^\infty(E)$ such that*

$$S \setminus K = graph(u) := \{x + u(x)\omega : x \in E\},$$

for some compact set $K \subset \mathbb{R}^{n+1}$.

2. *There exists constants $c_1, c_2 < \infty$, and $\beta \in (0,1)$ such that*

$$\sup_{x \in E} \Big(\max(1, |x|)^{-(1-\beta)} |u(x)| \Big) < c_1,$$

and

$$\sup_{x \in E} \Big(|x|^{n-1} |Du(x)| + |x|^n |D^2 u(x)| \Big) \leq c_2.$$

3. The outward unit normal γ of S coincides with the downward normal of u with respect to ω.

4. The mean curvature H_S of S is $\mathcal{H}^n$-integrable, i.e. $H_S \in L^1(\mathcal{H}^n \llcorner S)$.

We will refer to (ω, E, u) as asymptotically flat coordinates.

Definition 2.2. (asymptotically catenoidal hypersurface) *We say that an asymptotically flat hypersurface S in $\mathbb{R}^{n+1}$ (with asymptotically flat coordinates (ω, E, u)) is* asymptotically catenoidal *if there exists a number $M \in \mathbb{R}$ and a function $\mathcal{R} \in C^\infty(E)$ such that*

$$u(x) = \phi_M(|x|) + \mathcal{R}(x),$$

where

$$\phi_M(r) = \begin{cases} M \log(r) & , n = 2 \\ -\frac{M}{(n-2)r^{n-2}} & , n \geq 3, \end{cases}$$

and where

$$\sup_{x \in E} \Big(|x|^{n-1} |a - \mathcal{R}(x)| + |x|^n |D\mathcal{R}(x)| + |x|^{n+1} |D^2 \mathcal{R}(x)| \Big) \leq c_3,$$

for some constants $a \in \mathbb{R}$ and $c_3 < \infty$.

In this case we will refer to (ω, E, u) as asymptotically catenoidal coordinates, or for short just asymptotic coordinates. Note that in this case $\omega = -\lim_{x \in S, |x| \to \infty} \gamma(x)$.

Definition 2.3. (exterior hypersurface) *We say that an asymptotically flat hypersurface S in $\mathbb{R}^{n+1}$ with asymptotic coordinates (ω, E, u) is an* exterior hypersurface *if it satisfies the following condition:*

If ∂S is non-empty then there exist a compact orientable free boundary minimal surfaces N such that $\partial S = \partial N$, the outward unit normal γ of S coincides with the exterior unit conormal on N. There are no other compact free boundary minimal surfaces with respect to S. The union $S \cup N$ is the boundary ∂G of a domain $G \subset \mathbb{R}^{n+1}$.

In case S is asymptotically catenoidal, we call S an asymptotically catenoidal exterior hypersurface.

Notation
We denote by $\mathcal{F}_n(\beta, c_1, c_2)$ the class of asymptotically flat hypersurfaces $S \subset \mathbb{R}^{n+1}$ with constants β, c_1, c_2 as in Definition 2.1. We also set $\mathcal{F}_n(c_1, c_2) := \bigcup_{\beta \in (0,1)} \mathcal{F}_n(\beta, c_1, c_2)$.

We denote by $\mathcal{C}_n(M, c_3)$ the class of asymptotically catenoidal hypersurfaces $S \subset \mathbb{R}^{n+1}$ with constants M, c_3 as in Definition 2.2.

Example 2.4. *The model example of an asymptotically catenoidal exterior hypersurface is an n-dimensional upper half-catenoid.*

Similar to the ADM mass of asymptotically flat Riemannian manifolds we can assign a number to asymptotically flat hypersurfaces in $\mathbb{R}^{n+1}$.

Definition 2.5. (extrinsic mass) *Let S be an asymptotically flat hypersurface of $\mathbb{R}^{n+1}$ with asymptotic coordinates (ω, E, u). We define the* extrinsic mass $m(S)$ *of S by*

$$m(S) := \lim_{r \to \infty} \frac{1}{\omega_{n-1}} \int_{\partial B_r^\omega(0)} \frac{\partial u}{\partial \nu} \, d\mathcal{H}^{n-1}, \tag{2.1}$$

where ν denotes the euclidean outward unit normal to $B^\omega_r(0) \subset \mathbb{R}^n_\omega$ and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff-measure with respect to the euclidean metric in $\mathbb{R}^n_\omega$, and where $\omega_{n-1} = \mathcal{H}^{n-1}(\partial B^\omega_1(0))$.

Proposition 2.6. (well definition of extrinsic mass) *The extrinsic mass $m(S)$ of an asymptotically flat hypersurface S is well defined.*

Proof. Assume w.l.o.g. that $\omega = e_{n+1}$, and set $\Omega(r) := S \cap (B^n_r(0) \times \mathbb{R})$. For large enough r, we have that $\Omega(r)$ is a subset of S with C^2 boundary consisting of $S \cap (\partial B^n_r(0) \times \mathbb{R})$ and ∂S. The outward unit conormal $\eta_{\Omega(r)}$ of $\Omega(r)$ can be obtained by projecting the outward unit normal $(x'/|x'|, 0)$ of $B^n_r(0) \times \mathbb{R}$ onto S and normalizing. I.e.

$$\eta_{\Omega(r)} = \frac{((1+|Du|^2)x' - x' \cdot DuDu, x' \cdot Du)}{\sqrt{1+|Du|^2}\sqrt{(1+|Du|^2)|x'|^2 - (x' \cdot Du)^2}}.$$

We have that

$$e_{n+1} \cdot \eta_{\Omega(r)} = \frac{Du}{\sqrt{1+|Du|^2}\sqrt{(1+|Du|^2) - (x'/|x'| \cdot Du)^2}} \cdot \frac{x'}{|x'|}.$$

Assuming that the limit (2.1) exists, making use of the asymptotics, and using the area formula, we compute

$$\begin{aligned}
m(S) &= \lim_{r\to\infty} \frac{1}{\omega_{n-1}} \int_{\partial B^n_r(0)} \frac{Du}{\sqrt{1+|Du|^2}} \cdot \frac{x'}{|x'|}\, d\mathcal{H}^{n-1} \\
&= \lim_{r\to\infty} \frac{1}{\omega_{n-1}} \int_{(\partial B^n_r(0)\times\mathbb{R})\cap S} e_{n+1} \cdot \eta_{\Omega(r)}\, d\mathcal{H}^{n-1} \\
&= \lim_{r\to\infty} \frac{1}{\omega_{n-1}} \left(\int_{\partial\Omega(r)} e_{n+1} \cdot \eta_{\Omega(r)}\, d\mathcal{H}^{n-1} - \int_{\partial S} e_{n+1} \cdot \eta_{\Omega(r)}\, d\mathcal{H}^{n-1} \right) \\
&= \lim_{r\to\infty} \frac{1}{\omega_{n-1}} \left(\int_{\Omega(r)} \vec{H}_S \cdot e_{n+1}\, d\mathcal{H}^{n} - \int_{\partial S} e_{n+1} \cdot \eta_S\, d\mathcal{H}^{n-1} \right) \\
&= \frac{1}{\omega_{n-1}} \int_S \vec{H}_S \cdot e_{n+1}\, d\mathcal{H}^n - \frac{1}{\omega_{n-1}} \int_{\partial S} e_{n+1} \cdot \eta_S\, d\mathcal{H}^{n-1}. \qquad (2.2)
\end{aligned}$$

Since by assumption $H_S \in L^1(\mathcal{H}^n \llcorner S)$ we can read the above calculation in the reverse sense to conclude that indeed the limit (2.1) exists.

Now suppose that (ω, E, u) and (ω', E', u') are two sets of asymptotic coordinates for S. The rescaled surfaces $S_\lambda := \lambda \cdot S$, $\lambda \downarrow 0$, converge to $\mathbb{R}^n_\omega$ and $\mathbb{R}^n_{\omega'}$ locally uniformly away from the origin. This implies that $\omega = \pm\omega'$. By point 3. in Definition 2.1 $\omega = \omega'$. ☐

Remark 2.7. In case S is asymptotically catenoidal it is easy to see that $m(S) = M$.

Theorem 2.8. (Positive Mass Theorem) *Let $S \subset \mathbb{R}^{n+1}$ be an asymptotically catenoidal exterior hypersurface of non-negative mean curvature. Then*

$$m(S) \geq 0,$$

with equality if and only if S is a hyperplane.

Proof. Suppose $M \leq 0$. Since outside a compact set $K \subset \mathbb{R}^{n+1}$ we have that

$$S \setminus K = \operatorname{graph}(\phi_M(|x|) + \mathcal{O}(1)) \setminus K,$$

where ϕ_M is as in Definition 2.2, the function $x \mapsto x \cdot \omega$ restricted to $S \cup N$ attains its maximum a some point $x_0 \in S \cup N$. In other words we can touch $S \cup N$ from one side with the plane $\{x_0\} + \mathbb{R}^n_\omega$ at x_0.

Suppose the maximum is attained at a point in N. Since $\Delta_N(x \cdot \omega) = 0$ on N, the maximum principle implies that $x_0 \in \partial N$, and hence

$$0 \leq \frac{\partial(x \cdot \omega)}{\partial \eta_N}(x_0) = \omega \cdot \eta_N(x_0) = \omega \cdot \gamma(x_0).$$

This contradicts the fact that the plane $\{x_0\} + \mathbb{R}^n_\omega$ touches from the inside. Hence, the maximum has to be attained at some point $x_0 \in S \setminus \partial S$, and $\gamma(x_0) = -\omega$. Let S_0 be the connected component of the set $\{x \in S : \gamma(x) \cdot \omega < 0\}$ that contains x_0. By assumption $\Delta_S(x \cdot \omega) = \vec{H}_S \cdot \omega \geq 0$ on S_0. The strong maximum principle implies that $\gamma \cdot \omega \equiv \text{const}$ on S_0, and thus $S_0 \subset \{x_0\} + \mathbb{R}^n_\omega$. By connectivity and smoothness of S we have that $\partial S_0 = \partial S$. But clearly $\partial S = \emptyset$, since otherwise we could find a point $x_1 \in N$ with $x_1 \cdot \omega > x_0 \cdot \omega$. Whence $S = \{x_0\} + \mathbb{R}^n_\omega$. □

Proposition 2.9. (Extrinsic Penrose type inequality in the graphical case) *Let $S \subset \mathbb{R}^{n+1}$ be an asymptotically catenoidal exterior hypersurface of non-negative mean curvature that is entirely graphical, and such that $\eta = -\omega$ on $\partial S \neq \emptyset$. Then the following Penrose type inequality holds.*

$$m(S) \geq \left(\frac{n}{\omega_{n-1}}\right)^{\frac{n-1}{n}} \mathcal{H}^n(N)^{\frac{n-1}{n}}, \tag{2.3}$$

with equality if and only if S is an oriented half-catenoid.

Proof. The inequality follows immediately from the equation (2.2) and by applying Almgren's isoperimetric inequality [Alm86] in $\mathbb{R}^{n+1}$. Assuming equality the isoperimetric inequality implies that N is a a round n-ball and ∂S is a round $(n-1)$-sphere. Moreover, (2.2) implies that S is a minimal surface. A result of Kuwert [Kuw93, Theorem 2] implies that S is a half-catenoid. □

We see that in this context the half-catenoid with neck radius m plays the role of the spacial Schwarzschild manifold of mass m. It is now natural to make the following conjecture (due to Huisken): *The inequality* (2.3) *is true for any exterior hypersurface of non-negative mean curvature.*

In this context we mention the PhD thesis of Marquardt [Mar12], in which the author verifies that an extrinsic Hawking type mass in monotone under inverse mean curvature flow with Neumann boundary condition with respect to support surfaces of non-negative mean curvature, however assuming smooth existence of the flow. For convex support surfaces that are graphical and asymptotically cone-like Marquardt was able to prove the existence of weak solutions for inverse mean curvature flow with Neumann boundary condition.

In order to more easily emphasize the dependencies of constants in the following, we shall make use of the following technical lemma, the easy proof of which we shall obmit.

Lemma 2.10. (technical) *Let $S \in \mathcal{F}_n(\beta, c_1, c_2)$ be an asymptotically flat hypersurface of $\mathbb{R}^{n+1}$ with asymptotic coordinates (ω, E, u). There exists a radius $R_1 = R_1(\operatorname{diam}(E), \beta, c_1) \geq 2$ such that the following conditions are met:*

1. $\mathbb{R}^n_\omega \setminus B^\omega_{\frac{R_1}{2}} \subset E$

2. *For every $\sigma \geq R_1$: $x \in S \setminus \overline{B}_\sigma \Rightarrow |P_{\mathbb{R}^n_\omega} x| > \frac{\sigma}{2}$.*
   Here $P_{\mathbb{R}^n_\omega}$ denotes the orthogonal projection onto $\mathbb{R}^n_\omega$.

2.2 Noncompact free boundary minimal surfaces and free boundary constant mean curvature surfaces

Throughout this chapter a free boundary surface Σ with respect to an exterior surface $S \subset \mathbb{R}^3$ will mean, unless specified otherwise, a free boundary surface with respect to S such that $\Sigma \subset \overline{G}$ (with G as in Definition 2.3).

A free boundary minimal surface Σ with respect to a support surface S is a free boundary surface with respect to a support surface S with zero mean curvature. These surfaces are critical points for the area functional in the class of surfaces with boundaries that are confined to lie inside the support surface S.

Using min-max methods Grüter and Jost [GJ86a] proved the existence of free boundary minimal disks inside arbitrary strictly convex bounded domains in $\mathbb{R}^3$. Li [Li14a] was able to prove the existence of free boundary minimal surfaces inside arbitrary compact Riemannain 3-manifolds with boundary that do not contain closed minimal surfaces. Very recently, Maximo, Nunes, and Smith [MNS13] proved the existence of free boundary minimal annuli inside compact, strictly functionally convex Riemannian 3-manifolds of non-negative Ricci curvature.

Stable free boundary minimal surfaces are free boundary minimal surfaces that have non-negative second variation of area. Compact stable free boundary minimal surfaces inside mean convex domains were studied by Ros [Ros08].

Definition 2.11. *Let S be a smooth hypersurface of $\mathbb{R}^{n+1}$, and let Σ be a free boundary minimal hypersurface of $\mathbb{R}^{n+1}$ with respect to S. We say that Σ is* stable *if*

$$0 \leq \int_\Sigma |\nabla^\Sigma f|^2 \, d\mathcal{H}^n - \int_\Sigma |A|^2 f^2 \, d\mathcal{H}^n - \int_{\partial\Sigma} A_S(\nu, \nu) f^2 \, d\mathcal{H}^{n-1} \tag{2.4}$$

for every $f \in C^1_c(\Sigma)$. Here ∇^Σ is the tangential gradient along Σ.

Example 2.12. *Let $G \subset \mathbb{R}^{n+1}$ be a bounded strictly convex domain. Then there are no stable free boundary minimal hypersurfaces inside $\overline{G}$. This follows by testing equation (2.4) with $f \equiv 1$.*

In dimension $n + 1 = 3$ one can make use of the Gauss-Bonnet theorem to conclude that every stable free boundary minimal surface inside a mean convex domain G is a topological disk, cf. [Ros08, Proposition 2].

In this section we prove the following theorem about non-compact stable free boundary minimal surfaces, which is a free boundary analogue of [Car14, Theorem 1].

Theorem 2.13. *Let S be an asymptotically catenoidal exterior surface in $\mathbb{R}^3$ such that S has non-negative mean curvature. Let Σ be a non-compact stable free boundary minimal surface with respect to S. Then S is a plane and Σ is a half-plane meeting S orthogonally.*

To our knowledge this is the first result about non-compact free boundary minimal surfaces in the literature.

A free boundary constant mean curvature surface Σ with respect to a support surface S is a free boundary surface with respect to a support surface S with constant mean curvature. These surfaces are critical points for the area functional in the class of surfaces enclosing a prescribed volume and with boundaries that are confined to lie inside the support surface S.

Stable free boundary constant mean curvature surfaces are free boundary CMC surfaces that have non-negative second variation of area with respect to admissible volume-preserving variations. Compact stable free boundary CMC surfaces inside convex and mean convex domains were studied by Ros and Vergasta [RV95], and Ros [Ros08], respectively. For a survey on volume-preserving stable free boundary CMC surfaces inside convex domains we refer to [Ros05]. Surprisingly, it is still an open problem whether or not the unit ball in $\mathbb{R}^3$ contains volume-preserving stable free boundary CMC surfaces other than the round free boundary spherical caps or the flat free boundary unit disks. Ros and Vergasta [RV95] have shown that such a surface would have to have genus one and at most two connected boundary components.

Definition 2.14 ([RV95])**.** *Let S be a smooth hypersurface of $\mathbb{R}^{n+1}$, and let Σ be a free boundary constant mean curvature hypersurface of $\mathbb{R}^{n+1}$ with respect to S. We say that Σ is* volume-preserving stable *if*

$$0 \leq \int_\Sigma |\nabla^\Sigma f|^2 \, d\mathcal{H}^n - \int_\Sigma |A|^2 f^2 \, d\mathcal{H}^n - \int_{\partial\Sigma} A_S(\nu,\nu) f^2 \, d\mathcal{H}^{n-1} \tag{2.5}$$

for every $f \in C^1_c(\Sigma)$ with $\int_\Sigma f \, d\mathcal{H}^n = 0$.

We say that Σ is strongly stable *if* (2.5) *holds for every $f \in C^1_c(\Sigma)$.*

Remark 2.15. Volume-preserving stable free boundary CMC surfaces naturally arise as stable critical points of the relative isoperimetric problem, also known as Dido's problem.

In this section we prove the following properties about compact stable free boundary CMC surfaces. This is to some extend a free boundary analogue of [EM12, Theorem 1.5].

Theorem 2.16. *Let S be an asymptotically catenoidal exterior surface in $\mathbb{R}^3$ of non-negative mean curvature that is not a plane. For every compact set $K \subset \mathbb{R}^3$ and every $\Theta > 0$ there exists a constant $L = L(S, \Theta, K) > 0$ with the following property:*

Let Σ be a connected, compact volume-preserving stable free boundary constant mean curvature surface with respect to S with $\mathcal{H}^2(\Sigma \cap B_\sigma) \leq \Theta\, \sigma^2$ for all $\sigma \geq 1$ and with $\mathcal{H}^1(\partial\Sigma) \geq L$. Then $\Sigma \cap K = \emptyset$.

2.2.1 Preparatory results

In this subsection we derive the necessary results that are needed in order to prove Theorems 2.13 and 2.16. These include for example new curvature estimates for stable free boundary

CMC surfaces (Propositions 2.28 and 2.29) and new free boundary length estimates (Corollary 2.19 and Proposition 2.19).

As in the works of Huisken and Yau [HY96], Eichmair and Metzger [EM12], Chodosh [Cho14], and Chodosh, Eichmair, and Volkmann [CEV14] a crucial ingredient to studying sequences of connected volume-preserving stable free boundary CMC surfaces of diverging area is the following Christodoulou-Yau type inequality due to Ros and Vergasta [RV95].

Proposition 2.17 (Essentially [RV95, Theorem 5])**.** *Let Σ be a connected compact volume-preserving stable free boundary constant mean curvature surface with respect to a support surface $S \subset \mathbb{R}^3$. Then*

$$\int_{\partial\Sigma} H_S \, d\mathcal{H}^1 + \frac{3}{4}\int_{\Sigma} H^2 \, d\mathcal{H}^2 + \frac{1}{2}\int_{\Sigma} |A^\circ|^2 \, d\mathcal{H}^2 \leq 14\pi.$$

In case Σ is a topological disk the bound on the right hand side can be improved to 10π.

The following proposition will allow us to estimate boundary integrals by means of integrals over the interior of the surface. First length estimates for the free boundary of compact partially free boundary minimal surfaces were obtained by Hildebrandt and Nitsche [HN83, Theorem 1].

Recall that for $A \subset S$

$$\kappa_A(x) := \inf\left\{\kappa \geq 0 : |\gamma(x)\cdot(x-y)| \leq \frac{\kappa}{2}|x-y|^2 \text{ for all } y \in A\right\}.$$

It is easy to see that this definition agrees with the one given in Definition 1.11 of Chapter 1.

Lemma 2.18. (free boundary length estimate) *Let $G \subset \mathbb{R}^3$ be a domain with C^2 boundary $S = \partial G$ such that $\kappa_S \leq \Lambda < \infty$ for some $\Lambda \geq 0$. For every connected compact free boundary surface $\Sigma \subset \overline{G}$ one has the following estimate*

$$\mathcal{H}^1(\partial\Sigma) \leq 8\Lambda\,\mathcal{H}^2(\Sigma) + \int_{\Sigma} |H| \, d\mathcal{H}^2.$$

Proof. Let $A := \{x \in \overline{G} : \pi_S(x) \text{ is a well defined}\}$. By our assumption $\{x \in \overline{G} : d_S(x) < \Lambda^{-1}\} \subset A$. For all $x \in A$ we have that

$$x = \pi_S(x) + d_S(x) D d_S(x)$$

and therefore

$$\|D\pi_S\| \leq 2 + d_S\|D^2 d_S\|.$$

By [GT01, Lemma 14.17] we have the pointwise estimate

$$\|D^2 d_S(x)\| \leq \max_{|v|=1} \frac{|A_S(\pi_S(x))(v,v)|}{1 - |A_S(\pi_S(x))(v,v)| d_S(x)},$$

and hence for all $x \in A$ such that $d_S(x) \leq \frac{1}{2}\Lambda^{-1}$ we have that

$$\|D^2 d_S(x)\| \leq 2\max_{|v|=1} |A_S(\pi_S(x))(v,v)| \leq 2\Lambda.$$

We define a Lipschitz function φ_0 on A by

$$\varphi_0(x) := (1 - 3\Lambda d_S(x))^+$$

Then $|D\varphi_0|(x) \leq 3\Lambda$. Mollifying φ_0 we obtain a smooth function $\varphi \in C^2_c(A)$ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ on S, $\operatorname{spt}(\varphi) \subset \{d_S < (2\Lambda)^{-1}\}$, and $|D\varphi| \leq 4\Lambda$. Now set $\tilde{\gamma} := -\varphi\, Dd_S$. Since Σ is compact we may use $\tilde{\gamma}$ as a test function in the first variation identity to obtain

$$\begin{aligned}
\mathcal{H}^1(\partial\Sigma) &= -\int_\Sigma \operatorname{div}_\Sigma(\varphi D d_S)\, d\mathcal{H}^2 - \int_\Sigma \varphi \vec{H} \cdot Dd_S \, d\mathcal{H}^2 \\
&\leq \int_\Sigma |D\varphi|\, d\mathcal{H}^2 + \int_{\Sigma \cap \{d_S < (2\Lambda)^{-1}\}} |D^2 d_S|\, d\mathcal{H}^2 + \int_\Sigma \varphi |H|\, d\mathcal{H}^2 \\
&\leq 8\Lambda \mathcal{H}^2(\Sigma) + \int_\Sigma |H|\, d\mathcal{H}^2.
\end{aligned}$$

□

Combining Proposition 2.17 and Corollary 2.18 we obtain using Hölder's inequality:

Corollary 2.19. *Let $G \subset \mathbb{R}^3$ be a domain with C^2 boundary $S = \partial G$ of non-negative mean curvature H_S and such that $\kappa_S \leq \Lambda < \infty$. For every compact volume-preserving stable free boundary constant mean curvature surface Σ inside G one has the following estimate*

$$\mathcal{H}^1(\partial\Sigma) \leq 8\Lambda\, \mathcal{H}^2(\Sigma) + \left(\frac{56\pi}{3}\right)^{\frac{1}{2}} \mathcal{H}^2(\Sigma)^{\frac{1}{2}}.$$

Proposition 2.20. *Let S be an exterior surface in $\mathbb{R}^3$, and let $R_1 \geq 2$ be as in Lemma 2.10. Let Σ be a free boundary surface with respect to S such that $\Sigma \cap S = \partial\Sigma$. Then for any $p \in \mathbb{R}$ and a.e. $R_1 < \sigma < \rho < \infty$*

$$\begin{aligned}
\int_{\partial\Sigma \cap A_{\sigma,\rho}} r^{-p}\, d\mathcal{H}^1 \leq{}& C(p, c_2/R_1) \int_{\Sigma \cap A_{\sigma,\rho}} r^{-1-p}\, d\mathcal{H}^2 \\
&+ C(c_2/R_1) \int_{\Sigma \cap A_{\sigma,\rho}} r^{1-p} H^2\, d\mathcal{H}^2 \\
&+ C(c_2/R_1) \left(\rho^{-p}\mathcal{H}^1(\Sigma \cap \partial B_\rho) + \sigma^{-p}\mathcal{H}^1(\Sigma \cap \partial B_\sigma)\right),
\end{aligned}$$

where $A_{\sigma,\rho} = B_\rho \setminus \overline{B}_\sigma$, and c_2 is as in Definition 2.1.

Proof. We test the first variation identity with the vector field $-r^{-p}\omega$ to obtain for a.e. $0 < \sigma < \rho < \infty$

$$\begin{aligned}
&\int_{\Sigma \cap A_{\sigma,\rho}} p r^{-p-1} \nabla^\Sigma r \cdot \omega \, d\mathcal{H}^2 \\
&= \int_{\Sigma \cap A_{\sigma,\rho}} r^{-p} \vec{H} \cdot \omega\, d\mathcal{H}^2 - \int_{\partial A_{\sigma,\rho} \cap \Sigma} r^{-p} \omega \cdot \eta \, d\mathcal{H}^1 - \int_{\partial\Sigma \cap A_{\sigma,\rho}} r^{-p} \omega \cdot \gamma\, d\mathcal{H}^1,
\end{aligned}$$

where η denotes the outward unit conormal of $\Sigma \cap A_{\sigma,\rho}$. Applying Young's inequality yields

$$\begin{aligned}
\left| \int_{\partial\Sigma \cap A_{\sigma,\rho}} \frac{\omega \cdot \gamma}{r^p}\, d\mathcal{H}^1 \right| \leq{}& (\frac{1}{2} + p) \int_{\Sigma \cap A_{\sigma,\rho}} r^{-1-p}\, d\mathcal{H}^2 + \frac{1}{2} \int_{\Sigma \cap A_{\sigma,\rho}} r^{1-p} H^2\, d\mathcal{H}^2 \\
&+ \rho^{-p}\mathcal{H}^1(\Sigma \cap \partial B_\rho) + \sigma^{-p}\mathcal{H}^1(\Sigma \cap \partial B_\sigma).
\end{aligned}$$

On the other hand, we have for $\sigma \geq \sigma_0 (> 0)$ that

$$-\omega \cdot \gamma = \frac{1}{\sqrt{1 + |Du|^2}} \geq \frac{1}{\sqrt{1 + c_2^2 R_1^{-2}}}.$$

Whence

$$\begin{aligned}\int_{\partial\Sigma \cap A_{\sigma,\rho}} \frac{1}{r^p}\, d\mathcal{H}^1 \leq C(p, c_2/R_1) &\int_{\Sigma \cap A_{\sigma,\rho}} r^{-1-p}\, d\mathcal{H}^2 \\ + C(c_2/R_1) &\int_{\Sigma \cap A_{\sigma,\rho}} r^{1-p} H^2\, d\mathcal{H}^2 \\ + C(c_2/R_1) &\left(\rho^{-p}\mathcal{H}^1(\Sigma \cap \partial B_\rho) + \sigma^{-p}\mathcal{H}^1(\Sigma \cap \partial B_\sigma)\right).\end{aligned}$$

□

Corollary 2.21. *Let Σ be a free boundary surface with respect to an exterior surface S such that Σ has square-integrable mean curvature and such that $\mathcal{H}^2(\Sigma \cap B_\sigma) \leq \Theta\, \sigma^2$ for all $\sigma \geq 1$. Then for all $p > 1$*

$$\int_{\partial\Sigma} \frac{1}{r^p}\, d\mathcal{H}^1 < \infty.$$

Curvature estimates for stable free boundary CMC surfaces

Now we prove curvature estimates for volume-preserving stable free boundary constant mean curvature surfaces. Inspired by the work of Eichmair and Metzger [EM12] these will be obtained by blow up arguments and will rely on the following characterization of volume-preserving stable (free boundary) constant mean curvature immersions in $\mathbb{R}^3(\cap\{x^3 \geq 0\})$.

Theorem 2.22 ([BdC84,Pal86,DS87,LR89]). *Let M be a 2-dimensional orientable connected smooth manifold. Let $F : M \to \mathbb{R}^3$ be a complete immersion of constant mean curvature that is volume-preserving stable. Then $F(M) \subset \mathbb{R}^3$ is either a plane or a round sphere.*

At this point we also recall the following older characterization of stable minimal immersion in $\mathbb{R}^3$.

