CHAPTER 2
BOUNDARY VALUE PROBLEMS FOR SECOND ORDER PARTIAL DIFFEENTIAL EQUATIONS IN THE PLANE WITH FUCHS OPERATOR IN THE MAIN PART

In this chapter continuous solutions in an unbounded angular domain of the Dirichlet and the Neumann problem, an initial problem with given growth at infinity for some classes of partial differential equations of second order with Fuchs operator in the main part are constructed. Thus the varieties of continuous solutions constructed in the first section are used. These problems are regularly investigated here for the first time.

2.1 Dirichlet and Neumann problems with given growth at infinity for a model second order partial differential equation in the plane with Fuchs operator in the main part and specified right hand side

Dirichlet problem. Let us consider the Dirichlet problem for equation (1.1).

Problem D_1. *Let* $\beta \neq \alpha + \gamma$. *It is required to find the solution of the equation (0.1) from the class* $W_p^{\,2}(G)$, *where* $1 < p < \dfrac{2}{2-\lambda}$, *if* $\lambda < 2$ *and* $p > 1$, *if* $\lambda \geq 2$, *satisfying the conditions*

$$|V(r,\varphi)| = \mathrm{O}(r^{\lambda}), r \to \infty, \tag{2.1}$$

$$V(r,0) = b_1 r^{\lambda},\ V(r,\varphi_1) = b_2 r^{\lambda}, \tag{2.2}$$

where b_1, b_2 *are given complex numbers,* $\lambda > 0$ *is a real number.*

Solving the problem. For solving problem D_1 we use the formula (1.17). Then it automatically satisfies (2.1). From the forms of the functions $(BF)(\varphi)$, $P_{\nu,1}(\varphi)$ $P_{\nu,2}(\varphi)$, $Q_{\nu,1}(\varphi)$, $Q_{\nu,2}(\varphi)$ follow

$$(BF)(0) = 0,\ P_{\nu,1}(0) = 0,\ Q_{\nu,1}(0) = 0,$$

$$P_{\nu,2}(0)=\begin{cases}1\,, if\ \nu\neq 0\,,\\ 0\,, if\ \nu=0\,,\end{cases}\qquad Q_{\nu,2}(\varphi)=\begin{cases}1\,, if\ \nu\geq 0\,,\\ 0\,, if\ \nu<0\,.\end{cases}\tag{2.3}$$

From (1.17) in view of (2.3) we have

$$V(r,0)=(c_1\delta_{\nu,1}+c_2\delta_{\nu,2})r^{\lambda}\,,$$

where $\delta_{\nu,1}=\begin{cases}1\,, if\ \nu\neq 0\,,\\ 0\,, if\ \nu=0\,,\end{cases}\quad \delta_{\nu,2}=\begin{cases}1\,, if\ \nu\geq 0\,,\\ 0\,, if\ \nu<0\,.\end{cases}$

Therefore from the boundary conditions (2.2) depending on the sign of $\nu=\dfrac{(4\alpha\gamma-\beta^2)\lambda^2+2(\beta(\alpha+\gamma)-4\alpha\gamma)\lambda-(\alpha-\gamma)^2}{q^2}$ follow some algebraic system of equations for c_1 and c_2:

1) Let $\nu=0$. Then

$c_2=b_1,$

$P_{\nu,2}(\varphi_1)c_1+P_{\nu,1}(\varphi_1)\overline{c}_1=\Delta_1(\varphi_1),$

where $\Delta_1(\varphi_1)=b_2e^{-a\varphi_1}-(BF)(\varphi_1)-b_1Q_{\nu,2}(\varphi_1)-\overline{b}_1Q_{\nu,1}(\varphi_1)$.

2) Let $\nu<0$. Then

$c_1=b_1,$

$Q_{\nu,2}(\varphi_1)c_2+Q_{\nu,1}(\varphi_1)\overline{c}_2=\Delta_2(\varphi_1),$

where $\Delta_2(\varphi_1)=b_2e^{-a\varphi_1}-(BF)(\varphi_1)-b_1P_{\nu,2}(\varphi_1)-\overline{b}_1P_{\nu,1}(\varphi_1)$.

3) Let $\nu>0$. Then

$c_1+c_2=b_1,$

$(Q_{\nu,2}(\varphi_1)-P_{\nu,2}(\varphi_1))c_2+(Q_{\nu,1}(\varphi_1)-P_{\nu,1}(\varphi_1))\overline{c}_2=\Delta_2(\varphi_1).$

Each of these systems has unique solutions in case

$$\begin{aligned}&1)\ \nu=0,\ |P_{\nu,2}(\varphi_1)|^2\neq|P_{\nu,1}(\varphi_1)|^2,\\ &2)\ \nu<0,\ |Q_{\nu,2}(\varphi_1)|^2\neq|Q_{\nu,1}(\varphi_1)|^2,\\ &3)\ \nu>0,\ |Q_{\nu,2}(\varphi_1)-P_{\nu,2}(\varphi_1)|^2\neq|Q_{\nu,1}(\varphi_1)-P_{\nu,1}(\varphi_1)|^2.\end{aligned}\tag{2.4}$$

From these conditions the solution are found by the formulas:

1) If $\nu = 0$, $\left|P_{\nu,2}(\varphi_1)\right|^2 \neq \left|P_{\nu,1}(\varphi_1)\right|^2$, then

$$c_2 = b_1,\ c_1 = \frac{\Delta_1(\varphi_1)\overline{P_{\nu,2}(\varphi_1)} - \overline{\Delta_1(\varphi_1)}P_{\nu,1}(\varphi_1)}{\left|P_{\nu,2}(\varphi_1)\right|^2 - \left|P_{\nu,1}(\varphi_1)\right|^2}. \tag{2.5}$$

2) If $\nu < 0$, $\left|Q_{\nu,2}(\varphi_1)\right|^2 \neq \left|Q_{\nu,1}(\varphi_1)\right|^2$, then

$$c_1 = b_1,\ c_2 = \frac{\Delta_2(\varphi_1)\overline{Q_{\nu,2}(\varphi_1)} - \overline{\Delta_2(\varphi_1)}Q_{\nu,1}(\varphi_1)}{\left|Q_{\nu,2}(\varphi_1)\right|^2 - \left|Q_{\nu,1}(\varphi_1)\right|^2}. \tag{2.6}$$

3) If $\nu > 0$, $\left|Q_{\nu,2}(\varphi_1) - P_{\nu,2}(\varphi_1)\right|^2 \neq \left|Q_{\nu,1}(\varphi_1) - P_{\nu,1}(\varphi_1)\right|^2$, then

$$c_2 = \frac{\Delta_2(\varphi_1)(\overline{Q_{\nu,2}(\varphi_1)} - \overline{P_{\nu,2}(\varphi_1)}) - \overline{\Delta_2(\varphi_1)}(Q_{\nu,1}(\varphi_1) - P_{\nu,1}(\varphi_1))}{\left|Q_{\nu,2}(\varphi_1) - P_{\nu,2}(\varphi_1)\right|^2 - \left|Q_{\nu,1}(\varphi_1) - P_{\nu,1}(\varphi_1)\right|^2}, \tag{2.7}$$

$c_1 = b_1 - c_2$.

Hence, the following result holds.

Theorem 2.1. *When conditions (2.4) are fulfilled, problem D_1 has a unique solution. In this case the unique solution is then given by the formulas (1.17), (2.5)-(2.7).*

In case

$$\begin{aligned} &1)\ \nu = 0,\ \left|P_{\nu,2}(\varphi_1)\right|^2 = \left|P_{\nu,1}(\varphi_1)\right|^2, \\ &2)\ \nu < 0,\ \left|Q_{\nu,2}(\varphi_1)\right|^2 = \left|Q_{\nu,1}(\varphi_1)\right|^2, \\ &3)\ \nu > 0,\ \left|Q_{\nu,2}(\varphi_1) - P_{\nu,2}(\varphi_1)\right|^2 = \left|Q_{\nu,1}(\varphi_1) - P_{\nu,1}(\varphi_1)\right|^2 \end{aligned} \tag{2.8}$$

for the solvability of the indicated algebraic systems is necessary and sufficient that the following conditions are satisfied.

$$\begin{aligned} &1)\ \begin{array}{l} \mathrm{Re}(\overline{\Delta_1(\varphi_1)}(P_{\nu,2}(\varphi_1) - P_{\nu,1}(\varphi_1))) = 0, \\ \mathrm{Im}(\overline{\Delta_1(\varphi_1)}(P_{\nu,2}(\varphi_1) + P_{\nu,1}(\varphi_1))) = 0,\ \ for\ \ \nu = 0, \end{array} \\ &2)\ \begin{array}{l} \mathrm{Re}(\overline{\Delta_2(\varphi_1)}(Q_{\nu,2}(\varphi_1) - Q_{\nu,1}(\varphi_1))) = 0, \\ \mathrm{Im}(\overline{\Delta_2(\varphi_1)}(Q_{\nu,2}(\varphi_1) + Q_{\nu,1}(\varphi_1))) = 0,\ \ for\ \ \nu < 0, \end{array} \end{aligned} \tag{2.9}$$

$$3)\quad \begin{array}{l} \mathrm{Re}(\overline{\Delta_2(\varphi_1)}(Q_{\nu,2}(\varphi_1)-P_{\nu,2}(\varphi_1)-Q_{\nu,1}(\varphi_1)+P_{\nu,1}(\varphi_1)))=0, \\ \mathrm{Im}(\overline{\Delta_2(\varphi_1)}(Q_{\nu,2}(\varphi_1)-P_{\nu,2}(\varphi_1)+Q_{\nu,1}(\varphi_1)-P_{\nu,1}(\varphi_1)))=0, \;\; for \;\; \nu>0. \end{array}$$

When this conditions hold the solution of the algebraic system are given by the formulas

1) If $\nu=0$, $\left|P_{\nu,2}(\varphi_1)\right|^2=\left|P_{\nu,1}(\varphi_1)\right|^2$, then

