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Intrinsic Discretization Error Bounds for Geodesic Finite Elements

Dissertation zur Erlangung der Doktorwürde

March 2015

Mathematisches Institut
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Tag der Disputation:	16.07.2015

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Zusammenfassung Die vorliegende Arbeit beschäftigt sich mit dem Beweis optimaler Fehlerschranken für die Diskretisierung H^1-elliptischer Minimisierungsprobleme, deren Lösungen Werte in einer Riemannschen Mannigfaltigkeit annehmen. Die Diskretisierung wird dabei mithilfe Geodätischer Finiter Elemente durchgeführt, welche eine unter Isometrien invariante Methode beliebiger Ordnung darstellen. Der Diskretisierungsfehler wird sowohl intrinsisch in einer speziell eingeführten Sobolev-Distanz als auch extrinsisch betrachtet und es werden optimale Abschätzungen vom H^1- und L^2-Typ hergeleitet, die in vorausgegangenen Arbeiten anderer Autoren experimentell beobachtet wurden. Unter Verwendung der Rothe-Methode bestehend aus einem impliziten Eulerverfahren zur Zeitdiskretisierung und Geodätischen Finiten Elementen zur Ortsdiskretisierung werden zusätzlich Fehlerabschätzungen für L^2-Gradientenflüsse H^1-elliptischer Energien hergeleitet.

Kern der Arbeit bilden die Diskretisierungsfehlerabschätzungen für Minimisierungsprobleme in intrinsischen H^1- und L^2-Distanzen. Zu deren Herleitung werden zunächst inverse Abschätzungen sowie Interpolationsfehler für Geodätische Finite Elemente und deren diskrete Variationen gezeigt. Unter Verwendung eines nichtlinearen Céa-Lemmas werden daraus H^1-Diskretisierungsfehler für Minimierer H^1-elliptischer Energien hergeleitet. Mit Hilfe einer Verallgemeinerung des Aubin-Nitsche-Lemmas werden sodann für (im Wesentlichen) semilineare Energien auch optimale L^2-Fehlerschranken gezeigt, wobei aus technischen Gründen die Dimension des Definitionsgebietes des Minimierers auf $d < 4$ beschränkt wird. Alle Resultate werden anhand von harmonischen Abbildungen in eine glatte Riemannsche Mannigfaltigkeit, welche gewisse Krümmungsschranken erfüllt, illustriert.

Die Ideen, welche zu Abschätzungen vom H^1-Typ führen, wurden in Teilen schon im Vorfeld in [GHS14] veröffentlicht.

Contents

Chapter 0
Introduction

The numerical approximation of maps between Riemannian manifolds that minimize a certain energy is a developing field. Non-Euclidean domains, e.g., surfaces or more generally embedded Riemannian manifolds, which arise from measurements, assumptions, or solutions to geometric PDE-problems, have been studied quite extensively in [DDE05, DE07, DE12, DE13]. Methods for manifold codomains which for example arise from physically enforced symmetry, e.g., in the context of Cosserat materials or liquid chrystals, are often ad hoc contructions specific to an individual energy and manifold [BP07, Bar05, WG93, Mün07, MWN09, MNW11, SVQ86, SFR90]. Recently, geodesic finite elements have been developed and experimentally studied to address this shortcoming of other methods by providing an objective, i.e., isometry-invariant method of arbitrary order [San10, San12, San13, SNB14]. With the exception of [BP07, Bar05], approximation error estimates for methods used for manifold codomains are rarely addressed in the literature, even for simple energies. In this work, we aim to generalize some results from the basic theory of finite element methods for the minimization of $W^{1,2}$-elliptic energies to functions with Riemannian manifold codomains, in particular concentrating on geodesic finite elements for the discretization. First results in this direction have already been published in [GHS14]. We do not aim for completeness but for a basis on which more advanced theory can be built.

We consider minimization problems of the form

$$u : \Omega \to M, \qquad u = \operatorname*{arg\,min}_{w \in H} \mathfrak{J}(w), \tag{0.1}$$

where $\Omega \subset \mathbb{R}^d$, M is a smooth Riemannian manifold, and $\mathfrak{J} : H \to \mathbb{R}$ a nonlinear functional. The domain H of $\mathfrak{J}$ is a set of functions $\Omega \to M$ of $W^{1,2}$ smoothness, which we discuss in detail in Chapter 1.

As the codomain M is a Riemannian manifold, we discretize (0.1) by geodesic finite elements (GFEs), which are an adaption of Lagrangian finite elements of arbitrary order. Based on the Riemannian center of mass, they do not rely on an embedding of M into Euclidean space and in particular are equivariant under isometries of M, which leads to the desirable property of objectivity.

The Riemannian center of mass has already been used for interpolation of values on a manifold in [BF01, Moa02, PFA06]. In [Gro13b], interpolation errors have been estimated in the L^∞ norm.

By construction, GFEs form a conforming discretization, i.e., the set S_h of GFE functions for a given grid is a subset of H (see [San13, Thm. 5.1]). Restricting the minimization problem (0.1) to S_h then yields the discrete problem

$$u^h = \arg\min_{w^h \in S_h} \mathfrak{J}(w^h). \tag{0.2}$$

Details for a numerical solver for (0.2) have been described in [San12].

Besides the static solutions to (0.1) we will also consider corresponding L^2-gradient flows

$$\langle u'(t), V\rangle_{L^2(\Omega, u(t)^{-1}TM)} = -\frac{d}{ds}_{|s=0} \mathfrak{J}(\exp_{u(t)}(sV)) \qquad \forall V \in T_{u(t)}H, \tag{0.3}$$

and their discretization by a method of time layers consisting of an implicit Euler scheme for the time discretization and geodesic finite elements for the space discretization.

While it is well-known that solutions to (0.3) may develop singularities in finite time even for smooth initial conditions (see, e.g., [HW08, CG89, CDY92, Gro93]), we will only do an a priori error analysis for smooth solutions. In [Bar05] harmonic maps into spheres $S^2 \in \mathbb{R}^3$ are approximated using first order finite elements while constraining the vertex values to S^2. It is shown that even for non-regular solutions, there exists a subsequence of discrete solutions converging weakly to a harmonic map. Similar results for geodesic finite elements would be intriguing but beyond the scope of this work.

Note that the domain Ω in (0.1) and (0.3) is always a subset of $\mathbb{R}^d$. For the discretization of non-Euclidean domains there exist well-developed techniques [DDE05, DE07, DE12, DE13]. In [vD13], the Riemannian center of mass is investigated in the context of the discretization of non-Euclidean domains. A combination of either of these methods with our techniques for manifold codomains is planned.

We want to point out that only smooth Riemannian manifolds are considered as codomains in this work, although the construction of geodesic finite elements is meaningful and natural even in a metric space setting, in particular in the context of $CAT(0)$-spaces. At several instances the more general metric space view point will be depicted. A full generalization, however, is not studied here.

We begin this work by introducing Sobolev spaces for manifold codomains. We provide several different plausible definitions that we show to be equivalent for a numerically relevant subset H of continuous $W^{1,2}$-functions. In particular, we measure the regularity by a smoothness descriptor for manifold-valued functions introduced in [GHS14] for this purpose. In order to measure errors, we introduce an intrinsic

Sobolev distance which is equivalent to the error measured in an embedding for the functions of interest.

In the second chapter we recall the definition of geodesic finite elements from [San12, San13, Gro13b, GHS14]. In particular, we reprove well-posedness of the defining minimization problems by a simpler argument than the one given in [San13]. As a tool for further analysis of geodesic finite elements, we will discuss inverse estimates that turn out to be more subtle than in the Euclidean case as sufficiently high derivatives of geodesic finite elements do not vanish as those of polynomials do. We then recall the interpolation error estimates published in [GHS14] with some minor changes to the proof. Geodesic interpolation of vector fields corresponds to variations in the set of geodesic finite elements. We will amend the techniques of [GHS14] to obtain associated vector field interpolation error estimates needed in Chapter 3 for the proof of optimal L^2-error estimates.

The third chapter is dedicated to discretization error estimates. Using a nonlinear version of Céa's Lemma combined with the interpolation error estimates of Chapter 2, we obtain for elliptic energies a $W^{1,2}$-discretization error estimate for a solution restricted in a $W^{1,q}$-ball, as already described in [GHS14]. We then show that this restricted solution indeed correlates to a local solution. This improves on the result in [GHS14], where additional regularity of the continuous solution is required for a corresponding result. For dimensions $d < 4$, we then continue with an analysis of the L^2-error for (morally) semi-linear energies. Using a generalization of the Aubin–Nitsche lemma, we obtain optimal error estimates. Note that the results in this chapter are not restricted to geodesic finite elements but can be applied to other discretization methods as long as interpolation error estimates comparable to the ones in Chapter 2 can be derived.

In the fourth chapter we discuss the discretization of the gradient flow (0.3) by the method of time layers employing an implicit Euler scheme for the time- and first order GFEs for the space-discretization. Using error estimates from [AGS06] for the time discretization, we prove $W^{1,2}$- and L^2-error estimates for the fully discrete scheme.

We illustrate the discretization error estimates of Chapters 3 and 4 in the fifth chapter using the harmonic energy as an example. Under certain bounds on the curvature of M, we obtain the optimal error bounds for the static problem, which have been experimentally observed in [San12, San13]. The results for the harmonic map heat flow are of a more speculative nature.

I thank my advisors Klaus Ecker and Ralf Kornhuber for their support, Oliver Sander for suggesting the topic and day to day mentoring, Philipp Grohs and Oliver Sander for the collaboration on [GHS14], and my family for everything else.

Chapter 1
Sobolev Spaces with Riemannian Manifold Targets

In the context of finite elements for the approximation of elliptic and parabolic partial differential equations, the concept of weak solutions and Sobolev spaces arises naturally. In the course of this work we will discuss mainly partial differential equations whose solutions are mappings from some open subset $\Omega \in \mathbb{R}^d$ with piecewise Lipschitz boundary $\partial\Omega$ into a smooth Riemannian manifold (M,g) without boundary.

In order to gain a deeper insight into the topic, we will also consider mappings into a metric space (X,d) without boundary and only later confine ourselves to the special case of Riemannian manifolds. We will always assume that (X,d) is a complete (or strictly intrinsic) length space, i.e., that for each pair of points $p,q \in X$ there exists a connecting, length realizing rectifiable curve. We call these curves geodesics. Note that completeness in the metric sense, i.e., the fact that every Cauchy sequence has a limit, is related to the completeness of a length space. Specifically, a complete metric space is a complete length space if and only if for each $\varepsilon > 0$ and two points $p,q \in X$ there exists a finite sequence $p = x_1,\dots,x_k = q$ such that $d(x_i,x_{i+1}) \le \varepsilon$ and $\sum_{i=1}^{k-1} d(x_i,x_{i+1}) < d(q,p)+\varepsilon$ (see Corollary 2.4.17 in [BBI01]).

We can define curvature in a length space in several equivalent ways. Here, we will use local convexity and concavity of the distance d (cf. [BBI01, KS93]):

Definition 1.1. Let (X,d) be a complete length space. We say that X has nonpositive curvature if for any three points $p,q,r \in X$ and a constant speed geodesic $\gamma_{q,r} : [0,1] \to X$ connecting q to r, i.e.,

$$d(q,\gamma_{q,r}(t)) = t\, d(q,r), \qquad d(r,\gamma_{q,r}(t)) = (1-t)\, d(q,r) \qquad \forall t \in [0,1],$$

the squared distance with reference point p, $d^2(p,\cdot)$, is 2-convex along $\gamma_{q,r}$, i.e.,

$$d^2(p,\gamma_{q,r}(t)) \le (1-t)d^2(q,p) + td^2(r,p) - t(1-t)d^2(q,r) \qquad \forall t \in [0,1]. \quad (1.1)$$

If the inverse inequality in (1.1) holds, we say X has nonnegative curvature.

Further, we say that X has curvature bounded from above, if there exists a $0 \leq K \in \mathbb{R}$ such that

$$d^2(p,\gamma_{q,r}(t)) \leq (1-t)d^2(q,p)+td^2(r,p)-\frac{C(K)}{2}t(1-t)d^2(q,r) \qquad \forall t \in [0,1] \tag{1.2}$$

holds locally in X, where $C(K)$ is the convexity constant of the distance function on the constant curvature space with curvature K.

X has curvature bounded from below, if there exists a $0 \geq k \in \mathbb{R}$ such that

$$d^2(p,\gamma_{q,r}(t)) \geq (1-t)d^2(q,p)+td^2(r,p)-\frac{C(k)}{2}t(1-t)d^2(q,r) \qquad \forall t \in [0,1] \tag{1.3}$$

holds locally in X, where $C(k)$ is the concavity constant of the distance function on the constant curvature space with curvature k.

In the following we will restrict ourselves to the special case of (X,d) being a complete length space with curvature bounded from above and below. Note that in this case (X,d) indeed is a manifold (M,g) possessing a C^3-atlas such that in its charts the metric d can be defined by the metric tensor g whose coefficients are in class $W^{2,p}$ for every $p > 1$ (see [BBI01]). This allows the formal definition of Christoffel symbols, curvature tensor, and parallel transport having geometric meaning (for almost all paths).

There are several different possibilities to define Sobolev spaces of functions $f : \Omega \to (M,g)$ in this context. Some of these definitions are even valid for complete length spaces with non-positive curvature (NPC-spaces). In the following we will discuss some of these definitions and their relations to each other. As our goal is an a priori error analysis for the numerical approximation of solutions of PDEs, we will then discuss the concept of distances implied by the Sobolev structure. We will introduce an approximate distance yielding a notion of $W^{1,2}$-error. We will see that although it is not a distance in the strict sense, we recover several properties useful for assessing the quality of numerical approximation.

1.1 Different Definitions

Recall the standard definition of Sobolev spaces, taking values in $\mathbb{R}^n$.

Definition 1.2. For $k \in \mathbb{N}$ and $1 \leq p \leq +\infty$, the Sobolev space $W^{k,p}(\Omega)$ is the set of all functions $f : \Omega \to \mathbb{R}$ such that for every multi-index $\vec{\alpha}$ with $|\vec{\alpha}| \leq k$ the weak partial derivative $D^{\vec{\alpha}} f$ is locally integrable and in $L^p(\Omega)$, i.e.,

$$W^{k,p}(\Omega) := \left\{ u \in L^p(\Omega) : D^{\vec{\alpha}} u \in L^p(\Omega)\ \forall |\vec{\alpha}| \leq k \right\}.$$

The Sobolev norm is defined by

$$\|f\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\vec{\alpha}|\le k} \left\|D^{\vec{\alpha}}u\right\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}}, & 1 \le p < +\infty; \\ \max_{|\vec{\alpha}|\le k} \left\|D^{\vec{\alpha}}u\right\|_{L^\infty(\Omega)}, & p = +\infty. \end{cases}$$

Further, the Sobolev space $W^{k,p}(\Omega,\mathbb{R}^n)$ is defined by

$$W^{k,p}(\Omega,\mathbb{R}^n) := \left\{ u = (u_1,\dots,u_n) : u_i \in W^{k,p}(\Omega,\mathbb{R}) \quad \forall i = 1,\dots,n \right\}.$$

The corresponding Sobolev norm is defined by the usual combination of the components u_i, i.e.,

$$\|u\|_{W^{k,p}(\Omega,\mathbb{R}^n)} := \left(\sum_{i=1}^n \|u_i\|_{W^{k,p}(\Omega)}^p \right)^{\frac{1}{p}}.$$

For the case $p = 2$, we write $H^k(\Omega,\mathbb{R}^n)$ and note that we have a scalar product

$$\langle u,v\rangle_{H^k(\Omega,\mathbb{R}^n)} := \sum_{i=1}^n \sum_{|\vec{\alpha}|\le k} \langle D^{\vec{\alpha}}u_i, D^{\vec{\alpha}}v_i\rangle_{L^2}.$$

The definition of L^p-functions can be generalized in a straightforward fashion to functions taking their values in a complete length space (X,d) (see, e.g., [KS93]).

Definition 1.3. Let (X,d) be a complete length space and $1 \le p < \infty$. We define

$$L^p(\Omega,X) := \Big\{ u : \Omega \to X \mid u \text{ measurable}, \ u(\Omega) \text{ separable}, \\ \int_\Omega d^p(u(x),Q)\,dx < \infty \text{ for some } Q \in X \Big\}.$$

We further define a distance function d_{L^p} on $L^p(\Omega,X)$ by

$$d_{L^p}^p(u,v) := \int_\Omega d^p(u(x),v(x))\,dx.$$

Remark 1.4. It is easy to see that d_{L^p} is indeed well defined for $u,v \in L^p(\Omega,X)$, as $x \mapsto (u(x),v(x))$ is a measurable function to $X \times X$. Further, the definition of a function $u \in L^p(\Omega,X)$ is independent of the point $Q \in X$ in the definition, as the triangle inequalities for d and the real-valued L^p-distance imply for any other $P \in X$ that

$$\left(\int_\Omega d^p(u(x),P)\,dx \right)^{\frac{1}{p}} \le \left(\int_\Omega (d(u(x),Q) + d(Q,P))^p \ dx \right)^{\frac{1}{p}}$$

$$\le \left(\int_{\Omega} d^p(u(x),Q)\,dx\right)^{\frac{1}{p}} + |\Omega|^{\frac{1}{p}} d(Q,P)$$
$$< \infty.$$

The same argument also shows that d_{L^p} is finite for $u,v \in L^p(\Omega,X)$, and that it fulfills a triangle inequality.

Remark 1.5. Obviously, $(L^p(\Omega,X),d_{L^p})$ is a complete metric space, if (X,d) is complete. Note furthermore that Definition 1.3 also generalizes from subsets $\Omega \subset \mathbb{R}^d$ to any connected Riemannian manifold (Ω^d, g_Ω).

While the definition of $L^p(\Omega,X)$ is thus immediate, this is not true for $W^{k,p}(\Omega,X)$. In the following we will introduce an approach by Korevaar and Schoen [KS93] that works for X being a complete metric space. We will then discuss three other ways conceivable for Riemannian manifolds and compare them.

1.1.1 Sobolev Spaces with Metric Space Targets

For the general setting of functions taking their values in a complete metric space (X,d) a direct intrinsic definition of $W^{1,p}(\Omega,X)$ was introduced in [KS93]. We will recapitulate this definition in the following. Note that a similar approach was taken independently by Jost [Jos94]. In [Chi07] it was proven that both approaches are indeed equivalent.

Let (X,d) be a complete metric space and $u \in L^p(\Omega,X)$ for some $1 \le p < \infty$. Let further $V : \Omega \to \mathbb{R}^d$ denote a smooth vectorfield on Ω, and set for $\varepsilon > 0$

$$\Omega_\varepsilon := \{x \in \Omega \mid d(x,\partial\Omega) > \varepsilon\},$$
$$\Omega_\varepsilon^V := \{x \in \Omega \mid d(x,\partial\Omega) > \varepsilon|V|_\infty\}.$$

Note that the map $x \mapsto u(x+\varepsilon V(x))$ is in $L^p(\Omega_\varepsilon^V,X)$ as

$$\int_{\Omega_\varepsilon^V} d^p(u(x+\varepsilon V(x)),P)\,dx = \int_{y(\Omega_\varepsilon^V)} d^p(u(y),P)|\det Dy^{-1}(y(x))|\,dy$$
$$< \infty,$$

where $y : x \mapsto x+\varepsilon V(x)$ and $P \in X$, and thus

$$\det Dy^{-1}(y(x)) = (I+\varepsilon DV(x))^{-1} = \left(1+\varepsilon\,\mathrm{tr}\,DV(x)+O(\varepsilon^2)\right)^{-1}.$$

This also implies

$$\int_{\Omega_\varepsilon^V} d^p(u(x),u(x+\varepsilon V(x)))\,dx \le C,$$

independently of ε.

Definition 1.6. The spherically averaged ε-approximate energy density function is defined on Ω by

$$e_\varepsilon(x) := \begin{cases} \int_{\partial B_1(0)} \frac{d^p(u(x),u(x+\varepsilon\omega))}{\varepsilon^p}\, d\sigma(\omega) & \text{for } x \in \Omega_\varepsilon, \\ 0 & \text{else,} \end{cases}$$

where $d\sigma$ denotes the $(d-1)$-dimensional surface measure on $\partial B_1(0)$. Let ν be a Borel measure on the interval $(0,2)$ with

$$\nu \geq 0, \quad \nu((0,2)) = 1, \quad \int_0^2 \lambda^{-p} d\nu(\lambda) < \infty.$$

Then the ν, ε-approximate energy density function is given by

$${}_\nu e_\varepsilon(x) := \begin{cases} \int_0^2 e_{\lambda\varepsilon}(x)\, d\nu(\lambda) & \text{for } x \in \Omega_{2\varepsilon}, \\ 0 & \text{else.} \end{cases}$$

Remark 1.7. By Tonelli's and Fubini's theorem, ${}_\nu e_\varepsilon$ is a real-valued L^1-function with

$$\int_{\Omega_\varepsilon} {}_\nu e_\varepsilon(x)\, dx \leq C\varepsilon^{-p}.$$

Indeed, in [KS93] the more general case of Ω being a Riemannian domain is discussed where the proof of the corresponding equation uses an orthonormal frame and a partition of unity argument to simplify to the situation we reviewed here.

Remark 1.8. In [KS93] it is mentioned that for smooth $u : \Omega \to \mathbb{R}$ the energy density corresponds to $|Du|^p$ in the following sense

$$\lim_{\varepsilon \to 0} {}_\nu e_\varepsilon(x) = C(d,p)|Du(x)|^p,$$
$$C(d,p) = \int_{\partial B_1(0)} |\omega^1|^p\, d\sigma(\omega),$$

where $\omega = \left(\omega^1, \ldots, \omega^{d-1}\right) \in \partial B_1(0)$, and $d\sigma$ denotes the $(d-1)$-dimensional surface measure on $\partial B_1(0)$. In particular, $C(d,2) = \omega_d$, where ω_d is the volume of the unit d-ball.
For $p = 2$ this formula also holds for smooth Riemannian manifold targets. For $p \neq 2$ however, this is not generally true.

Using the approximate energy density, one can define an energy, and consequently the set $W^{1,p}(\Omega, X)$.

Definition 1.9. Let $1 \leq p < \infty$, $u \in L^p(\Omega, X)$, and ν as above. For $\varepsilon > 0$ and $f \in C_C(\Omega, \mathbb{R}) := \{\tilde{f} \in C(\Omega, \mathbb{R}) \;:\; \operatorname{supp} \tilde{f} \Subset \Omega\}$ define

$${}_\nu E_\varepsilon(f) := \int_\Omega f(x)\, {}_\nu e_\varepsilon(x)\, dx.$$

The energy of u is then defined by

$$_{\nu}E(u) := \sup_{\substack{f\in C_C(\Omega)\\ 0\le f\le 1}} \left(\limsup_{\varepsilon\to 0} {}_{\nu}E_{\varepsilon}(f) \right),$$

and we set

$$W^{1,p}(\Omega,X) := \{u \in L^p(\Omega,X) \mid {}_{\nu}E(u) < \infty \text{ for some } \nu\}.$$

Remark 1.10. It is proven in [KS93] that the above definition is independent of the choice of the measure ν. Furthermore, each measure ${}_{\nu}e_{\varepsilon}dx$ converges weakly to the same "energy density" measure de.

1.1.2 Sobolev Spaces with Riemannian Manifold Targets

We later concern ourselves mainly with metric spaces allowing a Riemannian manifold structure. For such spaces M other constructions of $W^{1,p}(\Omega,M)$ are possible.

The most common definition for Sobolev spaces with Riemannian manifold targets uses the Nash embedding theorem.

Theorem 1.11 (Nash Embedding Theorem). *Let (M,g) be an n-dimensional Riemannian manifold of class C^k. Then there exists a C^k isometric embedding $\iota : M \to \mathbb{R}^N$ for some $m = N(n)$.*

For a proof see [Nas56].

Many authors (see e.g. [HW08, Haj09, Hél02, Str85]) use the Nash embedding theorem to give the following definition of manifold valued Sobolev functions.

Definition 1.12. Let (M,g) be a n-dimensional Riemannian manifold of class C^k. Let $\iota : M \to \mathbb{R}^N$ be an isometric embedding into Euclidean space. We then define

$$W^{k,p}_{\iota}(\Omega,M) := \{v \in W^{k,p}(\Omega,\mathbb{R}^N) \mid v(x) \in \iota(M) \text{ a.e.}\}.$$

Remark 1.13. If Ω and M are compact, then $W^{k,p}_{\iota}(\Omega,M)$ is independent of ι (see, e.g., [HW08]). Thus, Definition 1.12 is mostly used in this setting. Properties are discussed in some detail in [HW08] and [Haj09].

Under the same assumption Definition 1.12 is equivalent to Definition 1.9 (see [KS93, Jos08, Haj09]).

An important question is whether Sobolev functions can be approximated by smooth functions. For $p > d$, the result indeed follows by the Sobolev embedding theorem (see, e.g., [Eva98]).

Theorem 1.14 (Sobolev Embedding Theorem). *Assume $u \in W^{k,p}(\Omega,\mathbb{R})$.*

1. *If $kp < d$, then $u \in L^q(\Omega,\mathbb{R})$, $q = \frac{dp}{d-kp}$ and there exists a constant $C_1 = C_1(k,d,p,\Omega)$ such that*

$$\|u\|_{L^q(\Omega,\mathbb{R})} \le C_1 \|u\|_{W^{k,p}(\Omega,\mathbb{R})}.$$

2. *If $kp > d$, then u is indeed in the Hölder space $C^{k-\lfloor \frac{d}{p} \rfloor - \frac{d}{p},\vartheta}(\Omega,\mathbb{R})$, where*

$$\vartheta := \begin{cases} \left\lfloor \frac{d}{p} \right\rfloor + 1 - \frac{d}{p}, & \text{if } \frac{d}{p} \text{ is not an integer}, \\ \text{any number in } (0,1), & \text{if } \frac{d}{p} \text{ is an integer}. \end{cases}$$

   There exists a constant $C_1 = C_1(k,d,p,\Omega)$ such that

$$\|u\|_{C^{k-\lfloor \frac{d}{p} \rfloor - \frac{d}{p},\vartheta}(\Omega,\mathbb{R})} \le C_1 \|u\|_{W^{k,p}(\Omega,\mathbb{R})}.$$

For $k = 1$, $p = d$ the answer is due to Schoen and Uhlenbeck [SU83].

Theorem 1.15. *If $p \ge d$, then $C^1(\Omega,M) \cap W_\iota^{1,p}(\Omega,M)$ is dense in $W_\iota^{1,p}(\Omega,M)$.*

If $p < d$, the corresponding result is no longer true in general. For a survey of known results see [HW08].

For smooth functions $u : \Omega \to M$ the definition of $W^{k,p}(\Omega,\mathbb{R})$ directly transfers to sections $u^{-1}TM$, i.e., to vector fields $V : \Omega \to TM$ such that $V(x) \in T_{u(x)}M$ for almost every $x \in \Omega$ (cf. [Jos08]).

Definition 1.16. Let (M,g) be an n-dimensional Riemannian manifold, $u \in C^\infty(\Omega,M)$, and $V \in L^p(\Omega,u^{-1}TM)$, i.e., $V : \Omega \to TM$ with $V(x) \in T_{u(x)}M$ almost everywhere in Ω, and

$$\int_\Omega |V(x)|^p_{g(u(x))}\, dx < \infty.$$

Let $\eta \in C_C^\infty(\Omega,u^{-1}TM)$. Then for $\alpha = 1,\dots,d$ the covariant derivative of η along u is a vector field along u defined by

$$\nabla_{du^\alpha}\eta(x) := \lim_{h\to 0} \frac{1}{h}\left(\pi_{u(x)}^{t\mapsto u(x+he_\alpha)}(\eta(x+he_\alpha)) - \eta(x)\right),$$

where $e_\alpha \in \mathbb{R}^d$ denotes the α-th Euclidean unit vector, and $\pi_{u(x)}^{t\mapsto u(x+he_\alpha)}$ is the parallel transport along the curve defined by $t \mapsto u(x+he_\alpha)$. In coordinates we can write

$$(\nabla_{du}\eta)^k_\alpha := \frac{\partial}{\partial x^\alpha}\eta^k(x) + \Gamma^k_{ij}(u(x))du^i_\alpha(x)\eta^j(x),$$

where greek indices range from 1 to d, latin indices range from 1 to n, and we sum over repeated indices.

We say that V is in $W^{1,p}(\Omega,u^{-1}TM)$, if the partial derivatives in the definition of $(\nabla_{du}V)^k_\alpha$ exist in a weak sense and are in $L^p(\Omega,u^{-1}TM)$. We denote $\nabla_\alpha V := (\nabla_{du}V)_\alpha$.

Remark 1.17. If u is smooth enough, the Sobolev embedding theorem for vector fields $V \in W^{k,p}(\Omega,u^{-1}TM)$ follows from the Euclidean case. Indeed, for $k=1$ we consider $f(x) := |V(x)|$. Then $f \in L^p(\Omega,\mathbb{R})$, and we can estimate

$$D_\alpha f(x) = \frac{\langle \nabla_\alpha V(x), V(x)\rangle}{|V(x)|} \leq |\nabla_\alpha V(x)|$$

almost everywhere. The Sobolev embedding theorem for V is then induced by the one for f. For higher k, the theorem can then be proven analogously to the Euclidean case (see, e.g., [Eva98]).

Remark 1.18. If $p > d$ every map in $W^{1,p}(\Omega,M)$ can be described as a pointwise small deformation of a smooth function. Thus, $W^{1,p}(\Omega,M)$ can be locally modelled as a Banach manifold over the space of $W^{1,p}$-deformations, i.e., $W^{1,p}(\Omega,u^{-1}TM)$. More on this construction can be found in [Pal68].

Remark 1.19. If $p > d$, we have equivalence of the characterization of $W^{1,p}(\Omega,M)$ described in Remark 1.18 to Definition 1.12 and Defintion 1.9. Indeed, $u \in W^{1,p}_\iota(\Omega,M)$ with $p > d$ implies that u is continuous and hence the image of Ω is contained in a compact ball B_R in M. By Remark 1.13, the definition of $W^{1,p}_\iota(\Omega,B_R)$ is independent of ι. Note however that the radius R depends on u.

Although $W^{1,2}(\Omega,M)$ is only a manifold for $d=1$, we can nevertheless consider the $W^{1,2}$-norm for vector fields along functions in $W^{1,q}(\Omega,M)$ if $q > \max\{2,d\}$ as those functions are continuous and thus local charts can be used to define covariant derivatives. For this norm we can show the following version of the Poincaré inequality.

Lemma 1.20 (Poincaré Inequality). *Let $u \in W^{1,q}(\Omega,M)$ with $q > \max\{2,d\}$, and assume that $W : \Omega \to u^{-1}TM$ with $W|_{\partial\Omega} = 0$. Then we have*

$$|W|^2_{L^2(\Omega,u^{-1}TM)} \leq C_2(\Omega) \sum_{\alpha=1}^d \int_\Omega |\nabla_\alpha W(x)|^2_{g(u(x))}\, dx,$$

with $C_2(\Omega)$ the Poincaré constant of the domain Ω.

Proof. By the Poincaré inequality for $f : x \mapsto |W(x)|_{g(u(x))} \in \mathbb{R}$ we get

$$|W|^2_{L^2(\Omega,u^{-1}TM)} = \int_\Omega |W(x)|^2_{g(u(x))}\, dx = \|f\|^2_{L^2} \leq C_2(\Omega) \sum_{\alpha=1}^d \left\| \frac{df}{dx^\alpha} \right\|^2_{L^2}.$$

Using the Cauchy inequality for g we may then calculate

$$\left|\frac{df}{dx^\alpha}(x)\right| = \frac{\langle W(x), \nabla_\alpha W\rangle_{g(u(x))}}{|W(x)|_{g(u(x))}} \leq |\nabla_\alpha W(x)|_{g(u(x))},$$

and the assertion follows. □

1.1.3 Traces of Sobolev Maps

In the following we will mostly concern ourselves with functions $u : \Omega \to M$ that are continuous. Thus, we do not really need to concern ourselves with traces of Sobolev maps. For completeness we nevertheless repeat a short overview given in [HW08].

Recall that

$$W^{1-\frac{1}{p},p}(\partial\Omega, \mathbb{R}^N) := \left\{ g \in L^p(\partial\Omega, \mathbb{R}^N) \;:\; \|g\|_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|g(x) - g(y)|^p}{|x-y|^{p+m-2}}\, dx\, dy \right)^{\frac{1}{p}} < \infty \right\}. \tag{1.4}$$

The trace operator $\mathrm{tr} : C^1(\Omega, \mathbb{R}^N) \to C^1(\partial\Omega, \mathbb{R}^N)$ be extended to a continuous and surjective operator $\mathrm{tr} : W^{1,p}(\Omega, \mathbb{R}^N) \to W^{1-\frac{1}{p},p}(\partial\Omega, \mathbb{R}^N)$ (see, e.g., [Eva98]).

Definition (1.4) can be extended to manifolds analogously to Definition 1.12 by setting

$$W^{1-\frac{1}{p},p}(\partial\Omega, M) := \left\{ g \in W^{1-\frac{1}{p},p}(\partial\Omega, \mathbb{R}^N) \;:\; g(x) \in M \text{ for a.e. } x \in \partial\Omega \right\}. \tag{1.5}$$

Then the trace of a map $u \in W^{1,p}(\Omega, M)$ is always contained in $W^{1-\frac{1}{p},p}(\partial\Omega, M)$ but the map $\mathrm{tr} : W^{1,p}(\Omega, M) \to W^{1-\frac{1}{p},p}(\partial\Omega, M)$ is in general not onto. This is also the case for continuous maps, and the question if tr is onto for given manifold and p is largely open. In [HW08] an overview of known results is given.

Given boundary and homotopy data $\phi : \Omega \to M$, we set $W^{k,p}_\phi(\Omega, M)$ to be those functions $v \in W^{k,p}(\Omega, M)$ such that $\mathrm{tr}(v) = \mathrm{tr}(\phi)$ and v and ϕ are of the same homotopy class, i.e., there exists a continuous homotopy connecting v and ϕ.