Theorem 2.23 ([dCP79, FCS80]). *Let M be a 2-dimensional orientable connected smooth manifold. Let $F : M \to \mathbb{R}^3$ be a complete stable minimal immersion. Then $F(M) \subset \mathbb{R}^3$ is a plane.*

We will also need the corresponding results for free boundary surfaces inside a half-space. The theorem for stable free boundary minimal surfaces may be directly inferred from Theorem 2.23 by a reflection argument. Theorem 2.22 on the other hand cannot be concluded from the corresponding theorem for closed surfaces by such an argument. However, it is straightforward (since all the boundary terms vanish) but rather lengthy to adapt the proof in [DS87] to the case below. To get an idea of the adaptations involved, we refer to Subsection 2.4.3 of this chapter's appendix. However, for the sake of brevity we shall omit the (complete) proof of the following theorem.

Theorem 2.24. *Let M be a 2-dimensional orientable connected smooth manifold with boundary $\partial M \neq \emptyset$. Let $F : M \to \mathbb{R}^3 \cap \{x^3 \geq 0\}$ be a complete immersion of constant mean curvature that is volume-preserving stable and has free boundary inside $\{x^3 = 0\}$. Then $F(M) \subset \mathbb{R}^3 \cap \{x^3 \geq 0\}$ is either a free boundary half-plane or a free boundary round half-sphere.*

Remark 2.25. We point out that the immersions in Theorems 2.22, 2.23, and 2.24 are not required to be proper. This will be relevant in the proof of Propositions 2.28 and 2.29 below.

Before we state our curvature estimates we need a technical lemma that gives an estimate on the ball curvature κ_S of asymptotically flat surfaces S outside a sufficiently large compact region.

Lemma 2.26. *Let $S \in \mathcal{F}_n(\beta, c_1, c_2)$ be an asymptotically flat hypersurface of $\mathbb{R}^{n+1}$ with asymptotic coordinates (ω, E, u). There exists a radius σ_2 depending only on $n, \operatorname{diam}(E), \beta, c_1, c_2$ such that for all $x \in S \setminus \overline{B}_{\sigma_2}$*

$$\kappa_S(x) < 8|x|^{-1}.$$

Proof. W.l.o.g. we assume that $\omega = e_{n+1}$ and $\mathbb{R}^n \setminus \overline{B}_{R_1} \subset E$, where R_1 is as in Lemma 2.10. For the duration of this proof x, z will denote points in $\mathbb{R}^n \setminus \overline{B}_{R_1}$. We set $F(x) := (x, u(x))$. Now let $x \in \mathbb{R}^n \setminus \overline{B}_{2R_1}$. Suppose that there exists a point $z \in \mathbb{R}^n \setminus \overline{B}_{R_1}$, $z \neq x$, such that $F(z) \in \partial B_R(p)$, for a tangent sphere $\partial B_R(p)$ to S at x, and with some radius $R \leq |x|/4$. Then

$$\begin{aligned}
0 &= |p - F(z)|^2 - R^2 \\
&= \left| F(x) - F(z) \pm R \frac{(-Du(x), 1)}{\sqrt{1 + |Du(x)|^2}} \right|^2 - R^2 \\
&= |F(x) - F(z)|^2 \pm 2R(F(x) - F(z)) \cdot \frac{(-Du(x), 1)}{\sqrt{1 + |Du(x)|^2}} \\
&= |x - z|^2 + |u(x) - u(z)|^2 \mp 2R \frac{(x - z) \cdot Du(x)}{\sqrt{1 + |Du(x)|^2}} \pm 2R \frac{u(x) - u(z)}{\sqrt{1 + |Du(x)|^2}}.
\end{aligned}$$

Hence,

$$1 + \frac{|u(x) - u(z)|^2}{|x - z|^2} = \frac{2R}{\sqrt{1 + |Du(x)|^2}} \frac{|(u(z) - u(x)) - Du(x) \cdot (z - x)|}{|x - z|^2}. \tag{2.6}$$

The right hand side of (2.6) is no greater than

$$2R \frac{|(u(z) - u(x)) - Du(x) \cdot (z - x) - \frac{1}{2} D^2 u(x)(z - x, z - x)|}{|x - z|^2} + R \frac{c_2}{|x|^n}. \tag{2.7}$$

Set $f(t) := u(x + t(z - x))$. Then there exists a $\tau \in [0, 1]$ such that

$$f(1) - f(0) - f'(0) - \frac{1}{2} f''(0) = \frac{1}{2} \left(f''(\tau) - f''(0) \right).$$

Hence,

$$\begin{aligned}
&u(z) - u(x) - Du(x) \cdot (z - x) - \frac{1}{2} D^2 u(x)(z - x, z - x) \\
&\qquad = \frac{1}{2} \Big(D^2 u(x + \tau(z - x)) - D^2 u(x) \Big) (z - x, z - x),
\end{aligned}$$

and we infer from (2.6) and (2.7) that

$$
\begin{aligned}
R^{-1} &\leq 2\frac{|(u(z)-u(x)) - Du(x)\cdot(z-x) - \frac{1}{2}D^2u(x)(z-x,z-x)|}{|x-z|^2} + \frac{c_2}{|x|^n} \\
&\leq \sup_{\tau\in[0,1]} |D^2u(x+\tau(z-x)) - D^2u(x)| + \frac{c_2}{|x|^n} \\
&\leq 2\sup_{y\in B_{2R}(x)} |D^2u(y)| + \frac{c_2}{|x|^n} \\
&\leq 2\sup_{y\in B_{2R}(x)} \frac{c_2}{|y|^n} + \frac{c_2}{|x|^n} \\
&\leq 2\frac{c_2}{||x|-2R|^n} + \frac{c_2}{|x|^n}.
\end{aligned}
$$

Using the fact that $R \leq \frac{|x|}{4}$ we get a contradiction for $|x| > (c_2(1+2^{n+1})/4)^{1/(n-1)}$. We conclude that for $|x| > \max(2R_1, (c_2(1+2^{n+1})/4)^{1/(n-1)})$

$$\kappa_S(F(x)) < \frac{4}{|x|}.$$

Using the estimate $|F(x)| \leq |x| + |u(x)| \leq 2|x|$, which holds for all $|x| > \max(R_1, c_1^{1/\beta})$, the claim follows. $\square$

Remark 2.27. Varying the proof slightly one can prove the existence of a constant $C = C(c_2) < \infty$ such that

$$\kappa_{S\setminus B_\sigma} < C\sigma^{-2}$$

for all $\sigma > R_1$.

We omit the proof of the following proposition as its proof is easier and may be derived similarly, to the one of Proposition 2.29.

Proposition 2.28. (free boundary CMC curvature estimate) *Let $S = \partial G \subset \mathbb{R}^3$ be a surface with $\kappa_S \leq \Lambda$, for some $\Lambda < \infty$. There exists a constant $C_1 = C_1(\Lambda) < \infty$ such that for every orientable, immersed volume-preserving stable free boundary constant mean curvature surface $\Sigma \subset \overline{G}$ with respect to S with $|H| \leq 1$ satisfies $\sup_\Sigma |A| \leq C_1$.*

Proposition 2.29. (free boundary CMC weighted curvature estimate) *Let S be an exterior surface and let $K \subset \mathbb{R}^3$ be a compact set. There exists a constant $C_2 > 0$ depending on S and K such that*

$$\sup_{x\in\Sigma} (\max\{|x|,1\}|A|(x)) \leq C_2$$

for every connected volume-preserving stable free boundary constant mean curvature surface Σ with respect to S that satisfies $|H| \leq 1$ and $\Sigma \cap K \neq \emptyset$.

Proof. Assume by contradiction that the theorem is false. Then there exists a surface S as in the statement and connected volume-preserving stable free boundary CMC surfaces Σ_k with $|H_k| \leq 1$ and $\Sigma_k \cap K \neq \emptyset$, and points $x_k \in \Sigma_k \setminus B_1$ (by Proposition 2.28) such that

$$\gamma_k := \max_{z\in\Sigma_k\cap B_{\frac{|x_k|}{2}}(x_k)} \left(\left(\frac{|x_k|}{2} - |z-x_k|\right)|A_k|(z)\right) \to \infty \quad \text{as } k\to\infty.$$

By Proposition 2.28 we have that $|x_k| \to \infty$. In fact we even have that $|x_k'| \to \infty$, since otherwise we could pass to a subsequence such that $B_{\frac{|x_k|}{2}}(x_k) \cap S = \emptyset$, which would contradict the interior curvature estimate [EM12, Proposition 2.3]. Now choose $z_k \in \Sigma_k \cap B_{\frac{|x_k|}{2}}(x_k)$ such that

$$\left(\frac{|x_k|}{2} - |z_k - x_k|\right) |A_k|(z_k) = \gamma_k,$$

and set $r_k := \frac{|x_k|}{2} - |z_k - x_k|$. Then, since for all $z \in B_{\frac{r_k}{2}}(z_k)$, $\frac{r_k}{2} \leq \frac{|x_k|}{2} - |z - x_k|$, and since $B_{\frac{r_k}{2}}(z_k) \subset B_{\frac{|x_k|}{2}}(x_k)$, we have

$$\begin{aligned}
\sup_{B_{\frac{r_k}{2}}(z_k)} |A_k| &\leq \frac{2}{r_k} \sup_{z \in \Sigma_k \cap B_{\frac{r_k}{2}}(z_k)} \left(\left(\frac{|x_k|}{2} - |z - x_k| \right) |A_k|(z) \right) \\
&\leq \frac{2}{r_k} \sup_{z \in \Sigma_k \cap B_{\frac{|x_k|}{2}}(x_k)} \left(\left(\frac{|x_k|}{2} - |z - x_k| \right) |A_k|(z) \right) \\
&= 2|A_k|(z_k).
\end{aligned}$$

On the other hand, we trivially have that

$$|A_k|(z_k) \leq \sup_{B_{\frac{r_k}{2}}(z_k)} |A_k|.$$

We consider the rescaled surfaces $\hat{\Sigma}_k := h_k(\Sigma_k)$ and $\hat{S}_k := h_k(S)$, where

$$h_k(x) := \frac{\gamma_k}{r_k}(x - z_k) = |A_k|(z_k)(x - z_k).$$

Then $0 \in \hat{\Sigma}_k$, $|\hat{A}_k|(0) = 1$, and

$$\sup_{B_{\frac{\gamma_k}{2}}(0)} |\hat{A}_k| \leq 2.$$

We consider two cases:

1. $\limsup_{k\to\infty} \operatorname{dist}(0, \partial\hat{\Sigma}_k) = \limsup_{k\to\infty} \frac{\gamma_k \operatorname{dist}(z_k, \partial\Sigma_k)}{r_k} = \infty$.

   In this case we have that $\partial\hat{\Sigma}_k \cap B_{\frac{\sigma_k}{2}}(0) = \emptyset$ for $\sigma_k = \min(\gamma_k, \operatorname{dist}(0, \partial\hat{\Sigma}_k)/2)$, and we can argue as in [EM12, Proposition 2.3], making use of Theorem 2.22, to obtain a contradiction.

2. $\limsup_{k\to\infty} \operatorname{dist}(0, \partial\hat{\Sigma}_k) = \limsup_{k\to\infty} \frac{\gamma_k \operatorname{dist}(z_k, \partial\Sigma_k)}{r_k} = \Lambda < \infty$.

   Noting that $\operatorname{dist}(B_{\frac{r_k}{2}}(z_k), 0) \to \infty$ as $k \to \infty$, we have that

   $$\hat{S}_k \cap B_{\frac{\gamma_k}{2}}(0) = \operatorname{graph}(\hat{u}_k) \cap B_{\frac{\gamma_k}{2}}(0),$$

   where

   $$\hat{u}_k(y') = \frac{\gamma_k}{r_k}\left(u\left(\frac{r_k}{\gamma_k}y' + z_k'\right) - z_k \cdot \omega\right), \quad y' \in \mathbb{R}^2_\omega.$$

   We aim to show that we can pass to a subsequence such that $\hat{u}_k$ converges to a constant function in $C^2_{loc}(\mathbb{R}^2_\omega)$.

Firstly, notice that since $|x_k'| \to \infty$ we also have that $|z_k'| \to \infty$. Now pick a sequence $p_k \in S$ of nearest points to z_k, i.e. $|z_k - p_k| = \operatorname{dist}(z_k, S)$. Since, in particular $\limsup_{k\to\infty} \frac{\operatorname{dist}(z_k,S)}{r_k} = 0$, we have (after passing to a subsequence) that

$$|p_k| \geq |z_k| - |z_k - p_k| \geq \left(r_k + \frac{|x_k|}{2}\right) - o(1) r_k. \tag{2.8}$$

Hence, $|p_k| \to \infty$ and therefore $p_k \in \operatorname{graph}(u)$ for sufficiently large k.

Moreover, our cases-assumption together with (2.8) implies that

$$|z_k - p_k| \leq o(1)|p_k|. \tag{2.9}$$

On the other hand, Lemma 2.26 implies that $|p_k| \leq 8\,\kappa_S(p_k)^{-1}$, which implies that $p_k = \pi_S(z_k)$ is the unique nearest point of S to z_k for all sufficiently large k. By asymptotic flatness of S we have

$$|p_k| \leq 2|p_k'| \quad \text{for } k \text{ sufficiently large.} \tag{2.10}$$

Using the identity $p_k = z_k \pm |z_k - p_k|\gamma(p_k)$ together with (2.9) and (2.10) we infer that

$$|z_k| \leq 8|z_k'| \quad \text{for } k \text{ sufficiently large.} \tag{2.11}$$

We are now ready to estimate the C^2-norms of the rescaled functions $\hat{u}_k$. On $B_\rho^\omega(0) \subset B^\omega_{\gamma_k/16}(0)$ we estimate using (2.11) and the fact that $r_k \leq |z_k'|$

$$\sup_{B_\rho^\omega(0)} |D\hat{u}_k| \leq \frac{c_2}{|z_k'| - \frac{r_k}{\gamma_k}\rho} \leq \frac{16c_2}{|z_k|} \quad \text{for } k \text{ sufficiently large}$$

and

$$\sup_{B_\rho^\omega(0)} |D^2\hat{u}_k| \leq \frac{r_k}{\gamma_k}\frac{16c_2}{|z_k|} \leq \frac{16c_2}{\gamma_k} \quad \text{for } k \text{ sufficiently large.}$$

Moreover, we have after passing to a subsequence

$$\begin{aligned}
|\hat{u}_k(0)| &\leq \frac{\gamma_k}{r_k}\,|u(z_k') - z_k \cdot \omega| \\
&\leq \frac{\gamma_k}{r_k}|u(z_k') - u(p_k')| + |(z_k - p_k)\cdot\omega| \\
&\leq \frac{\gamma_k}{r_k}\left(1 + \sup_{B^\omega_{o(1)r_k}(z_k')} |Du|\right)|z_k - p_k| \\
&\leq 2\Lambda\left(1 + \frac{16c_2}{|z_k|}\right) \quad \text{for } k \text{ sufficiently large.}
\end{aligned}$$

Hence,

$$\begin{aligned}
\sup_{B_\rho^\omega(0)} |\hat{u}_k| &\leq |\hat{u}_k(0)| + \rho \sup_{B_\rho^\omega(0)} |D\hat{u}_k| \\
&\leq 2\Lambda\left(1 + \frac{16c_2}{|z_k|}\right) + \rho\frac{16c_2}{|z_k|} \quad \text{for } k \text{ sufficiently large.}
\end{aligned}$$

We conclude that we can pass to a subsequence such that $\hat{u}_k$ converges to a constant function in $C^2_{loc}(\mathbb{R}^2_\omega)$, and $\hat{S}_k$ converges to a plane in C^2_{loc}. On the other hand, we can apply a standard argument as in [PR02, Theorem 4.39], but now together with Lemma 2.42, to see that we may pass to a subsequence such that the accumulation set of the sequence $\hat{\Sigma}_k \cap B_{\frac{\sigma_k}{2}}(0)$ contains a complete (not necessarily properly) embedded volume-preserving stable CMC surface Σ_∞ in $\mathbb{R}^3 \cap \{x^3 \geq 0\}$ with $|A_\infty|(0) = 1$ and (possibly empty) free boundary inside $\{x^3 = 0\}$. Theorem 2.24 or Theorem 2.22, respectively, implies that Σ_∞ is a plane, a free boundary half-plane, a sphere, or a free boundary half-sphere. The first two alternatives are excluded since by construction $|A_\infty|(0) = 1$. The latter two alternatives would imply that every Σ_k contains a far out spherical component in $B_{\frac{|x_k|}{2}}(x_k)$, contradicting the assumption that Σ_k is connected and $\Sigma_k \cap K \neq \emptyset$.

□

Graphicality and quadratic area growth

We now use the curvature estimates derived above to show, via blow-down, that non-compact stable free boundary minimal surfaces with respect to asymptotically catenoidal exterior surfaces only have finitely many ends which are each graphical on large 'annular' sets. The proof, which does not *assume* quadratic area growth (cf. [EM12, Lemma 3.5]), is inspired by the work of [Car14]. The price to pay it that we need to assume that the support surface be asymptotically *catenoidal* instead of only asymptotically *flat*.

Lemma 2.30. (graphical decomposition) *Let $S \subset \mathbb{R}^3$ be an asymptotically catenoidal exterior surface. Let Σ be a connected non-compact stable free boundary minimal surface with respect to S. For every sequence $\sigma_k \to \infty$ there exists a subsequence $\sigma'_k \to \infty$ and a plane Π such that the intersection of Σ with the normal cylinder above each annulus $\Pi \cap B_{3\sigma'_k} \setminus B_{\sigma'_k}$ is a union of finitely many disjoint graphs above connected subsets $U^l_{\sigma'_k}$ of this annulus. Moreover, denoting by $\{w^1_k, ..., w^N_k\}$ these functions we have that*

$$\sup_{U^l_{\sigma'_k}} \left(\frac{|w^l_k|}{|\cdot|} + |Dw^l_k| + |\cdot||D^2 w^l_k| \right) \to 0 \quad \text{as } \sigma'_k \to \infty.$$

Proof. W.l.o.g. we assume that we have asymptotic coordinates (ω, E, u) with $\omega = e_3$. Let R_1 be as in Lemma 2.10. Consider the rescaled surfaces $\Sigma_k := \sigma_k^{-1} \cdot \Sigma$ in $\mathbb{R}^3 \setminus B_{R_1/\sigma_k}(0)$, and the rescaled support surfaces $S_k := \sigma_k^{-1} \cdot S$. It is easy to see that for every index $l \geq 1$ we have that for $k \geq l$

$$S_k \cap \{|x'| > R_1/\sigma_l\} = \text{graph}(u_k) \cap \{|x'| > R_1/\sigma_l\},$$

where $u_k(x') = \sigma_k^{-1} u(\sigma_k x')$. We set $\phi_k(x) := (x', x^3 + u_k(x'))$. Then for all fixed $\varepsilon > 0$ and all $k \geq 1$ sufficiently large (such that $\sigma_k \geq R_1/\varepsilon$) we have that

$$\phi_k : (\{x^3 > 0\} \cap \{|x'| > \varepsilon\}, \phi_k^* \delta) \to (G \cap \{|x'| > \varepsilon\}, \delta)$$

is an isometry. Moreover, we have for $k \geq 1$ sufficiently large that

$$\sup_{|x'| \geq \varepsilon} |\phi_k^* \delta - \delta|_\delta \leq \sup_{|x'| \geq \varepsilon \sigma_k} |Du| \sqrt{2 + |Du|^2} < \infty,$$

and which goes to zero as $k \to \infty$. Similarly, one easily verifies that for the first derivative of the metric

$$\sup_{|x'|\geq\varepsilon} |D(\phi_k^*\delta)|_\delta < \infty,$$

which also goes to zero as $k \to \infty$. In particular, we see that for a given $\varepsilon > 0$ and sufficiently large k we have that

$$\sup_{|x'|>\varepsilon} |A_{\Sigma_k^0}| \leq 2 \sup_{|x'|>\varepsilon} |A_{\Sigma_k^0,g_k}|_{g_k} = 2 \sup_{|x'|>\varepsilon} |A_k|,$$

where we set $g_k := \phi_k^*\delta$ and where $\Sigma_k^0 := \phi_k^{-1}(\Sigma_k \cap \{|x'| \geq R_1/\sigma_k\})$. Using the curvature estimate of Proposition 2.29 we arrive at

$$\sup_{|x'|>\varepsilon} |A_{\Sigma_k^0}| \leq \frac{2C_2}{\varepsilon} \quad \text{for all } k \text{ sufficiently large.}$$

Moreover, we have on $\partial\Sigma_k^0 \cap \{|x'| > \varepsilon\}$ that

$$\sup_{|x'|>\varepsilon} |\eta_{\Sigma_k^0} \cdot e_3| \geq 1 - o(1) \quad \text{as } k \to \infty.$$

We claim that we can pass to a subsequence, obtain a limit, and perform a reflection to obtain a stable minimal lamination of $\mathbb{R}^3 \setminus \{0\}$.

To see this, let $x_0 \in (\mathbb{R}^3 \setminus \{0\}) \cap \{x^3 = 0\}$ and let $\rho \in (0, |x_0|/2)$. Then for k sufficiently large, depending on $|x_0|$, $\Sigma_k^0 \cap B_\rho(x_0)$ are (possibly disconnected surfaces) with boundary inside $(\partial B_\rho(x_0) \cap \{x^3 > 0\}) \cup \{x^3 = 0\}$, and such that

$$\sup_{B_{\frac{|x_0|}{2}}(x_0)} |A_{\Sigma_k^0}| \leq \frac{4C_2}{|x_0|} \quad \text{for all } k \text{ sufficiently large.}$$

For the duration of this proof we set $Z_R^v(x) := \{z \in \mathbb{R}^3 : |z - z \cdot vv - x| < R\}$ for $R > 0$ and $|v| = 1$. Moreover, we denote by n_k the outward unit normal of $\partial\Sigma_k^0 \cap \{x^3 = 0\}$ as a submanifold of $\{x^3 = 0\}$.

For k sufficiently large, we apply Lemma 2.42 with $\lambda \geq 1/2$ and infer that for all $x \in \partial\Sigma_k^0 \cap B_{|x_0|/8}(x_0)$ the component of $\Sigma_k^0 \cap Z_R^{n_k}(x)$ through x is the graph of a function w_k over the half-disk $B_R^{n_k}(x) \cap \{x^3 \geq 0\}$ of radius $R = \lambda\frac{|x_0|}{8} \min\left\{3, \frac{\lambda}{2C_2}\right\} \geq \frac{|x_0|}{16\max\{1,4C_2\}}$ such that for all $y' \in B_R^{n_k}(x) \cap \{x^3 \geq 0\}$

$$\frac{|w_k(y')|}{|y' - x_0'|} + |Dw_k(y')| + |y' - x_0'||D^2w_k(y')| \leq \left(8\frac{4C_2}{|x_0|}\lambda^{-3}|y' - x_0'| + \lambda^{-1}\sqrt{1-\lambda^2}\right).$$

Hence, there exists a universal constant $\theta \in (0,1)$ such that all connected components of $\Sigma_k^0 \cap B_{2\theta R}(x_0)$ with $\partial\Sigma_k^0 \cap B_{\theta R}(x_0) \neq \emptyset$ can be written as graphs of functions $w_k^1, ..., w_k^{N_k} \in C^2(\overline{B_{\theta R}^{v_k}(x_0) \cap \{x^3 > 0\}})$ for some unit vector $v_k \in \{x^3 = 0\}$, and with uniform C^2-bounds (only depending on $|x_0|$ and C_2). Here N_k is a finite non-negative integer depending on $|x_0|$ and k. Now let $x \in \Sigma_k^0 \cap B_{\theta R}(x_0)$ such that $x \notin \bigcup_{i=1}^{N_k} \operatorname{graph}(w_k^i)$. Then $\operatorname{dist}_{\Sigma_k^0}(x, \partial\Sigma_k^0) \geq \theta R$, and hence we may apply a standard graphical argument (see e.g. [PR02, Lemma 4.1.1]) to conclude that, after possibly choosing θ slightly smaller, all components of $\Sigma_k^0 \cap B_{\theta R}(x_0) \setminus \bigcup_{i=1}^{N_k} \operatorname{graph}(w_k^i)$ that run through $B_{\theta R/2}(x_0)$ are graphical above $B_{\theta R/2}(x_0) \cap \{x^3 = 0\}$.

Using the curvature estimates of Theorem 2.29 and a diagonal subsequence argument, we conclude that there exists a subsequence σ_k' so that $\Sigma_k' := (\sigma_k')^{-1} \cdot \Sigma$ converges to a (free boundary) minimal limit lamination $\mathcal{L}$ of $\{x^3 \geq 0\} \setminus \{0\}$ with respect to the support plane $\{x^3 = 0\}$. Note that the free boundary of all the leaves of $\mathcal{L}$ is empty if and only if $\partial\Sigma$ is compact. Here one may argue as in [CM04, Appendix B], but now (in case $\partial\Sigma$ is non-compact) also using the boundary Schauder estimates [GT01, Theorem 6.30] and boundary Harnack inequality [Lie01, Theorem 4.3]. Thus, we may use a reflection argument across $\{x^3 = 0\}$ and obtain a minimal lamination $\hat{\mathcal{L}}$ of $\mathbb{R}^3 \setminus \{0\}$ that is complete away from the origin. Also notice that each leaf in the foliation $\hat{\mathcal{L}}$ is stable (after possibly lifting to its universal cover). By a removable singularity theorem of Gulliver and Lawson [GL86] this lamination must be complete in $\mathbb{R}^3$. Theorem 2.23 implies that $\hat{\mathcal{L}}$ is a union of flat planes. Let us first assume that $\partial\Sigma$ is non-compact. After possibly applying a rotation about the x^3-axis we may assume that $\hat{\mathcal{L}} = Y \times \mathbb{R}^2$ for a closed set $Y \subset \mathbb{R}$. Since planes in $\mathbb{R}^3$ are totally geodesic the local C^2-convergence implies that in fact $|A(x)| \leq o(1)|x|^{-1}$ as $|x| \to \infty$. This implies that for σ large enough we have that $|A(x)| \leq \frac{1}{4}|x|^{-1}$.

Now let $f : \Sigma \to \mathbb{R}$, $f(x) = |x|^2/2$, then

$$\Delta_\Sigma f = 2 > 0 \text{ on } \Sigma \quad \text{and} \quad \nabla^\Sigma_\gamma f = x \cdot \gamma \text{ on } \partial\Sigma.$$

On $\partial\Sigma \setminus \overline{B}_{R_1}$ we have that

$$\begin{aligned} x \cdot \gamma(x) &= \frac{x' \cdot Du(x')}{\sqrt{1 + |Du(x')|^2}} - \frac{u(x')}{\sqrt{1 + |Du(x')|^2}} \\ &\leq \frac{|a| + M + c_3/|x'|}{\sqrt{1 + |Du(x')|^2}} - \frac{M \log |x'|}{\sqrt{1 + |Du(x')|^2}} \\ &< 0, \end{aligned}$$

provided $|x'| \geq \sigma_0 = \sigma_0(|a|, M, c_3, R_1) \geq R_1$. That is, f does not have a local maximum on $\Sigma \setminus \overline{B}_{\sigma_0}$. We also readily verify that

$$\begin{aligned} \Delta_{\partial\Sigma} f &= \Delta f - D^2 f(\nu, \nu) - D^2 f(\gamma, \gamma) - Df \cdot \gamma A_S(\tau, \tau) - Df \cdot \nu A(\tau, \tau) \\ &= 1 - x \cdot \nu \kappa_n - x \cdot \gamma \kappa_g = 1 + x \cdot \vec{\kappa}. \end{aligned}$$

Here $\tau \in T(\partial\Sigma)$ with $|\tau| = 1$. We estimate

$$\begin{aligned} |\vec{\kappa}| &\leq |\kappa_n| + |\kappa_g| = |A(\tau, \tau)| + |A_S(\tau, \tau)| \\ &\leq \frac{1}{4|x|} + \frac{2M + c_3}{|x|^2}, \end{aligned}$$

which is no greater than $\frac{1}{2}|x|^{-1}$ for $|x| \geq \sigma_0$. Hence,

$$\Delta_{\partial\Sigma} f \geq \frac{1}{2}.$$

By a standard Morse theoretic lemma the sets $\{x \in \Sigma : f(x) \leq t\}$ do not change topology for all $t \geq \sigma_0$. Owing the structure of the limit lamination and using the properness assumption we see that $\Sigma \cap B_\rho \setminus \overline{B}_{\sigma_0}$ is diffeomorphic to the disjoint union $\bigcup_{j=1}^{l} [2j - 1, 2j] \times (\sigma_0, \rho)$ for all $\rho > \sigma_0$ and some fixed number $l \geq 1$, independent of $\rho > \sigma_0$. The following arguments can be made for each component separately, so we shall assume now w.l.o.g. that $l = 1$.