$c_2=b_1$,

$$c_1=\begin{cases} \dfrac{\mathrm{Re}\,\Delta_1(\varphi_1)+i\alpha(\overline{P_{\nu,2}(\varphi_1)}+P_{\nu,1}(\varphi_1))}{\mathrm{Re}(P_{\nu,2}(\varphi_1)+P_{\nu,1}(\varphi_1))}, if \; \mathrm{Re}(P_{\nu,2}(\varphi_1)+P_{\nu,1}(\varphi_1))\neq 0, \\ i\dfrac{\mathrm{Re}\,\Delta_1(\varphi_1)-\alpha(\overline{P_{\nu,2}(\varphi_1)}+P_{\nu,1}(\varphi_1))}{\mathrm{Im}(P_{\nu,1}(\varphi_1)-P_{\nu,2}(\varphi_1))}, if \; \mathrm{Im}(P_{\nu,1}(\varphi_1)-P_{\nu,2}(\varphi_1))\neq 0, \\ c_3, if \; \mathrm{Re}(P_{\nu,2}(\varphi_1)+P_{\nu,1}(\varphi_1))=0, \mathrm{Im}(P_{\nu,1}(\varphi_1)-P_{\nu,2}(\varphi_1))=0. \end{cases} \tag{2.10}$$

If $\nu<0$, $\left|Q_{\nu,2}(\varphi_1)\right|^2=\left|Q_{\nu,1}(\varphi_1)\right|^2$, then

$c_1=b_1$,

$$c_2=\begin{cases} \dfrac{\mathrm{Re}\,\Delta_2(\varphi_1)+i\alpha(\overline{Q_{\nu,2}(\varphi_1)}+Q_{\nu,1}(\varphi_1))}{\mathrm{Re}(Q_{\nu,2}(\varphi_1)+Q_{\nu,1}(\varphi_1))}, if \; \mathrm{Re}(Q_{\nu,2}(\varphi_1)+Q_{\nu,1}(\varphi_1))\neq 0, \\ i\dfrac{\mathrm{Re}\,\Delta_2(\varphi_1)-\alpha(\overline{Q_{\nu,2}(\varphi_1)}+Q_{\nu,1}(\varphi_1))}{\mathrm{Im}(Q_{\nu,1}(\varphi_1)-Q_{\nu,2}(\varphi_1))}, if \; \mathrm{Im}(Q_{\nu,1}(\varphi_1)-Q_{\nu,2}(\varphi_1))\neq 0, \\ c_3, if \; \mathrm{Re}(Q_{\nu,2}(\varphi_1)+Q_{\nu,1}(\varphi_1))=0, \mathrm{Im}(Q_{\nu,1}(\varphi_1)-Q_{\nu,2}(\varphi_1))=0. \end{cases} \tag{2.11}$$

3) If $\nu>0$, $\left|Q_{\nu,2}(\varphi_1)-P_{\nu,2}(\varphi_1)\right|^2=\left|Q_{\nu,1}(\varphi_1)-P_{\nu,1}(\varphi_1)\right|^2$, then

$c_1=b_1-c_2$,

$$c_2 = \begin{cases} \dfrac{\operatorname{Re}\Delta_2(\varphi_1) + i\alpha(\overline{Q_{\nu,2}(\varphi_1)} - \overline{P_{\nu,2}(\varphi_1)} + Q_{\nu,1}(\varphi_1) + P_{\nu,1}(\varphi_1))}{\operatorname{Re}(Q_{\nu,2}(\varphi_1) - P_{\nu,2}(\varphi_1) + Q_{\nu,1}(\varphi_1) - P_{\nu,1}(\varphi_1))}, \\ \text{if } \operatorname{Re}(Q_{\nu,2}(\varphi_1) - P_{\nu,2}(\varphi_1) + Q_{\nu,1}(\varphi_1) - P_{\nu,1}(\varphi_1)) \neq 0, \\ i\dfrac{\operatorname{Re}\Delta_2(\varphi_1) - \alpha(\overline{Q_{\nu,2}(\varphi_1)} - \overline{P_{\nu,2}(\varphi_1)} + Q_{\nu,1}(\varphi_1) - P_{\nu,1}(\varphi_1))}{\operatorname{Im}(-Q_{\nu,2}(\varphi_1) + P_{\nu,2}(\varphi_1) + Q_{\nu,1}(\varphi_1) - P_{\nu,1}(\varphi_1))}, \\ \text{if } \operatorname{Im}(-Q_{\nu,2}(\varphi_1) + P_{\nu,2}(\varphi_1) + Q_{\nu,1}(\varphi_1) - P_{\nu,1}(\varphi_1)) \neq 0, \\ c_3, \text{if } \operatorname{Re}(Q_{\nu,2}(\varphi_1) - P_{\nu,2}(\varphi_1) + Q_{\nu,1}(\varphi_1) - P_{\nu,1}(\varphi_1)) = 0, \\ \qquad \operatorname{Im}(-Q_{\nu,2}(\varphi_1) + P_{\nu,2}(\varphi_1) + Q_{\nu,1}(\varphi_1) - P_{\nu,1}(\varphi_1)) = 0, \end{cases} \tag{2.12}$$

where α is any real, c_3 is any complex number.

Hence, the following results holds

Theorem 2.2. *In cases (2.8) for the solvability of problem* D_1 *the conditions (2.9) are necessary and sufficient. Under these conditions the problem has infinitely many solutions. These solutions are given by the formulas (1.17), (2.10)-(2.12).*

Problem D_2. *Let* $\lambda \neq 1, \alpha \neq \gamma$ *and* $\beta = \alpha + \gamma$. *It is required to find the solution of equation (1.1) from the class (1.3), satisfying the conditions*

$$|V(r,\varphi)| = O(r^{\lambda}), r \to \infty, \tag{2.13}$$

$$V(r,\varphi_1) = b_1 r^{\lambda}, \tag{2.14}$$

where b_1 *is a given complex number,* $\lambda > 0$ *is a given real number.*

Solution of the problem. For the solvability of problem D_2 formula (1.33) is used. Then (2.13) automatically holds. Substituting (1.33) in the condition (2.14), we have

$$cP_2(\varphi_1) + \bar{c}P_1(\varphi_1) = \Delta(\varphi_1), \tag{2.15}$$

where $\Delta(\varphi_1) = b_1 - (BF)(\varphi_1)$.

This equation for c has a unique solution when the condition

$$\left|P_2(\varphi_1)\right|^2 \neq \left|P_1(\varphi_1)\right|^2 \tag{2.16}$$

is satisfied. The solution is given by the formula

$$c = \frac{\overline{P_2(\varphi_1)}\Delta(\varphi_1) - P_1(\varphi_1)\overline{\Delta(\varphi_1)}}{\left|P_2(\varphi_1)\right|^2 - \left|P_1(\varphi_1)\right|^2}. \tag{2.17}$$

Hence, the following result holds.

Theorem 2.3. *When condition (2.16) is satisfied problem D_2 has a unique solution. The unique solution is given by formulas (1.33) and (2.17).*

When

$$\left|P_2(\varphi_1)\right|^2 = \left|P_1(\varphi_1)\right|^2 \tag{2.18}$$

for the solvability of equation (2.15) the following conditions are necessary and sufficient

$$\mathrm{Re}(\overline{\Delta(\varphi_1)}(P_2(\varphi_1) - P_1(\varphi_1))) = 0,\ \mathrm{Im}(\overline{\Delta(\varphi_1)}(P_2(\varphi_1) + P_1(\varphi_1))) = 0. \tag{2.19}$$

When these conditions are fulfilled the solutions of equation (2.15) is given by the formula

$$c = \begin{cases} \dfrac{\mathrm{Re}\,\Delta(\varphi_1) + i\alpha(\overline{P_2(\varphi_1)} + P_1(\varphi_1))}{\mathrm{Re}(P_2(\varphi_1) + P_1(\varphi_1))}, \textit{ if } \mathrm{Re}(P_2(\varphi_1) + P_1(\varphi_1)) \neq 0, \\ i\dfrac{\mathrm{Re}\,\Delta(\varphi_1) - \alpha(\overline{P_2(\varphi_1)} + P_1(\varphi_1))}{\mathrm{Im}(-P_2(\varphi_1) + P_1(\varphi_1))}, \textit{ if } \mathrm{Im}(-P_2(\varphi_1) + P_1(\varphi_1)) \neq 0, \\ c_1, \textit{ if } \mathrm{Re}(P_2(\varphi_1) + P_1(\varphi_1)) = 0,\ \mathrm{Im}(-P_2(\varphi_1) + P_1(\varphi_1)) = 0, \end{cases} \tag{2.20}$$

where α is any real, c_1 is any complex number.

Hence, the following result holds.