1.1.4 Smoothness Descriptors

We want to characterize Sobolev functions by having finite energy. If the function we study is continuous, we can make sense of weak covariant derivatives by using local charts on the target manifold (M, g). This restriction to continuous functions is the reason why Definition 1.12 uses the Nash Embedding Theorem rather than local charts. In the case where weak covariant derivatives can be defined, any given characterization should be equivalent to the Sobolev norm in the defining embedding. This makes it necessary to track more terms than the usual Sobolev energy consists of in the Euclidean setting. To this purpose we define the so-called smoothness descriptor (cf. [GHS14]).

For covariant differentiation we cannot use the usual multi-index notation as covariant derivatives do not commute. In the following we use for multiple covariant derivatives the multi-index notation

$$\nabla^{\vec{\beta}} u := \nabla_{du_{\beta_k}} \dots \nabla_{du_{\beta_2}} du_{\beta_1}, \qquad \vec{\beta} \in \{1, \dots, d\}^k,\ k \in \mathbb{N}_0, \tag{1.6}$$

where $du_\beta = du(\frac{\partial}{\partial x_\beta})$, and ∇_{du} denotes the covariant derivative along u as defined in Definition 1.16.

For a shorter notation we set $|\vec{\beta}| = \dim \vec{\beta}$ and

$$[d] := \{1, \dots, d\}.$$

Additionally, we set $\nabla^{\vec{\beta}} u := 1 \in \mathbb{R}$ if $|\vec{\beta}| = 0$. For a subfamily $\vec{\alpha} = (\beta_j)_{j \in J}$ of a multi-index $\vec{\beta} = (\beta_i)_{i \in I}$ we use the notation $\vec{\alpha} \subset \vec{\beta}$. We set $\vec{\beta} \backslash \vec{\alpha} = (\beta_i)_{i \in I \backslash J}$.

Analogously to the Euclidean setting, we set for $k \geq 1$

$$\|\nabla^k u\|_{L^p} := \left(\sum_{|\vec{\beta}|=k} \int_\Omega |\nabla^{\vec{\beta}} u(x)|^p \, dx \right)^{\frac{1}{p}}. \tag{1.7}$$

This term is an obvious candidate for a Sobolev half-norm. However, recalling Definition 1.12 we see that in light of the chain rule the correct notion of Sobolev half-norm has to include lower order terms in order to be equivalent to $|\iota \circ u|_{W^{k,p}(\Omega, \mathbb{R}^N)}$. This motivates the following definition.

Definition 1.21 (Smoothness Descriptor). Let $k \geq 1$, $p \in [1, \infty]$. The homogeneous k-th order smoothness descriptor of a function $u \in C(\Omega, M) \cap W^{k,p}(\Omega, M)$ is defined by

$$\dot{\theta}_{k,p,\Omega}(u) := \left(\sum_{\substack{\vec{\beta}_j \in [d]^{m_j},\ j=1,\dots,l \\ \Sigma_{j=1}^l m_j = k}} \int_\Omega \prod_{j=1}^l \left| \nabla^{\vec{\beta}_j} u(x) \right|_{g(u(x))}^p dx \right)^{1/p},$$

with the usual modifications for $p = \infty$. For $k = 0$, and a fixed reference point $Q \in M$, we set

$$\dot{\theta}_{0,p,\Omega;Q}(u) := \left(\int_\Omega d^p(u(x), Q) \, dx \right)^{1/p}.$$

Further, we set

$$\dot{\theta}_{0,p,\Omega}(u) := \min_{Q \in M} \dot{\theta}_{0,p,\Omega;Q}(u).$$

The corresponding inhomogeneous smoothness descriptor is defined by

$$\theta_{k,p,\Omega}(u) := \left(\sum_{i=0}^{k} \dot{\theta}^p_{i,p,\Omega}(u) \right)^{\frac{1}{p}}.$$

In Definition 1.21, the condition $u \in C(\Omega, M) \cap W^{k,p}(\Omega, M)$ can be interpreted in terms of weak derivatives in coordinates (cf. [Jos08]). We will see in Proposition 1.24 that for continuous functions finiteness of the smoothness descriptor (when defined) is indeed equivalent to finiteness of the Sobolev norm in an isometric embedding, and thus there is no ambiguity in the definition of $W^{k,p}(\Omega, M)$ (if $kp > d$). In fact, the definition itself of the smoothness descriptor is motivated by Proposition 1.24 which follows from the chain rule. Technically, the first order and kth order terms are enough to characterize the smoothness descriptor. Indeed, these terms bound all all other terms of the smoothness descriptor as well as terms that scale similarly.

Proposition 1.22. *Let $u \in C(\Omega, M) \cap W^{k,p}(\Omega, M)$, and let $\vec{\alpha}$ be a multi-index in the sense of* (1.6) *with $|\vec{\alpha}| = l+1$, $0 \le l \le k-1$. Then*

$$\left(\int_\Omega |\nabla^{\vec{\alpha}} u|^{\frac{kp}{l+1}} \, dx \right)^{\frac{1}{p}} \le C \left(\int_\Omega |\nabla^k u|^p \, dx + \int_\Omega |du|^{kp} \, dx \right)^{\frac{1}{p}},$$

where

$$\int_\Omega |\nabla^k u|^p \, dx = \sum_{|\vec{\beta}|=k} \int_\Omega \left| \nabla^{\vec{\beta}} u(x) \right|^p_{g(u(x))} dx$$

$$\int_\Omega |du|^{kp} \, dx = \sum_{\vec{\beta}_j \in [d],\ j=1,\dots,k} \int_\Omega \prod_{j=1}^{k} \left| d^{\vec{\beta}_j} u(x) \right|^p_{g(u(x))} dx.$$

In particular this implies

$$\left(\int_\Omega |\nabla^k u|^p \, dx + \int_\Omega |du|^{kp} \, dx \right)^{\frac{1}{p}} \le \dot{\theta}_{k,p,\Omega}(u) \le C \left(\int_\Omega |\nabla^k u|^p \, dx + \int_\Omega |du|^{kp} \, dx \right)^{\frac{1}{p}}.$$

Proof. Set $W := d^{\alpha_1} u \in W^{k-1,p}(T, u^{-1}TM)$. By the Gagliardo–Nirenberg interpolation inequality, we can estimate for $|\vec{\beta}| = l$

$$\|\nabla^{\vec{\beta}} W\|_{L^{\frac{kp}{l+1}}} \le C \left(\|\nabla^{k-1} W\|_{L^p}^{\frac{l}{k-1}} \|W\|_{L^{kp}}^{\frac{k-1-l}{k-1}} + \|W\|_{L^{\frac{kp}{l+1}}} \right).$$

Using Young's inequality with $\frac{kl}{(k-1)(l+1)} + \frac{k-1-l}{(k-1)(l+1)} = 1$ and Hölder's inequality, we can estimate

$$\|\nabla^{\vec{\beta}} W\|_{L^{\frac{kp}{l+1}}} \leq C\left(\|\nabla^{k-1} W\|_{L^p}^{\frac{l+1}{k}} + \|W\|_{L^{kp}}^{l+1}\right) + \|W\|_{L^{\frac{kp}{l+1}}}$$
$$\leq C\left(\|\nabla^{k-1} W\|_{L^p}^{\frac{l+1}{k}} + \|W\|_{L^{kp}}^{l+1}\right).$$

Thus, we have indeed shown

$$\|\nabla^{\vec{\alpha}} u\|_{L^{\frac{kp}{l+1}}} \leq C\left(\|\nabla^k u\|_{L^p}^{\frac{l+1}{k}} + \|du\|_{L^{kp}}^{l+1}\right) \leq C\,\dot{\theta}_{k,p,T}^{\frac{l+1}{k}}.$$

This implies the assertion. □

In the Euclidean setting, i.e., $M = \mathbb{R}^n$, the smoothness descriptor does not coincide with the Sobolev norm, as already shown in [GHS14]. Instead, they relate in the following way.

Proposition 1.23. *Let $u \in W^{k,p}(\Omega, \mathbb{R}^n)$, $k \geq 1$. Then*

$$|u|_{k,p,\Omega} \leq \dot{\theta}_{k,p,\Omega}(u) \leq C\left(|u|_{k,p,\Omega} + \|du\|_{0,kp,\Omega}^k\right) \leq C\|u\|_{k,p,\Omega}^k.$$

Proof. The proof follows by Proposition 1.22 and the Sobolev embedding theorem. □

We can compare the smoothness descriptor of a function $u \in C(\Omega, M) \cap W_\iota^{k,p}(\Omega, M)$ to the smoothness descriptor of the embedded function $\iota \circ u \in W^{k,p}(\Omega, \mathbb{R}^N)$.

Proposition 1.24. *Let (M,g) be compact and of class C^k, and $\iota : M \to \mathbb{R}^N$ an isometric embedding of class C^k such that $0 \in \iota(M)$. Then for $k \geq 1$ there exist constants C_3,C_4 depending on $\|\iota\|_{C^k}$ such that*

$$C_3\,\dot{\theta}_{k,p,\Omega}(\iota \circ u) \leq \dot{\theta}_{k,p,\Omega}(u) \leq C_4 \dot{\theta}_{k,p,\Omega}(\iota \circ u) \tag{1.8}$$

holds for all $u \in W^{k,p}(\Omega, M) \cap C(\Omega, M)$. For $kp > d$ we have

$$W_\iota^{k,p}(\Omega, M) = \left\{v \in C(\Omega, M) \;:\; \theta_{k,p,\Omega}(v) \text{ is well-defined and } < \infty\right\}, \tag{1.9}$$

which is independent of ι.

Proof. First note that as ι is assumed to be isometric, we have

$$\dot{\theta}_{1,o,\Omega}(\iota \circ u) = \dot{\theta}_{1,o,\Omega}(u)$$

for all $o \in \mathbb{N}$ such that the terms are finite for $u \in W^{k,p}(\Omega, M)$. The chain rule directly implies

$$\nabla^k(\iota \circ u) = \sum_{\substack{\vec{\beta}_j \in [d]^{m_j},\ j=1,\ldots,l \\ \Sigma_{j=1}^l m_j = k}} \left(\left(\nabla^l \iota\right) \circ u\right) \prod_{j=1}^l \nabla^{\vec{\beta}_j} u(x),$$

and thus

$$|\iota \circ u|_{k,p,\Omega} \le C\,\|\iota\|_{C^k}\,\dot\theta_{k,p,\Omega}(u).$$

Note that indeed all terms of the smoothness descriptor appear on the right hand side when applying the chain rule. Applying Proposition 1.22 then yields the upper estimate in (1.8).

The lower estimate follows by the same argument applied to ι^{-1}, where $\|\iota^{-1}\|_{C^k}$ can be estimated using the implicit function theorem, since $\|\iota^{-1}\|_{C^1} = 1$.

In order to show (1.9), we still need to estimate the terms of order $k = 0$.

$$\begin{aligned}\dot\theta^p_{0,p,\Omega}(\iota \circ u) &\le \int_\Omega \|\iota(u(x)) - \iota(Q)\|^p\,dx\\ &\le \int_\Omega d^p(u(x),Q)\,dx\end{aligned}$$

for all $Q \in M$. This implies

$$\dot\theta_{0,p,\Omega}(\iota \circ u) \le \dot\theta_{0,p,\Omega}(u).$$

Furthermore, if M is compact, we have

$$\|\iota \circ u\|_{L^p(\Omega,\mathbb{R}^N)} \le \dot\theta_{0,p,\Omega}(\iota \circ u) + C(M).$$

As M is compact, the distance on M is Lipschitz continuous with respect to the embedded distance, i.e., there exists a constant C such that

$$d(p,q) \le C\,|i(p) - i(q)|_{\mathbb{R}^N} \qquad \forall p,q \in M.$$

Indeed, if this did not hold, there would exist sequences p_j and q_j with

$$\frac{d(p_j,q_j)}{|i(p_j) - i(q_j)|_{\mathbb{R}^N}} \to \infty.$$

As ι is an embedding, this can only happen if $d(p_j,q_j) \to 0$, and as M is compact this implies $p_j, q_j \to P \in M$. Locally near P, we can find a map $\psi : B_r(\iota(P)) \to \mathbb{R}^N$ with

$$\psi(x) = (y_1,\dots,y_n,0,\dots,0) \;\Leftrightarrow\; x \in B_r(\iota(P)) \cap \iota(M).$$

This defines a smooth local projection map

$$\pi_\psi(y_1,\dots,y_n,y_{n+1},\dots,y_N) = \psi^{-1}(y_1,\dots,y_n,0,\dots,0).$$

Given p and q, we can define a connecting path by

$$\alpha(t) := (\iota^{-1} \circ \pi_\psi)((1-t)\iota(p) + t(\iota(q))), \tag{1.10}$$

and obtain

$$d(p,q) \le \int_0^1 |\dot{\alpha}(t)|\, dt \le \int_0^1 |D(\iota^{-1} \circ \pi_\Psi)|\, |\iota(q) - \iota(p)|\, dt \le C\, |\iota(q) - \iota(p)|.$$

Hence, we can estimate

$$\begin{aligned} \dot{\theta}_{0,p,\Omega}(u) &\le \left(\int_\Omega d^p(u(x), \iota^{-1}(0))\, dx \right)^{\frac{1}{p}} \\ &\le C \|\iota \circ u\|_{L^p(\Omega,\mathbb{R}^N)}. \end{aligned}$$

The identity (1.9) now follows from Proposition 1.23 as we obtain the estimates

$$\theta_{k,p,\Omega}(u) \le C(\|\iota\|_{C^k})\theta_{k,p,\Omega}(\iota \circ u) \le C(\|\iota\|_{C^k})\|\iota \circ u\|_{k,p,\Omega}^k,$$

and

$$\|\iota \circ u\|_{k,p,\Omega} \le C(\iota, M)\dot{\theta}_{k,p,\Omega}(u).$$

□

For vector fields a similar construction can be introduced. Note that, while vector fields are a linear concept in the sense that $u^{-1}TM$ is a vector space for each u, we can also view them as functions in the tangent bundle. Thus, they are dependent on their base functions and do not form a vector space.

Definition 1.25. Let $u \in W^{k,b}(\Omega, M) \cap C(\Omega, M)$, and $V \in W^{k,p}(\Omega, u^{-1}TM)$, where

$$b := \begin{cases} p & \text{for } kp > d \\ p+1 & \text{for } kp = d \\ \frac{d}{k} & \text{for } kp < d. \end{cases}$$

We define a k-th order homogenous smoothness descriptor for vector fields by

$$\begin{aligned} \dot{\Theta}_{k,p,\Omega}(V) &:= \|V\|_{L^a(\Omega,M)} \dot{\theta}_{k,b,\Omega}(u) \\ &\quad + \left(\sum_{\substack{0 \le o \le k,\, \vec{\beta}_j \in [d]^{m_j} \\ \sum_{j=0}^o m_j = k}} \int_\Omega |\nabla^{\vec{\beta}_0} V(x)|_{g(u(x))}^p \prod_{j=1}^o |\nabla^{\vec{\beta}_j} u(x)|_{g(u(x))}^p \, dx \right)^{1/p}, \end{aligned}$$

where

$$\frac{1}{a} = \frac{1}{p} - \frac{1}{b}.$$

1.2 Distances

Core to this work are error estimates and thus distances. Closely related to the concept of distance is the concept of geodesic. As it is essential for our work we will specify what we mean by a geodesic in the following although it is a well-known concept and a working definition has already been given in Definition 1.1.

Definition 1.26. Let (X,d) be a metric space, and let $\gamma : I \to X$ be an absolutely continuous curve, i.e., $\gamma \in AC(I,X)$. We set

$$|\gamma'|(t) := \lim_{s\to t} \frac{d(\gamma(s),\gamma(t))}{|s-t|}.$$

Note that this limit exists for L^1-a.e. $t \in I$, and $t \mapsto |\gamma'|(t)$ belongs to $L^1(I,\mathbb{R})$ (see, e.g., [AGS06]). The length of γ is then defined as

$$L(\gamma) := \int_I |\gamma'|(t)\, dt.$$

We call $\gamma \in AC(I,X)$ a length minimizing geodesic connecting $p \in X$ to $q \in X$ if

$$\gamma \in \operatorname*{arg\,min}_{\substack{\alpha\in AC(I,X)\\ \alpha(0)=p,\alpha(1)=q}} L(\alpha).$$

We call $\gamma \in AC(I,X)$ a geodesic if $\gamma_{|[t,s]}$ is a length minimizing geodesic for small enough intervals $[t,s] \subset I$. We say that $\gamma \in AC(I,X)$ is a constant speed geodesic connecting $p \in X$ to $q \in X$, if

$$d(\gamma(s),\gamma(t)) \equiv |s-t| d(p,q)$$

L^1-a.e. in I.

Remark 1.27. In a Riemannian manifold a first variation shows that constant speed geodesics γ correspond to solutions of the geodesic equation, in coordinates

$$\left(\nabla_{\dot\gamma}\dot\gamma\right)^k := \frac{d^2\gamma^k}{dt^2} + \Gamma^k_{ij} \circ \gamma \,\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} \equiv 0.$$

1.2.1 L^p-Distances for Manifold Targets

We have already defined the L^p-distance for metric space targets, and thus for Riemannian manifolds, in Definition 1.3. We want to compare geodesics on M with geodesics in $L^p(\Omega,M)$. For this purpose we need the concept of geodesic homotopy.

Definition 1.28. Let $u,v \in C(\Omega,M)$. We call a continuous map $\Gamma : \Omega \times I \to M$ a *geodesic homotopy* connecting u to v if for every $x \in \Omega$ the track curve γ_x defined by $\gamma_x(t) := \Gamma(x,t)$ is a constant speed geodesic connecting $u(x)$ to $v(x)$.

For $u,v \in L^p(\Omega,M)$ a geodesic homotopy is defined by requiring the above for almost every $x \in \Omega$.

We can identify L^p-geodesics with geodesic homotopies in the following way.

Lemma 1.29. *Let $u,v \in L^p(\Omega,M) \cap C(\Omega,M)$ and Γ be a geodesic homotopy connecting u to v. Then $\Gamma(t,\cdot) \in L^p(\Omega,M)$ for every $t \in I$ and $\hat{\Gamma} : I \to L^p(\Omega,M)$ defined by $\hat{\Gamma}(t) := \Gamma(t,\cdot)$ is a constant speed geodesic with respect to the L^p-distance.*

Proof. Let $u,v \in L^p(\Omega,M) \cap C(\Omega,M)$. Let Γ be a geodesic homotopy connecting u to v, and let $Q \in M$. To see that $\Gamma(t,\cdot) \in L^p(\Omega,M)$ for every $t \in I$, we estimate

$$\begin{aligned}\int_\Omega d^p(\Gamma(t,x),Q)\,dx &\le \int_\Omega \left(d(\gamma_x(t),u(x)) + d(u(x),Q)\right)^p\,dx \\ &\le \int_\Omega \left(d(v(x),u(x)) + d(u(x),Q)\right)^p\,dx \\ &< \infty.\end{aligned}$$

Thus $\Gamma(t,\cdot) \in L^p(\Omega,M)$ for every $t \in I$.

Set $\hat{\Gamma}(t) := \Gamma(t,\cdot)$. To show that $\hat{\Gamma}$ is a constant speed geodesic with respect to the L^p-distance, we calculate for $0 < s \le t < 1$

$$\begin{aligned}d_{L^p}(\hat{\Gamma}(s),\hat{\Gamma}(t)) &= \left(\int_\Omega d^p(\Gamma(s,x),\Gamma(t,x))\,dx\right)^{\frac{1}{p}} \\ &= \left(\int_\Omega (t-s)^p d^p(u(x),v(x))\,dx\right)^{\frac{1}{p}} \\ &= (t-s)\,d_{L^p}(u,v).\end{aligned}$$

Thus $\hat{\Gamma}$ is indeed a constant speed geodesic in $L^p(\Omega,M)$. □

1.2.2 Metrics on the Tangent Bundle TM

The difference of two (close enough) points on a manifold is characterized by the vector $(p,\log_p q) \in TM$, where $\log_p : B_{\mathrm{inj}(p)} \to T_pM$ denotes the inverse of the exponential map $\exp_p : T_pM \to M$, and $\mathrm{inj}(p)$ stands for the injectivity radius at $p \in M$.

The difference of two functions $u,v \in C(\Omega,M)$ is then characterized by the pointwise difference $(u(x),\log_{u(x)} v(x)) \in TM$. In order to characterize the distance between the differentials of two functions $u,v \in C^1(\Omega,M)$, i.e., $(u(x),d^\alpha u(x)),(v(x),d^\alpha v(x)) \in TM$, we consider the tangent bundle itself as a manifold.

There are several natural metrics on the tangent bundle. A complete classification has been provided in [KS97]. We will only introduce two classical constructions, namely the Sasaki metric, which is a Riemannian metric on TM, and the horizontal (or complete) lift, which is a pseudo-Riemannian metric on TM.

Let $\tau : TM \to M$ denote the canonical projection. Denote by $(TU, \xi^k, \xi^{\bar{k}})$ the natural tangent chart, i.e., unbarred indices $k = 1, \dots, n$ correspond to indices in the base M, barred indices $\bar{k} = 1, \dots, n$ correspond to indices in the fibers of TM.

The tangent bundle of TM at any point (p, V) splits into the horizontal and the vertical subspace with respect to ∇

$$T_{(p,V)}TM = H_{(p,V)} \oplus V_{(p,V)},$$

where the vertical subspace is defined as the kernel of $d\tau_{(p,V)}$.

For any vector $W \in T_pM$ there exists a unique vector $W^h(p,V) \in H_{(p,V)}$ such that $d\tau(W^h) = W$. This vector W^h is called the horizontal lift. In local coordinates, W^h can be expressed as

$$W^h = \xi^k W_k - V^i \Gamma_{ij}^{\bar{k}} \xi^j W_{\bar{k}}.$$

The vertical lift of W is the vector $W^v(p,V) \in V_{(p,V)}$ such that $W^v(df) = Wf$ for all functions on M. In local coordinates

$$W^v = \xi^{\bar{k}} W_{\bar{k}}.$$

Note that the horizontal and vertical lifts define isomorphisms between the vector spaces T_pM and $H_{(p,V)}$, and T_pM and $V_{(p,V)}$ repectively.

1.2.2.1 The Sasaki metric

The Sasaki metric g^S on TM is derived from the metric g by

$$\begin{aligned} g^S_{(p,V)}(X^h, Y^h) &= g_p(X,Y), \\ g^S_{(p,V)}(X^v, Y^h) &= 0, \\ g^S_{(p,V)}(X^v, Y^v) &= g_p(X,Y). \end{aligned}$$

It is a Riemannian metric on TM. In local coordinates it is defined by

$$g^S_{(p,V)} = \begin{pmatrix} V^a V^b \Gamma_{ai}^k \Gamma_{bj}^l g_{kl} + g_{ij} & V^a \Gamma_{ai}^k g_{kj} \\ V^a \Gamma_{aj}^k g_{ki} & g_{ij} \end{pmatrix}.$$

Remark 1.30. As a Riemannian metric the Sasaki metric induces the distance

$${}^S D^2((p,V_p),(q,V_q)) = \inf_\gamma \left(L^2(\gamma) + \|\pi^\gamma_{q\mapsto p} V_q - V_p\|^2_{g(p)} \right)$$

for $(q,V_q),(p,V_p) \in TM$, where the infimum is taken over all smooth curves connecting p and q in M, and $\pi^{\gamma}_{p\mapsto q} : T_pM \to T_qM$ denotes the parallel transport along γ.

Remark 1.31. Geodesics of g^S are in general complicated objects. Especially the projections onto the manifold are in general not geodesics in M. This property of geodesics in the tangent bundle projecting to M-geodesics is desirable for a host of reasons, among them a natural splitting of distances into a part on M and a vector part.

Remark 1.32. A restriction to manifolds, where g^S-geodesics project onto g-geodesics is not reasonable. In [BBNV03] in fact a classification of all such manifolds is given. To do this, the authors consider curves in the unit tangent bundle TM_1 corresponding to a base curve γ in M and a unit vector field Γ along the curve. The geodesic equation of g^S reads

$$\begin{aligned} \ddot{\gamma} &= -R(V,\dot{V})\dot{\gamma} \\ \ddot{V} &= -\|\dot{V}\|^2 V, \end{aligned}$$

where $\|\dot{V}\|^2$ is constant. Given $(p,V_p) \in (TM_1, g^S)$, and $(W_1,W_2) \in T_{(p,V_p)}TM \cong (T_pM)^2$, g^S-geodesics starting in (p,V_p) in direction (W_1,W_2) arise in three different types, namely

1. If $W_1 = 0$, then (γ,Γ) is a so-called vertical geodesic, and $\gamma(t) \equiv p$ and Γ is a great circle in T_pM_1.
2. If $W_2 = 0$, then (γ,Γ) is a so-called horizontal geodesic, and $\gamma(t)$ is a geodesic in M and $\Gamma(t)$ a parallel vector field along γ.
3. If $W_1 \neq 0$ and $W_2 \neq 0$, then (γ,Γ) is a so-called oblique geodesic.

In [BBNV03], the authors have shown that all g^S-geodesics project to a geodesic or a great circle if and only if the manifold M is either flat or a two-dimensional space of constant curvature.

1.2.2.2 The horizonal lift

The horizontal lift g^h on TM is derived from the metric g by

$$\begin{aligned} g^h_{(p,V)}(X^h,Y^h) &= 0, \\ g^h_{(p,V)}(X^v,Y^h) &= g_p(X,Y), \\ g^h_{(p,V)}(X^v,Y^v) &= 0. \end{aligned}$$

It is a (non-degenerate) pseudo-Riemannian metric on TM of signature (n,n). In local coordinates it is defined by

$$g^h_{(p,V)} = \begin{pmatrix} V^a\Gamma^k_{ai}g_{kj} + V^a\Gamma^k_{aj}g_{ki} & g_{ij} \\ g_{ij} & 0 \end{pmatrix}.$$

Note that the horizontal lift of a Riemannian metric coincides with the so called complete lift [KS97, CF97].

Lemma 1.33. *Let* $(p,V_p) \in (TM, g^h)$. *Geodesics of* g^h *correspond to Jacobi fields along geodesics in* M. *The exponential map on* TM *is defined by*

$$^h\exp_{(p,V)}(W_1,W_2) = \big(\exp_p(W_1), d\exp_p(W_1)(W_2) + d_2\exp_p(W_1)(V)\big)$$

for $(W_1,W_2) \in T_{(p,V_p)}TM \cong (T_pM)^2$. *Its inverse is defined by*

$$^h\log_{(p,V_p)}(q,V_q) = \big(\log_p q, d\log_p q(V_q) + \; d_2\log_p q(V_p)\big)$$

for $(q,V_q) \in TM$ *such that* $d(p,q) \leq \mathrm{inj}_M(p)$.

Proof. It is known that geodesics in TM with respect to g^h project over geodesics in M and define Jacobi fields along them (see, e.g., [CF97]).
Consider then the family of geodesics

$$c(t,s) := \exp_{\exp_p(sV_p)}(t(W_1 + sW_2)).$$

Set $\gamma(t) = c(t,0)$, and

$$J(t) := \frac{\partial}{\partial s}_{|s=0} c(t,s) = d\exp_p(tW_1)(tW_2) + d_2\exp_p(tW_1)(V).$$

Then (γ, J) is a g^h-geodesic, and

$$\begin{aligned}(\gamma,J)(0) &= (p,V_p)\\ (\dot\gamma,\dot J)(0) &= (W_1,W_2).\end{aligned}$$

The definition of exp follows. As the chain rule implies

$$\begin{aligned}(d\exp_p(\log_p q))^{-1} &= d\log_p q,\\ d_2\exp_p\log_p q &= -d\exp_p(\log_p q)\; d_2\log_p q,\end{aligned}$$

we can directly calculate the logarithm as the inverse of the exponential map. □

Remark 1.34. The horizontal lift arises naturally when we consider the change of the distance between two curves γ and μ in M as we can calculate

$$\frac{d}{dt}_{|t=0} \left|\log_{\gamma(t)} \mu(t)\right|_g^2 = \left|\, ^h\log_{(\gamma(0),\dot\gamma(0))}\left(\mu(0),\dot\mu(0)\right)\right|_{^hg}^2.$$

As g^h is only a pseudo-Riemannian metric, it is not meaningful to consider lengths with respect to g^h. We can however consider the g^S-length of g^h-geodesics and - as an approximation - of the g^h-logarithm.

Lemma 1.35. *Let* $(p,V_p),(q,V_q) \in TM$, *such that* $d(p,q) < \mathrm{inj}_M(p)$, *and*

$$\|V_q\|_g^2 + \|V_p\|_g^2 \le \frac{1}{|\operatorname{Rm}|_\infty^2 d^2(p,q)},$$

where Rm *denote the Riemannian curvature tensor. Let* $(\gamma, J) : I \to TM$ *be the* g^h*-geodesic connecting* (p, V_p) *to* (q, V_q)*. Set*

$${}^h\tilde{D}^2((p,V_p),(q,V_q)) := {}^S E(\gamma, J),$$

where ${}^S E(\gamma, J)$ *denotes the energy*

$${}^S E(\gamma, J) := \int_I {}^S g\left(\frac{d}{dt}(\gamma, J), \frac{d}{dt}(\gamma, J)\right) dt,$$

and

$${}^h D^2((p,V_p),(q,V_q)) := \|{}^h \log_{(p,V_p)}(q,V_q)\|^2_{g^S(p,V_p)}.$$

Then there exists a constant C_5 *depending on the curvature of* M *such that*

$$\frac{1}{C_5}\, {}^h D^2((p,V_p),(q,V_q)) \le {}^h\tilde{D}^2((p,V_p),(q,V_q)) \le C_5\, {}^h D^2((p,V_p),(q,V_q)). \quad (1.11)$$

Further, we can estimate

$$\frac{1}{2}\, {}^S D^2((p,V_p),(q,V_q)) \le {}^h D^2((p,V_p),(q,V_q)) \le 4\, {}^S D^2((p,V_p),(q,V_q)). \quad (1.12)$$

Proof. First note that the base curve γ of the g^h-geodesic (γ, J) is a g-geodesic, and that J is a Jacobi field along γ. We can write

$${}^h\tilde{D}^2((p,V_p),(q,V_q)) = {}^S E(\gamma, J) = \int_0^1 \|\dot\gamma(t)\|_g^2 + \|\dot J(t)\|_g^2\, dt,$$

and

$${}^h D^2((p,V_p),(q,V_q)) = \|{}^h \log_{(p,V_p)}(q,V_q)\|^2_{g^S(p,V_p)} = \|\dot\gamma(0)\|_g^2 + \|\dot J(0)\|_g^2.$$

Note that $\|\dot\gamma(t)\|$ is constant in t. We consider

$$\begin{aligned} \frac{d}{dt}\|\dot J(t)\| &= \frac{\langle \ddot J(t), \dot J(t)\rangle}{\|\dot J(t)\|} \\ &= -\frac{\langle R(J(t), \dot\gamma(t))\dot\gamma(t), \dot J(t)\rangle}{\|\dot J(t)\|}. \end{aligned}$$

As J is a Jacobi field, and $\|J(0)\|^2 + \|J(1)\|^2 \le \frac{1}{|\operatorname{Rm}|_\infty^2 d^2(p,q)}$, by Rauch comparison (see, e.g., [Jos08, Chapter 5.5]) there exists a constant $C(M)$ such that

$$\|J(t)\| \leq \frac{C(M)}{|\mathrm{Rm}|_\infty d(p,q)} \qquad \forall t \in I.$$

Thus, we can estimate for any $t, s \in I$

$$\begin{aligned}
\|\dot{J}(t)\| &= \|\dot{J}(s)\| + \int_s^t \frac{d}{d\tau}\|\dot{J}(\tau)\|\, d\tau \\
&\leq \|\dot{J}(s)\| + \int_s^t |\mathrm{Rm}|_\infty \, \|J(\tau)\| \, \|\dot{\gamma}(\tau)\|^2 \, d\tau \\
&\leq \|\dot{J}(s)\| + C(M)d(p,q)|t-s|.
\end{aligned}$$

Using Young's inequality, we can estimate

$$\begin{aligned}
{}^h\tilde{D}^2((p,V_p),(q,V_q)) &= d^2(p,q) + \int_0^1 \|\dot{J}(t)\|^2 \, dt \\
&\leq \left(1 + \frac{\sqrt{3}}{6}C(M)\right) d^2(p,q) + \frac{\sqrt{3}}{6}C(M)\|\dot{J}(0)\|^2 \\
&\leq \left(1 + \frac{\sqrt{3}}{6}C(M)\right) {}^hD^2((p,V_p),(q,V_q)).
\end{aligned}$$

Analogously, we can estimate

$$\begin{aligned}
{}^hD^2((p,V_p),(q,V_q)) &= d^2(p,q) + \int_0^1 \|\dot{J}(0)\|^2 \\
&\leq \left(1 + \frac{\sqrt{3}}{6}C(M)\right) d^2(p,q) + \frac{\sqrt{3}}{6}C(M)\int_0^1 \|\dot{J}(t)\|^2 \, dt \\
&\leq \left(1 + \frac{\sqrt{3}}{6}C(M)\right) {}^h\tilde{D}^2((p,V_p),(q,V_q)).
\end{aligned}$$

This concludes the proof of (1.11).
To see (1.12), note that we can write

$$\begin{aligned}
{}^hD^2((p,V_p),(q,V_q)) &= \|{}^h\log_{(p,V_p)}(q,V_q)\|^2_{g^S(p,V_p)} \\
&= \|\log_p q\|^2_{g(p)} + \|d\log_p q(V_q) + \, d_2\log_p q(V_p)\|^2_{g(p)} \\
&= \|\log_p q\|^2_{g(p)} \\
&\quad + \|\pi_{q\mapsto p}V_q - V_p + (d\log_p q - \pi_{q\mapsto p})(V_q) + (Id + \, d_2\log_p q)(V_p)\|^2_{g(p)},
\end{aligned}$$

where $\pi_{q\mapsto p}$ denotes the parallel transport along the geodesic connecting p and q.
To obtain the estimate from below, we observe using Proposition A.1

$${}^hD^2((p,V_p),(q,V_q)) \geq \|\log_p q\|^2 + \frac{1}{2}\|\pi_{q\mapsto p}V_q - V_p\|^2$$

$$
\begin{aligned}
&\quad - 2\left(\|d\log_p q - \pi_{q\mapsto p}\|^2\|V_q\|^2 + \|Id + \ d_2\log_p q\|^2\|V_p\|^2\right)\\
&\geq \|\log_p q\|^2 + \frac{1}{2}\|\pi_{q\mapsto p}V_q - V_p\|^2\\
&\quad - \frac{1}{2}|\operatorname{Rm}|_\infty^2\|\log_p q\|^4\left(\|V_q\|^2 + \|V_p\|^2\right)\\
&\geq \frac{1}{2}\left(\|\log_p q\|^2 + \|\pi_{q\mapsto p}V_q - V_p\|^2\right)\\
&\geq \frac{1}{2}\,{}^S D^2((p,V_p),(q,V_q)).
\end{aligned}
$$

Analogously, we can estimate from above

$$
\begin{aligned}
{}^h D^2((p,V_p),(q,V_q)) &\leq \|\log_p q\|^2 + 2\|\pi_{q\mapsto p}V_q - V_p\|^2\\
&\quad + 4\left(\|d\log_p q - \pi_{q\mapsto p}\|^2\|V_q\|^2 + \|Id + \ d_2\log_p q\|^2\|V_p)\|^2\right)\\
&\leq \|\log_p q\|^2 + 2\|\pi_{q\mapsto p}V_q - V_p\|^2\\
&\quad + |\operatorname{Rm}|_\infty\|\log_p q\|^4\left(\|V_q\|^2 + \|V_p\|^2\right)\\
&\leq 2\left(\|\log_p q\|^2 + \|\pi_{q\mapsto p}V_q - V_p\|^2\right).
\end{aligned}
$$

For any smooth curve γ connecting p and q the change of the parallel transport along the closed curve consisting of the geodesic between p and q and the curve γ can be estimated by the curvature and the area enclosed (see, e.g., [MTW73, Chapter 11.4]), i.e.,

$$
\|V_q - \pi_{p\mapsto q}\pi^\gamma_{q\mapsto p}V_q\| \leq \|\operatorname{Rm}\|_\infty d(p,q)L(\gamma)\|V_q\| \leq L(\gamma).
$$

Thus, we can estimate

$$
\|\pi_{q\mapsto p}V_q - V_p\| \leq \|\pi_{q\mapsto p}V_q - \pi^\gamma_{q\mapsto p}V_q\| \leq L(\gamma) + \|\pi^\gamma_{q\mapsto p}V_q - V_p\|.
$$

This implies the estimate from above in (1.11). □

Obviously, ${}^h\tilde{D}$ and ${}^h D$ are not distances but are compatible with one. We define the following notion of approximate distance.