Since $\sigma_k' \to \infty$, we have that $\{0\} \times \mathbb{R}^2$ is a leaf of the lamination $\hat{\mathcal{L}}$. We can argue similarly as in [Car14] to conclude that $\{0\} \times \mathbb{R}^2$ is the only leaf of $\hat{\mathcal{L}}$: By [PR02, Lemma 4.1.1] and Lemma 2.42 we have that for any $\sigma > 0$ there is an index k_0 such that for all $k \geq k_0$ the surfaces Σ_k' have a graphical component inside $B_{3\sigma} \setminus \overline{B}_\sigma$ with C^2-norms going to zero as $k \to \infty$. Now assume that there exists another leaf of $\mathcal{L}$ of the form $(\{t\} \times \mathbb{R}^2) \cap \{x^3 \geq 0\}$ for some $t \neq 0$. Then Σ_k' will also have a graphical component inside $B_{|t|/4}(t,0,0) \setminus \overline{B}_{|t|/8}(t,0,0)$ for k large enough and with Lipschitz constants going to zero as $k \to \infty$. For $\sigma = |t|/2$ we have that the sphere $\partial B_{7|t|/6}$ intersects both $B_{3\sigma} \setminus \overline{B}_\sigma \cap \{x^1 = 0\}$ and $B_{|t|/4}(t,0,0) \setminus \overline{B}_{|t|/8}(t,0,0)$ in a circle, which implies that it will eventually also intersect both graphical components in an almost half circle. This however contradicts the fact that $l = 1$.

The argument in case $\partial\Sigma$ is compact is similar but easier. $\square$

Corollary 2.31. *Let $S \subset \mathbb{R}^3$ be an asymptotically catenoidal exterior surface. Let Σ be a connected non-compact stable free boundary minimal surface with respect to S. Then there are constants $\sigma_0 < \infty$ and $\Theta < \infty$ such that for every $\sigma \geq \sigma_0$*

$$\mathcal{H}^1(\Sigma \cap \partial B_\sigma) \leq \Theta\, \sigma.$$

Moreover, Σ has quadratic area growth.

Proof. Let Σ_k be as in Proposition 2.30. There exists an index k_0 such that for every $k \geq k_0$ there exist m-valued functions φ_k (m independent of k) with $\operatorname{graph}(\varphi_k) = \Sigma_k \cap B_2 \setminus \overline{B}_1$ and $|\varphi_k| + |D\varphi_k| \leq 1$, and $|\nu_k \cdot x| \leq |x|/2$. Suppose by contradiction, that the claim is false. Then there exists a sequence $s_j \to \infty$ as $j \to \infty$ such that

$$\mathcal{H}^1(\Sigma \cap \partial B_{s_j}) > j\, s_j.$$

For a subsequence k_j we have $s_j/\sigma_{k_j} \geq 1$. But this implies

$$\mathcal{H}^1(\Sigma_k \cap \partial B_{s_j/\sigma_k}) > j,$$

which contradicts the graph property of Σ_k for large k.

Moreover, we have that

$$\begin{aligned}\mathcal{H}^2(\Sigma \cap B_\sigma \setminus \overline{B}_1) &= \int_1^\sigma \int_{\Sigma \cap \partial B_\sigma} \frac{1}{|\nabla^\Sigma r|}\, d\mathcal{H}^1\, dt \\ &\leq 2 \int_1^\sigma \mathcal{H}^1(\Sigma \cap \partial B_\sigma)\, dt \leq \Theta\, \sigma^2.\end{aligned}$$

$\square$

Stable free boundary surfaces and positive mean curvature

We now prove that non-compact stable free boundary minimal surfaces with respect to exterior surfaces of non-negative mean curvature are totally geodesic, which is the main step in the proof of Theorems 2.13 and 2.16.

Lemma 2.32 ([EM12, Lemma 3.2])**.** *Let $\Sigma \subset \mathbb{R}^3$ be a non-compact surface with bounded mean curvature, and such that $\mathcal{H}^2(\Sigma \cap B_\sigma) \leq \Theta\, \sigma^2$ for all $\sigma \geq 1$. For every $\varepsilon > 0$ there exists a Lipschitz function χ_ε defined on Σ such that (i) χ_ε has compact support and $\operatorname{spt}(\chi_\varepsilon) \cap B_{\varepsilon^{-1}} = \emptyset$, (ii) $\int_\Sigma |\nabla^\Sigma \chi_\varepsilon|^2\, d\mathcal{H}^2 \leq \varepsilon$, and such that (iii) $0 \leq \chi_\varepsilon \leq 1$ and $\int_\Sigma \chi_\varepsilon\, d\mathcal{H}^2 = 1$.*

The following proposition establishes finite total curvature, and proves that non-compact volume-preserving free boundary CMC surfaces with respect to exterior surfaces are in fact *strongly* stable and minimal (cf. [EM12, Proposition 3.3]).

Proposition 2.33. *Let $S \subset \mathbb{R}^3$ be an exterior surface. Let Σ be a non-compact volume-preserving stable free boundary constant mean curvature surface with respect to S such that $\mathcal{H}^2(\Sigma \cap B_\sigma) \leq \Theta\,\sigma^2$ for all $\sigma \geq 1$. Then Σ is a (strongly) stable minimal surface and*

$$\int_\Sigma |A|^2 \, d\mathcal{H}^2 \leq -\int_{\partial\Sigma} A_S(\nu,\nu)\, d\mathcal{H}^1 < \infty.$$

In particular,

$$\int_\Sigma |K| \, d\mathcal{H}^2 + \int_{\partial\Sigma} |\kappa_g| \, d\mathcal{H}^1 < \infty.$$

Proof. First note that by Corollary 2.21 $\int_{\partial\Sigma} |A_S(\nu,\nu)|\, d\mathcal{H}^1 < \infty$. This is because

$$\sup_{x\in S} \Big(\max(1,|x|)^2 |A_S|\Big) < \infty. \tag{2.12}$$

Moreover, Corollary 2.21 implies that there exists a sequence $\varepsilon_i \downarrow 0$ as $i \to \infty$ such that

$$\int_{\partial\Sigma\setminus B_{\varepsilon_i^{-1}}} |A_S(\nu,\nu)|\, d\mathcal{H}^1 = \mathcal{O}(\varepsilon_i) \quad \text{as } i\to\infty. \tag{2.13}$$

Now, fix $f \in C^1_c(\Sigma)$ and let i be large enough such that $\operatorname{spt}(f) \subset B_{\varepsilon_i^{-1}}$. Let $\alpha := \int_\Sigma f\, d\mathcal{H}^2$. Then $f_i := f - \alpha\chi_{\varepsilon_i}$, with χ_{ε_i} as in Lemma 2.32, is Lipschitz with compact support and has mean zero. Hence,

$$\int_\Sigma |\nabla^\Sigma f_i|^2 \, d\mathcal{H}^2 \geq \int_\Sigma |A|^2 f_\varepsilon^2 \, d\mathcal{H}^2 + \int_{\partial\Sigma} A_S(\nu,\nu) f_i^2 \, d\mathcal{H}^1.$$

As in [EM12, Proposition 3.3] we have

$$\int_\Sigma |\nabla^\Sigma f_i|^2 \, d\mathcal{H}^2 = \int_\Sigma |\nabla^\Sigma f|^2 \, d\mathcal{H}^2 + \int_\Sigma \alpha^2 |\nabla^\Sigma \chi_{\varepsilon_i}|^2 \, d\mathcal{H}^2 = \int_\Sigma |\nabla^\Sigma f|^2 \, d\mathcal{H}^2 + o(1),$$

and using (2.13) we have

$$\begin{aligned}\int_\Sigma &|A|^2 f_i^2 \, d\mathcal{H}^2 + \int_{\partial\Sigma} A_S(\nu,\nu) f_i^2 \, d\mathcal{H}^1 \\ &\geq \int_\Sigma |A|^2 f^2 \, d\mathcal{H}^2 + \int_{\partial\Sigma} A_S(\nu,\nu) f^2 \, d\mathcal{H}^1 - \mathcal{O}(\varepsilon_i).\end{aligned}$$

Hence, letting $i \to \infty$ we obtain

$$\int_\Sigma |\nabla^\Sigma f|^2 \, d\mathcal{H}^2 \geq \int_\Sigma |A|^2 f^2 \, d\mathcal{H}^2 + \int_{\partial\Sigma} A_S(\nu,\nu) f^2 \, d\mathcal{H}^1.$$

That is, Σ is strongly stable. The fact that $\int_\Sigma |A|^2 \, d\mathcal{H}^2 + \int_{\partial\Sigma} A_S(\nu,\nu)\, d\mathcal{H}^1 \leq 0$ follows from the log-cut-off trick:

For $\sigma > 1$ set $\Sigma_\sigma := \Sigma \cap B_\sigma$. Let φ be the logarithmic-cut-off function (as in [SY79b]), i.e.

$$\varphi(x) := \begin{cases} 1 & , \Sigma_\sigma \\ \frac{\log(\sigma^2/|x|)}{\log(\sigma)} & , \Sigma_{\sigma^2} \setminus \Sigma_\sigma \\ 0 & , \Sigma \setminus \Sigma_{\sigma^2}. \end{cases} \tag{2.14}$$

Then have

$$|\nabla^\Sigma \varphi| \le \frac{|\nabla^\Sigma |x||}{|x|} \frac{1}{\log(\sigma)} \le \frac{1}{|x|} \frac{1}{\log(\sigma)}. \tag{2.15}$$

Now, let g be a Lipschitz function on Σ such that $|g| \le 1$ and $g = 1$ outside a compact set contained in Σ_σ. Since Σ is strongly stable we may test with $f = \varphi g$ to obtain (for a.e. σ)

$$\begin{aligned} \int_\Sigma |A|^2 \varphi^2 g^2 \, d\mathcal{H}^2 + \int_{\partial\Sigma} A_S(\nu, \nu) \varphi^2 g^2 \, d\mathcal{H}^1 &\le \int_\Sigma |\nabla^\Sigma \varphi|^2 g^2 + |\nabla^\Sigma g|^2 \varphi^2 \, d\mathcal{H}^2 \\ &\le \frac{1}{\log(\sigma)^2} \int_{\Sigma_{\sigma^2} \setminus \Sigma_\sigma} \frac{1}{|x|^2} \, d\mathcal{H}^2 + \int_\Sigma |\nabla^\Sigma g|^2 \varphi^2 \, d\mathcal{H}^2. \end{aligned}$$

Using the estimate [SY79b, (2.11)] and the estimate (2.12) with Corollary 2.21, we obtain upon letting $\sigma \to \infty$ that

$$\int_\Sigma |A|^2 g^2 \, d\mathcal{H}^2 \le \int_\Sigma |\nabla^\Sigma g|^2 \, d\mathcal{H}^2 + \int_{\partial\Sigma} |A_S(\nu, \nu)| g^2 \, d\mathcal{H}^1.$$

Choosing $g \equiv 1$ we get

$$\int_\Sigma |A|^2 \, d\mathcal{H}^2 < \infty.$$

Similarly, we derive that

$$\int_\Sigma |A|^2 \, d\mathcal{H}^2 \le - \int_{\partial\Sigma} A_S(\nu, \nu) \, d\mathcal{H}^1 < \infty.$$

Since by Young's inequality $2|K| \le |A|^2$, and trivially $|\kappa_g| \le |A_S|$, we also get that

$$\int_\Sigma |K| \, d\mathcal{H}^2 + \int_{\partial\Sigma} |\kappa_g| \, d\mathcal{H}^1 < \infty. \tag{2.16}$$

Using the identities

$$A_S(\nu, \nu) = H_S - \kappa_g \quad \text{and} \quad \frac{1}{2}|A|^2 = \frac{1}{2} H^2 - K,$$

we infer that

$$\frac{1}{2} \int_\Sigma |A|^2 + H^2 \, d\mathcal{H}^2 + \int_{\partial\Sigma} H_S \, d\mathcal{H}^1 \le \int_\Sigma K \, d\mathcal{H}^2 + \int_{\partial\Sigma} \kappa_g \, d\mathcal{H}^1. \tag{2.17}$$

Since Σ has infinite area, we see that $H = 0$. $\square$

Proposition 2.34. *Let $S \subset \mathbb{R}^3$ be an asymptotically catenoidal exterior surface. Let $\Sigma \subset \mathbb{R}^3$ be a connected non-compact stable free boundary minimal surface with respect to S. Then*

$$\frac{1}{2} \int_\Sigma |A|^2 \, d\mathcal{H}^2 + \int_{\partial\Sigma} H_S \, d\mathcal{H}^1 \le 0.$$

Proof. First note that by Corollary 2.31 we know that Σ has quadratic area growth. Assume by contradiction that $\frac{1}{2}\int_\Sigma |A|^2\, d\mathcal{H}^2 + \int_{\partial\Sigma} H_S\, d\mathcal{H}^1 > 0$. We claim that this implies that $\Sigma \cong \{z \in \mathbb{C} : \operatorname{Im}(z) \geq 0\}$.

By (2.16) Huber's theorem with boundary (cf. [SST03]) implies that Σ is finitely connected. The boundary version of the Cohen-Vossen theorem [SST03, Theorem 2.2.1] gives

$$2\pi\chi(\Sigma) - \left(\int_\Sigma K\, d\mathcal{H}_g^2 + \int_{\partial\Sigma} \kappa_g\, d\mathcal{H}_g^1\right) \geq \pi\chi(\partial\Sigma).$$

Since $\partial\Sigma \neq \emptyset$ we have that $\chi(\Sigma) \leq 1$, and therefore $\chi(\partial\Sigma) \leq 1$ by (2.17). On the other hand, since Σ in non-compact we have that $\chi(\partial\Sigma) \geq 1$. As such $\chi(\Sigma) = \chi(\partial\Sigma) = 1$, and Σ is homeomorphic to a half-plane. In particular, we have that $\partial\Sigma$ is non-compact.

Consider a sequence $\sigma_k' \to \infty$ as in Lemma 2.30. W.l.o.g. we may assume that the asymptotic plane Π is equal to $\{x^1 = 0\}$. For $\sigma \geq 1$ let $P_\sigma := \{x \in \mathbb{R}^3 : x_2^2 + x_3^2 \leq \sigma^2\}$. Lemma 2.30 implies that $\Sigma \cap \partial P_{\sigma_k'}$ consists of a union of disjoint simply connected curves that are all graphical above Π and are C^2 close to half-circles. The number of these curves equals the multiplicity of Π in the blow down limit.

A similar argument to the one we gave in Lemma 2.30 implies that each of these curves bounds a topological disk in $\Sigma \cap P_{2\sigma_k'}$.

By Lemma 2.30, these curves converge to round half-circles in Π upon blow down. Hence, by scaling invariance, the total geodesic curvature of each of these curves in $\Sigma \cap P_{2\sigma_k'}$ converge to π as $k \to \infty$. Moreover, since $\eta_{\sigma_k'}(S) \to \{x^1 = 0\}$ as $k \to \infty$ we see that the contact angles of these curves converge to $\pi/2$. By the Gaus-Bonnet theorem, this implies that

$$\begin{aligned}
0 &< \lim_{k\to\infty} \left(\int_{\Sigma\cap P_{\sigma_k'}} K\, d\mathcal{H}^2 + \int_{\partial\Sigma\cap P_{\sigma_k'}} \kappa_g\, d\mathcal{H}^1\right) \\
&= \lim_{k\to\infty} \left(2\pi\chi(\Sigma \cap P_{\sigma_k'}) - \int_{C_{\sigma_k'}} \kappa_g^{C_{\sigma_k'}}\, d\mathcal{H}^1 - \pi + o(1)\right) \\
&= 2\pi - \pi - \pi = 0,
\end{aligned}$$

a contradiction. $\square$

2.2.2 Proof of Theorem 2.13

Proposition 2.34 implies that Σ is totally geodesic and that H_S vanishes along $\partial\Sigma$. In particular, we have that ν is a constant vector. Moreover, the argument in the proof of Proposition 2.34 shows that

$$\int_{\partial\Sigma} \kappa_g\, d\mathcal{H}^1 = 0.$$

We modify an idea of Fischer-Colbrie and Schoen [FCS80] to obtain a contradiction: Fix a point $x_0 \in \Sigma$, and set $\Omega_\sigma :=$ connected component of $(\Sigma \setminus \partial\Sigma) \cap B_\sigma(x_0)$ that contains x_0. For a.e. $\sigma > 0$ we have that Ω_σ is a domain with piecewise smooth boundary such that $\overline{\Omega_\sigma} \cap S \subset \partial\Sigma \cap \overline{B}_\sigma(x_0)$. Since Σ is stable, and $A \equiv 0$, we know that

$$0 \leq \varsigma_1(\Omega_\sigma) := \inf\left\{R(f, \Omega_\sigma) : f \in W_0^{1,2}(\Omega_\sigma \cup \partial^{\mathcal{N}}\Omega_\sigma), \int_{\partial^{\mathcal{N}}\Omega_\sigma} f^2\, d\mathcal{H}^1 = 1\right\},$$

where

$$R(f,\Omega_\sigma) := \frac{\int_{\Omega_\sigma} |\nabla^\Sigma f|^2 \, d\mathcal{H}^2 - \int_{\partial^\mathcal{N}\Omega_\sigma} A_S(\nu,\nu) f^2 \, d\mathcal{H}^1}{\int_{\partial^\mathcal{N}\Omega_\sigma} f^2 \, d\mathcal{H}^1},$$

and $\partial^\mathcal{D}\Omega_\sigma := \overline{\partial\Omega_\sigma \setminus S}$ and $\partial^\mathcal{N}\Omega_\sigma := \partial\Omega_\sigma \setminus \partial^\mathcal{D}\Omega_\sigma$. Note that on the vertex $\partial^\mathcal{D}\Omega_\sigma \cap S$ we have

$$\eta_{\Omega_\sigma} \cdot \gamma = \frac{x \cdot \gamma}{\sqrt{\sigma^2 - (x\cdot\nu)^2}} < 0, \tag{2.18}$$

provided $\sigma \geq \sigma_0$.

Since Σ is non-compact a standard contradiction argument using the Harnack inequality implies that in fact $\varsigma_1(\Omega_\sigma) > 0$, and the Fredholm-alternative (cf. [Lie86]) implies the existence of a solution $f \in C^2(\Omega_\sigma \cup \partial^\mathcal{N}\Omega_\sigma) \cap C^0(\overline{\Omega_\sigma})$ of

$$\begin{cases} \Delta_\Sigma f = 0 & \text{in } \Omega_\sigma \\ f = 0 & \text{on } \partial^\mathcal{D}\Omega_\sigma \\ \nabla^\Sigma_\gamma f - A_S(\nu,\nu) f = A_S(\nu,\nu) & \text{on } \partial^\mathcal{N}\Omega_\sigma. \end{cases} \tag{2.19}$$

We make the important remark that view of (2.18), [Lie89] implies that in fact $f \in C^1(\overline{\Omega_\sigma})$, which in turn implies that the classical (Perron) solution of (2.19) agrees with its weak solution. Standard elliptic Schauder estimates imply that f is smooth away from the vertex $\partial^\mathcal{D}\Omega_\sigma \cap S$. Then $v := 1 + f$ is a solution of

$$\begin{cases} \Delta_\Sigma v = 0 & \text{in } \Omega_\sigma \\ v = 1 & \text{on } \partial^\mathcal{D}\Omega_\sigma \\ \nabla^\Sigma_\gamma v - A_S(\nu,\nu) v = 0 & \text{on } \partial^\mathcal{N}\Omega_\sigma. \end{cases}$$

We claim that $v > 0$. It follows from the strong maximum principle that if $v \geq 0$ on Ω_σ we have that $v > 0$ on Ω_σ. Suppose now that $\Omega^* \subset \Omega_\sigma := \{x \in \Omega_\sigma : v(x) < 0\} \neq \emptyset$. Hence, $\Omega^* \subset \Omega_\sigma$ is a bounded domain, and thus $\varsigma_1(\Omega^*) > 0$. However, $\Delta_\Sigma v = 0$ in Ω^* and $v = 0$ on $\partial\Omega^*$, and so $v = 0$ on Ω^*, contradicting the unique continuation property. This implies that $v > 0$.

Setting $g_\sigma := v(x_0)^{-1} v$, we see that

$$\begin{cases} \Delta_\Sigma g_\sigma = 0 & \text{in } \Omega_\sigma \\ g_\sigma(x_0) = 1, \quad g_\sigma > 0 & \text{on } \Omega_\sigma \\ \nabla^\Sigma_\gamma g_\sigma - A_S(\nu,\nu) g_\sigma = 0 & \text{on } \partial^\mathcal{N}\Omega_\sigma. \end{cases}$$

The (interior) Harnack inequality [GT01, Theorem 8.20] together with the boundary Harnack inequality [Lie01, Theorem 4.3] locally near S imply the existence of a constant $C = C(\Sigma, S, \sigma) < \infty$ such that for a.e. $\rho > 2\sigma$

$$g_\rho \leq C \quad \text{on } \Omega_\sigma(x_0).$$

Standard elliptic theory [GT01, Theorem 6.2] and [GT01, Lemma 6.29] implies that all derivatives of g_ρ are bounded uniformly in ρ on compact subsets of Σ. By the Arzelà-Ascoli theorem and a diagonal sequence argument we see that we may choose a sequence $\rho_i \to \infty$ such that g_{ρ_i} converges along with its derivatives on any compact subset of Σ to a function g satisfying

$\Delta_\Sigma g = 0$ and $g \geq 0$ in Σ, $\nabla^\Sigma_\gamma g - A_S(\nu,\nu) g = 0$ on $\partial\Sigma$, and $g(x_0) = 1$. The strong maximum principle together with the Hopf boundary point lemma imply that $g > 0$ on Σ.

Hence, we may set $w := \log(g)$. Then $-\Delta_\Sigma w = |\nabla^\Sigma w|^2$ in Σ and $\nabla^\Sigma_\gamma w = A_S(\nu,\nu)$ on $\partial\Sigma$. Now let ζ be a smooth cut-off function such that

$$\begin{cases} \zeta = 1 \quad \text{in } B_{\rho/2} & , \zeta = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B}_\rho \\ \zeta \geq 0 \quad \text{in } \mathbb{R}^3 & , |D\zeta| \leq c\,\rho^{-1} \quad \text{in } \mathbb{R}^3. \end{cases}$$

Multiplying by ζ^2, integrating by parts, and using Cauchy-Schwarz' inequality we get

$$\frac{3}{4}\int_\Sigma |\nabla^\Sigma w|^2 \zeta^2 \, d\mathcal{H}^2 \leq 4\int_\Sigma |\nabla^\Sigma \zeta|^2 \, d\mathcal{H}^2 - \int_{\partial\Sigma} A_S(\nu,\nu)\zeta^2 \, d\mathcal{H}^1.$$

Letting $\rho \to \infty$ we arrive at

$$\frac{3}{4}\int_\Sigma |\nabla^\Sigma w|^2 \, d\mathcal{H}^2 \leq -\int_{\partial\Sigma} A_S(\nu,\nu)\, d\mathcal{H}^1 = 0.$$

Hence, $w = \text{const}$ and $A_S(\nu,\nu) = 0$. Here we used that $A_S(\nu,\nu) + \kappa_g = H_S = 0$ along $\partial\Sigma$.

On the other hand, an explicit expansion shows that unless $M = 0$, $A_S(\nu,\nu) > 0$ outside a compact set: Assuming w.l.o.g. that we have asymptotic coordinates (ω, E, u) of S with $\omega = e_3$ we see that $\nu \cdot e_3 = 0$, and w.l.o.g. $\nu = e_1$. Hence, we have on $\partial\Sigma \setminus K$ (for some compact set $K \supset B_1$)

$$\begin{aligned} A_S(\nu,\nu) &= \frac{1}{\sqrt{1+|Du|^2}}\left(D^2u(\nu,\nu) - \frac{D^2u(Du,\nu)Du\cdot\nu}{1+|Du|^2}\right) \\ &\geq \frac{1}{\sqrt{1+|Du|^2}}\frac{M}{r^2}\left(1 - 2\left(\frac{x}{r}\cdot\nu\right)^2\right) - \frac{C(c_3,M)}{r^3}. \end{aligned}$$

Now notice that $\partial\Sigma \setminus K = \{\lambda_0\, e_1 + t e_2 : t \in \mathbb{R}\setminus I\}$, for some number $\lambda_0 \in \mathbb{R}$ and some compact interval I. We infer that

$$A_S(\nu,\nu) \geq \frac{1}{\sqrt{1+|Du|^2}}\frac{M}{r^2}\left(1 - 2\frac{\lambda_0^2}{r^2}\right) - \frac{C(c_3,M)}{r^3} \geq \frac{1}{2}\frac{M}{r^2}, \tag{2.20}$$

provided $|x| = r \geq \sigma_0(\lambda_0, c_3, M)$. □

2.2.3 Proof of Theorem 2.16

Assume by contradiction that there exists a sequence $\{\Sigma_i\}$ of connected compact volume-preserving stable free boundary constant mean curvature surfaces with respect to S such that $\mathcal{H}^2(\Sigma_i \cap B_\sigma) \leq \Theta\,\sigma^2$ for all $\sigma \geq 1$ and all $i \in \mathbb{N}$, and such that $\mathcal{H}^1(\partial\Sigma_i) \geq i$, but with the property that $\Sigma_i \cap K \neq \emptyset$. It follows from Corollary 2.19 that also $\mathcal{H}^2(\Sigma_i) \uparrow \infty$ as $i \to \infty$.

These assumptions imply that $\max_{x\in\Sigma_i} |x| \to \infty$ as $i \to \infty$. From Corollary 2.17 we have that

$$\mathcal{H}^2(\Sigma_i) H^2_{\Sigma_i} \leq \frac{56\pi}{3},$$

which implies that $H_{\Sigma_i} \to 0$ as $i \to \infty$. Together with Proposition 2.20 we also get that $\max_{x\in\partial\Sigma_i} |x| \to \infty$ as $i \to \infty$.

Using the curvature estimates from Theorem 2.29 we can pass to a subsequential limit Σ_∞ (of possibly higher multiplicity) locally in C^2. The components of Σ_∞ are all unbounded. Let $\hat{\Sigma}_\infty$ be a connected component of Σ_∞ such that $\hat{\Sigma}_\infty \cap K \neq \emptyset$. Clearly $\hat{\Sigma}_\infty$ is a complete non-compact embedded orientable volume-preserving stable minimal surface of quadratic area growth, either without boundary or with free boundary inside S. If $\partial\hat{\Sigma}_\infty = \emptyset$, then $\hat{\Sigma}_\infty$ is a plane by Theorem 2.22. Since by assumption S is not a plane, Theorem 2.8 implies that $M > 0$, which yields a contradiction. Hence, $\partial\hat{\Sigma}_\infty \neq \emptyset$, however contradicting Theorem 2.13. $\square$

2.3 Relative isoperimetric mass

Let S is an exterior hypersurface of $\mathbb{R}^3$ and denote by G the domain as in Definition 2.3.

The *relative isoperimetric profile* A_S of S is defined by

$$\begin{aligned} A_S(V) := \inf\{\mathcal{H}^2(\partial^*_G\Omega) : \Omega \subset G \text{ is a Borel set of finite perimeter} \\ \text{with } \mathcal{L}^3(\Omega) = V \text{ and } \chi^+_\Omega = 1 \text{ on } N\}. \end{aligned} \tag{2.21}$$

Here $\partial^*_G\Omega$ denotes the reduced boundary of Ω inside G and χ^+_Ω denotes the inner trace of Ω on N. Minimizers of (2.21) are called *relative isoperimetric regions*.

The main goal of this section is to prove the following theorem.

Theorem 2.35. (existence of arbitrary large isoperimetric regions) *Let S be an asymptotically catenoidal exterior hypersurface of $\mathbb{R}^3$. There exists a sequence of relative isoperimetric regions $\Omega_i \subset G$ with $\mathcal{L}^3(\Omega_i) \to \infty$.*

Remark 2.36. Theorem 2.35 in particular implies the existence of arbitrary large stable free boundary CMC surfaces with respect to S.

Inspired by works of Huisken [Hui06, Hui09], and Eichmair and Metzger [EM13b] we make the following definition.

Definition 2.37. (relative isoperimetic mass) *We define the* relative isoperimetric mass *of S as*

$$m_{iso}(S) := \limsup_{V\to\infty} \frac{4}{A_S(V)}\left(V - \frac{1}{\sqrt{18\pi}} A_S(V)^{\frac{3}{2}}\right).$$

Theorem 2.38. *Let S be an asymptotically catenoidal exterior hypersurface of $\mathbb{R}^3$ with $M > 0$. Then*

$$m_{iso}(S) \geq m(S).$$

Proof. The proof works by an explicit comparison argument, somewhat inspired by the works of Fan, Shi, and Tam [FST09].