Theorem 2.4. *When the conditions (2.18) hold, for the solvability of problem D_2 the equality (2.19) is necessary and sufficient. Under these conditions the problem has infinitely many solutions. These solutions are given by the formulas (1.33) and (2.20).*

Neumann problem

Problem N_1. *Let* $\beta \neq \alpha + \gamma$. *It is required to find the solution of equation (1.1) from the class (1.3), satisfying the conditions*

$$|V(r,\varphi)| = \mathrm{O}(r^{\lambda}),\, r \to \infty, \tag{2.21}$$

$$\left.\frac{\partial V(r,\varphi)}{\partial \varphi}\right|_{\varphi=0} = b_1 r^{\lambda}\,,\ \left.\frac{\partial V(r,\varphi)}{\partial \varphi}\right|_{\varphi=\varphi_1} = b_2 r^{\lambda}, \tag{2.22}$$

where b_1, b_2 *are given complex numbers,* $\lambda > 0$ *is a given real number.*

Solution of the problem. For the solvability of problem N_1 formulas (1.17) are used. Then (2.21) automatically holds. The functions

$$(BF)(\varphi) = \int_0^{\varphi} f(\varphi,\gamma)d\gamma + \int_0^{\varphi} b(\varphi,\gamma)\overline{(BF)(\gamma)}d\gamma, \quad P_{\nu,2}(\varphi) = I_{\nu,0}(\varphi) + \int_0^{\varphi} b(\varphi,\gamma)\overline{P_{\nu,1}(\gamma)}d\gamma,$$

$$P_{\nu,1}(\varphi) = \int_0^{\varphi} b(\varphi,\gamma)\overline{P_{\nu,2}(\gamma)}d\gamma\,,\ Q_{\nu,2}(\varphi) = J_{\nu,0}(\varphi) + \int_0^{\varphi} b(\varphi,\gamma)\overline{Q_{\nu,1}(\gamma)}d\gamma\,,$$

$$Q_{\nu,1}(\varphi) = \int_0^{\varphi} b(\varphi,\gamma)\overline{Q_{\nu,2}(\gamma)}d\gamma$$

have the differential properties

$$\frac{\partial (BF)(\varphi)}{\partial \varphi} = \begin{cases} \int_0^{\varphi} f_1(\gamma)ch(\sqrt{\nu}(\varphi-\gamma))d\gamma + \int_0^{\varphi} b_1(\gamma)ch(\sqrt{\nu}(\varphi-\gamma))\overline{(BF)(\gamma)}d\gamma, \, if\ \nu > 0, \\ \int_0^{\varphi} f_1(\gamma)\cos(\sqrt{-\nu}(\varphi-\gamma))d\gamma + \\ + \int_0^{\varphi} b_1(\gamma)\cos(\sqrt{-\nu}(\varphi-\gamma))\overline{(BF)(\gamma)}d\gamma, \, if\ \nu < 0, \\ \int_0^{\varphi} f_1(\gamma)d\gamma + \int_0^{\varphi} b_1(\gamma)\overline{(BF)(\gamma)}d\gamma, \, if\ \nu = 0, \end{cases}$$

$$\frac{\partial P_{\nu,2}(\varphi)}{\partial\varphi}=\begin{cases}\sqrt{\nu}\exp(\sqrt{\nu}\varphi)+\int\limits_0^{\varphi}b_1(\gamma)ch(\sqrt{\nu}(\varphi-\gamma))\overline{P_{\nu,1}(\gamma)}d\gamma, \, if \ \nu>0,\\ -\sqrt{-\nu}\sin(\sqrt{-\nu}\varphi)+\int\limits_0^{\varphi}b_1(\gamma)\cos(\sqrt{-\nu}(\varphi-\gamma))\overline{P_{\nu,1}(\gamma)}d\gamma, \, if \nu<0,\\ 1+\int\limits_0^{\varphi}b_1(\gamma)\overline{P_{\nu,1}(\gamma)}d\gamma, \, if \ \nu=0,\end{cases}$$

$$\frac{\partial P_{\nu,1}(\varphi)}{\partial\varphi}=\begin{cases}\int\limits_0^{\varphi}b_1(\gamma)ch(\sqrt{\nu}(\varphi-\gamma))\overline{P_{\nu,2}(\gamma)}d\gamma, \, if \ \nu>0,\\ \int\limits_0^{\varphi}b_1(\gamma)\cos(\sqrt{-\nu}(\varphi-\gamma))\overline{P_{k,2}(\gamma)}d\gamma, \, if \ \nu<0,\\ \int\limits_0^{\varphi}b_1(\gamma)\overline{P_{\nu,2}(\gamma)}d\gamma, \, if \ \nu=0,\end{cases}$$

$$\frac{\partial Q_{\nu,2}(\varphi)}{\partial\varphi}=\begin{cases}-\sqrt{\nu}\exp(-\sqrt{\nu}\varphi)+\int\limits_0^{\varphi}b_1(\gamma)ch(\sqrt{\nu}(\varphi-\gamma))\overline{Q_{\nu,1}(\gamma)}d\gamma, \, if \ \nu>0,\\ \sqrt{-\nu}\cos(\sqrt{-\nu}\varphi)+\int\limits_0^{\varphi}b_1(\gamma)\cos(\sqrt{-\nu}(\varphi-\gamma))\overline{Q_{\nu,1}(\gamma)}d\gamma, \, if \ \nu<0,\\ \int\limits_0^{\varphi}b_1(\gamma)\overline{Q_{\nu,1}(\gamma)}d\gamma, \, if \ \nu=0,\end{cases}$$

$$\frac{\partial Q_{\nu,1}(\varphi)}{\partial\varphi}=\begin{cases}\int\limits_0^{\varphi}b_1(\gamma)ch(\sqrt{\nu}(\varphi-\gamma))\overline{Q_{\nu,2}(\gamma)}d\gamma, \, if \ \nu>0,\\ \int\limits_0^{\varphi}b_1(\gamma)\cos(\sqrt{-\nu}(\varphi-\gamma))\overline{Q_{\nu,2}(\gamma)}d\gamma, \, if \ \nu<0,\\ \int\limits_0^{\varphi}b_1(\gamma)\overline{Q_{\nu,2}(\gamma)}d\gamma, \, if \ \nu=0.\end{cases}$$

Substituting (1.17) in the first of the boundary conditions (2.22), and using preceding formulas $(BF)(\varphi)$, $P_{\nu,2}(\varphi)$, $P_{\nu,1}(\varphi)$, $Q_{\nu,2}(\varphi)$, $Q_{\nu,1}(\varphi)$, we have

$$P_{\nu,2}(0)=\begin{cases}1, \, if \ \nu>0,\\ 1, \, if \ \nu<0,\\ 0, \, if \ \nu=0,\end{cases} \qquad Q_{\nu,2}(0)=\begin{cases}1, \, if \ \nu>0,\\ 0, \, if \ \nu<0,\\ 1, \, if \ \nu=0,\end{cases}$$

$$\left.\frac{\partial (BF)(\varphi)}{\partial \varphi}\right|_{\varphi=0} = \left.\frac{\partial P_{\nu,1}(\varphi)}{\partial \varphi}\right|_{\varphi=0} = \left.\frac{\partial Q_{\nu,1}(\varphi)}{\partial \varphi}\right|_{\varphi=0} = 0,$$

$$\left.\frac{\partial P_{\nu,2}(\varphi)}{\partial \varphi}\right|_{\varphi=0} = \begin{cases} \sqrt{\nu}\ , if\ \nu > 0, \\ 0, if\ \nu < 0, \\ 1, if\ \nu = 0, \end{cases} \qquad \left.\frac{\partial Q_{\nu,2}(\varphi)}{\partial \varphi}\right|_{\varphi=0} = \begin{cases} -\sqrt{\nu}, if\ \nu > 0, \\ \sqrt{-\nu}, if\ \nu < 0, \\ 0, if\ \nu = 0. \end{cases}$$

$$\frac{\partial V(r,\varphi)}{\partial \varphi} = r^{\lambda} e^{a\varphi} a((BF)(\varphi) + c_1 P_{\nu,2}(\varphi) + c_2 Q_{\nu,2}(\varphi) + \bar{c}_1 P_{\nu,1}(\varphi) + \bar{c}_2 Q_{\nu,1}(\varphi)) +$$
$$+ r^{\lambda} e^{a\varphi}\left(\frac{\partial (BF)(\varphi)}{\partial \varphi} + c_1 \frac{\partial P_{\nu,2}(\varphi)}{\partial \varphi} + c_2 \frac{\partial Q_{\nu,2}(\varphi)}{\partial \varphi} + \bar{c}_1 \frac{\partial P_{\nu,1}(\varphi)}{\partial \varphi} + \bar{c}_2 \frac{\partial Q_{\nu,1}(\varphi)}{\partial \varphi}\right).$$

Therefore from the boundary conditions (2.22) in dependence of the sign of $\nu = \dfrac{(4\alpha\gamma - \beta^2)\lambda^2 + 2(\beta(\alpha+\gamma) - 4\alpha\gamma)\lambda - (\alpha-\gamma)^2}{q^2}$ the following different algebraic systems for c_1 and c_2 occur.

1) If $\nu > 0$, then

$$c_1 = \frac{b_1 - (a - \sqrt{\nu})c_2}{(a + \sqrt{\nu})}, \quad c_2 T_1(\varphi_1) + \bar{c}_2 T_2(\varphi_1) = \Delta(\varphi_1),$$

$$where \quad \Delta(\varphi_1) = b_2 \exp(-a\varphi) - a(BF)(\varphi_1) - \frac{\partial (BF)(\varphi_1)}{\partial \varphi} -$$
$$- \frac{b_1}{(a+\sqrt{\nu})}\left(aP_{\nu,2}(\varphi_1) + \frac{\partial P_{\nu,2}(\varphi_1)}{\partial \varphi}\right) - \frac{\overline{b_1}}{(-a+\sqrt{\nu})}\left(aP_{\nu,1}(\varphi_1) + \frac{\partial P_{\nu,1}(\varphi_1)}{\partial \varphi}\right),$$

$$T_1(\varphi_1) = \left(\left(aQ_{\nu,2}(\varphi_1) + \frac{\partial Q_{\nu,2}(\varphi_1)}{\partial \varphi}\right) - \frac{(a-\sqrt{\nu})}{(a+\sqrt{\nu})}\left(aP_{\nu,2}(\varphi_1) + \frac{\partial P_{\nu,2}(\varphi_1)}{\partial \varphi}\right)\right),$$

$$T_2(\varphi_1) = \left(\left(aQ_{\nu,1}(\varphi_1) + \frac{\partial Q_{\nu,1}(\varphi_1)}{\partial \varphi}\right) + \frac{(a+\sqrt{\nu})}{(-a+\sqrt{\nu})}\left(aP_{\nu,1}(\varphi_1) + \frac{\partial P_{\nu,1}(\varphi_1)}{\partial \varphi}\right)\right).$$

