## CHAPTER 1

## CONSTRUCTION OF CONTINUOUS SOLUTIONS OF SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS IN THE PLANE WITH FUCHS OPERATOR IN THE MAIN PART

In the present chapter varieties of continuous solutions of Fuchs type equations are constructed. Particular kinds of this equation arise in the theory of infinitesimal bendings of surfaces of positive curvature with flattening point and are considered in the works [28,29,30].

If solutions of the investigated equation are looked for in the form of a double series $\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_{kl} z^k \bar{z}^l$, so with the relation $z = re^{i\varphi}$ we have $\sum_{k=0}^{\infty} V_k(\varphi) r^k$. Therefore solutions are searched by A. Tungatarov's methods. The received variety of solutions are used in second chapter to solve boundary value problems.

### 1.1. Model second order partial differential equations in the plane with Fuchs operator in the main part and specified right hand side.

Let $0 < \varphi_1 \le 2\pi$ and $G = \{z = re^{i\varphi} : 0 \le r < \infty, \quad 0 < \varphi < \varphi_1\}$. Consider the equation

$$4\alpha\bar{z}^2 V_{\bar{z}\bar{z}} + 4\beta z\bar{z} V_{z\bar{z}} + 4\gamma z^2 V_{zz} + b(\varphi)\overline{V} = f(\varphi) r^{\lambda}, \quad z \in G, \tag{1.1}$$

in $G$ when $b(\varphi),\ f(\varphi) \in C[0, \varphi_1]$ and $\alpha, \beta, \gamma, \lambda > 0$ are real parameters.

Particular kinds of equation (1.1) can be applied in the theory of infinitesimal bendings of surfaces and are studied in [5,20].

Here

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$

$$\frac{\partial^2}{\partial \bar{z}\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}\left(\frac{\partial}{\partial \bar{z}}\right), \quad \frac{\partial^2}{\partial z\partial \bar{z}} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial \bar{z}}\right), \quad \frac{\partial^2}{\partial z\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\right).$$

Following, we using these operators in polar coordinates:

$$\frac{\partial}{\partial \bar{z}} = \frac{e^{i\varphi}}{2}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \varphi}\right), \quad \frac{\partial}{\partial z} = \frac{e^{-i\varphi}}{2}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \varphi}\right),$$

$$\frac{\partial^2}{\partial \bar{z}\partial \bar{z}} = \frac{e^{2i\varphi}}{4}\left(\frac{\partial^2}{\partial r^2} - \frac{2i}{r^2}\frac{\partial}{\partial \varphi} + \frac{2i}{r}\frac{\partial^2}{\partial r\partial \varphi} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right), \tag{1.2}$$

$$\frac{\partial^2}{\partial \bar{z}\partial z} = \frac{1}{4}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right),$$

$$\frac{\partial^2}{\partial z\partial z} = \frac{e^{-2i\varphi}}{4}\left(\frac{\partial^2}{\partial r^2} + \frac{2i}{r^2}\frac{\partial}{\partial \varphi} - \frac{2i}{r}\frac{\partial^2}{\partial \varphi\partial r} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right)$$

In work [21] the Dirichlet problem for equation (1.1) in the unit disk is investigated, where the parameters $\alpha > 0, \beta > 0, \gamma = 0$ and $b(\varphi) \equiv f(\varphi) \equiv 0$ are concerned.

The solutions of equation (1.1) are searched for in the S.L. Sobolev class [21]

$$W_p^{\ 2}(G) \tag{1.3}$$

where $1 < p < \dfrac{2}{2-\lambda}$, *if* $\lambda < 2$ *and* $p > 1$, *if* $\lambda \geq 2$.

From the Sobolev imbedding theorem follows $W_p^{\ 2}(G) \subset C(\overline{G})$.

1. Let $\beta \neq \alpha + \gamma$. Using formulas (1.2), equation (1.1) in polar coordinates is written in the form

$$\begin{aligned}&(\alpha+\beta+\gamma)r^2\frac{\partial^2 V}{\partial r^2} - (\alpha-\gamma)2i\frac{\partial V}{\partial \varphi} + (\alpha-\gamma)2ir\frac{\partial^2 V}{\partial r\partial \varphi} - \\ &-(\alpha-\beta+\gamma)r\frac{\partial V}{\partial r} - (\alpha-\beta+\gamma)\frac{\partial^2 V}{\partial \varphi^2} + b(\varphi)\overline{V} = f(\varphi)r^\lambda\end{aligned} \tag{1.4}$$

For solving equation (1.4) the method of separation of variables is applied. Let

$$V(r,\varphi) = r^\lambda \psi(\varphi), \tag{1.5}$$

where $\psi(\varphi)$ is a new unknown function from $C^2[0,\varphi_1]$, satisfying the equation

$$-(\alpha-\beta+\gamma)\psi''-2i(\alpha-\gamma)(1-\lambda)\psi'+$$
$$+((\alpha+\beta+\gamma)(\lambda^2-\lambda)-\lambda(\alpha-\beta+\gamma))\psi = f(\varphi)-b(\varphi)\overline{\psi}$$

Substituting

$$\psi(\varphi)=\exp(a\varphi)P(\varphi), \tag{1.6}$$

where $a=\dfrac{i(\alpha-\gamma)(\lambda-1)}{q}$, $q=\alpha-\beta+\gamma$ and $P(\varphi)$ is the new unknown function from $C^2[0,\varphi_1]$, the last equation becomes

$$P''(\varphi)-\nu P(\varphi)=b_1(\varphi)\overline{P(\varphi)}+f_1(\varphi), \tag{1.7}$$

where

$$\nu=\frac{(4\alpha\gamma-\beta^2)\lambda^2+2(\beta(\alpha+\gamma)-4\alpha\gamma)\lambda-(\alpha-\gamma)^2}{q^2},$$

$$b_1(\varphi)=\frac{b(\varphi)\exp\left(\dfrac{-2i(\alpha-\gamma)(\lambda-1)\varphi}{q}\right)}{q},\quad f_1(\varphi)=-\frac{f(\varphi)\exp\left(-\dfrac{i(\alpha-\gamma)(\lambda-1)\varphi}{q}\right)}{q}.$$

Solving equation (1.7) by applying the method of variation of constant, we have

$$P(\varphi)=\int_0^{\varphi} b(\varphi,\gamma)\overline{P(\gamma)}d\gamma+\int_0^{\varphi} f(\varphi,\gamma)d\gamma+c_1I_{\nu,0}(\varphi)+c_2J_{\nu,0}(\varphi), \tag{1.8}$$

where

$$b(\varphi,\gamma)=\begin{cases}\dfrac{b_1(\gamma)}{\sqrt{\nu}}sh(\sqrt{\nu}(\varphi-\gamma)),\ if\ \nu>0,\\ \dfrac{b_1(\gamma)}{\sqrt{-\nu}}\sin(\sqrt{-\nu}(\varphi-\gamma)),\ if\ \nu<0,\\ b_1(\gamma)(\varphi-\gamma),\ if\ \nu=0,\end{cases}$$

$$f(\varphi,\gamma)=\begin{cases}\dfrac{f_1(\gamma)}{\sqrt{\nu}}sh(\sqrt{\nu}(\varphi-\gamma)),\ if\ \nu>0,\\ \dfrac{f_1(\gamma)}{\sqrt{-\nu}}\sin(\sqrt{-\nu}(\varphi-\gamma)),\ if\ \nu<0,\\ f_1(\gamma)(\varphi-\gamma),\ if\ \nu=0,\end{cases}$$

$$I_{\nu,0}(\varphi)=\begin{cases} e^{\sqrt{\nu}\varphi}, if\ \nu>0, \\ \cos(\sqrt{-\nu}\varphi), if\ \nu<0,, \\ \varphi, if\ \nu=0, \end{cases} \qquad J_{\nu,0}(\varphi)=\begin{cases} e^{-\sqrt{\nu}\varphi}, if\ \nu>0, \\ \sin(\sqrt{-\nu}\varphi), if\ \nu<0, \\ 1, if\ \nu=0. \end{cases}$$

For the construction of solutions of equation (1.8) the following functions and operators are used:

$$I_{\nu,k}(\varphi)=\int_0^{\varphi} b(\varphi,\gamma)\overline{I_{\nu,k-1}}(\gamma)d\gamma,\ J_{\nu,k}(\varphi)=\int_0^{\varphi} b(\varphi,\gamma)\overline{J_{\nu,k-1}}(\gamma)d\gamma,\ 1\le k,$$

$$(Bf)(\varphi)=\int_0^{\varphi} b(\varphi,\gamma)\overline{f(\gamma)}d\gamma,\ f_2(\varphi)=\int_0^{\varphi} f(\varphi,\gamma)d\gamma,$$

$$(B^k f)(\varphi)=B(B^{k-1}f)(\varphi),\ 2\le k,\ (B^1 f)(\varphi)=(Bf)(\varphi)$$

For these functions it is easy to check that

$$(B(cI_{\nu,k}(\varphi)))(\varphi)=\bar{c}I_{\nu,k+1}(\varphi),\ (B(cJ_{\nu,k}(\varphi)))(\varphi)=\bar{c}J_{\nu,k+1}(\varphi), \tag{1.9}$$

$$\left|(B^k f)(\varphi)\right|\le\begin{cases} \left(\dfrac{sh(\sqrt{\nu}\varphi_1)|b|_0\varphi}{|q|\sqrt{\nu}}\right)^k \dfrac{|f|_0}{k!}, if\ \nu>0, \\ \left(\dfrac{|b|_0\varphi}{|q|\sqrt{-\nu}}\right)^k \dfrac{|f|_0}{k!}, if\ \nu<0, \\ \left(\dfrac{\sqrt{|b|_0}\varphi}{\sqrt{|q|}}\right)^{2k} \dfrac{|f|_0}{(2k)!}, if\ \nu=0, \end{cases}$$

$$\left|(B^k f_2)(\varphi)\right|\le\begin{cases} \left(\dfrac{sh(\sqrt{\nu}\varphi_1)|b|_0\varphi}{|q|\sqrt{\nu}}\right)^{k+1} \dfrac{|f|_0}{|b|_0(k+1)!}, if\ \nu>0, \\ \left(\dfrac{|b|_0\varphi}{|q|\sqrt{-\nu}}\right)^{k+1} \dfrac{|f|_0}{|b|_0(k+1)!}, if\ \nu<0, \\ \dfrac{|b|_0^{\,k}|f|_0\varphi^{2k+2}}{|q|^{k+1}(2k+2)!}, if\ \nu=0, \end{cases} \tag{1.10}$$

$$
\left|I_{\nu,k}(\varphi)\right| \le \begin{cases} \left( \dfrac{sh(\sqrt{\nu}\varphi_1)|b|_0 \varphi}{|q|\sqrt{\nu}} \right)^k \dfrac{\exp(\sqrt{\nu}\varphi_1)}{k!}, \; if \;\; \nu > 0, \\ \left( \dfrac{|b|_0 \varphi}{|q|\sqrt{-\nu}} \right)^k \dfrac{1}{k!}, \; if \;\; \nu < 0, \\ \dfrac{|b|_0^{\,k} \varphi^{2k+1}}{|q|^k (2k+1)!}, \; if \;\; \nu = 0, \end{cases}
$$

$$
\left|J_{\nu,k}(\varphi)\right| \le \begin{cases} \left( \dfrac{sh(\sqrt{\nu}\varphi_1)|b|_0 \varphi}{|q|\sqrt{\nu}} \right)^k \cdot \dfrac{1}{k!}, \; if \;\; \nu > 0, \\ \left( \dfrac{|b|_0 \varphi}{|q|\sqrt{-\nu}} \right)^k \dfrac{1}{k!}, \; if \;\; \nu < 0, \\ \dfrac{|b|_0^{\,k} \varphi^{2k}}{|q|^k (2k)!}, \; if \;\; \nu = 0, \end{cases}
$$

where c is any complex number, $|f|_0 = \|f\|_{C[0,\varphi_1]}$.