W.l.o.g. we use assume that we have asymptotical coordinates (ω, E, u) of S are such that $\omega = e_3$ and $\mathbb{R}^2 \setminus B^2_{R_1/2}(0) \subset E$. For $r \geq R_1$ consider the surface Σ_r defined by

$$\Sigma_r := \{a(r)e_{n+1} + \rho(r)v : v \in \mathbb{S}^2\} \cap \overline{G},$$

where $a(r) := a + \phi_M(r) - r\phi'_M(r)$, where a and ϕ_M are as in Definition 2.2, and where $\rho(r) = r\sqrt{1 + \phi'_M(r)^2} = \sqrt{r^2 + M^2}$. (In the special case where $u = a + \phi_M$ the surfaces Σ_r are spherical caps meeting S orthogonally.) W.l.o.g. we may assume that $\partial\Sigma_r \cap (B^2_{R_1/2}(0)\times\mathbb{R}) = \emptyset$ for all $r \geq R_1$. Let $x \in \Sigma_r \setminus (B^2_{R_1/2}(0)\times\mathbb{R})$, then $u(x') \leq x^3$. Moreover, $x \in \partial\Sigma_r$ if and only if $u(x') = x^3$. There exists a small number $\varepsilon_0 \in (0,1)$ such that for every $v \in \mathbb{S}^1$ and every $\sin(\theta) \in [0,\varepsilon_0]$ we have that

$$\Phi(r,\theta,v) := a(r)e_3 + \rho(r)(\sin(\theta)v, \cos(\theta)) \in \Sigma_r,$$

whenever $r \geq R_1$. For $r \geq \sigma_0 = \sigma_0(R_1,\varepsilon_0)$ and $\sin(\theta) \in [\varepsilon_0, 1]$ we have, by Definition 2.2, the estimate

$$a + \phi_M(\rho(r)\sin(\theta))(+/-)\frac{c_3}{\rho(r)\sin(\theta)}\ (\geq / \leq)\ u(\rho(r)\sin(\theta)v).$$

We define for $l \in (\varepsilon_0, 1)$

$$f_r^{\pm}(l) = \phi_M(\rho(r)l) - \phi_M(r) + M \pm \frac{c_3}{\rho(r)l} - \rho(r)\sqrt{1-l^2}.$$

With this definition we have that

$$\begin{aligned}\Sigma_r \subset & \Big\{\Phi(r,\theta,v) : v \in \mathbb{S}^1 \text{ and } \sin(\theta) \in [\varepsilon_0,1] \text{ is such that } f_r^-(\sin(\theta)) \leq 0\Big\} \\ & \cup \Big\{\Phi(r,\theta,v) : v \in \mathbb{S}^1 \text{ and } \sin(\theta) \in [0,\varepsilon_0)\ \Big\},\end{aligned}$$

and

$$\begin{aligned}\Sigma_r \supset & \Big\{\Phi(r,\theta,v) : v \in \mathbb{S}^1 \text{ and } \sin(\theta) \in [\varepsilon_0,1] \text{ is such that } f_r^+(\sin(\theta)) \leq 0\Big\} \\ & \cup \Big\{\Phi(r,\theta,v) : v \in \mathbb{S}^1 \text{ and } \sin(\theta) \in [0,\varepsilon_0)\ \Big\}.\end{aligned}$$

For $r \geq \sigma_0(\varepsilon_0, c_3)$ we have $(f_r^{\pm})'(l) > 0$ on $[\varepsilon_0, 1)$. For $\varepsilon \in (0,1)$ we have

$$\begin{aligned} f_r^{\pm}\left(\frac{r \mp \varepsilon r^{-1}}{\rho(r)}\right) &= \phi_M(r \mp \varepsilon r^{-1}) - \phi_M(r) \\ &\quad + M \pm \frac{c_3}{r - \varepsilon r^{-1}} - \sqrt{\rho(r)^2 - (r - \varepsilon r^{-1})^2}.\end{aligned}$$

Since $\phi_M' > 0$ we see that

$$f_r^+\left(\frac{r - \varepsilon r^{-1}}{\rho(r)}\right) \leq \frac{c_3}{r - \varepsilon r^{-1}} - \frac{2\varepsilon - \varepsilon^2 r^{-2}}{M + \sqrt{M^2 + 2\varepsilon - \varepsilon^2 r^{-2}}},$$

which is ≤ 0, provided r is sufficiently large depending on c_3, M, ε.

Similarly, we have

$$f_r^-\left(\frac{r + \varepsilon r^{-1}}{\rho(r)}\right) \geq -\frac{c_3}{r + \varepsilon r^{-1}} + \frac{2\varepsilon + \varepsilon^2 r^{-2}}{M + \sqrt{M^2 - 2\varepsilon - \varepsilon^2 r^{-2}}},$$

which is ≥ 0, provided $0 < \varepsilon < M^2/2$, and r is sufficiently large depending on c_3, M, ε. For the enclosed volume we have we have

$$\mathcal{L}^3\Big(\bigcup_{r \in (R_1,\sigma)} \Sigma_r\Big) = \int_{R_1}^{\sigma}\int_{\mathbb{S}^1}\int_{I(r,v)} J(\Phi)\, d\theta\, d\mathcal{H}^1\, dr,$$

for subsets $[0, \varepsilon_0] \subset I(r, v) \subset [0, \pi/2)$, and where $J(\Phi) = \rho(r)^2 \sin(\theta)(\rho'(r) + a'(r)\cos(\theta))$. Noting that

$$\rho(r)^2 \rho'(r) = r^2 \sqrt{1 + \frac{M^2}{r^2}} \quad \text{and} \quad \rho(r)^2 a'(r) = Mr \left(1 + \frac{M^2}{r^2}\right)$$

one easily verifies that

$$\begin{aligned}
&\int_0^{\arcsin(\frac{r - \varepsilon r^{-1}}{\rho(r)})} J(\Phi)\, d\theta \\
&= r^2 \sqrt{1 + \frac{M^2}{r^2}} \left(1 - \sqrt{1 - \frac{(r - \varepsilon r^{-1})^2}{r^2 + M^2}}\right) + \frac{M(r - \varepsilon r^{-1})^2}{2r} \\
&= r^2 \sqrt{1 + \frac{M^2}{r^2}} - r\sqrt{M^2 + 2\varepsilon - \varepsilon^2 r^{-2}} + \frac{Mr}{2} - \frac{M\varepsilon}{r} + \frac{M\varepsilon^2}{2r^3}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathcal{L}^3\Big(\bigcup_{r \in (R_1, \sigma)} \Sigma_r\Big) \\
&\geq 2\pi \int_{\sigma_0}^{\sigma} \int_0^{\arcsin(\frac{r - \varepsilon r^{-1}}{\rho(r)})} J(\Phi)\, d\theta\, dr \\
&= -C(\sigma_0, \varepsilon) + \frac{2\pi}{3}\sigma^3 \left(1 + \frac{M^2}{\sigma^2}\right)^{\frac{3}{2}} + \frac{\pi M}{2}\sigma^2 - \pi\sigma^2 \sqrt{M^2 + 2\varepsilon - \varepsilon^2 \sigma^{-2}} \\
&\quad - 2\pi\varepsilon M \log(\sigma) - \frac{\pi M}{2\sigma^2}\varepsilon^2 \\
&\quad + \pi \frac{\varepsilon^2 \log\left[\sigma\left(M^2 + 2\varepsilon + \sqrt{M^2 + 2\varepsilon}\sqrt{M^2 + 2\varepsilon - \varepsilon^2\sigma^{-2}}\right)\right]}{\sqrt{M^2 + 2\varepsilon}} \\
&\geq -C(\sigma_0, \varepsilon) + \frac{2\pi}{3}\sigma^3 \left(1 + \frac{M^2}{\sigma^2}\right)^{\frac{3}{2}} + \frac{\pi M}{2}\sigma^2 - \pi\sigma^2\sqrt{M^2 + 2\varepsilon} \\
&\quad - o(\sigma) \quad \text{as } \sigma \to \infty \text{ uniformly in } \varepsilon \in (0, 1].
\end{aligned}$$

For the area we estimate

$$\mathcal{H}^2(\Sigma_\sigma) \leq 2\pi \int_0^{\arcsin(\frac{\sigma + \varepsilon\sigma^{-1}}{\rho(\sigma)})} J(\Phi_\sigma)\, d\theta,$$

where $\Phi_r(\theta, v) \equiv \Phi(r, \theta, v)$. We have $J(\Phi_r) = \rho(r)^2 \sin(\theta)$. Hence,

$$\mathcal{H}^2(\Sigma_\sigma) \leq 2\pi\sigma^2 \left(1 + \frac{M^2}{\sigma^2}\right)\left(1 - \sqrt{\frac{M^2 - 2\varepsilon - \varepsilon^2\sigma^{-2}}{M^2 + \sigma^2}}\right).$$

Using these two estimates we infer

$$\begin{aligned}
m_{iso}(S) &\geq \limsup_{\sigma \to \infty} \frac{4}{\mathcal{H}^2(\Sigma_\sigma)} \left(\mathcal{L}^3(\Omega_\sigma) - \frac{1}{\sqrt{18\pi}}\mathcal{H}^2(\Sigma_\sigma)^{\frac{3}{2}}\right) \\
&\geq M + 2\sqrt{M^2 - 2\varepsilon} - 2\sqrt{M^2 + 2\varepsilon}.
\end{aligned}$$

The claim follows by letting $\varepsilon \downarrow 0$. $\square$

Lemma 2.39. *Let S be an exterior hypersurface of $\mathbb{R}^{n+1}$ with domain G as in Definition 2.3. There exists a constant $C = C(G) < \infty$ such that for any bounded Borel set $\Omega \subset G$ of finite perimeter in G we have that*

$$\mathcal{L}^{n+1}(\Omega) \leq C\, \mathcal{H}^n(\partial_G^* \Omega)^{\frac{n+1}{n}}.$$

Proof. The inequality follows in the usual way from the Sobolev inequality

$$\left(\int_G |f|^{\frac{n+1}{n}}\, d\mathcal{L}^{n+1}\right)^{\frac{n}{n+1}} \leq C \int_G |Df|\, d\mathcal{L}^{n+1} \quad \text{for all } f \in C_c^1(\overline{G}).$$

It is straightforward to adapt the proof of [EM13a, Lemma 2.4] (see also [SY79b, Lemma 2.3]) to prove the latter inequality. Here one combines, in a contradiction argument, the euclidean Sobolev inequality of the form

$$\left(\int_{\mathbb{R}^{n+1}_+ \setminus B_1} |f|^{\frac{n+1}{n}}\, d\mathcal{L}^{n+1}\right)^{\frac{n}{n+1}} \leq C' \int_{\mathbb{R}^{n+1}_+ \setminus B_1} |Df|\, d\mathcal{L}^{n+1} \quad \text{for all } f \in C_c^1(\overline{\mathbb{R}^{n+1}_+}),$$

where $\mathbb{R}^{n+1}_+ := \mathbb{R}^{n+1} \cap \{x^{n+1} > 0\}$, with Poincaré type inequalities on boundary coordinate charts. □

The following Proposition is a slight extension of [RR04, Theorem 2.1] and is inspired by [EM13a, Proposition 4.2] in which the authors consider isoperimetric hypersurfaces in asymptotically flat Riemannian manifolds.

Proposition 2.40. *Let S be an exterior hypersurface of $\mathbb{R}^{n+1}$ with domain G as in Definition 2.3. For given $V > 0$ there is $\rho > 0$ and an relative isoperimetric region $\Omega \subset G$ such that the following hold:*

(i) $\frac{\omega_n \rho^{n+1}}{2(n+1)} + \mathcal{L}^{n+1}(\Omega) = V$

(ii) $\frac{\omega_n \rho^n}{2} + \mathcal{H}^n(\partial_G^* \Omega) = A_S(V)$.

Moreover, if $\rho > 0$ and $\mathcal{L}^{n+1}(\Omega) > 0$ the mean curvature of $\partial_G^* \Omega$ equals $\frac{n}{\rho}$.

Proof. The idea is to consider a minimizing sequence $\{\Omega_i\}_{i=1}^\infty$ for the relative isoperimetric problem of volume V. Let

$$V_0 = \lim_{r \to \infty} \liminf_{i \to \infty} \mathcal{L}^{n+1}(\Omega_i \cap B_r).$$

Standard arguments from geometric measure theory, just as in [RR04, Theorem 2.1], show that there is a relative isoperimetric region Ω in G of volume V_0. Let $\rho \geq 0$ be such that

$$\frac{\omega_n \rho^{n+1}}{2(n+1)} = V - V_0.$$

Since S is asymptotically flat we may apply the relative isoperimetric inequality of the euclidean half-space to replace the original minimizing sequence for volume V by a sequence of the form $\{\Omega \cup R_i\}_{i=1}^\infty$ where $R_i = B_{r_i}(x_i) \cap G$ for $x_i \in \partial G$ diverging to infinity, and suitable $\sigma_i \to \rho$ as $i \to \infty$. By an argument involving the coarea formula (cf. [RR04, Theorem 2.1]) the additional perimeter that is created in this cut and paste procedure can be made to tend to zero as $i \to \infty$.

To see the second statement, first note that $\partial^*_G \Omega$ is a (smooth) free boundary CMC hypersurface with mean curvature H such that the singular set has Hausdorff-dimension at most $n-7$ (see [Grü87]). Now, $n+1 \leq 7$, consider a vector field $X \in C^1_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ such that $X \cdot \gamma = 0$ on ∂G and $X \cdot \nu = 1$ on $\partial^*_G E$. We denote by $\{\phi_t\}$ the flow associated to X. There exists a smooth function $t \mapsto \rho_t$ such that

$$\frac{\omega_n \rho_t^{n+1}}{2(n+1)} + \mathcal{L}^{n+1}(\phi_t(\Omega)) \equiv V. \tag{2.22}$$

By the minimizing property we have that

$$0 = \frac{d}{dt}\left[\frac{\omega_n \rho_t^n}{2} + \mathcal{H}^n(\partial^*_G \phi_t(\Omega))\right]|_{t=0} = \frac{n\omega_n \rho^{n-1}}{2}\frac{d}{dt}\rho_t|_{t=0} + H\mathcal{H}^n(\partial^*_G \Omega).$$

Differentiating (2.22) at $t=0$ and substituting into the above yields the claim. For higher dimensions one may use a cut-off function argument (as in [SZ99]) since the singular set is small. □

Proof. (of Theorem 2.35) In view of Theorem 2.38 we may assume that $m_{iso}(S) > 0$. Let $V > 0$. By Proposition 2.40 there exists a, possibly empty, relative isoperimetric region $\Omega \subset G$ and a sequence of balls $B_{\sigma_i}(x_i)$ centered on ∂G with $|x_i| \to \infty$ and $0 \leq \sigma_i \to \sigma \in [0, \infty)$ as $i \to \infty$ such that

(i) $\mathcal{L}^3(\Omega) + \mathcal{L}^3(B_{\sigma_i}(x_i) \cap G) = V$ for all i

(ii) $\mathcal{H}^2(\partial^*_G \Omega) + \mathcal{H}^2(\partial_G B_{\sigma_i}(x_i)) \to A_S(V)$ as $i \to \infty$.

Our goal is to show that for any given threshold $\Lambda > 0$ we can choose $V > 0$ sufficiently large such that $\mathcal{L}^3(\Omega) > \Lambda$. Now let $V > 0$ be sufficiently large such that

$$\frac{4}{A_S(V)}\left(V - \frac{1}{\sqrt{18\pi}} A_S(V)^{\frac{3}{2}}\right) > \frac{m_{iso}(S)}{2}.$$

Using the lower bound $A_S(V) \geq 2\pi\,\sigma^2$ we obtain

$$\mathcal{L}^3(\Omega) = V - \frac{2\pi}{3}\sigma^3 > \frac{m_{iso}(S)}{2}\frac{A_S(V)}{4}.$$

The fact that, by Lemma 2.39, $\lim_{V\to\infty} A_S(V) = \infty$ finishes the proof. □

2.4 Appendix

2.4.1 Graphical estimates

Definition 2.41. (convergence of free boundary hypersurfaces) *Let $G \subset \mathbb{R}^{n+1}$ be a domain. Let Σ_i be a sequence of free boundary hypersurfaces inside $\overline{G}$. We say that Σ_i converge in $C^k_{loc}(U)$, for some open set $U \subset \mathbb{R}^{n+1}$, to a free boundary hypersurface Σ inside $\overline{G}$ if the following conditions are satisfied:*

1. *The surfaces $\Sigma_i \setminus \partial\Sigma_i$ converge to $\Sigma \setminus \partial\Sigma$ in $C^k_{loc}(U)$ in the usual sense of graphs.*
2. *For every point $x \in \partial\Sigma \cap U$ the exists an open set $V \subset U$ with $x \in V$ and a C^k diffeomorphism $\phi : V \to B$ with the property that $\phi(G \cap V) = B \cap \{x^1 > 0\}$ such that $\phi(\Sigma_i \cap V)$ converge to $\phi(\Sigma \cap V)$ in $C^k_{loc}(B)$ in the sense of graphs over $B \cap \{x^1 \geq 0\}$.*

Lemma 2.42 (cf. [PR02, Lemma 4.1.1])**.** *Let $U \subset \mathbb{R}^3$ be an open set, and let $\Sigma \subset U \cap \{x^1 \geq 0\}$ be a surface with $\Sigma \cap U \cap \{x^1 = 1\} = \partial\Sigma \cap U$ such that $\sup_{\Sigma\cap U} |A| \leq c_0$ for some constant $c_0 < \infty$. For $x_0 \in \partial\Sigma \cap U$ such that $\eta(x_0) \cdot e_1 \geq \lambda > 0$, $\nu(x_0) \cdot e_3 \geq \lambda$, and*

$$R = \lambda \min\left\{\frac{\operatorname{dist}(x_0, \partial U)}{2}, \frac{\lambda}{4c_0}\right\},$$

the following holds:

The component of $\Sigma \cap (B^2_R(x_0) \times \mathbb{R})$ through x_0 is the graph of a function $f \in C^\infty(B^2_R(x_0) \cap \{x^1 \geq 0\})$. The function f satisfies the following estimates on $B^2_R(x_0) \cap \{x^1 \geq 0\}$:

1. $|f(y')| \leq \left(8c_0\lambda^{-3}|y' - x_0'| + \lambda^{-1}\sqrt{1-\lambda^2}\right)|y' - x_0'|$
2. $|Df(y')| \leq 8c_0\lambda^{-3}|y' - x_0'| + \lambda^{-1}\sqrt{1-\lambda^2}$
3. $|D^2 f| \leq 8c_0\lambda^{-3}$

Proof. W.l.o.g. assume that $x_0 = 0$. Hence, there exists a radius $R > 0$ with the following properties:

1. Σ can locally be written as a graph of a function $f \in C^\infty(B^{e_3}_R \cap \{x^1 \geq 0\})$.
2. $\hat{\nu} \cdot e_3 = (1 + |Df|^2)^{-1/2} > \lambda/2$ in $B^{e_3}_R \cap \{x^1 \geq 0\}$,

where $\hat{\nu} := \nu \circ F$ and $F(x') := (x', f(x'))$. We have

$$|D_i\hat{\nu}| \leq |A| \circ F\sqrt{1 + |Df|^2} < 2c_0\lambda^{-1}. \tag{2.23}$$

Assume now that $R > 0$ is the maximal radius such that 1. and 2. are satisfied at $x_0 = 0$. Note that if f were defined on $\partial B^{e_3}_R \cap \{x^1 \geq 1\}$ and $\hat{\nu} \cdot e_3 > \lambda/2$ were true along $\partial B^{e_3}_R \cap \{x^1 \geq 1\}$, we could extend f to a larger radius, which would contradict the maximality of R. Hence, one of the following possibilities must hold:

(a) The function f extends smoothly to a larger radius and there exists a point $y' \in \partial B^{e_3}_R \cap \{x^1 \geq 1\}$ such that $\hat{\nu}(y') \cdot e_3 = \lambda/2$.

(b) There exists a sequence $y'_k \in B^{e_3}_R \cap \{x^1 \geq 1\}$ with $\operatorname{dist}(F(y'_k), \partial U) \to 0$ as $k \to \infty$.

In case (a) we have that for some $\tau \in [0,1]$

$$\lambda/2 \leq |(\hat{\nu}(0) - \hat{\nu}(y')) \cdot e_3| \leq |D\hat{\nu}|(\tau y')|y'| < 2c_0\lambda^{-1}R.$$

In case (b) we have, setting $[0, y'_k] := \{ty'_k : t \in [0,1]\}$, that

$$\begin{aligned} |F(y'_k)| \leq \mathcal{H}^1(F([0,y'_k])) &= \int_0^1 |DF(ty'_k) \cdot y'_k| \, dt \\ &\leq |y'_k| \int_0^1 \sqrt{1 + |Df|^2} \, dt \\ &< 2\lambda^{-1}R. \end{aligned}$$

Hence, $\operatorname{dist}(0, \partial U) \leq |F(y'_k)| + \operatorname{dist}(F(y'_k), \partial U) < 2\lambda^{-1}R + o(1)$ as $k \to \infty$, which establishes the first claim.

Moreover, we have using (2.23) that

$$\frac{|D^2_{ij}f|}{\sqrt{1+|Df|^2}} = |\hat{\nu} \cdot D^2_{ij}F| = |D_j\hat{\nu} \cdot D_iF| \leq c_0(1 + |Df|^2),$$

and hence,

$$|D^2 f| \leq c_0(\hat{\nu} \cdot e_3)^{-3} \leq 8c_0\lambda^{-3}.$$

Using the mean value theorem we get

$$|Df(y')| \leq |Df(y') - Df(x'_0)| + |Df(x'_0)| \leq 8c_0\lambda^{-3}|y' - x'_0| + \lambda^{-1}\sqrt{1-\lambda^2}.$$

Finally,

$$|f(y')| = |f(y') - f(x'_0)| \leq \left(8c_0\lambda^{-3}|y' - x'_0| + \lambda^{-1}\sqrt{1-\lambda^2}\right)|y' - x'_0|.$$

□

2.4.2 Cohn-Vossen theorem for manifolds with boundary

Definition 2.43. *We say that Σ is* finitely connected *if there exist a compact 2-manifold N and finitely many points $p_1, ..., p_k \in N$ for $k \geq 1$ such that Σ is homeomorphic to $N \setminus \{p_1, ..., p_k\}$.*

If Σ is homeomorphic to $N \setminus \{p_1, ..., p_k\}$, if $p_1, ..., p_l \in \operatorname{int}(N)$ and if $p_{l+1}, ..., p_k \in \partial N$ then the Euler characteristic $\chi(\Sigma)$ of Σ is

$$\chi(\Sigma) := \chi(N) - l.$$

We have that

$$\chi(\partial\Sigma) = k - l,$$

where $\chi(\partial\Sigma)$ is the number of unbounded components of $\partial\Sigma$.

Theorem 2.44 ([SST03, Theorem 2.2.1])**.** *Let (Σ, g) be a connected non-compact finitely connected complete Riemannian 2-manifold. If Σ admits a total curvature and $\partial\Sigma$ a total geodesic curvature and if $\int_\Sigma K \, d\mathcal{H}^2_g = -\int_{\partial\Sigma} \kappa_g \, d\mathcal{H}^1_g = \pm\infty$ does* not *hold, then*

$$2\pi\chi(\Sigma) - \left(\int_\Sigma K \, d\mathcal{H}^2_g + \int_{\partial\Sigma} \kappa_g \, d\mathcal{H}^1_g\right) \geq \pi\chi(\partial\Sigma).$$

2.4.3 Complete non-compact finite index free boundary surfaces inside a half-space

Let $F : (M, g) \to \mathbb{R}^3 \cap \{x^3 \geq 0\}$ be an isometric immersion of constant mean curvature of a 2-dimensional complete non-compact connected manifold with boundary ∂M, such that $F(\partial M) \subset \{x^3 = 0\}$ and $\eta \cdot e_3 = -1$ on ∂M.

For a given function $q \in C^\infty(M)$ we consider the operator $L = \Delta_g + q - K$. For a bounded Lipschitz domain $\Omega \subset M \setminus \partial M$ we let $\operatorname{Ind}(L, \Omega)$ be defined as the index of the operator

$$L : W_0^{1,2}(\Omega \cup \partial^{\mathcal{N}}\Omega) \to W_0^{1,2}(\Omega \cup \partial^{\mathcal{N}}\Omega)',$$

where $\partial^{\mathcal{N}}\Omega := \partial\Omega \setminus \partial^{\mathcal{D}}\Omega$ and $\partial^{\mathcal{D}}\Omega := \overline{\partial\Omega \setminus \partial M}$.

Proposition 2.45. *If M has finite index then there is a compact set C in M so that $M \setminus C$ is stable and there exists a positive function u on M so that $Lu = 0$ on $M \setminus C$ and $\frac{\partial u}{\partial \gamma} = 0$ on $\partial M \setminus C$.*

Proof. Fix $p_0 \in \partial M$, let $\check{M}$ denote the doubled surface, and let $\check{F} : \check{M} \to \mathbb{R}^3$ denote the reflected immersion given by $\check{F}|_M = F$ and $\check{F}|_{\check{M}\setminus M} = R \circ F$, where $R(x) = (x^1, x^2, -x^3)$. Hence, $\check{F} : (\check{M}, \check{g} := \check{F}^*\delta) \to \mathbb{R}^3$ is an isometric immersion of constant mean curvature of a 2-dimensional complete connected manifold (without boundary). Note that since ∂M is totally geodesic $\check{M}$ carries a canonical smooth atlas, the smoothness of $\check{g}$ follows from elliptic regularity since $F : M \to \mathbb{R}^3 \cap \{x^3 \geq 0\}$ is assumed to have constant mean curvature. Now let $\mathcal{B}_\rho$ denote the geodesic ball with respect to the metric $\check{g}$ of radius ρ centered at p_0. Moreover, we let $\mathcal{B}_\rho^+ := \mathcal{B}_\rho \cap M$. Then $\mathcal{B}_\rho^+$ has piecewise smooth boundary.

Now we can argue exactly as in [FC85, Proposition 1], only replacing $\mathcal{B}_\rho$ by $\mathcal{B}_\rho^+$ and replacing the auxiliary function v_R by the solution of the following mixed boundary value problem:

$$\begin{cases} v_R > 0 & \text{on } (\mathcal{B}_R^+ \setminus \overline{\mathcal{B}_{R_0}^+}) \cap \Omega_i \\ Lv_R = 0 & \text{on } \mathcal{B}_R^+ \setminus \overline{\mathcal{B}_{R_0}^+} \\ v_R = 1 & \text{on } \partial^{\mathcal{D}}(\mathcal{B}_R^+ \setminus \overline{\mathcal{B}_{R_0}^+}) \\ \frac{\partial v_R}{\partial \gamma} = 0 & \text{on } \partial^{\mathcal{N}}(\mathcal{B}_R^+ \setminus \overline{\mathcal{B}_{R_0}^+}). \end{cases}$$

By a simple reflection argument one can easily verify that v_R is smooth on all of $\overline{\mathcal{B}_R^+ \setminus \mathcal{B}_{R_0}^+}$. $\square$

Theorem 2.46. *Let $q \geq 0$ and let $\operatorname{Ind}(L) < \infty$. Then M is conformally equivalent to a compact Riemann surface with boundary punctured at a finite number of points, and*

$$\int_M q \, d\mathcal{H}_g^2 < \infty.$$

Proof. Let u be the function from Proposition 2.45. The function $\check{u} : \check{M} \to \mathbb{R}$, given by $\check{u}|_M = u$ and $\check{u}|_{\check{M}\setminus M}(x^1, x^2, x^3) \equiv u(x^1, x^2, -x^3)$, is a smooth solution of $L\check{u} = 0$ on $\check{M}\setminus\check{C}$. We can argue exactly as in [FC85, Theorem 1] to conclude that $\tilde{\check{g}} := \check{u}^2\, ds^2$ and hence $\tilde{g} := u^2\, ds^2$ is a complete metric of non-negative Gaussian curvature on $\check{M}$ and M, respectively. More precisely,

$$K_{\tilde{\check{g}}} = \check{u}^{-2} \left(q + \frac{|\nabla_{\check{g}} \check{u}|_{\check{g}}^2}{\check{u}^2} \right) \geq 0. \tag{2.24}$$

What is more, we have that $\kappa_{\tilde{g}} = u^{-1}\kappa_g + u^{-2}\frac{\partial u}{\partial \gamma} = 0$ on $\partial M \setminus C$. It follows from [Hub57, Theorem 13] that $\check{M}$, and thus also M are finitely connected. Hence, M is conformal to a Riemann surface with a finite number of discs and half-disks deleted.