2) If $\nu < 0$, then

$$c_2 = \frac{b_1 - ac_1}{\sqrt{-\nu}}, \quad c_1 T_3(\varphi_1) + \bar{c}_1 T_4(\varphi_1) = \Delta_1(\varphi_1),$$

$$where \quad \Delta_1(\varphi_1) = b_2 \exp\left(-a\varphi\right) - a(BF)(\varphi_1) - \frac{\partial(BF)(\varphi_1)}{\partial\varphi} -$$

$$-\frac{b_1}{\sqrt{-\nu}}\left(aQ_{\nu,2}(\varphi_1) + \frac{\partial Q_{\nu,2}(\varphi_1)}{\partial\varphi}\right) - \frac{\overline{b_1}}{\sqrt{-\nu}}\left(aQ_{\nu,1}(\varphi_1) + \frac{\partial Q_{\nu,1}(\varphi_1)}{\partial\varphi}\right),$$

$$T_3(\varphi_1) = \left(\left(aP_{\nu,2}(\varphi_1) + \frac{\partial P_{\nu,2}(\varphi_1)}{\partial\varphi}\right) - \frac{a}{\sqrt{-\nu}}\left(aQ_{\nu,2}(\varphi_1) + \frac{\partial Q_{\nu,2}(\varphi_1)}{\partial\varphi}\right)\right),$$

$$T_4(\varphi_1) = \left(\left(aP_{\nu,1}(\varphi_1) + \frac{\partial P_{\nu,1}(\varphi_1)}{\partial\varphi}\right) + \frac{a}{\sqrt{-\nu}}\left(aQ_{\nu,1}(\varphi_1) + \frac{\partial Q_{\nu,1}(\varphi_1)}{\partial\varphi}\right)\right).$$

3) If $\nu = 0$, then

$$c_1 = b_1 - ac_2, \quad c_2 T_5(\varphi_1) + \bar{c}_2 T_6(\varphi_1) = \Delta_2(\varphi_1),$$

$$where \quad \Delta_2(\varphi_1) = b_2 \exp(-a\varphi) - a(BF)(\varphi_1) - \frac{\partial(BF)(\varphi_1)}{\partial\varphi} -$$

$$-b_1\left(aP_{\nu,2}(\varphi_1) + \frac{\partial P_{\nu,2}(\varphi_1)}{\partial\varphi}\right) - \overline{b_1}\left(aP_{\nu,1}(\varphi_1) + \frac{\partial P_{\nu,1}(\varphi_1)}{\partial\varphi}\right),$$

$$T_5(\varphi_1) = \left(\left(aQ_{\nu,2}(\varphi_1) + \frac{\partial Q_{\nu,2}(\varphi_1)}{\partial\varphi}\right) - a\left(aP_{\nu,2}(\varphi_1) + \frac{\partial P_{\nu,2}(\varphi_1)}{\partial\varphi}\right)\right),$$

$$T_6(\varphi_1) = \left(\left(aQ_{\nu,1}(\varphi_1) + \frac{\partial Q_{\nu,1}(\varphi_1)}{\partial\varphi}\right) + a\left(aP_{\nu,1}(\varphi_1) + \frac{\partial P_{\nu,1}(\varphi_1)}{\partial\varphi}\right)\right).$$

Each of these systems has a unique solution when the conditions

1) $\nu > 0$, $|T_1(\varphi_1)|^2 \neq |T_2(\varphi_1)|^2$,

2) $\nu < 0$, $|T_3(\varphi_1)|^2 \neq |T_4(\varphi_1)|^2$, (2.23)

3) $\nu = 0$, $|T_5(\varphi_1)|^2 \neq |T_6(\varphi_1)|^2$,

hold respectively. Under these conditions the solutions are given by the formulas

1) If $\nu > 0$, $|T_1(\varphi_1)|^2 \neq |T_2(\varphi_1)|^2$, then

$$c_1 = \frac{b_1 - (a - \sqrt{\nu})c_2}{(a + \sqrt{\nu})}, \; c_2 = \frac{\overline{\Delta(\varphi_1)} T_2(\varphi_1) - \Delta(\varphi_1)\overline{T_1(\varphi_1)}}{|T_2(\varphi_1)|^2 - |T_1(\varphi_1)|^2}. \tag{2.24}$$

2) If $\nu < 0$, $|T_3(\varphi_1)|^2 \neq |T_4(\varphi_1)|^2$, then

$$c_2 = \frac{b_1 - ac_1}{\sqrt{-\nu}},\ c_1 = \frac{\overline{\Delta_1(\varphi_1)}T_4(\varphi_1) - \Delta_1(\varphi_1)\overline{T_3(\varphi_1)}}{|T_4(\varphi_1)|^2 - |T_3(\varphi_1)|^2}. \tag{2.25}$$

3) If $\nu = 0$, $|T_5(\varphi_1)|^2 \neq |T_6(\varphi_1)|^2$, then

$$c_2 = \frac{\overline{\Delta_2(\varphi_1)}T_6(\varphi_1) - \Delta_2(\varphi_1)\overline{T_5(\varphi_1)}}{|T_6(\varphi_1)|^2 - |T_5(\varphi_1)|^2},\ \ c_1 = b_1 - ac_2. \tag{2.26}$$

Hence, the following result holds

Theorem 2.5. *Problem N_1 has a unique solution when one of the conditions (2.23) holds. In this case the unique solutions are given by the formulas (1.17), (2.24)-(2.26), respectively.*

In the cases

$$\begin{aligned}
&1)\ \ \nu > 0,\ |T_1(\varphi_1)|^2 = |T_2(\varphi_1)|^2,\\
&2)\ \ \nu < 0,\ |T_3(\varphi_1)|^2 = |T_4(\varphi_1)|^2,\\
&3)\ \ \nu = 0,\ |T_5(\varphi_1)|^2 = |T_6(\varphi_1)|^2
\end{aligned} \tag{2.27}$$

For the solvability of the indicated algebraic system the following conditions are necessary and suffusient

$$\begin{aligned}
&1)\ \begin{aligned}&\mathrm{Re}\left(\overline{\Delta(\varphi_1)}\left(T_1(\varphi_1) - T_2(\varphi_1)\right)\right) = 0,\\ &\mathrm{Im}\left(\overline{\Delta(\varphi_1)}\left(T_1(\varphi_1) + T_2(\varphi_1)\right)\right) = 0,\ \ if\ \ \nu > 0,\end{aligned}\\
&2)\ \begin{aligned}&\mathrm{Re}\left(\overline{\Delta_1(\varphi_1)}\left(T_3(\varphi_1) - T_4(\varphi_1)\right)\right) = 0,\\ &\mathrm{Im}\left(\overline{\Delta_1(\varphi_1)}\left(T_3(\varphi_1) + T_4(\varphi_1)\right)\right) = 0,\ \ if\ \ \nu < 0,\end{aligned}\\
&3)\ \begin{aligned}&\mathrm{Re}\left(\overline{\Delta_2(\varphi_1)}\left(T_5(\varphi_1) - T_6(\varphi_1)\right)\right) = 0,\\ &\mathrm{Im}\left(\overline{\Delta_2(\varphi_1)}\left(T_5(\varphi_1) + T_6(\varphi_1)\right)\right) = 0,\ \ if\ \ \nu = 0.\end{aligned}
\end{aligned} \tag{2.28}$$

When these conditions hold then the solutions of the indicated algebraic systems are given by the formulas

1) If $\nu > 0$, $|T_1(\varphi_1)|^2 = |T_2(\varphi_1)|^2$, then

$$c_1 = \frac{b_1 - \left(a - \sqrt{\nu}\right)c_2}{\left(a + \sqrt{\nu}\right)},$$

$$c_2 = \begin{cases} \dfrac{\operatorname{Re}\Delta(\varphi_1) + i\alpha(\overline{T_1(\varphi_1)} + T_2(\varphi_1))}{\operatorname{Re}\left(T_1(\varphi_1) + T_2(\varphi_1)\right)}, \mathit{if}\ \operatorname{Re}\left(T_1(\varphi_1) + T_2(\varphi_1)\right) \neq 0, \\ i\dfrac{\operatorname{Re}\Delta(\varphi_1) - \alpha(\overline{T_1(\varphi_1)} + T_2(\varphi_1))}{\operatorname{Im}\left(T_2(\varphi_1) - T_1(\varphi_1)\right)}, \mathit{if}\ \operatorname{Im}\left(T_1(\varphi_1) - T_2(\varphi_1)\right) \neq 0, \\ c_3, \mathit{if}\ \operatorname{Re}\left(T_1(\varphi_1) + T_2(\varphi_1)\right) = 0, \operatorname{Im}\left(T_1(\varphi_1) - T_2(\varphi_1)\right) = 0. \end{cases} \tag{2.29}$$

2) If $\nu < 0$, $\left|T_3(\varphi_1)\right|^2 = \left|T_4(\varphi_1)\right|^2$, then

$$c_2 = \frac{b_1 - ac_1}{\sqrt{-\nu}},$$

$$c_1 = \begin{cases} \dfrac{\operatorname{Re}\Delta_1(\varphi_1) + i\alpha(\overline{T_3(\varphi_1)} + T_4(\varphi_1))}{\operatorname{Re}\left(T_3(\varphi_1) + T_4(\varphi_1)\right)}, \mathit{if}\ \operatorname{Re}\left(T_3(\varphi_1) + T_4(\varphi_1)\right) \neq 0, \\ i\dfrac{\operatorname{Re}\Delta_1(\varphi_1) - \alpha(\overline{T_3(\varphi_1)} + T_4(\varphi_1))}{\operatorname{Im}\left(T_4(\varphi_1) - T_3(\varphi_1)\right)}, \mathit{if}\ \operatorname{Im}\left(T_3(\varphi_1) - T_4(\varphi_1)\right) \neq 0, \\ c_3, \mathit{if}\ \operatorname{Re}\left(T_3(\varphi_1) + T_4(\varphi_1)\right) = 0, \operatorname{Im}\left(T_3(\varphi_1) - T_4(\varphi_1)\right) = 0. \end{cases} \tag{2.30}$$

3) If $\nu = 0$, $\left|T_5(\varphi_1)\right|^2 = \left|T_6(\varphi_1)\right|^2$, then

$$c_1 = b_1 - ac_2,$$

$$c_2 = \begin{cases} \dfrac{\operatorname{Re}\Delta_2(\varphi_1) + i\alpha(\overline{T_5(\varphi_1)} + T_6(\varphi_1))}{\operatorname{Re}\left(T_5(\varphi_1) + T_6(\varphi_1)\right)}, \mathit{if}\ \operatorname{Re}\left(T_5(\varphi_1) + T_6(\varphi_1)\right) \neq 0, \\ i\dfrac{\operatorname{Re}\Delta_2(\varphi_1) - \alpha(\overline{T_5(\varphi_1)} + T_6(\varphi_1))}{\operatorname{Im}\left(T_6(\varphi_1) - T_5(\varphi_1)\right)}, \mathit{if}\ \operatorname{Im}\left(T_5(\varphi_1) - T_6(\varphi_1)\right) \neq 0, \\ c_3, \mathit{if}\ \operatorname{Re}\left(T_5(\varphi_1) + T_6(\varphi_1)\right) = 0, \operatorname{Im}\left(T_5(\varphi_1) - T_6(\varphi_1)\right) = 0, \end{cases} \tag{2.31}$$

where α is any real, c_3 is any complex number.