Definition 1.36. Let S be a set and $D : S \times S \to \mathbb{R}$ be a positive definite mapping. We say that D is an inframetric on S, if D is symmetric and there exists a constant $C_6 > 1$, such that D fulfills a C_6-relaxed triangle inequality

$$
D(x,y) \leq C_6\left(D(x,z) + D(z,y)\right) \qquad \forall x,y,z \in S.
$$

If D is only symmetric up to a constant C_7, i.e.,

$$
D(x,y) \leq C_7 D(y,x) \qquad \forall x,y \in S,
$$

we call D a quasi-inframetric.

Remark 1.37. Lemma 1.35 implies that although ${}^h\tilde{D}$ and ${}^hD$ are not distances, they form (quasi)-inframetrics on those subsets U of TM, where for any pair $(p,V_p),(q,V_q)$, we have $d(p,q) < \operatorname{inj}_M(p)$, and $\|V_q\|_g^2 + \|V_p\|_g^2 \le \frac{1}{|\mathrm{Rm}|_\infty^2 d^2(p,q)}$. In particular, positive definiteness follows by estimating from below by ${}^SD$. Moreover, ${}^h\tilde{D}$ is symmetric, and ${}^hD$ is symmetric on U up to the constant $2\sqrt{2}$. Further, ${}^hD$ fulfills a relaxed triangle inequality on U with respect to the constant $2\sqrt{2}$, which also implies a $(2\sqrt{2}C_5^2)$-relaxed triangle inequality for ${}^h\tilde{D}$.

The length of g^h-geodesics is of particular interest, as the derivatives of geodesic homotopies are g^h-geodesic homotopies in the following sense.

Lemma 1.38. *Let $u,v \in C^1(\Omega,M)$ such that $d_{L^\infty}(u,v) \le \operatorname{inj}_M(p)$ for all points $p \in u(\Omega)\cup v(\Omega) \subset M$. Let $\Gamma : I\times\Omega \to M$ be the geodesic homotopy connecting u to v. Then $\Gamma(t,\cdot) \in C^1(\Omega,M)$, and for any $\alpha \in \{1,\dots,d\}$ the curve $d^\alpha\Gamma(\cdot,x) : I \to TM$ describes a g^h-geodesic in TM connecting $d^\alpha u(x)$ to $d^\alpha v(x)$.*

Proof. The geodesic homotopy Γ is defined by $\Gamma(t,x) = \exp_{u(x)}(t\log_{u(x)} v(x))$. As we stay within the injectivity radius, exp is a diffeomorphism. Thus, by the chain rule, $\Gamma(t,\cdot) \in C^1(\Omega,M)$ and $d^\alpha\Gamma(\cdot,x) : I \to TM$ is well defined. Obviously, the vector field $J := d^\alpha\Gamma(\cdot,x)$ connects $d^\alpha u(x)$ to $d^\alpha v(x)$ along $\gamma := \Gamma(\cdot,x)$. We need to show, that J is a Jacobi field. We calculate

$$\begin{aligned}
\ddot{J} &= \nabla_{\frac{d}{dt}}\nabla_{\frac{d}{dt}}\nabla_{\frac{d}{dx^\alpha}}\gamma \\
&= \nabla_{\frac{d}{dt}}\nabla_{\frac{d}{dx^\alpha}}\nabla_{\frac{d}{dt}}\gamma \\
&= \nabla_{\frac{d}{dx^\alpha}}\nabla_{\frac{d}{dt}}\nabla_{\frac{d}{dt}}\gamma - R(\nabla_{\frac{d}{dx^\alpha}}\gamma, \nabla_{\frac{d}{dt}}\gamma)\nabla_{\frac{d}{dt}}\gamma \\
&= -R(J,\dot{\gamma})\dot{\gamma}.
\end{aligned}$$

□

Remark 1.39. Geodesic homotopies also inherit the weak differentiability of their endpoint functions. Indeed, if exp and log are in C^k in their arguments and M admits a C^k-embedding into Euclidean space, $u,v \in W^{k,p}(\Omega,M)\cap C(\Omega,M)$, and $d_{L^\infty}(u,v) \le \operatorname{inj}_M(p)$ for all $p \in u(\Omega)\cup v(\Omega) \subset M$, then the geodesic homotopy connecting u to v lies in $W^{k,p}(\Omega,M)\cap C(\Omega,M)$. This follows by the chain rule.

1.2.3 Sobolev Distances for Manifold Targets

Definition 1.12 implies a notion of Sobolev distance implied by an embedding $\iota : M \to \mathbb{R}^N$

$$d_{W_\iota^{1,p}(\Omega,M)}(u,v) := \|\iota\circ u - \iota\circ v\|_{W^{1,p}(\Omega,\mathbb{R}^N)}. \tag{1.13}$$

Geodesics for this distance depend on the embedding. Our goal is to introduce an equivalent concept which is intrinsic. In particular we like to keep one class of distance realizing curves for all $W^{k,p}$-distances. We start by considering smooth functions.

Definition 1.40. Let $u,v \in C^1(\Omega,M)$, and consider $d^\alpha u$, $d^\alpha v$ for $\alpha = 1,\dots,d$ as vector fields along u and v, respectively. Then we set

$$
\begin{aligned}
{}^S D_{1,p}^p(u,v) &:= \sum_{\alpha=1}^{d} \int_\Omega {}^S D^p((u(x),d^\alpha u(x)),(v(x),d^\alpha v(x)))\,dx \\
{}^h \tilde{D}_{1,p}^p(u,v) &:= \sum_{\alpha=1}^{d} \int_\Omega {}^h \tilde{D}^p((u(x),d^\alpha u(x)),(v(x),d^\alpha v(x)))\,dx \\
{}^h D_{1,p}^p(u,v) &:= \sum_{\alpha=1}^{d} \int_\Omega {}^h D^p((u(x),d^\alpha u(x)),(v(x),d^\alpha v(x)))\,dx.
\end{aligned}
$$

Proposition 1.41. *Let $q > \max\{2,d\}$, and let K and L be two constants such that $L \le \mathrm{inj}(M)$ and $KL \le \frac{1}{|\mathrm{Rm}|_\infty}$. We set*

$$
W_K^{1,q} := \left\{ v \in W^{1,q}(\Omega,M) \,:\, \theta_{1,q,\Omega}(v) \le K \right\}, \tag{1.14}
$$

and denote by $H_{K,L}^{1,2,q}$ a L-ball w.r.t. L^s in $W_K^{1,q}$, i.e., for all pairs $u,v \in H_{K,L}^{1,2,q}$ holds $d_{L^s(\Omega,M)}(u,v) \le L$, where

$$
s := \begin{cases} \frac{2q}{q-2} & \text{for } d = 1 \\ \frac{4q}{q-2} & \text{for } d = 2 \\ \frac{dq}{q-d} & \text{for } d > 2. \end{cases}
$$

Then ${}^S D_{1,2}(u,v)$, ${}^h \tilde{D}_{1,2}(u,v)$, and ${}^h D_{1,2}(u,v)$ as defined in Definition 1.40 are equivalent on $H_{K,L}^{1,2,q}$. In particular, ${}^h D_{1,2}$ defines a quasi-inframetric on $H_{K,L}^{1,2,q}$ (cf. Definition 1.36).

Proof. The proof follows along the same argumentation as that of Lemma 1.35. Hölder's inequality is used to replace the $W^{1,\infty}$-bounds by the weaker ones that functions in $H_{K,L}^{1,2,q}$ fulfill. More specifically, we estimate terms of the form

$$
\begin{aligned}
\|Rm\|_\infty \left(\int_\Omega |\log_{u(x)} v(x)|^4 |d^\alpha u|^2 \,dx \right)^{\frac{1}{2}} &\le |\mathrm{Rm}|_\infty d_{L^s}(u,v)\, d_{L^r}(u,v)\, \|d^\alpha u\|_{L^q} \\
&\le d_{L^r}(u,v),
\end{aligned}
$$

where

$$
\frac{1}{r} := \frac{1}{2} - \frac{1}{q} - \frac{1}{s},
$$

and thus

$$r = \begin{cases} \infty & \text{for } d = 1 \\ \frac{4q}{q-2} & \text{for } d = 2 \\ \frac{2d}{d-2} & \text{for } d > 2. \end{cases}$$

Note that s is defined in such a way that we can estimate

$$d_{L^r}(u,v) \leq C\, {}^h D_{1,2}(u,v).$$

by the Sobolev embedding theorem. □

Proposition 1.41 essentially states that ${}^h D_{1,2}$ inherits on $H^{1,2,q}_{K,L}$ the properties ${}^h D$ has on a suitable ball in TM. As ${}^h\tilde{D}^2_{1,2}(u,v)$ is realized by geodesic homotopies, i.e.,

$${}^h\tilde{D}^2_{1,2}(u,v) = \int_0^1 \|\dot{\Gamma}(\cdot,t)\|^2_{W^{1,2}(\Omega,\Gamma(\cdot,t)^{-1}TM)}\, dt,$$

where Γ is the geodesic homotopy connecting u and v in $H^{1,2,q}_{K,L}$, Proposition 1.41 includes an estimate along geodesic homotopies of the form

$$\max_{t\in[0,1]} \|\dot{\Gamma}(\cdot,t)\|_{W^{1,2}(\Omega,\Gamma(\cdot,t)^{-1}TM)} \leq C \min_{t\in[0,1]} \|\dot{\Gamma}(\cdot,t)\|_{W^{1,2}(\Omega,\Gamma(\cdot,t)^{-1}TM)}\, dx,$$

where the constant depends on the curvature of M. In particular, we obtain a similar result for parallel vector fields along Γ.

Lemma 1.42. *Let $u,v \in H^{1,2,q}_{K,L}$ as defined in Proposition 1.41, and let Γ be the geodesic homotopy connecting u to v. Consider a parallel vector field $V \in W^{1,2}(\Omega \times I, \Gamma^{-1}TM) \cap C(\Omega \times I, \Gamma^{-1}TM)$ along Γ. Then there exists a constant C_8 depending on the curvature of M, the Sobolev constant, and d such that*

$$\begin{aligned} \frac{1}{1+C_8 t}\|V(\cdot,0)\|_{W^{1,2}(\Omega,\Gamma(\cdot,t)^{-1}TM)} &\leq \|V(\cdot,t)\|_{W^{1,2}(\Omega,\Gamma(\cdot,t)^{-1}TM)} \\ &\leq (1+C_8 t)\|V(\cdot,0)\|_{W^{1,2}(\Omega,\Gamma(\cdot,t)^{-1}TM)} \end{aligned}$$

holds for all $t \in I$.

Proof. As V is parallel, $\|V(\cdot,t)\|_{L^2(\Omega,\Gamma(\cdot,t)^{-1}TM)}$ is constant. Thus, we just need to consider

$$U^2(t) := \sum_{\alpha=1}^{d} \|\nabla_{d^\alpha \Gamma} V(\cdot,t)\|^2_{L^2(\Omega,\Gamma(\cdot,t)^{-1}TM)}.$$

We calculate

$$\begin{aligned}\frac{d}{dt}U^2(t) &= 2\sum_{\alpha=1}^{d}\int_\Omega \langle \nabla_{d^\alpha\Gamma}V, \nabla_{\dot\Gamma}\nabla_{d^\alpha\Gamma}V\rangle\, dx \\ &= 2\sum_{\alpha=1}^{d}\int_\Omega \langle \nabla_{d^\alpha\Gamma}V, R(\dot\Gamma, d^\alpha\Gamma)V\rangle\, dx \\ &\le 2\sum_{\alpha=1}^{d} |\operatorname{Rm}|_\infty \int_\Omega \|\nabla_{d^\alpha\Gamma}V\|\,\|\dot\Gamma\|\,\|d^\alpha\Gamma\|\,\|V\|\, dx.\end{aligned}$$

Using Hölder's inequality, we can estimate

$$\frac{d}{dt}U^2(t) \le 2\sqrt{d}|\operatorname{Rm}|_\infty \|d^\alpha\Gamma\|_{L^q}\|\dot\Gamma\|_{L^s}\|V\|_{L^r}U(t),$$

where

$$\frac{1}{r} = \frac{1}{2} - \frac{1}{q} - \frac{1}{s}$$

as in the proof of Proposition 1.41. We can rewrite this to

$$\begin{aligned}\frac{d}{dt}U(t) &\le \sqrt{d}|\operatorname{Rm}|_\infty C(M)KL\|V\|_{L^r} \\ &\le \sqrt{d}C(M)\|V\|_{L^r}.\end{aligned}$$

Note that the right hand side is constant in t, as V is parallel. Thus we can estimate for any $t, s \in I$ using the Sobolev embedding theorem

$$\begin{aligned}\|V(t)\|_{L^2} + U(t) &\le \|V(s)\|_{L^2} + U(s) + \sqrt{d}C(M)\|V(s)\|_{L^r}|t-s| \\ &\le (1+\sqrt{d}C(M)C_1)\,(U(s) + \|V(s)\|_{L^2}).\end{aligned}$$

This implies the assertion. □

It follows that ${}^h\tilde{D}_{1,2}$, and thus ${}^hD_{1,2}$, is equivalent to the norm implied by an embedding ι on the restricted ball $H^{1,2;q}_{K,L}$.

Proposition 1.43. *Let $u, v \in H^{1,2;q}_{K,L}$ as defined in Proposition 1.41, and let $\iota : M \to \mathbb{R}^N$ be a smooth isometric embedding. Then there exists a constant C_9 depending on the curvature of M, $\|\iota\|_{C^2}$, and K, such that*

$$\|\iota \circ u - \iota \circ v\|_{W^{1,2}(\Omega,\mathbb{R}^N)} \le C_9\, {}^h\tilde{D}_{1,2}(u,v).$$

If additionally $d_{L^\infty}(u,v) \le \operatorname{inj}_M$, there exists a constant C_{10} such that

$${}^h\tilde{D}_{1,2}(u,v) \le C_{10}\|\iota \circ u - \iota \circ v\|_{W^{1,2}(\Omega,\mathbb{R}^N)}.$$

Proof. To see the first estimate, we set

$$\beta(x,t) := \iota(\Gamma(x,t)),$$

where Γ denotes the geodesic homotopy connecting u to v. Then

$$\begin{aligned}
\|\iota\circ u-\iota\circ v\|^2_{W^{1,2}(\Omega,\mathbb{R}^N)} &\le \int_0^1\int_\Omega \|\dot\beta\|^2+\|D_x\dot\beta\|^2\,dx\,dt\\
&\le \int_0^1\int_\Omega \|\dot\Gamma\|^2+2\|\nabla_x\dot\Gamma\|^2+2|d^2\iota|^2_\infty\|\dot\Gamma\|^2\|\nabla_x\Gamma\|^2\,dx\,dt\\
&\le \int_0^1\int_\Omega \|\dot\Gamma\|^2+2\|\nabla_x\dot\Gamma\|^2\,dx\,dt\\
&\quad +2|d^2\iota|^2_\infty\int_0^1\left(\int_\Omega\|\nabla_x\Gamma\|^q\,dx\right)^{\frac{2}{q}}\left(\|\dot\Gamma\|^{\frac{q}{q-2}}\,dx\right)^{\frac{2q-4}{q}}\,dt\\
&\le 2(1+C\,|d^2\iota|^2_\infty K^2)\int_0^1\int_\Omega\|\dot\Gamma\|^2+\|\nabla_x\dot\Gamma\|^2\,dx\,dt.
\end{aligned}$$

For the second estimate, note that Proposition 1.41 implies for any homotopy γ connecting u to v

$${}^h\tilde D^2_{1,2}(u,v)\le C\int_0^1\int_\Omega\|\dot\gamma\|^2+\|\nabla_x\dot\gamma\|^2\,dx\,dt.$$

As u and v are close, we can connect them pointwise by curves defined by (1.10) using a local projection map π. Then chain rule implies

$$\int_0^1\int_\Omega\|\dot\alpha\|^2+\|\nabla_x\dot\alpha\|^2\,dx\,dt\le C\,(1+\|D^2(\iota^{-1}\circ\pi)\|^2_\infty K^2)\|\iota\circ u-\iota\circ v\|_{W^{1,2}(\Omega,\mathbb{R}^N)},$$

which concludes the proof. □

Remark 1.44. In this section we have defined ${}^hD_{1,2}(u,v)$ as the $W^{1,2}$-length of the direction of a geodesic homotopy connecting u to v at 0. Geodesic homotopies are families of geodesics in M. Thus, ${}^hD_{1,2}(u,v)$ is a simple intrinsic concept.

We have also shown that ${}^hD_{1,2}(u,v)$ is equivalent to the Sobolev distance for $W^{1,p}_\iota(\Omega,M)$ on a restricted ball $H^{1,2,q}_{K,L}$. In the following chapters we will always consider functions in such balls. We will omit the superscript h from now on and write $D_{1,2}$ instead of ${}^hD_{1,2}$. In view of the equivalence statements in Propositions 1.41 and 1.43 this is justifiable.

As a quasi-inframetric $D_{1,2}$ fulfills a relaxed triangle inequality, i.e., there exists a constant such that

$$D_{1,2}(u,v)\le C(D_{1,2}(u,w)+D_{1,2}(w,v))$$

holds for any three functions $u,v,w\in H^{1,2,q}_{K,L}$ (cf. Proposition 1.41).
Additionally we will need the following triangle inequality with respect to the smoothness descriptor.

Proposition 1.45. *Let $u,v \in H^{1,2,q}_{K,L}$ as defined in Proposition 1.41. Then there exists a constant C such that*

$$\dot{\theta}_{1,2,\Omega}(v) \le \dot{\theta}_{1,2,\Omega}(u) + C\, D_{1,2}(u,v). \tag{1.15}$$

Proof. Let $u,v \in H^{1,2,q}_{K,L}$, and let Γ be the geodesic homotopy connecting u to v. Then

$$\begin{aligned}\dot{\theta}_{1,2,\Omega}(v) - \dot{\theta}_{1,2,\Omega}(u) &= \int_0^1 \frac{d}{dt}\dot{\theta}_{1,2,\Omega}(\Gamma(t))\, dt \\ &\le C^{\,h}\tilde{D}_{1,2}(u,v) \\ &\le C\, D_{1,2}(u,v).\end{aligned}$$ □

Remark 1.46. All proofs in this section can be done for general p instead of $p=2$ as well. In particular, we have to choose $q > \max\{p,d\}$,

$$s := \begin{cases} \frac{pq}{q-p} & \text{for } d < p1 \\ \frac{2pq}{q-p} & \text{for } d = p \\ \frac{dq}{q-d} & \text{for } d > p, \end{cases}$$

and

$$\frac{1}{r} := \frac{1}{p} - \frac{1}{q} - \frac{1}{s}.$$

For the case $p = q > d$, we can set $r = s = \infty$ and still obtain the same results.

1.3 Scaling Properties

In the classical error analysis of numerical discretization schemes, the homogeneity of the Sobolev half-norms with respect to scaling of the domain $\Omega \in \mathbb{R}^d$ is utilized. In [GHS14] a similar behavior of the smoothness descriptor and the Sobolev infra-metric $D_{1,p}$ is shown, which we repeat here for completeness.

Definition 1.47. Let T_1, T_2 be two domains in $\mathbb{R}^d$, and $F : T_1 \to T_2$ a C^∞-diffeomorphism. For $l \in \mathbb{N}_0$ we say that F scales with h of order l if we have

$$\begin{aligned}\sup_{x\in T_2}\left|\partial^{\vec{\beta}} F^{-1}(x)\right| &\le C\, h^k \qquad &&\text{for all } \vec{\beta} \in [d]^k,\ k = 0,\dots,l, \\ |\det(DF(x))| &\sim h^{-d} &&\text{for all } x \in T_1 \ (\text{where } DF \text{ is the Jacobian of } F), \\ \sup_{x\in T_1}\left|\frac{\partial}{\partial x^\alpha}F(x)\right| &\le C\, h^{-1} &&\text{for all } \alpha = 1,\dots,d.\end{aligned}$$

Note that as derivatives commute the multi-indices defined by (1.6) can be equivalently replaced by ordinary multi-indices for $\mathbb{R}^d$.

Such F arise as transformations of an element of a discretization of Ω to a reference element and back. The smoothness descriptor scales in the following manner.

Lemma 1.48. *Let T_1, T_2 be two domains in $\mathbb{R}^d$, and $F : T_1 \to T_2$ a map that scales with h of order l. Consider $u \in W^{k,p}(T_1, M)$ with $1 \le k \le l$ and $p \in [1, \infty]$. Then*

$$\begin{aligned}\dot{\theta}_{k,p,T_2}(u \circ F^{-1}) &\le C\, h^{k-\frac{d}{p}} \left(\sum_{l=1}^{k} \dot{\theta}^p_{l,p,T_1}(u) \right)^{\frac{1}{p}} \\ &\le C\, h^{k-\frac{d}{p}}\, \theta_{k,p,T_1}(u).\end{aligned}$$

Proof. The proof follows directly by chain rule and the transformation of the integrals. The details can be found in [GHS14]. □

The smoothness descriptor for vector fields scales similarly.

Lemma 1.49. *Let T_1, T_2 be two domains in $\mathbb{R}^d$, and $F : T_1 \to T_2$ a map that scales with h of order l. Consider $u \in W^{k,p}(T_1, M)$ with $1 \le k \le l$ and $p \in [1, \infty]$, and $V \in W^{k,p}(T_1, u^{-1}TM)$. Then*

$$\begin{aligned}\dot{\Theta}_{k,p,T_2}(V \circ F^{-1}) &\le C\, h^{k-\frac{d}{p}} \left(\sum_{l=1}^{k} \dot{\Theta}^p_{l,p,T_1}(V) \right)^{\frac{1}{p}} \\ &\le C\, h^{k-\frac{d}{p}}\, \Theta_{k,p,T_1}(V).\end{aligned}$$

Proof. Again, the proof follows directly by chain rule and the transformation of the integrals. □

Remark 1.50. In Lemmas 1.48 and 1.49 the homogenous smoothness descriptor is bounded by the inhomogenous one.

The third assumption of Definition 1.47 is not needed for the proof of Lemmas 1.48 and 1.49. It is needed for the following 'inverse' estimate for the Sobolev error measure $D_{1,p}$.

Lemma 1.51. *Let T_1, T_2 be two domains in $\mathbb{R}^d$, and $F : T_1 \to T_2$ a map that scales with h of order l. Consider $u, v \in W^{1,p}(T_1, M) \cap C(T_1, M)$ with $p \in [1, \infty]$. Then*

$$\begin{aligned}d_{L^p}(u, v) &\le C\, h^{\frac{d}{p}}\, d_{L^p}(u \circ F^{-1}, v \circ F^{-1}) \\ D_{1,p}(u, v) &\le C\, h^{\frac{d}{p}-1} D_{1,p}(u \circ F^{-1}, v \circ F^{-1}).\end{aligned}$$

Proof. The proof follows directly follows from Definition 1.47, the chain rule, and integral transformation. □

The same argument also provides the following inverse estimates for the smoothness descriptor.

Lemma 1.52. *Let T_1, T_2 be two domains in $\mathbb{R}^d$, and $F : T_1 \to T_2$ a map that scales with h of order l. Consider $u \in W^{1,p}(T_1, M)$ with $p \in [1, \infty]$. Then*

$$\dot{\theta}_{1,p,T_1}(u) \le C\, h^{-1+\frac{d}{p}}\, \dot{\theta}_{1,p,T_2}(u \circ F^{-1}).$$

Chapter 2
Geodesic Finite Elements and Approximation Error Estimates

Geodesic finite elements have been introduced as a means to interpolate data in a Riemannian manifold [Gro13b], and approximate solutions of minimization problems over functions with Riemannian manifold targets [San12, San13]. An analysis of the interpolation error has been given in [GHS14].

In the following we will recall the definition of geodesic finite elements and some of their properties. The structure of standard finite elements usually allows a class of inverse estimates. Their translations to geodesic finite elements will be discussed in Section 2.1.1. We will then recall the interpolation error estimates published in [GHS14] with some minor changes, leading to a Bramble–Hilbert Lemma for geodesic finite elements. Lastly, we will consider discrete variations of geodesic finite elements which induce geodesic vector field interpolation. Corresponding interpolation error estimates will be derived.

Note that all estimates will be in terms of the smoothness descriptors as introduced in Section 1.1.4 and the Sobolev distance $D_{1,p}$ motivated in Section 1.2.3.

If not stated otherwise, Ω will denote a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary, and (M,g) will denote a (smooth) Riemannian manifold with curvature bounded from above and below.

We will need some estimates on the derivatives of the exponential map $\exp_p : T_pM \to M$ and its inverse $\log_p : B_{\mathrm{inj}(p)} \to T_pM$. The notation used and the estimates themselves can be found in Appendix A.

2.1 Definition and General Properties

Let $T \subset \mathbb{R}^d$ be a reference element, e.g., the reference simplex

$$T := \left\{ x = (x^1, \dots x^d) \in \mathbb{R}^d \;:\; x^\alpha \geq 0,\; \sum_{\alpha=1}^{d} x^\alpha \leq 1 \right\}.$$

Definition 2.1. We say that a conforming grid G for the domain $\Omega \subset \mathbb{R}^d$ is of width h and order m, if for each element T_h of G there exists a map $F_{T_h} : T_h \to T$ that scales with h of order m.

For a given order parameter m, let $a_i \in T$, $i = 1, \dots, l$ denote a set of Lagrangian nodes, and let $\lambda_i : T \to \mathbb{R}$ denote Lagrangian polynomials of order m such that

$$\lambda_i(a_j) = \delta_{ij} \qquad \forall 1 \leq i, j, \leq l,$$

and

$$\sum_{i=1}^{l} \lambda_i(x) = 1 \qquad \forall x \in T.$$

The following generalization of geodesic interpolation was given and motivated in [San13].

Definition 2.2. Let $\{\lambda_i, i = 1, \dots, l\}$ be a set of m-th order scalar Lagrangian shape functions, and let $v_i \in M$, $i = 1, \dots, l$ be values at the corresponding Lagrange nodes. We call

$$\Upsilon \,:\, M^l \times T \to M$$
$$\Upsilon(v_1, \dots, v_l; x) = \arg\min_{q \in M} \sum_{i=1}^{l} \lambda_i(x) d(v_i, q)^2 \tag{2.1}$$

m-th order geodesic interpolation on M.

The set of all such functions will be denoted by $P_m(T, M)$.

It is easy to verify that this definition reduces to m-th order Lagrangian interpolation if $M = \mathbb{R}^n$ and $d(\cdot, \cdot)$ denotes the standard distance.

For manifolds with either negative sectional curvature or certain restrictions on the curvature and the v_i, well-posedness of the definition for $m = 1$ is a classic result by Karcher [Kar77]. For $m \geq 2$, where the λ_i can become negative, well-posedness has been proven in [San13]. We will do a simpler proof of well-posedness here as it will include a bound on the diameter of interpolating functions, which will be needed later.

Lemma 2.3. *For $i = 1, \dots, l$ let $v_i \in M$ with $d(v_i, v_1) \leq \rho$ for all $i = 1, \dots, l$. Then there exists a solution $\Upsilon(v_1, \dots, v_m; x)$ to the minimization problem* (2.1) *for all x, and there exists a constant $C_{11} \leq 6l \max_i \|\lambda_i\|_\infty$ depending on the shape functions λ_i such that for each $x \in T$ all solutions $\Upsilon(v_1, \dots, v_m; x)$ lie in $B_{C_{11}\rho}(v_1)$.*

If ρ is small enough depending on the curvature, the solution $v_I(x) := \Upsilon(v_1, \dots, v_l; x)$ is unique and $v_I : T \to M$ is smooth.

Proof. We denote by $F_i : M \to \mathbb{R}$ the squared distance function to v_i, i.e., $F_i(q) := d^2(v_i, q)$. Further we set $F : T \times M \to \mathbb{R}$

$$F(x,q) := \sum_{i=1}^{l} \lambda_i(x) F_i(q).$$

Let $C_m \geq 1$ be a bound for the shape functions of order m, i.e., $|\lambda_i(x)| \leq C_m$ for all $i = 1, \dots, l$ and all $x \in T$. Then

$$F(x, v_1) = \sum_{i=1}^{l} \lambda_i(x) d^2(v_1, v_i) \leq C_m l \rho^2.$$

Let $q \in M$ with $d(q, v_1) > 6C_m l \rho$. Then for this q we have

$$1 - \frac{1}{6C_m l} \leq 1 - \frac{\rho}{d(q,v_1)} \leq \frac{d(q,v_i)}{d(q,v_1)} \leq 1 + \frac{\rho}{d(q,v_1)} \leq 1 + \frac{1}{6C_m l},$$

and thus

$$\left| 1 - \frac{d^2(q,v_i)}{d^2(q,v_1)} \right| \leq \frac{1}{6C_m l} \left(2 + \frac{1}{6C_m l} \right) \leq \frac{1}{2C_m l}.$$

Therefore, we have for such q

$$\begin{aligned} F(x,q) &= \sum_{i=1}^{l} \lambda_i(x) d^2(q, v_i) \\ &= \sum_{i=1}^{l} \lambda_i(x) d^2(q, v_1) \left(1 - \left(1 - \frac{d^2(q,v_i)}{d^2(q,v_1)} \right) \right) \\ &= d^2(q, v_1) \left(1 - \sum_{i=1}^{l} \lambda_i(x) \left(1 - \frac{d^2(q,v_i)}{d^2(q,v_1)} \right) \right) \\ &\geq \frac{1}{2} d^2(q, v_1) \\ &\geq 18 C_m^2 l^2 \rho^2 \\ &> F(x, v_1). \end{aligned}$$

Note further, that for p with $d(p, v_1) \leq 6C_m l \rho$, we have

$$F(x,p) = \sum_{i=1}^{l} \lambda_i(x) d^2(p, v_i) \geq -2C_m l (1 + 6C_m l)^2 \rho^2.$$

As M is complete, $B := \overline{B_{6C_m l \rho}(v_1)}$ is compact. Thus, there exists at least one minimizer of $F(x, \cdot)$ in B, and all minimizers are in B.

In general this minimizer is not unique. If we assume that $(1 + 6C_m l)\rho \leq \operatorname{inj}(v_i)$ for each i, then the functions F_i are smooth, and

$$dF_i(q)(V) = 2\langle \log_{v_i} q, d\log_{v_i} q(V) \rangle$$

$$d^2F_i(q)(V,W) = 2\left(\langle d\log_{v_i} q(V), d\log_{v_i} q(W)\rangle + \langle \log_{v_i} q, d^2\log_{v_i} q(V,W)\rangle\right).$$

Thus, at $q = v_i$, we have $d^2F_i(v_i) = 2g_{v_i}$, where g_{v_i} denotes the metric of M at v_i. By continuity, there exists a constant $0 < \delta \leq \operatorname{inj}(v_i)$ such that

$$\|d^2F_i(q) - 2g_q\| \leq \frac{1}{2C_m l} \qquad \forall q \in B_\delta(v_i).$$

If we assume $(1+6C_m l)\rho \leq \delta$, this implies for all $q \in B$ strict convexity of $F(x,\cdot)$ at q, i.e. for $V \in T_qM$

$$\begin{aligned} d^2F(x,q)(V,V) &= \sum_{i=1}^{l} \lambda_i(x) d^2F_i(q)(V,V) \\ &= 2|V|^2 - \sum_{i=1}^{l} \lambda_i(x)(2g_q - d^2F_i(q))(V,V) \\ &\geq 2|V|^2 - C_m l\|d^2F_i(q) - 2g_q\|\,|V|^2 \\ &\geq \frac{3}{2}|V|^2. \end{aligned}$$

The strict convexity then implies uniqueness of the minimizer. As F is smooth in B, and d^2F is invertible as a small perturbance of the metric (in particular $\|d^2F(q) - 2g_q\| \leq \frac{1}{2}$), smoothness of v_I follows by the implicit function theorem. □

Since the values of Υ are defined as solutions of a minimization problem, we can also characterize them by the corresponding first-order optimality condition (see, for instance, [Kar77]).