We now prove that $\int_M q \, d\mathcal{H}^2_g$ is finite. Since M is finitely connected, $\kappa_{\tilde{g}}$ vanishes on $\partial M \setminus C$ and $K_{\tilde{g}}$ is non-negative on $M \setminus C$, we may apply the Cohn-Vossen inequality for surfaces with boundary (here Theorem 2.44) to conclude that

$$\int_M K_{\tilde{g}} \, d\mathcal{H}^2_{\tilde{g}} \leq 2\pi\chi(M) - \pi\chi(\partial M) - \int_{\partial M \cap C} \kappa_{\tilde{g}} \, d\mathcal{H}^1_{\tilde{g}} < \infty.$$

It follow from the above and (2.24) that

$$\int_{M \setminus C} q \, d\mathcal{H}^2_g \leq \int_{M \setminus C} K_{\tilde{g}} \, d\mathcal{H}^2_{\tilde{g}} < \infty,$$

and since C is compact that $\int_M q \, d\mathcal{H}^2_g$ is finite. □

2.4.4 Integral decay estimates

Lemma 2.47. *Let $S \in \mathcal{F}_2(\beta, c_1, c_2)$ be an asymptotically flat surface in $\mathbb{R}^3$. There is a radius σ_0 and a constant $C_3 < \infty$ both depending only on S such that for every $\sigma \geq \sigma_0$ for which ∂B_σ and S are transversal, and every compact surface $\Sigma \subset \mathbb{R}^3 \setminus B_\sigma$ with $\Sigma \cap (S \cup \partial B_\sigma) = \partial\Sigma$ such that $\partial\Sigma$ and S meet orthogonally one has the estimate*

$$\int_{\partial\Sigma \setminus \overline{B}_\sigma} \frac{1}{r^{1+\beta}} \, d\mathcal{H}^1 \leq C_3 \, \sigma^{-\beta} \left(\int_{\Sigma \setminus B_\sigma} H^2 \, d\mathcal{H}^2 + \frac{\mathcal{H}^1(\Sigma \cap \partial B_\sigma)}{\sigma} \right).$$

Proof. By Proposition 2.20 we have that

$$\begin{aligned} \int_{\partial\Sigma \cap S} \frac{1}{r^{1+\beta}} \, d\mathcal{H}^1 \leq C(\beta, c_2/R_1) \int_\Sigma \frac{1}{r^{2+\beta}} \, d\mathcal{H}^2 + \frac{C(c_2/R_1)}{\sigma^\beta} \int_\Sigma H^2 \, d\mathcal{H}^2 \\ + C(c_2/R_1) \frac{\mathcal{H}^1(\Sigma \cap \partial B_\sigma)}{\sigma^{1+\beta}}. \end{aligned} \tag{2.25}$$

In order to estimate the second term on the right hand side of (2.25) we slightly modify an idea of Huisken and Yau [HY96, Lemma 5.2]: For $p \in \mathbb{R}$ we have

$$\operatorname{div}_\Sigma(r^{-p}x) = (2-p)r^{-p} + pr^{-p-2}|x^\perp|^2.$$

Testing the first variation identity with the vector field $r^{-p}x$ we obtain

$$\begin{aligned} &\int_\Sigma (2-p)r^{-p} + pr^{-p-2}|x^\perp|^2 \, d\mathcal{H}^2 \\ &\quad = -\int_\Sigma r^{-p}\vec{H} \cdot x \, d\mathcal{H}^2 + \int_{\partial\Sigma \cap S} r^{-p} x \cdot \gamma \, d\mathcal{H}^1 + \int_{\partial\Sigma \cap \partial B_\sigma} r^{-p} x \cdot \eta \, d\mathcal{H}^1, \end{aligned}$$

where η denotes the outward unit conormal of Σ. For $p = 2$ we thus have upon applying Young's inequality and absorbing

$$\begin{aligned} &\int_\Sigma \frac{|x^\perp|^2}{r^4} \, d\mathcal{H}^2 \\ &\quad \leq \frac{1}{4} \int_\Sigma H^2 \, d\mathcal{H}^2 + \int_{\partial\Sigma \cap S} \frac{|x \cdot \gamma|}{r^2} \, d\mathcal{H}^1 + \frac{\mathcal{H}^1(\Sigma \cap \partial B_\sigma)}{\sigma}. \end{aligned}$$

On the other hand, choosing $p = 2 + \beta$, we can estimate to arrive at

$$\begin{aligned}\int_\Sigma \frac{\beta}{r^{2+\beta}}\, d\mathcal{H}^2 &\le (3+\beta)\int_\Sigma \frac{|x^\perp|^2}{r^{4+\beta}}\, d\mathcal{H}^2 + \frac{1}{4}\int_\Sigma \frac{H^2}{r^\beta}\, d\mathcal{H}^2 \\ &\quad + \int_{\partial\Sigma\cap S} \frac{|x\cdot\gamma|}{r^{2+\beta}}\, d\mathcal{H}^1 + \frac{\mathcal{H}^1(\Sigma\cap\partial B_\sigma)}{\sigma^{1+\beta}} \\ &\le \frac{4}{\sigma^\beta}\int_\Sigma \frac{|x^\perp|^2}{r^4}\, d\mathcal{H}^2 + \frac{1}{4\sigma^\beta}\int_\Sigma H^2\, d\mathcal{H}^2 \\ &\quad + \frac{1}{\sigma^\beta}\int_{\partial\Sigma\cap S} \frac{|x\cdot\gamma|}{r^2}\, d\mathcal{H}^1 + \frac{\mathcal{H}^1(\Sigma\cap\partial B_\sigma)}{\sigma^{1+\beta}}.\end{aligned}$$

Combining the last two estimates we get

$$\begin{aligned}\int_\Sigma \frac{1}{r^{2+\beta}}\, d\mathcal{H}^2 &\le \frac{5}{4\beta\sigma^\beta}\int_\Sigma H^2\, d\mathcal{H}^2 \\ &\quad + \frac{5}{\beta\sigma^\beta}\int_{\partial\Sigma\cap S} \frac{|x\cdot\gamma|}{r^2}\, d\mathcal{H}^1 + \frac{5}{\beta}\frac{\mathcal{H}^1(\Sigma\cap\partial B_\sigma)}{\sigma^{1+\beta}}.\end{aligned}$$

Inserting the above estimate into (2.25) we obtain

$$\begin{aligned}\int_{\partial\Sigma\cap S} \frac{1}{r^{1+\beta}}\, d\mathcal{H}^1 &\le \frac{C(\beta, c_2/R_1)}{\sigma^\beta}\int_{\partial\Sigma\cap S} \frac{|x\cdot\gamma|}{r^2}\, d\mathcal{H}^1 + \frac{C(\beta, c_2/R_1)}{\sigma^\beta}\int_\Sigma H^2\, d\mathcal{H}^2 \\ &\quad + C(\beta, c_2/R_1)\frac{\mathcal{H}^1(\Sigma\cap\partial B_\sigma)}{\sigma^{1+\beta}}.\end{aligned}$$

Now for $\sigma \ge R_1$ we have that

$$|x\cdot\gamma| \le c_2 + c_1 r^{1-\beta},$$

so we obtain

$$\begin{aligned}\int_{\partial\Sigma\cap S} \frac{1}{r^{1+\beta}}\, d\mathcal{H}^1 &\le \frac{C(\beta, c_1, c_2, R_1)}{\sigma^\beta}\int_{\partial\Sigma\cap S} \frac{1}{r^{1+\beta}}\, d\mathcal{H}^1 + \frac{C(\beta, c_2, R_1)}{\sigma^\beta}\int_\Sigma H^2\, d\mathcal{H}^2 \\ &\quad + C(\beta, c_2, R_1)\frac{\mathcal{H}^1(\Sigma\cap\partial B_\sigma)}{\sigma^{1+\beta}}.\end{aligned}$$

Choosing $\sigma \ge \sigma_0 = \sigma_0(\beta, c_1, c_2, R_1)$ and absorbing yields the desired estimate. □

2.4.5 Bending energy and area growth

Lemma 2.48. *Let $S \subset \mathbb{R}^3$ be an asymptotically flat surface. There exists a radius $\sigma_0 > 0$ depending only on S such that for all $\sigma_0 < \sigma < \rho$ and every bounded surface $\Sigma \subset B_\rho \setminus B_\sigma$ with $\Sigma \cap (\partial B_\sigma \cup S) = \partial\Sigma$ and $\partial\Sigma$ meet S orthogonally, one has that*

$$\mathcal{H}^2(\Sigma) \le \rho^2 \int_\Sigma H^2\, d\mathcal{H}^2 + 2\sigma\,\mathcal{H}^1(\Sigma\cap\partial B_\sigma).$$

Proof. Let R_1 be as in Lemma 2.10, and assume that $\sigma > R_1$. We test the first variation identity with the position vector field, and apply Cauchy-Schwarz', Hölder's, and Young's inequality to obtain

$$\begin{aligned}2\mathcal{H}^2(\Sigma) &\le \rho\,\mathcal{H}^2(\Sigma)^{\frac{1}{2}}\left(\int_\Sigma H^2\, d\mathcal{H}^2\right)^{\frac{1}{2}} + \sigma\,\mathcal{H}^1(\partial\Sigma\cap\partial B_\sigma) \\ &\quad + \int_{\partial\Sigma\setminus\overline{B}_\sigma} |x\cdot\gamma|\, d\mathcal{H}^1.\end{aligned}$$

Using the estimate $|x \cdot \gamma| \leq c_2 + c_1 r^{1-\beta}$, which holds on $S \setminus B_{R_1}$, we can make use of Proposition 2.20 to arrive at

$$
\begin{aligned}
2\mathcal{H}^2(\Sigma) \leq\, & \rho\, \mathcal{H}^2(\Sigma)^{\frac{1}{2}} \left(\int_\Sigma H^2 \, d\mathcal{H}^2 \right)^{\frac{1}{2}} + \sigma\, \mathcal{H}^1(\Sigma \cap \partial B_\sigma) \\
& + C(\beta, c_1, c_2, R_1) \sigma^{-\beta} \mathcal{H}^2(\Sigma) \\
& + C(\beta, c_1, c_2, R_1) \rho^{2-\beta} \int_\Sigma H^2 \, d\mathcal{H}^2 \\
& + C(\beta, c_1, c_2, R_1) \sigma^{1-\beta} \mathcal{H}^1(\Sigma \cap \partial B_\sigma).
\end{aligned}
$$

Choosing $\sigma \geq \sigma_0 = \sigma_0(\beta, c_1, c_2, R_1)$ and absorbing yields the desired estimate. □

Lemma 2.49. *Let $S \subset \mathbb{R}^3$ be an asymptotically flat surface. There exists a radius σ_0 and a constant $C_4 < \infty$ both depending only on S with the following property: Let $\Sigma \subset \mathbb{R}^3 \setminus B_R$ be a compact surface with $\Sigma \cap (S \cup \partial B_R) = \partial\Sigma$ such that $\partial\Sigma$ and S meet orthogonally, and where $R \geq \sigma_0$ is a fixed radius. Then for all $R \leq \sigma < \rho < \infty$*

$$
\frac{\mathcal{H}^2(\Sigma \cap B_\sigma)}{\sigma^2} \leq C_4 \left(\frac{\mathcal{H}^2(\Sigma \cap B_\rho)}{\rho^2} + \int_\Sigma H^2 \, d\mathcal{H}^2 + \frac{\mathcal{H}^1(\Sigma \cap \partial B_R)}{\sigma} \right).
$$

Proof. By a slight variation of Simon's monotonicity formula (cf. [EM12, Lemma B.2], see also Chapter 1) we have that

$$
\begin{aligned}
\frac{\mathcal{H}^2(\Sigma \cap B_\sigma)}{\sigma^2} \leq\, & c \frac{\mathcal{H}^2(\Sigma \cap B_\rho)}{\rho^2} + c \int_{\Sigma \cap B_\rho} H^2 \, d\mathcal{H}^2 \\
& + c \int_{\partial\Sigma \cap B_\rho \setminus \overline{B}_\sigma} \left(\frac{1}{r^2} - \frac{1}{\rho^2} \right) |x \cdot \gamma| \, d\mathcal{H}^1 \\
& + c \left(\frac{1}{\sigma^2} - \frac{1}{\rho^2} \right) \int_{\partial\Sigma \cap S \cap B_\sigma} |x \cdot \gamma| \, d\mathcal{H}^1 \\
& + c \left(\frac{1}{\sigma^2} - \frac{1}{\rho^2} \right) R\, \mathcal{H}^1(\Sigma \cap \partial B_R).
\end{aligned}
$$

Here $c < \infty$ denotes a universal constant. Now assume that $R \geq R_1$, where R_1 is as in Lemma 2.10. Then

$$
|x \cdot \gamma| \leq c_2 + c_1 r^{1-\beta},
$$

and we infer

$$
\begin{aligned}
\frac{\mathcal{H}^2(\Sigma \cap B_\sigma)}{\sigma^2} \leq\, & c \frac{\mathcal{H}^2(\Sigma \cap B_\rho)}{\rho^2} + c \int_{\Sigma \cap B_\rho} H^2 \, d\mathcal{H}^2 \\
& + c \int_{\partial\Sigma \setminus \overline{B}_R} \frac{1}{r^{1+\beta}} \, d\mathcal{H}^1 + c \frac{\mathcal{H}^1(\Sigma \cap \partial B_R)}{\sigma}.
\end{aligned}
$$

Lemma 2.47 implies the desired estimate. □

Corollary 2.50. *Let S be an asymptotically flat surface of non-negative mean curvature. There exist constants C, σ_0 depending only on S with the following property: Let Σ be a* *connected compact volume-preserving stable free boundary constant mean curvature surface with respect to S. Let $\sigma_0 \leq s \leq \sigma$ and assume that Σ intersects ∂B_s transversally. Then*

$$\frac{\mathcal{H}^2(\Sigma \cap (B_\sigma \setminus B_s))}{\sigma^2} \leq C \left(1 + \frac{\mathcal{H}^1(\Sigma \cap \partial B_s)}{s}\right).$$

If S has strictly positive mean curvature everywhere, then

$$\mathcal{H}^1(\partial\Sigma \cap B_\sigma) \inf_{\partial\Sigma \cap B_\sigma} H_S \leq 14\pi \quad \textit{for all } \sigma \geq \sigma_0.$$

Proof. Applying Lemma 2.49 to the surface $\Sigma \setminus B_s$, i.e. $R = s$, and $\rho = 2 \max_\Sigma |x|$ yields

$$\frac{\mathcal{H}^2(\Sigma \cap (B_\sigma \setminus B_s))}{\sigma^2} \leq C_2 \left(\frac{\mathcal{H}^2(\Sigma)}{\rho^2} + \int_\Sigma H^2 \, d\mathcal{H}^2 + \frac{\mathcal{H}^1(\Sigma \cap \partial B_s)}{\sigma}\right).$$

Lemma 2.48 implies that

$$\mathcal{H}^2(\Sigma) \leq \rho^2 \int_\Sigma H^2 \, d\mathcal{H}^2 + 2s \, \mathcal{H}^1(\Sigma \cap \partial B_s).$$

Combining the last two inequalities we arrive at

$$\frac{\mathcal{H}^2(\Sigma \cap (B_\sigma \setminus B_s))}{\sigma^2} \leq 3C_2 \left(\int_\Sigma H^2 \, d\mathcal{H}^2 + \frac{\mathcal{H}^1(\Sigma \cap \partial B_s)}{s}\right).$$

Now using the assumption that $H_S \geq 0$, Proposition 2.17 implies the first claim.

The second claim follows from Proposition 2.17. □

Remark 2.51. Corollary 2.50 establishes an a priori bound on the free boundary length growth of Σ in case $H_S > 0$ everywhere. We suspect that it is possible to prove an a priori bound on the quantity $\frac{\mathcal{H}^1(\Sigma \cap \partial B_s)}{s}$, which would let us conclude quadratic area growth of Σ instead of having to assume it in Theorem 2.16 (cf. [EM12, Theorem 1.6]).

3 Nonlinear mean curvature flow with Neumann boundary condition

Let M^n be a compact, orientable manifold with non-empty boundary ∂M, and let $F_0 : M \to \mathbb{R}^{n+1}$ be a smooth immersion. Further, let $S = \partial G$ be the boundary of a domain $G \subset \mathbb{R}^{n+1}$. Upon defining $M_0 := F_0(M)$ we assume that

$$M_0 \cap S = \partial M_0 \quad \text{and} \quad \langle \nu_0, \gamma \circ F_0 \rangle(p) = 0 \quad \text{for all } p \in \partial M.$$

Here ν_0 and γ denote the outer unit normal to M_0 and S, respectively.

Definition 3.1. **(H^k-flow with Neumann boundary condition)** *Let $k \geq 1$. A family of immersions $F_t = F(\cdot, t) : M^n \to \mathbb{R}^{n+1}$, $t \in [0, T)$, is an H^k-flow with Neumann boundary condition with respect to $S = \partial G$, if the following equations are satisfied:*

$$(\star) \begin{cases} \frac{d}{dt} F(p,t) = -H(p,t)^k \, \nu(p,t) & , (p,t) \in M \times (0,T) \\ F(\cdot, 0) = F_0 & \\ F(p,t) \in S & , (p,t) \in \partial M \times [0,T) \\ \langle \nu, \gamma \circ F \rangle (p,t) = 0 & , (p,t) \in \partial M \times [0,T) \\ F(p,t) \in G & , (p,t) \in \operatorname{int}(M) \times [0,T). \end{cases}$$

Here $H(\cdot, t)$ denotes the mean curvature and $\nu(\cdot, t)$ denotes a unit normal field of the immersion F_t such that $-H(\cdot, t)\, \nu(\cdot, t)$ equals the mean curvature vector $\vec{H}(\cdot, t)$.

Remark 3.2. For $k = 1$ our definition is what Koeller [Koe10] calls mean curvature flow with Neumann free boundary condition on the *solid* support surface S. Removing the last condition in $(\star)$ coincides with the original definition of Stahl [Sta96b], and is what Koeller [Koe12] calls mean curvature flow with Neumann free boundary condition on the *transversable* support surface S.

Now suppose that $\{F_t\}_{t \in [0,T)}$ is a smooth solution of $(\star)$ for some $k \geq n - 1$. Denote by $A(t)$ the area of M_t, and denote by $V(t)$ the enclosed volume inside G. Assuming that there exists a positive constant $c_0 > 0$ such that

$$\inf_{t \in (0,T)} \left(\int_{M_t} |H|^n \, d\mu_t \right)^{\frac{1}{n}} \geq \frac{n}{n+1} c_0,$$

we have that the relative isoperimetric difference

$$A(t)^{\frac{n+1}{n}} - c_0 V(t)$$

is non-increasing. In the closed case, i.e. for the H^k-flow without Neumann boundary condition, this monotonicity was first observed by Huisken and later exploited by Schulze [Sch08] to prove the standard isoperimetric inequalites in $\mathbb{R}^{n+1}$ for $n \leq 7$ and to give a new proof of the 3-dimensional case of the Aubin-Cartan-Hadamard conjecture, the original proof of which was put forward by Kleiner [Kle92]. The proof of this monotonicity in our case is identical to the one given in [Sch08] for closed surfaces since all boundary terms vanish due to the orthogonality condition.

If the flow existed until $V(t)$ converged to zero, the monotonicity of the relative isoperimetric difference would imply that

$$c_0 \operatorname{vol}(\Omega) \leq \operatorname{area}(M_0)^{\frac{n+1}{n}},$$

where Ω denotes the bounded region enclosed by M_0 in G. Even though smooth solutions under special geometric assumptions exists until the enclosed volume goes to zero cf. [Sta96a], in general singularities may occur in the interior as well as on the supporting hypersurface (cf. [Koe10]) before the enclosed volume tends to zero. In order to continue the flow past those singularities we replace $(\star)$ by the following level set formulation:

Let Ω be an bounded open subset of G such that its boundary $\partial\Omega$ may be decomposed into a disjoint union of $\partial^{\mathcal{D}}\Omega := \overline{\partial\Omega \cap G}$ and $\partial^{\mathcal{N}}\Omega := \partial\Omega \setminus \partial^{\mathcal{D}}\Omega$ with the following properties. $\partial^{\mathcal{D}}\Omega$ and $\partial^{\mathcal{N}}\Omega$ are smooth hypersurfaces with and without boundary, respectively, and in the "vertex" $V := \partial^{\mathcal{D}}\Omega \cap S (= \partial^{\mathcal{D}}\Omega \cap \overline{\partial^{\mathcal{N}}\Omega})$ we have that $\langle \nu, \gamma \rangle = 0$, where ν denotes the outward unit normal to $\partial^{\mathcal{D}}\Omega$. Moreover, assume that the mean curvature of $\partial^{\mathcal{D}}\Omega$ is strictly positive. The evolving hypersurfaces are then given by the relative boundaries of the superlevel sets of a function $u : \overline{\Omega} \to \mathbb{R}_{\geq 0}$, $u = 0$ on $\partial^{\mathcal{D}}\Omega$ via

$$M_t = \overline{\partial\{x \in \Omega : u(x) > t\} \cap G},$$

and $(\star)$ is replaced by the following degenerate elliptic mixed boundary value problem.

$$(\star\star) \begin{cases} \operatorname{div}\left(\frac{Du}{|Du|}\right) = -|Du|^{-\frac{1}{k}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial^{\mathcal{D}}\Omega \text{ and} \\ \frac{\partial u}{\partial \gamma} := \gamma^i D_i u = 0 & \text{on } \partial^{\mathcal{N}}\Omega. \end{cases}$$

This formulation is inspired by the work of Schulze [Sch08] for H^k-flow, which in turn was inspired by the work of Evans and Spruck [ES91] and Chen, Giga, and Goto [CGG89] on mean curvature flow and by work of Huisken and Ilmanen [HI01] on the inverse mean curvature flow. A level set formulation for inverse mean curvature flow with Neumann boundary condition was put forward independently by Marquardt [Mar12].

As in [Sch08] we use the method of elliptic regularization to define a family of approximating problems to $(\star\star)$. Since the linear theory for mixed boundary value problem only yields solutions that are Hölder continuous up to the vertex it is not immediately possible to ensure the existence of solutions of the regularized problems by means of a standard linearization approach. Therefore, we use yet another family of approximating problems. More precisely, we approximate the domain Ω from the outside by a family of domains, the Dirichlet boundaries of which have a contact angle with their Neumann boundaries of strictly less than $\pi/2$, thereby ensuring $C^{1,\alpha}$-solutions of the associated linearized problems. We then prove existence to the approximating problem, which satisfy a uniform a priori sub- and gradient-bound, and hence subconverge to a Lipschitz continuous function on $\overline{\Omega}$. We define any such limit function to be a weak solution to $(\star\star)$, and call it a weak H^k-flow with Neumann boundary condition generated by the pair (Ω, G).

3.1 Short time existence in the smooth case

The short time existence of $(\star)$ for the case $k = 1$ was proved by Stahl [Sta96b]. Stahl defines generalized Gaussian coordinates that are adjusted to the geometry of the supporting surface S to be able to write the evolving surfaces $\{M_t\}$ as a graph over M_0 for a short time, thereby reducing the parabolic system $(\star)$ to a *scalar* parabolic Neumann problem, which can be solved for a short time by standard results (see [LSU68]). This is a modification of an idea of Ecker and Huisken [EH91] to prove the short time existence of mean curvature flow.

The same proof works to prove short time existence of $(\star)$ for the case $k > 0$ under the additional assumption that $H_0 > 0$. This comes from the fact that the coefficients of the leading order term in the scalar problem is of the form $a^{ij} = kH^{k-1}g^{ij}$, where H and g are the mean curvature and the first fundamental form of the evolving graphs, respectively, and which is uniformly elliptic in case $H_0 > 0$.

See also [Mar13] for a discussion on the short time existence of inverse mean curvature flow with Neumann boundary condition.

Summarizing we have the following theorem.

Theorem 3.3. (Short time existence) *Suppose that $k = 1$, or $k > 0$ and the mean curvature H_0 of M_0 is strictly positive. There exists a $T > 0$ and unique solution $F \in C^{2+\alpha,1+\frac{\alpha}{2}}(M \times [0,T], \mathbb{R}^{n+1}) \cap C^\infty(M \times (0,T], \mathbb{R}^{n+1})$ of $(\star)$.*

3.2 Elliptic regularization and existence of weak solution

In order to define a weak solution of $(\star\star)$ let us assume that the evolving surfaces M_t are given as the level sets of a function $u : \overline{\Omega} \to \mathbb{R}$, i.e.

$$\Omega_t := \{x \in \Omega : u(x) > t\} \quad \text{and} \quad M_t := \overline{\partial_G \Omega_t},$$

where we use the notation $\partial_G E := \partial E \cap G$ for a set $E \subset \mathbb{R}^{n+1}$.

Then the H^k-flow with Neumann boundary condition with respect to $S = \partial G$ is described by the following degenerate elliptic PDE with mixed boundary values.

$$(\star\star)\begin{cases} \operatorname{div}\left(\frac{Du}{|Du|}\right) = -|Du|^{-\frac{1}{k}} & \text{in } \Omega \\ u = 0 & \text{on } \partial^{\mathcal{D}}\Omega \\ \frac{\partial u}{\partial \gamma} = 0 & \text{on } \partial^{\mathcal{N}}\Omega, \end{cases}$$

where $\partial^{\mathcal{D}}\Omega := \overline{\partial_G \Omega}$ and $\partial^{\mathcal{N}}\Omega := \partial\Omega \setminus \partial^{\mathcal{D}}\Omega$ denote the Dirichlet and the Neumann boundary, respectively.

In order to solve $(\star\star)$ we first define the following approximating equations known as the elliptic regularization of $(\star\star)$.

$$(\star\star)_\varepsilon\begin{cases} \operatorname{div}\left(\frac{Du^\varepsilon}{\sqrt{\varepsilon^2+|Du^\varepsilon|^2}}\right) = -(\varepsilon^2 + |Du^\varepsilon|^2)^{-\frac{1}{2k}} & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial^{\mathcal{D}}\Omega \\ \frac{\partial u^\varepsilon}{\partial \gamma} = 0 & \text{on } \partial^{\mathcal{N}}\Omega. \end{cases}$$

As in the closed case we may interpret these approximating equations in a geometric way: Given a solution u^ε of $(\star)_\varepsilon$, and setting $\hat{u}^\varepsilon := \frac{1}{\varepsilon}u^\varepsilon$ we see that $\hat{u}^\varepsilon$ satisfies the following elliptic mixed boundary value problem:

$$\begin{cases} \operatorname{div}\left(\frac{D\hat{u}^\varepsilon}{\sqrt{1+|D\hat{u}^\varepsilon|^2}}\right) = -\varepsilon^{-\frac{1}{k}}(1+|D\hat{u}^\varepsilon|^2)^{-\frac{1}{2k}} & \text{in } \Omega \\ \hat{u}^\varepsilon = 0 & \text{on } \partial^{\mathcal{D}}\Omega \\ \frac{\partial \hat{u}^\varepsilon}{\partial\gamma} = 0 & \text{on } \partial^{\mathcal{N}}\Omega. \end{cases}$$

This equation means that the translating graphs

$$N_t^\varepsilon := \left\{\left(x, \hat{u}^\varepsilon(x) - \frac{t}{\varepsilon}\right) : x \in \overline{\Omega}\right\}, \quad t \in \mathbb{R}$$

solve the H^k-flow in $\Omega \times \mathbb{R}$ with partial Neumann boundary condition

$$\partial^{\mathcal{N}} N_t^\varepsilon := \left\{\left(x, \hat{u}^\varepsilon(x) - \frac{t}{\varepsilon}\right) : x \in \partial^{\mathcal{N}}\Omega\right\} \subset S \times \mathbb{R}.$$

We refer to [Eck04, Chapter 2] for details. Unfortunately, in contrast to the closed case, the linear theory for mixed boundary value problem only yields solutions that are Hölder continuous up to the vertex it is not immediately possible to ensure the existence of solutions of the regularized problems by means of a standard linearization approach. Therefore, we use yet another family of approximating problems. More precisely, we approximate the domain Ω from the outside by a family of domains, the Dirichlet boundaries of which have a contact angle with their Neumann boundaries of strictly less than $\pi/2$, thereby ensuring $C^{1,\alpha}$-solutions of the associated linearized problems.