Hence, the following result holds.

Theorem 2.6. *In the cases (2.27) for the solvability of problem* N_1 *condition (2.28) is necessary and sufficient. Under these conditions the problem has infinitely many solutions. These solutions are given by the formulas (1.17), (2.29)-(2.31), respectively.*

Problem N_2. *Let* $\lambda \neq 1, \alpha \neq \gamma$ *and* $\beta = \alpha + \gamma$. *It is required to find the solution of equation (1.1), from the class (1.3), satisfying the conditions*

$$|V(r,\varphi)| = \mathrm{O}(r^{\lambda}),\, r \to \infty, \tag{2.32}$$

$$\left.\frac{\partial V(r,\varphi)}{\partial \varphi}\right|_{\varphi=\varphi_1} = b_2 r^{\lambda}\ , \tag{2.33}$$

where b_2 *is a given complex number,* $\lambda > 0$ *is a given real number.*

For the solution of problem N_2 formula (1.33) is used. Then (2.32) automatically holds. From the form of the functions $(BF)(\varphi)$, $P_1(\varphi)$, $P_2(\varphi)$ follows

$$\frac{\partial (BF)(\varphi)}{\partial \varphi} = f_1(\varphi) + b_1(\varphi)\overline{(BF)(\varphi)} - i\nu (BF)(\varphi),$$

$$\frac{\partial P_2(\varphi)}{\partial \varphi} = b_1(\varphi)\overline{P_1(\varphi)} - i\nu P_2(\varphi), \tag{2.34}$$

$$\frac{\partial P_1(\varphi)}{\partial \varphi} = b_1(\varphi)\overline{P_2(\varphi)} - i\nu P_1(\varphi).$$

Substituting (1.33) in the boundary condition (2.33), we have in view of (2.34)

$$cT_2(\varphi_1) + \bar{c}T_1(\varphi_1) = \Delta_1(\varphi_1), \tag{2.35}$$

where

$$T_2(\varphi_1) = \left(b_1(\varphi_1)\overline{P_1(\varphi_1)} - i\nu P_2(\varphi_1)\right),\ T_1(\varphi_1) = \left(b_1(\varphi_1)\overline{P_2(\varphi_1)} - i\nu P_1(\varphi_1)\right),$$

$$\Delta_1(\varphi_1) = b_2 - f_1(\varphi_1) - b_1(\varphi_1)\overline{(BF)(\varphi_1)} + i\nu (BF)(\varphi_1).$$

Equation (2.35) for the unknown c has a unique solution when

$$|T_1(\varphi_1)|^2 \neq |T_2(\varphi_1)|^2, \tag{2.36}$$

which is given by the formula

$$c=\frac{\Delta_1(\varphi_1)\overline{T_2(\varphi_1)}-\overline{\Delta_1(\varphi_1)}T_1(\varphi_1)}{\left|T_2(\varphi_1)\right|^2-\left|T_1(\varphi_1)\right|^2}. \tag{2.37}$$

Thus, the following result holds.

Theorem 2.7. *When the condition (2.36) is satisfied, problem N_2 has a unique solution. This solution is given by the formulas (1.33), (2.37).*

In case, when

$$\left|T_2(\varphi_1)\right|^2=\left|T_1(\varphi_1)\right|^2 \tag{2.38}$$

for the solvability of equation (2.35) the conditions

$$\operatorname{Re}\left(\overline{\Delta_1(\varphi_1)}(T_2(\varphi_1)-T_1(\varphi_1))\right)=0,\ \operatorname{Im}\left(\overline{\Delta_1(\varphi_1)}(T_2(\varphi_1)+T_1(\varphi_1))\right)=0. \tag{2.39}$$

are necessary and sufficient. When these conditions hold the solution of equation (2.35) is given by the formulas

$$c=\begin{cases}\dfrac{\operatorname{Re}\Delta_1+i\alpha(\overline{T_2(\varphi_1)}+T_1(\varphi_1))}{\operatorname{Re}\left(T_2(\varphi_1)+T_1(\varphi_1)\right)},\ if\ \operatorname{Re}\left(T_2(\varphi_1)+T_1(\varphi_1)\right)\neq 0,\\ i\dfrac{\operatorname{Re}\Delta_1-\alpha(\overline{T_2(\varphi_1)}+T_1(\varphi_1))}{\operatorname{Im}\left(-T_2(\varphi_1)+T_1(\varphi_1)\right)},\ if\ \operatorname{Im}\left(T_2(\varphi_1)-T_1(\varphi_1)\right)\neq 0,\\ c_2,\ if\ \operatorname{Re}\left(T_2(\varphi_1)+T_1(\varphi_1)\right)=0,\ \operatorname{Im}\left(T_2(\varphi_1)-T_1(\varphi_1)\right)=0,\end{cases} \tag{2.40}$$

where α is any real, c_2 is any complex number.

Thus, the following result holds.

Theorem 2.8. *When the condition (2.38) holds, for the solvability of problem N_2 equalities (2.39) are necessary and sufficient. Under these conditions the problem has infinitely many solutions. These solutions are given by the formulas (1.33) and (2.40).*

2.2 Initial boundary problem for nonhomogeneous model second order partial differential equations in the plane with Fuchs operator in the main part

Problem K_1. *Let* $\beta \neq \alpha + \gamma$. *It is required to find the solution of equation (1.34) from the class (1.35), satisfying the conditions*

$$\left.\frac{\partial^k}{\partial p^k} V(r,\varphi)\right|_{\substack{r=0\\ \varphi=0}} = a_k, \qquad 0 \le k, \tag{2.41}$$

$$\left.\frac{\partial^k}{\partial p^k} \frac{\partial V(r,\varphi)}{\partial \varphi}\right|_{\substack{r=0\\ \varphi=0}} = b_k, \qquad 0 \le k, \tag{2.42}$$

where $p = r^{\nu}$; a_k, b_k, $0 \le k$ *are given complex numbers, so that the series* $\sum_{k=1}^{\infty} \frac{a_k}{k!} r^{\nu k}$, $\sum_{k=1}^{\infty} \frac{b_k}{k!} r^{\nu k}$ *are convergent in* G.

Solution of the problem

The functions $(BF)_k(\varphi) = \int_0^{\varphi} f_k(\varphi,\gamma) d\gamma + \int_0^{\varphi} b_k(\varphi,\gamma) \overline{(BF)_k(\gamma)} d\gamma$,

$P_{k,2}(\varphi) = I_{k,0}(\varphi) + \int_0^{\varphi} b_k(\varphi,\gamma) \overline{P_{k,1}(\gamma)} d\gamma$, $P_{k,1}(\varphi) = \int_0^{\varphi} b_k(\varphi,\gamma) \overline{P_{k,2}(\gamma)} d\gamma$,

$Q_{k,2}(\varphi) = J_{k,0}(\varphi) + \int_0^{\varphi} b_k(\varphi,\gamma) \overline{Q_{k,1}(\gamma)} d\gamma$, $Q_{k,1}(\varphi) = \int_0^{\varphi} b_k(\varphi,\gamma) \overline{Q_{k,2}(\gamma)} d\gamma$

have the differential properties

$$\frac{\partial (BF)_0(\varphi)}{\partial \varphi} = \int_0^{\varphi} f_{0,1}(\gamma) \exp(\tau_0(\varphi-\gamma)) d\gamma + \int_0^{\varphi} b_0(\gamma) \exp(\tau_0(\varphi-\gamma)) \overline{(BF)_0(\gamma)} d\gamma,$$

$$\frac{\partial P_{0,2}(\varphi)}{\partial \varphi} = \tau_0 \exp(\tau_0 \varphi) + \int_0^{\varphi} b_0(\gamma) \exp(\tau_0(\varphi-\gamma)) \overline{P_{0,1}(\gamma)} d\gamma,$$

$$\frac{\partial P_{0,1}(\varphi)}{\partial \varphi} = \int_0^{\varphi} b_0(\gamma)\exp(\tau_0(\varphi-\gamma))\overline{P_{0,2}(\gamma)}d\gamma,$$

$$\frac{\partial Q_{0,2}(\varphi)}{\partial \varphi} = \int_0^{\varphi} b_0(\gamma)\exp(\tau_0(\varphi-\gamma))\overline{Q_{0,1}(\gamma)}d\gamma,$$

$$\frac{\partial Q_{0,1}(\varphi)}{\partial \varphi} = \int_0^{\varphi} b_0(\gamma)\exp(\tau_0(\varphi-\gamma))\overline{Q_{0,2}(\gamma)}d\gamma,$$