Lemma 2.4. *The minimizer $q^* := \Upsilon(v_1,\dots,v_m;x)$ is (locally uniquely) characterized by the first-order condition*

$$\sum_{i=1}^{l} \lambda_i(x) \log_{q^*} v_i = 0 \ \in \ T_{q^*}M. \tag{2.2}$$

Proof. Let $q^* := \Upsilon(v_1,\dots,v_m;x)$, $V \in T_{q^*}M$, and $\gamma_V(t) = \exp_{q^*}(tV)$. Then

$$\begin{aligned} 0 &= \frac{d}{dt}_{|t=0} \sum_i \lambda_i(x) d^2(v_i, \gamma_V(t)) \\ &= 2\sum_i \lambda_i(x)\langle \log_{v_i} q^*, d\log_{v_i} q^*(V)\rangle \\ &= 2\sum_i \lambda_i(x)\langle \log_{v_i} q^*, \left(d\exp_{v_i}(\log_{v_i} q^*)\right)^{-1}(V)\rangle \\ &= 2\sum_i \lambda_i(x)\langle d\exp_{v_i}(\log_{v_i} q^*)(\log_{v_i} q^*), V\rangle \\ &= -2\sum_i \lambda_i(x)\langle \log_{q^*} v_i, V\rangle, \end{aligned}$$

where we have used $d\log_p q = \big(d\exp_p(\log_p q)\big)^{-1}$, the Gauss Lemma

$$\langle V,W\rangle = \langle d\exp_p(V)(V), d\exp_p(V)(W)\rangle \qquad \forall V,W \in T_pM,$$

and

$$d\exp_p(\log_p q)(\log_p q) = \frac{d}{dt}_{|t=1}\exp_p(t\log_p q) = -\log_q p.$$

Thus (2.2) corresponds indeed to the variational formulation of (2.1). □

As in the linear setting, we define global finite elements as continuous functions that are interpolants on each grid element.

Definition 2.5 (Geodesic finite elements). Let M be a Riemannian manifold and G a grid for a d-dimensional domain Ω, $d \geq 1$. A geodesic finite element function is a continuous function $v_h : \Omega \to M$ such that for each element T of G, $v_h|_T \in P_m(T,M)$. The space of all such functions will be called S_h^m.

Remark 2.6. Geodesic finite elements can be used to interpolate continuous function $v \in C(\Omega,M)$. Indeed, if G_h is a grid for Ω of width h and order m (cf. Definition 2.1), by continuity of v there exists an h_0 such that for all $h \leq h_0$ the interpolation nodes for each element are contained in a ball of radius ρ, and thus interpolation $v_I \in S_h^m$ is well-defined.

Note furthermore that we can always control the diameter $\operatorname{diam}(v_I(T_h))$ of the image of an element T_h of G_h und v_I by choosing h_0 small enough.

Already in [San12] it was observed, that geodesic finite elements are conforming and objective. We state the result here for completeness.

Lemma 2.7 (Properties of geodesic finite elements). *Geodesic finite elements are*

1. *locally smooth on each triangle*

$$P_m(T,M) \subset C^\infty(T,M).$$

2. *conforming*

$$S_h^m \subset W^{1,\infty}(\Omega,M) \subset W^{1,2}(\Omega,M) \cap C(\Omega,M),$$

   where $W^{1,p}(\Omega,M)$ is defined in Definition 1.12.
3. *objective, i.e., if $\phi : M \to M$ is an isometry, then for $v_h(x) = \Upsilon(v_1,\dots,v_m;x)$*

$$\phi(v_h(x)) = \Upsilon(\phi(v_1),\dots,\phi(v_m);x).$$

Proof. 1. The local smoothness of geodesic interpolants is already discussed in Lemma 2.3.

2. A function $v_h \in S_h^m$ is piecewise smooth and globally continuous. Let $\iota : M \to \mathbb{R}^N$ be a smooth embedding. Then $\iota \circ v_h$ is a piecewise smooth and globally continuous function from $\Omega \subset \mathbb{R}^d$ to $\mathbb{R}^N$. This implies $\iota \circ v_h \in W^{1,\infty}(\Omega, \mathbb{R}^N)$, and thus $v_h \in W^{1,\infty}(\Omega, M)$.
3. Objectivity follows directly by the invariance of the distance d under isometries. □

2.1.1 Inverse Estimates

We want to use the structure of geodesic finite elements to investigate so-called inverse estimates. By this term we mean estimates of higher order derivatives by lower order ones.

Throughout this section, let $\Omega \subset \mathbb{R}^d$ be a domain, G a grid of width h and order m on Ω (cf. Definition 2.1), $T_h \in G$ an element of the grid, and $T \subset \mathbb{R}^d$ the reference element. Moreover let $\rho > 0$ be small enough such that geodesic interpolation is well-defined and unique for values on M that lie within a ball of radius ρ on M (cf. Lemma 2.3).

Remark 2.8. For finite elements in the Euclidean setting, i.e. piecewise polynomials, the $(m+1)$th derivative vanishes on each element, i.e., for $v \in P_m(T, \mathbb{R}^n)$ we have

$$D^{m+1} v(x) \equiv 0.$$

Furthermore, due to the finite dimension of $P_m(T, \mathbb{R}^n)$, we have for $v \in P_m(T, \mathbb{R}^n)$, $0 \leq j \leq k$, and $p, q \in [0, \infty]$

$$\|D^k v\|_{L^p(T,\mathbb{R}^n)} \leq C \, \|D^j v\|_{L^q(T,\mathbb{R}^n)},$$

and thus for $v_h \in P_m(T_h, \mathbb{R}^n)$ by rescaling

$$\|D^k v_h\|_{L^p(T_h,\mathbb{R}^n)} \leq C \, h^{j-k+d\left(\frac{1}{p}-\frac{1}{q}\right)} \, \|D^j v_h\|_{L^q(T_h,\mathbb{R}^n)}.$$

Globally, we obtain for finite elements $\tilde{v}_h \in S_h$

$$\|D^k \tilde{v}_h\|_{L_h^p(\Omega,\mathbb{R}^n)} \leq C \, h^{j-k+d\left(\frac{1}{p}-\frac{1}{q}\right)} \|D^j \tilde{v}_h\|_{L_h^q(\Omega,\mathbb{R}^n)},$$

where the mesh dependent norm

$$\|D^j \tilde{v}_h\|_{L_h^p(\Omega,\mathbb{R}^n)} := \left(\sum_{T_h \in G} \|D^j \tilde{v}_h\|_{L^p(T_h,\mathbb{R}^n)}^p \right)^{\frac{1}{p}},$$

is well-defined for functions in S_h even though they are in general not globally in $W^{j,p}(\Omega, \mathbb{R}^n)$ (see, e.g., [Bra07]).

We would like to obtain similar estimates for geodesic finite elements.

Functions $v \in P_m(T,M)$ are defined as solutions to a minimization problem. In order to obtain bounds on the derivatives, we will first characterize these derivatives.

Proposition 2.9. *Let $v \in P_m(T,M)$ and $\vec{\alpha}$ a multi-index in the sense of Section 1.1.4 with $|\vec{\alpha}| = k$, $0 \le k \le m+1$. Then the $\vec{\alpha}$th derivative of v can be written as*

$$\begin{aligned}\nabla^{\vec{\alpha}} v(x) = \sum_{i=1}^{l} \lambda_i(x)(Id + d_2 \log_{v(x)} v_i)(\nabla^{\vec{\alpha}} v(x)) \\ + \sum_{\substack{\vec{\beta}\subset\vec{\alpha} \\ 2\le|\vec{\beta}|\le k}} \sum_{i=1}^{l} D^{\vec{\alpha}\setminus\vec{\beta}} \lambda_i(x) \left(\nabla^{\vec{\beta}} \log_{v(x)} v_i - d_2 \log_{v(x)} v_i(\nabla^{\vec{\beta}} v(x))\right) \\ + \sum_{\substack{\vec{\beta}\subset\vec{\alpha} \\ 1\le|\vec{\beta}|\le k-1}} \sum_{i=1}^{l} D^{\vec{\alpha}\setminus\vec{\beta}} \lambda_i(x)(Id + d_2 \log_{v(x)} v_i)(\nabla^{\vec{\beta}} v(x)) \\ + \sum_{i=1}^{l} D^{\vec{\alpha}} \lambda_i(x) \log_{v(x)} v_i. \end{aligned} \tag{2.3}$$

Proof. Consider (2.2) for v

$$0 = \sum_{i=1}^{l} \lambda_i(x) \log_{v(x)} v_i.$$

Differentiating yields (2.3), where we used that that $\sum_{i=1}^{l} \lambda_i(x) \equiv 1$, and hence, $\sum_{i=1}^{l} D^{\vec{\beta}} \lambda_i(x) \equiv 0$ for all $\vec{\beta} \neq 0$. □

The shape functions $\lambda_i : T \to \mathbb{R}$ are polynomials. Thus, their $(m+1)$th order derivatives vanish.

Example 2.10. Let $d = 1$ and $m = 1$. We consider $v \in P_1(I,M)$. Then v is a constant speed geodesic (see, e.g., [San10]). It is well known, that the second derivatives of geodesics vanish.

In general, however, we cannot expect the $(m+1)$th order derivatives of a function $v \in P_m(T,M)$ to vanish. Particularly, if a map $v : T \to M$ with a 2-dimensional image fulfills $\nabla^2 v \equiv 0$ this already implies that the image is flat.

Remark 2.11. It is not meaningful to aim for estimates in terms of Sobolev half-norms. Consider for $d \ge 2$, $\alpha, \beta \in [d]$, $v \in P_1(T,M)$, and a smooth isometry $\phi : M \to M$ the following derivative

$$\nabla^{\beta} \nabla^{\alpha} (\phi \circ v) = d^2 \phi \left(\nabla^{\alpha} v, \nabla^{\beta} v\right) + d\phi \left(\nabla^{\beta} \nabla^{\alpha} v\right).$$

For an objective method all estimates should change equivalently under isometries. Thus, we will phrase estimates in terms of the smoothness descriptor as it is compatible with the chain rule by definition (cf. Section 1.1.4). Note that this is even true, if we just expect the same kind of compatibility with embeddings of M into Euclidean space instead of isometries on M.

In order to obtain an estimate for the higher order smoothness descriptor of a geodesic interpolant u_I, we cannot simply use the fact that the $m+1$-th derivatives of the shape functions λ_i vanish. Instead, we will need to estimate the smoothness descriptor of the geodesic interpolant u_I by the smoothness descriptor of the continuous function u it interpolates.

Proposition 2.12. *Let M possess a C^k-atlas with $kp > d$, and let $m \geq k-1$. Then there exists a constant C_{12} such that for all $u \in W^{k,p}(T,M)$ with $u(T) \subset B_\rho \subset M$*

$$\dot{\theta}_{k,p}(u_I) \leq C_{12}\ \dot{\theta}_{k,p}(u),$$

holds for the interpolation $u_I \in P_m(T,M)$ of u.

Proof. Assume the converse, i.e. for every $K > 0$ there exists a set of points $\{u_i(K)\} \subset B_\rho$ on M and a function $v^K \in W^{k,p}(T,M)$ with $v^K(a_i) = u_i(K)$ for $i = 1,\dots,l$, such that

$$\dot{\theta}_{k,p}(v^K) < \frac{1}{K}\ \dot{\theta}_{k,p}(v_I^K).$$

This implies that for each K there are at least two distinct points in $\{u_i(K)\}$. As $\{u_i(K)\} \subset B_\rho$ and $\overline{B_\rho}$ is compact, we can assume w.l.o.g. that the sequences of points converge

$$u_i(K) \to u_i^\star \qquad \text{for } K \to \infty$$

for all $i = 1,\dots,l$. Let $u_I^\star \in P_m(T,M)$ denote the interpolation of $\{u_i^\star\}$. As the interpolation operator Υ is C^∞ in all its arguments as evident from the implicit function theorem (cf. [San13]), we can estimate

$$\dot{\theta}_{k,p}(v_I^K) \leq C\dot{\theta}_{k,p}(u_I^\star)$$

independently of K. Further, Lemma 2.3 shows that $v_I^K(T), u_I^\star \subset B_{C\rho}$. Let $\xi : B_{C\rho} \to \mathbb{R}^n$ be a local chart for M. As ξ is continuous, we have for $\hat{u}_i(K) := \xi(u_i(K))$, $\hat{u}_i(K) \to \hat{u}_i^\star := \xi(u_i^\star)$ in $\mathbb{R}^n$ for $K \to \infty$ and all $i = 1,\dots,l$. The regularity $\xi \in C^k$ implies $\hat{v}^K := \xi \circ v^K \in W^{k,p}(T,\mathbb{R}^n)$. The $\hat{v}^K$ are a Cauchy sequence in $W^{k,p}(T,\mathbb{R}^n)$:

$$\begin{aligned}
\|\hat{v}^K - \hat{v}^L\|_{W^{k,p}(T,\mathbb{R}^n)} &\leq C\left(|\hat{v}^K - \hat{v}^L|_{W^{k,p}(T,\mathbb{R}^n)} + \sum_{i=1}^{l} |(\hat{v}^K - \hat{v}^L)(a_i)|\right)\\
&\leq C\left(|\hat{v}^K|_{W^{k,p}(T,\mathbb{R}^n)} + |\hat{v}^L|_{W^{k,p}(T,\mathbb{R}^n)} + \sum_{i=1}^{l} |\hat{u}_i(K) - \hat{u}_i(L)|\right)
\end{aligned}$$

$$\leq C\left(\frac{C}{K}+\frac{C}{K}+\sum_{i=1}^{l}|\hat{u}_i(K)-\hat{u}_i(L)|\right)$$
$$\to 0, \qquad \text{as } K,L\to\infty,$$

where we have used $|\xi\circ v|_{W^{k,p}(T,\mathbb{R}^n)} \leq \dot{\theta}_{k,p,T}(\xi\circ v) \leq C\ \dot{\theta}_{k,p,T}(v)$ (see Propositions 1.23 and 1.24).

Thus, the $\hat{v}^K$ converge to a limit $v^\star \in W^{k,p}(T,\mathbb{R}^n)$. As $kp>d$, the Sobolev embedding implies that $\hat{v}^K$ converges to $v^\star$ pointwise, and in particular $v^\star(a_i)=\hat{u}_i^\star$ for $i=1,\dots,l$. Further, $\dot{\theta}_{k,p,T}(\hat{v}^K)\to 0$ for $K\to\infty$ implies that $v^\star$ is constant. This is a contradiction. □

As the homogeneous smoothness descriptor consists of mixed order derivatives, inverse estimates are more complicated than those for standard Sobolev half-norms. An exception are the first order smoothness descriptors. Analogously to the Euclidean case, we have the following first order inverse estimates.

Proposition 2.13. *Let $v\in P_m(T,M)$ with $v(T)\subset B_\rho$, ρ small enough (cf. Lemma 2.3), and in particular $\rho\leq \frac{1}{\sqrt{|R|_\infty \|\sum_i|\lambda_i|\|_\infty}}$. Let $p,q\in[1,\infty]$. Then there exist constants C_{13} and C_{14} such that*

$$\dot{\theta}_{1,p,T}(v)\leq C_{13}\ \dot{\theta}_{0,q,T;Q}(v), \tag{2.4}$$
$$\dot{\theta}_{1,p,T}(v)\leq C_{14}\ \dot{\theta}_{1,q,T}(v) \tag{2.5}$$

where Q is the v_i such that $\dot{\theta}_{0,p,T;v_i}(v)$ is maximal.

Thus, after rescaling we have for $v_h\in P_m(T_h,M)$

$$\dot{\theta}_{1,p,T_h}(v_h)\leq C\,h^{-1+d(\frac{1}{p}-\frac{1}{q})}\ \dot{\theta}_{0,q,T_h;Q}(v), \tag{2.6}$$
$$\dot{\theta}_{1,p,T_h}(v_h)\leq C\,h^{d(\frac{1}{p}-\frac{1}{q})}\ \dot{\theta}_{1,q,T_h}(v_h). \tag{2.7}$$

Globally, for $v_h\in S_h^m$, we can then estimate

$$\dot{\theta}_{1,p,\Omega}(v_h)\leq C\,h^{-1+d(\frac{1}{p}-\frac{1}{q})}\min_{Q\in v(\Omega)}\dot{\theta}_{0,q,\Omega;Q}(v), \tag{2.8}$$
$$\dot{\theta}_{1,p,\Omega}(v_h)\leq C\,h^{d(\frac{1}{p}-\frac{1}{q})}\ \dot{\theta}_{1,q,\Omega}(v_h). \tag{2.9}$$

Proof. As $v\in P_m(T,M)$, we can estimate for all $p,q\in[1,\infty]$ and v_j

$$\dot{\theta}_{0,q,T;v_j}(v)\leq C\ \max_i \theta_{0,p,T;v_i}(v). \tag{2.10}$$

Indeed, we have shown in Lemma 2.3, that

$$\dot{\theta}_{0,\infty,T;v_j}(v)=\max_{x\in\Omega} d(v(x),v_j)\leq C\ \max_i d(v_1,v_i).$$

Norm equivalence in $\mathbb{R}^l$ yields

$$\dot{\theta}_{0,\infty,T;v_j}(v) \le C\sum_{i=1}^{l} d(v_1,v_i) \le C\int_{\Omega}(d(v_1,v(x))+d(v(x),v_i))\,dx \le C\,\max_i \theta_{0,1,T;v_i}(v).$$

This implies (2.10).

Consider (2.3) for $k=1$

$$d^{\alpha}v(x) = \sum_{i=1}^{l}\lambda_i(x)(Id+d_2\log_{v(x)}v_i)(d^{\alpha}v(x)) + \sum_{i=1}^{l}\partial_{\alpha}\lambda_i(x)\log_{v(x)}v_i.$$

We estimate using (A.1)

$$|d^{\alpha}v(x)| = \sum_{i=1}^{l}|\lambda_i(x)|\frac{1}{2}|R|_{\infty}d^2(v(x),v_i)|d^{\alpha}v(x)| + \sum_{i=1}^{l}|\partial_{\alpha}\lambda_i(x)|d(v(x),v_i).$$

Using $d^2(v(x),v_i) \le \rho^2 \le \frac{1}{|R|_{\infty}\|\sum_i|\lambda_i|\|_{\infty}}$, we obtain

$$|d^{\alpha}v(x)| \le 2\sum_{i=1}^{l}|\partial_{\alpha}\lambda_i(x)|d(v_i,v(x)).$$

Integration and (2.10) yield the estimates.

Lemmas 1.48 and 1.52 then imply the rescaled estimates. □

Similarly to Proposition 2.13, we can obtain higher order inverse estimates.

Proposition 2.14. *Let $v \in P_m(T,M)$ with $v(T) \subset B_{\rho}$, ρ small enough (cf. Lemma 2.3), and in particular $\rho \le \frac{1}{\sqrt{|R|_{\infty}\|\sum_i|\lambda_i|\|_{\infty}}}$. Let $p,q \in [1,\infty]$. Then there exists a constant C_{15} such that*

$$\dot{\theta}_{2,p,T}(v) \le \dot{\theta}^2_{1,2p,T}(v) + C_{15}\,\dot{\theta}_{1,q,T}(v). \tag{2.11}$$

If we additionally assume that $F_h^{-1}: T \to T_h$ scales with order 2, we have after rescaling for $v_h \in P_m(T_h,M)$

$$\dot{\theta}_{2,p,T_h}(v_h) \le C\,\dot{\theta}^2_{1,2p,T_h}(v_h) + C\,h^{-1+d(\frac{1}{p}-\frac{1}{q})}\,\dot{\theta}_{1,q,T_h}(v_h). \tag{2.12}$$

Proof. Inequality (2.11) follows from (2.3) and (2.10) in the same manner as Proposition 2.13.

To see (2.12), we estimate using Lemma 1.48

$$\begin{aligned}\dot{\theta}_{2,p,T_h}(v_h) &\le C\,h^{-2+\frac{d}{q}}\left(\dot{\theta}_{2,p,T}(v_h\circ F_h^{-1}) + \dot{\theta}_{1,p,T}(v_h\circ F_h^{-1})\right)\\ &\le C\,h^{-2+\frac{d}{q}}\left(\dot{\theta}^2_{1,2p,T}(v_h\circ F_h^{-1}) + 2C_{14}\,\dot{\theta}_{1,q,T}(v_h\circ F_h^{-1})\right).\end{aligned}$$

Applying Lemma 1.48 again yields (2.12). □

2.2 Interpolation Error Estimates

In Euclidean theory interpolation error estimates for finite element approximations of real valued functions on a reference element T are closely related to the Bramble–Hilbert Lemma. Throughout this section we will assume $\rho > 0$ to be small enough such that Lemma 2.3 and Propositions 2.12 and 2.13 hold for the interpolation of values on a manifold that lie within a ball of radius ρ.

Lemma 2.15 (Classical Interpolation Error Estimate). *Let $(m+1)p > d$, $u \in W^{m+1,p}(T,\mathbb{R})$, and let $u_I \in P_m(T,\mathbb{R})$ be the nodal interpolation of u. Then there exists a constant C independent of u, such that*

$$\|u - u_I\|_{W^{1,2}(T,\mathbb{R})} \leq C\|D^{m+1}u\|_{L^2(T,\mathbb{R})}.$$

There are several possible ways to prove Lemma 2.15 (see e.g. [Bra07, Cia78]). In [Bra07] the equivalence of the norms

$$\|v\| := |v|_{W^{m+1,p}(\Omega,\mathbb{R})} + \sum_{i=1}^{l} |v(a_i)| \overset{!}{\sim} \|v\|_{W^{m+1,p}(\Omega,\mathbb{R})}$$

is shown and employed for the difference $v = u - u_I$. One of the main obstacles to generalize this method of proof to the manifold setting is the fact that differences are realized through vector fields, and as such depend on the base function u. A generalization of the mentioned norm equivalence is possible as $u^{-1}TM$ is a vector space, but the constants arising implicitely depend on the function u. We, however, require scalable estimates. Other methods of proof lead to similar difficulties.

To overcome this problem a direct estimate using a Taylor expansion was chosen in [GHS14] to estimate the L^2- and the $W^{1,2}$- errors (measured in $D_{1,2}$; cf. Section 1.2.3) of geodesic interpolation in terms of the smoothness operator. The scaling properties described in Section 1.3 then yield estimates of the same order as the estimates in the Euclidean setting.

We will repeat this approach, and then generalize it to the approximation of vector fields by vector fields which arise as discrete variations of geodesic finite elements. This will later enable us to prove optimal L^2-error estimates for geodesic finite element solutions of $W^{1,2}$-elliptic minimization problems.

2.2.1 A Bramble-Hilbert Lemma for Geodesic Finite Elements

We will present a generalization of Lemma 2.15 to geodesic finite elements. The result and the proof presented are very similar to the one in [GHS14], but with a slightly better control over the nonlinearity in the $W^{1,p}$-estimate.

As we will use a Taylor expansion to obtain the interpolation error estimates, we need the following technical tool in order to estimate the remainder terms. It can also be found similarly in the appendix of [GHS14].

Proposition 2.16. *Let $f \in L^p(\Omega,\mathbb{R})$, $\vec{\beta}$ a multi-index with $|\vec{\beta}| = k > \frac{d}{p}$. Then*

$$\left\| \int_0^1 t^{k-1} |f(tx)| x^{\vec{\beta}}\, dt \right\|_{L^p(\Omega,\mathbb{R})} \le C(\operatorname{diam}(\Omega)) \|f\|_{L^p(\Omega,\mathbb{R})}.$$

Proof. We assume w.l.o.g. that Ω is contained in a ball of radius 1. Using polar coordinates $(\omega, r) \in S^{d-1} \times (0,1)$, we write

$$\begin{aligned} \left\| \int_0^1 t^{k-1} |f(tx)| x^{\vec{\beta}}\, dt \right\|^p_{L^p(\Omega,\mathbb{R})} &= \int_0^1 \int_{S^{d-1}} \left(\int_0^1 r^k t^{k-1} |f(tr\omega)|\, dt \right)^p dS(\omega)\, dr \\ &= \int_0^1 \int_{S^{d-1}} \left(\int_0^r \tau^{k-1} |f(\tau\omega)|\, d\tau \right)^p dS(\omega)\, dr. \end{aligned}$$

Using Hölder's inequality on the inner integral, we obtain

$$\begin{aligned} &\left\| \int_0^1 t^{k-1} |f(tx)| x^{\vec{\beta}}\, dt \right\|^p_{L^p(\Omega,\mathbb{R})} \\ &\quad \le \int_0^1 \left(\int_{S^{d-1}} \int_0^r \tau^{d-1} |f(\tau\omega)|^p\, d\tau\, dS(\omega) \right) \left(\int_0^r \tau^{\frac{kp-d}{p-1}}\, d\tau \right)^{p-1} dr \\ &\quad \le \|f\|^p_{L^p(\Omega,\mathbb{R})}, \end{aligned}$$

as $kp > d$. □

We now prove an interpolation error estimate for geodesic finite elements.

Lemma 2.17. *Let $kp > d$, $m \ge k-1$, and $u \in W^{k,p}(T,M)$ with $u(T) \subset B_\rho \subset M$. Let $u_I \in P_m(T,M)$ denote the geodesic interpolation of u. Then there exists a constant C_{16} such that*

$$d_{L^p(T,M)}(u,u_I) \le C_{16} C_{1,u}(T)\, \dot{\theta}_{k,p,T}(u) \tag{2.13}$$

$$D_{1,p,T}(u,u_I) \le C_{16} \left(C^p_{1,u}(T) + C^p_{2,u}(T) \right)^{\frac{1}{p}} \dot{\theta}_{k,p,T}(u), \tag{2.14}$$

where

$$\begin{aligned} C_{1,u}(T) &:= \sup_{1 \le j \le k} \sup_{\substack{p \in u_I(T) \\ q \in u(T)}} \|d^j \log_p q\| \\ C_{2,u}(T) &:= \sup_{1 \le j \le k-1} \sup_{\substack{p \in u_I(T) \\ q \in u(T)}} \|d_2 d^j \log_p q\|, \end{aligned}$$

and the constant C_{16} depends on the shape functions λ_i, and but is independent of u and M.

Proof. The proof is based on the first order condition (cf. Lemma 2.4)

$$\sum_{i=1}^{l} \lambda_i(x) \log_{u_I(x)} u(a_i) = 0$$

for all $x \in T$. We can rewrite this to

$$\log_{u_I(x)} u(x) = \sum_{i=1}^{l} \lambda_i(x) \left(\log_{u_I(x)}(u(x)) - \log_{u_I(x)}(u(a_i)) \right).$$

Setting

$$G(x,y) := \log_{u_I(x)} u(y),$$

and performing a Taylor expansion of G in the second argument, we obtain

$$G(x,a_i) = \sum_{|\vec{\alpha}|<k} \frac{(a_i - x)^{\vec{\alpha}}}{\vec{\alpha}!} \partial_y^{\vec{\alpha}} G(x,x) + R_k(x,x,a_i)$$

almost everywhere in T, where

$$R_k(x,z,a_i) = \sum_{|\vec{\alpha}|=k} \frac{k}{\vec{\alpha}!} (a_i - z)^{\vec{\alpha}} \int_0^1 (1-t)^{k-1} \partial_y^{\vec{\alpha}} G(x, z + t(a_i - z))\, dt.$$

Thus, we can write

$$\begin{aligned} -\log_{u_I(x)} u(x) &= \sum_{i=1}^{l} \lambda_i(x) \left(G(x,a_i) - G(x,x) \right) \\ &= \sum_{1<|\vec{\alpha}|<k} \sum_{i=1}^{l} \lambda_i(x) \frac{(a_i - x)^{\vec{\alpha}}}{\vec{\alpha}!} \partial_y^{\vec{\alpha}} G(x,x) + \sum_{i=1}^{l} \lambda_i(x) R_k(x,x,a_i). \end{aligned}$$

As the shape functions are exact on polynomials of degree less then $k-1$, we have for each $\vec{\alpha}$ with $|\vec{\alpha}| \le k-1$

$$\sum_{i=1}^{l} \lambda_i(x) \frac{(a_i - x)^{\vec{\alpha}}}{\vec{\alpha}!} \partial_y^{\vec{\alpha}} G(x,x) = 0.$$

It remains the expression

$$-\log_{u_I(x)} u(x) = \sum_{i=1}^{l} \lambda_i(x) R_k(x,x,a_i). \tag{2.15}$$

After a suitable translation, we may assume $a_i = 0$, and estimate (almost everywhere in T)

$$|R_k(x,x,0)| \le \sum_{|\vec{\alpha}|=k} \frac{k}{\vec{\alpha}!} x^{\vec{\alpha}} \int_0^1 t^{k-1} \left| \partial_y^{\vec{\alpha}} G(x,tx) \right| \, dt$$

$$
\begin{aligned}
&\leq \sum_{|\vec{\alpha}|=k} \frac{k}{\vec{\alpha}!} x^{\vec{\alpha}} \int_0^1 t^{k-1} \sum_{\substack{1\leq o\leq k,\, \vec{\beta}_j\in[d]^{m_j} \\ \sum_{j=1}^o m_j = k}} \|d^o \log_{u_I(x)} u(tx)\| \prod_{j=1}^{o} |\nabla^{\vec{\beta}_j} u(tx)|\, dt \\
&\leq C(k) C_{1,u}(T) \sum_{\substack{1\leq o\leq k,\, \vec{\beta}_j\in[d]^{m_j} \\ \sum_{j=1}^o m_j = k}} \int_0^1 t^{k-1} \prod_{j=1}^{o} |\nabla^{\vec{\beta}_j} u(tx)| x^{\vec{\beta}}\, dt. \qquad (2.16)
\end{aligned}
$$

Integration yields

$$
\begin{aligned}
&\left(\int_T |\log_{u_I(x)} u(x)|^p \, dx \right)^{\frac{1}{p}} \\
&\quad \leq C(\lambda,k) C_{1,u}(T) \sum_{\substack{1\leq o\leq k,\, \vec{\beta}_j\in[d]^{m_j} \\ \sum_{j=1}^o m_j = k}} \left(\int_T \left(\int_0^1 t^{k-1} \prod_{j=1}^{o} |\nabla^{\vec{\beta}_j} u(tx)| x^{\vec{\beta}}\, dt \right)^p dx \right)^{\frac{1}{p}} \\
&\quad \leq C(\lambda,k,T) C_{1,u}(T) \sum_{\substack{1\leq o\leq k,\, \vec{\beta}_j\in[d]^{m_j} \\ \sum_{j=1}^o m_j = k}} \left(\int_T \prod_{j=1}^{o} |\nabla^{\vec{\beta}_j} u(tx)|^p \, dx \right)^{\frac{1}{p}},
\end{aligned}
$$

where the last estimate follows from Proposition 2.16. Recalling the definition of the smoothness descriptor (cf. Definition 1.21) this proves (2.13). Let $\vec{e}$ be a multi-index with $|\vec{e}| = 1$. To obtain (2.14), we split $\nabla^{\vec{e}} \log_{u_I(x)} u(x)$ in the following way:

$$
\begin{aligned}
\nabla^{\vec{e}} \log_{u_I(x)} u(x) &= d_2 \log_{u_I(x)} u(x) (\nabla^{\vec{e}} u_I(x)) + d \log_{u_I(x)} u(x) (\nabla^{\vec{e}} u(x)) \\
&= \sum_{i=1}^{l} \lambda_i(x) \Big(d_2 \log_{u_I(x)} u(x) (\nabla^{\vec{e}} u_I(x)) - d_2 \log_{u_I(x)} u(a_i) (\nabla^{\vec{e}} u_I(x)) \Big) \\
&\qquad + d \log_{u_I(x)} u(x) (\nabla^{\vec{e}} u(x)) - \sum_{i=1}^{l} \partial^{\vec{e}} \lambda_i(x) \log_{u_I(x)} u(a_i).
\end{aligned}
$$

We set

$$
H(x,y) := d_2 \log_{u_I(x)} u(y) (\nabla^{\vec{e}} u_I(x)),
$$

and write

$$
\nabla^{\vec{e}} \log_{u_I(x)} u(x) = \sum_{i=1}^{l} \lambda_i(x) \left(H(x,x) - H(x,a_i) \right) + \partial_y^{\vec{e}} \left(\sum_{i=1}^{l} \lambda_i(y) (G(x,y) - G(x,a_i)) \right)_{|y=x}.
$$

We can handle the H term as we did the the G term before, using a Taylor expansion of H up to $k-1$ to obtain

$$\sum_{i=1}^{l} \lambda_i(x)\left(H(x,x) - H(x,a_i)\right) = -\sum_{i=1} \lambda_i(x) R^H_{k-1}(x,x,a_i),$$

and

$$\begin{aligned}
&|R^H_{k-1}(x,x,0)| \\
&\leq \sum_{|\vec{\alpha}|=k-1} \frac{k-1}{\vec{\alpha}!} x^{\vec{\alpha}} \int_0^1 t^{k-2} \left|\partial_y^{\vec{\alpha}} H(x,tx)\right| \, dt \\
&\leq \sum_{|\vec{\alpha}|=k-1} \frac{k-1}{\vec{\alpha}!} x^{\vec{\alpha}} \int_0^1 t^{k-2} \sum_{\substack{1\leq o\leq k-1,\ \vec{\beta}_j\in[d]^{m_j} \\ \sum_{j=1}^{o} m_j = k-1}} \|d^o d_2 \log_{u_I(x)} u(tx)\| \, |\nabla^{\vec{e}} u_I(x)| \prod_{j=1}^{o} |\nabla^{\vec{\beta}_j} u(tx)| \, dt \\
&\leq C_{2,u}(T) \sum_{|\vec{\alpha}|=k-1} \frac{k-1}{\vec{\alpha}!} x^{\vec{\alpha}} \int_0^1 t^{k-2} \sum_{\substack{1\leq o\leq k-1,\ \vec{\beta}_j\in[d]^{m_j} \\ \sum_{j=1}^{o} m_j = k-1}} |\nabla^{\vec{e}} u_I(x)| \prod_{j=1}^{o} |\nabla^{\vec{\beta}_j} u(tx)| \, dt.
\end{aligned} \tag{2.17}$$

Continuing as before, we need to estimate terms of the form

$$A := \left(\int_\Omega \left(|\nabla^{\vec{e}} u_I(x)| \int_0^1 t^{k-2} x^{\vec{\alpha}} \sum_{\substack{1\leq o\leq k-1,\ \vec{\beta}_j\in[d]^{m_j} \\ \sum_{j=1}^{o} m_j = k-1}} \prod_{j=1}^{o} |\nabla^{\vec{\beta}_j} u(tx)| \, dt \right)^p dx \right)^{\frac{1}{p}}.$$