Using the specified notations, equation (1.8) is written in the form

$$
P(\varphi) = (BP)(\varphi) + f_2(\varphi) + c_1 I_{\nu,0}(\varphi) + c_2 J_{\nu,0}(\varphi) \tag{1.11}
$$

If we apply the operator $B$ to both sides of equation (1.11), we have in view of (1.9)

$$
(BP)(\varphi) = (B^2 P)(\varphi) + (Bf_2)(\varphi) + \bar{c}_1 I_{\nu,1}(\varphi) + \bar{c}_2 J_{\nu,1}(\varphi) \tag{1.12}
$$

From (1.11), (1.12) it follows

$$
\begin{aligned} P(\varphi) &= (B^2 P)(\varphi) + (Bf_2)(\varphi) + f_2(\varphi) + \bar{c}_1 I_{\nu,1}(\varphi) + \bar{c}_2 J_{\nu,1}(\varphi) + \\ &+ c_1 I_{\nu,0}(\varphi) + c_2 J_{\nu,0}(\varphi) \end{aligned} \tag{1.13}
$$

If we again apply the operator $B$ to both sides of equality (1.13), we have in view of (1.9)

$$
\begin{aligned} (BP)(\varphi) &= (B^3 P)(\varphi) + (B^2 f_2)(\varphi) + (Bf_2)(\varphi) + c_1 I_{\nu,2}(\varphi) + c_2 J_{\nu,2}(\varphi) + \\ &+ \bar{c}_1 I_{\nu,1}(\varphi) + \bar{c}_2 J_{\nu,1}(\varphi) \end{aligned} \tag{1.14}
$$

From (1.11) and (1.14) it follows

$$P(\varphi)=(B^3P)(\varphi)+(B^2f_2)(\varphi)+(Bf_2)(\varphi)+f_2(\varphi)+c_1(I_{\nu,2}(\varphi)+I_{\nu,0}(\varphi))+$$
$$+c_2(J_{\nu,2}(\varphi)+J_{\nu,0}(\varphi))+\bar{c}_1I_{\nu,1}(\varphi)+\bar{c}_2J_{\nu,1}(\varphi)$$

Continuing this process 2n-1 and 2n-2 times, respectively we receive at the following representations for solutions of equation (1.7)

$$P(\varphi)=(B^{2n}P)(\varphi)+\sum_{k=0}^{2n-1}B^kf_2(\varphi)+c_1\sum_{k=0}^{n-1}I_{\nu,2k}(\varphi)+c_2\sum_{k=0}^{n-1}J_{\nu,2k}(\varphi)+$$
$$+\bar{c}_1\sum_{k=1}^{n}I_{\nu,2k-1}(\varphi)+\bar{c}_2\sum_{k=1}^{n}J_{\nu,2k-1}(\varphi)$$

and (1.15)

$$P(\varphi)=(B^{2n+1}P)(\varphi)+\sum_{k=0}^{2n}B^kf_2(\varphi)+c_1\sum_{k=0}^{n}I_{\nu,2k}(\varphi)+c_2\sum_{k=0}^{n}J_{\nu,2k}(\varphi)+$$
$$+\bar{c}_1\sum_{k=1}^{n}I_{\nu,2k-1}(\varphi)+\bar{c}_2\sum_{k=1}^{n}J_{\nu,2k-1}(\varphi)$$

If we pass to the limit as $n\to\infty$ in the representations (1.15), by virtue of (1.10) we receive

$$P(\varphi)=(BF)(\varphi)+c_1P_{\nu,2}(\varphi)+c_2Q_{\nu,2}(\varphi)+\bar{c}_1P_{\nu,1}(\varphi)+\bar{c}_2Q_{\nu,1}(\varphi), \qquad (1.16)$$

where $(BF)(\varphi)=\sum_{k=0}^{\infty}B^kf_2(\varphi)$, $P_{\nu,2}(\varphi)=\sum_{k=0}^{\infty}I_{\nu,2k}(\varphi)$, $P_{\nu,1}(\varphi)=\sum_{k=1}^{\infty}I_{\nu,2k-1}(\varphi)$,

$$Q_{\nu,2}(\varphi)=\sum_{k=0}^{\infty}J_{\nu,2k}(\varphi),\ Q_{\nu,1}(\varphi)=\sum_{k=1}^{\infty}J_{\nu,2k-1}(\varphi).$$

Using the inequalities (1.10), we receive the estimates

$$\left|(BF)(\varphi)\right|\le\begin{cases}\dfrac{|f|_0}{|b|_0}\left(\exp\left(\dfrac{|b|_0 sh(\sqrt{\nu}\varphi_1)}{|q|\sqrt{\nu}}\varphi\right)-1\right), if\ \nu>0,\\[2ex] \dfrac{|f|_0}{|b|_0}\left(\exp\left(\dfrac{|b|_0}{|q|\sqrt{-\nu}}\varphi\right)-1\right), if\ \nu<0,\\[2ex] \dfrac{|f|_0}{|b|_0}\left(\exp\left(\dfrac{\sqrt{|b|_0}\varphi}{\sqrt{|q|}}\right)-1\right), if\ \nu=0,\end{cases}$$

$$\left|P_{\nu,2}(\varphi)\right| \le \begin{cases} \exp\left(\sqrt{\nu}\varphi_1\right) ch\left(\dfrac{|b|_0\, sh(\sqrt{\nu}\varphi_1)}{|q|\sqrt{\nu}}\varphi\right), if\ \nu > 0, \\ ch\left(\dfrac{|b|_0\varphi}{|q|\sqrt{-\nu}}\right), if\ \nu < 0, \\ \dfrac{\sqrt{|q|}}{\sqrt{|b|_0}} sh\left(\dfrac{\varphi\sqrt{|b|_0}}{\sqrt{|q|}}\right), if\ \nu = 0, \end{cases}$$

$$\left|P_{\nu,1}(\varphi)\right| \le \begin{cases} \exp\left(\sqrt{\nu}\varphi_1\right) sh\left(\dfrac{|b|_0\, sh(\sqrt{\nu}\varphi_1)}{|q|\sqrt{\nu}}\varphi\right), if\ \nu > 0, \\ sh\left(\dfrac{|b|_0\varphi}{|q|\sqrt{-\nu}}\right), if\ \nu < 0, \\ \dfrac{\sqrt{|q|}}{\sqrt{|b|_0}} sh\left(\dfrac{\varphi\sqrt{|b|_0}}{\sqrt{|q|}}\right), if\ \nu = 0, \end{cases}$$

$$\left|Q_{\nu,2}(\varphi)\right| \le \begin{cases} \exp(\varphi_1\sqrt{\nu}) ch\left(\dfrac{|b|_0\, sh(\sqrt{\nu}\varphi_1)}{|q|\sqrt{\nu}}\varphi\right), if\ \nu > 0, \\ ch\left(\dfrac{|b|_0\varphi}{|q|\sqrt{-\nu}}\right), if\ \nu < 0, \\ ch\left(\dfrac{\varphi\sqrt{|b|_0}}{\sqrt{|q|}}\right), if\ \nu = 0, \end{cases}$$

$$\left|Q_{\nu,1}(\varphi)\right| \le \begin{cases} \exp(\varphi_1\sqrt{\nu}) sh\left(\dfrac{|b|_0\, sh(\sqrt{\nu}\varphi_1)}{|q|\sqrt{\nu}}\varphi\right), if\ \nu > 0, \\ sh\left(\dfrac{|b|_0\varphi}{|q|\sqrt{-\nu}}\right), if\ \nu < 0, \\ ch\left(\dfrac{\varphi\sqrt{|b|_0}}{\sqrt{|q|}}\right), if\ \nu = 0, \end{cases}$$

By means of these estimates it is easy to show, that the function $P(\varphi)$, given by formula (1.16), is a solution of equation (1.7) from the class $C^2[0,\varphi_1]$.

From (1.16), (1.6) and (1.5) we find

$$V(r,\varphi)=r^{\lambda}e^{a\varphi}((BF)(\varphi)+c_1P_{\nu,2}(\varphi)+c_2Q_{\nu,2}(\varphi)+\overline{c}_1P_{\nu,1}(\varphi)+\overline{c}_2Q_{\nu,1}(\varphi)) \tag{1.17}$$

Thus, the following result holds

***Theorem 1.1.*** *When* $\beta\neq\alpha+\gamma$, $\lambda>0$ *the equation (1.1) is solvable in the class (1.3). The general solution of equation (1.1) from the class (1.3) is given by formula (1.17).*

2. Let $\beta=\alpha+\gamma$. Using formulas (1.2), equation (1.1) in polar coordinates is written in the form

$$(\alpha+\beta+\gamma)r^2\frac{\partial^2V}{\partial r^2}-(\alpha-\gamma)2i\frac{\partial V}{\partial\varphi}+(\alpha-\gamma)2ir\frac{\partial^2V}{\partial r\partial\varphi}+b(\varphi)\overline{V}=f(\varphi)r^{\lambda} \tag{1.18}$$

If finding the solution of equation (1.18) with substituting (1.5), then

$$-2i(\alpha-\gamma)(1-\lambda)\psi'+(\alpha+\beta+\gamma)(\lambda^2-\lambda)\psi=f(\varphi)-b(\varphi)\overline{\psi} \tag{1.19}$$

Consider varies cases of (1.19) with respect to $\alpha,\gamma,\lambda$:

1. If $\alpha=\gamma$ or $\lambda=1$, then equation (1.19) has the view

$$A\psi=f(\varphi)-b(\varphi)\overline{\psi} \tag{1.20}$$

where $A=(\alpha+\beta+\gamma)(\lambda^2-\lambda)$.

Solving the equation (1.20) when $|b(\varphi)|\neq|A|$, obtain:

$$\psi(\varphi)=\frac{Af(\varphi)-b(\varphi)\overline{f(\varphi)}}{A^2-|b(\varphi)|^2} \tag{1.21}$$

In case, when $|b(\varphi)|=|A|$, for the solvability of equation (1.20) it is sufficient and necessary that the conditions

$$\operatorname{Re}(\overline{f(\varphi)}(-b(\varphi)+A))=0,\ \operatorname{Im}(\overline{f(\varphi)}(b(\varphi)+A))=0, \tag{1.22}$$

are fulfilled. When the condition are fulfilled the solution are given by the formulas

$$\psi(\varphi)=\begin{cases}\dfrac{\operatorname{Re} f(\varphi)+i\theta(\varphi)(A+b(\varphi))}{\operatorname{Re}(b(\varphi)+A)}, \textit{if } \operatorname{Re}(b(\varphi)+A)\neq 0,\\ i\dfrac{\operatorname{Re} f(\varphi)-\theta(\varphi)(A+b(\varphi))}{\operatorname{Im}(b(\varphi)-A)}, \textit{if } \operatorname{Im}(b(\varphi)-A)\neq 0,\\ \psi_1(\varphi),\ \textit{if } \operatorname{Re}(b(\varphi)+A)=0\ \textit{and}\ \operatorname{Im}(b(\varphi)-A)=0,\end{cases} \tag{1.23}$$

where $\theta(\varphi)$ is any real function, $\psi_1(\varphi)$ is any complex function.

2. If $\alpha\neq\gamma$ and $\lambda\neq 1$, then equation (1.19) is written as

$$\psi'+i\nu\psi=b_1(\varphi)\overline{\psi}+f_1(\varphi), \tag{1.24}$$

where

$$\nu=-\frac{(\alpha+\beta+\gamma)\lambda}{2(\alpha-\gamma)},\ b_1(\varphi)=i\frac{b(\varphi)}{2(\alpha-\gamma)(\lambda-1)},\ f_1(\varphi)=-i\frac{f(\varphi)}{2(\alpha-\gamma)(\lambda-1)}$$

Solving equation (1.24) by applying the method of variation of constant, we have

$$\psi(\varphi)=\int_0^{\varphi} b(\varphi,\gamma)\overline{\psi(\gamma)}d\gamma+\int_0^{\varphi} f(\varphi,\gamma)d\gamma+cI_0(\varphi), \tag{1.25}$$

where c is an arbitrary constant,

$b(\varphi,\gamma)=b_1(\gamma)\exp(i\nu(\gamma-\varphi))$, $f(\varphi,\gamma)=f_1(\gamma)\exp(i\nu(\gamma-\varphi))$,

$I_0(\varphi)=\exp(-i\nu\varphi)$.

For the construction of a solution of equation (1.25) the following functions and operators are used:

$$I_k(\varphi)=\int_0^{\varphi} b(\varphi,\gamma)\overline{I_{k-1}}(\gamma)d\gamma\ ,\ 1\leq k,\ (Bf)(\varphi)=\int_0^{\varphi} b(\varphi,\gamma)\overline{f(\gamma)}d\gamma, f_2(\varphi)=\int_0^{\varphi} f(\varphi,\gamma)d\gamma,$$

$(B^k f)(\varphi)=B(B^{k-1}f)(\varphi)$, $2\leq k$, $(B^1 f)(\varphi)=(Bf)(\varphi)$.