$$\frac{\partial (BF)_k(\varphi)}{\partial \varphi} = \begin{cases} \int_0^{\varphi} f_{k,1}(\gamma)ch(\sqrt{\tau_k}(\varphi-\gamma))d\gamma + \\ + \int_0^{\varphi} b_k(\gamma)ch(\sqrt{\tau_k}(\varphi-\gamma))\overline{(BF)_k(\gamma)}d\gamma, \ if \ \tau_k > 0, \\ \int_0^{\varphi} f_{k,1}(\gamma)\cos(\sqrt{-\tau_k}(\varphi-\gamma))d\gamma + \\ + \int_0^{\varphi} b_k(\gamma)\cos(\sqrt{-\tau_k}(\varphi-\gamma))\overline{(BF)_k(\gamma)}d\gamma, \ if \ \ \tau_k < 0, \\ \int_0^{\varphi} f_{k,1}(\gamma)d\gamma + \int_0^{\varphi} b_k(\gamma)\overline{(BF)_k(\gamma)}d\gamma, \ if \ \tau_k = 0, \end{cases}$$

$$\frac{\partial P_{k,2}(\varphi)}{\partial \varphi} = \begin{cases} \sqrt{\tau_k}\exp(\sqrt{\tau_k}\varphi) + \int_0^{\varphi} b_k(\gamma)ch(\sqrt{\tau_k}(\varphi-\gamma))\overline{P_{k,1}(\gamma)}d\gamma, if \ \tau_k > 0, \\ -\sqrt{-\tau_k}\sin(\sqrt{-\tau_k}\varphi) + \int_0^{\varphi} b_k(\gamma)\cos(\sqrt{-\tau_k}(\varphi-\gamma))\overline{P_{k,1}(\gamma)}d\gamma, if \ \tau_k < 0, \\ 1 + \int_0^{\varphi} b_k(\gamma)\overline{P_{k,1}(\gamma)}d\gamma, if \ \tau_k = 0, \end{cases}$$

$$\frac{\partial P_{k,1}(\varphi)}{\partial \varphi} = \begin{cases} \int_0^{\varphi} b_k(\gamma)ch(\sqrt{\tau_k}(\varphi-\gamma))\overline{P_{k,2}(\gamma)}d\gamma, if \ \tau_k > 0, \\ \int_0^{\varphi} b_k(\gamma)\cos(\sqrt{-\tau_k}(\varphi-\gamma))\overline{P_{k,2}(\gamma)}d\gamma, if \ \tau_k < 0, \\ \int_0^{\varphi} b_k(\gamma)\overline{P_{k,2}(\gamma)}d\gamma, if \ \tau_k = 0, \end{cases}$$

$$\frac{\partial Q_{k,2}(\varphi)}{\partial\varphi}=\begin{cases}-\sqrt{\tau_k}\exp(\sqrt{\tau_k}\varphi)+\int\limits_0^{\varphi}b_k(\gamma)ch(\sqrt{\tau_k}(\varphi-\gamma))\overline{Q_{k,1}(\gamma)}d\gamma, \, if\ \tau_k>0,\\ \sqrt{-\tau_k}\cos(\sqrt{-\tau_k}\varphi)+\int\limits_0^{\varphi}b_k(\gamma)\cos(\sqrt{-\tau_k}(\varphi-\gamma))\overline{Q_{k,1}(\gamma)}d\gamma, \, if\ \tau_k<0,\\ \int\limits_0^{\varphi}b_k(\gamma)\overline{Q_{k,1}(\gamma)}d\gamma, \, if\ \tau_k=0,\end{cases}$$

$$\frac{\partial Q_{k,1}(\varphi)}{\partial\varphi}=\begin{cases}\int\limits_0^{\varphi}b_k(\gamma)ch(\sqrt{\tau_k}(\varphi-\gamma))\overline{Q_{k,2}(\gamma)}d\gamma, \, if\ \tau_k>0,\\ \int\limits_0^{\varphi}b_k(\gamma)\cos(\sqrt{-\tau_k}(\varphi-\gamma))\overline{Q_{k,2}(\gamma)}d\gamma, \, if\ \tau_k<0,\\ \int\limits_0^{\varphi}b_k(\gamma)\overline{Q_{k,2}(\gamma)}d\gamma, \, if\ \tau_k=0, \qquad 1\le k.\end{cases}$$

From these equalities and the functions $(BF)_k(\varphi)$, $P_{k,2}(\varphi)$, $P_{k,1}(\varphi)$, $Q_{k,2}(\varphi)$, $Q_{k,1}(\varphi)$, also the numbers

$\tau_k=\dfrac{(vk)^2(4\alpha\gamma-\beta^2)-vk(8\alpha\gamma-2\alpha\beta-2\beta\gamma)-(\alpha-\gamma)^2}{q^2}$, $1\le k$, are determined:

$(BF)_0(0)=P_{0,1}(0)=Q_{0,1}(0)=0$, $P_{0,2}(0)=Q_{0,2}(0)=1$,

$(BF)_k(0)=P_{k,1}(0)=Q_{k,1}(0)=0$,

$$P_{k,2}(0)=\begin{cases}1, \, if\ \tau_k>0,\\ 1, \, if\ \tau_k<0,\\ 0, \, if\ \tau_k=0,\end{cases}\qquad Q_{k,2}(0)=\begin{cases}1, \, if\ \tau_k>0,\\ 0, \, if\ \tau_k<0,\\ 1, \, if\ \tau_k=0, \quad 1\le k,\end{cases}$$

$$\left.\frac{\partial(BF)_0(\varphi)}{\partial\varphi}\right|_{\varphi=0}=\left.\frac{\partial P_{0,1}(\varphi)}{\partial\varphi}\right|_{\varphi=0}=\left.\frac{\partial Q_{0,1}(\varphi)}{\partial\varphi}\right|_{\varphi=0}=\left.\frac{\partial Q_{0,2}(\varphi)}{\partial\varphi}\right|_{\varphi=0}=0,$$

$$\left.\frac{\partial P_{0,2}(\varphi)}{\partial\varphi}\right|_{\varphi=0}=\tau_0,\quad \left.\frac{\partial(BF)_k(\varphi)}{\partial\varphi}\right|_{\varphi=0}=\left.\frac{\partial P_{k,1}(\varphi)}{\partial\varphi}\right|_{\varphi=0}=\left.\frac{\partial Q_{k,1}(\varphi)}{\partial\varphi}\right|_{\varphi=0}=0, \tag{2.43}$$

$$\left.\frac{\partial P_{k,2}(\varphi)}{\partial\varphi}\right|_{\varphi=0}=\begin{cases}\sqrt{\tau_k}, \, if\ \tau_k>0,\\ 0, \, if\ \tau_k<0,\\ 1, \, if\ \tau_k=0,\end{cases}\quad \left.\frac{\partial Q_{k,2}(\varphi)}{\partial\varphi}\right|_{\varphi=0}=\begin{cases}-\sqrt{\tau_k}, \, if\ \tau_k>0,\\ \sqrt{-\tau_k}, \, if\ \tau_k<0,\\ 0, \, if\ \tau_k=0, \quad 1\le k.\end{cases}$$

The coefficients $c_{k,1}$ *and* $c_{k,2}$, $0 \le k$, we take so that the function is given by formula (1.50) satisfying the conditions (2.41) and (2.42). For this, substituting (1.50) in (2.41) and (2.42), we have in view of (2.43)

$$c_{0,1} = \frac{b_0}{\tau_0}\ , \ c_{0,2} = a_0 - \frac{b_0}{\tau_0},$$

$$c_{k,1} = \begin{cases} \dfrac{a_k\sqrt{\tau_k} + b_k}{2\sqrt{\tau_k}}, if\ \tau_k > 0, \\ a_k, if\ \tau_k < 0, \\ b_k, if\ \tau_k = 0, \end{cases} \qquad c_{k,2} = \begin{cases} \dfrac{a_k\sqrt{\tau_k} - b_k}{2\sqrt{\tau_k}}, if\ \tau_k > 0, \\ \dfrac{b_k}{\sqrt{\tau_k}}, if\ \tau_k < 0, \\ a_k, if\ \tau_k = 0, \end{cases} \tag{2.44}$$

$1 \le k.$

Thus, the following result holds.

Theorem 2.9. *Let* $\beta \ne \alpha + \gamma$. *Then the problem* K_1 *has a unique solution. The unique solution is given by formulas (1.50) and (2.44).*

Problem K_2. *Let* $\beta = \alpha + \gamma$. *It is required to find the solution of equation (1.34) from the class (1.35), satisfying the conditions*

$$\left.\frac{\partial^k}{\partial p^k} V(r,\varphi)\right|_{\substack{r=0 \\ \varphi=1}} = a_k, \qquad 0 \le k, \tag{2.45}$$

where $p = r^{\nu}$, $a_k, 0 \le k$, *are given complex numbers, so that the series* $\sum_{k=1}^{\infty} \frac{a_k}{k!} r^{\nu k}$ *is convergent in* G.

Solution of the problem. Substituting (2.45) in (1.68), we have

$$c_k P_{k,2}(\varphi_1) + \bar{c}_k P_{k,1}(\varphi_1) = \Delta_k(\varphi_1), \tag{2.46}$$

where $\Delta_k(\varphi_1) = a_k - F_{k,1}(\varphi_1)$.