Using Hölder's and Young's inequality, we obtain

$$\begin{aligned}
A &\leq \left(\int_\Omega |\nabla^{\vec{e}} u_I(x)|^{kp} \, dx \right)^{\frac{1}{kp}} \left\| \int_0^1 t^{k-2} x^{\vec{\alpha}} \sum_{\substack{1\leq o\leq k-1,\ \vec{\beta}_j\in[d]^{m_j} \\ \sum_{j=1}^{o} m_j = k-1}} \prod_{j=1}^{o} |\nabla^{\vec{\beta}_j} u(tx)| \, dt \right\|_{L^{\frac{kp}{k-1}}(\Omega,\mathbb{R})} \\
&\leq C \left(\dot{\theta}_{k,p,T}(u_I) + \left\| \int_0^1 t^{k-2} x^{\vec{\alpha}} \sum_{\substack{1\leq o\leq k-1,\ \vec{\beta}_j\in[d]^{m_j} \\ \sum_{j=1}^{o} m_j = k-1}} \prod_{j=1}^{o} |\nabla^{\vec{\beta}_j} u(tx)| \, dt \right\|_{L^{\frac{kp}{k-1}}(\Omega,\mathbb{R})}^{\frac{k}{k-1}} \right).
\end{aligned}$$

We can again use Proposition 2.16 with $\tilde{k} = k-1$, $\tilde{p} = \frac{kp}{k-1}$ to estimate

$$\begin{aligned}
&\left\| \int_0^1 t^{k-2} x^{\vec{\alpha}} \sum_{\substack{1\le o\le k-1,\, \vec{\beta}_j\in[d]^{m_j}\\ \sum_{j=1}^o m_j=k-1}} \prod_{j=1}^{o} |\nabla^{\vec{\beta}_j} u(tx)|\, dt \right\|_{L^{\frac{kp}{k-1}}(\Omega,\mathbb{R})}^{\frac{k}{k-1}} \\
&\qquad\qquad \le C \left\| \sum_{\substack{1\le o\le k-1,\, \vec{\beta}_j\in[d]^{m_j}\\ \sum_{j=1}^o m_j=k-1}} \prod_{j=1}^{o} |\nabla^{\vec{\beta}_j} u(x)| \right\|_{L^{\frac{kp}{p-1}}(T,\mathbb{R})}^{\frac{k}{k-1}} \\
&\qquad\qquad \le C\, \dot{\theta}_{k,p,T}(u),
\end{aligned}$$

where we used Proposition 1.22 for the last estimate.
Proposition 2.12 then implies

$$\| \sum_{i=1}^{l} \lambda_i(x)\, (H(x,x) - H(x,a_i))\, \|_{L^p(T,u_I^{-1}TM)} \le C\, C_{2,u}(T)\; \dot{\theta}_{k,p,T}(u).$$

For the remaining terms we can write

$$\begin{aligned}
&\partial_y^{\vec{e}} \left(\sum_{i=1}^{l} \lambda_i(y)(G(x,y) - G(x,a_i)) \right)_{|y=x} \\
&\qquad = -\partial_y^{\vec{e}} \left(\sum_{i=1}^{l} \lambda_i(y) R_k^G(x,y,a_i) \right)_{|y=x} \\
&\qquad = -\sum_{i=1}^{l} \partial^{\vec{e}} \lambda_i(x) R_k^G(x,x,a_i) - \left(\sum_{i=1}^{l} \lambda_i(y) \partial_y^{\vec{e}} R_k^G(x,y,a_i) \right)_{|y=x}.
\end{aligned}$$

The first term can be estimated as before. As described in [GHS14] the second term can be rewritten as

$$\begin{aligned}
-\sum_{i=1}^{l} \lambda_i(y) \partial_y^{\vec{e}} R_k^G(x,y,a_i) &= -\sum_{i=1}^{l} \lambda_i(y) \partial_y^{\vec{e}} \left(G(x,a_i) - \sum_{|\vec{\alpha}|<k} \frac{(a_i - y)^{\vec{\alpha}}}{\vec{\alpha}!} \partial_y^{\vec{\alpha}} G(x,y) \right) \\
&= \sum_{i=1}^{l} \lambda_i(y) \partial_y^{\vec{e}} \sum_{|\vec{\alpha}|<k} \frac{(a_i - y)^{\vec{\alpha}}}{\vec{\alpha}!} \partial_y^{\vec{\alpha}} G(x,y) \\
&= -\sum_{i=1}^{l} \lambda_i(y) \sum_{|\vec{\alpha}|<k} \frac{(a_i - y)^{\vec{\alpha}-\vec{e}}}{(\vec{\alpha}-\vec{e})!} \partial_y^{\vec{\alpha}} G(x,y) \\
&\quad + \sum_{i=1}^{l} \lambda_i(y) \sum_{|\vec{\alpha}|<k} \frac{(a_i - y)^{\vec{\alpha}}}{\vec{\alpha}!} \partial_y^{\vec{\alpha}+\vec{e}} G(x,y)
\end{aligned}$$

$$= \sum_{i=1}^{l} \lambda_i(y) \sum_{|\vec{\alpha}|=k-1} \frac{(a_i - y)^{\vec{\alpha}}}{\vec{\alpha}!} \partial_y^{\vec{\alpha}+\vec{e}} G(x,y).$$

Thus, we obtain

$$\partial_y^{\vec{e}} \left(\sum_{i=1}^{l} \lambda_i(y)(G(x,y) - G(x,a_i)) \right)_{|y=x}$$
$$= -\sum_{i=1}^{l} \partial^{\vec{e}} \lambda_i(x) R_k^G(x,x,a_i) + \sum_{i=1}^{l} \lambda_i(x) \sum_{|\vec{\alpha}|=k-1} \frac{(a_i - x)^{\vec{\alpha}}}{\vec{\alpha}!} \partial_y^{\vec{\alpha}+\vec{e}} G(x,x),$$

and

$$\left\| \partial_y^{\vec{e}} \left(\sum_{i=1}^{l} \lambda_i(y)(G(x,y) - G(x,a_i)) \right)_{|y=x} \right\|_{L^p(T, u_I^{-1}TM)} \leq C\, C_{1,u}(T)\; \dot{\theta}_{k,p,T}(u),$$

which concludes the proof. □

Remark 2.18. When estimating the remainder terms of the Taylor expansions in the above proof, the smoothness descriptor arises naturally when using the chain rule for derivatives of the auxillary functions G and H in (2.16) and (2.17), respectively. In the Euclidean setting $M = \mathbb{R}^n$ the inverse of the exponential map log corresponds to simple subtraction, and thus only the first derivative of log contributes to the constants, which implies $C_{1,u}(T) = 1, C_{2,u}(T) = 0$. If we consider this before taking the supremum in (2.16) and (2.17), we also see that the Sobolev half-norm captures all derivatives of u on the right hand side. Thus, we retrieve the standard interpolation error estimate for finite elements (cf. Lemma 2.15).

Remark 2.19. In [GHS14] the constant $C_{2,u}(T)$ depends on derivatives of log up to $(k+1)$th order, while in Lemma 2.17 only derivatives up to kth order appear. In particular for $k = 2$ this yields geometric constants, i.e., using the estimate on the derivatives of log in Appendix A we can estimate

$$C_{1,u}(T) \leq \sup_{\substack{p \in u_I(T) \\ q \in u(T)}} \max \left\{ 1 + C(\mathrm{Rm}) d^2(p,q), C_{30}(\mathrm{Rm}) d(p,q) \right\} \leq 1 + C\rho^2,$$
$$C_{2,u}(T) \leq \sup_{\substack{p \in u_I(T) \\ q \in u(T)}} C_{30}(\mathrm{Rm}) d(p,q) \leq C\rho,$$

if ρ is small enough depending on $C_{30}(\mathrm{Rm})$, the constant in Proposition A.2. This is consistent with the intuition that in very small neighborhoods the constants behave like in the flat case.

Furthermore, the proof in [GHS14] uses that u_I has uniformly bounded derivatives independent of u and M, which is valid in view of Proposition 2.13 and the condition on ρ. In the approach presented above, derivatives of u_I need only be as good as the ones of u (cf. Proposition 2.12). This difference in the approach to [GHS14] makes generalizations to higher derivatives possible, although we will not include these estimates in this work.

As in Euclidean theory, Lemma 2.17 lets us state a Bramble–Hilbert Lemma for geodesic finite element interpolation.

Corollary 2.20. *Let $kp > d$, $m \geq k-1$, and Y denote a vector space with norm $\|\cdot\|$. Let $L: W^{k,p}(T,M) \to Y$ be a map such that $P_m(T,M) \subset \ker L$, and assume that for all $u \in W^{k,p}(T,M)$ the map $\tilde{L}_u : W^{1,p}(T,u^{-1}TM) \to Y$ defined by $\tilde{L}_u(V) = Lu - L(\exp_u V)$ is bounded, i.e., $\|\tilde{L}_u(V)\| \leq \|\tilde{L}_u\| \, \|V\|_{W^{1,p}(T,u^{-1}TM)}$. Then*

$$\|Lv\| \leq C\|\tilde{L}_v\| \; \dot{\theta}_{k,p,T}(v)$$

for all $v \in W^{k,p}(T,M)$ with $v(T) \subset B_\rho$.

Proof. Let $v \in W^{k,p}(T,M)$ with $v(T) \subset B_\rho$. Then the interpolation $v_I \in P_m(T,M)$ is well defined, $Lv_I = 0$, and $\log_v v_I \in W^{1,p}(T,v^{-1}TM)$. By definition of $\tilde{L}_v$, we have

$$\begin{aligned} \|Lv\| = \|Lv - Lv_I\| &= \|Lv - L(\exp_v \log_v v_I)\| = \|\tilde{L}_v(\log_v v_I)\| \\ &\leq \|\tilde{L}_v\| \, \|\log_v v_I\|_{W^{1,p}(T,v^{-1}TM)}. \end{aligned}$$

Lemma 2.17 then yields the assertion. □

Remark 2.21. The Euclidean result analogous to Corollary 2.20, namely a bound on linear functionals, is used in [Cia78] to discuss the effect of numerical integration in the application of finite element methods to elliptic second order problems. The question whether Corollary 2.20 can fulfill the same role in the study of geodesic finite elements will not be answered in the course of this work, but is of interest in view of so-called variational crimes, in particular quadrature error estimates (cf. [Str72]).

2.2.2 Interpolation Error Estimates for Geodesic Finite Elements

Lemma 2.17 bounds the interpolation error on a reference element T. We now derive estimates for domains in $\Omega \subset \mathbb{R}^d$ discretized by a conforming grid of width h and order m (cf. Definition 2.1).

By scaling we have the following local estimate.

Theorem 2.22. *Let T be a reference element, and $T_h \in \mathbb{R}^d$ be a domain such that there exists a map $F_h : T_h \to T$ that scales with h of order m (cf. Definition 1.47). Let $kp > d$, $m \geq k-1$, and $u \in W^{k,p}(T_h,M)$ with $u(T_h) \subset B_\rho \subset M$.*
Then there exists a constant C_{17} such that we have for the geodesic interpolation $u_I \in P_m(T_h,M)$ of u the elementwise estimates

$$d_{L^p}(u,u_I) \le C_{17}\, h^k\, C_{1,u}(T)\; \theta_{k,p,T_h}(u)$$
$$D_{1,p}(u,u_I) \le C_{17}\, \left(C_{1,u}(T) + C_{1,u}(T)\right)\; h^{k-1}\; \theta_{k,p,T_h}(u).$$

Proof. Set $\hat{u} := u \circ F_h^{-1}$, and note that the geodesic interpolation of $\hat{u}$ has the representation

$$\hat{u}_I = u_I \circ F_h^{-1}.$$

Using the scaling Lemmas 1.51 and 1.48 together with the estimate on the reference element in Lemma 2.17 yields the assertion, taking into account, that

$$C_{1,u}(T_h) = C_{1,\hat{u}}(T)$$
$$C_{2,u}(T_h) = C_{2,\hat{u}}(T).$$

□

As in standard theory a global estimate follows by summation.

Theorem 2.23. *Suppose G is a conforming grid of width h and order m for the domain $\Omega \subset \mathbb{R}^d$. Let $kp > d$, $m \ge k-1$, and $u_I \in S_h^m(\Omega, M)$ be the geodesic finite element function interpolating $u \in W^{k,p}(\Omega, M)$ on G.*

If h is small enough depending on M, then

$$d_{L^p}(u,u_I) \le C_{17}\, C_{M,G,1}(u)\, h^k\; \theta_{k,p,\Omega}(u)$$
$$D_{1,p}(u,u_I) \le C_{17}\, h^{k-1}(C_{M,G,1}(u) + C_{M,G,2}(u))\; \theta_{k,p,\Omega}(u),$$

where C_{17} is the constant in Theorem 2.22,

$$C_{M,G,i}(u) := \sup_{T_h \in G} C_{1,u}(T_h)$$

for $i = 1,2$, and

$$\lim_{h\to 0} C_{M,G,1}(u) = \sup_{1\le j\le k}\, \sup_{q\in u(\Omega)} \|d^j \log_q q\|$$
$$\lim_{h\to 0} C_{M,G,2}(u) = \sup_{1\le j\le k-1}\, \sup_{q\in u(\Omega)} \|d_2 d^j \log_q q\|.$$

Proof. As u is continuous, we can choose h small enough depending on the curvature of M such that the restrictions $u_{|T_h}$ on each element T_h of G fulfill the assumptions of Theorem 2.22. Summation then yields the result. The continuity of u also implies that the sets $u(T_h)$ and $u_I(T_h)$ converge to single points as $h \to 0$, which implies the limit behaviour of the constant. □

2.2.3 Vector Field Approximation

The variation of geodesic interpolants through geodesic interpolants induces a natural definition of the interpolation of vector fields along a geodesic interpolant.

Definition 2.24. Let $\hat{v} \in P_m(T,M)$, and let $V^i \in T_{\hat{v}(a_i)}M$ be vectors given at the Lagrange nodes. Set $v_i(t) := \exp_{\hat{v}(a_i)}(tV^i)$ for $i = 1,\ldots,l$. The interpolating vector field V_I along $\hat{v}$ is then defined by

$$V_I(x) := \frac{d}{dt}|_{t=0}\Upsilon(v_1(t),\ldots,v_l(t);x).$$

We denote the space of all interpolating vector fields along $\hat{v}$ by $IV(T,\hat{v}^{-1}TM)$.

Note that the interpolating vector fields are generalized Jacobi fields in the same sense as geodesic finite elements are generalized geodesics.

Remark 2.25. The interpolation of vector fields along a discrete function $\hat{v}$ is well defined as long as geodesic interpolation of the points $\exp_{\hat{v}(a_i)}(tV^i)$ is well defined and smooth for small t. Smoothness also follows by smoothness of the geodesic finite element interpolation. Indeed, we can differentiate (2.2) for $\Upsilon(v_1(t),\ldots,v_l(t);x)$ with respect to t and obtain

$$V_I(x) = \sum_{i=1}^{l}\lambda_i(x)\left(Id + d_2\log_{\hat{v}(x)}\hat{v}_i\right)(V_I(x)) + \sum_{i=1}^{l}\lambda_i(x)d\log_{\hat{v}(x)}\hat{v}_i(V_i) \qquad (2.18)$$

as an implicit formula for V_I. For $\operatorname{diam}(\hat{v}(T)) \leq \frac{1}{\sqrt{|R|_\infty \|\sum_i |\lambda_i|\|_\infty}}$, this yields in particular

$$|V_I(x)| \leq C\max_i |V_i|. \qquad (2.19)$$

Note further that for a constant function $\hat{v}$, vector field interpolation corresponds to polynomial interpolation in $\mathbb{R}^n$.

Remark 2.26. Geodesic vector field interpolation is defined by variation of geodesic interpolants. However, we can also see it as a variational form of geodesic interpolation on TM with respect to the pseudo-Riemannian metric defined by the horizonal lift g^h (see Section 1.2.2.2). By this we mean that if (u_i,V^i) denote values in TM, u_I the geodesic interpolation of u_i in M, and V_I the interpolation of the V^i in the sense of Definition 2.24, we have

$$\begin{aligned}
&\sum_{i=1}^{l}\lambda_i(x)^h\log_{(u_I(x),V_I(x))}(u_i,V^i)\\
&\qquad = \sum_{i=1}^{l}\lambda_i(x)\left[\log_{u_I(x)}u_i, d\log_{u_I(x)}u_i(V_i) + \ d_2\log_{u_I(x)}u_i(V_I(x))\right]\\
&\qquad = [0,0] \ \in (T_{u_I(x)}M)^2.
\end{aligned}$$

Note that we do not obtain a minimization formulation of geodesic vector field interpolation as ${}^{h}g$ is only a pseudo-metric.

Having defined geodesic vector field interpolation, we discuss interpolation error estimates. In particular, we prove an analogon of Lemma 2.17 for vector fields.

Lemma 2.27. *Let $kp > d$, $m \geq k-1$, and $(u,V) \in W^{k,p}(T,TM)$ with $u(T) \subset B_\rho \subset M$. Let (u_I,V_I) denote the geodesic interpolation of (u,V). Then there exists a constant C_{18} depending on the shape functions but independent of u and M such that*

$$\| {}^{h}\log_{(u_I,V_I)}(u,V)\|_{L^p(T,TM)} \leq C_{18}\, C_{u,k+1}(T)\, \left(\dot{\theta}_{k,p,T}(u) + \dot{\Theta}_{k,p,T}(V)\right) \tag{2.20}$$

$$| {}^{h}\log_{(u_I,V_I)}(u,V)|_{W^{1,p}(T,TM)} \leq C_{18}\, C_{u,k+1}(T)\, \left(\dot{\theta}_{k,p,T}(u) + \dot{\Theta}_{k,p,T}(V)\right), \tag{2.21}$$

where $C_{u,k+1}(T)$ is the supremum of derivatives up to $k+1$th order of the log.

Proof. The proof is analogous to the one of Lemma 2.17. We replace the first order condition by the one derived in Remark 2.26. Instead of the auxillary function $G : T^2 \to TM$, we obtain $G := (G_1, G_2)$ where the $G_i : T^2 \to TM$ are defined by

$$\begin{aligned} G_1(x,y) &:= \log_{u_I(x)} u(y) \\ G_2(x,y) &:= d\log_{u_I(x)} u(y)(V(y)) + d_2 \log_{u_I(x)} u(y)(V_I(x)). \end{aligned}$$

After a Taylor expansion of G, we need to estimate for $|\vec{\alpha}| = k$

$$|\partial_y^{\vec{\alpha}} G(x,y)|_{g^S} \leq C\, \left(|\partial_y^{\vec{\alpha}} G_1(x,y)|_g + |\partial_y^{\vec{\alpha}} G_2(x,y)|_g\right).$$

As G_1 is already discussed in Lemma 2.17, we just estimate the corresponding term for G_2:

$$\begin{aligned} |\partial_y^{\vec{\alpha}} G_2(x,y)| \leq C & \sum_{\substack{1\leq o\leq k,\, \vec{\beta}_j \in [d]^{m_j} \\ \sum_{j=1}^{o} m_j = k}} \|d^o d_2 \log_{u_I(x)} u(y)\|\, |V_I(x)| \prod_{j=1}^{o} |\nabla^{\vec{\beta}_j} u(y)| \\ & + C \sum_{\substack{1\leq o\leq k,\, \vec{\beta}_j \in [d]^{m_j} \\ \sum_{j=1}^{o} m_j = k}} \|d^{o+1} \log_{u_I(x)} u(y)\|\, |V(y)| \prod_{j=1}^{o} |\nabla^{\vec{\beta}_j} u(y)| \\ & + C \sum_{\substack{0\leq o\leq k,\, \vec{\beta}_j \in [d]^{m_j} \\ \sum_{j=0}^{o} m_j = k}} \|d^{o+1} \log_{u_I(x)} u(y)\|\, |\nabla^{\vec{\beta}_0} V(y)| \prod_{j=1}^{o} |\nabla^{\vec{\beta}_j} u(y)| \\ \leq C\, C_{u,k+1}(T) & \sum_{\substack{1\leq o\leq k,\, \vec{\beta}_j \in [d]^{m_j} \\ \sum_{j=1}^{o} m_j = k}} |V_I(x)| \prod_{j=1}^{o} |\nabla^{\vec{\beta}_j} u(y)| \end{aligned}$$

$$+C\,C_{u,k+1}(T) \sum_{\substack{1\le o\le k,\, \vec{\beta}_j\in[d]^{m_j} \\ \sum_{j=1}^{o} m_j=k}} |V(x)| \prod_{j=1}^{o} |\nabla^{\vec{\beta}_j} u(y)|$$

$$+C\,C_{u,k+1}(T) \sum_{\substack{0\le o\le k,\, \vec{\beta}_j\in[d]^{m_j} \\ \sum_{j=0}^{o} m_j=k}} |\nabla^{\vec{\beta}_0} V(y)| \prod_{j=1}^{o} |\nabla^{\vec{\beta}_j} u(y)|.$$

We can estimate $\|V_I\|_{L^\infty} \le C\|V\|_{L^\infty}$ in view of (2.19). Doing so, then proceeding as in the proof of Lemma 2.17, and recalling the definition of $\dot{\Theta}_{k,p,T}(V)$ yields

$$\|\,{}^h\log_{(u_I,V_I)}(u,V)\|_{L^p(T,TM)} \le C\,C_{u,k}(T)\dot{\theta}_{k,p,T}(u) \\ + C\,C_{u,k+1}(T)\left(\|V\|_{L^\infty(T,u^{-1}TM)}\dot{\theta}_{k,p,T}(u) + \dot{\Theta}_{k,p,T}(V)\right),$$

and thus (2.20).

To see (2.21), we decompose the second component of $\nabla^{\vec{e}}\,{}^h\log_{(u_I,V_I)}(u,V)$ for $|\vec{e}|=1$ as follows

$$\nabla^{\vec{e}}\left(d\log_{u_I(x)} u(x)(V(x)) + d_2\log_{u_I(x)} u(x)(V_I(x))\right)$$
$$= \nabla^{\vec{e}} \sum_{i=1}^{l} \lambda_i(y)\left(G_2(x,y) - G_2(x,a_i)\right)_{|y=x}$$
$$+ \sum_{i=1}^{l} \lambda_i(H_1(x,x) - H_1(x,a_i) + H_2(x,x) - H_2(x,a_i) + H_3(x,x) - H_3(x,a_i)),$$

where

$$H_1(x,y) = d_2\log_{u_I(x)} u(y)(\nabla^{\vec{e}} V_I(x))$$
$$H_2(x,y) = d_2^2\log_{u_I(x)} u(y)(V_I(x), \nabla^{\vec{e}} u_I(x))$$
$$H_3(x,y) = d_2 d\log_{u_I(x)} u(y)(V(y), \nabla^{\vec{e}} u_I(x)).$$

We can estimate

$$|\nabla^{\vec{e}} V_I(x)| \le C\,(\|V\|_{L^\infty(T,u^{-1}TM)} + |V(x)|\,|\nabla^{\vec{e}} u_I(x)|).$$

Proceeding as in the proof of Lemma 2.17 yields the assertion. □

The error $\|\,{}^h\log_{(u_I,V_I)}(u,V)\|_{W^{1,p}(T,TM)}$ can be split into two parts, one consisting solely of the interpolation error of the function u, and one homogenously depending on the vector field V. An estimate for the vector field component only should also be homogenous in V. This amplification of the estimates (2.20) and (2.21) can be obtained by scaling.

Corollary 2.28. *Let $kp > d$, $m \geq k-1$, and $(u,V) \in W^{k,p}(T,TM)$ with $u(T) \subset B_\rho \subset M$. Let (u_I,V_I) denote the geodesic interpolation of (u,V). Then there exists a constant C_{19} such that*

$$\|d\log_{u_I(x)} u(x)(V(x)) + d_2\log_{u_I(x)} u(x)(V_I(x))\|_{W^{1,p}(T,u_I^{-1}TM)} \leq C_{19}\left(\|V\|_{L^\infty(\Omega,u^{-1}TM)}\dot{\theta}_{k,p,\Omega}(u) + \dot{\Theta}_{k,p,\Omega}(V)\right) \tag{2.22}$$

Proof. Lemma 2.27 implies

$$\|d\log_{u_I(x)} u(x)(V(x)) + d_2\log_{u_I(x)} u(x)(V_I(x))\|_{W^{1,p}(T,u_I^{-1}TM)} \leq C\,\dot{\theta}_{k,p,T}(u) + C\left(\|V\|_{L^\infty(\Omega,u^{-1}TM)}\dot{\theta}_{k,p,\Omega}(u) + \dot{\Theta}_{k,p,\Omega}(V)\right). \tag{2.23}$$

In particular, the terms

$$r_1(V) := \|d\log_{u_I(x)} u(x)(V(x)) + d_2\log_{u_I(x)} u(x)(V_I(x))\|_{W^{1,p}(T,u_I^{-1}TM)}$$
$$r_2(V) := \left(\|V\|_{L^\infty(\Omega,u^{-1}TM)}\dot{\theta}_{k,p,\Omega}(u) + \dot{\Theta}_{k,p,\Omega}(V)\right)$$

are homogenous, i.e.,

$$r_1(\lambda V) = |\lambda| r_1(V)$$
$$r_2(\lambda V) = |\lambda| r_2(V)$$

for $\lambda \in \mathbb{R}$. Setting $A := C\,\dot{\theta}_{k,p,T}(u)$ we can rewrite (2.23)

$$r_1(V) \leq A + r_2(V).$$

Then for $\lambda > 0$, we have

$$\lambda r_1(V) = r_1(\lambda V) \leq A + r_2(\lambda V) = A + \lambda r_2(V),$$

and thus,

$$r_1(V) \leq \frac{A}{\lambda} + r_2(V).$$

For a fixed V and $\lambda \to \infty$ this implies the assertion. □

As for functions, we obtain local and global estimates for discretized domains via the scaling properties.

Theorem 2.29. *Suppose G is a conforming grid of width h for the domain $\Omega \subset \mathbb{R}^d$. Let $kp > d$, $m \geq k-1$, and $(u,V) \in W^{k,p}(\Omega,TM)$. If h is small enough depending on M, for the geodesic interpolation (u_I,V_I) of (u,V) holds*

$$h\| {}^h\log_{(u_I,V_I)}(u,V)\|_{L^p(T,TM)} + |{}^h\log_{(u_I,V_I)}(u,V)|_{W^{1,p}(T,TM)} \le C_{20}h^{k-1}\Theta_{k,p,T}(u,V). \tag{2.24}$$

Chapter 3
Discretization Error Bounds for $W^{1,2}$-Elliptic Problems of Second Order

As in Chapter 2, we consider a smooth Riemannian manifold (M,g) with curvature bounded from above and below, and a bounded domain Ω in $\mathbb{R}^d$ with Lipschitz boundary. For a given order parameter m, let G be a conforming grid of width h with a reference element T, elements denoted by T_h, and the mth order geodesic finite element space denoted by S_h^m (cf. Definition 2.5). We will often drop the superscript m in the notation.

We consider the minimization of $W^{1,2}$-elliptic energies $\mathfrak{J}$ in $H \subset W^{1,2}(\Omega,M)$

$$u \in H : \qquad \mathfrak{J}(u) \leq \mathfrak{J}(v) \qquad \forall v \in H. \tag{3.1}$$

Let u_h be the corresponding geodesic finite element solution

$$u_h \in S_h^m : \qquad \mathfrak{J}(u_h) \leq \mathfrak{J}(v_h) \qquad \forall v_h \in S_h^m. \tag{3.2}$$

Our goal is to show a priori discretization error bounds in this setting. In particular we show an estimate of the form

$$h\, D_{1,2}(u,u_h) + d_{L^2}(u,u_h) \leq C\, h^k,$$

where $2k > d$, and the solution $u \in W^{k,2}(\Omega,M)$. We expect these results due to numerical tests in [San13].

The first corresponding error estimate for $D_{1,2}(u,u_h)$ was given in [GHS14]. In order to deal with the nonlinearity of M, the error estimate is first shown in a restricted set of functions, namely in a $W^{1,q}$-ball for $q > \max\{2,d\}$. This in particular leads to the same restriction of the discrete functions. We then show, that we can choose q and the ball's radius such that the discrete solution u_h lies in the interior of the ball. This corresponds to an estimate without the additional restriction. While in [GHS14] additional regularity is needed for a corresponding estimate, we do not have this restriction here.

We will continue with an analysis of $d_{L^2}(u,u_h)$. In particular we aim for a generalization of the Aubin–Nitsche lemma to the manifold setting. The Aubin–Nitsche lemma uses the H^2-regularity of the so-called adjoint problem and the interpolation error estimate for its solution. This imposes a restriction on the dimension to $d < 4$, as we only provided interpolation error estimates for this case in Chapter 2. Note further that the Aubin–Nitsche lemma is only suitable for quadratic energies, and the manifold introduces a nonlinearity. We will assume that the energy is "predominantly quadratic", which we define as a bound on the third variation. For $d < 4$ this condition can be interpreted as semi-linearity of the corresponding PDE. Under this condition, we can then indeed generalize the Aubin–Nitsche lemma, and obtain an estimate in a restricted space.

It might be useful for the reader to keep the harmonic energy in mind while reading through the assumptions of the theorems. Indeed, we will show in Chapter 5 that the harmonic energy fulfills the assumptions needed.

3.1 $W^{1,2}$-Discretization Error Estimates

In order to obtain a bound on $D_{1,2}(u,u_h)$, we follow the classical Galerkin approach, i.e., we consider a generalized version of Céa's lemma, show that we can apply it to an elliptic energy, and then combine this with the approximation error estimates of Chapter 2. This leads to $W^{1,2}$-error bounds for discrete functions restricted in a $W^{1,q}$-ball. Note that this result and the procedure have already been described in [GHS14].

As a priori first derivatives of geodesic finite elements may deteriorate with a negative power of the mesh size h (as standard finite elements for Euclidean space do), this restriction is problematic. In order to overcome this, in [GHS14] a dependence of the radius of the $W^{1,q}$-ball on the mesh size is allowed, which then yields an unrestricted result provided that the continuous solution u fulfills stronger regularity assumptions. We will prove a similar result without these additional assumptions.

In order to formulate Céa's lemma in a metric space (H,D) we need to specify the assumptions on the energy functional.

Definition 3.1. Let (H,D) be a metric space, $v_0, v_1 \in H$, and $\gamma : [0,1] \to H$ a curve connecting v_0 to v_1. We say that an energy functional $\mathfrak{J} : H \to \mathbb{R}$ is λ-convex with respect to D along γ for some $\lambda \in \mathbb{R}$ if

$$\mathfrak{J}(\gamma(t)) \leq (1-t)\mathfrak{J}(v_0) + t\mathfrak{J}(v_1) - \frac{\lambda}{2} t(1-t) D^2(v_0, v_1) \qquad \forall t \in [0,1]. \tag{3.3}$$

We call $\mathfrak{J}$ just λ-convex if for every $v_0, v_1 \in D(\mathfrak{J}) = \{v \in H : \ \mathfrak{J}(v) < \infty\}$ there exists a curve $\gamma : [0,1] \mapsto H$ with $\gamma(0) = v_0$ and $\gamma(1) = v_1$ such that $\mathfrak{J}$ is λ-convex along γ.

Let $u \in H$ be a minimizer of an energy functional $\mathfrak{J}$ and $V \subset H$. If there exists a $\Lambda > 0$ such that

$$\mathfrak{J}(v) - \mathfrak{J}(u) \leq \frac{\Lambda}{2} D^2(v,u) \qquad \forall v \in V, \tag{3.4}$$

we call $\mathfrak{J}$ quadratically bounded in V with respect to D and Λ.

For convex and quadratically bounded energies with positive constants λ and Λ, we can now state the following.

Lemma 3.2 (Céa's Lemma in Metric Spaces). *Let (H,D) be a metric space, and $u \in H$ be a minimizer of the functional $\mathfrak{J} : H \to \mathbb{R}$. Consider a subset $V \subset H$ such that $w \in \arg\min_{v\in V} \mathfrak{J}(v)$ exists. We assume that $\mathfrak{J}$ is quadratically bounded in $V \subset H$ around u with respect to D and $\Lambda > 0$, and that for any $v \in V$ there exists a curve γ in H connecting u to v along which $\mathfrak{J}$ is λ-convex with respect to D for some $\lambda > 0$. Then we can estimate*

$$D^2(w,u) \leq \frac{\Lambda}{\lambda} \inf_{v\in V} D^2(v,u). \tag{3.5}$$

Proof. Note that (3.3) implies for any $\in (0,1)$

$$\mathfrak{J}(u) \leq t\mathfrak{J}(v) + (1-t)\mathfrak{J}(u) - \frac{\lambda}{2} t(1-t)D^2(v,u)$$

and thus

$$\mathfrak{J}(v) - \mathfrak{J}(u) \geq \frac{\lambda}{2}(1-t)D^2(v,u).$$

Letting $t \to 0$, we obtain

$$\mathfrak{J}(v) - \mathfrak{J}(u) \geq \frac{\lambda}{2} D^2(v,u)$$

for any $v \in V$. Using (3.4) we obtain for all $v \in V$

$$\frac{\lambda}{2} D^2(w,u) \leq \mathfrak{J}(w) - \mathfrak{J}(u) \leq \mathfrak{J}(v) - \mathfrak{J}(u) \leq \frac{\Lambda}{2} D^2(v,u)$$

which concludes the proof. □

Remark 3.3. We do not need D to be a metric as we do not use any properties of metrics in the proof. In particular, we consider the quasi-inframetric $D_{1,2}$ (cf. Section 1.2.3) instead of a metric D, and set

$$H := W^{1,q}_{\phi}(\Omega, M) \tag{3.6}$$

for some $q > \max\{2,d\}$, where ϕ denotes suitable boundary data (see Section 1.1.3 for the precise definition). Note that $H \subset C(\Omega,M) \cap W^{1,2}(\Omega,M)$.

To continue, we need to specify what we mean by a $W^{1,2}$-elliptic energy functional $\mathfrak{J} : H \to \mathbb{R}$.