For these functions it is easy to check that

$$\left|(B^k f)(\varphi)\right|\leq(|b_1|_0\varphi)^{\kappa}\frac{|f|_0}{k!},$$

$$\left|(B^k f_2)(\varphi)\right| \le (|b_1|_0 \varphi)^{k+1} \frac{|f_1|_0}{|b_1|_0 (k+1)!}, \tag{1.26}$$

$$|I_k(\varphi)| \le (|b_1|_0 \varphi)^k \frac{1}{k!}, \quad 0 \le k.$$

Here $|f|_0 = \|f\|_{C[0,\varphi_1]}$.

Using the specified notations, equation (1.25) is written in the form

$$\psi(\varphi) = (B\psi)(\varphi) + f_2(\varphi) + cI_0(\varphi) \tag{1.27}$$

If we apply the operator $B$ to both sides of equation (1.27), we have in view of (1.9)

$$(B\psi)(\varphi) = (B^2\psi)(\varphi) + (Bf_2)(\varphi) + \bar{c}I_1(\varphi) \tag{1.28}$$

From (1.27), (1.28) it follows

$$\psi(\varphi) = (B^2\psi)(\varphi) + (Bf_2)(\varphi) + f_2(\varphi) + \bar{c}I_1(\varphi) + cI_0(\varphi) \tag{1.29}$$

If we again apply the operator $B$ to both sides of equality (1.29), we have in view of (1.9)

$$(B\psi)(\varphi) = (B^3\psi)(\varphi) + (B^2 f_2)(\varphi) + (Bf_2)(\varphi) + cI_2(\varphi) + \bar{c}I_1(\varphi) \tag{1.30}$$

From (1.27) and (1.30) it follows

$$\psi(\varphi) = (B^3\psi)(\varphi) + (B^2 f_2)(\varphi) + (Bf_2)(\varphi) + f_2(\varphi) + c(I_2(\varphi) + I_0(\varphi)) + \bar{c}I_1(\varphi).$$

Continuing this process 2n-1 and 2n-2 times, respectively we receive the representations

$$\psi(\varphi) = (B^{2n}\psi)(\varphi) + \sum_{k=0}^{2n-1} B^k f_2(\varphi) + c\sum_{k=0}^{n-1} I_{2k}(\varphi) + \bar{c}\sum_{k=1}^{n} I_{2k-1}(\varphi)$$

and (1.31)

$$\psi(\varphi) = (B^{2n-1}\psi)(\varphi) + \sum_{k=0}^{2n-2} B^k f_2(\varphi) + c\sum_{k=0}^{n-1} I_{2k}(\varphi) + \bar{c}\sum_{k=1}^{n-1} I_{2k-1}(\varphi)$$

for solutions of equation (1.24). If we pass to the limit as $n \to \infty$ in the representation (1.31), by virtue of (1.26) we receive

$$\psi(\varphi) = (BF)(\varphi) + cP_2(\varphi) + \bar{c}P_1(\varphi), \tag{1.32}$$

where

$$(BF)(\varphi)=\sum_{k=0}^{\infty}B^k f_2(\varphi),\ \ P_2(\varphi)=\sum_{k=0}^{\infty}I_{2k}(\varphi),\ P_1(\varphi)=\sum_{k=1}^{\infty}I_{2k-1}(\varphi),$$

$c$ is any complex number.

Using the inequalities (1.26), we receive the estimates

$$|(BF)(\varphi)|\le\frac{|f_1|_0}{|b_1|_0}(\exp(|b_1|_0)-1),\ \ |P_2(\varphi)|\le ch(|b_1|_0\varphi),\ \ |P_1(\varphi)|\le sh(|b_1|_0\varphi),$$

By means of these estimates it is easy to show, that the function $\psi(\varphi)$, given by formula (1.32), is a solution of equation (1.24) from the class $C^1[0,\varphi_1]$.

From (1.32) and (1.5) we receive

$$V(r,\varphi)=r^{\lambda}((BF)(\varphi)+cP_2(\varphi)+\bar{c}P_1(\varphi)) \tag{1.33}$$

Thus, the following results hold.

***Theorem 1.2.*** *When one of the condition are satisfied: 1)* $\beta=\alpha+\gamma$, $\alpha=\gamma$ *and* $|b(\varphi)|\ne|A|$ *or* $\beta=\alpha+\gamma$, $\lambda=1$ *and* $|b(\varphi)|\ne|A|$ *and 2)* $\alpha\ne\gamma$, $\lambda\ne 1$ *the equation (1.1) is solvable in the class (1.3). When 1) holds then equation (1.1) has a unique solution in the class (1.3). This solution is given by the formulas (1.5), (1.21). When 2) holds the equation (1.1) has a general solution. This solution is given by the formulas (1.5), (1.32). When* $\alpha=\gamma$ *or* $\lambda=1$ *and* $|b(\varphi)|=|A|$ *in some point* $0\le\varphi\le\varphi_1$ *for the solvability of equation (1.1) in the class (1.3) condition (1.22) is necessary and sufficient. In this case the solution of equation (1.1) from the class (1.3) is given by the formulas (1.5), (1.23).*

### 1.2. Nonhomogeneous model second order partial differential equations in the plane with Fuchs operator in the main part

Let $0<\varphi_1\le 2\pi$ and $G=\{z=re^{i\varphi}:0\le r<\infty,\ \ 0<\varphi<\varphi_1\}$. Consider in $G$ the equation

$$4\alpha\bar{z}^2V_{\bar{z}\bar{z}}+4\beta z\bar{z}V_{z\bar{z}}+4\gamma z^2V_{zz}+b(\varphi)\overline{V}=g(r,\varphi),\quad z\in G, \tag{1.34}$$

where $b(\varphi)\in C[0,\varphi_1]$; $\alpha,\beta,\gamma$ are real parameters.

Assume, what the function $g(r,\varphi)$ is satisfying conditions (A): in the domain $G$ the function $g(r,\varphi)$ is given in the form $g(r,\varphi)=\sum_{k=0}^{\infty}\frac{g_k(\varphi)r^{\nu k}}{k!}$, where $\nu$ is some positive constant, $g_k(\varphi)\in C[0,\varphi_1]$, $0\le k$ and the series $g(r,\varphi)=\sum_{k=0}^{\infty}\frac{|g_k(\varphi)|r^{\nu k}}{k!}$ converges in $G$.

Particular kinds of equation (1.34) are studied in [8,9].

The solution of equation (1.34) are searched for in the S.L. Sobolev class [21]

$$W_p^{\ 2}(G) \tag{1.35}$$

where $1<p<\frac{2}{2-\nu}, if\ \ \nu<2\ \ and\ \ p>1,\ if\ \ \nu\ge 2$.

From the Sobolev imbedding theorem follows $W_p^{\ 2}(G)\subset C(G)$.

1. Let $\beta\neq\alpha+\gamma$. Using formulas (1.2), equation (1.34) in polar coordinates is written in the form

$$\begin{aligned}&(\alpha+\beta+\gamma)r^2\frac{\partial^2V}{\partial r^2}-(\alpha-\gamma)2i\frac{\partial V}{\partial\varphi}+(\alpha-\gamma)2ir\frac{\partial^2V}{\partial r\partial\varphi}-\\&-(\alpha-\beta+\gamma)r\frac{\partial V}{\partial r}-(\alpha-\beta+\gamma)\frac{\partial^2V}{\partial\varphi^2}+b(\varphi)\overline{V}=g(r,\varphi)\end{aligned} \tag{1.36}$$

Solutions of equation (1.36) from the class (1.35) are searched in the form

$$V(r,\varphi)=\sum_{k=0}^{\infty}\frac{V_k(\varphi)r^{\nu k}}{k!}, \tag{1.37}$$

where $V_k(\varphi)$, $0\le k$ are new unknown functions from $C^2[0,\varphi_1]$ such that the given series and the series differentiated with respect to $r$ and $\varphi$ as far as appearing in equation (1.37) converge in $G$.

Substituting (1.37) in (1.36) and comparing coefficients of $r^{\nu k}$ for $k=0$ and $k>0$ we have

$$
\begin{gathered}
-2i(\alpha-\gamma)V_0'(\varphi)-(\alpha-\beta+\gamma)V_0''(\varphi)+b(\varphi)\overline{V_0}=g_0(\varphi),\\
\big((\alpha+\beta+\gamma)(\nu k-1)-(\alpha-\beta+\gamma)\big)\nu kV_k(\varphi)+\\
+2i(\alpha-\gamma)(\nu k-1)V_k'(\varphi)-(\alpha-\beta+\gamma)V_k''(\varphi)=g_k(\varphi)-b(\varphi)\overline{V_k(\varphi)}
\end{gathered}
\tag{1.38}
$$

Substituting

$$
V_k(\varphi)=\exp(a_k\varphi)P_k(\varphi),\ 1\le k, \tag{1.39}
$$

where

$$
a_k=\frac{i(\alpha-\gamma)(\nu k-1)}{\alpha-\beta+\gamma},
$$

and $P_k(\varphi)$ are new unknown functions from $C^2[0,\varphi_1]$, the system of equations (1.38), reduce to the form

$$
\begin{gathered}
V_0''(\varphi)-\tau_0V_0'(\varphi)=b_0(\varphi)\overline{V_0}+f_{0,1}(\varphi),\\
P_k''-\tau_kP_k=b_k(\varphi)\overline{P_k}+f_{k,1}(\varphi),\ 1\le k,
\end{gathered}
\tag{1.40}
$$

where

$$
\tau_0=-\frac{2i(\alpha-\gamma)}{q},\ b_0(\varphi)=\frac{b(\varphi)}{q},\ f_{0,1}(\varphi)=-\frac{g_0(\varphi)}{q},\ q=\alpha-\beta+\gamma,
$$

$$
\tau_k=\frac{(\nu k)^2(4\alpha\gamma-\beta^2)-\nu k(8\alpha\gamma-2\alpha\beta-2\beta\gamma)-(\alpha-\gamma)^2}{q^2},
$$

$$
b_k(\varphi)=\frac{b(\varphi)\exp\left(\dfrac{-2i(\alpha-\gamma)(\nu k-1)\varphi}{q}\right)}{q},
$$

$$
f_{k,1}(\varphi)=-\frac{g_k(\varphi)\exp\left(-\dfrac{i(\alpha-\gamma)(\nu k-1)\varphi}{q}\right)}{q}.
$$

Applying the method of variation of constant, we have for systems (1.40) the equivalent system of integral equations

$$
\begin{gathered}
V_0(\varphi)=\int_0^{\varphi}b_0(\varphi,\gamma)\overline{V_0(\gamma)}d\gamma+\int_0^{\varphi}f_0(\varphi,\gamma)d\gamma+c_{0,1}I_{0,0}(\varphi)+c_{0,2}J_{0,0}(\varphi),\\
P_k(\varphi)=\int_0^{\varphi}b_k(\varphi,\gamma)\overline{P_k(\gamma)}d\gamma+\int_0^{\varphi}f_k(\varphi,\gamma)d\gamma+c_{k,1}I_{k,0}(\varphi)+c_{k,2}J_{k,0}(\varphi),
\end{gathered}
\tag{1.41}
$$

where

$$b_0(\varphi,\gamma)=\frac{b_0(\gamma)}{\tau_0}(\exp(\tau_0(\varphi-\gamma))-1), \quad f_0(\varphi,\gamma)=\frac{f_{0,1}(\gamma)}{\tau_0}(\exp(\tau_0(\varphi-\gamma))-1),$$