For a family $\{\Omega^\tau\}_{\tau\in[0,\tau_0]}$ of domains $\Omega^\tau \subset G$, to be specified below, we define the following family of approximating problems.

$$(\star\star)_{\varepsilon,\tau} \begin{cases} \operatorname{div}\left(\frac{Du^{\varepsilon,\tau}}{\sqrt{\varepsilon^2+|Du^{\varepsilon,\tau}|^2}}\right) = -(\varepsilon^2+|Du^{\varepsilon,\tau}|^2)^{-\frac{1}{2k}} & \text{in } \Omega^\tau \\ u^{\varepsilon,\tau} = 0 & \text{on } \partial^{\mathcal{D}}\Omega^\tau \\ \frac{\partial u^{\varepsilon,\tau}}{\partial\gamma} = 0 & \text{on } \partial^{\mathcal{N}}\Omega^\tau. \end{cases}$$

Approximating the domain

Let $\Omega \subset G$ be an open bounded set such that $\partial^{\mathcal{D}}\Omega$ and $\partial^{\mathcal{N}}\Omega$ are smooth hypersurfaces with and without boundary, respectively, that meet orthogonally, i.e. $\nu_{\partial^{\mathcal{D}}\Omega} \cdot \gamma = 0$ on $V := \partial^{\mathcal{D}}\Omega \cap \overline{\partial^{\mathcal{N}}\Omega}$. Let $H_{\partial^{\mathcal{D}}\Omega} \geq \delta_0$ for some positive $\delta_0 > 0$.

Using a partition of unity and local graph representations of $\partial^{\mathcal{D}}\Omega$ near the vertex V, we may employ a standard extension Lemma (see [GT01, Lemma 6.37]) to extend the surface $\partial^{\mathcal{D}}\Omega$ to a smooth compact hypersurface $\widetilde{\partial^{\mathcal{D}}\Omega}$ with boundary across S such that the following three conditions are met:

1. $\sup_{\partial^{\mathcal{N}}\Omega^\tau} |A_S| \leq \frac{3}{2}\sup_{\partial^{\mathcal{N}}\Omega} |A_S|$
2. $H_{\widetilde{\partial^{\mathcal{D}}\Omega}} \geq \frac{2}{3}\delta_0$
3. For some $\delta_D \in (0,1)$ the "signed distance function" f defined by

$$f(x) := \begin{cases} -d(x) & x \in \Omega \\ d(x) & x \in G \setminus \Omega, \end{cases}$$

with $d := \operatorname{dist}(\widetilde{\partial^{\mathcal{D}}\Omega}, \cdot)$, is smooth on $\{p \in G : d(p) < 2\delta_D\} \setminus \Omega$.

We shall also assume that

$$|Dd \cdot \gamma| \le c_1\, d \quad \text{on } S \cap (\Omega \setminus \Omega_{2\delta_D}) \tag{3.1}$$

for some positive constant $c_1 < \infty$, depending on the curvatures of $\widetilde{\partial^{\mathcal{D}}\Omega}$ and $S \cap (\Omega \setminus \Omega_{2\delta_D})$. Now, let $\delta_N \in (0,1]$ be such that the signed distance function f_S is smooth in $\{d_S < 2\delta_N\} \cap K$, where K is a compact set with $\operatorname{dist}(\Omega, \mathbb{R}^{n+1} \setminus K) > 0$. We set $\phi(s) := \delta_D\, \phi_0(2s/\delta_N)$, where ϕ_0 is the mollification of the function $s \mapsto (1-s)^+$ with the standard mollifier $\eta(s) = c_0 \exp((s^2-1)^{-1})\chi_{(-1,1)}(s)$ with $\int_{\mathbb{R}} \eta\, ds = 1$. Note that

$$\|\phi\|_{C^0(0,\infty)} \le \delta_D, \quad \|\phi'\|_{C^0(\mathbb{R})} \le 2\frac{\delta_D}{\delta_N}, \quad \text{and} \quad \|\phi''\|_{C^0(\mathbb{R})} \le 4c_0 \frac{\delta_D}{\delta_N^2}.$$

For $\tau \in [0,1]$ we now define the approximating domains Ω^τ as follows:

$$\Omega^\tau := \{x \in G : f(x) < \tau\phi(d_S(x))\}.$$

It is obvious from the definition that $\partial^{\mathcal{D}}\Omega^\tau$ converge to $\partial^{\mathcal{D}}\Omega$ in C^2 as $\tau \to 0$. Hence, we may assume w.l.o.g., after possibly multiplying the function ϕ by a small constant, that for all $\tau \in [0,1]$ $d_\tau := \operatorname{dist}(\widetilde{\partial^{\mathcal{D}}\Omega^\tau}, \cdot)$ smooth on $\overline{\{x \in G : 0 < d_\tau(x) < \delta_D\}}$, where

$$\widetilde{\partial^{\mathcal{D}}\Omega^\tau} = \{x + \tau\phi(d_S(x))\nu_{\widetilde{\partial^{\mathcal{D}}\Omega}}(x) : x \in \widetilde{\partial^{\mathcal{D}}\Omega}\}.$$

Moreover, we may assume that $H_{\widetilde{\partial^{\mathcal{D}}\Omega^\tau}} \ge \frac{1}{2}\delta_0$, $\|A_{\widetilde{\partial^{\mathcal{D}}\Omega^\tau}}\|_{C^0(\partial^{\mathcal{D}}\Omega^\tau)} \le 2\|A\|_{C^0(\partial^{\mathcal{D}}\Omega)}$, and $\mathcal{L}^{n+1}(\Omega^\tau) \le 2\mathcal{L}^{n+1}(\Omega)$ for all $\tau \in [0,1]$. On $\partial^{\mathcal{D}}\Omega^\tau$ we have (with obvious notation)

$$\nu_\tau = Df_\tau = \lambda_\tau \left(Df - \tau\phi'(d_S)Dd_S\right),$$

where $\lambda_\tau = |Df - \tau\phi'(d_S)Dd_S|^{-1}$. So in particular,

$$\nu_\tau = \lambda_\tau \left(Df - \tau\frac{2\delta_D}{\delta_N}\gamma\right) \quad \text{on } V^\tau := \partial^{\mathcal{D}}\Omega^\tau \cap S.$$

We wish to choose parameters as to guarantee that

$$\nu_\tau \cdot \gamma \le -\frac{\tau}{2} \quad \text{on } V^\tau \tag{3.2}$$

for every sufficiently small $\tau > 0$. Making use of (3.1) we estimate on V^τ

$$\begin{aligned}
\nu_\tau \cdot \gamma &= \lambda_\tau \left(Df \cdot \gamma - \frac{2\tau\delta_D}{\delta_N}\right) \\
&\le \lambda_\tau \left(c_1 d - \frac{2\tau\delta_D}{\delta_N}\right) \\
&\le \lambda_\tau \left(c_1 - \frac{2}{\delta_N}\right)\tau\delta_D \\
&\le -\lambda_\tau\tau,
\end{aligned}$$

provided $\delta_N \le \frac{2\delta_D}{1+c_1\delta_D}$, where we used that $d(x_0) \le \tau\delta_D$ whenever $x_0 \in V^\tau$. By the triangle inequality, we have that $\lambda_\tau \ge 1/2$ for all $\tau \le \delta_N/(2\delta_D)$. We conclude that (3.2) holds for all $\tau \le \delta_N/(2\delta_D)$.

Summarizing, we have constructed a family of domains $\{\Omega^\tau\}_{\tau\in[0,\tau_0]}$, $\tau_0 \le \delta_N/(2\delta_D)$, with the following properties:

1. $\Omega^\tau \subset G$ and have smooth boundaries $\partial^{\mathcal{D}}\Omega^\tau$ and $\partial^{\mathcal{N}}\Omega^\tau$

2. $\Omega \subset \Omega^\tau$ for all τ

3. $\mathcal{L}^{n+1}(\Omega^\tau) \leq 2\mathcal{L}^{n+1}(\Omega)$ for all τ

4. $\mathrm{diam}(\Omega^\tau) \leq 2\mathrm{diam}(\Omega)$ for all τ

5. $\partial^{\mathcal{D}}\Omega^\tau \to \partial^{\mathcal{D}}\Omega$ in C^2 as $\tau \to \infty$

6. $\nu_\tau \cdot \gamma \leq -\frac{\tau}{2}$, where ν_τ denotes the outward unit normal to $\partial^{\mathcal{D}}\Omega^\tau$

7. $H_{\partial^{\mathcal{D}}\Omega^\tau} \geq \frac{1}{2}\delta_0$ and $\sup_{\partial^{\mathcal{D}}\Omega^\tau} |A^\tau| \leq 2\sup_{\partial^{\mathcal{D}}\Omega} |A|$

8. $\sup_{\partial^{\mathcal{N}}\Omega^\tau} |A_S| \leq 2\sup_{\partial^{\mathcal{N}}\Omega} |A_S|$

We remark that the special form of the approximating sequence is not relevant for the sequel, but rather that the above properties are satisfied.

A priori estimates for $(\star\star)_{\varepsilon,\tau}$

In order to prove existence of solutions to $(\star\star)_{\varepsilon,\tau}$ we need to prove a priori sup- and gradient-bounds.

Lemma 3.4. (sup-estimate) *Suppose there exists a foliation $\{N_t\}_{t\in I}$ of an open neighborhood U of Ω, $\Omega \subset\subset U$, consisting of C^2-hypersurfaces $N_t = \{w = t\}$ for some function $w \in C^2(\overline{U})$ such that the mean curvature $H_{N_t} = \mathrm{div}\,(\nu_{N_t})$ of each surface N_t is bounded from below by some positive constant $\theta_0 > 0$ and such that they intersect S with a non-positive angle, i.e. $\nu_{N_t} \cdot \gamma \leq 0$, and where $\nu_{N_t} = Dw/|Dw|$, then*

$$\sup_{\Omega^\tau} |u^{\varepsilon,\tau}| \leq C_1 = C_1 \left(k, \theta_0, |I|, \inf_U |Dw|, \||Dw|^{-1}D^2w\|_{C^0(U)}\right) \tag{3.3}$$

for any solution $u^{\varepsilon,\tau} \in C^2(\Omega^\tau) \cap C^1(\overline{\Omega^\tau})$ of $(\star\star)_{\varepsilon,\tau}$ with $\tau \leq \tau_0$ such that $\Omega^\tau \subset U$, and with $\varepsilon \in (0,1]$.

Proof. Since for every $0 < \varepsilon \leq 1$, $0 < \tau \leq \tau_0$ the constant zero function is a subsolution of $(\star\star)_{\varepsilon,\tau}$ the maximum principle implies that $u^{\varepsilon,\tau} \geq 0$. It remains to construct a supersolution of $(\star\star)_{\varepsilon,\tau}$. For the mean curvature H_{N_t} of N_t we have

$$H_{N_t} = \mathrm{div}\left(\frac{Dw}{|Dw|}\right) = \frac{\Delta w}{|Dw|} - \left(\frac{D^2w}{|Dw|}\right)\left(\frac{Dw}{|Dw|}, \frac{Dw}{|Dw|}\right).$$

We make the ansatz $\Phi(x) := \psi(w(x))$ for some function ψ with $\psi' < 0$ to be determined, and we compute

$$
\begin{aligned}
&\operatorname{div}\left(\frac{D\Phi}{\sqrt{\varepsilon^2+|D\Phi|^2}}\right)\\
&= \frac{\psi'(w)}{\sqrt{\varepsilon^2+\psi'(w)^2|Dw|^2}}\Delta w + \left\langle D\left(\frac{\psi'(w)}{\sqrt{\varepsilon^2+\psi'(w)^2|Dw|^2}}\right), Dw\right\rangle\\
&= \frac{\psi'(w)}{\sqrt{\varepsilon^2+\psi'(w)^2|Dw|^2}}\left(H_w|Dw| + (D^2w)\left(\frac{Dw}{|Dw|},\frac{Dw}{|Dw|}\right)\right)\\
&\quad + \frac{\varepsilon^2\psi''(w)|Dw|^2}{(\varepsilon^2+\psi'(w)^2|Dw|^2)^{\frac{3}{2}}} - \frac{\psi'(w)^3(D^2w)\,(Dw,Dw)}{(\varepsilon^2+\psi'(w)^2|Dw|^2)^{\frac{3}{2}}}\\
&= \frac{\psi'(w)|Dw|}{\sqrt{\varepsilon^2+\psi'(w)^2|Dw|^2}}H_w + \frac{\varepsilon^2\psi''(w)|Dw|^2}{(\varepsilon^2+\psi'(w)^2|Dw|^2)^{\frac{3}{2}}}\\
&\quad + \frac{\varepsilon^2\psi'(w)|Dw|}{(\varepsilon^2+\psi'(w)^2|Dw|^2)^{\frac{3}{2}}}\left(\frac{D^2w}{|Dw|}\right)\left(\frac{Dw}{|Dw|},\frac{Dw}{|Dw|}\right). \qquad (3.4)
\end{aligned}
$$

This should be less than $-(\varepsilon^2+\psi'(w)^2|Dw|^2)^{-\frac{1}{2k}}$ for Φ to be a supersolution. We use the assumption on the positivity of the mean curvatures H_w and the assumption $\psi' < 0$ to see that a sufficient condition for this is that

$$
\begin{aligned}
\theta_0 \geq &-\frac{\varepsilon^2\psi''(w)|Dw|}{\psi'(w)(\varepsilon^2+\psi'(w)^2|Dw|^2)} - \frac{\varepsilon^2}{\varepsilon^2+\psi'(w)^2|Dw|^2}\left(\frac{D^2w}{|Dw|}\right)\left(\frac{Dw}{|Dw|},\frac{Dw}{|Dw|}\right)\\
&- \frac{1}{\psi'(w)|Dw|}(\varepsilon^2+\psi'(w)^2|Dw|^2)^{\frac{k-1}{2k}}. \qquad (3.5)
\end{aligned}
$$

Clearly, we may assume w.l.o.g. that $w \geq 1$ and that $|Dw| \geq 1$. Let $\sigma > 0$ be a constant, to be chosen later, and take $\psi(t) = \sigma(k+1)^{-1}(T_0^{k+1} - t^{k+1})$, which gives

$$
\psi'(t) = -\sigma t^k \quad \text{and} \quad \psi''(t) = -k\sigma t^{k-1}.
$$

The inequality (3.5) then becomes

$$
\begin{aligned}
\theta_0 \geq &-\frac{\varepsilon^2 k|Dw|}{w(\varepsilon^2+\sigma^2w^{2k}|Dw|^2)} - \frac{\varepsilon^2}{\varepsilon^2+\sigma^2w^{2k}|Dw|^2}\left(\frac{D^2w}{|Dw|}\right)\left(\frac{Dw}{|Dw|},\frac{Dw}{|Dw|}\right)\\
&+ \frac{1}{\sigma w^k|Dw|}(\varepsilon^2+\sigma^2w^{2k}|Dw|^2)^{\frac{k-1}{2k}}. \qquad (3.6)
\end{aligned}
$$

Dropping the first term on the RHS, a sufficient condition for (3.6), and hence for Φ to be a supersolution, is that

$$
\theta_0 \geq \frac{1}{1+\sigma^2}\||Dw|^{-1}D^2w\|_{C^0(U)} + \frac{1}{\sigma^{\frac{1}{k}}}\left(\frac{\varepsilon^2}{\sigma^2w^{2k}|Dw|^2}+1\right)^{\frac{k-1}{2k}}.
$$

Hence, we can choose σ large enough depending on k, θ_0 and $\||Dw|^{-1}D^2w\|_{C^0(U)}$, but independent of ε and τ, such that the above condition is satisfied for all points in Ω^{τ_0} and $0 < \varepsilon \leq 1$. To ensure that Φ is a supersolution of $(\star\star)_{\varepsilon,\tau}$ we need to make sure that

$$
\frac{\partial\Phi}{\partial\gamma} \geq 0 \quad \text{on } \partial^{\mathcal{N}}\Omega^\tau.
$$

We compute

$$\frac{\partial\Phi}{\partial\gamma} = \psi'(w)\frac{\partial w}{\partial\gamma} = -\sigma w^k\, Dw\cdot\gamma = -\sigma w^k |Dw|\,\nu_{N_w}\cdot\gamma,$$

which is non-negative on $\partial^{\mathcal{N}}\Omega^\tau$ by assumption. Choosing $T_0 = \|w\|_{C^0(U)}$ we see that $\Phi \geq 0$ on $\partial^{\mathcal{D}}\Omega^\tau$ for all τ. Thus Φ is a positive supersolution on every Ω^τ and all $0 < \varepsilon \leq 1$. The maximum principle implies the desired C^0-estimate. □

A sufficient condition for which a foliation as in the assumptions of Lemma 3.4 exists is that $\partial^{\mathcal{N}}\Omega$ is compactly contained in a graph over some ball that is disjoint from Ω:

Lemma 3.5. *Suppose that there exists a point $x_0 \in \mathbb{R}^{n+1}\setminus\overline{\Omega}$ such that*

$$(x - x_0)\cdot\gamma < 0 \quad \textit{on } \overline{\partial^{\mathcal{N}}\Omega}.$$

Then there exists a foliation $\{N_t\}_{t\in I}$ as in the statement of Lemma 3.4.

Proof. Setting $U := \mathbb{R}^{n+1}$ and $w(x) := |x - x_0|^2$, the requirements of Lemma 3.4 are easily seen to be satisfied. □

Remark 3.6. A necessary condition to obtain a sup-bound which for $\varepsilon = 0$ corresponds to a finite existence time in the parabolic flow is that Ω does not contain a free boundary minimal surface with respect to S. Consider for example S =catenoid and M_0 =spherical cap with boundary meeting S orthogonally. The MCF with Neumann condition of M_0 exists for all times and converges to the free boundary minimal disk inside the catenoidal neck [Whe14b]. We suspect that this condition is also sufficient to obtain a sup-bound.
This example also shows how the Neumann boundary strongly influences the evolution of surfaces flowing by MCF with NBC. The spherical cap in the above example would self-similarly shrink to a point in finite time under ordinary MCF, i.e. without a Neumann condition.

For the gradient-estimate we aim to apply the maximum principle for $|u^{\varepsilon,\tau}|$. Instead of trying to derive an equation for $|u^{\varepsilon,\tau}|$ we use a more geometric approach (like in [Sch08]). Setting $v := \sqrt{1 + |D\hat{u}^{\varepsilon,\tau}|^2}$ one computes, see for example [Sch08] for details, that

$$\Delta_{\mathcal{M}} v = \frac{2}{v}|\nabla^{\mathcal{M}} v|^2 - v^2\langle\nabla^{\mathcal{M}} H, \omega\rangle_{\mathbb{R}^{n+2}} + v|A|^2, \tag{3.7}$$

where $\omega = (0, ..., 0, 1) \in \mathbb{R}^{n+1}\times\mathbb{R}$, and where H and A are the mean curvature and the second fundamental form, respectively, of $\mathcal{M} := \text{graph}(\hat{u}^{\varepsilon,\tau})$ in $\mathbb{R}^{n+1}\times\mathbb{R}$ with respect to the upward unit normal ν. Here we have identified the function v with the function $\widetilde{v}$ given by $\widetilde{v}(x,z) \equiv v(x)$ on $\overline{\Omega^\tau}\times\mathbb{R}$. Then $\mathcal{M}$ is orthogonal to $S' := S\times\mathbb{R}$. On the boundary $\partial^{\mathcal{N}}\Omega^\tau\times\mathbb{R}$ we compute

$$\begin{aligned}
\frac{\partial v}{\partial\gamma} &= D_{(\gamma,0)}(\langle\nu,\omega\rangle^{-1}) \\
&= -v^2\langle D_{(\gamma,0)}\nu, \omega\rangle \\
&= -v^2 A_{\mathcal{M}}((\gamma,0), \omega^{T\mathcal{M}}) \\
&= v^2 A_{S'}(\nu, \omega^{T\mathcal{M}}) \\
&= -v^{-1} A_S(D\hat{u}^{\varepsilon,\tau}, D\hat{u}^{\varepsilon,\tau}) \\
&= -v^{-1} A_S(\nabla^S\hat{u}^{\varepsilon,\tau}, \nabla^S\hat{u}^{\varepsilon,\tau}),
\end{aligned} \tag{3.8}$$

where we used the fact that $A_{S'}(\nu,\zeta) = -A_{\mathcal{M}}((\gamma,0),\zeta)$ for any vector $\zeta \in T\mathcal{M} \cap TS'$, which follows from differentiating the identity $\langle \nu, (\gamma,0)\rangle = 0$ in the direction of ζ (cf. [Sta96a, Proposition 2.2]).

In case S is a convex hypersurface the right hand side of (3.8) is non-positive. The Hopf boundary point lemma then implies that a boundary-gradient-estimate only needs to be established on the Dirichlet boundary $\partial^{\mathcal{D}}\Omega^\tau$. This observation goes back to work of Miersemann [Mie84].

In order to establish a gradient-estimate for arbitrary support surfaces we use a modified test function: We define $w := \exp(-\eta\, z)$ and $\phi := \exp(\beta\varrho)$ for some $\eta, \beta \geq 0$ to be determined, and where ϱ is a "regularized" distance function on $\overline{\Omega^1}$, i.e. $\varrho = d_S$ in a neighborhood of $\partial^{\mathcal{N}}\Omega^1$, $0 \leq \varrho \leq 2d_S$, $|D\varrho| \leq 2$, and $D^2_{ij}\varrho \geq -C_B\,\delta_{ij}$, for a constant $C_B = C_B(\overline{\kappa}_{\partial^{\mathcal{N}}\Omega}) \geq 0$.

We denote by $\overline{D}$, $\overline{D}^2$, $\overline{\Delta}$ the gradient, the Hessian, and the Laplacian in $\mathbb{R}^{n+2}$, respectively. We have

$$\nabla^{\mathcal{M}} w = -\eta w(\omega - v^{-1}\nu), \qquad \Delta_{\mathcal{M}} w = \eta^2 w(1-v^{-2}) + \eta v^{-1} H w$$
$$\nabla^{\mathcal{M}}\phi = \beta\phi\nabla^{\mathcal{M}}\varrho, \qquad \Delta_{\mathcal{M}}\phi = \beta^2\phi|\nabla^{\mathcal{M}}\varrho|^2 + \beta\phi\Delta_{\mathcal{M}}\varrho.$$

Hence,

$$\begin{aligned}\Delta_{\mathcal{M}}(w\phi) &= \phi\Delta_{\mathcal{M}} w + w\Delta_{\mathcal{M}}\phi + 2\nabla^{\mathcal{M}} w\cdot\nabla^{\mathcal{M}}\phi \\ &= w\phi\left(\eta^2(1-v^{-2}) + \eta v^{-1}H + \beta^2|\nabla^{\mathcal{M}}\varrho|^2 + \beta\Delta_{\mathcal{M}}\varrho - 2\eta\beta\nabla^{\mathcal{M}}\varrho\cdot\omega\right).\end{aligned} \tag{3.9}$$

Combining (3.7) and (3.9) yields

$$\begin{aligned}\Delta_{\mathcal{M}}(vw\phi) &= w\phi\Delta_{\mathcal{M}} v + v\Delta_{\mathcal{M}}(w\phi) + \frac{2}{v}\nabla^{\mathcal{M}}(vw\phi)\cdot\nabla^{\mathcal{M}} v - \frac{2w\phi}{v}|\nabla^{\mathcal{M}} v|^2 \\ &= vw\phi\left(-v\nabla^{\mathcal{M}} H\cdot\omega + |A|^2\right) \\ &\quad + vw\phi\left(\eta^2(1-v^{-2}) + \eta v^{-1}H + \beta^2|\nabla^{\mathcal{M}}\varrho|^2 + \beta\Delta_{\mathcal{M}}\varrho - 2\eta\beta\nabla^{\mathcal{M}}\varrho\cdot\omega\right) \\ &\quad + \frac{2}{v}\nabla^{\mathcal{M}}(vw\phi)\cdot\nabla^{\mathcal{M}} v.\end{aligned}$$

Whence,

$$\begin{aligned}\Delta_{\mathcal{M}}\log(vw\phi) = &-v\nabla^{\mathcal{M}} H\cdot\omega + |A|^2 \\ &+ \eta^2(1-v^{-2}) + \eta v^{-1}H + \beta^2|\nabla^{\mathcal{M}}\varrho|^2 + \beta\Delta_{\mathcal{M}}\varrho - 2\eta\beta\nabla^{\mathcal{M}}\varrho\cdot\omega \\ &+ 2\nabla^{\mathcal{M}}\log(vw\phi)\cdot\nabla^{\mathcal{M}}\log(v) - |\nabla^{\mathcal{M}}\log(vw\phi)|^2.\end{aligned}$$

On $\partial^{\mathcal{N}}\Omega^\tau$ we have in view of (3.8) and the Neumann condition

$$\frac{\partial\log(vw\phi)}{\partial\gamma_{S'}} = -v^{-2}A_S(\nabla^S\hat{u}^{\varepsilon,\tau}, \nabla^S\hat{u}^{\varepsilon,\tau}) - \beta. \tag{3.10}$$

Note that

$$\nabla^{\mathcal{M}}\varrho = \overline{D}\varrho - \overline{D}\varrho\cdot\nu\nu = (D\varrho, 0) + v^{-1}D\varrho\cdot Du\nu$$

and

$$\begin{aligned}\Delta_{\mathcal{M}}\varrho &= \overline{\Delta}\varrho - (\overline{D}^2\varrho)(\nu,\nu) - H\overline{D}d_{S'}\cdot\nu \\ &= \Delta\varrho - v^{-2}(D^2\varrho)(Du,Du) + v^{-1}HD\varrho\cdot Du \\ &\geq -nC_B + v^{-1}HD\varrho\cdot Du.\end{aligned}$$

Hence,

$$\nabla^{\mathcal{M}}\varrho \cdot \omega = v^{-2} D\varrho \cdot Du$$

and

$$\nabla^{\mathcal{M}}\varrho \cdot \gamma_{S'} = \Big((D\varrho, 0) + v^{-1} D\varrho \cdot Du\nu\Big) \cdot (\gamma, 0) = -1 \quad \text{on } \partial^{\mathcal{N}}\Omega^{\tau}.$$

Furthermore,

$$\begin{aligned}
\nabla^{\mathcal{M}} H &= -\frac{1}{k} H \frac{\nabla^{\mathcal{M}} v}{v} \\
&= -\frac{1}{k} H \frac{\nabla^{\mathcal{M}}(vw\phi)}{vw\phi} + \frac{1}{k} H \frac{\nabla^{\mathcal{M}} w}{w} + \frac{1}{k} H \frac{\nabla^{\mathcal{M}}\phi}{\phi} \\
&= -\frac{1}{k} H \nabla^{\mathcal{M}} \log(vw\phi) - \frac{1}{k} H\eta(\omega - v^{-1}\nu) + \frac{1}{k} H\beta\nabla^{\mathcal{M}}\varrho.
\end{aligned}$$

Set

$$C_d := \sup_{\partial^{\mathcal{D}}\Omega^{\tau}} \left(\sqrt{\varepsilon^2 + |Du^{\varepsilon,\tau}|^2}\, \exp(\beta\varrho)\right).$$

Let us suppose, by contradiction, that

$$\sup_{\Omega^{\tau}} \left(\sqrt{\varepsilon^2 + |Du^{\varepsilon,\tau}|^2}\, \exp(-\frac{\eta}{\varepsilon} u^{\varepsilon,\tau} + \beta\varrho)\right) > \max\{C_d, \exp(4\beta \mathrm{diam}(\Omega))\}.$$

We now choose $\beta := \max\Big\{0, -2\inf\{A_S(\zeta,\zeta) : \zeta \in T_x S, |\zeta| = 1, x \in \partial^{\mathcal{N}}\Omega\}\Big\}$. By the Hopf boundary point lemma and in view of (3.10) the supremum of $vw\phi$ is attained at an interior point $x_0 \in \Omega$. We compute at x_0:

$$\begin{aligned}
\Delta_{\mathcal{M}} \log(vw\phi) &= -v\nabla^{\mathcal{M}} H \cdot \omega + |A|^2 \\
&\quad + \eta^2(1 - v^{-2}) + \eta v^{-1} H + \beta^2 |\nabla^{\mathcal{M}}\varrho|^2 + \beta\Delta_{\mathcal{M}}\varrho - 2\eta\beta\nabla^{\mathcal{M}}\varrho \cdot \omega \\
&= \eta\frac{1}{k} vH + \beta v^{-1} H(1 - \frac{1}{k}) D\varrho \cdot Du + |A|^2 \\
&\quad + \eta^2(1 - v^{-2}) + \eta v^{-1} H(1 - \frac{1}{k}) + \beta^2 |\nabla^{\mathcal{M}}\varrho|^2 \\
&\quad + \beta\Delta\varrho - \beta v^{-2}(D^2\varrho)(Du, Du) - 2\eta\beta v^{-2} D\varrho \cdot Du \\
&> \eta\frac{1}{k} vH - 2\beta H \left|1 - \frac{1}{k}\right| - \beta n C_B - 4\eta\beta \qquad (3.11)\\
&= \frac{\eta}{\varepsilon}\frac{1}{k}(\varepsilon v)^{1-\frac{1}{k}} - (nC_B + 4\eta + 2\frac{k-1}{k}(\varepsilon v)^{-\frac{1}{k}})\beta.
\end{aligned}$$

By our assumption we have that $\varepsilon v \geq 1$. Choosing $\eta = 2\varepsilon k\beta\,(2\frac{k-1}{k} + nC_B)$ we obtain

$$\Delta_{\mathcal{M}} \log(vw\phi) > \beta\left(2\frac{k-1}{k} + nC_B\right)(1 - 8\varepsilon k\beta),$$

which is a contradiction for all $\varepsilon \leq (8k\beta)^{-1}$. We conclude that

$$\sup_{\Omega^{\tau}} \sqrt{\varepsilon^2 + |Du^{\varepsilon,\tau}|^2} \leq \sup_{\partial^{\mathcal{D}}\Omega^{\tau}} \left(1 + \sqrt{\varepsilon^2 + |Du^{\varepsilon,\tau}|^2}\right) \exp(4\beta\,[\mathrm{diam}(\Omega) + (k - 1 + nkC_B)u^{\varepsilon,\tau}]). \qquad (3.12)$$

Remark 3.7. In case $\partial^{\mathcal{N}}\Omega^1$ is convex we may choose $\beta = 0$. In case $G = \mathbb{R}^{n+1} \setminus K$ for a closed convex set K we may choose $C_B = 0$, and if on top $k = 1$ the bound (3.12) does not depend on a sup-bound.