Definition 3.4. We say that $\mathfrak{J} : H \to \mathbb{R}$ is $W^{1,2}$-elliptic with respect to geodesic homotopies, if it is twice continuously differentiable along geodesic homotopies, $W^{1,2}$-coercive, i.e., there exists a constant $\lambda > 0$ such that for all $v \in H$ and $V \in W^{1,2}_0(\Omega, v^{-1}TM)$ we have

$$\lambda \|V\|^2_{W^{1,2}(\Omega,v^{-1}TM)} \leq \frac{d^2}{ds^2}_{|s=0} \mathfrak{J}(\exp_v(sV)), \tag{3.7}$$

and $W^{1,2}$-bounded, i.e., there exists a constant $\Lambda > 0$ such that for all $v \in H$ and for all $V, W \in W_0^{1,2}(\Omega, v^{-1}TM)$ we have

$$\left| \frac{d^2}{dr\,ds}_{|(r,s)=(0,0)} \mathfrak{J}(\exp_v(sV + rW)) \right| \leq \Lambda \ \|V\|_{W^{1,2}(\Omega,v^{-1}TM)} \|W\|_{W^{1,2}(\Omega,v^{-1}TM)}. \tag{3.8}$$

We now apply the Céa-Lemma to a subset of $H^{1,2,q}_{K,L}$ as defined in Proposition 1.41 endowed with the quasi-inframetric $D_{1,2}$. We will see that λ-convexity (3.3) along geodesic homotopies and quadratic boundedness (3.4) follow from ellipticity.

Lemma 3.5. *Let $q > \max\{2, d\}$ and H be defined by* (3.6). *Assume that $u \in H$ is a minimizer of $\mathfrak{J} : H \to \mathbb{R}$ w.r.t. variations along geodesic homotopies in H, and that $\mathfrak{J}$ is elliptic along geodesic homotopies starting in u.*

For $K > \theta_{1,q,\Omega}(u)$, $L \leq \mathrm{inj}(M)$, and $KL \leq \frac{1}{|\mathrm{Rm}|_\infty}$ let $W^{1,q}_K$ and $H^{1,2,q}_{K,L}$ be defined as in Proposition 1.41, and set

$$H_{K,L} := H \cap H^{1,2,q}_{K,L}.$$

Consider a subset $V \subset H_{K,L}$ such that

$$w = \arg\min_{v \in V} \mathfrak{J}(v)$$

exists.

Then

$$D_{1,2}(u,w) \leq (1 + C_8)^2 \sqrt{\frac{\Lambda}{\lambda}} \inf_{v \in V} D_{1,2}(u,v) \tag{3.9}$$

holds, where $D_{1,2}$ is defined as in Section 1.2.3, and C_8 is the constant appearing in Lemma 1.42.

Proof. In order to apply Lemma 3.2 we need to show that the ellipticity of $\mathfrak{J}$ starting in u implies convexity (3.3) with respect to $D_{1,2}$ along geodesic homotopies as well as quadratic boundedness in V.

Let $v \in V$, and let Γ_v denote the geodesic homotopy connecting u to v. As u is a stationary map for $\mathfrak{J}$, we can write

$$\begin{aligned}
\mathfrak{J}(v) - \mathfrak{J}(u) &= \int_0^1 \frac{d}{dt} \mathfrak{J}(\Gamma_v(t))\, dt \\
&= \int_0^1 \frac{d}{dt} \mathfrak{J}(\Gamma_v(t))\, dt - \int_0^1 \frac{d}{dt} \mathfrak{J}(\Gamma_v(0))\, dt \\
&= \int_0^1 \int_0^t \frac{d^2}{dt^2} \mathfrak{J}(\Gamma_v(s))\, ds\, dt
\end{aligned}$$

$$= \int_0^1 (1-t) \frac{d^2}{dt^2} \mathfrak{J}(\Gamma_v(t))\, dt.$$

Using the ellipticity and Lemma 1.42, we obtain the estimate

$$\begin{aligned}
\mathfrak{J}(v) - \mathfrak{J}(u) &\le \Lambda \int_0^1 (1-t) \|\Gamma_v(t)\|^2_{W^{1,2}(\Omega,\Gamma_v(\cdot,t)^{-1}TM)}\, dt \\
&\le \Lambda \int_0^1 (1-t)\, dt\, (1+C_8)^2 D_{1,2}^2(u,v) \\
&= \frac{1}{2} \Lambda (1+C_8)^2 D_{1,2}^2(u,v),
\end{aligned}$$

which shows that $\mathfrak{J}$ is quadratically bounded with respect to $D_{1,2}$ and $\big(\Lambda(1+C_8)^2\big)$.

In order to see the convexity, we set $f(s) := \mathfrak{J}(\Gamma_v(s))$. Then the ellipticity and Lemma 1.42 imply $f \in C^2([0,1],\mathbb{R})$ with

$$f''(t) \ge \lambda \frac{1}{(1+C_8)^2} D_{1,2}^2(u,v).$$

Hence, we obtain

$$f(t) \le (1-t) f(0) + t f(1) - \frac{\lambda}{2(1+C_8)^2} D_{1,2}^2(u,v)$$

for all $t \in [0,1]$, which means $\mathfrak{J}$ is $\big(\lambda(1+C_8)^{-2}\big)$-convex along geodesic homotopies.

The application of Lemma 3.2 then proves (3.9). □

We can now combine this version of Céa's lemma with the approximation error estimates of Chapter 2 in order to obtain a discretization error bound in the restricted space $S_h^m \cap H_{K,L}$.

Theorem 3.6. *Let $2k > d$, $q > \max\{2,d\}$, and $m \ge k-1$. Assume that $u \in H \cap W^{k,2}(\Omega,M)$ is a minimizer of $\mathfrak{J} : H \to \mathbb{R}$ w.r.t. variations along geodesic homotopies in H, and that $\mathfrak{J}$ is elliptic along geodesic homotopies starting in u.*

For a conforming grid G of width h and order m (cf. Definition 2.1) set $V_h := H \cap S_h$. Assume that the boundary data ϕ is such that V_h is not empty.

Let K be a constant such that

$$K \ge C_{12} \theta_{1,q,\Omega}(u), \tag{3.10}$$

where C_{12} is the constant of Proposition 2.12, and θ the smoothness descriptor defined in Definition 1.21. Assume that h is small enough such that $u_I \in H_{K,L}$, where $H_{K,L}$ is defined as in Lemma 3.5.

Then the discrete minimizer

$$u_h = \operatorname*{arg\,min}_{v_h \in V_h \cap H_{K,L}} \mathfrak{J}(v_h)$$

fulfills the a priori error estimate

$$D_{1,2}(u,u_h) \leq C_{21} h^{k-1} \theta_{k,2,\Omega}(u). \tag{3.11}$$

If the error is measured in an isometric embedding $\iota : M \to \mathbb{R}^N$*, we have*

$$\|\iota \circ u - \iota \circ u_h\|_{W^{1,2}(\Omega,\mathbb{R}^N)} \leq C\, h^{k-1} \theta_{k,2,\Omega}(\iota \circ u) \leq C\, h^{k-1} \|\iota \circ u\|^k_{k,p,\Omega}. \tag{3.12}$$

Proof. Assumption (3.10) and h small enough imply that $u_I \in V_h \cap H_{K,L}$. Thus, we can apply Lemma 3.5 and Theorem 2.23 to obtain the estimate (3.11).

The bound (3.12) then follows from Propositions 1.43, 1.24, and 1.23. □

Remark 3.7. The assumption of Theorem 3.6 that the boundary data ϕ can be represented exactly in S_h may be waived and replaced by a standard approximation argument for u_h interpolating smooth boundary data.

Note that Theorem 3.6 just provides a discretization error bound in the restricted set $V_{h;K,L}$. In particular, this includes the a priori assumptions

$$\theta_{1,q,\Omega}(u_h) \leq K$$
$$d_{L^s}(u,u_h) \leq L$$

for $KL \leq \frac{1}{|\mathrm{Rm}|_\infty}$, $q > \max\{2,d\}$, and

$$s := \begin{cases} \frac{2q}{q-2} & \text{for } d = 1 \\ \frac{4q}{q-2} & \text{for } d = 2 \\ \frac{dq}{q-d} & \text{for } d > 2. \end{cases}$$

We need to show that we indeed stay away from these a priori bounds to obtain a true error estimate.

Theorem 3.8. *Let* $2k > d$*,* $k \geq 2$*, and* $m \geq k-1$*. Assume that* $u \in H \cap W^{k,2}(\Omega,M)$ *is a minimizer of* $\mathfrak{J} : H \to \mathbb{R}$ *w.r.t. variations along geodesic homotopies in H, and that* $\mathfrak{J}$ *is elliptic along geodesic homotopies starting in u.*

For a conforming grid G of width h and order m (cf. Definition 2.1) set $V_h := H \cap S_h$*. Assume that the boundary data* ϕ *is such that* V_h *is not empty.*

If h is small enough, there exists a $q > \max\{2,d\}$ *and constants K and L, such that the minimizer* u_h *in* $V_{h;K,L}$ *as in Theorem 3.6 is indeed a local minimizer in* S_h *which fulfills*

$$D_{1,2}(u,u_h) \leq C_{22} h^{k-1} \theta_{k,2,\Omega}(u). \tag{3.13}$$

Measured in an isometric embedding $\iota : M \to \mathbb{R}^N$*, we have*

$$\|\iota \circ u - \iota \circ u_h\|_{W^{1,2}(\Omega,\mathbb{R}^N)} \leq C\, h^{k-1} \theta_{k,2,\Omega}(\iota \circ u) \leq C\, h^{k-1} \|\iota \circ u\|^k_{k,p,\Omega}. \tag{3.14}$$

Proof. We have to show that we can choose q, K, L, and h such that we can apply Theorem 3.6 and

$$\begin{aligned} \theta_{1,q,\Omega}(u_h) &< K \\ d_{L^\infty}(u,u_h) &< L \end{aligned}$$

hold for the discrete solution u_h.

By Proposition 1.45 and Remark 1.46 we can estimate

$$\begin{aligned} \dot{\theta}_{1,q,\Omega}(u_h) &\le \dot{\theta}_{1,q,\Omega}(u) + C\, D_{1,q}(u,u_h), \\ d_{L^\infty}(u,u_h) &\le C\, D_{1,q}(u,u_h). \end{aligned}$$

Thus, if we can show that there exists a $q > \max\{2,d\}$ such that

$$D_{1,q}(u,u_h) \le C\, h^{\delta}$$

holds for the solution in the restricted set $V_{h;K,L}$ for some $\delta > 0$, the assertion follows.

Let $r > \max\{2,d\}$ be such that $W^{k,2} \subset W^{1,r}$, i.e.,

$$r := \begin{cases} \infty & \text{for } 2(k-1) > d \\ \frac{4-2\varepsilon}{\varepsilon} & \text{for } 2(k-1) = d \\ \frac{2d}{2-(2k-d)} & \text{for } 2(k-1) < d \end{cases}$$

for any small $\varepsilon > 0$. Note that indeed $r > d$ in the case $2k > d > 2(k-1)$.

We have for all $\max\{2,d\} < q < r$

$$\begin{aligned} D_{1,q}(u,u_h) &\le h^{\delta-k+1} D_{1,2}(u,u_h) + h^{\frac{k-1-\delta}{\mu}} D_{1,r}(u,u_h) \\ &\le C\, h^{\delta}\, \theta_{k,2,\Omega}(u) + C\, h^{\frac{k-1-\delta}{\mu}}\, \dot{\theta}_{k,2,\Omega}(u) + h^{\frac{k-1-\delta}{\mu}}\, \dot{\theta}_{1,r,\Omega}(u_h), \end{aligned}$$

where

$$\mu = \left(\frac{1}{2} - \frac{1}{q}\right)\left(\frac{1}{q} - \frac{1}{r}\right)^{-1}.$$

We can use the inverse estimate in Proposition 2.13 to get

$$\dot{\theta}_{1,r,\Omega}(u_h) \le C\, h^{-d\left(\frac{1}{q}-\frac{1}{r}\right)}\, \dot{\theta}_{1,q,\Omega}(u_h) \le C\, h^{-d\left(\frac{1}{q}-\frac{1}{r}\right)}.$$

Thus, we obtain

$$D_{1,q}(u,u_h) \le C\, \left(h^{\delta} + h^{\frac{k-1-\delta}{\mu} - d\left(\frac{1}{q}-\frac{1}{r}\right)}\right).$$

As for all $\max\{2,d\} < q < r$ we have

$$(k-1)\frac{2q}{q-2} > d,$$

there exists a $\delta > 0$ such that

$$\frac{k-1-\delta}{\mu} - d\left(\frac{1}{q} - \frac{1}{r}\right) > 0.$$

The assertion follows by choosing K, L and h such that the assumptions of Theorem 3.6 are fulfilled, and then applying said theorem. □

Remark 3.9. Note that while Theorem 3.8 looks like corresponding results for the Euclidean case (see e.g. [Cia78]), the nonlinearity induced by the manifold is hidden in the smallness of the mesh size h. This corresponds to the intuition that a Riemannian manifold behaves almost like flat space in small neighborhoods.

3.2 L^2-Discretization Error Estimates

Our goal in this chapter is to show that for $W^{1,2}$-elliptic minimization problems the L^2-discretization error is in $O(h^k)$.

We recall the Aubin–Nitsche lemma for the approximation of a quadratic minimization problem in $H = H_0^1(\Omega, \mathbb{R})$ by standard finite elements. This means that we consider the energy $J(v) = \frac{1}{2}a(v,v) - (f,v)$, the variational equalities

$$\begin{aligned} u \in H: &\qquad a(u,v) = (f,v) &\qquad \forall v \in H, \\ u_h \in S_h: &\qquad a(u_h,v_h) = (f,v_h) &\qquad \forall v_h \in S_h, \end{aligned}$$

and the adjoint problem

$$w \in H: \qquad a(v,w) = (g,v) \qquad \forall v \in H,$$

where $g := u - u_h$. We assume H^2-regularity of the adjoint problem, i.e., $|w|_{H^2} \leq C\|g\|_{L^2}$. Using Galerkin orthogonality and the H^1-ellipticity of $a(\cdot,\cdot)$, we can then estimate

$$\begin{aligned} \|u-u_h\|_{L^2}^2 &= (g, u-u_h) = a(u-u_h, w) = a(u-u_h, w-w_I) \\ &\leq \Lambda \|u-u_h\|_{H^1} \|w-w_I\|_{H^1} \\ &\leq Ch^{k-1}|u|_{H^k}\, h\, |w|_{H^2} \\ &\leq Ch^k |u|_{H^2} \|u-u_h\|_{L^2}. \end{aligned}$$

We want to emulate this proof for the L^2-error for geodesic finite elements.

However, this proof of an L^2-error estimate only works for quadratic energies as J above, i.e., for so-called linear PDEs. Since sets of functions into a manifold do not form a vector space, the entire concept of linear PDEs is meaningless in this setting. Euclidean techniques to obtain error estimates for nonlinear energies as found in [DR80] rely on the deformation to a linear PDE and weighted norms. A generalization of these to geodesic finite elements is desirable, in particular since they also provide L^∞-error estimates. We, however, follow our general approach to geometrically generalize the concept of linearity rather than use a linearization. A combination with a deformation argument in order to obtain error estimates for more general energies is conceivable, but beyond the scope of this work.

In particular, we will assume that the energy is "predominantly quadratic". Note that this is still a restriction on the energy, i.e., the PDE, not on the manifold. We mean by "predominantly quadratic" the following bound on the third variation of the energy.

Definition 3.10. Let $q > \max\{d,2\}$ and $\mathfrak{J} : H \to \mathbb{R}$ be an energy functional. We say that $\mathfrak{J}$ is predominantly quadratic if $\mathfrak{J}$ is C^3 along geodesic homotopies, and for any $v \in H \cap W_K^{1,q}$, and vector fields U, V along v

$$|\delta^3 \mathfrak{J}(v)(U,V,V)| \leq C(K,M) \|U\|_{W^{1,p}(\Omega, v^{-1}TM)} \|V\|^2_{W^{1,2}(\Omega, v^{-1}TM)}, \tag{3.15}$$

where p is such that $W^{2,2} \subset W^{1,p}$, i.e.

$$p := \begin{cases} \infty & \text{for } d = 1 \\ \varepsilon^{-1} & \text{for } d = 2 \\ \frac{2d}{d-2} & \text{for } d > 2, \end{cases}$$

with $\varepsilon > 0$ arbitrarily small.

Remark 3.11. In the Euclidean case $M = \mathbb{R}^n$, quadratic energies are obviously predominantly quadratic, as the third variation vanishes. As long as the coefficient functions of a semi-linear PDE coming from a minimization problem are smooth enough and bounded, the third variation of the energy will have a bound of the form

$$|\delta^3 \mathfrak{J}(v)(U,V,V)| \leq C \int_\Omega |U|(|V| + |\nabla V|)^2 \, dx.$$

Thus, for $d < 4$ a such an energy is also predominantly quadratic.

The leading term of the third variation of the energy for a typical quasi-linear equation, e.g., the minimal surface energy for graphs $\mathfrak{J}(u) = \int_\Omega \sqrt{1 + |Du|^2} \, dx$, has the form

$$|\delta^3 \mathfrak{J}(v)(U,V,V)| \leq C \int_\Omega |\nabla U| \, |\nabla V|^2 \, dx.$$

Thus, for $d = 1$ such an energy is predominantly quadratic.

For a Riemannian manifold M, the harmonic energy is predominantly quadratic, if $d < 4$ (see Chapter 5).

We consider the variational formulation of the problems (3.1) and (3.2)

$$u \in H: \qquad \frac{d}{dt}_{|t=0} \mathfrak{J}(\exp_{u(\cdot)}(tV(\cdot))) = 0 \qquad \forall V \in W_0^{1,2}(\Omega, u^{-1}TM), \tag{3.16}$$

and

$$u_h \in S^h: \qquad \frac{d}{dt}_{|t=0} \mathfrak{J}(\exp_{u_h(\cdot)}(tV_h(\cdot))) = 0 \qquad \forall V_h \in IV_0(\Omega, u_h^{-1}TM), \tag{3.17}$$

where $IV_0(\Omega, u_h^{-1}TM)$ is the set of all interpolating vector fields along u_h as defined in Definition 2.24 with boundary values 0. The concept of Galerkin orthogonality can be generalized as follows.

Proposition 3.12. *Let u and u_h be solutions to* (3.16) *and* (3.17)*, respectively, and let Γ be the geodesic homotopy joining u and u_h.*
Then for the parallel transport $V_h(t)$ along Γ of any discrete vector field $V_{h,1} \in IV(\Omega, u_h^{-1}TM)$ holds

$$\int_0^1 \delta^2 \mathfrak{J}(\Gamma(t))(V_h(t), \dot{\Gamma}(t))\, dt = 0.$$

Proof. Indeed, as V_h is parallel along Γ, and u and u_h fulfill (3.16) and (3.17), respectively, we have

$$\begin{aligned}\int_0^1 \delta^2 \mathfrak{J}(\Gamma(t))(V_h(t), \dot{\Gamma}(t))\, dt &= \int_0^1 \frac{d}{dt} \delta \mathfrak{J}(\Gamma(t,\cdot)(V_h(t,\cdot)) - \delta \mathfrak{J}(\Gamma(t,\cdot))(\partial_t V_h(t,\cdot))\, dt \\ &= \delta \mathfrak{J}(u_h(\cdot))(V_h(1,\cdot)) - \delta \mathfrak{J}(u(\cdot))(V_h(0,\cdot)) \\ &= 0. \qquad \square\end{aligned}$$

We now define a generalization of the adjoint problem in this context. As the manifold M induces a nonlinearity, we need to linearize the energy in order to obtain a bilinear form. Let u and u_h be the solutions to (3.16) and (3.17), respectively. The resulting linearized problem is: Find $(w,W) \in W^{1,2}(\Omega, TM)$ such that

$$\delta^2 \mathfrak{J}(w)(W, V_1) + \delta \mathfrak{J}(w)(V_2) = -(V_1, \log_w u_h - \log_w u)_{L^2(\Omega, w^{-1}TM)} \tag{3.18}$$

holds for all tangent vectors $(V_1, V_2) \in T_{(w,W)}TM = (W^{1,2}(\Omega, w^{-1}TM))^2$. This bilinear form, the second variation of the energy, acts on deformations of functions, i.e. on vector fields. We therefore call it the deformation problem.

Remark 3.13. One can easily check by inserting test vector fields of the form $(V_1, 0)$ that the solution (w,W) of (3.18) projects over the solution u of (3.1). Thus, (3.18) is equivalent to the system consisting of (3.1) and finding $W \in W^{1,2}(\Omega, u^{-1}TM)$ such that

$$\delta^2 \mathfrak{J}(u)(W, V) = -(V, \log_u u_h)_{L^2(\Omega, u^{-1}TM)} \qquad \forall V \in W^{1,2}(\Omega, u^{-1}TM). \tag{3.19}$$

One major difference to the Euclidean setting is that if we interpolate the solution (u, W), we obtain a discrete vector field not along u_h but along u_I. In order to obtain a discrete vector field along u_h, and thus a valid test vector field for (3.17), we need to first transport the W to u_h and then interpolate along u_h.

In order to preserve bounds through this transport, we will need the following technical estimate.

Proposition 3.14. *Let $q > \max\{d, 2\}$, and a, b be as in Definition 1.25 for $k = 2$, $p = 2$, i.e.,*

$$a := \begin{cases} \infty & \text{for } d < 4 \\ 6 & \text{for } d = 4 \\ \frac{2d}{d-4} & \text{for } d > 4, \end{cases}$$

and

$$\frac{1}{b} = \frac{1}{2} - \frac{1}{a}.$$

Let Γ be a geodesic homotopy such that $\Gamma(s) \in H \cap W^{2,b}(\Omega, M)$. We set $K_1 = \max_s \dot{\theta}_{1,q,\Omega}(\Gamma(s))$, and $K_2 = \max_s \dot{\theta}_{2,b,\Omega}(\Gamma(s))$.

Then there exists a constant C_{23} depending on K_1, K_2, and M such that

$$\Theta_{2,2,\Omega}(W(1)) + \|W(1)\|_{L^a(\Omega,\Gamma(1)^{-1}TM)} \theta_{2,b,\Omega}(\Gamma(1)) \le C_{23} \|W(0)\|_{W^{2,2}(\Omega,\Gamma(0)^{-1}TM)} \tag{3.20}$$

holds for any parallel vector field $W(s) \in W^{2,2}(\Omega, \Gamma(s)^{-1}TM)$.

Proof. Note that the choice of q, a, and b, allows for the following estimates

$$\begin{aligned} \|V\|_{L^\infty} &\le C\, \|V\|_{W^{1,q}} \\ \|V\|_{W^{1,\frac{2q}{q-2}}} &\le C\, \|V\|_{W^{2,2}} \\ \|V\|_{L^a} &\le C\, \|V\|_{W^{2,2}}. \end{aligned}$$

As W is parallel along Γ, we can use Lemma 1.42 to obtain

$$\Theta_{1,2,\Omega}(W(1)) + \|W(1)\|_{L^a(\Omega,\Gamma(1)^{-1}TM)} \theta_{2,b,\Omega}(\Gamma(1)) \le C\, \|W(0)\|_{W^{2,2}(\Omega,\Gamma(0)^{-1}TM)}.$$

We still need to estimate

$$\begin{aligned} \dot{\Theta}_{2,2,\Omega}(W(1)) &= \left(\sum_{\alpha=1}^{d} \| |\nabla_\alpha W(1)|\, |d^\alpha \Gamma(1)| \|^2_{L^2(\Omega,\mathbb{R})} + |W(1)|^2_{W^{2,2}(\Omega,\Gamma(1)^{-1}TM)} \right)^{\frac{1}{2}} \\ &\le C\, \|W(1)\|_{W^{1,\frac{2q}{q-2}}} \dot{\theta}_{1,q,\Omega}(\Gamma(1)) + C\, |W(1)|_{W^{2,2}}. \end{aligned}$$

The first term can be estimated using Lemma 1.42 again.

We cannot directly apply Lemma 1.42 to the last term $|W(1)|_{W^{2,2}(\Omega,\Gamma(1)^{-1}TM}$, but as in the proof of Lemma 1.42 we can differentiate to obtain

$$\frac{d}{dt}|W(t)|_{W^{2,2}(\Omega,\Gamma(t)^{-1}TM}$$
$$\leq C(M)\left(\dot{\theta}_{2,b,\Omega}(\Gamma(t))\|\dot{\Gamma}(t)\|_{L^\infty}\|W\|_{L^a}+\dot{\theta}_{1,q,\Omega}(\Gamma(t))\|\dot{\Gamma}(t)\|_{L^\infty}\|W(t)\|_{W^{1,\frac{2q}{q-2}}}\right).$$

Using Lemma 1.42 and the estimates on Γ, we obtain

$$\begin{aligned}\frac{d}{dt}|W(t)|_{W^{2,2}(\Omega,\Gamma(t)^{-1}TM} &\leq C\left(\|W(0)\|_{L^a}+\|W(0)\|_{W^{1,\frac{2q}{q-2}}}\right)\\ &\leq C\,\|W(0)\|_{W^{2,2}}.\end{aligned}$$

We insert this estimate into

$$\begin{aligned}|W(1)|_{W^{2,2}(\Omega,\Gamma(t)^{-1}TM)} &= |W(0)|_{W^{2,2}(\Omega,\Gamma(0)^{-1}TM)}+\int_0^1\frac{d}{dt}|W(t)|_{W^{2,2}(\Omega,\Gamma(t)^{-1}TM)}\,dt\\ &\leq C\,|W(0)|_{W^{2,2}(\Omega,\Gamma(0)^{-1}TM)},\end{aligned}$$

which yields the assertion. □

We can now prove the L^2-error estimate.

Theorem 3.15. *Let $d<4$ and the assumptions of Theorem 3.8 be fulfilled. Let additionally $\mathfrak{J}$ be predominantly quadratic in the sense of Definition 3.10 with respect to q chosen as in the proof of Theorem 3.8.*

Assume that the discrete solution u_h fulfills on each element $T_h\in G$

$$\theta_{2,2,T_h}(u_h)\leq K_2 \tag{3.21}$$

for a constant K_2.

Finally, suppose that (3.19) *is H^2-regular, i.e., that the solution W fulfills*

$$\|W\|_{W^{2,2}(\Omega,u^{-1}TM)}\leq C\,\|\log_u u_h\|_{L^2(\Omega,u^{-1}TM)}. \tag{3.22}$$

Then there exists a constant C_{24}, such that

$$d_{L^2}(u,u_h)\leq C_{24}\,h^k\theta^2_{k,2,\Omega}(u).$$

Proof. We insert $V:=\log_u u_h$ into (3.19), and obtain

$$d^2_{L^2}(u,u_h)=-\delta^2\mathfrak{J}(u)(W,\log_u u_h),$$

where W is the solution of (3.19).

Let Γ denote the geodesic homotopy joining u and u_h, and $W(t)$ the parallel transport of W along Γ. Let $W(1)_I$ be the interpolation of $W(1)$ along u_h, and let $W_I(t)$ denote its parallel transport. Note that $W_I(0)$ is not the interpolation of W along u.

By Proposition 3.12 we have

$$\begin{aligned} d_{L^2}^2(u,u_h) &= -\delta^2\mathfrak{J}(u)(W,\log_u u_h) + \int_0^1 \delta^2\mathfrak{J}(\Gamma(t))(W_I(t),\dot{\Gamma}(t))\,dt \\ &= \int_0^1\int_0^t \frac{d}{ds}\delta^2\mathfrak{J}(\Gamma(s))(\frac{s}{t}W_I(s)+(1-\frac{s}{t})W(s),\dot{\Gamma}(s))\,ds\,dt \\ &= \int_0^1\int_0^t \delta^3\mathfrak{J}(\Gamma(s))(\frac{s}{t}W_I(s)+(1-\frac{s}{t})W(s),\dot{\Gamma}(s),\dot{\Gamma}(s))\,ds\,dt \\ &\quad + \int_0^1\int_0^t \frac{1}{t}\delta^2\mathfrak{J}(\Gamma(s))(W_I(s)-W(s),\dot{\Gamma}(s))\,ds\,dt. \end{aligned} \tag{3.23}$$

We can estimate the second integral in (3.23) using the ellipticity assumption (3.8)

$$\begin{aligned} &\int_0^1\int_0^t \frac{1}{t}\delta^2\mathfrak{J}(\Gamma(s))(W_I(s)-W(s),\dot{\Gamma}(s))\,ds\,dt \\ &\quad\le \Lambda \int_0^1\int_0^t \frac{1}{t}\|W_I(s)-W(s)\|_{W^{1,2}(\Omega,\Gamma(s)^{-1}TM)}\|\dot{\Gamma}(s)\|_{W^{1,2}(\Omega,\Gamma(s)^{-1}TM)}\,ds\,dt. \end{aligned}$$

As the vector fields W_I, W, $\dot{\Gamma}$ are parallel along Γ, we can further estimate using Lemma 1.42

$$\begin{aligned} &\int_0^1\int_0^t \frac{1}{t}\delta^2\mathfrak{J}(\Gamma(s))(W_h(s)-W(s),\dot{\Gamma}(s))\,ds\,dt \\ &\qquad\qquad \le C\,\|W(1)_I - W(1)\|_{W^{1,2}(\Omega,u_h^{-1}TM)}\|\log_u u_h\|_{W^{1,2}(\Omega,u^{-1}TM)}. \end{aligned}$$

Corollary 2.28, Proposition 3.14, and Assumption (3.21) imply on each element $T_h \in G$

$$\begin{aligned} &\|W(1)_I - W(1)\|_{W^{1,2}(T_h,u_h^{-1}TM)} \\ &\qquad \le C\,h\,\Big(\Theta_{2,2,T_h}(W(1)) + \|W(1)\|_{L^\infty(T_h,u_h^{-1}TM)}\theta_{2,2,T_h}(u_h)\Big) \\ &\qquad \le C\,h\,\|W\|_{W^{2,2}(T_h,u^{-1}TM)}. \end{aligned}$$

Summing over all elements and using the H^2-regularity we obtain

$$\|W(1)_I - W(1)\|_{W^{1,2}(\Omega,u_h^{-1}TM)} \le C\,h\,\|W\|_{W^{2,2}(\Omega,u^{-1}TM)} \le C\,h\,d_{L^2}(u,u_h).$$

Thus, we obtain using Theorem 3.6

$$\int_0^1\int_0^t \frac{1}{t}\delta^2\mathfrak{J}(\Gamma(s))(W_I(s)-W(s),\dot{\Gamma}(s))\,ds\,dt \le C\,h^k\,d_{L^2}(u,u_h)\,\theta_{k,2,\Omega}(u).$$

In order to estimate the first integral term in (3.23) we use that $\mathfrak{J}$ is predominantly quadratic. Since $p > d$ in Definition 3.10 we can estimate

$$\|\tau W_I + (1-\tau)W\|_{W^{1,p}} \le C\,\|W\|_{W^{1,p}} \le C\,\Theta_{2,2,\Omega}(W).$$

Thus we obtain using the H^2-regularity of the deformation problem (3.19)

$$
\begin{aligned}
|\delta^3\mathfrak{J}(\Gamma(s))(\tau W_I + (1-\tau)W, \dot\Gamma, \dot\Gamma)| \\
&\le C(K,M)\|\log_u u_h\|^2_{W^{1,2}}\,(\tau\|W_I\|_{W^{1,p}} + (1-\tau)\|W\|_{W^{1,p}}) \\
&\le C\,h^{2k-2}\theta^2_{k,2,\Omega}(u)\;\Theta_{2,2,\Omega}(W) \\
&\le C\,h^{2k-2}\theta^2_{k,2,\Omega}(u)\;d_{L^2}(u,u_h).
\end{aligned}
$$

Note that $2k-2 \ge k$ for $k \ge 2$. This yields the assertion. □

Remark 3.16. Theorem 3.15 assumes (3.21), i.e., a bound on second derivatives of the discrete solution u_h. This is a strong restriction on the set of discrete functions, as we can in generally not show convergence of the second derivatives. This restriction can possibly be avoided by using that u_h solves (3.2).

Remark 3.17. Theorem 3.15 is restricted to $d < 4$. This is due to the lack of an interpolation error estimate for vector fields unless $W^{2,2} \subset C^0$.

Assume for $d \ge 4$ that there exist estimates of the form

$$
\begin{aligned}
\|W(1)_I - W(1)\|_{W^{1,2}(T_h,u_h^{-1}TM)} \\
&\le C\,h\,\left(\Theta_{2,2,T_h}(W(1)) + \|W(1)\|_{L^a(T_h,u_h^{-1}TM)}\theta_{2,b,T_h}(u_h)\right),
\end{aligned}
\tag{3.24}
$$

with a and b defined as in Proposition 3.14, and

$$\|W_I\|_{1,p} \le C\|W\|_{1,p} \tag{3.25}$$

for $p = \frac{2d}{d-2}$.

Under these additional assumptions we can replace (3.21) by

$$\theta_{2,b,T_h}(u_h) \le K_2, \tag{3.26}$$

and Definition 3.10 by the condition

$$|\delta^3\mathfrak{J}(v)(U,V,V)| \le C(K,M)\|U\|_{W^{1,p}(\Omega,v^{-1}TM)}\|V\|_{W^{1,2}(\Omega,v^{-1}TM)}\|V\|_{L^r(\Omega,v^{-1}TM)} \tag{3.27}$$

for $v \in H \cap W^{1,q}_K$, $q > \max\{2,d\}$, $p \le r \le d$. Note that (3.27) is again in $\mathbb{R}^n$ fulfilled for semi-linear PDEs.

Using (3.27) and L^p-interpolation with $\varepsilon = h$ then yields

$$
\begin{aligned}
&|\delta^3\mathfrak{J}(\Gamma(s))(\tau W_I + (1-\tau)W, \dot\Gamma, \dot\Gamma)| \\
&\le C\,\|\dot\Gamma(0)\|_{W^{1,2}}\,(\tau\|W_I\|_{W^{1,2}} + (1-\tau)\|W\|_{W^{1,2}})\left(h\,\|\dot\Gamma(0)\|_{L^\infty} + h^{1-\frac{r}{p}}\|\dot\Gamma(0)\|_{W^{1,2}}\right) \\
&\le C\,h^{2(k-1)+1-\frac{r}{p}}\,d_{L^2}(u,u_h).
\end{aligned}
$$

As $k \geq \frac{d}{2}$ and $d \geq r$, we have $k - \frac{r}{p} \geq 1$. Thus, we obtain also for $d \geq 4$

$$|\delta^3 \mathfrak{J}(\Gamma(s))(\tau W_I + (1-\tau)W, \dot{\Gamma}, \dot{\Gamma})| \leq C\, h^k d_{L^2}(u, u_h).$$

In total, if one can show (3.24) and (3.25) for example by adapting techniques from Clément interpolation to this case, the L^2-error estimate follows for arbitrary dimensions. Unfortunately, this is beyond the scope of this work.