$I_{0,0}(\varphi)=\exp(\tau_0\varphi)$, $J_{0,0}(\varphi)=1$,

$$b_k(\varphi,\gamma)=\begin{cases}\frac{b_k(\gamma)}{\sqrt{\tau_k}}sh(\sqrt{\tau_k}(\varphi-\gamma)), \text{ if } \tau_k>0,\\ \frac{b_k(\gamma)}{\sqrt{-\tau_k}}\sin(\sqrt{-\tau_k}(\varphi-\gamma)), \text{ if } \tau_k<0,\\ b_k(\gamma)(\varphi-\gamma), \text{ if } \tau_k=0,\end{cases}$$

$$f_k(\varphi,\gamma)=\begin{cases}\frac{f_{k,1}(\gamma)}{\sqrt{\tau_k}}sh(\sqrt{\tau_k}(\varphi-\gamma)), \text{ if } \tau_k>0,\\ \frac{f_{k,1}(\gamma)}{\sqrt{-\tau_k}}\sin(\sqrt{-\tau_k}(\varphi-\gamma)), \text{ if } \tau_k<0,\\ f_{k,1}(\gamma)(\varphi-\gamma), \text{ if } \tau_k=0,\end{cases}$$

$$I_{k,0}(\varphi)=\begin{cases}\exp(\sqrt{\tau_k}\varphi), \text{ if } \tau_k>0,\\ \cos(\sqrt{-\tau_k}\varphi), \text{ if } \tau_k<0,\\ \varphi, \text{ if } \tau_k=0,\end{cases} \quad J_{k,0}(\varphi)=\begin{cases}\exp(-\sqrt{\tau_k}\varphi), \text{ if } \tau_k>0,\\ \sin(\sqrt{-\tau_k}\varphi), \text{ if } \tau_k<0,\\ 1, \text{ if } \tau_k=0.\end{cases}$$

For the construction of solutions of equation (1.41) the following functions and operators are used:

$$I_{k,s}(\varphi)=\int_0^{\varphi}b_k(\varphi,\gamma)\overline{I_{k,s-1}(\gamma)}d\gamma, \; J_{k,s}(\varphi)=\int_0^{\varphi}b_k(\varphi,\gamma)\overline{J_{k,s-1}(\gamma)}d\gamma,$$

$$(B_kf_k)(\varphi)=\int_0^{\varphi}b_k(\varphi,\gamma)\overline{f_k(\gamma)}d\gamma, \; f_{k,2}(\varphi)=\int_0^{\varphi}f_k(\varphi,\gamma)d\gamma, \; 1\le s,$$

$$(B_k^s f)(\varphi)=B_k(B_k^{s-1}f)(\varphi), \; 2\le s, \; (B_k^1 f)(\varphi)=(B_kf)(\varphi), \; 0\le k.$$

For this functions and operator it is easy to check that:

$$(B_k(cI_{k,s}(\varphi)))(\varphi)=\bar{c}I_{k,s+1}(\varphi), \; (B_k(cJ_{k,s}(\varphi)))(\varphi)=\bar{c}J_{k,s+1}(\varphi), \tag{1.42}$$

$$\left|I_{0,s}(\varphi)\right| \le 2^s \frac{\left|b_0\right|_0^s \varphi^s}{\left|\tau_0\right|^s s!}, \quad \left|J_{0,s}(\varphi)\right| \le 2^s \frac{\left|b_0\right|_0^s \varphi^s}{\left|\tau_0\right|^s s!},$$

$$\left|(B_0^s f_{0,2})(\varphi)\right| \le 2^{s+1} \frac{\left|b_0\right|_0^s \left|f_{0,1}\right| \varphi^{s+1}}{\left|\tau_0\right|^{s+1} (s+1)!}, \quad \left|(B_0^s f)(\varphi)\right| \le 2^s \frac{\left|b_0\right|_0^s \left|f\right|_0 \varphi^s}{\left|\tau_0\right|^s (s)!},$$

$$\left|(B_k^s P_k)(\varphi)\right| \le \begin{cases} \left(\dfrac{sh(\sqrt{\tau_k}\varphi_1)\left|b_k\right|_0 \varphi}{\sqrt{\tau_k}}\right)^s \dfrac{\left|P_k\right|_0}{s!}, if\ \tau_k > 0, \\ \left(\dfrac{\left|b_k\right|_0 \varphi}{\sqrt{-\tau_k}}\right)^s \dfrac{\left|P_k\right|_0}{s!}, if\ \tau_k < 0, \\ \left(\sqrt{\left|b_k\right|_0}\varphi\right)^{2s} \dfrac{\left|P_k\right|_0}{(2s)!}, if\ \tau_k = 0, \end{cases}$$

$$\left|(B_k^s f_{k,2})(\varphi)\right| \le \begin{cases} \left(\dfrac{sh(\sqrt{\tau_k}\varphi_1)\left|b_k\right|_0 \varphi}{\sqrt{\tau_k}}\right)^{s+1} \dfrac{\left|f_{k,1}\right|_0}{\left|b_k\right|_0 (s+1)!}, if\ \tau_k > 0, \\ \left(\dfrac{\left|b_k\right|_0 \varphi}{\sqrt{-\tau_k}}\right)^{s+1} \dfrac{\left|f_{k,1}\right|_0}{\left|b_k\right|_0 (s+1)!}, if\ \tau_k < 0, \\ \dfrac{\left|b_k\right|_0^{\ s} \left|f_{k,1}\right|_0 \varphi^{2s+2}}{(2s+2)!}, if\ \tau_k = 0, \end{cases} \tag{1.43}$$

$$\left|I_{k,s}(\varphi)\right| \le \begin{cases} \left(\dfrac{sh(\sqrt{\tau_k}\varphi_1)\left|b_k\right|_0 \varphi}{\sqrt{\tau_k}}\right)^s \cdot \dfrac{\exp(\sqrt{\tau_k}\varphi_1)}{s!}, if\ \tau_k > 0, \\ \left(\dfrac{\left|b_k\right|_0 \varphi}{\sqrt{-\tau_k}}\right)^s \dfrac{1}{s!}, \ if\ \tau_k < 0, \\ \dfrac{\left|b_k\right|_0^{\ s} \varphi^{2s+1}}{(2s+1)!}, if\ \tau_k = 0, \end{cases}$$

$$
\left|J_{k,s}(\varphi)\right| \le \begin{cases} \left(\dfrac{sh(\sqrt{\tau_k}\varphi_1)|b_k|_0\varphi}{\sqrt{\tau_k}}\right)^s \dfrac{1}{s!}, if\ \tau_k > 0, \\ \left(\dfrac{|b_k|_0\varphi}{\sqrt{-\tau_k}}\right)^s \dfrac{1}{s!}, \ if\ \tau_k < 0, \\ \dfrac{|b_k|_0^{\ s}\varphi^{2s}}{(2s)!}, if\ \tau_k = 0, \end{cases}
$$

$0 \le s$, $1 \le k$.

Here $|f|_0 = \|f\|_{C[0,\varphi_1]}$.

Using the specified notations and consider $V_0(\varphi) = P_0(\varphi)$, the system of equations (1.41) is written in the form

$$
P_k(\varphi) = (B_k P_k)(\varphi) + f_{k,2}(\varphi) + c_{k,1} I_{k,0}(\varphi) + c_{k,2} J_{k,0}(\varphi). \tag{1.44}
$$

If we apply the operator $B_k$ to both sides of equation (1.44) we have in view of (1.42)

$$
(B_k P_k)(\varphi) = (B_k^2 P_k)(\varphi) + (B_k f_{k,2})(\varphi) + \bar{c}_{k,1} I_{k,1}(\varphi) + \bar{c}_{k,2} J_{k,1}(\varphi). \tag{1.45}
$$

From (1.44), (1.45) it follows

$$
\begin{aligned} P_k(\varphi) = (B_k^2 P_k)(\varphi) + (B_k f_{k,2})(\varphi) + f_{k,2}(\varphi) + \bar{c}_{k,1} I_{k,1}(\varphi) + \bar{c}_{k,2} J_{k,1}(\varphi) + \\ + c_{k,1} I_{k,0}(\varphi) + c_{k,2} J_{k,0}(\varphi). \end{aligned} \tag{1.46}
$$

If we again apply the operator $B_k$ to both sides of equality (1.46), we have in view of (1.42)

$$
\begin{aligned} (B_k P_k)(\varphi) = (B_k^3 P_k)(\varphi) + (B_k^2 f_{k,2})(\varphi) + (B_k f_{k,2})(\varphi) + c_{k,1} I_{k,2}(\varphi) + \\ + c_{k,2} J_{k,2}(\varphi) + \bar{c}_{k,1} I_{k,1}(\varphi) + \bar{c}_{k,2} J_{k,1}(\varphi). \end{aligned} \tag{1.47}
$$

From (1.44) and (1.47) it follows

$$
\begin{aligned} P_k(\varphi) = (B_k^3 P_k)(\varphi) + (B_k^2 f_{k,2})(\varphi) + (B_k f_{k,2})(\varphi) + f_{k,2}(\varphi) + \\ + c_{k,1} I_{k,2}(\varphi) + c_{k,1} I_{k,0}(\varphi) + c_{k,2} J_{k,2}(\varphi) + c_{k,2} J_{k,0}(\varphi) + \bar{c}_{k,1} I_{k,1}(\varphi) + \bar{c}_{k,2} J_{k,1}(\varphi) \end{aligned}
$$

Continuing this process 2n-1 and 2n-2 times, respectively we receive the following representations for solutions of equation (1.41)

$$P_k(\varphi) = (B_k^{2n} P_k)(\varphi) + \sum_{s=0}^{2n-1} (B_k^s f_{k,2})(\varphi) + c_{k,1} \sum_{s=0}^{n-1} I_{k,2s}(\varphi) +$$

$$+ c_{k,2} \sum_{s=0}^{n-1} J_{k,2s}(\varphi) + \bar{c}_{k,1} \sum_{s=1}^{n} I_{k,2s-1}(\varphi) + \bar{c}_{k,2} \sum_{s=1}^{n} J_{k,2s-1}(\varphi)$$

and (1.48)

$$P_k(\varphi) = (B_k^{2n+1} P_k)(\varphi) + \sum_{s=0}^{2n} (B_k^s f_{k,2})(\varphi) + c_{k,1} \sum_{s=0}^{n} I_{k,2s}(\varphi) +$$

$$+ c_{k,2} \sum_{s=0}^{n} J_{k,2s}(\varphi) + \bar{c}_{k,1} \sum_{s=1}^{n} I_{k,2s-1}(\varphi) + \bar{c}_{k,2} \sum_{s=1}^{n} J_{k,2s-1}(\varphi).$$

If we pass to the limit as $n \to \infty$ in the representations (1.48), by virtue of (1.43) we receive

$$P_k(\varphi) = (BF)_k(\varphi) + c_{k,1} P_{k,2}(\varphi) + c_{k,2} Q_{k,2}(\varphi) + \bar{c}_{k,1} P_{k,1}(\varphi) + \bar{c}_{k,2} Q_{k,1}(\varphi), \tag{1.49}$$

where

$$(BF)_k(\varphi) = \sum_{s=0}^{\infty} (B_k^s f_{k,2})(\varphi), \quad P_{k,2}(\varphi) = \sum_{s=0}^{\infty} I_{k,2s}(\varphi), \quad P_{k,1}(\varphi) = \sum_{s=1}^{\infty} I_{k,2s-1}(\varphi),$$

$$Q_{k,2}(\varphi) = \sum_{s=0}^{\infty} J_{k,2s}(\varphi), \quad Q_{k,1}(\varphi) = \sum_{s=1}^{\infty} J_{k,2s-1}(\varphi).$$