Equation (2.46) for c_k has a unique solution when

$$\Delta_k(\varphi_1) = \left|P_{k,2}(\varphi_1)\right|^2 - \left|P_{k,1}(\varphi_1)\right|^2 \neq 0. \tag{2.47}$$

This solution is given by the following formula

$$c_k = \frac{\overline{P_{k,2}(\varphi_1)} \cdot \Delta_k(\varphi_1) - P_{k,1}(\varphi_1) \cdot \overline{\Delta_k(\varphi_1)}}{\left|P_{k,2}(\varphi_1)\right|^2 - \left|P_{k,1}(\varphi_1)\right|^2}. \tag{2.48}$$

When

$$\Delta_k(\varphi_1) = 0 \tag{2.49}$$

for the solvability of equation (2.46) the conditions

$$\mathrm{Re}(\overline{\Delta_k(\varphi_1)}(P_{k,2}(\varphi_1) - P_{k,1}(\varphi_1))) = 0, \mathrm{Im}(\overline{\Delta_k(\varphi_1)}(P_{k,2}(\varphi_1) + P_{k,1}(\varphi_1))) = 0 \tag{2.50}$$

are necessary and sufficient. When these conditions hold the solution of equation (2.46) is

$$c_k = \begin{cases} \dfrac{\mathrm{Re}\,\Delta_k(\varphi_1) + i\alpha_k(\overline{P_{k,2}(\varphi_1)} + P_{k,1}(\varphi_1))}{\mathrm{Re}(P_{k,2}(\varphi_1) + P_{k,1}(\varphi_1))}, \mathit{if}\ \mathrm{Re}(P_{k,2}(\varphi_1) + P_{k,1}(\varphi_1)) \neq 0, \\ i\dfrac{\mathrm{Re}\,\Delta_k(\varphi_1) - \alpha_k(\overline{P_{k,2}(\varphi_1)} + P_{k,1}(\varphi_1))}{\mathrm{Im}(-P_{k,2}(\varphi_1) + P_{k,1}(\varphi_1))}, \mathit{if}\ \mathrm{Im}(-P_{k,2}(\varphi_1) + P_{k,1}(\varphi_1)) \neq 0, \\ c_{k,1}, \mathit{if}\ \mathrm{Re}(P_{k,2}(\varphi_1) + P_{k,1}(\varphi_1)) = 0, \mathrm{Im}(-P_{k,2}(\varphi_1) + P_{k,1}(\varphi_1)) = 0, \end{cases} \tag{2.51}$$

where α_k are any real, $c_{k,1}$ are any complex numbers.

Thus, the following result holds.

Theorem 2.10. *1) When $\Delta_k(\varphi_1) \neq 0$, $0 \leq k$, the problem K_2 has a unique solution. This solution is given by formulas (1.68) and (2.48).*

2) If for some index k the equality $\Delta_k(\varphi_1) = 0$ holds, then for the solvability of problem K_2 the condition (2.50) for this particular index k is necessary and sufficient. In this case the problem has infinitely many solutions. These solutions are given by formula (1.68), where c_κ is defined by formulas (2.48) if $\Delta_k(\varphi_1) \neq 0$ and by formulas (2.51) if $\Delta_k(\varphi_1) = 0$.

2.3 Dirichlet problem with given growth at infinity for second order partial differential equations in the plane with Fuchs operator in the main part and specified right hand side.

Consider the Dirichlet problem for equation (1.69).

Problem D_3. *Let* $b(\varphi) \neq a(\varphi) + c(\varphi)$. *It is required to find the solution of equation (1.69) from the class (1.3), satisfying the conditions*

$$|V(r,\varphi)| = \mathrm{O}(r^{\lambda}),\ r \to \infty, \tag{2.52}$$

$$V(r,0) = b_1 r^{\lambda},\ V(r,\varphi_1) = b_2 r^{\lambda}, \tag{2.53}$$

where b_1, b_2 *are given complex numbers,* $\lambda > 0$ *is a given real number.*

Solution of the problem. For solving problem D_3 formula (1.80) is used. Then (2.52) automatically holds. For the functions $F_1(\varphi)$, $P_1(\varphi)$, $P_2(\varphi)$, $Q_1(\varphi)$, $Q_2(\varphi)$ follows

$$F_1(0) = 0,\ P_1(0) = 0,\ Q_1(0) = 0,\ \ P_2(0) = \psi_1(0),\ Q_2(0) = \psi_2(0).$$

Therefore from the boundary conditions (2.53) follows the algebraic system of equations for c_1 and c_2,

$$\begin{aligned} c_1\psi_1(0) + c_2\psi_2(0) &= b_1, \\ c_2 T_1(\varphi_1) + \bar{c}_2 T_2(\varphi_1) &= \Delta_1(\varphi_1), \end{aligned} \tag{2.54}$$

where *if* $\psi_1(0) \neq 0$,

$$\Delta_1(\varphi_1) = b_2 - F_1(\varphi_1) - \frac{b_1}{\psi_1(0)} P_2(\varphi_1) - \frac{\overline{b_1}}{\overline{\psi_1(0)}} P_1(\varphi_1),$$

$$T_1(\varphi_1) = Q_2(\varphi_1) - \frac{\psi_2(0)}{\psi_1(0)} P_2(\varphi_1), \quad T_2(\varphi_1) = Q_1(\varphi_1) - \frac{\overline{\psi_2(0)}}{\overline{\psi_1(0)}} P_1(\varphi_1).$$

System (2.54) has a unique solution in case, when

$$\Lambda_1(\varphi_1) = |T_1(\varphi_1)|^2 - |T_2(\varphi_1)|^2 \neq 0. \tag{2.55}$$

Under this condition the solution is given as

$$c_2 = \frac{\Delta_1(\varphi_1)\overline{T_1(\varphi_1)} - \overline{\Delta_1(\varphi_1)}T_2(\varphi_1)}{|T_1(\varphi_1)|^2 - |T_2(\varphi_1)|^2}, \qquad c_1 = \frac{b_1 - c_2\psi_2(0)}{\psi_1(0)}. \tag{2.56}$$

In case, when

$$\Lambda_1(\varphi_1) = 0 \tag{2.57}$$

for the solvability of the algebraic system (2.54) the conditions

$$\begin{aligned} &\mathrm{Re}(\overline{\Delta_1}(T_1(\varphi_1) - T_2(\varphi_1))) = 0, \\ &\mathrm{Im}(\overline{\Delta_1}(T_1(\varphi_1) + T_2(\varphi_1))) = 0, \end{aligned} \tag{2.58}$$

are necessary and sufficient. When these conditions hold the solutions of the algebraic system (2.54) are

$$c_2 = \begin{cases} \dfrac{\mathrm{Re}\,\Delta_1(\varphi_1) + i\alpha(\overline{T_1(\varphi_1)} + T_2(\varphi_1))}{\mathrm{Re}(T_1(\varphi_1) + T_2(\varphi_1))}, \ \mathit{if} \ \ \mathrm{Re}(T_1(\varphi_1) + T_2(\varphi_1)) \neq 0, \\ i\dfrac{\mathrm{Re}\,\Delta_1(\varphi_1) - \alpha(\overline{T_1(\varphi_1)} + T_2(\varphi_1))}{\mathrm{Im}(-T_1(\varphi_1) + T_2(\varphi_1))}, \mathit{if} \ \ \mathrm{Im}(-T_1(\varphi_1) + T_2(\varphi_1)) \neq 0, \\ c_3, \ \mathit{if} \quad \mathrm{Re}(T_1(\varphi_1) + T_2(\varphi_1)) = 0, \ \ \mathrm{Im}(-T_1(\varphi_1) + T_2(\varphi_1)) = 0, \end{cases} \tag{2.59}$$

where α is any real, c_3 is any complex number.

When $\psi_1(0) = 0$ *but* $\psi_2(0) \neq 0$ we have

$$\begin{aligned} &c_2 = \frac{b_1}{\psi_2(0)}, \\ &c_1 P_2(\varphi_1) + \bar{c}_2 P_1(\varphi_1) = \Delta_2(\varphi_1), \end{aligned} \tag{2.60}$$

where $\Delta_2(\varphi_1) = b_2 - F_1(\varphi_1) - \dfrac{b_1}{\psi_2(0)} Q_2(\varphi_1) - \dfrac{\overline{b_1}}{\overline{\psi_2(0)}} Q_1(\varphi_1),$

If $\psi_1(0) = 0, \ \ \psi_2(0) = 0$ for the solvability of systems (2.54) the equality $b_1(0) = 0$ is sufficient. Under this condition we choose the constant c_2 arbitrarily. Then from (2.54) we have

$$c_1 P_2(\varphi) + \bar{c}_1 P_1(\varphi) = \Delta_2(\varphi_1), \tag{2.61}$$

where

$\Delta_2(\varphi_1)=b_2-(BF)(\varphi_1)-c_2Q_2(\varphi_1)-\bar{c}_2Q_1(\varphi_1)$.

Equation (2.61) has a unique solution in case, when

$$\Lambda_2(\varphi_1)=|P_2(\varphi_1)|^2-|P_1(\varphi_1)|^2\neq 0. \qquad (2.62)$$

Under this condition the solution of equation (2.61) is

$$c_1=\frac{\overline{P_2(\varphi_1)}\Delta_2(\varphi_1)-P_1(\varphi_1)\overline{\Delta_2(\varphi_1)}}{|P_2(\varphi_1)|^2-|P_1(\varphi_1)|^2}, \qquad (2.63)$$

where $\Delta_2(\varphi_1)=b_2-(BF)(\varphi_1)-c_2Q_2(\varphi_1)-\bar{c}_2Q_1(\varphi_1)$.

The solvability of equation (2.60) is similar.

In case, when

$$\Lambda_2(\varphi_1)=0 \qquad (2.64)$$

for the solvability of (2.61) the conditions

$$\mathrm{Re}(\overline{\Delta_2(\varphi_1)}(P_2(\varphi_1)-P_1(\varphi_1)))=0,\ \mathrm{Im}(\overline{\Delta_2(\varphi_1)}(P_2(\varphi_1)+P_1(\varphi_1)))=0 \qquad (2.65)$$

are necessary and sufficient. When these conditions hold the solution of equation (2.61) is

$$c_1=\begin{cases}\dfrac{\mathrm{Re}\,\Delta_2(\varphi_1)+i\alpha(\overline{P_2(\varphi_1)}+P_1(\varphi_1))}{\mathrm{Re}(P_2(\varphi_1)+P_1(\varphi_1))}, \textit{if } \mathrm{Re}(P_2(\varphi_1)+P_1(\varphi_1))\neq 0,\\ i\dfrac{\mathrm{Re}\,\Delta_2(\varphi_1)-\alpha(\overline{P_2(\varphi_1)}+P_1(\varphi_1))}{\mathrm{Im}(-P_2(\varphi_1)+P_1(\varphi_1))}, \textit{if } \mathrm{Im}(-P_2(\varphi_1)+P_1(\varphi_1))\neq 0,\\ c_3, \textit{if } \mathrm{Re}(P_2(\varphi_1)+P_1(\varphi_1))=0, \mathrm{Im}(-P_2(\varphi_1)+P_1(\varphi_1))=0,\end{cases} \qquad (2.66)$$

where α is any real, c_3 is any complex number.