Remark 3.18. In [Gro13a, GHS14] it is proposed to replace exp and log by so-called retraction pairs, that may lead to numerically easier implementation of geodesic finite elements. Similar to [GHS14, Theorem 5.6.], we believe that our results remain valid for general retraction pairs as long as they fulfill the estimates on the first and second derivatives proven in Appendix A for log.

Chapter 4
L^2-Gradient Flows for $W^{1,2}$-Elliptic Energies

Let $\mathfrak{J} : W^{1,2}(\Omega, M) \mapsto (-\infty, \infty]$ be a proper $W^{1,2}$-elliptic functional (cf. Definition 3.4).
We consider the L^2-gradient flow

$$\langle u'(t), V\rangle_{L^2(\Omega, u(t)^{-1}TM)} = -\frac{d}{ds}_{|s=0} \mathfrak{J}(\exp_{u(t)}(sV)) \qquad V \in W^{1,2}(\Omega, u(t)^{-1}TM), \tag{4.1}$$

with some initial condition $u(t) = u_0$, and Dirichlet boundary conditions on $\partial\Omega$ (note that these single out a homotopy class; see Section 1.1.3). We assume that there exists a solution $u \in L^2((0,T), H^{2,2}(\Omega, M)) \cap W^{1,\infty}((0,T), L^2(\Omega, M))$.

We want to approximate a solution to (4.1) by the method of time layers, i.e., we will discretize time by an implicit Euler scheme and then approximate the resulting time discrete problems using geodesic finite elements.

We are interested in a priori discretization error estimates in the time step width τ and the spatial grid parameter h.

The time discretization by the implicit Euler scheme for a given partition $\{0 = t_0 < t_1 < \ldots < t_M < T\}$ with $\tau_i = t_i - t_{i-1}$ leads to variational equalities of the form

$$\left\langle \frac{1}{\tau_i} \log_{U^i} U^{i-1}, V \right\rangle_{L^2(\Omega, (U^i)^{-1}TM)} = -\frac{d}{ds}_{|s=0} \mathfrak{J}(\exp_{U^i}(sV)) \tag{4.2}$$

for all $V \in W^{1,2}(\Omega, U^{i;-1}TM)$. This can be equivalently written as an energy minimization problem

$$U^i \in W^{1,2}(\Omega, M) : \qquad \mathfrak{J}_{\tau_i, U^{i-1}}(V) \to \min,$$

where

$$\mathfrak{J}_{\tau_i, U^{i-1}}(V) := \frac{1}{2\tau_i} d^2_{L^2}(V, U^{i-1}) + \mathfrak{J}(V). \tag{4.3}$$

We define the discrete solution $u_\tau(t)$ by piecewise constant interpolation

$$u_\tau(t) = U^i \qquad \forall t \in (t_{i-1}, t_i].$$

We then use geodesic finite elements to approximate u_τ by a fully discrete solution $u_{\tau,h}$.

It is well-known that solutions of (4.1) can blow up in finite or infinite time (see e.g. [HW08, CG89, CDY92, Gro93] for the harmonic energy). In order to discretize the problem and show a priori discretization error bounds, however, we will always assume global existence and smoothness of the continuous solution up to some time T. A reverse statement, i.e., that we can find such a smooth continuous solution whenever we have a proper discrete solution, as well as numerical studies of the behaviour of discrete solutions approaching a singularity, are of great interest but beyond the scope of this work.

In order to analyze the discretization error of the method of time layers, we will first look at generalized gradient flows in metric spaces. We will see that we can use known results from this field to obtain discretization error estimates for $d_{L^2}(u(t), u_\tau(t))$ and $D_{1,2}(u(t), u_\tau(t))$ for every t. The theory for elliptic problems as outlined in Chapter 3 will then provide estimates for $d_{L^2}(u(t), u_{\tau,h}(t))$ and $D_{1,2}(u(t), u_{\tau,h}(t))$.

For simplicity we restrict ourselves to first order geodesic finite elements and hence dimension $d < 4$.

4.1 Gradient Flow in Metric Spaces

In this section we will consider gradient flows in metric spaces following the exposition in [AGS06]. We will restate and reprove some of the results in [AGS06] as we need to customize them to our setting later.

Let in the following (S,d) be a complete metric space and $\mathfrak{J} : S \mapsto (-\infty, +\infty]$ a proper, coercive, l.s.c. functional, i.e.,

$$\exists v \in D(\mathfrak{J}), r > 0: \quad m := \inf\{\mathfrak{J}(w) : \; w \in S, \; d(w,v) \leq r\} > -\infty, \tag{4.4}$$

$$d(u_n, u) \to 0 \quad \Rightarrow \quad \liminf_{n\to\infty} \mathfrak{J}(u_n) > \mathfrak{J}(u), \tag{4.5}$$

where $D(\mathfrak{J}) := \{v \in S \; : \; \mathfrak{J}(v) < \infty\}$ denotes the domain of $\mathfrak{J}$. Assume furthermore that $\mathfrak{J}_{\tau,w}$ is defined by

$$\mathfrak{J}_{\tau,w}(v) = \frac{1}{2\tau} d^2(v,w) + \mathfrak{J}(v) \tag{4.6}$$

is $(\lambda + \tau^{-1})$-convex with respect to d (cf. Definition 3.1) for every $\tau > 0$ such that $\lambda\tau > -1$.

Remark 4.1. If $\mathfrak{J}$ is λ-convex along geodesics, then $\mathfrak{J}_{\tau,w}$ is $(\lambda+\tau^{-1})$-convex along geodesics if $d^2(w,\cdot)$ is 2-convex along geodesics, i.e., if (S,d) is nonpositively curved. For the positively curved space of probability measures endowed with the L^2-Wasserstein metric it is shown in [AGS06] that the squared distance is 2-convex along a class of so-called generalized geodesics that are then used instead of ordinary geodesics.

In order to state the gradient flow problem in this setting we need some basic tools from the analysis in metric spaces which can be found in [AGS06, Chapter 1]:

1. A metric version of Rademacher's theorem states, that for any absolutely continuous curve $v:(a,b)\to S$ the limit
$$|v'|(t) := \lim_{h\to 0} \frac{d(v(t+h),v(t))}{|h|}$$
exists L^1-almost everywhere, and
$$d(v(s),v(t)) \le \int_s^t |v'|(r)\,dr.$$
2. The function $g: S\to[0,+\infty]$ is called the strong upper gradient of $\mathfrak{J}$ if for every absolutely continuous curve v the function $g\circ v$ is Borel and fulfills
$$|\mathfrak{J}(v(t))-\mathfrak{J}(v(s))| \le \int_s^t g(v(r))|v'|(r)\,dr.$$
This implies that if $(g\circ v)|v'|\in L^1(a,b)$, then $\mathfrak{J}\circ v$ is absolutely continuous and
$$|(\mathfrak{J}\circ v)'|(t) \le g(v(t))|v'|(t)$$
almost everywhere.
3. The function g is called a weak upper gradient if only the second inequality holds, where $(\mathfrak{J}\circ v)'$ denotes the approximate derivative if $\mathfrak{J}\circ v$ is a function of bounded variation.
4. A canonical choice of weak upper gradient is the local slope
$$|\partial\mathfrak{J}|(v) := \limsup_{w\to v} \frac{(\mathfrak{J}(v)-\mathfrak{J}(w))^+}{d(v,w)}.$$
5. For any λ-convex $\mathfrak{J}$ the local slope is also a strong upper gradient and it is lower semi-continuous.

Definition 4.2. A curve u is of maximal slope with respect to an upper gradient g if
$$\frac{1}{2}\int_s^t |u'|^2(r)+|g(u(r))|^2\,dr \le \mathfrak{J}(u(s))-\mathfrak{J}(u(t))$$
almost everywhere with $s\le t$.

For a strong upper gradient, a curve of maximal slope fulfills the equality, i.e.,

$$\frac{1}{2}\int_s^t |u'|^2(r) + |\partial\mathfrak{J}|^2(u(r))\,dr = \mathfrak{J}(u(s)) - \mathfrak{J}(u(t)).$$

Existence of curves of maximal slope is proved in [AGS06] by showing the existence of a continuum limit of an implicit Euler scheme.

Given an initial datum $u_0 \in S$, a partition $P_\tau = \{0 = t_0 < t_1 < \ldots < t_M < T\}$ with $\tau_n = t_n - t_{n-1}$, and an approximate initial datum U_τ^0, we define recursively

$$U^n \in S: \qquad \mathfrak{J}_{\tau_n,U^{n-1}}(U^n) \le \mathfrak{J}_{\tau_n,U^{n-1}}(V) \qquad \forall V \in S, \tag{4.7}$$

where $\mathfrak{J}_{\tau,W}$ is defined by (4.6). Existence and uniqueness of the U^n is discussed and affirmed if $U^0 \in \overline{D(\mathfrak{J})}$ and $\lambda\tau > -1$. A discrete solution u_τ can then be defined by piecewise constant interpolation

$$\overline{U}_\tau(t) = U^n \qquad \forall t \in (t^{n-1}, t^n].$$

In this setting the following theorem was proven in [AGS06].

Theorem 4.3. *Let (S,d) be a complete metric space and $\mathfrak{J} : S \mapsto (-\infty,+\infty]$ a proper, coercive, l.s.c. functional, such that $\mathfrak{J}_{\tau,w}$ is λ-convex for some $\lambda \in \mathbb{R}$. Then for each $u_0 \in \overline{D(\mathfrak{J})}$ there exists a unique limit*

$$u(t) = \lim_{n\to\infty}\left(\Phi_{\frac{t}{n}}\right)^n [u_0],$$

where $\Phi_\tau[v] := \arg\min_{u\in S}\mathfrak{J}_{\tau,v}(u)$.

Further, u is locally a Lipschitz curve of maximal slope with $u(t) \in D(|\partial\mathfrak{J}|) \subset D(\mathfrak{J})$ for $t > 0$ fulfilling for $\lambda \ge 0$ the a priori bounds

$$\mathfrak{J}(u(t)) \le \mathfrak{J}(v) + \frac{1}{2t}d^2(v,u_0) \quad \forall v \in D(\mathfrak{J})$$
$$|\partial\mathfrak{J}|^2(u(t)) \le |\partial\mathfrak{J}|^2(v) + \frac{1}{t^2}d^2(v,u_0) \quad \forall v \in D(|\partial\mathfrak{J}|).$$

In particular, u is the unique solution of the evolution variational inequality

$$\frac{1}{2}\frac{d}{dt}d^2(u(t),v) + \frac{1}{2}\lambda d^2(u(t),v) + \mathfrak{J}(u(t)) \le \mathfrak{J}(v) \qquad L^1\text{-a.e. } t > 0,\ \forall v \in D(\mathfrak{J}) \tag{4.8}$$

among the locally absolutely continuous curves such that $\lim_{t\to 0} u(t) = u_0$. The corresponding flow operator generates a λ-contracting semigroup, i.e.,

$$d(u(t),v(t)) \le e^{-\lambda t} d(u_0,v_0) \qquad \forall u_0, v_0 \in \overline{D(\mathfrak{J})}.$$

In particular, the following theorem on optimal a priori error bounds is proven in [AGS06]:

Theorem 4.4. *Let (S,d) be a complete metric space and $\mathfrak{J} : S \mapsto (-\infty,+\infty]$ a proper, coercive, l.s.c. functional, such that $\mathfrak{J}_{\tau,w}$ is λ-convex for some $\lambda > 0$. Let u denote the unique solution of* (4.8) *and let $\overline{U}_\tau$ be a discrete solution associated to a partition $P_\tau = \{0 = t_0 < t_1 < \ldots < t_M < T\}$ with $\tau := \sup_n \tau_n$. Define*

$$\lambda_\tau := \frac{\log(1+\lambda\tau)}{\tau}.$$

Let $t \in (0,T)$ and let $U_\tau^0 = u_0 \in D(\mathfrak{J})$, then there exists a constant C_{25} such that

$$d^2(\overline{U}_\tau(t),u(t)) \le C_{25}\tau\left(\mathfrak{J}(u_0) - \inf_S \mathfrak{J}\right) e^{-2\lambda_\tau t}. \tag{4.9}$$

If $u_0 \in D(|\partial\mathfrak{J}|)$ we have

$$d^2(\overline{U}_\tau(t),u(t)) \le C_{26}\frac{\tau^2}{2}|\partial\mathfrak{J}|^2(u_0)e^{-2\lambda_\tau t}. \tag{4.10}$$

Note that we restricted ourselves to stating the result for $\lambda > 0$. For $\lambda \le 0$ similar results can also be found in [AGS06].

The discretization scheme (4.7) is used in [AGS06] to address the existence of the solution to the continuous problem. As we are mainly interested in error bounds we will only review the proof of 4.4 here and simplify it by using the results of Theorem 4.3, while in [AGS06] these actually follow from the convergence of (4.7). Although still quite voluminous, we recapitulate the proof here in detail instead of just citing the result, as we need to adapt it later (see Corollary 4.10).

Proof. The proof relies on a Gronwall-type argument.

Note that the U^n are defined by a convex minimization problem (4.7). We can thus estimate for any V

$$\mathfrak{J}_{\tau_n,U^{n-1}}(U^n) \le (1-t)\mathfrak{J}_{\tau_n,U^{n-1}}(U^n) + t\mathfrak{J}_{\tau_n,U^{n-1}}(V) - \frac{1+\lambda\tau_n}{2\tau_n}t(1-t)d^2(U^n,V)$$

for all $t \in (0,1)$. Dividing by t and letting $t \to 0$ yields the variational inequality

$$\frac{1}{2\tau_n}(d^2(U^n,V) - d^2(U^{n-1},V)) + \frac{1}{2}\lambda d^2(U^n,V) \le \mathfrak{J}(V) - \mathfrak{J}_{\tau_n,U^{n-1}}(U^n). \tag{4.11}$$

In order to obtain a corresponding variational inequality along the discrete solution u_τ, rather than evaluating functionals along an interpolation of the functions (U^i), the values of the functionals in $\mathbb{R}$ are interpolated linearly, i.e., we set

$$d_\tau^2(t;V) := \frac{t_n - t}{\tau_n}d^2(U^{n-1},V) + \frac{t - t_{n-1}}{\tau_n}d^2(U^n,V) \qquad t \in (t_{n-1},t_n]$$
$$\mathfrak{J}_\tau(t) := \frac{t_n - t}{\tau_n}\mathfrak{J}(U^{n-1}) + \frac{t - t_{n-1}}{\tau_n}\mathfrak{J}(U^n) \qquad t \in (t_{n-1},t_n].$$

Using this notation, we obtain for all $t \in (0,T)\backslash P_\tau$ the discrete variational inequalities

$$\frac{1}{2}\frac{d}{dt}d_\tau^2(t) + \frac{\lambda}{2}d^2(U^n,V) + \mathfrak{J}_\tau(t) - \mathfrak{J}(V) \\ \leq \mathfrak{J}_\tau(t) - \mathfrak{J}_{\tau_n,U^{n-1}}(U^n) =: \frac{1}{2}R_\tau(t) \qquad \forall V \in D(\mathfrak{J}) \quad (4.12)$$

by simply rewriting (4.11). Setting $\delta^2(t,s) := d_\tau^2(t;u(s))$ we obtain from (4.8)

$$\frac{1}{2}\frac{\partial}{\partial s}\delta^2(t,s) + \frac{\lambda}{2}d_\tau^2(t;u(s)) - \mathfrak{J}_\tau(t) + \mathfrak{J}(u(s)) \leq 0$$

and from (4.12)

$$\frac{1}{2}\frac{\partial}{\partial t}\delta^2(t,s) + \frac{\lambda}{2}d^2(U^n,u(s)) + \mathfrak{J}_\tau(t) - \mathfrak{J}(u(s)) \leq \frac{1}{2}R_\tau(t).$$

Adding these, we obtain

$$\frac{d}{dt}\delta^2(t,t) + 2\lambda\delta^2(t,t) \leq R_\tau(t) + \lambda\left(\delta^2(t,t) - d^2(U^n,u(t))\right).$$

Note that for $t \in (t_{n-1},t_n)$

$$\begin{aligned}\tau_n&\left(\delta^2(t,t) - d^2(U^n,u(t))\right)\\ &= (t_n - t)\left(d^2(U^{n-1},u(t)) - d^2(U^n,u(t))\right)\\ &\leq (t_n - t)d(U^{n-1},U^n)\left(d(U^{n-1},u(t)) + d(U^n,u(t))\right)\\ &\leq (t_n - t)d(U^{n-1},U^n)\sqrt{\frac{\tau_n}{t - t_{n-1}}d(U^{n-1},u(t))^2 + \frac{\tau_n}{t_n - t}d(U^n,u(t))^2}\\ &= \tau_n\sqrt{\frac{t_n - t}{t - t_{n-1}}}d(U^{n-1},U^n)\delta(t,t).\end{aligned}$$

A Gronwall-type lemma [AGS06, Lemma 4.1.8] then yields for any $t > 0$

$$e^{\lambda t}\delta(t,t) \leq \left(d^2(U^0,u_0) + \sup_{s\in[0,t]}\int_0^s e^{2\lambda r}R_\tau(r)\,dr\right)^{\frac{1}{2}} + 2\int_0^t \lambda e^{\lambda s}D_\tau(s)\,ds, \quad (4.13)$$

where $D_\tau(t) = \frac{1}{2}\sqrt{\frac{t_n - t}{t - t_{n-1}}}d(U^{n-1},U^n)$ for $t \in (t_{n-1},t_n]$. We need to estimate the terms appearing on the right hand sight. Note that since $\lambda_\tau \leq \lambda$ we obtain analogous bounds by replacing λ by λ_τ.

We assume from now on that $U^0 = u_0$. Furthermore, we will make the simplification that $t = t_N$. Inequality (4.13) then reads

$$e^{\lambda_\tau t_N} d^2(U^N, u(t_N)) \le \left(\sup_{s\in[0,t_N]} \int_0^s e^{2\lambda_\tau r} R_\tau(r)\, dr \right)^{\frac{1}{2}} + 2\int_0^{t_N} \lambda e^{\lambda_\tau s} D_\tau(s)\, ds, \quad (4.14)$$

Recalling the definition of R_τ, we can write for $t \in (t_{n-1}, t_n]$

$$\begin{aligned} R_\tau(t) &= 2(\mathfrak{J}_\tau(t) - \mathfrak{J}_{\tau_n,U^{n-1}}(U^n)) \\ &= 2\frac{t^n - t}{\tau_n}\left(\mathfrak{J}(U^{n-1}) - \mathfrak{J}(U^n) - \frac{d^2(U^n,U^{n-1})}{2\tau_n}\right) - \frac{t-t_{n-1}}{\tau_n^2} d^2(U^n,U^{n-1}). \end{aligned}$$

As

$$\mathfrak{J}(U^{n-1}) - \mathfrak{J}(U^n) - \frac{d^2(U^n,U^{n-1})}{2\tau_n} = \mathfrak{J}_{\tau,U^{n-1}}(U^{n-1}) - \mathfrak{J}_{\tau_n,U^{n-1}}(U^n) \ge 0,$$

we can estimate

$$\begin{aligned} \int_{t^{n-1}}^{t_n} e^{2\lambda_\tau r} R_\tau(r)\, dr &\le e^{2\lambda_\tau t_n} \int_{t^{n-1}}^{t_n} (R_\tau(r))^+\, dr \\ &\le \tau_n e^{2\lambda_\tau t_n}\left(\mathfrak{J}(U^{n-1}) - \mathfrak{J}(U^n) - \frac{d^2(U^n,U^{n-1})}{2\tau_n}\right). \end{aligned} \quad (4.15)$$

Note further that since $\mathfrak{J}$ is l.s.c. and convex, we have for any $u, v \in S$

$$\mathfrak{J}(u) - \mathfrak{J}(v) \le |\partial\mathfrak{J}|(u) d(u,v) - \frac{\lambda}{2} d^2(u,v) \le \frac{1}{2\lambda}|\partial\mathfrak{J}|^2(u), \quad (4.16)$$

and thus

$$\mathfrak{J}(u) - \inf_{v\in S}\mathfrak{J}(v) \le \frac{1}{2\lambda}|\partial\mathfrak{J}|^2(u).$$

Since U^n minimizes $\mathfrak{J}_{\tau,U^{n-1}}$, we can estimate for any $V \in S$

$$\begin{aligned} \mathfrak{J}(U^n) - \mathfrak{J}(V) &\le \frac{1}{2\tau}\left(d^2(V,U^{n-1}) - d^2(U^n,U^{n-1})\right) \\ &\le \frac{1}{2\tau} d(V,U^n)\left(d(V,U^{n-1}) + d(U^n,U^{n-1})\right). \end{aligned}$$

Dividing by $d(V,U^n)$ and letting $V \to U^n$, we obtain the slope estimate

$$|\partial\mathfrak{J}|(U^n) \le \frac{d(U^n,U^{n-1})}{\tau}.$$

Combining this with (4.16) yields

$$\frac{d^2(U^n,U^{n-1})}{\tau^2} \ge |\partial\mathfrak{J}|^2(U^n) \ge 2\lambda\left(\mathfrak{J}(U^n) - \inf_{v\in S}\mathfrak{J}(v)\right). \quad (4.17)$$

We rescale $\mathfrak{J}$ to have

$$\inf_{v\in S}\mathfrak{J}(v)=0.$$

Inserting (4.17) into (4.15) yields

$$\int_{t^{n-1}}^{t_n} e^{2\lambda_\tau r}R_\tau(r)\,dr \le \tau_n e^{2\lambda_\tau t_n}\left(\mathfrak{J}(U^{n-1})-(1+\lambda\tau_n)\mathfrak{J}(U^n)\right). \tag{4.18}$$

The convexity of $\mathfrak{J}_{t_n,U^{n-1}}$ implies for the minimizer U^n that

$$0\ge\left(\frac{\lambda}{2}+\frac{1}{\tau}\right)td^2(U^n,U^{n-1})+\mathfrak{J}(U^n)-\mathfrak{J}(U^{n-1})$$

for all $t\in(0,1)$. Letting $t\to 1$ yields

$$\left(\frac{\lambda}{2}+\frac{1}{\tau}\right)d^2(U^{n-1},U^n)-\mathfrak{J}(U^{n-1})+\mathfrak{J}(U^n)\le 0. \tag{4.19}$$

Combining this with (4.17) yields

$$\frac{\mathfrak{J}(U^{n-1})-\mathfrak{J}(U^n)}{\tau_n}\ge\left(1+\frac{\lambda\tau_n}{2}\right)|\partial\mathfrak{J}|^2(U^n)\ge 2\lambda\left(1+\frac{\lambda\tau_n}{2}\right)\mathfrak{J}(U^n),$$

and thus

$$\mathfrak{J}(U^{n-1})\ge(1+\lambda\tau)^2\mathfrak{J}(U^n).$$

By concavity of the logarithm, we have

$$\frac{1}{1+\lambda\tau_n}\le e^{-\lambda_\tau\tau_n}. \tag{4.20}$$

This implies

$$e^{2\lambda_\tau t^n}\mathfrak{J}(U^n)\le e^{2\lambda_\tau t^n-2\lambda_\tau\tau_n}\mathfrak{J}(U^{n-1})=e^{2\lambda_\tau t^{n-1}}\mathfrak{J}(U^{n-1}).$$

Using this we estimate

$$\begin{aligned}
&\int_{t^{n-1}}^{t_n} e^{2\lambda_\tau r}R_\tau(r)\,dr\\
&\quad\le \tau_n e^{2\lambda_\tau t_n}(1+\lambda\tau_n)\left(\frac{\lambda\tau}{(1+\lambda\tau_n)^2}\mathfrak{J}(U^{n-1})+\frac{1}{(1+\lambda\tau_n)^2}\mathfrak{J}(U^{n-1})-\mathfrak{J}(U^n)\right)\\
&\quad\le \tau_n(1+\lambda\tau_n)\left(\lambda\tau_n e^{2\lambda_\tau t_{n-1}}\mathfrak{J}(U^{n-1})+e^{2\lambda_\tau t_{n-1}}\mathfrak{J}(U^{n-1})-e^{2\lambda_\tau t_n}\mathfrak{J}(U^n)\right)\\
&\quad\le \tau_n(1+\lambda\tau_n)\left(\lambda\tau_n\mathfrak{J}(U^0)+e^{2\lambda_\tau t_{n-1}}\mathfrak{J}(U^{n-1})-e^{2\lambda_\tau t_n}\mathfrak{J}(U^n)\right).
\end{aligned}$$

Summing up, we obtain

$$\int_0^{t_N} e^{2\lambda_\tau r} R_\tau(r)\,dr \le \tau(1+\lambda\tau)\Big((1+\lambda t_N)\mathfrak{J}(U^0) - e^{2\lambda_\tau t_N}\mathfrak{J}(U^N)\Big) \\ \le \tau(1+\lambda\tau)(1+\lambda t_N)\mathfrak{J}(U^0). \tag{4.21}$$

For the second integral term in (4.14) note, that

$$2\int_{t_{n-1}}^{t_n} e^{\lambda_\tau r} D_\tau(r)\,dr = d(U^{n-1},U^n)\int_{t_{n-1}}^{t_n} e^{\lambda_\tau r}\sqrt{\frac{t_n - r}{r - t_{n-1}}}\,dr \le \frac{\pi}{2} e^{\lambda_\tau t_n}\tau d(U^{n-1},U^n).$$

Summing up, we obtain

$$2\int_0^{t_N} e^{\lambda_\tau r} D_\tau(r)\,dr \le \tau\frac{\pi}{2}\sum_{n=1}^N e^{\lambda_\tau t_n} d(U^{n-1},U^n) \\ \le \tau\frac{\pi}{2}\sqrt{N}\left(\sum_{n=1}^N e^{2\lambda_\tau t_n} d^2(U^{n-1},U^n)\right)^{\frac{1}{2}}.$$

By (4.19) and (4.17), we can estimate

$$\frac{1+\lambda\tau_n}{2\tau_n} d^2(U^{n-1},U^n) \le \mathfrak{J}_{\tau,U^{n-1}}(U^{n-1}) - \mathfrak{J}_{\tau,U^{n-1}}(U^n) \le \mathfrak{J}(U^{n-1}) - (1+\lambda\tau_n)\mathfrak{J}(U^n).$$

Inserting yields

$$2\int_0^{t_N} e^{\lambda_\tau r} D_\tau(r)\,dr \le \tau\frac{\pi}{2}\left(\frac{2N}{1+\lambda\tau}\sum_{n=1}^N \tau_n e^{2\lambda_\tau t_n}\left(\mathfrak{J}(U^{n-1}) - (1+\lambda\tau_n)\mathfrak{J}(U^n)\right)\right)^{\frac{1}{2}}.$$

As in (4.21) we can estimate

$$2\int_0^{t_N} e^{\lambda_\tau r} D_\tau(r)\,dr \le \tau\frac{\pi}{2}\left(2t_N(1+\lambda t_N)\mathfrak{J}(U^0)\right)^{\frac{1}{2}}. \tag{4.22}$$

Combing (4.14) with (4.21) and (4.22) yields (4.9) for $t = t_N$. We omit the proof for general $t \in (0,T)$, which can also be found in [AGS06].

To obtain (4.10), note that if $U_0 \in D(|\partial\mathfrak{J}|)$, we can estimate from (4.15) using (4.17), (4.16), and (4.20)

$$
\begin{aligned}
&\int_{t^{n-1}}^{t_n} e^{2\lambda_\tau r} R_\tau(r)\,dr \\
&\quad \leq \tau_n e^{2\lambda_\tau t_n}(\mathfrak{J}(U^{n-1}) - \mathfrak{J}(U^n) - \frac{d^2(U^n, U^{n-1})}{2\tau_n}) \\
&\quad \leq \frac{\tau_n^2}{2} e^{2\lambda_\tau t_n}\left(\frac{1}{\lambda\tau_n}|\partial\mathfrak{J}|^2(U^{n-1}) - |\partial\mathfrak{J}|^2(U^n)\right) \\
&\quad \leq \frac{\tau_n^2}{2}\left(\lambda\tau e^{-2\lambda_\tau t_{n-1}}|\partial\mathfrak{J}|^2(U^{n-1}) + e^{2\lambda_\tau t_{n-1}}|\partial\mathfrak{J}|^2(U^{n-1}) - e^{2\lambda_\tau t_n}|\partial\mathfrak{J}|^2(U^n)\right)
\end{aligned}
$$

instead of (4.18). Proceeding as before, we then obtain

$$
\int_0^{t_N} e^{2\lambda_\tau r} R_\tau(r)\,dr \leq \frac{\tau^2}{2}|\partial\mathfrak{J}|^2(U^0)(1+\lambda t_N) \tag{4.23}
$$

instead of (4.21). Analogously, we obtain

$$
2\int_0^{t_N} e^{\lambda_\tau r} D_\tau(r)\,dr \leq \frac{\pi}{2}\tau\left(N\tau^2\frac{1+\lambda t_N}{1+\lambda\tau}|\partial\mathfrak{J}|^2(U^0)\right)\frac{1}{2} \leq \frac{\pi}{2}t_N\tau|\partial\mathfrak{J}|(U^0).
$$

instead of (4.22) as $\frac{1+\lambda t_N}{1+\lambda\tau} \leq \frac{t_N}{\tau}$. This concludes the proof of Theorem 4.4. □

4.2 Time Discretization

In the context of gradient flows on Riemannian manifolds, i.e., Problem (4.1), we want to use Theorem 4.3 in order to obtain error estimates for the time discretized solution.

First, we need to choose the correct metric space to work in.

Proposition 4.5. *Let M be complete and $q > d$. Set $H = W^{1,q}_{K;\phi}(\Omega, M)$. Then (H, d_{L^2}) is a complete metric space.*

Proof. By the Rellich–Kondrachov compactness theorem $W^{1,q}_{K;\phi}(\Omega,\mathbb{R}^N)$ with $d_{L^2(\Omega,\mathbb{R}^N)}$ is a complete metric space. Since M is complete, (H, d_{L^2}) inherits this property. □

Remark 4.6. We will assume that $\mathfrak{J}$ is $W^{1,2}$-elliptic and bounded from below. Note that these are stronger assumptions than (4.4) and (4.5). In particular the $W^{1,2}$-ellipticity of $\mathfrak{J}$ implies strong L^2-convexity along geodesics by the Poincaré inequality, i.e.

$$
\mathfrak{J}(\Gamma(t)) \leq (1-t)\mathfrak{J}(\Gamma(0)) + t\mathfrak{J}(\Gamma(1)) - \frac{\lambda}{2C_2(\Omega)}t(1-t)d^2_{L^2}(\Gamma(0),\Gamma(1)).
$$

The convexity of $\mathfrak{J}_{\tau,w}$ depends not only the convexity of $\mathfrak{J}$ but also on the convexity of the distance d_{L^2}. In view of Remark 4.1, d_{L^2} is 2-convex if M has non-positive curvature. In this case $\mathfrak{J}_{\tau,w}$ is $(\tilde{\lambda}+\tau^{-1})$-convex along geodesic homotopies with respect to d_{L^2}, where $\tilde{\lambda}=\frac{\lambda}{C_2(\Omega)}$.

In general, we can discuss convexity of the energy functionals with respect to three different notions, namely L^2-convexity, $W^{1,2}$-convexity and a mixed $W^{1,2}$–L^2-convexity.

Definition 4.7. We say that an energy functional $\mathfrak{J}: H \to \mathbb{R}$ is (λ_1,λ_2)-$W^{1,2}$–L^2-convex along a curve $\Gamma:[0,1]\mapsto H$ if

$$\begin{aligned}\mathfrak{J}(\Gamma(t)) &\le (1-t)\mathfrak{J}(\Gamma(0))+t\mathfrak{J}(\Gamma(1))\\ &\quad -\frac{1}{2}t(1-t)\left(\lambda_1 D_{1,2}^2(\Gamma(0),\Gamma(1))+\lambda_2 d_{L^2}^2(\Gamma(0),\Gamma(1))\right) \qquad \forall t\in[0,1],\end{aligned}$$

where $D_{1,2}$ is defined in Section 1.2.3.

Proposition 4.8. *Let the sectional curvature of M be bounded from above by $k_0 < \frac{1}{C_2}$. Set $H = W^{1,q}_{K;\phi}(\Omega,M)$ and assume that $\mathfrak{J}$ is $W^{1,2}$-elliptic.*
Then $\mathfrak{J}_{\tau,w}(v)$ as defined by (4.3) is $(\lambda,\frac{1-C_2k_0}{C_2\tau})$-$W^{1,2}$–$L^2$-convex along geodesic homotopies. In particular, this implies λ-$W^{1,2}$-convexity and $\frac{\lambda\tau+1-C_2k_0}{C_2\tau}$-$L^2$-convexity.

Proof. Let $w,v_0,v_1\in D(\mathfrak{J})$. Let $\Gamma:[0,1]\mapsto H$ denote a constant speed geodesic homotopy with $\Gamma(0)=v_0$ and $\Gamma(1)=v_1$. Let furthermore $\alpha(s,\cdot)$ be a constant speed geodesic homotopy with $\alpha(s,0)=\Gamma(s)$ and $\alpha(s,1)=w$. Then

$$\frac{d}{ds}\frac{1}{2}d_{L^2}^2(\Gamma(s),w)=\int_\Omega \langle \partial_t\alpha(s,0),\nabla_s\partial_t\alpha(s,0)\rangle\,dx,$$

and

$$\begin{aligned}&\frac{d^2}{ds^2}\frac{1}{2}d_{L^2}^2(\Gamma(s),w)\\ &=\int_\Omega |\nabla_s\partial_t\alpha(s,0)|^2\,dx-\int_\Omega \mathrm{Rm}(\partial_t\alpha(s,0),\partial_s\alpha(s,0),\partial_s\alpha(s,0),\partial_t\alpha(s,0))\,dx\\ &\ge \int_\Omega |\nabla_s\partial_t\alpha(s,0)|^2\,dx-k_0\int_\Omega |\partial_t\alpha(s,0)|\,|\partial_s\alpha(s,0)|\,dx.\end{aligned}$$

If $k_0\le 0$, then this shows convexity of $d_{L^2}^2$. If $0\le k_0$, we estimate for $\varepsilon>0$ and $r\in[0,1]$

$$\frac{d^2}{ds^2}\frac{1}{2}d^2(\Gamma(s),w)\ge\left(\frac{r-\varepsilon k_0C_2}{C_2}\right)\|\dot{\Gamma}\|_{L^2}^2+\left(\frac{1-r}{C_2}-\frac{k_0}{4\varepsilon}\right)d_{L^2}^2(\Gamma(s),w).$$

Choosing $\varepsilon=\frac{1}{2}$ and $r=1-\frac{k_0C_2}{2}$ implies

$$\frac{r-\varepsilon k_0 C_2}{C_2} = \frac{1}{C_2} - k_0 > 0$$
$$\frac{1-r}{C_2} - \frac{k_0}{4\varepsilon} = 0,$$

which yields the assertion. □

Remark 4.9. Note that the results of [AGS06] as described in Section 4.1 are also true for convexity along non-geodesic curves. This is even used in [AGS06] for metric spaces with concave distances. In view of the upper and lower curvature bounds we already used for the elliptic theory in Chapter 3, we will not generalize in this direction.