Using the inequalities (1.43), we receive the following estimates:

$$|(BF)_0(\varphi)| \le \frac{|f_{0,1}|_0}{|b_0|_0} \left( \exp\left( \frac{2|b_0|_0}{|\tau_0|} \varphi \right) - 1 \right), \quad |P_{0,2}(\varphi)| \le ch\left( \frac{2|b_0|_0}{|\tau_0|} \varphi \right),$$

$$|P_{0,1}(\varphi)| \le sh\left( \frac{2|b_0|_0}{|\tau_0|} \varphi \right), \quad |Q_{0,2}(\varphi)| \le ch\left( \frac{2|b_0|_0}{|\tau_0|} \varphi \right), \quad |Q_{0,1}(\varphi)| \le sh\left( \frac{2|b_0|_0}{|\tau_0|} \varphi \right),$$

$$|(BF)_k(\varphi)| \le \begin{cases} \dfrac{|f_{k,1}|_0}{|b_k|_0} \left( \exp\left( \dfrac{|b_k|_0 sh(\sqrt{\tau_k} \varphi_1)}{\sqrt{\tau_k}} \varphi \right) - 1 \right), if\ \tau_k > 0, \\ \dfrac{|f_{k,1}|_0}{|b_k|_0} \left( \exp\left( \dfrac{|b_k|_0}{\sqrt{-\tau_k}} \varphi \right) - 1 \right), if\ \tau_k < 0, \\ \dfrac{|f_{k,1}|_0}{|b_k|_0} (\exp(\sqrt{|b_k|_0} \varphi) - 1), if\ \tau_k = 0, \end{cases}$$

$$
\left|P_{k,2}(\varphi)\right| \le \begin{cases} \exp(\sqrt{\tau_k}\varphi_1) ch\left( \dfrac{|b_k|_0 sh(\sqrt{\tau_k}\varphi_1)}{\sqrt{\tau_k}} \varphi \right), if\ \tau_k > 0, \\ ch\left( \dfrac{|b_k|_0}{\sqrt{-\tau_k}} \varphi \right), if\ \tau_k < 0, \\ \dfrac{1}{\sqrt{|b_k|_0}} sh(\sqrt{|b_k|_0}\varphi), if\ \tau_k = 0, \end{cases}
$$

$$
\left|P_{k,1}(\varphi)\right| \le \begin{cases} \exp(\sqrt{\tau_k}\varphi_1) sh\left( \dfrac{|b_k|_0 sh(\sqrt{\tau_k}\varphi_1)}{\sqrt{\tau_k}} \varphi \right), if\ \tau_k > 0, \\ sh\left( \dfrac{|b_k|_0}{\sqrt{-\tau_k}} \varphi \right), if\ \tau_k < 0, \\ \dfrac{1}{\sqrt{|b_k|_0}} sh(\sqrt{|b_k|_0}\varphi), if\ \tau_k = 0, \end{cases}
$$

$$
\left|Q_{k,2}(\varphi)\right| \le \begin{cases} \exp(\sqrt{\tau_k}\varphi_1) ch\left( \dfrac{|b_k|_0 sh(\sqrt{\tau_k}\varphi_1)}{\sqrt{\tau_k}} \varphi \right), if\ \tau_k > 0, \\ ch\left( \dfrac{|b_k|_0}{\sqrt{-\tau_k}} \varphi \right), if\ \tau_k < 0, \\ ch(\sqrt{|b_k|_0}\varphi), if\ \tau_k = 0, \end{cases}
$$

$$
\left|Q_{k,1}(\varphi)\right| \le \begin{cases} \exp(\sqrt{\tau_k}\varphi_1) sh\left( \dfrac{|b_k|_0 sh(\sqrt{\tau_k}\varphi_1)}{\sqrt{\tau_k}} \varphi \right), if\ \tau_k > 0, \\ sh\left( \dfrac{|b_k|_0}{\sqrt{-\tau_k}} \varphi \right), if\ \tau_k < 0, \\ ch(\sqrt{|b_k|_0}\varphi), if\ \tau_k = 0, \end{cases}
$$

By means of these estimates it is easy to show, that the function $\mathrm{P}_k(\varphi)$, given by formula (1.49), is a solution of equation (1.40) from the class $C^2[0, \varphi_1]$.

From (1.37), (1.39) and (1.49) we receive

$$V(r,\varphi)=\sum_{k=0}^{\infty}\exp(a_k\varphi)[(BF)_k(\varphi)+c_{k,1}P_{k,2}(\varphi)+c_{k,2}Q_{k,2}(\varphi)+$$
$$+\bar{c}_{k,1}P_{k,1}(\varphi)+\bar{c}_{k,2}Q_{k,1}(\varphi)]\frac{r^{\nu k}}{k!}, \qquad (1.50)$$

where $a_k=\begin{cases}0, \ if \ \ k=0,\\ \dfrac{i(\alpha-\gamma)(\nu k-1)}{\alpha-\beta+\gamma}, \ if \ \ 1\le k.\end{cases}$

It is easy to check, that the function, given by formula (1.50), is a solution of equation (1.34) from the class (1.35).

Thus, the following result holds.

***Theorem 1.3.*** *When $\beta\neq\alpha+\gamma$ the equation (1.34) is solvable in the class (1.35).The general solution of equation (1.34), from the class (1.35) is given by formula (1.50).*

2. Let $\beta=\alpha+\gamma$. Using formulas (1.2), equation (1.34) is written in polar coordinates in the form

$$(\alpha+\beta+\gamma)r^2\frac{\partial^2 V}{\partial r^2}-(\alpha-\gamma)2i\frac{\partial V}{\partial\varphi}+(\alpha-\gamma)2ir\frac{\partial^2 V}{\partial r\partial\varphi}+b(\varphi)\overline{V}=g(r,\varphi). \qquad (1.51)$$

The solution of equation (1.51), from the class (1.35) are searched for in the form (1.37).

Substituting (1.37) in (1.51) and comparing coefficients of the same power of $r^{\nu}$ lead for $0\le k$ to the equation

$$2i(\alpha-\gamma)(\nu k-1)V_k'(\varphi)+(\alpha+\beta+\gamma)\nu k(\nu k-1)V_k(\varphi)=g_k(\varphi)-b(\varphi)\overline{V_k(\varphi)}. \qquad (1.52)$$

This equation is considered in various cases for the parameters $\alpha,\beta,\gamma$ *and* $\nu$.

a) If $\alpha=\gamma$ or $\nu k=1$, then equation (1.52) has the form

$$\upsilon_k V_k(\varphi)+b(\varphi)\overline{V_k(\varphi)}=g_k(\varphi), \qquad (1.53)$$

where

$$\upsilon_k = (\alpha + \beta + \gamma)\nu k(\nu k - 1)$$

This equation has unique solution just only, when

$$|b(\varphi)|^2 \neq |\upsilon_k|^2 \quad (1.54)$$

By this condition the solution of equation (1.53) is found in the form

$$V_k(\varphi) = \frac{g_k(\varphi)\overline{\upsilon_k} - \overline{g_k(\varphi)}b(\varphi)}{|\upsilon_k|^2 - |b(\varphi)|^2}. \quad (1.55)$$

b) In case

$$|b(\varphi)|^2 = |\upsilon_k|^2 \quad (1.56)$$

for the solvability of equation (1.53) it is necessary and sufficient that the conditions

$$\begin{aligned} &\mathrm{Re}(\overline{g_k(\varphi)}(\upsilon_k - b(\varphi))) = 0, \\ &\mathrm{Im}(\overline{g_k(\varphi)}(\upsilon_k + b(\varphi))) = 0 \end{aligned} \quad (1.57)$$

are fulfilled. If these conditions are satisfied the solution of equation (1.53) is found as

$$V_k(\varphi) = \begin{cases} \dfrac{\mathrm{Re}\, g_k(\varphi) + i\theta_k(\varphi)(\upsilon_k + b(\varphi))}{\mathrm{Re}(\upsilon_k + b(\varphi))}, if \ \mathrm{Re}(\upsilon_k + b(\varphi)) \neq 0, \\ i\dfrac{\mathrm{Re}\, g_k(\varphi) - \theta_k(\varphi)(\upsilon_k + b(\varphi))}{\mathrm{Im}(-\upsilon_k + b(\varphi))}, if \ \mathrm{Im}(-\upsilon_k + b(\varphi)) \neq 0, \\ c_k(\varphi), if \ \mathrm{Re}(\upsilon_k + b(\varphi)) = 0, \ \mathrm{Im}(-\upsilon_k + b(\varphi)) = 0, \end{cases} \quad (1.58)$$

where $\theta_k(\varphi)$ is any real, $c_k(\varphi)$ is any complex function.

c) When $\alpha \neq \gamma$ and $\nu k \neq 1$ equation (1.52) has the form

$$V_k'(\varphi) - \tau_k V_k(\varphi) = b_k(\varphi)\overline{V_k(\varphi)} + g_{k,0}(\varphi), \quad (1.59)$$

where

$$\tau_k = i\frac{(\alpha + \beta + \gamma)\nu k}{2(\alpha - \beta)}, \quad g_{k,0}(\varphi) = -i\frac{g_k(\varphi)}{2(\alpha - \gamma)(\nu k - 1)},$$

$$b_k(\varphi) = i\frac{b(\varphi)}{2(\alpha-\gamma)(\nu k-1)}$$

Solving equation (1.59) by applying the method of variation of constant, we obtain the integral equation

$$V_k(\varphi) = \int_0^{\varphi} b_k(\varphi,\gamma)\overline{V_k(\gamma)}d\gamma + \int_0^{\varphi} f_k(\varphi,\gamma)d\gamma + c_k I_{k,0}(\varphi), \qquad (1.60)$$

where

$b_k(\varphi,\gamma) = b_k(\gamma)\exp(\tau_k(\varphi-\gamma))$ , $f_k(\varphi,\gamma) = g_{\mathrm{k},0}(\gamma)\exp(\tau_k(\varphi-\gamma))$,

$I_{k,0}(\varphi) = \exp(\tau_k\varphi)$

For the construction of solutions of equation (1.60) the following functions and operators are used:

$$I_{k,s}(\varphi) = \int_0^{\varphi} b_k(\varphi,\gamma)\overline{I_{k,s-1}(\gamma)}d\gamma, \quad (B_k f_k)(\varphi) = \int_0^{\varphi} b_k(\varphi,\gamma)\overline{f_k(\gamma)}d\gamma,$$

$$(B_k^0 F_k)(\varphi) = F_k(\varphi) = \int_0^{\varphi} f_k(\varphi,\gamma)d\gamma, \quad (B_k^s f)(\varphi) = B_k(B_k^{s-1} f)(\varphi), \ 1 \le s.$$

For these functions it is easy to check that

$$\left|I_{k,s}(\varphi)\right| \le \frac{\left|b_k\right|_0^s}{\left|\tau_k\right|^s}\exp(\tau_k\varphi), \quad \left|B_k^s F_k(\varphi)\right| \le \frac{\left|b_k\right|_0^s \left|g_{k,0}\right|_0 \varphi^{s+1}}{(s+1)!}, \qquad (1.61)$$

$$\left|B_k^s f(\varphi)\right| \le \frac{\left|b_k\right|_0^s \left|f\right|_0 \varphi^s}{s!}.$$

Here $\left|f\right|_0 = \left\|f\right\|_{C[0,\varphi_1]}$.

Using the specified notations, equation (1.60) is written in the form

$$V_k(\varphi) = (B_k V_k)(\varphi) + F_k(\varphi) + c_k I_{k,0}(\varphi) \qquad (1.62)$$

If we apply the operator $B_k$ to both sides of equation (1.62), we have in view of (1.42)

$$(B_k V_k)(\varphi) = (B_k^2 V_k)(\varphi) + (B_k F_k)(\varphi) + \overline{c}_k I_{k,1}(\varphi) \qquad (1.63)$$

From (1.62), (1.63) it follows

$$V_k(\varphi) = (B_k^2 V_k)(\varphi) + (B_k F_k)(\varphi) + F_k(\varphi) + \overline{c}_k I_{k,1}(\varphi) + c_k I_{k,0}(\varphi) \quad (1.64)$$

If we again apply the operator $B_k$ to both sides of equality (1.64), we have in view of (1.42)

$$(B_k V_k)(\varphi) = (B_k^3 V_k)(\varphi) + (B_k^2 F_k)(\varphi) + (B_k F_k)(\varphi) + c_k I_{k,2}(\varphi) + \overline{c}_k I_{k,1}(\varphi) \quad (1.65)$$

From (1.62) and (1.65) it follows

$$V_k(\varphi) = (B_k^3 V_k)(\varphi) + (B_k^2 F_k)(\varphi) + (B_k F_k)(\varphi) + F_k(\varphi) + c_k(I_{k,2}(\varphi) + I_{k,0}(\varphi)) + \overline{c}_k I_{k,1}(\varphi).$$

Continuing this process 2n-1 and 2n-2 time, respectively we receive the representations for solutions of equation (1.60)

$$V_k(\varphi) = (B_k^{2n} V_k)(\varphi) + \sum_{s=0}^{2n-1} (B_k^s F_k)(\varphi) + c_k \sum_{s=0}^{n-1} I_{k,2s}(\varphi) + \overline{c}_k \sum_{s=1}^{n} I_{k,2s-1}(\varphi)$$

and (1.66)

$$V_k(\varphi) = (B_k^{2n+1} V_k)(\varphi) + \sum_{s=0}^{2n} (B_k^s F_k)(\varphi) + c_k \sum_{s=0}^{n} I_{k,2s}(\varphi) + \overline{c}_k \sum_{s=1}^{n} I_{k,2s-1}(\varphi).$$

If we pass to the limit as $n \to \infty$ in the representations (1.66), by virtue of (1.61) we receive

$$V_k(\varphi) = F_{k,1}(\varphi) + c_k P_{k,2}(\varphi) + \overline{c}_k P_{k,1}(\varphi), \quad (1.67)$$

where $F_{k,1}(\varphi) = \sum_{s=0}^{\infty} (B_k^s F_k)(\varphi)$, $P_{k,2}(\varphi) = \sum_{s=0}^{\infty} I_{k,2s}(\varphi)$, $P_{k,1}(\varphi) = \sum_{s=1}^{\infty} I_{k,2s-1}(\varphi)$,

$c_k$, $0 \le k$, are any complex numbers.