Thus, the following results holds.

Theorem 2.11. *1) When* $\psi_1^2(0)+\psi_2^2(0)\neq 0$, *problem* D_3 *has a unique solution. This solution is given by formula (1.80), where the numbers* c_1, c_2 *are defined by formula (2.56), when* $\psi_1(0)\neq 0$ *and by formula (2.63) when* $\psi_2(0)\neq 0$.

2) If $\Lambda_1(\varphi_1)=0$ *and* $\psi_1(0)\neq 0$ *or* $\Lambda_2(\varphi_1)=0$ *and* $\psi_1(0)=0$, $\psi_2(0)\neq 0$, *then for the solvability of problem* D_3 *the condition (2.58) when* $\psi_1(0)\neq 0$ *and*

(2.65) when $\psi_1(0)=0$*,* $\psi_2(0)\neq 0$ *is necessary and sufficient. When the conditions are satisfied problem* D_3 *has infinitely many solutions. These solutions are given by the formulas (1.80), where* c_1 *and* c_2 *are defined by formula (2.59), if* $\Lambda_1(\varphi_1)=0$*,* $\psi_1(0)\neq 0$ *and by formula (2.66), if* $\Lambda_2(\varphi_1)=0$*,* $\psi_1(0)=0$*,* $\psi_2(0)\neq 0$.

3) If $\psi_1(0)=0, \psi_2(0)=0$*, then for the solvability of problem* D_3 *necessarily holds* $b_1=0$*. In this case problem* D_3 *has infinitely many solutions. These solutions are given by formula (2.61), where one of the numbers* c_1*,* c_2 *are arbitrary.*

Let c_2 *be arbitrarily chosen. Then* c_1 *is given by formula (2.63), if* $\Lambda_2(\varphi_1)\neq 0$ *and by formula (2.66), if* $\Lambda_2(\varphi_1)=0$ *and condition (2.65) holds.*

Problem D_4*. Let* $b(\varphi)=a(\varphi)+c(\varphi)$*,* $a(\varphi)\neq c(\varphi)$ *and* $\lambda\neq 1$*. It is required to find the solution of equation (1.69) from the class (1.3), satisfying the conditions*

$$|V(r,\varphi)| = \mathrm{O}(r^{\lambda}),\, r\to\infty, \tag{2.67}$$

$$V(r,\varphi_1)=b_1 r^{\lambda}, \tag{2.68}$$

where b_1 *is a given complex number,* $\lambda>0$ *is a given real number.*

Solution of the problem. For solving problem D_4 formula (1.96) is used. Then (2.67) automatically holds. From the boundary condition (2.68) we get the equation for determining c:

$$cP_2(\varphi_1)+\bar{c}P_1(\varphi_1)=\Delta(\varphi_1), \tag{2.69}$$

where $\Delta(\varphi_1)=b_1-F_1(\varphi_1)$.

Equation (2.69) has a unique solution in case, when

$$|P_2(\varphi_1)|^2 \neq |P_1(\varphi_1)|^2 . \tag{2.70}$$

Under this condition the solution is

$$c=\frac{\Delta(\varphi_1)\overline{P_2(\varphi_1)}-\overline{\Delta_1(\varphi_1)}P_1(\varphi_1)}{|P_2(\varphi_1)|^2-|P_1(\varphi_1)|^2}, \tag{2.71}$$

Hence, the following result holds.

Theorem 2.12 *When the condition (2.70) is fulfilled problem* D_4 *has a unique solution. This solution is given by formulas (1.96) and (2.71).*

In case, when

$$|P_2(\varphi_1)|^2 = |P_1(\varphi_1)|^2 \tag{2.72}$$

for the solvability of equation (2.69) the conditions

$$\mathrm{Re}(\overline{\Delta(\varphi_1)}(P_2(\varphi_1) - P_1(\varphi_1))) = 0, \mathrm{Im}(\overline{\Delta(\varphi_1)}(P_2(\varphi_1) + P_1(\varphi_1))) = 0. \tag{2.73}$$

are necessary and sufficient. When these conditions hold, the solution of equation (2.69) is given by the formulas

$$c = \begin{cases} \dfrac{\mathrm{Re}\,\Delta(\varphi_1) + i\alpha(\overline{P_2(\varphi_1)} + P_1(\varphi_1))}{\mathrm{Re}(P_2(\varphi_1) + P_1(\varphi_1))}, \textit{if } \mathrm{Re}(P_2(\varphi_1) + P_1(\varphi_1)) \neq 0, \\ i\dfrac{\mathrm{Re}\,\Delta(\varphi_1) - \alpha(\overline{P_2(\varphi_1)} + P_1(\varphi_1))}{\mathrm{Im}(-P_2(\varphi_1) + P_1(\varphi_1))}, \textit{if } \mathrm{Im}(-P_2(\varphi_1) + P_1(\varphi_1)) \neq 0, \\ c_1, \textit{if } \mathrm{Re}(P_2(\varphi_1) + P_1(\varphi_1)) = 0, \mathrm{Im}(-P_2(\varphi_1) + P_1(\varphi_1)) = 0, \end{cases} \tag{2.74}$$

where α is any real, c_1 is any complex number.

Thus, the following result holds.

Theorem 2.13. *In case, when (2.72) holds for the solvability of problem* D_4 *conditions (2.73) are necessary and sufficient. Under these conditions the problem has infinitely many solutions. These solutions are given by formulas (1.96), (2.74).*

2.4 Dirichlet problem for nonhomogeneous second order partial differential equations in the plane with Fuchs operator in the main part

Problem D_5. *Let* $a(\varphi) \neq c(\varphi)$. *It is required to find the solution of equation (1.97) from the class (1.35), satisfying the conditions*

$$V(r,\varphi_1)=t(r), \tag{2.75}$$

where $t(r)=\sum_{k=0}^{\infty} t_k r^{\nu k}$ *is convergent in* G , t_k *are given complex numbers.*

Solution of the problem. For solving problem D_5 formula (1.109) is used. Then, from the boundary condition (2.75) follows the equation for c_k,

$$c_k P_{k,2}(\varphi_1)+\bar{c}_k P_{k,1}(\varphi_1)=\Delta_k, \tag{2.76}$$

where $\Delta_k = t_k - F_{k,1}(\varphi_1)$.

Equation (2.76) has a unique solution in case, when

$$\Delta_\kappa(\varphi_1)=\left|P_{k,2}(\varphi_1)\right|^2-\left|P_{k,1}(\varphi_1)\right|^2\neq 0. \tag{2.77}$$

Under this condition the solution of equation (2.76) are given by the formula

$$c_k=\frac{\Delta_k\overline{P_{k,2}(\varphi_1)}-\overline{\Delta_k}P_{k,1}(\varphi_1)}{\left|P_{k,2}(\varphi_1)\right|^2-\left|P_{k,1}(\varphi_1)\right|^2} \tag{2.78}$$

In case, when

$$\Delta_\kappa(\varphi_1)=0 \tag{2.79}$$

for the solvability of equation (2.76) the conditions

$$\operatorname{Re}(\overline{\Delta_k}(P_{k,2}(\varphi_1)-P_{k,1}(\varphi_1)))=0,\ \operatorname{Im}(\overline{\Delta_k}(P_{k,2}(\varphi_1)+P_{k,1}(\varphi_1)))=0. \tag{2.80}$$

are necessary and sufficient. When these conditions hold the solution of equation (2.76) is given by

$$c_k=\begin{cases}\dfrac{\operatorname{Re}\Delta_k+i\alpha_k(\overline{P_{k,2}(\varphi_1)}+P_{k,1}(\varphi_1))}{\operatorname{Re}(P_{k,2}(\varphi_1)+P_{k,1}(\varphi_1))},\textit{if}\quad \operatorname{Re}(P_{k,2}(\varphi_1)+P_{k,1}(\varphi_1))\neq 0,\\ i\dfrac{\operatorname{Re}\Delta_k-\alpha_k(\overline{P_{k,2}(\varphi_1)}+P_{k,1}(\varphi_1))}{\operatorname{Im}(-P_{k,2}(\varphi_1)+P_{k,1}(\varphi_1))},\textit{if}\quad \operatorname{Im}(-P_{k,2}(\varphi_1)+P_{k,1}(\varphi_1))\neq 0,\\ c_{k,1},\textit{if}\ \operatorname{Re}(P_{k,2}(\varphi_1)+P_{k,1}(\varphi_1))=0,\quad \operatorname{Im}(-P_{k,2}(\varphi_1)+P_{k,1}(\varphi_1))=0,\end{cases} \tag{2.81}$$

where α_k is any real, $c_{k,1}$ is any complex number.

Thus, the following results holds.

Theorem 2.14. *1) When* $\Delta_k(\varphi_1) \neq 0$, $0 \leq k$ *problem* D_5 *has a unique solution. This solution is given by the formulas (1.109) and (2.78).*

2) If for some index k *the equality* $\Delta_k(\varphi_1) = 0$ *holds, then for the solvability of problem* D_5 *conditions (2.80) for this index* k *are necessary and sufficient. In this case the problem has an infinitely many solutions. These solutions are given by formulas (1.109), where* c_κ *is given by formulas (2.78) if* $\Delta_k(\varphi_1) \neq 0$ *and by formulas (2.81) if* $\Delta_k(\varphi_1) = 0$.