The idea for the choice of our test function $vw\phi$ is the following: We use the function ϕ to push maxima of our test function away from the Neumann boundary by increasing the test function exponentially with increasing distance to the Neumann boundary. This works fine when flowing by mean curvature flow ($k = 1$) outside a convex set. In the general case however, since free boundary minimal surfaces may exist inside Ω, we need to make explicit use of a sup-bound (corresponding to finite existence time and non-existence of free boundary minimal surfaces) via the function w to obtain our estimate.

Summarizing we have the following lemma.

Lemma 3.8. *For any solution $u^{\varepsilon,\tau} \in C^3(\Omega^\tau) \cap C^2(\Omega^\tau \cup \partial^{\mathcal{N}}\Omega^\tau) \cap C^1(\overline{\Omega^\tau})$ of $(\star\star)_{\varepsilon,\tau}$ with $\varepsilon \in (0, \varepsilon_0]$ and $\tau \in (0, 1]$ we have*

$$\begin{aligned}\sup_{\Omega^\tau}&\sqrt{\varepsilon^2 + |Du^{\varepsilon,\tau}|^2}\\ &\le \sup_{\partial^{\mathcal{D}}\Omega^\tau}\left(1 + \sqrt{\varepsilon^2 + |Du^{\varepsilon,\tau}|^2}\right)\exp(4C_A\left[\operatorname{diam}(\Omega) + (k - 1 + nkC_B)u^{\varepsilon,\tau}\right]),\end{aligned} \tag{3.13}$$

where $C_A := \max\left\{0, -2\inf\{A_S(\zeta,\zeta) : \zeta \in T_xS, |\zeta| = 1, x \in \partial^{\mathcal{N}}\Omega\}\right\}$, $C_B = C_B(\overline{\kappa}_{\partial^{\mathcal{N}}\Omega})$, and $\varepsilon_0 := (8kC_A)^{-1}$.

Lemma 3.8 above reduces the gradient-estimate to a boundary-gradient-estimate.

Before we come to the boundary gradient-estimate we stress the point that since $Dd_\tau \cdot \gamma \ge 0$ on the vertex V_τ we can find a non-negative constant $\Lambda < \infty$, depending only on $\sup_{\partial^{\mathcal{D}}\Omega^\tau}|A^\tau|$ and $\sup_{\partial^{\mathcal{N}}\Omega^\tau}|A_S|$ (which by construction/assumption only depend on $\sup_{\partial^{\mathcal{D}}\Omega}|A|$ and $\sup_{\partial^{\mathcal{N}}\Omega}|A_S|$) such that for all $\tau \in [0, 1]$

$$Dd_\tau \cdot \gamma \ge -\Lambda\, d_\tau \quad \text{on } \partial^{\mathcal{N}}\Omega^\tau. \tag{3.14}$$

We begin by estimating the gradient on the Dirichlet boundary $\partial\Omega^\tau$:

Lemma 3.9. (Dirichlet boundary gradient-estimate) *Any solution $u^{\varepsilon,\tau} \in C^2(\Omega^\tau) \cap C^1(\overline{\Omega^\tau})$ of $(\star\star)_{\varepsilon,\tau}$ with $\varepsilon \in (0, 1]$, $\tau \in [0, \tau_0]$ satisfies the estimate*

$$\sup_{\partial^{\mathcal{D}}\Omega^\tau}|Du^{\varepsilon,\tau}| \le C_2 = C_2(n, k, \delta_0, \sup_{\partial^{\mathcal{D}}\Omega}|A|, \sup_{\partial^{\mathcal{N}}\Omega}|A_S|, C_1), \tag{3.15}$$

where C_1 is the sup-bound from Lemma 3.4.

Proof. For a given $\tau \in (0, \tau_0]$ we construct a barrier at the Dirichlet boundary $\partial^{\mathcal{D}}\Omega^\tau$. Since $u^{\varepsilon,\tau} \ge 0$ we only need to construct an upper barrier. We would like to use the standard barrier $x \mapsto \alpha\, d_\tau(x)$ for some suitable $\alpha > 0$. However, the Neumann condition that needs to be satisfied for this function to be a supersolution might not be satisfied away from the vertex. The idea is to "bend up" the graph of ψ locally near S to obtain the desired barrier.

We make the following ansatz:

$$\Phi_\tau(x) := \psi(d_\tau(x)) \cdot \psi_S(d_S(x)),$$

where $\psi(s) = \alpha s$ for some $\alpha > 0$ to be determined later. Then

$$\frac{\partial \Phi_\tau}{\partial \gamma} = \psi'(d_\tau)\psi_S(0)\, Dd_\tau \cdot \gamma - \psi_S'(0)\psi(d_\tau) \quad \text{on } \partial^{\mathcal{N}}\Omega^\tau,$$

which should be non-negative for Φ_τ to be a supersolution. We make the following ansatz:

$$\psi_S(s) = 1 + \phi(s),$$

where $\phi(s) := \phi_0(\frac{2s}{\delta_2})$ for some $\delta_2 \leq \delta_N$ that shall be chosen appropriately later. Here ϕ_0 denotes a mollified version of the function $s \mapsto \max(0, 1-s)$ as above. Just note that

$$\|\phi\|_{C^0((0,\infty))} \leq 1, \quad \|\phi'\|_{C^0(\mathbb{R})} \leq \frac{2}{\delta_2}, \quad \text{and} \quad \|\phi''\|_{C^0(\mathbb{R})} \leq \frac{4c_0}{\delta_2^2}.$$

Then, using (3.14) we get that

$$\frac{\partial \Phi_\tau}{\partial \gamma} = \alpha\, Dd_\tau \cdot \gamma + \frac{2}{\delta_2}\alpha\, d_\tau \geq \alpha(2{\delta_2}^{-1} - \Lambda)d_\tau,$$

which is non-negative provided $\delta_2 \leq \Lambda^{-1}$. Observe that in $\Omega^\tau \setminus \Omega^\tau_{\delta_D}$ we have $\Delta d_\tau = -H_{M^\tau_{d_\tau}}$, where $H_{M^\tau_r}$ is the mean curvature of the hypersurface $M^\tau_r := \{d_\tau = r\}$. Also notice that in $\{d_S < \delta_N\} \cap \overline{\Omega^\tau}$, $\tau \in [0,1]$, we have $\Delta d_S = -H_{S_{d_S}}$, where H_{S_r} is the mean curvature of the hypersurface $S_r := \{d_S = r\}$. Suppressing the arguments in the notation we calculate

$$\begin{aligned}\frac{1}{2}D(|\Phi_\tau|^2) = {}& \psi_S^2\psi'\psi'' Dd_\tau + \psi\psi'(\psi_S')^2 Dd_\tau + \psi^2\psi_S'\psi_S'' Dd_S + \psi_S(\psi')^2\psi_S' Dd_S \\ &+ \Big(\psi_S(\psi')^2\psi_S' Dd_\tau + \psi\psi_S\psi_S'\psi'' Dd_\tau\Big)\, Dd_\tau \cdot Dd_S \\ &+ \Big(\psi\psi'(\psi_S')^2 Dd_S + \psi\psi_S\psi'\psi_S'' Dd_S\Big)\, Dd_\tau \cdot Dd_S \\ &+ \psi\psi_S\psi'\psi_S' \Big((D^2 d_\tau)(\cdot, Dd_S) + (D^2 d_S)(Dd_\tau, \cdot)\Big).\end{aligned}$$

We note that in $\Omega^\tau \setminus \Omega^\tau_{\delta_D}$ we have that $(D^2 d_\tau)(Dd_S, Dd_S) = -A^{\tau,r}(\nabla^{M^\tau_r} d_S, \nabla^{M^\tau_r} d_S)|_{r=d_\tau}$, where $A^{\tau,r} = \{h^{\tau,r}_{\alpha\beta}\}$ is the second fundamental form of the hypersurface M^τ_r. Also notice that in $\{d_S < \delta_N\} \cap \overline{\Omega^\tau}$, $\tau \in [0,1]$, we have $(D^2 d_S)(Dd_\tau, Dd_\tau) = -A_{S_r}(\nabla^{S_r} d_\tau, \nabla^{S_r} d_\tau)|_{r=d_S}$, where $A_{S_r} = \{h^{S_r}_{\alpha\beta}\}$ is the second fundamental form of the hypersurface S_r. Reasoning similarly to Lemma 3.4 we compute on $\Omega^\tau \setminus \Omega^\tau_{\delta_D}$

$$\begin{aligned}&\operatorname{div}\left(\frac{D\Phi_\tau}{\sqrt{\varepsilon^2 + |D\Phi_\tau|^2}}\right) \\ &= -\frac{\psi'\psi_S}{\sqrt{\varepsilon^2 + |D\Phi_\tau|^2}} H_{M^\tau_{d_\tau}} - \frac{\psi\psi_S'}{\sqrt{\varepsilon^2 + |D\Phi_\tau|^2}} H_{S_{d_S}} \\ &\quad + \frac{\varepsilon^2(\psi''\psi_S + \psi\psi_S'')}{(\varepsilon^2 + |D\Phi_\tau|^2)^{\frac{3}{2}}} + \frac{2\varepsilon^2\psi'\psi_S'}{(\varepsilon^2 + |D\Phi_\tau|^2)^{\frac{3}{2}}}\, Dd_\tau \cdot Dd_S \\ &\quad + \frac{\psi\psi_S^2(\psi')^2\psi_S'' + \psi^2\psi_S(\psi_S')^2\psi''}{(\varepsilon^2 + |D\Phi_\tau|^2)^{\frac{3}{2}}}\Big(1 - (Dd_\tau \cdot Dd_S)^2\Big) \\ &\quad - \frac{2\psi\psi_S(\psi')^2(\psi_S')^2}{(\varepsilon^2 + |D\Phi_\tau|^2)^{\frac{3}{2}}}\Big(1 - (Dd_\tau \cdot Dd_S)^2\Big) \\ &\quad - \frac{\psi\psi_S\psi'\psi_S'}{(\varepsilon^2 + |D\Phi_\tau|^2)^{\frac{3}{2}}}\Big(\psi\psi_S'(D^2 d_\tau)(Dd_S, Dd_S) + \psi_S\psi'(D^2 d_S)(Dd_\tau, Dd_\tau)\Big),\end{aligned}$$

where we used that $(D^2 d_\tau)(Dd_\tau,\cdot)=0$ and $(D^2 d_S)(Dd_S,\cdot)=0$. Inserting our ansatz this is equal to

$$\begin{aligned}
&-\frac{\alpha(1+\phi(d_S))}{\sqrt{\varepsilon^2+|D\Phi_\tau|^2}}H_{M^\tau_{d_\tau}}-\frac{\alpha\phi'(d_S)}{\sqrt{\varepsilon^2+|D\Phi_\tau|^2}}H_{S_{d_S}}d_\tau\\
&+\frac{\varepsilon^2\alpha\phi''(d_S)}{(\varepsilon^2+|D\Phi_\tau|^2)^{\frac{3}{2}}}d_\tau+\frac{2\varepsilon^2\alpha\phi'(d_S)}{(\varepsilon^2+|D\Phi_\tau|^2)^{\frac{3}{2}}}Dd_\tau\cdot Dd_S\\
&+\frac{\alpha^3(1+\phi(d_S))^2\phi''(d_S)}{(\varepsilon^2+|D\Phi_\tau|^2)^{\frac{3}{2}}}\left(1-(Dd_\tau\cdot Dd_S)^2\right)d_\tau\\
&-\frac{2\alpha^3(1+\phi(d_S))\phi'(d_S)^2}{(\varepsilon^2+|D\Phi_\tau|^2)^{\frac{3}{2}}}\left(1-(Dd_\tau\cdot Dd_S)^2\right)d_\tau\\
&+\frac{\alpha^3(1+\phi(d_S))\phi'(d_S)}{(\varepsilon^2+|D\Phi_\tau|^2)^{\frac{3}{2}}}\\
&\quad\cdot\left(\phi'(d_S)d_\tau\, A^{\tau,r}(\nabla^{M^\tau_r}d_S,\nabla^{M^\tau_r}d_S)|_{r=d_\tau}+(1+\phi(d_S))A_{S_r}(\nabla^{S_r}d_\tau,\nabla^{S_r}d_\tau)|_{r=d_S}\right)d_\tau.
\end{aligned}$$

This should be less than $-(\varepsilon^2+|D\Phi_\tau|^2)^{-\frac{1}{2k}}$ in $\Omega^\tau\setminus\Omega^\tau_{\delta_1}$ for some $\delta_1\le\delta_D$ to be determined later. The evolution equation of $H_{M^\tau_r}$ along a geodesic is given by

$$\frac{\partial}{\partial r}H_{M^\tau_r}=|A_{M^\tau_r}|^2\ge 0,$$

which implies that for $0\le r\le\delta_1$

$$H_{M^\tau_r}\ge H_{\widetilde{\partial^{\mathcal{D}}\Omega^\tau}}\ge\frac{\delta_0}{2}.$$

Let $c_d:=2\|A\|_{C^0(\partial^{\mathcal{D}}\Omega)}$, $c_N:=\|A_S\|_{C^0(\partial^{\mathcal{D}}\Omega^1)}$, and $c_D:=\sup_{\tau\in[0,1]}\|A^\tau\|_{C^0(\partial^{\mathcal{D}}\Omega^\tau)}$. Suppose w.l.o.g. that $\sup_{\tau\in[0,1]}\|A^{\tau,r}\|_{C^0(M^\tau_r)}\le 2c_D$ for all $r\le\delta_D$ and $\|A_{S_r}\|_{C^0((\partial^{\mathcal{D}}\Omega^1)_r)}\le 2c_N$ for all $r\le\delta_N$. Hence, a sufficient condition to ensure that Φ_τ is a supersolution is that

$$\begin{aligned}
&-\frac{\delta_0}{2}+\frac{4n}{\delta_2}c_Nd_\tau+\frac{4c_0}{\delta_2^2}d_\tau+\frac{4}{\delta_2}\frac{\varepsilon^2}{\varepsilon^2+|D\Phi_\tau|^2}\\
&+\frac{4\alpha^2}{\varepsilon^2+|D\Phi_\tau|^2}\frac{4c_0}{\delta_2^2}d_\tau+\frac{2\alpha^2}{\varepsilon^2+|D\Phi_\tau|^2}\frac{4}{\delta_2}\left(\frac{2}{\delta_2}c_Dd_\tau+2c_N\right)d_\tau\\
&\le-\frac{1}{\alpha}(\varepsilon^2+|D\Phi_\tau|^2)^{\frac{k-1}{2k}}\quad\text{on }\Omega^\tau\setminus\Omega^\tau_{\delta_1}.
\end{aligned}\tag{3.16}$$

Note that

$$|D\Phi_\tau|^2\ge\alpha^2(1-8\frac{\delta_1}{\delta_2})\ge\frac{1}{2}\alpha^2,$$

provided $\frac{\delta_1}{\delta_2}\le\frac{1}{16}$. Then a sufficient condition again for (3.16) is that

$$\begin{aligned}
&-\frac{\delta_0}{2}+\frac{4n}{\delta_2}c_Nd_\tau+\frac{4c_0}{\delta_2^2}d_\tau+\frac{8}{\delta_2}\frac{1}{\alpha^2}\\
&+\frac{32c_0}{\delta_2^2}d_\tau+\frac{32c_D}{\delta_2^2}d_\tau^2+\frac{32c_N}{\delta_2}d_\tau\\
&\le-\frac{1}{\alpha}(\varepsilon^2+|D\Phi_\tau|^2)^{\frac{k-1}{2k}}\quad\text{on }\Omega^\tau\setminus\Omega^\tau_{\delta_1}.
\end{aligned}\tag{3.17}$$

We estimate the right hand side of (3.17) on $\Omega^\tau \setminus \Omega^\tau_{\delta_1}$:

$$\begin{aligned}
\frac{1}{\alpha}(\varepsilon^2 + |D\Phi_\tau|^2)^{\frac{k-1}{2k}} &\leq \frac{1}{\alpha} 2^{\frac{k-1}{2k}} |D\Phi_\tau|^{\frac{k-1}{k}}, \quad \text{if } |D\Phi_\tau| \geq 1 \\
&\leq \frac{1}{\alpha}\frac{1}{16}\delta_0 |D\Phi_\tau|, \quad \text{if } |D\Phi_\tau| \geq 2^{\frac{k-1}{2}} 16^k \delta_0^{-k} \\
&\leq \frac{1}{16}\delta_0((1+\phi(d_S)) + |\phi'(d_S)| d_\tau) \\
&\leq \frac{1}{8}\delta_0(1+\delta_1\delta_2^{-1}) \\
&\leq \frac{1}{4}\delta_0,
\end{aligned}$$

where in the last line we again used the provisor that $\delta_1 \leq \delta_2/16$. A sufficient condition for (3.17) is then that

$$36n\left(\frac{c_0}{\delta_2^2} + \frac{c_D + c_N}{\delta_2}\right) d_\tau + \frac{8}{\delta_2}\frac{1}{\alpha^2} \leq \frac{1}{4}\delta_0.$$

Since $\delta_D, \delta_N \leq 1$ it is sufficient that

$$\frac{144n}{\delta_2^2}\left\{(c_0 + c_D + c_N)\delta_1 + \frac{1}{\alpha^2}\right\} \leq \delta_0. \tag{3.18}$$

We proceed as follows: first we choose $\delta_2 \leq \min\{\delta_N, \Lambda^{-1}\}$, and then we choose

$$\delta_1 \leq \delta_2 \min\left\{\frac{1}{16}, \delta_0 \frac{\delta_2}{288n}(c_0 + c_D + c_N)^{-1}\right\}.$$

Hence,

$$|D\Phi_\tau| \geq \frac{\alpha}{\sqrt{2}}.$$

So choosing

$$\alpha \geq \max\{12\sqrt{2}\delta_2^{-2}, 2^{\frac{9k}{2}}\delta_0^{-k}\}$$

we see that condition (3.18) is satisfied (for all $\varepsilon, \tau \in [0,1]$). If on top

$$\alpha := \max\{12\sqrt{2}\delta_2^{-2}, 2^{\frac{9k}{2}}\delta_0^{-k}, C_1\delta_1^{-1}\},$$

then $\Phi_\tau \geq u^{\varepsilon,\tau}$ on $\partial(\Omega^\tau \setminus \Omega^\tau_{\delta_1})$, and the gradient-estimate follows from the maximum principle. $\square$

Approximate existence

We now use the a priori estimates that we derived above to prove existence of solutions to the approximating problems $(\star\star)_{\varepsilon,\tau}$ via the method of continuity. These solutions will be constructed in weighted Hölder spaces $H^{(-1-\alpha)}_{2+\alpha}(\Omega^\tau)$ for $\alpha \in (0,1)$ depending on $\varepsilon, \tau > 0$. For a precise definition of these spaces we refer to the appendix. We only remark here that we have the following inclusions:

$$C^{2,\alpha}(\overline{\Omega^\tau}) \subset H^{(-1-\alpha)}_{2+\alpha}(\Omega^\tau) \subset C^{1,\alpha}(\overline{\Omega^\tau}) \cap C^{2,\alpha}_{loc}(\Omega^\tau).$$

To show the existence of solutions to $(\star\star)_{\varepsilon,\tau}$ we study solutions of the following family of equations

$$(\star\star)_{\varepsilon,\tau,s}\begin{cases}\operatorname{div}\left(\frac{Du^{\varepsilon,\tau,s}}{\sqrt{\varepsilon^2+|Du^{\varepsilon,\tau,s}|^2}}\right)=-s(\varepsilon^2+|Du^{\varepsilon,\tau,s}|^2)^{-\frac{1}{2k}} & \text{in } \Omega^\tau\\ u^{\varepsilon,\tau,s}=0 & \text{on } \partial^{\mathcal{D}}\Omega^\tau\\ \frac{\partial u^{\varepsilon,\tau,s}}{\partial\gamma}=0 & \text{on } \partial^{\mathcal{N}}\Omega^\tau.\end{cases}$$

for $s\in[0,1]$, $\varepsilon\in(0,\varepsilon_0]$, and $\tau\in(0,\tau_0]$. In the following we show that for any fixed pair $\varepsilon\in(0,\varepsilon_0]$ and $\tau\in(0,\tau_0]$ we have uniform a priori sup- and gradient-estimates in s. Since $s\leq 1$ it is easy to check that (3.3) and (3.15) also hold for solutions $u\in C^2(\Omega^\tau)\cap C^1(\overline{\Omega^\tau})$ of $(\star,\star)_{\varepsilon,\tau,s}$:

$$\sup_{\Omega^\tau}|u^{\varepsilon,\tau,s}|\leq C_1 \quad\text{and}\quad \sup_{\partial^{\mathcal{D}}\Omega^\tau}|Du^{\varepsilon,\tau,s}|\leq C_2, \tag{3.19}$$

for all $s\in[0,1]$. Here C_1 is the constant from Lemma 3.4 and C_2 is the constant form Lemma 3.9. For the global gradient-estimate we fix a pair $\varepsilon\in(0,\varepsilon_0]$ and $\tau\in(0,\tau_0]$, and consider the hypersurface

$$\mathcal{M}:=\operatorname{graph}\Big(\frac{u^{\varepsilon,\tau,s}}{\varepsilon}\Big)$$

for a solution $u^{\varepsilon,\tau,s}\in C^3(\Omega^\tau)\cap C^2(\Omega^\tau\cup\partial^{\mathcal{N}}\Omega^\tau)\cap C^1(\overline{\Omega^\tau})$ of $(\star\star)_{\varepsilon,\tau,s}$. Equation $(\star\star)_{\varepsilon,\tau,s}$ then implies that the mean curvature H of $\mathcal{M}$ is given by

$$H=\frac{s}{\varepsilon^{\frac{1}{k}}v^{\frac{1}{k}}}.$$

As in the proof of Lemma 3.8, but now keeping the good term $\eta^2(1-v^{-2})$ in (3.11), we obtain that at an interior maximum of $vw\phi$, we have the inequality

$$\begin{aligned}0&>\eta\frac{1}{k}vH+\eta^2(1-v^{-2})-2\beta H\left|1-\frac{1}{k}\right|-C_AnC_B-4\eta C_A\\&\geq\eta^2(1-v^{-2})-2sC_A(\varepsilon v)^{-\frac{1}{k}}\frac{k-1}{k}-nC_AC_B-4\eta C_A\\&\geq\eta^2(1-v^{-2})-(nC_B+4\eta+2\varepsilon^{-1})C_A,\end{aligned}$$

which yields a contradiction for $\eta\geq 10C_A+nC_B+2\varepsilon^{-1}$ and $v>\sqrt{2}$. On the other hand, the same calculation as in (3.10) implies that the maximum of $vw\phi$ cannot be attained on $\partial^{\mathcal{N}}\Omega^\tau$. We infer that

$$\begin{aligned}&\sup_{\Omega^\tau}\sqrt{\varepsilon^2+|Du^{\varepsilon,\tau,s}|^2}\\&\leq\sup_{\partial^{\mathcal{D}}\Omega^\tau}\left(1+\sqrt{\varepsilon^2+|Du^{\varepsilon,\tau,s}|^2}\right)\exp\left(4C_A\operatorname{diam}(\Omega)+\varepsilon^{-1}(10C_A+nC_B+2\varepsilon^{-1})u^{\varepsilon,\tau,s}\right)\end{aligned}\tag{3.20}$$

for all $s\in[0,1]$.

Lemma 3.10. (approximate existence) *Under the assumptions of Lemma 3.4 there exists an $\alpha=\alpha(\varepsilon,\tau)\in(0,1)$ and a unique solution $u^{\varepsilon,\tau}\in H_{2+\alpha}^{(-1-\alpha)}(\Omega^\tau)$ of $(\star\star)_{\varepsilon,\tau}$.*

Proof. We wish to apply the method of continuity. We fixed $\varepsilon \in (0, \varepsilon_0]$ and $\tau \in (0, \tau_0]$, and define the following Banach spaces:

$$X := \left\{ u \in H_{2+\alpha}^{(-1-\alpha)}(\Omega^\tau) : u|_{\partial^{\mathcal{D}}\Omega^\tau} = 0 \ \text{ and } \ \frac{\partial u}{\partial \gamma}|_{\partial^{\mathcal{N}}\Omega^\tau} = 0 \right\}$$

and

$$Y := H_\alpha^{(1-\alpha)}(\Omega^\tau).$$

Here $\alpha \in (0,1)$, to be chosen below, will depend on the contact angle between $\partial^{\mathcal{D}}\Omega^\tau$ and $\partial^{\mathcal{N}}\Omega^\tau$. For $s \in [0,1]$ we define the operator Q^s, given by

$$\begin{aligned} Q^s(u) &:= \operatorname{div}\left(\frac{Du}{\sqrt{\varepsilon^2 + |Du|^2}} \right) + s(\varepsilon^2 + |Du|^2)^{-\frac{1}{2k}} \\ &= \frac{1}{\sqrt{\varepsilon^2 + |Du|^2}} \left(\delta^{ij} - \frac{D^i u D^j u}{\varepsilon^2 + |Du|^2} \right) D_{ij}^2 u + s(\varepsilon^2 + |Du|^2)^{-\frac{1}{2k}} \\ &=^{def} a^{ij}(Du) D_{ij}^2 u + B(Du, s), \end{aligned}$$

and the map $F : X \times \mathbb{R} \to Y$ defined by $F(u,s) := Q^s(u)$. The well-definition of F follows from Lemma 3.22 by the following argument: $u \in X$ implies $Du \in H_{1+\alpha}^{(-\alpha)}(\Omega^\tau)$ and Lemma 3.22 yields $\psi(|Du|) \in H_{1+\alpha}^{(-\alpha)}(\Omega^\tau)$ for any $\psi \in C^\infty(\mathbb{R})$, which implies that $Q^s(u) \in H_\alpha^{(1-\alpha)}(\Omega^\tau)$. It is then clear that $F \in C^1(X \times \mathbb{R}, Y)$. Let

$$I := \{ s \in [0,1] : \exists u \in X \text{ s.t. } F(u,s) = 0 \}.$$

Then clearly $0 \in I$. We show that I is relatively open an closed:

To see that I is relatively open we linearize F about a solution: Let $s \in I$ and $u \in X$ such that $F(u,s) = 0$. The linearization of Q^s about u in the direction $\varphi \in X$ is given by

$$\begin{aligned} DQ^s(u)\varphi &= (1 + |Du|^2)^{-\frac{1}{2}} \left(\delta^{ij} - \frac{D^i u D^j u}{1 + |Du|^2} \right) D_{ij}^2 \varphi \\ &\quad + \frac{\partial}{\partial p_l} \left[(1 + |p|^2)^{-\frac{1}{2}} \left(\delta^{ij} - \frac{p^i p^j}{1 + |p|^2} \right) \right] |_{p = Du} D_{ij}^2 u \, D_l \varphi \\ &\quad + s\varepsilon^{-\frac{1}{2k}} \frac{\partial}{\partial p_l} \left[(1 + |p|^2)^{-\frac{1}{2k}} \right] |_{p = Du} D_l \varphi \\ &=^{def} a^{ij}(Du) D_{ij}^2 \varphi + b^i(Du, D^2 u, s) D_i \varphi. \end{aligned}$$

We want to apply the inverse function theorem to the operator Q^s. We need to check that $DQ^s(u)$ is a homeomorphism.

The maximum principle together with the Hopf boundary point Lemma, imply that the linearization is injective. The linear existence theory and Schauder estimates for mixed boundary value problems [Lie86] and [Lie89], together with the C^0-estimate [GT01, Theorem 3.7] imply that $DQ^s(u)$ is a surjective in case $\alpha \in (0,1)$ is sufficiently small depending on τ. Here we used the fact that $\gamma \cdot \nu_\tau \leq -\tau/2 (< 0)$ on the vertex V^τ. Since F is linear in the second argument the implicit function theorem implies that the set I is open.