Using the unequalities (1.61), we get the estimates

$$|F_{k,1}(\varphi)| \le \frac{|g_{k,0}|_0}{|b_k|_0} \exp(|b_k|_0 \varphi), \quad |P_{k,2}(\varphi)| \le \exp(\tau_k \varphi) ch\left(\frac{|b_k|_0}{|\tau_k|_0}\right),$$

$$|P_{k,1}(\varphi)| \le \exp(\tau_k \varphi) sh\left(\frac{|b_k|_0}{|\tau_k|_0}\right).$$

By means of these estimates it is easy to show, that the functions $V_k(\varphi)$, given by formula (1.67), is a solution of equation (1.59) from the class $C^1[0, \varphi_1]$.

From (1.67) and (1.37) we find

$$V(r,\varphi)=\sum_{k=0}^{\infty}(F_{k,1}(\varphi)+c_k P_{k,2}(\varphi)+\bar{c}_k P_{k,1}(\varphi))\frac{r^{\nu k}}{k!} \qquad (1.68)$$

Thus, the following results holds.

***Theorem 1.4.*** *Let $\beta=\alpha+\gamma$. Of one of the condition holds: 1) $\alpha=\gamma$ and $|b(\varphi)|\neq|\upsilon_k|$ or $\nu k=1$ and $|b(\varphi)|\neq|\upsilon_k|$ 2) $\alpha\neq\gamma, \nu k\neq 1$ equation (1.34) is solvable in the class (1.35). If 1) is satisfied equation (1.34) has a unique solution in the class (1.35). This solution is found by the formulas (1.37), (1.55). If 2) is satisfied equation (1.34) has a general solution. This solution is given by the formulas (1.37), (1.67). If $\alpha=\gamma$ or $\nu k=1$ and $|b(\varphi)|=|\upsilon_k|$ in some point $0\leq\varphi\leq\varphi_1$ for the solvability of equation (1.34) in the class (1.35) condition (1.57) is necessary and sufficient. In this case the solution of equation (1.34) in the class (1.35) is given by the formulas (1.37), (1.58).*

## 1.3 Second order partial differential equations in the plane with Fuchs operator in the main part and specified right hand side

Let $0<\varphi_1\leq 2\pi$ and $G=\{z=re^{i\varphi}:0\leq r<\infty, 0<\varphi<\varphi_1\}$. Consider the equation

$$4a(\varphi)\bar{z}^2V_{\bar{z}\bar{z}}+4b(\varphi)z\bar{z}V_{z\bar{z}}+4c(\varphi)z^2V_{zz}+d(\varphi)\overline{V}=f(\varphi)r^{\lambda}, \quad z\in G,\ z=re^{i\varphi}, \qquad (1.69)$$

in $G$, when $a(\varphi), b(\varphi), c(\varphi), d(\varphi), f(\varphi)\in C[0,\varphi_1]$, $\lambda$ is a real parameters. The equation (1.69) when $a(\varphi)=const.$, $b(\varphi)=const.$ and $c(\varphi)=const.$ are studied in the section 1.1.

The solution of equation (1.69) are searched for in the class (1.3), where $1<p<\dfrac{2}{2-\lambda}$, *if* $\lambda<2$ *and* $p>1$, *if* $\lambda\geq 2$.

1. Let $b(\varphi)\neq a(\varphi)+c(\varphi)$. Using formulas (1.2), equation (1.69) in polar coordinates is written in the form

$$\begin{aligned}&(a(\varphi)+b(\varphi)+c(\varphi))r^2\frac{\partial^2 V}{\partial r^2}-2i(a(\varphi)-c(\varphi))\frac{\partial V}{\partial\varphi}+2i(a(\varphi)-c(\varphi))r\frac{\partial^2 V}{\partial r\partial\varphi}-\\&-(a(\varphi)-b(\varphi)+c(\varphi))r\frac{\partial V}{\partial r}-(a(\varphi)-b(\varphi)+c(\varphi))\frac{\partial^2 V}{\partial\varphi^2}+d(\varphi)\overline{V}=f(\varphi)r^{\lambda}.\end{aligned}\qquad(1.70)$$

We are searching for the solution of equation (1.70) in the form (1.5), where $\psi(\varphi)$ is a new unknown function from $C^2[0,\varphi_1]$, satisfying the equation

$$\begin{aligned}&-(a(\varphi)-b(\varphi)+c(\varphi))\psi''-2i(a(\varphi)-c(\varphi))(1-\lambda)\psi'+\\&+((a(\varphi)+b(\varphi)+c(\varphi))(\lambda^2-\lambda)-\lambda(a(\varphi)-b(\varphi)+c(\varphi)))\psi=\\&=f(\varphi)-d(\varphi)\overline{\psi}.\end{aligned}\qquad(1.71)$$

Let $\psi_1(\varphi)$, $\psi_2(\varphi)$ be two linearly independent solutions of the related homogeneous equation

$$\begin{aligned}&-(a(\varphi)-b(\varphi)+c(\varphi))\psi''-2i(a(\varphi)-c(\varphi))(1-\lambda)\psi'+\\&+((a(\varphi)+b(\varphi)+c(\varphi))(\lambda^2-\lambda)-\lambda(a(\varphi)-b(\varphi)+c(\varphi)))\psi=0.\end{aligned}$$

Solving equation (1.71) by applying the method of variation of constant we obtain the integral equation

$$\psi(\varphi)=\int_0^{\varphi}b(\varphi,\gamma)\overline{\psi(\gamma)}d\gamma+\int_0^{\varphi}f(\varphi,\gamma)d\gamma+c_1I_0(\varphi)+c_2J_0(\varphi),\qquad(1.72)$$

where

$$b(\varphi,\gamma)=\frac{d(\gamma)(\psi_2(\varphi)\psi_1(\gamma)-\psi_1(\varphi)\psi_2(\gamma))}{(a(\gamma)-b(\gamma)+c(\gamma))(\psi_1(\gamma)\psi_2'(\gamma)-\psi_1'(\gamma)\psi_2(\gamma))},$$

$$f(\varphi,\gamma)=\frac{f(\gamma)(\psi_1(\varphi)\psi_2(\gamma)-\psi_2(\varphi)\psi_1(\gamma))}{(a(\gamma)-b(\gamma)+c(\gamma))(\psi_1(\gamma)\psi_2'(\gamma)-\psi_1'(\gamma)\psi_2(\gamma))}.$$

For the construction of solutions of equation (1.72) the following functions and operators are used:

$$I_0(\varphi)=\psi_1(\varphi),\quad J_0(\varphi)=\psi_2(\varphi),\quad I_k(\varphi)=\int_0^{\varphi}b(\varphi,\gamma)\overline{I_{k-1}}(\gamma)d\gamma,$$

$$J_k(\varphi)=\int_0^{\varphi}b(\varphi,\gamma)\overline{J_{k-1}}(\gamma)d\gamma,\quad 1\le k,$$

$$(Bf)(\varphi)=\int_0^{\varphi} b(\varphi,\gamma)\overline{f(\gamma)}d\gamma,\ F(\varphi)=\int_0^{\varphi} f(\varphi,\gamma)d\gamma,\ (B^0F)(\varphi)=\int_0^{\varphi} f(\varphi,\gamma)d\gamma$$

$(B^k f)(\varphi)=B(B^{k-1}f)(\varphi)$, $2\le k$, $(B^1 f)(\varphi)=(Bf)(\varphi)$.

For these functions it is easy to check that

$$\left|(B^k f)(\varphi)\right|\le |b|_0^k\frac{\varphi^k}{k!},\ \left|(B^k F)(\varphi)\right|\le |f|_0|b|_0^k\frac{\varphi^{k+1}}{(k+1)!},$$

$$|I_k(\varphi)|\le |b|_0^k|\psi_1|_0\frac{\varphi^k}{k!},\ |J_k(\varphi)|\le |b|_0^k|\psi_2|_0\frac{\varphi^k}{k!},\ 0\le k. \tag{1.73}$$

Here $|f|_0=\|f\|_{C[0,\varphi_1]}$.

Using the specified notations, equation (1.72) is written in the form

$$\psi(\varphi)=(B\psi)(\varphi)+F(\varphi)+c_1I_0(\varphi)+c_2J_0(\varphi). \tag{1.74}$$

If we apply the operator $B$ to both sides of equation (1.74), we have in view of (1.9)

$$(B\psi)(\varphi)=(B^2\psi)(\varphi)+(BF)(\varphi)+\bar{c}_1I_1(\varphi)+\bar{c}_2J_1(\varphi). \tag{1.75}$$

From (1.74), (1.75) it follows

$$\psi(\varphi)=(B^2\psi)(\varphi)+(BF)(\varphi)+F(\varphi)+\bar{c}_1I_1(\varphi)+\bar{c}_2J_1(\varphi)+ \\ +c_1I_0(\varphi)+c_2J_0(\varphi). \tag{1.76}$$

If we again apply the operator $B$ to both sides of equality (1.76), we have in view of (1.9)

$$(B\psi)(\varphi)=(B^3\psi)(\varphi)+(B^2F)(\varphi)+(BF)(\varphi)+c_1I_2(\varphi)+c_2J_2(\varphi)+ \\ +\bar{c}_1I_1(\varphi)+\bar{c}_2J_1(\varphi). \tag{1.77}$$

From (1.74) and (1.77) it follows

$$\psi(\varphi)=(B^3\psi)(\varphi)+(B^2F)(\varphi)+(BF)(\varphi)+F(\varphi)+c_1(I_2(\varphi)+I_0(\varphi))+ \\ +c_2(J_2(\varphi)+J_0(\varphi))+\bar{c}_1I_1(\varphi)+\bar{c}_2J_1(\varphi).$$

Continuing this process 2n-1 and 2n-2 times, respectively we receive the representations for solutions of equation (1.71)

$$\psi(\varphi)=(B^{2n}\psi)(\varphi)+\sum_{k=0}^{2n-1}B^kF(\varphi)+c_1\sum_{k=0}^{n-1}I_{2k}(\varphi)+c_2\sum_{k=0}^{n-1}J_{2k}(\varphi)+$$

$$+\bar{c}_1\sum_{k=1}^{n}I_{2k-1}(\varphi)+\bar{c}_2\sum_{k=1}^{n}J_{2k-1}(\varphi)$$

and (1.78)

$$\psi(\varphi)=(B^{2n+1}\psi)(\varphi)+\sum_{k=0}^{2n}B^kF(\varphi)+c_1\sum_{k=0}^{n}I_{2k}(\varphi)+c_2\sum_{k=0}^{n}J_{2k}(\varphi)+$$

$$+\bar{c}_1\sum_{k=1}^{n}I_{2k-1}(\varphi)+\bar{c}_2\sum_{k=1}^{n}J_{2k-1}(\varphi)$$

If we pass to the limit as $n\to\infty$ in the representations (1.78), by virtue of (1.73) we receive

$$\psi(\varphi)=F_1(\varphi)+c_1P_2(\varphi)+c_2Q_2(\varphi)+\bar{c}_1P_1(\varphi)+\bar{c}_2Q_1(\varphi), \tag{1.79}$$

where

$$F_1(\varphi)=\sum_{k=0}^{\infty}B^kF(\varphi),\ P_2(\varphi)=\sum_{k=0}^{\infty}I_{2k}(\varphi),\quad P_1(\varphi)=\sum_{k=1}^{\infty}I_{2k-1}(\varphi),\ Q_2(\varphi)=\sum_{k=0}^{\infty}J_{2k}(\varphi),$$

$$Q_1(\varphi)=\sum_{k=1}^{\infty}J_{2k-1}(\varphi).$$

Using inequalities (1.73), we receive the estimates

$$|F_1(\varphi)|\le\frac{|f|_0}{|b|_0}\exp(|b|_0\varphi),\ |P_2(\varphi)|\le|\psi_1|ch(|b|_0\varphi),\ |P_1(\varphi)|\le|\psi_1|sh(|b|_0\varphi),$$

$$|Q_2(\varphi)|\le|\psi_2|ch(|b|_0\varphi),\ |Q_1(\varphi)|\le|\psi_2|sh(|b|_0\varphi).$$

By means of these estimates it is easy to show, that the function $\psi(\varphi)$, given by formula (1.79), is a solution of equation (1.71) from the class $C^2[0,\varphi_1]$.

From (1.5) and (1.79) we find

$$V(r,\varphi)=r^{\lambda}(F_1(\varphi)+c_1P_2(\varphi)+c_2Q_2(\varphi)+\bar{c}_1P_1(\varphi)+\bar{c}_2Q_1(\varphi)) \tag{1.80}$$

Thus, the following result holds.

***Theorem 1.5.*** *When* $b(\varphi) \neq a(\varphi) + c(\varphi)$ *the equation (1.69) is solvable in the class (1.3). The general solution of equation (1.69) from the class (1.3) is given by formula (1.80).*

2. Let $b(\varphi) = a(\varphi) + c(\varphi)$. Using formulas (1.2), equation (1.69) in polar coordinates is written in the form

$$(a(\varphi)+b(\varphi)+c(\varphi))r^2\frac{\partial^2 V}{\partial r^2} - 2i(a(\varphi)-c(\varphi))\frac{\partial V}{\partial \varphi} + \\ + 2i(a(\varphi)-c(\varphi))r\frac{\partial^2 V}{\partial r \partial \varphi} + d(\varphi)\overline{V} = f(\varphi)r^{\lambda} \tag{1.81}$$

We are searching the solution of equation (1.81) in the form (1.5), and receive

$$-2i(a(\varphi)-c(\varphi))(1-\lambda)\psi' + (a(\varphi)+b(\varphi)+c(\varphi))(\lambda^2-\lambda)\psi = \\ = f(\varphi) - d(\varphi)\overline{\psi(\varphi)} \tag{1.82}$$

Consider any case of equation (1.82) depending on the values of the functions $a(\varphi), b(\varphi), c(\varphi)$ and the parameter $\lambda$.

1. If $a(\varphi) = c(\varphi)$ or $\lambda = 1$, then equation (1.82) has the form

$$A(\varphi)\psi = f(\varphi) - d(\varphi)\overline{\psi}, \tag{1.83}$$

where $A(\varphi) = (a(\varphi)+b(\varphi)+c(\varphi))(\lambda^2-\lambda)$.

Solving equation (1.83) when $|d(\varphi)| \neq |A(\varphi)|$, we have

$$\psi(\varphi) = \frac{A(\varphi)f(\varphi) - d(\varphi)\overline{f(\varphi)}}{|A(\varphi)| - |d(\varphi)|^2} \tag{1.84}$$

In case, when $|d(\varphi)| = |A(\varphi)|$, for the solvability of equation (1.83) the condition

$$\mathrm{Re}(\overline{f(\varphi)}(-d(\varphi) + A(\varphi))) = 0, \mathrm{Im}(\overline{f(\varphi)}(d(\varphi) + A(\varphi))) = 0 \tag{1.85}$$

are necessary and sufficient. When these conditions are fulfilled the solution can be found by the formula

$$\psi(\varphi)=\begin{cases}\dfrac{\operatorname{Re} f(\varphi)+i\theta(\varphi)(\overline{A(\varphi)}+d(\varphi))}{\operatorname{Re}(d(\varphi)+A(\varphi))},\ \textit{if}\ \operatorname{Re}(d(\varphi)+A(\varphi))\neq 0,\\ i\dfrac{\operatorname{Re} f(\varphi)-\theta(\varphi)(\overline{A(\varphi)}+d(\varphi))}{\operatorname{Im}(d(\varphi)-A(\varphi))},\textit{if}\ \operatorname{Im}(d(\varphi)-A(\varphi))\neq 0,\\ \psi_1(\varphi),\textit{if}\ \operatorname{Re}(d(\varphi)+A(\varphi))=0\ \ \textit{and}\ \ \operatorname{Im}(d(\varphi)-A(\varphi))=0,\end{cases} \tag{1.86}$$

where $\theta(\varphi)$, $\psi_1(\varphi)$ are arbitrary functions.

2. If $a(\varphi)\neq c(\varphi)$ and $\lambda\neq 1$, then the equation (1.82) is written in the form

$$\psi'+\tau(\varphi)\psi=d_1(\varphi)\overline{\psi}+f_1(\varphi), \tag{1.87}$$

where

$$\tau(\varphi)=-i\frac{(a(\varphi)+b(\varphi)+c(\varphi))\lambda}{2(a(\varphi)-c(\varphi))},\quad d_1(\varphi)=i\frac{d(\varphi)}{2(a(\varphi)-c(\varphi))(\lambda-1)},$$

$$f_1(\varphi)=-i\frac{f(\varphi)}{2(\alpha(\varphi)-\gamma(\varphi))(\lambda-1)}.$$

Solving equation (1.87) by applying the method of variation of constant, we have the integral equation

$$\psi(\varphi)=\int_0^{\varphi} b(\varphi,\gamma)\overline{\psi(\gamma)}d\gamma+\int_0^{\varphi} f(\varphi,\gamma)d\gamma+cI_0(\varphi), \tag{1.88}$$

where

$$b(\varphi,\gamma)=d_1(\gamma)\exp\left(-\int_{\gamma}^{\varphi}\tau(\gamma_1)d\gamma_1\right),\ f(\varphi,\gamma)=f_1(\gamma)\exp\left(-\int_{\gamma}^{\varphi}\tau(\gamma_1)d\gamma_1\right),$$

$$I_0(\varphi)=\exp\left(-\int_0^{\varphi}\tau(\gamma)d\gamma\right).$$

For the construction of a solution of (1.88) the following functions and operators are used

$$I_k(\varphi)=\int_0^{\varphi} b(\varphi,\gamma)\overline{I_{k-1}(\gamma)}d\gamma,\ (Bf)(\varphi)=\int_0^{\varphi} b(\varphi,\gamma)\overline{f(\gamma)}d\gamma,$$

$$(BF)(\varphi)=F(\varphi)=\int_0^{\varphi} f(\varphi,\gamma)d\gamma\,,\;\; (B^k f)(\varphi)=B(B^{k-1}f)(\varphi)\,,\;\; 1\le k.$$

For these functions it is easy to check that

$$\left|I_k(\varphi)\right|\le \frac{1}{k!}\left|d_1\right|_0^k \varphi^k \exp\left(-k\int_{\gamma}^{\varphi}\operatorname{Im}\frac{(a(\gamma_1)+b(\gamma_1)+c(\gamma_1))\lambda}{2(a(\gamma_1)-c(\gamma_1))}d\gamma_1\right)\cdot$$

$$\cdot\exp\left(-\int_0^{\varphi}\operatorname{Im}\frac{(a(\gamma_1)+b(\gamma_1)+c(\gamma_1))\lambda}{2(a(\gamma_1)-c(\gamma_1))}d\gamma_1\right),$$

$$\left|(B^k F)(\varphi)\right|\le \frac{1}{(k+1)!}\left|d_1\right|_0^k\left|f_1\right|_0\varphi^{k+1}\cdot$$

$$\cdot\exp\left(-(k+1)\int_{\gamma}^{\varphi}\operatorname{Im}\frac{(a(\gamma_1)+b(\gamma_1)+c(\gamma_1))\lambda}{2(a(\gamma_1)-c(\gamma_1))}d\gamma_1\right), \tag{1.89}$$

$$\left|(B^k f)(\varphi)\right|\le \frac{1}{k!}\left|d_1\right|_0^k\left|f\right|_0\varphi^k \exp\left(-k\int_{\gamma}^{\varphi}\operatorname{Im}\frac{(a(\gamma_1)+b(\gamma_1)+c(\gamma_1))\lambda}{2(a(\gamma_1)-c(\gamma_1))}d\gamma_1\right).$$

Using these specified notations, equation (1.88) is written in the form

$$\psi(\varphi)=(B\psi)(\varphi)+F(\varphi)+cI_0(\varphi). \tag{1.90}$$

If we apply the operator $B$ to both sides of equation (1.90), we have in view of (1.9)

$$(B\psi)(\varphi)=(B^2\psi)(\varphi)+(BF)(\varphi)+\bar{c}I_1(\varphi). \tag{1.91}$$

From (1.90), (1.91) it follows

$$\psi(\varphi)=(B^2\psi)(\varphi)+(BF)(\varphi)+F(\varphi)+\bar{c}I_1(\varphi)+cI_0(\varphi). \tag{1.92}$$

If we again apply the operator $B$ to both sides of equality (1.92), we have in view of (1.9)

$$(B\psi)(\varphi)=(B^3\psi)(\varphi)+(B^2F)(\varphi)+(BF)(\varphi)+cI_2(\varphi)+\bar{c}I_1(\varphi) \tag{1.93}$$

From (1.90) and (1.93) it follows

$$\psi(\varphi)=(B^3\psi)(\varphi)+(B^2F)(\varphi)+(BF)(\varphi)+F(\varphi)+c(I_2(\varphi)+I_0(\varphi))+\bar{c}I_1(\varphi).$$

Continuing this process 2n-1 and 2n-2 time, respectively we receive the following representations for solutions of equation (1.87)

$$\psi(\varphi)=(B^k\psi)(\varphi)+\sum_{k=0}^{2n-1}(B^kF)(\varphi)+c\sum_{k=0}^{n-1}I_{2k}(\varphi)+\bar{c}\sum_{k=1}^{n}I_{2k-1}(\varphi)$$

and (1.94)

$$\psi(\varphi)=(B^{2k+1}\psi)(\varphi)+\sum_{k=0}^{2n}(B^kF)(\varphi)+c\sum_{k=0}^{n}I_{2k}(\varphi)+\bar{c}\sum_{k=1}^{n}I_{2k-1}(\varphi)$$

If we pass to the limit as $n\to\infty$ in the representations (1.94), by virtue of (1.89) we receive

$$\psi(\varphi)=F_1(\varphi)+cP_2(\varphi)+\bar{c}P_1(\varphi), \tag{1.95}$$

where $F_1(\varphi)=\sum_{k=0}^{\infty}(B^kF)(\varphi)$, $P_2(\varphi)=\sum_{k=0}^{\infty}I_{2k}(\varphi)$, $P_1(\varphi)=\sum_{k=1}^{\infty}I_{2k-1}(\varphi)$,

$c$ an arbitrary complex number.

Using the inequalities (1.89), we receive the estimates

$$|F_1(\varphi)|\le\frac{|f_1|_0}{|d_1|_0}\exp\left(|d_1|_0\exp\left((-\int_\gamma^\varphi\operatorname{Im}\frac{(a(\gamma_1)+b(\gamma_1)+c(\gamma_1))\lambda}{2(a(\gamma_1)-c(\gamma_1))}d\gamma_1)\varphi\right)\right),$$

$$|P_2(\varphi)|\le\exp(-\int_0^\varphi\operatorname{Im}\frac{(a(\gamma_1)+b(\gamma_1)+c(\gamma_1))\lambda}{2(a(\gamma_1)-c(\gamma_1))}d\gamma_1)$$

$$ch\left(|d_1|_0\exp\left((-\int_\gamma^\varphi\operatorname{Im}\frac{(a(\gamma_1)+b(\gamma_1)+c(\gamma_1))\lambda}{2(a(\gamma_1)-c(\gamma_1))}d\gamma_1)\varphi\right)\right),$$

$$|P_1(\varphi)|\le\exp(-\int_0^\varphi\operatorname{Im}\frac{(a(\gamma_1)+b(\gamma_1)+c(\gamma_1))\lambda}{2(a(\gamma_1)-c(\gamma_1))}d\gamma_1)$$

$$sh\left(|d_1|_0\exp\left((-\int_\gamma^\varphi\operatorname{Im}\frac{(a(\gamma_1)+b(\gamma_1)+c(\gamma_1))\lambda}{2(a(\gamma_1)-c(\gamma_1))}d\gamma_1)\varphi\right)\right).$$

By means of these estimates it is easy to show, that the function $\psi(\varphi)$, given by formula (1.95), is a solution of equation (1.87) from the class $C^1[0,\varphi_1]$.

From (1.5) and (1.95) we find

$$V(r,\varphi)=r^\lambda(F_1(\varphi)+cP_2(\varphi)+\bar{c}P_1(\varphi)). \tag{1.96}$$

Thus, the following result holds.