To see that I is closed we note that the a priori estimates (3.19) and (3.20) imply uniform $C^1(\overline{\Omega^\tau})$-bounds (independent of s and τ) and thus the operators Q^s are uniformly elliptic. Here, we also used that solutions of $(\star\star)_{\varepsilon,\tau,s}$ are of class C^3 up to the Neumann boundary $\partial^{\mathcal{N}}\Omega^\tau$ by standard local Schauder estimates. Moreover, we have for the coefficients $a^{ij} \in H^{(-\alpha)}_{1+\alpha}(\Omega^\tau)$ and $b^i \in H^{(1-\alpha)}_{\alpha}(\Omega^\tau)$. The local (interior) De Giorgi-Nash-Moser estimates (see [GT01, Theorem 13.6]) then yield

$$[Du^s]_{\beta,\Omega^\tau_\delta} \leq C(n, C_1, C_3, \operatorname{diam}(\Omega), \varepsilon)\delta^{-\beta},$$

where $\beta = \beta(n, C_1, C_3, \varepsilon)$. Hence, $Du^s \in H^{(0)}_{\alpha'}(\Omega^\tau)$ and therefore $a^{ij} \in H^{(0)}_{\alpha'}(\Omega^\tau)$, where $\alpha' = \min\{\alpha, \beta\} \in (0,1)$. Applying Lieberman's Schauder estimates [Lie89, Theorem 4] we obtain uniform (in s) bounds in $H^{(-1-\alpha')}_{2+\alpha'}(\Omega^\tau)$. By the Arzelà-Ascoli theorem for weighted Hölder spaces [Lie85], and using once more [Lie89, Theorem 4], we infer that I is closed. $\square$

Existence

Let $\Omega \subset G$ be a bounded domain that satisfies the condition of Lemma 3.4, such that its relative boundary $\overline{\partial\Omega \cap G}$ is a smooth free boundary surface with respect to $S = \partial G$ of strictly positive mean curvature $H \geq \delta_0 > 0$. Lemma 3.10 ensures the existence of solutions $u^{\varepsilon,\tau}$ of $(\star\star)_{\varepsilon,\tau}$ for all sufficiently small $\varepsilon, \tau > 0$. The a priori estimates (3.3), (3.13), and (3.15) guarantee uniform bounds in $C^1(\overline{\Omega})$, independent of ε and τ. Thus, given any sequence $(\varepsilon_i, \tau_i) \to (0,0)$, we can pass to a subsequence such that

$$u^{\varepsilon_i,\tau_i} \to u$$

in $C^0(\overline{\Omega})$ to a function $u \in C^{0,1}(\overline{\Omega})$. This suggests the following definition (cf. [Sch08, Definition 3.11]):

Definition 3.11. (weak H^k-flow with Neumann boundary condition) *Let $(\varepsilon_i, \tau_i) \to (0,0)$ and corresponding solutions u^{ε_i,τ_i} to $(\star\star)_{\varepsilon_i,\tau_i}$ be given. Assume that $u^{\varepsilon_i,\tau_i} \to u$ in $C^0(\overline{\Omega})$, where $u, u^{\varepsilon_i,\tau_i}$ are uniformly bounded in $C^{0,1}(\overline{\Omega})$. We then call u a weak H^k-flow with Neumann boundary condition generated by the pair (Ω, G).*

Hence, we proved the following theorem.

Theorem 3.12. (existence) *Let $G \subset \mathbb{R}^{n+1}$ be a smooth domain and let $\Omega \subset G$ be such that its relative boundary $\overline{\partial\Omega \cap G}$ is a smooth strictly mean convex free boundary surface with respect to $S = \partial G$. Assume that Ω satisfies the condition of Lemma 3.5. Then there exists a weak H^k-flow with Neumann boundary condition generated by the pair (Ω, G).*

3.3 Further properties of weak solutions

In this section we investigate further properties of the (super) level sets of u. We follow the exposition of [Sch08] and making necessary changes to deal with the Neumann boundary condition. In contrast to the work of Schulze [Sch08] we have to pay slight attention to whether we speak about the level sets of u as a function on Ω or on $\overline{\Omega}$. We use the following notation

$$\{u^{\varepsilon,\tau} > t\} := \{x \in \overline{\Omega^\tau} : u^{\varepsilon,\tau}(x) > t\} \quad \text{and} \quad \{u^{\varepsilon,\tau} = t\} := \{x \in \overline{\Omega^\tau} : u^{\varepsilon,\tau}(x) = t\}.$$

Throughout this section let $\Omega \subset G \subset \mathbb{R}^{n+1}$ be a fixed open and bounded set with smooth Dirichlet boundary $\partial^{\mathcal{D}}\Omega$ such that $H = H_{\partial^{\mathcal{D}}\Omega} > 0$. Let $u \in C^{0,1}(\overline{\Omega})$, $u \geq 0$, be a weak H^k-flow with Neumann boundary condition generated by the pair (Ω, G), i.e. there exist sequences $\varepsilon_i \searrow 0$, $\tau_i \searrow 0$ and solutions $u^{\varepsilon_i,\tau_i} \in C^1(\overline{\Omega^{\tau_i}}) \cap C^\infty(\overline{\Omega^{\tau_i}} \setminus V^{\tau_i})$ of $(\star\star)_{\varepsilon_i,\tau_i}$ that are uniformly bounded in $C^{0,1}(\overline{\Omega}; \mathbb{R}_{\geq 0})$ converging to u in $C^0(\overline{\Omega})$. Then the hypersurfaces $N^i_t \subset \overline{G} \times \mathbb{R}$, defined by

$$N^i_t := N^{\varepsilon_i,\tau_i}_t = \left\{ \left(x, \frac{u^{\varepsilon_i,\tau_i}(x)}{\varepsilon_i} - \frac{t}{\varepsilon_i} \right) : x \in \overline{\Omega} \right\},$$

which are level sets $\{U^{\varepsilon_i,\tau_i} = t\}$ of the function $U^{\varepsilon_i,\tau_i}(x,z) = u^{\varepsilon_i,\tau_i}(x) - \varepsilon_i z$ on $\overline{\Omega^{\tau_i}} \times \mathbb{R}$, are translating solutions of the H^k-flow with (partial) Neumann condition $(\star)$. Equation $(\star\star)_{\varepsilon_i,\tau_i}$ implies that the mean curvature H^i_t of N^i_t is given by

$$H^i_t = \frac{1}{(\varepsilon_i^2 + |Du^{\varepsilon_i,\tau_i}|^2)^{\frac{1}{2k}}}.$$

To fix some further notation define the following subsets of $\Omega^{\tau_i} \times \mathbb{R}$:

$$E^i_t := \{U^{\varepsilon_i,\tau_i} > t\} \cap (\Omega^{\tau_i} \times \mathbb{R}), \quad E'_t := \{U > t\} \cap (\Omega^{\tau_i} \times \mathbb{R}),$$

where $U(x,z) := u(x)$ on $\overline{\Omega^{\tau_i}} \times \mathbb{R}$. The set E'_t can be written as $E'_t = E_t \times \mathbb{R}$, where $E_t := \{u > t\} \cap \Omega \subset \Omega$. A first observation is that the sets E^i_t are minimzing hulls in $\Omega^{\tau_i} \times \mathbb{R}$ relative to $G' := G \times \mathbb{R}$.

Lemma 3.13. *The sets E^i_t are perimeter minimizing from outside relative to G' in $\Omega' := \Omega \times \mathbb{R}$, that is*

$$|\partial^*_{G'} E^i_t \cap K| \leq |\partial^*_{G'} F \cap K|$$

for measurable F with $E^i_t \subset F$, $F \setminus E^i_t \subset K \subset (\Omega^{\tau_i} \cup \partial^{\mathcal{N}}\Omega^{\tau_i}) \times \mathbb{R}$, where K is compact.

Proof. The outward unit normal to the surface N^i_t, which is given by $\nu := -DU^{\varepsilon_i,\tau_i}/|DU^{\varepsilon_i,\tau_i}|$ is a smooth vector field on $(\Omega \cup \partial^{\mathcal{N}}\Omega) \times \mathbb{R}$ with $\operatorname{div}(\nu)(x) = H_{U^{\varepsilon_i,\tau_i}(x)} = |DU^{\varepsilon_i,\tau_i}|^{-\frac{1}{k}}(x) > 0$. Suppose w.l.o.g that K is of class $C^{0,1}$. We may also suppose w.l.o.g. that $|\partial^*_{G'} E'_t \cap \partial K| = 0$. Employing the trace theorem and using ν as a calibration we derive:

$$\begin{aligned}
&|\partial^*_{G'} E^i_t \cap K| - \int_{E^i_t \cap K} |DU^{\varepsilon_i,\tau_i}|^{-\frac{1}{k}} \, d\mathcal{L}^{n+2} \\
&= \int_{G' \cap K} \langle \nu_{E^i_t}, \nu \rangle \, d\mu_{E^i_t} - \int_{E^i_t \cap G' \cap K} \operatorname{div}(\nu) \, d\mathcal{L}^{n+2} \\
&= \int_{\partial(G' \cap K)} \chi^+_{E^i_t} \langle \nu_{G' \cap K}, \nu \rangle \, d\mathcal{H}^{n+1} \\
&= \int_{\partial G' \cap K} \chi^+_{E^i_t} \langle \nu_{G'}, \nu \rangle \, d\mathcal{H}^{n+1} + \int_{\partial K \cap G'} \chi^+_{E^i_t} \langle \nu_K, \nu \rangle \, d\mathcal{H}^{n+1} \\
&= \int_{\partial K \cap G'} \chi^+_F \langle \nu_K, \nu \rangle \, d\mathcal{H}^{n+1} \\
&= \int_{G' \cap K} \langle \nu_F, \nu \rangle \, d\mu_F - \int_{F \cap G' \cap K} \operatorname{div}(\nu) \, d\mathcal{L}^{n+2} - \int_{\partial G' \cap K} \chi^+_F \langle \nu_{G'}, \nu \rangle \, d\mathcal{H}^{n+1} \\
&\leq |\partial^*_{G'} F \cap K| - \int_{F \cap G' \cap K} |DU^{\varepsilon_i,\tau_i}|^{-\frac{1}{k}} \, d\mathcal{L}^{n+2},
\end{aligned}$$

where we used that $\langle \nu_{G'}, \nu \rangle = 0$. This follows from the fact that $\nu_{G'} = (\nu_G, 0)$. Since $E_t^i \subset F$ this yields

$$|\partial_{G'}^* E_t^i \cap K| + \int_{(F \setminus E_t^i) \cap G' \cap K} |DU^{\varepsilon_i, \tau_i}|^{-\frac{1}{k}} \, d\mathcal{L}^{n+2} \leq |\partial_{G'}^* F \cap K|.$$

□

Corollary 3.14. (mass bound) *Let $\Omega'' := \Omega \times [a, b]$. Then*

$$\mathcal{H}^{n+1}(N_t^i \cap \Omega'') \leq (b - a)\mathcal{H}^n(\partial_G \Omega) + 2\mathcal{H}^{n+1}(\Omega) \tag{3.21}$$

for all $t \in \mathbb{R}$.

Proof. Let $\Omega_j \subset\subset \Omega \cup \partial^{\mathcal{N}}\Omega$ be a sequence of relatively open sets, such that $\overline{\partial_G \Omega_j} \to \overline{\partial_G \Omega} = \partial^{\mathcal{D}}\Omega$ in C^1. Then $F_j := \Omega_j \times (a, b) \cup E_t^i$ are valid comparison sets and the above lemma gives

$$\begin{aligned}
\mathcal{H}^{n+1}(N_t^i \cap \Omega'') &= |\partial_{G'}^* E_t^i \cap \Omega''| \\
&\leq \mathcal{H}^{n+1}(\partial_{G'}^* F_j \cap \Omega'') \\
&\leq \mathcal{H}^{n+1}((\partial \Omega_j \times (a, b)) \cap G' \cap \Omega'') \\
&\quad + \mathcal{H}^{n+1}((\Omega_j \times \{a, b\}) \cap G' \cap \Omega'') \\
&\quad + \mathcal{H}^{n+1}((\partial^* E_t^i \setminus (\overline{\Omega}_j \times [a, b])) \cap G' \cap \Omega'') \\
&= (b - a)\mathcal{H}^n(\partial_G \Omega_j) + 2\mathcal{H}^{n+1}(\Omega_j) \\
&\quad + \mathcal{H}^{n+1}(\partial^* E_t^i \cap G' \cap \Omega'' \setminus (\overline{\Omega}_j \times [a, b])) \\
&\to (b - a)\mathcal{H}^n(\partial_G \Omega) + 2\mathcal{H}^{n+1}(\Omega),
\end{aligned}$$

as $j \to \infty$. Here we used the continuity from above for Radon measures and the fact that

$$\partial^* F_j = (\partial \Omega_j \times (a, b)) \setminus E_t^i \cup (\Omega_j \times \{a, b\}) \setminus E_t^i \cup \partial^* E_t^i \setminus (\overline{\Omega}_j \times [a, b]).$$

□

Like in [Sch08] we can use this a priori mass bound and the lower bound on the mean curvature of $\partial^{\mathcal{D}}\Omega$ together with evolution equations to deduce space-time bounds, independent of ε_i.

Lemma 3.15. *Let $I := [a, b] \subset \mathbb{R}$ be a bounded interval. Then*

$$\int_{-\infty}^{\infty} \int_{N_t^i \cap (\Omega \times I)} H_i^{k+1} \, d\mathcal{H}^{n+1} \, dt \leq (b - a)\mathcal{H}^n(\partial_G \Omega) + 2\mathcal{H}^{n+1}(\Omega). \tag{3.22}$$

Proof. Observe that by the coarea formula

$$\begin{aligned}
\int_{\Omega \times I} |DU^{\varepsilon_i, \tau_i}|^{-\frac{1}{k}} \, d\mathcal{L}^{n+2} &= \int_{-\infty}^{\infty} \int_{\{U^{\varepsilon_i, \tau_i} = t\} \cap (\Omega \times I)} |DU^{\varepsilon_i, \tau_i}|^{-\frac{1}{k} - 1} \, d\mathcal{H}^{n+1} \, dt \\
&= \int_{-\infty}^{\infty} \int_{\{U^{\varepsilon_i, \tau_i} = t\} \cap (\Omega \times I)} (\varepsilon_i^2 + |Du^{\varepsilon_i, \tau_i}|^2)^{-\frac{k+1}{2k}} \, d\mathcal{H}^{n+1} \, dt \\
&= \int_{-\infty}^{\infty} \int_{N_t^i \cap (\Omega \times I)} H_i^{k+1} \, d\mathcal{H}^{n+1} \, dt.
\end{aligned}$$

On the other hand, using the comparison sets Ω_j from Corollary 3.14 and proceeding as in Lemma 3.13, we get for all $t \in \mathbb{R}$ that

$$\int_{(\Omega\times I)\setminus E_t^i} |DU^{\varepsilon_i,\tau_i}|^{-\frac{1}{k}}\, d\mathcal{L}^{n+2} \le (b-a)\mathcal{H}^n(\partial_G\Omega) + 2\mathcal{H}^{n+1}(\Omega).$$

Choosing t sufficiently large such that $(\Omega \times I) \cap E_t^i = \emptyset$ proves the claim. $\square$

Lemma 3.16. *The weak H^k-flow with Neumann boundary condition u is non-fattening, i.e. $\mathcal{L}^{n+1}(\{u = t\}) = 0$ for all $t \in [0, T]$, where $T = \sup_\Omega u$.*

Proof. Let $\Omega'' := \Omega \times (0, 1)$. Then for any $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$ we have by the coarea formula together with (3.21), (3.22), and Hölder's inequality that

$$\begin{aligned}
|\mathcal{L}^{n+2}(E_{t_1}^i \cap \Omega'') - \mathcal{L}^{n+2}(E_{t_2}^i \cap \Omega'')| &= \int_{t_1}^{t_2} \int_{N_t^i\cap\Omega''} H_i^k\, d\mathcal{H}^{n+1}\, dt \\
&\le |t_2 - t_1|^{\frac{1}{k+1}} \left(\int_{t_1}^{t_2} \left(\int_{N_t^i\cap\Omega''} H_i^k\, d\mathcal{H}^{n+1} \right)^{\frac{k+1}{k}} dt \right)^{\frac{k}{k+1}} \\
&\le |t_2 - t_1|^{\frac{1}{k+1}} \left(\int_{t_1}^{t_2} \int_{N_t^i\cap\Omega''} H_i^{k+1}\, d\mathcal{H}^{n+1}\, dt \right)^{\frac{k}{k+1}} \mathcal{H}^{n+1}(N_t^i \cap \Omega'')^{\frac{k}{(k+1)^2}} \\
&\le \Big(\mathcal{H}^n(\partial_G\Omega) + 2\mathcal{L}^{n+1}(\Omega)\Big)^{1-\frac{1}{(k+1)^2}} |t_2 - t_1|^{\frac{1}{k+1}}.
\end{aligned}$$

Observe that since $U^{\varepsilon_i,\tau_i} \to U$ in $C^0_{loc}((\Omega \cup \partial^{\mathcal{N}}\Omega) \times \mathbb{R})$ we have that

$$E_t^i \to E_t' \tag{3.23}$$

in $L^1_{loc}((\Omega \cup \partial^{\mathcal{N}}\Omega) \times \mathbb{R})$, provided that $\mathcal{L}^{n+2}(\{U = t\}) = 0$. Thus (3.23) holds for all t up to a countable set $S_0 = \{t \in [0, T] : \mathcal{L}^{n+2}(\{U = t\}) > 0\}$. Taking the limit we have

$$|\mathcal{L}^{n+2}(E_{t_1}' \cap \Omega'') - \mathcal{L}^{n+2}(E_{t_2}' \cap \Omega'')| \le C|t_2 - t_1|^{\frac{1}{k+1}},$$

for all $t_1, t_2 \in \mathbb{R} \setminus S_0$, where $C = \big(\mathcal{H}^n(\partial_G\Omega) + 2\mathcal{L}^{n+1}(\Omega)\big)^{1-\frac{1}{(k+1)^2}}$. Now let $t_0 \in S_0$ and pick sequences $t_j^- \nearrow t_0$, $t_j^+ \searrow t_0$, where $t_j^-, t_j^+ \in \mathbb{R} \setminus S_0$. Since

$$E_{t_j^-}' \to \{U \ge t_0\}, \quad E_{t_j^+}' \to \{U > t_0\}$$

this implies that $\mathcal{L}^{n+2}(\{U = t_0\}) = 0$, and thus $S_0 = \emptyset$. $\square$

We have seen before that the sets E_t^i are minimizing area from outside in $\Omega \times \mathbb{R}$ relative to $G \times \mathbb{R}$. As in the boundaryless case [Sch08] this property passes to limit.

Lemma 3.17. *Let $U \subset G \subset \mathbb{R}^{n+1}$ be open and $E_h \subset U$ a sequence of Caccioppoli sets in U, which converge in $L^1_{loc}(U \cup S)$ to $E \subset U$ such that $|\partial_G^* E_h \cap K| \le C(K)$ for all $K \subset U \cup S$, K compact, independent of h. If all the E_h are minimizing area from outside in U relative to G then so does E.*

Proof. Exactly as in [Sch08, Lemma 5.6]. $\square$

Corollary 3.18. *The sets E'_t are minimizing area from the outside in $\Omega \times \mathbb{R}$ relative to $G \times \mathbb{R}$ for all $t \in (0, T)$. Moreover, the sets E_t are minimizing area from the outside in Ω relative to G for all $t \in (0, T)$.*

Proof. The first statement follows from Lemma 3.17. For the second statement let F be a valid comparison set for E_t in Ω, i.e. $E_t \subset F$, $F \setminus E_t \subset K \subset \Omega \cup \partial^{\mathcal{N}}\Omega$. Define $F' := (F \times (-l, l)) \cup E'_t$ which is a valid comparison set for E'_t. Thus for $K' := K \times [-l+1, l+1]$ we have

$$|\partial^*_{G'} E'_t \cap K'| \leq |\partial^*_{G'} F' \cap K'|,$$

and hence

$$2l|\partial^*_G E_t \cap K| \leq 2l|\partial^*_G F \cap K| + 2\mathcal{L}^{n+1}(F \setminus E_t).$$

The second claim now follows by letting $l \to \infty$. □

Corollary 3.19. *The function $t \mapsto |\partial^*_G E_t|$, $t \in [0, T)$, is monotonically decreasing.*

Proof. Use E_s, $s > t$, as a comparison set in Corollary 3.18. □

Lemma 3.20. *There exists a set $B \subset [0, T]$ of full measure such that*

$$\mathcal{H}^n(\{u = t\} \setminus \partial^*_G E_t) = 0$$

for all $t \in B$.

Proof. First note that since $\mathcal{H}^n(\partial^{\mathcal{N}}\Omega) < \infty$ and

$$\{t > 0 : \mathcal{H}^n(\{u = t\} \setminus \Omega) > 0\} \subset \bigcup_{l=1}^{\infty} \left\{t > 0 : \mathcal{H}^n(\{u = t\} \setminus \Omega) > \frac{1}{l}\right\},$$

we see that the set of times $t \in [0, T]$ such that $\mathcal{H}^n(\{u = t\} \setminus \Omega) > 0$, is countable. Since $u \in C^{0,1}(\overline{\Omega}) \subset BV(\Omega)$ we can compare the coarea formula for BV-functions and Lipschitz-functions to get

$$\int_0^T \mathcal{H}^n(\partial^*_G E_t)\, dt = \int_\Omega |Du|\, dx = \int_0^T \mathcal{H}^n(\{u = t\} \cap \Omega)\, dt = \int_0^T \mathcal{H}^n(\{u = t\})\, dt.$$

Since the integrals are finite, this yields

$$\int_0^T \mathcal{H}^n((\{u = t\}) \setminus \partial^*_G E_t)\, dt,$$

which implies the statement. □

3.4 Appendix

For basic facts about weighted Hölder spaces we refer to [Lie13]. Here we only give their definition and prove one technical lemma that is needed in the proof of Lemma 3.10.

Definition 3.21. (weighted Hölder spaces) *Let $U \subset \mathbb{R}^{n+1}$ be a bounded domain and let $k, l \in \mathbb{N}_0$, $\alpha, \gamma \in (0,1]$ be such that $l + \gamma \leq k + \alpha$.*

$$|u|^{(-l-\gamma)}_{k+\alpha;U} := \sup_{\delta>0}\{\delta^{k-l+\alpha-\gamma}\|u\|_{C^{k,\alpha}(U_\delta)}\},$$

where $U_\delta := \{x \in U : dist(x, \partial U) > \delta\}$. We define the following weighted Hölder spaces*:*

$$H^{(-l-\gamma)}_{k+\alpha}(U) := \left\{u \in C^{k,\alpha}_{loc}(U) : |u|^{(-l-\gamma)}_{k+\alpha;U} < \infty\right\}.$$

It is not hard to see that $C^{k,\alpha}(\overline{U}) \subset H^{(-l-\gamma)}_{k+\alpha}(U) \subset C^{l,\gamma}(\overline{U}) \cap C^{k,\alpha}_{loc}(U)$.

Lemma 3.22. *Suppose $0 < \alpha \leq \beta < 1$, and let $\psi \in C^{1,\frac{\alpha}{\beta}}_{loc}(\mathbb{R})$ and $u \in H^{(-\beta)}_{1+\alpha}(U)$. Then $\psi \circ u \in H^{(-\beta)}_{1+\alpha}(U)$.*

Proof. Firstly, note that since $u \in C^0(\overline{U})$ its image $u(\overline{U}) =: K$ is a compact set. Let $\delta \in (0, \mathrm{diam}(U))$, and set $d := \mathrm{diam}(U)$. We have:

$$\delta^{1+\alpha-\beta}|\psi \circ u|_{0;U_\delta} \leq \|\psi\|_{C^0(K)} d^{1+\alpha-\beta} \quad \text{and}$$
$$\delta^{1+\alpha-\beta}|D(\psi \circ u)|_{0;U_\delta} \leq \|\psi'\|_{C^0(K)} |u|^{(-\beta)}_{1+\alpha;U}.$$

Now let $x, y \in U_\delta$, then

$$|D(\psi \circ u)(x) - D(\psi \circ u)(y)| \leq \|\psi'\|_{C^0(K)}|Du(x) - Du(y)| + [\psi']_{\frac{\alpha}{\beta},K}|u(x) - u(y)|^{\frac{\alpha}{\beta}}|Du|_{0;U_\delta}.$$

Hence,

$$[D(\psi \circ u)]_{\alpha;U_\delta} \leq \|\psi'\|_{C^0(K)}[Du]_{\alpha;U_\delta} + [\psi']_{\frac{\alpha}{\beta},K}[u]^{\frac{\alpha}{\beta}}_{\beta;U_\delta}|Du|_{0;U_\delta}.$$

Noticing that $C^{0,\beta}(\overline{U}) = H^{(-\beta)}_{\beta}(U) \subset H^{(-\beta)}_{1+\alpha}(U)$, we conclude that

$$|\psi \circ u|^{(-\beta)}_{1+\alpha;U} \leq \|\psi\|_{C^0(K)} d^{1+\alpha-\beta} + (2 + [u]^{\frac{\alpha}{\beta}}_{\beta;U})\|\psi\|_{C^{1,\frac{\alpha}{\beta}}(K)}|u|^{(-\beta)}_{1+\alpha;U} < \infty.$$

$\square$

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German abstract

In der vorliegenden Arbeit betrachten wir drei freie Randwertprobleme für (Hyper-)Flächen welche durch die mittlere Krümmung der (Hyper-)Fläche beschrieben werden:

1. *Eine Monotonieformel für Flächen mit freiem Rand bezüglich der Einheitskugel*

   Wir beweisen eine Monotonieidentität für kompakte Flächen mit freien Rändern in dem Rand der Einheitskugel des $\mathbb{R}^n$ welche quadratisch integrierbare mittlere Krümmung besitzen. Als eine Konsequenz erhalten wir eine Ungleichung vom Li-Yau Typ für diesen Fall, wodurch wir Resultate von Oliveira und Soret [RV95, Proposition 3], und Fraser und Schoen [FS11, Theorem 5.4] verallgemeinern. Im Anschluss leiten wir einige scharfe geometrische Ungleichungen für kompakte Flächen mit freien Rändern in beliebigen orientierbaren Stützflächen der Klasse C^2 her. Außerdem erhalten wir eine scharfe untere Schranke an die L^1-Tangentialpunktenergie für geschlossene Kurven im $\mathbb{R}^3$, wodurch wir eine Frage von Strzelecki, Szumańska, und von der Mosel [SSvdM13] beantworten.

2. *Relative isoperimetrische Eigenschaften asymptotisch flacher Stützflächen*

   Wir definieren einen Massebegriff von asymptotisch flachen Hyperflächen S des euklidischen Raums und beweisen ein positive-Masse-Theorem in allen Dimensionen. Im Anschluss leiten wir eine freie-Randwert-Version einer Obstruktion her, welche von Schoen und Yau in ihrem Beweis des positive-Masse-Theorems [SY79b] entdeckt, und durch Eichmair und Metzger [EM12], und sehr kürzlich von Carlotto [Car14] verfeinert wurde: positive mittlere Krümmung von $S \subset \mathbb{R}^3$ ist nicht kompatibel mit der Existenz (gewisser) stabiler Minimalflächen mit freiem Rand. Wir benutzen dies dann um zu zeigen, dass für gegebenes Kompaktum K des $\mathbb{R}^3$, alle stabilen Flächen mit konstanter mittlerer Krümmung und freiem Rand bezüglich S mit hinreichend großer Randkurvenlänge K entgehen, wodurch wir eine freie-Randwert-Version des Hauptresultats in [EM12] erhalten. Schließlich, inspiriert durch Ideen von Eichmair und Metzger [EM13b], beweisen wir die Existenz von beliebig großen isoperimetrischen Mengen relativ zu S.

3. *Schwache Lösungen vom nichtlinearen Mittleren Krümmungsfluss mit Neumann Randwerten*

   Wir schlagen einen neuen Flussansatz vor um relative isoperimetrische Ungleichungen zu erhalten. Als ersten Schritt dieses Programms entwickeln wir ein schwache Niveauflächenformulierung für den Fluss entlang der mittleren Krümmung und entlang positiver Potenzen der mittleren Krümmung mit Neumann Randwerten. Wir beweisen die Existenz von schwachen Lösungen unter natürlichen Bedingungen an die Stützfläche und leiten einige Eigenschaften der evolvierenden Flächen her. Der Fall für Flächen ohne Rand wurde von Schulze [Sch08] behandelt.

Eigenständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Hilfsmittel verwendet habe. Die Arbeit wird zum ersten Mal in einem Promotionsverfahren eingereicht.

Alexander Volkmann