***Theorem 1.6.*** *Let* $b(\varphi)=a(\varphi)+c(\varphi)$. *When one of the following conditions holds: 1)* $a(\varphi)=c(\varphi)$, $|b(\varphi)|\neq|A(\varphi)|$ *or* $\lambda=1$, $|b(\varphi)|\neq|A(\varphi)|$ *and 2)* $a(\varphi)\neq c(\varphi)$, $\lambda\neq 1$ *the equation (1.69) is solvable in the class (1.3). When condition 1) holds equation (1.69) has a unique solution in the class (1.3). This solution is given by the formulas (1.5), (1.84). When condition 2) holds equation (1.69) has a general solution. This solution is given by the formulas (1.5), (1.95). When* $a(\varphi)=c(\varphi)$ *or* $\lambda=1$ *and* $|b(\varphi)|=|A(\varphi)|$ *in some point* $0\leq\varphi\leq\varphi_1$ *for the solvability of equation (1.69) in the class (1.3) condition (1.85) is necessary and sufficient. In this case the solution of equation (1.69) from the class (1.3) is given by the formulas (1.5), (1.86).*

### 1.4 Nonhomogenuous second order partial differential equations in the plane with Fuchs operator in the main part

Let $0<\varphi_1\leq 2\pi$ and $G=\{z=re^{i\varphi}:0\leq r<\infty,\quad 0<\varphi<\varphi_1\}$. Consider the equation

$$4a(\varphi)\bar{z}^2V_{\bar{z}\bar{z}}+4(a(\varphi)+c(\varphi))z\bar{z}V_{z\bar{z}}+4c(\varphi)z^2V_{zz}+d(\varphi)\overline{V}=g(r,\varphi),\ z\in G, \tag{1.97}$$

in $G$, when $a(\varphi),\ c(\varphi),\ d(\varphi)\in C[0,\varphi_1]$, $a(\varphi)\neq c(\varphi)$, $\operatorname{Im}\dfrac{a(\varphi)+c(\varphi)}{a(\varphi)-c(\varphi)}\geq 0$.

The function $g(r,\varphi)$ satisfies the conditions: in $G$ it has the form $g(r,\varphi)=\sum_{k=0}^{\infty}g_k(\varphi)r^{\nu k}$, where $g_k(\varphi)\in C[0,\varphi_1]$, $\nu>0$ is a real parameter, $0\leq k$ and the series $g(r,\varphi)=\sum_{k=0}^{\infty}|g_k(\varphi)|r^{\nu k}$ is convergent in $G$.

The solution of equation (1.97) is searched for in the class (1.35), where $1<p<\dfrac{2}{2-\nu}$, *if* $\nu<2$ *and* $p>1$, *if* $\nu\geq 2$.

Using formulas (1.2), equation (1.97) in polar coordinates is written in the form

$$2(a(\varphi)+c(\varphi))r^2V_{rr}-2i(a(\varphi)-c(\varphi))V_\varphi+2i(a(\varphi)-c(\varphi))rV_{r\varphi}+ \\ +d(\varphi)\overline{V}=g(r,\varphi) \tag{1.98}$$

Solutions of equation (1.98) are searched for from the class (1.35) in the form

$$V(r,\varphi)=\sum_{k=0}^{\infty}V_k(\varphi)r^{\nu k}\ , \tag{1.99}$$

where $V_k(\varphi)$, $0\le k$ are new unknown functions from the class $C^2[0,\varphi_1]$, so that the series (1.99) is convergent in $G$.

Substituting (1.99) in (1.98) and compare the coefficients at same power of $r^\nu$, we have

$$V_k'(\varphi)+\tau_k(\varphi)V_k(\varphi)=d_k(\varphi)\overline{V_k(\varphi)}+g_{k,0}(\varphi), \tag{1.100}$$

where

$$\tau_k(\varphi)=-i\frac{(a(\varphi)+c(\varphi))\nu k}{a(\varphi)-c(\varphi)},\ \ g_{k,0}(\varphi)=-i\frac{g_k(\varphi)}{2(a(\varphi)-c(\varphi))(\nu k-1)},$$

$$d_k(\varphi)=i\frac{d(\varphi)}{2(a(\varphi)-c(\varphi))(\nu k-1)}.$$

Solving equation (1.100) by applying the method of variation of constant, we get the integral equation

$$V_k(\varphi)=\int_0^\varphi b_k(\varphi,\gamma)\overline{V_k(\gamma)}d\gamma+\int_0^\varphi f_k(\varphi,\gamma)d\gamma+c_kI_{k,0}(\varphi), \tag{1.101}$$

where

$$b_k(\varphi,\gamma)=d_{\mathrm{k}}(\gamma)\exp\left(-\int_\gamma^\varphi\tau_k(\gamma_1)d\gamma_1\right),\ f_k(\varphi,\gamma)=g_{\mathrm{k},0}(\gamma)\exp\left(-\int_\gamma^\varphi\tau_k(\gamma_1)d\gamma_1\right),$$

$$I_{k,0}(\varphi)=\exp\left(-\int_0^\varphi\tau_k(\gamma)d\gamma\right).$$

For constructing solutions of equation (1.101) the following functions and operators are used:

$$I_{k,s}(\varphi)=\int_0^{\varphi} b_k(\varphi,\gamma)\overline{I_{k,s-1}(\gamma)}d\gamma\,,\ (B_k f_k)(\varphi)=\int_0^{\varphi} b_k(\varphi,\gamma)\overline{f_k(\gamma)}d\gamma\,,$$

$$(B_k^0 F_k)(\varphi)=F_k(\varphi)=\int_0^{\varphi} f_k(\varphi,\gamma)d\gamma\,,\ \ (B_k^s f)(\varphi)=B_k(B_k^{s-1}f)(\varphi)\,,\ \ 1\le s\,.$$

For these functions it is easy to check that

$$\left|I_{k,s}(\varphi)\right|\le\frac{1}{s!}\left|d_k\right|_0^s\varphi^s\exp\left(-s\int_{\gamma}^{\varphi}\operatorname{Im}\frac{(a(\gamma_1)+c(\gamma_1))\nu k}{a(\gamma_1)-c(\gamma_1)}d\gamma_1\right)\cdot$$

$$\cdot\exp\left(-\int_0^{\varphi}\operatorname{Im}\frac{(a(\gamma_1)+c(\gamma_1))\nu k}{a(\gamma_1)-c(\gamma_1)}d\gamma_1\right),$$

$$\left|(B_k^s F_k)(\varphi)\right|\le\frac{1}{(s+1)!}\left|d_k\right|_0^s\left|g_{k,0}\right|_0\varphi^{s+1}\exp\left(-(s+1)\int_{\gamma}^{\varphi}\operatorname{Im}\frac{(a(\gamma_1)+c(\gamma_1))\nu k}{a(\gamma_1)-c(\gamma_1)}d\gamma_1\right),\qquad(1.102)$$

$$\left|(B_k^s f)(\varphi)\right|\le\frac{1}{s!}\left|d_k\right|_0^s\left|f\right|_0\varphi^s\exp\left(-s\int_{\gamma}^{\varphi}\operatorname{Im}\frac{(a(\gamma_1)+c(\gamma_1))\nu k}{a(\gamma_1)-c(\gamma_1)}d\gamma_1\right).$$

Here $\left|f\right|_0=\left\|f\right\|_{C[0,\varphi_1]}$.

Using the specified notations, equation (1.101) is written in the form

$$V_k(\varphi)=(B_kV_k)(\varphi)+F_k(\varphi)+c_kI_{k,0}(\varphi).\qquad(1.103)$$

If we apply the operator $B_k$ to both sides of equation (1.103), we have in view of (1.42)

$$(B_kV_k)(\varphi)=(B_k^2V_k)(\varphi)+(B_kF_k)(\varphi)+\overline{c}_kI_{k,1}(\varphi).\qquad(1.104)$$

From (1.103), (1.104) it follows

$$V_k(\varphi)=(B_k^2V_k)(\varphi)+(B_kF_k)(\varphi)+F_k(\varphi)+\overline{c}_kI_{k,1}(\varphi)+c_kI_{k,0}(\varphi).\qquad(1.105)$$

If we again apply the operator $B_k$ to both sides of equation (1.105), we have in view of (1.42)

$$(B_kV_k)(\varphi)=(B_k^3V_k)(\varphi)+(B_k^2F_k)(\varphi)+(B_kF_k)(\varphi)+c_kI_{k,2}(\varphi)+\overline{c}_kI_{k,1}(\varphi).\qquad(1.106)$$

From (1.103) and (1.106) it follows

$$V_k(\varphi)=(B_k^3V_k)(\varphi)+(B_k^2F_k)(\varphi)+(B_kF_k)(\varphi)+F_k(\varphi)+c_k(I_{k,2}(\varphi)+I_{k,0}(\varphi))+\overline{c}_kI_{k,1}(\varphi).$$

Continuing this process 2n-1 and 2n-2 times, respectively we receive the following representations for solutions of equation (1.100)

$$V_k(\varphi) = (B_k^{2n} V_k)(\varphi) + \sum_{s=0}^{2n-1} (B_k^s F_k)(\varphi) + c_k \sum_{s=0}^{n-1} I_{k,2s}(\varphi) + \bar{c}_k \sum_{s=1}^{n} I_{k,2s-1}(\varphi)$$

and (1.107)

$$V_k(\varphi) = (B_k^{2n+1} V_k)(\varphi) + \sum_{s=0}^{2n} (B_k^s F_k)(\varphi) + c_k \sum_{s=0}^{n} I_{k,2s}(\varphi) + \bar{c}_k \sum_{s=1}^{n} I_{k,2s-1}(\varphi).$$

If we pass to the limit as $n \to \infty$ in the representations (1.107), by virtue of (1.102) we receive

$$V_k(\varphi) = F_{k,1}(\varphi) + c_k P_{k,2}(\varphi) + \bar{c}_k P_{k,1}(\varphi),\ 0 \le k, \tag{1.108}$$

where $F_{k,1}(\varphi) = \sum_{s=0}^{\infty} (B_k^s F_k)(\varphi)$, $P_{k,2}(\varphi) = \sum_{s=0}^{\infty} I_{k,2s}(\varphi)$, $P_{k,1}(\varphi) = \sum_{s=1}^{\infty} I_{k,2s-1}(\varphi)$,

$c_k, 0 \le k$, are any complex numbers.

Using the inequalities (1.102), we receive the estimates

$$|F_{k,1}(\varphi)| \le \frac{|g_{k,0}|_0}{|d_k|_0} \exp\left(|d_k|_0 \exp\left((-\int_{\gamma}^{\varphi} \operatorname{Im} \frac{(a(\gamma_1)+c(\gamma_1))\nu k}{a(\gamma_1)-c(\gamma_1)} d\gamma_1)\varphi\right)\right),$$

$$|P_{k,2}(\varphi)| \le \exp\left(-\int_0^{\varphi} \operatorname{Im} \frac{(a(\gamma_1)+c(\gamma_1))\nu k}{a(\gamma_1)-c(\gamma_1)} d\gamma_1\right) \cdot$$

$$\cdot ch\left(|d_k|_0 \exp\left((-\int_{\gamma}^{\varphi} \operatorname{Im} \frac{(a(\gamma_1)+c(\gamma_1))\nu k}{a(\gamma_1)-c(\gamma_1)} d\gamma_1)\varphi\right)\right),$$

$$|P_{k,1}(\varphi)| \le \exp\left(-\int_0^{\varphi} \operatorname{Im} \frac{(a(\gamma_1)+c(\gamma_1))\nu k}{a(\gamma_1)-c(\gamma_1)} d\gamma_1\right) \cdot$$

$$\cdot sh\left(|d_k|_0 \exp\left((-\int_{\gamma}^{\varphi} \operatorname{Im} \frac{(a(\gamma_1)+c(\gamma_1))\nu k}{a(\gamma_1)-c(\gamma_1)} d\gamma_1)\varphi\right)\right).$$

By means of these estimates it is easy to show, that the function $V_k(\varphi)$, given by formula (1.108), is a solution of equation (1.100) from the class $C^1[0, \varphi_1]$.

From (1.99) and (1.108) we find

$$V(r,\varphi)=\sum_{k=0}^{\infty}(F_{k,1}(\varphi)+c_kP_{k,2}(\varphi)+\bar{c}_kP_{k,1}(\varphi))r^{\nu k} \tag{1.109}$$

Thus, the following result hold.

***Theorem 1.7.*** *When* $b(\varphi)=a(\varphi)+c(\varphi)$ *the equation (1.97) is solvable in the class (1.35). The general solution of equation (1.97) from the class (1.35) is given by formula (1.109).*