We can now state the following corollary of Theorem 4.3.

Corollary 4.10. *Let the sectional curvature of M be bounded from above by $k_0 < \frac{1}{C_2}$. Set $H = W^{1,q}_{K;\phi}(\Omega, M)$ and assume that $\mathfrak{J}$ is $W^{1,2}$-elliptic and bounded from below. For $u_0 \in D(|\partial\mathfrak{J}|)$, let $u \in L^2((0,T),H) \cap W^{1,\infty}((0,T),L^2(\Omega,M))$ be a solution* (4.1). *For a given partition $P_\tau = \{0 = t_0 < t_1 < \ldots < t_M < T\}$ with $\tau = t_n - t_{n-1}$ we set $U^0 = u_0$ and define recursively*

$$U^n = \underset{V \in H}{\arg\min}\, \mathfrak{J}_{\tau,U^{n-1}}(V).$$

Then there exist constants C_{27} and C_{28} such that for $t_n \in P_\tau$

$$d_{L^2}(u(t_n),U^n) \le C_{27}\ \tau, \tag{4.24}$$
$$D_{1,2}(u(t_n),U^n) \le C_{28}\ \sqrt{\tau}. \tag{4.25}$$

Proof. In view of Remark 4.6 and Proposition 4.8 we can apply Theorem 4.4 directly to obtain (4.24) if M has non-positive curvature.

For positive, bounded curvature, we can repeat the proof for Theorem 4.4 with the $\frac{\lambda\tau+1-C_2k_0}{C_2\tau}$-$L^2$-convexity of $\mathfrak{J}_{\tau,w}$. The constant C_{27} will then depend on $\frac{C_2}{1-k_0C_2}$.

Using $W^{1,2}$-ellipticity rather than L^2-convexity we can obtain (4.25). For this let Γ be the geodesic homotopy connecting $u(t_n)$ to U^n. Then

$$\begin{aligned}
\lambda D^2_{1,2}(u(t_n),U^n) &\le \int_0^1 \delta^2\mathfrak{J}(\Gamma(s))(\dot\Gamma(s),\dot\Gamma(s))\, ds \\
&= \int_0^1 \frac{d}{ds}\delta\mathfrak{J}(\Gamma(s))(\dot\Gamma(s))\, ds \\
&= \delta\mathfrak{J}(u(t_n))(\log_{u(t_n)} U^n) + \delta\mathfrak{J}(U^n)(\log_{U^n} u(t_n)) \\
&= (-u'(t), \log_{u(t_n)} U^n)_{L^2} + (\frac{1}{\tau}\log_{U^n} U^{n-1}, \log_{U^n} u(t_n))_{L^2} \\
&\le \left(\|u'(t)\|_{L^2} + \frac{1}{\tau} d_{L^2}(U^n,U^{n-1}) \right) d_{L^2}(u(t_n),U^n)
\end{aligned}$$

$$\leq \left(\|u'(t)\|_{L^2} + \frac{1}{\tau}\left(d_{L^2}(u(t_n),u(t_{n-1})) + 2C_{27}\tau\right)\right) d_{L^2}(u(t_n),U^n).$$

Using (4.24), we obtain (4.25), where the constant C_{28} depends on C_{27} and the Lipschitz constant of u. □

4.3 Space Discretization

Let $H = W^{1,q}_{K;\phi}(\Omega,M)$ with q and K as in the proof of Theorem 3.8. Let $W \in H$. We use first order geodesic finite elements as space discretization for the problem

$$w_\tau = \arg\min_{v\in H} \mathfrak{J}_{\tau,W}(v). \tag{4.26}$$

This leads to the discrete problems

$$w_{\tau,h} = \arg\min_{v_h\in V^h} \mathfrak{J}_{\tau,W}(v_h), \tag{4.27}$$

where $V^h = H \cap S_h$.

If we assume that M fulfills curvature bounds, we can show that $\mathfrak{J}_{\tau,W}$ is elliptic, i.e., that it fulfills (3.7) and (3.8). We have seen (3.7) already in Proposition 4.8. In particular, we have observed that a mixed error measure of the form $D^2_{1,2} + \tau^{-1}d^2_{L^2}$ is natural to the energy $\mathfrak{J}_{\tau,W}$.

Proposition 4.11. *Let $k_0, k_1 \geq 0$ such that the sectional curvature of M is bounded from above by k_0, and from below by $-k_1$. Assume that $k_0 < \frac{1}{C(\Omega)}$, where $C(\Omega)$ denotes the constant in Poincaré's inequality.*

If $\mathfrak{J}$ is $W^{1,2}$-elliptic with constants λ and Λ, then there exist constants $C(k_0)$ and $C(k_1)$ such that $\mathfrak{J}_{\tau,W}$ fulfills

$$\lambda\|\dot{\Gamma}\|^2_{W^{1,2}} + C(k_0)\tau^{-1}\|\dot{\Gamma}\|^2_{L^2} \leq \frac{d^2}{ds^2}\mathfrak{J}_{\tau,w}(\Gamma(s)) \leq \Lambda\|\dot{\Gamma}\|^2_{W^{1,2}} + C(k_1)\tau^{-1}\|\dot{\Gamma}\|^2_{L^2}. \tag{4.28}$$

Proof. The result follows directly from Definitions 1.1 and 3.4. □

We could indeed infer the $W^{1,2}$-ellipticity of $\mathfrak{J}_{\tau,W}$ from Proposition 4.11 by using Poincaré's inequality. However, this would lead to an upper ellipticity constant degenerating with smaller τ, while the ellipticity constants $\min\{\lambda, C(k_0)\}$ and $\max\{\Lambda, C(k_1)\}$ for the mixed norm $\|\cdot\|^2_{W^{1,2}} + \tau^{-1}\|\cdot\|^2_{L^2}$ are independent of τ.

We can obtain optimal error estimates under the additional assumption that $h^2 \leq \kappa\tau$ for some κ sufficiently small. This essentially uses the fact that we gain an additional order of h in the L^2-error that compensates for the τ^{-1}-weight of this term.

Lemma 4.12. *Assume that $d < 4$ and let $\mathfrak{J} : H \to \mathbb{R}$ be an $W^{1,2}$-elliptic energy functional. Let $w_\tau \in H \cap W^{2,2}(\Omega, M)$ be a solution to* (4.26).

For a conforming grid G of width h and order $m = 1$ (cf. Definition 2.1), set $V_h := H \cap S_h$, and let $w_{\tau,h}$ be a solution of (4.27).

Under the assumptions of this section and $h^2 \leq \kappa\tau$, we have

$$d_{L^2}(W_\tau, W_{\tau,h}) \leq C\left(\tau + h^2\right)$$
$$D_{1,2}(W_\tau, W_{\tau,h}) \leq C\left(\sqrt{\tau} + h\right).$$

Proof. We can use the mixed error measure $D_{1,2}^2 + \tau^{-1} d_{L^2}^2$ in the Céa lemma. In particular, for the solutions w_τ and $w_{\tau,h}$ of (4.26) and (4.27) we obtain the estimate

$$\begin{aligned} D_{1,2}^2(w_\tau, w_{\tau,h}) + \tau^{-1} d_{L^2}^2(w_\tau, w_{\tau,h}) &\leq C\left(D_{1,2}^2(w_\tau, w_{\tau,I}) + \tau^{-1} d_{L^2}^2(w_\tau, w_{\tau,I})\right) \\ &\leq C\,(h^2 + \tau^{-1} h^4). \end{aligned}$$

The assertion follows. □

4.4 Discretization Error Estimate

We will now discuss the error estimate for the fully discrete time layer scheme consisting of the implicit Euler method and geodesic finite elements.

The fully discrete scheme is given by

$$U_{\tau,h}^n = \operatorname*{arg\,min}_{v_h \in V^h} \mathfrak{J}_{\tau_n, U_{\tau,h}^{n-1}}. \tag{4.29}$$

Theorem 4.13. *Let $d < 4$, M be complete with bounded curvature, $\mathfrak{J}$ be $W^{1,2}$-elliptic, and suppose that for any $W \in H$ and τ there exist a minimizer of $\mathfrak{J}_{\tau,W}$, that has a bounded smoothness descriptor $\theta_{2,2,\Omega}$ independent of τ.*

For a conforming grid G of width h and order $m = 1$, set $V_h := H \cap S_h$.

Let $P_\tau = \{0 = t_0 < t_1 < \ldots < t_M < T\}$ be a uniform partition of $(0, T)$ with $\tau = t_n - t_{n-1}$ and $h^2 \leq \kappa\tau$.

Given $u_0 \in D(|\partial\mathfrak{J}|)$, let $u \in L^2((0,T), H^{2,2}(\Omega, M)) \cap W^{1,\infty}((0,T), L^2(\Omega, M))$ be a solution (4.1)*, and let $U_{\tau,h}^n$ be defined by* (4.29).

Then there exists a constant C_{29} such that for $t_n \in P_\tau$

$$d_{L^2}(u(t_n), U_{\tau,h}^n) \leq C_{29}\,(\tau + h^2) \tag{4.30}$$
$$D_{1,2}(u(t_n), U_{\tau,h}^n) \leq C_{29}(\sqrt{\tau} + h). \tag{4.31}$$

Proof. In order to obtain an error estimate for the fully discrete scheme, we introduce the following two semi-discrete schemes.

Given $U^0 = u_0$, we define

$$U_\tau^n = \operatorname*{arg\,min}_{V\in H} \mathfrak{J}_{\tau,U_\tau^{n-1}}(V), \tag{4.32}$$

and given $U_{\tau,h}^{n-1} \in V_h$, we define

$$V^n = \operatorname*{arg\,min}_{V\in H} \mathfrak{J}_{\tau,U_{\tau,h}^{n-1}}(V).$$

Lemma 4.12 implies

$$D_{1,2}^2(V^n, U_{\tau,h}^n) + \tau^{-1} d_{L^2}^2(V^n, U_{\tau,h}^n) \le C\,(h^2 + \tau^{-1}h^4),$$

and using the condition $h^2 \le \kappa\tau$, we thus obtain consistency

$$d_{L^2}(V^n, U_{\tau,h}^n) \le C\sqrt{\tau}h, \tag{4.33}$$

$$D_{1,2}(V^n, U_{\tau,h}^n) \le C\,h. \tag{4.34}$$

We now compare V^n and U_τ^n. First note that if $W, V \in H$ and $W_\tau \in H$ denotes the minimizer of $\mathfrak{J}_{\tau,W}$, the $W^{1,2}$–L^2-convexity shown in Proposition 4.8 yields

$$\mathfrak{J}_{\tau,W}(W_\tau) \le (1-t)\mathfrak{J}_{\tau,W}(W_\tau) + t\mathfrak{J}_{\tau,W}(V) - \frac{\tilde{\lambda}}{2} t(1-t)\left(D_{1,2}^2(V,W_\tau) + \tau^{-1} d_{L^2}^2(V,W_\tau)\right).$$

Rearranging, dividing by t, and then letting $t \to 0$ we obtain

$$\frac{\tilde{\lambda}}{2}\left(D_{1,2}^2(V,W_\tau) + \tau^{-1} d_{L^2}^2(V,W_\tau)\right) \le \mathfrak{J}(V) - \mathfrak{J}(W_\tau) + \frac{1}{2\tau}\left(d_{L^2}^2(W,V) - d_{L^2}^2(W,W_\tau)\right)$$

for all $V \in H$. As V^n and U_τ^n are minimizers of $\mathfrak{J}_{\tau,U_\tau^{n-1}}$ and $\mathfrak{J}_{\tau,U_{\tau,h}^{n-1}}$, respectively, we obtain the estimates

$$\begin{aligned}&\frac{\tilde{\lambda}}{2}\left(D_{1,2}^2(V,V^n) + \tau^{-1} d_{L^2}^2(V,V^n)\right)\\ &\qquad \le \mathfrak{J}(V) - \mathfrak{J}(V^n) + \frac{1}{2\tau}\left(d_{L^2}^2(U_{\tau,h}^{n-1},V) - d_{L^2}^2(U_{\tau,h}^{n-1},V^n)\right) \qquad \forall V \in H,\end{aligned}$$

and

$$\begin{aligned}&\frac{\tilde{\lambda}}{2}\left(D_{1,2}^2(V,U_\tau^n) + \tau^{-1} d_{L^2}^2(V,U_\tau^n)\right)\\ &\qquad \le \mathfrak{J}(V) - \mathfrak{J}(U_\tau^n) + \frac{1}{2\tau}\left(d_{L^2}^2(U_\tau^{n-1},V) - d_{L^2}^2(U_\tau^{n-1},U_\tau^n)\right) \qquad \forall V \in H.\end{aligned}$$

Inserting V^n as test function for U_τ^n and vice versa, we obtain

$$
\begin{aligned}
&\tilde{\lambda}\left(D^2_{1,2}(U^n_\tau,V^n)+\tau^{-1}d^2_{L^2}(U^n_\tau,V^n)\right)\\
&\quad\le \frac{1}{2\tau}\Big(d^2_{L^2}(U^{n-1}_{\tau,h},U^n_\tau)-d^2_{L^2}(U^{n-1}_{\tau,h},V^n)+d^2_{L^2}(U^{n-1}_\tau,V^n)-d^2_{L^2}(U^{n-1}_\tau,U^n_\tau)\Big)\\
&\quad\le \tau^{-1}\, d_{L^2}(U^n_\tau,V^n)\, d_{L^2}(U^{n-1}_\tau,V^{n-1}).
\end{aligned}
$$

Young's inequality implies

$$
D^2_{1,2}(U^n_\tau,V^n)+\tau^{-1}d^2_{L^2}(U^n_\tau,V^n)\le C\tau^{-1}d_{L^2}(U^{n-1}_\tau,U^{n-1}_{\tau,h}). \tag{4.35}
$$

Combining the (approximate) triangle inequalities for d_{L^2} and $D_{1,2}$ with (4.33), (4.34), and (4.35) yields

$$
d_{L^2}(U^n_\tau,U^n_{\tau,h})\le d_{L^2}(U^n_\tau,V^n)+d_{L^2}(V^n,U^n_{\tau,h})\le C\,d_{L^2}(U^{n-1}_\tau,U^{n-1}_{\tau,h})+C\sqrt{\tau}h,
$$

and

$$
D_{1,2}(U^n_\tau,U^n_{\tau,h})\le C\,D_{1,2}(U^n_\tau,V^n)+C\,D_{1,2}(V^n,U^n_{\tau,h})\le C\,\tau^{-\frac{1}{2}}\,d_{L^2}(U^{n-1}_\tau,U^{n-1}_{\tau,h})+C\,h.
$$

Assuming that $U^0_{\tau,h}$ is the GFE interpolation of U^0_τ, we have

$$
d_{L^2}(U^0_\tau,U^0_{\tau,h})\le C\,h^2,
$$

and thus, using again $h^2\le\kappa\tau$,

$$
\begin{aligned}
d_{L^2}(U^n_\tau,U^n_{\tau,h})&\le C\,h^2+C\,\tau\\
D_{1,2}(U^n_\tau,U^n_{\tau,h})&\le C\,h+C\,\sqrt{\tau}.
\end{aligned}
$$

Combining these estimates with Corollary 4.10, we obtain

$$
\begin{aligned}
d_{L^2}(u(t_n),U^n_{\tau,h})&\le d_{L^2}(u(t_n),U^n_\tau)+d_{L^2}(U^n_\tau,U^n_{\tau,h})\le C\,(h^2+\tau)\\
D_{1,2}(u(t_n),U^n_{\tau,h})&\le C\left(D_{1,2}(u(t_n),U^n_\tau)+D_{1,2}(U^n_\tau,U^n_{\tau,h})\right)\le C\,h+C\,\sqrt{\tau}.
\end{aligned}
$$

□

Chapter 5
Example: Harmonic Maps

In the study of discretization error bounds for geodesic finite elements in Chapters 3 and 4, we have made some fairly strong assumptions on the problem. In this section we will illustrate the results of these chapters by applying them to the minimization of the harmonic energy. In particular, we will show that under certain assumptions on the manifold, the harmonic energy fulfills the assumptions of Theorems 3.6, 3.15, and 4.13, such that we obtain a priori discretization error bounds.
The study of more general energies is left to future work.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $\partial\Omega$ be in C^2 and (M,g) a smooth Riemannian manifold. We discretize Ω by a grid G of width h and order m (cf. Definition 2.1). Let $\phi : \partial\Omega \to M$ be continuous. For simplicity we assume that ϕ can be attained exactly by mth order geodesic finite elements. This restriction can be waived by suitable approximation arguments. For $q > \max(2,d)$, we set

$$W^{1,q}_{K;\phi} := \left\{ v \in W^{1,q}(\Omega,M) \ : \ \theta_{1,q,\Omega}(v) \leq K, \ v_{|\partial\Omega} = \phi \right\}. \tag{5.1}$$

We study the harmonic energy $\mathfrak{J} : W^{1,2}(\Omega,M) \to \mathbb{R}$ defined by

$$\begin{aligned} \mathfrak{J}(v) &:= \frac{1}{2} \int_\Omega |\nabla u(x)|^2_{g(u(x))} \, dx \\ &= \frac{1}{2} \sum_{\alpha=1}^{d} \int_\Omega g_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha}(x) \frac{\partial u^j}{\partial x^\alpha}(x) \, dx. \end{aligned} \tag{5.2}$$

The theory of stationary points of this energy, so-called harmonic maps, is well-developed (see e.g. [EL78, EL88, Jos08]). The corresponding L^2-gradient flow has been studied, e.g., in [Str85, ES64, Har67, CG89, CDY92, Gro93]. A more recently published survey can be found in [HW08].

The harmonic energy has been studied numerically using geodesic finite elements in [San12] and [San13]. Other discretization methods have been employed in [BP07, Bar10, Alo97, LL89].

5.1 $W^{1,2}$-Discretization Error Bounds

The harmonic energy can be viewed as the standard example for an elliptic energy. As such it has also served as an example in [GHS14] and the content of this section can also be found there.

Lemma 5.1. *Let $q > \max\{2,d\}$, $W^{1,q}_{K;\phi}$ be defined by* (5.1)*, and $\mathfrak{J} : W^{1,2}(\Omega,M) \to \mathbb{R}$ be defined by* (5.2)*. Assume that either M has nonpositive sectional curvature, or that*

$$1 - K^2 \|\operatorname{Rm}\|_g C_1(q,\Omega)^2 > 0 \tag{5.3}$$

holds, where $C_1(q,\Omega)$ denotes the Sobolev constant for the embedding $W^{1,2}(\Omega) \subset L^{\frac{2q}{q-2}}(\Omega)$.
Then $\mathfrak{J}$ is elliptic in the sense of Definition 3.4 along geodesic homotopies in $W^{1,q}_{K;\phi}$.

Proof. The following calculations are standard (see e.g. [EL78, Jos08]).

Let $u \in W^{1,q}_{K;\phi}$, and let $\Gamma : (-1,1) \to W^{1,q}_{K;\phi}$ denote a geodesic homotopy with $\Gamma(0) = u$, i.e.,

$$\Gamma(t,x) = \exp_{u(x)}(tV(x)),$$

where $V \in W^{1,q}_0(\Omega, u^{-1}TM)$.

We calculate the first variation of $\mathfrak{J}$ along Γ

$$\begin{aligned}\delta\mathfrak{J}(u)(V) &= \frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla\Gamma(t,x)|^2_{g(\Gamma(t,x))}\, dx_{|t=0}\\ &= \int_\Omega \langle \nabla_x u, \nabla_x V\rangle_{g(u(x))}\, dx.\end{aligned}$$

Let $c : [-1,1]^2 \to W^{1,q}_{K;\phi}$ denote a family of curves, such that $c(t,\cdot)$, and $c(s,\cdot)$ are geodesic homotopies for each t and s, and $c(0,0) = u$, i.e.,

$$c(t,s) = \exp_{u(x)}(tV(x) + sW(x)),$$

where $V, W \in W^{1,q}_0(\Omega, u^{-1}TM)$. We calculate the second variation of $\mathfrak{J}$

$$\begin{aligned}&\delta^2\mathfrak{J}(u)(V,W)\\ =&\frac{d^2}{dsdt}\frac{1}{2}\int_\Omega |\nabla c(t,s,x)|^2_{g(c(t,s,x))}\, dx_{|(t,s)=(0,0)}\\ =&\frac{d}{ds}\int_\Omega \langle \nabla_x c(t,s,x), \nabla_x\nabla_t c(t,s,x)\rangle_{g(c(t,s,x))}\, dx_{|(t,s)=(0,0)}\\ =&\int_\Omega \langle \nabla_x\nabla_s c(t,s,x), \nabla_x\nabla_t c(t,s,x)\rangle_{g(c(t,s,x))}\, dx_{|(t,s)=(0,0)}\\ &\quad - \int_\Omega \langle \nabla_x c(t,s,x), R(\nabla_x c(t,s,x), \nabla_s c(t,s,x))\nabla_t c(t,s,x)\rangle_{g(c(t,s,x))}\, dx_{|(t,s)=(0,0)}\end{aligned}$$

$$= \int_\Omega \langle \nabla_x W(x), \nabla_x V(x) \rangle_{g(u(x))} \, dx$$
$$- \int_\Omega \langle \nabla_x u(x), R(\nabla_x u(x), W(x)) V(x) \rangle_{g(u(x))} \, dx.$$

Using Hölder's inequality and the Sobolev embedding theorem, we can easily see that $\mathfrak{J}$ fulfills (3.8) with $\Lambda = 1 + K^2 \| \operatorname{Rm} \|_g C_1(q,\Omega)^2$. If M has nonpositive sectional curvature, the curvature term has a sign, so that we obtain (3.7) with λ depending on the Poincaré constant C_2 of the domain Ω given by $\lambda = \frac{1}{C_2^2}$. Otherwise, we obtain (3.7) analogously to (3.8) with $\lambda = \frac{1}{C_2^2}\left(1 - K^2 \| \operatorname{Rm} \|_g C_2^2 C_1(q,\Omega)^2\right) > 0$. Thus, $\mathfrak{J}$ is indeed elliptic. □

Remark 5.2. Note that under the curvature assumptions on M of Lemma 5.1, stationary points of $\mathfrak{J}$ are indeed stable critical points. The assumed upper bound for positive curvature is fairly strong and may be weakened.

The discretization error bounds presented in Chapter 3 always assume existence of solutions with a certain regularity. The topic of existence and regularity of harmonic maps is extensively studied in the literature. For an overview see for example [EL78, EL88, HW08]. In particular, we have the following.

Lemma 5.3. *A harmonic map $u : \Omega \to M$ with continuous boundary data ϕ is in C^∞, if either M has nonpositive sectional curvature, or if $d \in \{1,2\}$, or if the image of ϕ is contained in a geodesically convex ball.*

We can now prove the following convergence theorem for the discretization of harmonic maps by geodesic finite elements.

Theorem 5.4. *Let u be a local minimizer of the harmonic energy $\mathfrak{J}$ on $W^{1,2}_\phi(\Omega, M)$, where M has either nonpositive sectional curvature, or* (5.3) *holds. If M does not have nonpositive sectional curvature, we additionally assume that either $d \in \{1,2\}$, or that $\phi(\Omega)$ is contained in a geodesically convex ball.*

For $m \geq 1$ fulfilling $2(m+1) > d$ let G be a conforming grid of width h and order m.

If h is small enough, there exists a local minimizer u_h of $\mathfrak{J}$ in S_h subject to the boundary condition fulfilling

$$D_{1,2}(u,u_h) \leq C\, h^m\, \theta_{m+1,2,\Omega}(u). \tag{5.4}$$

Measured in an isometric embedding $\iota : M \to \mathbb{R}^N$, we have

$$\| \iota \circ u - \iota \circ u_h \|_{W^{1,2}(\Omega,\mathbb{R}^N)} \leq C\, h^m \theta_{m+1,2,\Omega}(\iota \circ u) \leq C\, h^m \| \iota \circ u \|^{m+1}_{m+1,p,\Omega}. \tag{5.5}$$

Proof. As u is smooth by Lemma 5.3, we can apply Theorem 3.8 with $k = m+1$. □

Remark 5.5. In [San12], Theorem 5.4 is numerically confirmed for a test case with $M = S^2$, $d = 3$, and $m = 1$. Note that the assumptions of Lemma 5.3 did not hold there. Higher order geodesic finite elements have been studied in [San13]. Theorem 5.4 is numerically confirmed for a test case with $M = S^2$, $d = 2$, and $m \in 1,2,3$.

5.2 L^2-Discretization Error Bounds

The harmonic energy for functions into $\mathbb{R}^n$ is the prototypic example of a quadratic energy and thus of a linear second order PDE. We now want to show that it fulfills all the assumptions of Theorem 3.15 for a certain class of manifolds M. Note that Theorem 3.15 is only valid for $d < 4$. In view of Remark 3.17, we consider the case $d \geq 4$ for the technical prerequisites although they do not lead to an L^2-error estimate as long as there is no suitable interpolation theory for $W^{2,2}$-vector fields in this case.

In Section 3.2, we restricted ourselves to predominantly quadratic energy functionals (cf. Definition 3.10). The harmonic energy belongs to this group.

Lemma 5.6. *Let $q > \max\{2,d\}$, and let $W^{1,q}_{K;\phi}$ be defined by* (5.1). *Consider the harmonic energy $\mathfrak{J} : W^{1,2}(\Omega, M) \to \mathbb{R}$ defined by* (5.2). *Assume that* Rm *and* ∇Rm *of M are bounded.*

Then, if $d < 4$, $\mathfrak{J}$ is predominantly quadratic in the sense of Definition 3.10 on $W^{1,q}_{K;\phi}$. If $d \geq 4$, $\mathfrak{J}$ fulfills (3.27) *on $W^{1,q}_{K;\phi}$.*

Proof. We need to consider third variations of $\mathfrak{J}$. Let $c : [-1,1]^3 \to W^{1,q}_{K;\phi}$ denote a family of curves defined by

$$c(t,s,r) = \exp_{u(x)}(tU(x) + sV(x) + rW(x)),$$

where $U,V,W \in W^{1,\infty}_0(\Omega, u^{-1}TM)$. We calculate

$$\begin{aligned}
&\delta^3\mathfrak{J}(u)(U,V,W)\\
&\quad= \frac{d^3}{drdsdt}\frac{1}{2}\int_\Omega |\nabla c(t,s,r,x)|^2_{g(c(t,s,r,x))}\, dx_{|(t,s,r)=(0,0,0)}\\
&\quad= \int_\Omega \langle \nabla_r\nabla_x\nabla_t c(t,s,r,x), \nabla_x\nabla_s c(t,s,r,x)\rangle_{g(c(t,s,r,x))}\, dx_{|(t,s,r)=(0,0,0)}\\
&\qquad+ \int_\Omega \langle \nabla_x\nabla_t c(t,s,r,x), \nabla_r\nabla_x\nabla_s c(t,s,r,x)\rangle_{g(c(t,s,r,x))}\, dx_{|(t,s,r)=(0,0,0)}\\
&\qquad+ \int_\Omega \langle \nabla_s\nabla_x\nabla_t c(t,s,r,x), \nabla_x\nabla_r c(t,s,r,x)\rangle_{g(c(t,s,r,x))}\, dx_{|(t,s,r)=(0,0,0)}\\
&\qquad+ \int_\Omega \langle \nabla_r\nabla_s\nabla_x\nabla_t c(t,s,r,x), \nabla_x c(t,s,r,x)\rangle_{g(c(t,s,r,x))}\, dx_{|(t,s,r)=(0,0,0)}
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \mathrm{Rm}(W,du,U,\nabla V)\,dx + \int_\Omega \mathrm{Rm}(W,du,V,\nabla U)\,dx \\
&\quad + \int_\Omega \mathrm{Rm}(V,du,U,\nabla W)\,dx + \int_\Omega \mathrm{Rm}(V,\nabla W,U,du)\,dx \\
&\quad - \int_\Omega \nabla\mathrm{Rm}(du,V,U,du,W)\,dx.
\end{aligned}
$$

For $W=V$ and using the bounds on Rm and ∇Rm, we obtain

$$
\begin{aligned}
&|\delta^3\mathfrak{J}(u)(U,V,V)| \\
&\le C\left(\int_\Omega |du|^2\,|U|\,|V|^2\,dx + \int_\Omega |du|\,|\nabla U|\,|V|^2\,dx + \int_\Omega |du|\,|\nabla V|\,|U|\,|V|\,dx\right).
\end{aligned}
$$

Let p be defined as in Definition 3.10. For $d<4$, set $r=p$, for $d\ge 4$ set $r=d$. Then we can estimate using Hölder's inequality and the $W^{1,q}$-bound on u

$$
|\delta^3\mathfrak{J}(u)(U,V,V)| \le C\|U\|_{W^{1,p}}\|V\|_{W^{1,2}}\|V\|_{L^r}.
$$

This implies the assertion. □

We also need to show H^2-regularity of the solution $W\in W_0^{1,2}(\Omega,u^{-1}TM)$ of the deformation problem

$$
\int_\Omega \langle\nabla W,\nabla V\rangle\,dx = \int_\Omega \mathrm{Rm}(du,W,V,du) - \langle V,U\rangle\,dx \quad \forall V\in W_0^{1,2}(\Omega,u^{-1}TM), \tag{5.6}
$$

where $U\in L^\infty(\Omega,u^{-1}TM)$.

Lemma 5.7. *Let $q>\max\{2,d\}$ and $W^{1;q}_{K;\phi}$ be defined by* (5.1). *Assume that* Rm *of M is bounded, and let $u\in W^{1;q}_{K;\phi}\cap W^{2,2}(\Omega,M)$ be a harmonic map.*
Then the deformation problem (5.6) *is H^2-regular in the sense of* (3.22).

The proof is discussed in Appendix B. We need also to show that the discrete minimizer u_h of $\mathfrak{J}$ in S_h fulfills the a priori bound (3.21). We restrict ourselves to certain cases that are easy to show.

Proposition 5.8. *Let $m\ge 1$ and $2(m+1)>d$. If $d=1$, set $m=1$. Define b as in Lemma 3.14 as*

$$
b = \begin{cases} 2 & \text{for } d<4 \\ 3 & \text{for } d=4 \\ \frac{d}{2} & \text{for } d>4, \end{cases}
$$

and let $q>\max\{2,d\}$ fulfill $q\ge 2b$.
If $d>1$, we pose the additional assumption on the grid G on Ω that $F_h^{-1}: T\to T_h$ scales with order 2 for all elements T_h.

For $v \in C^\infty(\Omega, M)$, and $v_h \in S_h \cap W_K^{1,q}$ with $d_{L^s}(v_h, v) \le L$, where s is defined as in Proposition 1.41, we assume the relation

$$D_{1,2}(v, v_h) \le Ch^m \theta_{m+1,2,\Omega}(u). \tag{5.7}$$

Then there exists a constant K_2 depending on v and K but independent of h such that

$$\theta_{2,b,T_h}(v_h) \le K_2$$

on every element $T_h \in G$.

Proof. As $b \le q$ it is enough to estimate $\dot\theta_{2,b,T_h}(v_h)$.

For $d = m = 1$ the estimate is trivial as $\dot\theta_{2,2,T_h}(v_h) = \dot\theta^2_{1,4,T_h}(v_h) \le K^2$ (cf. Example 2.10).

For $d > 1$, we use Proposition 2.14 and $2b \le q$ to estimate

$$\begin{aligned}\dot\theta_{2,b,T_h}(v_h) &\le C\,\dot\theta^2_{1,2b,T_h}(v_h) + C\,h^{-1-d(\frac{1}{2}-\frac{1}{b})}\dot\theta_{1,2,T_h}(v_h)\\ &\le C\,K^2 + C\,h^{-1-d(\frac{1}{2}-\frac{1}{b})}\dot\theta_{1,2,T_h}(v_h).\end{aligned}$$

We apply Proposition 1.45 to estimate

$$\begin{aligned}h^{-1-d(\frac{1}{2}-\frac{1}{b})}\dot\theta_{1,2,T_h}(v_h) &\le C\,h^{-1-d(\frac{1}{2}-\frac{1}{b})}\dot\theta_{1,2,T_h}(v) + C\,h^{-1-d(\frac{1}{2}-\frac{1}{b})}D_{1,2}(v, v_h)\\ &\le C\,h^{-1-d(\frac{1}{2}-\frac{1}{b})}\dot\theta_{1,\infty,T_h}(v)|T_h|^{\frac{1}{2}} + C\,h^{-1-d(\frac{1}{2}-\frac{1}{b})}D_{1,2}(v, v_h).\end{aligned}$$

Note that the elements of the grid scale with $|T_h| \sim h^d$. Using this and assumption (5.7), we obtain

$$\dot\theta_{2,b,T_h}(v_h) \le C\,K^2 + C\,h^{\frac{d}{b}-1} + C\,h^{m-\frac{d}{2}+\frac{d}{b}-1}.$$

We have

$$\frac{d}{b} - 1 = \begin{cases} 0 & \text{for } d = 2\\ \frac{1}{2} & \text{for } d = 3\\ \frac{1}{3} & \text{for } d = 4\\ 1 & \text{for } d > 4,\end{cases}$$

and $m > \frac{d}{2} - 1$. As $m \ge 1$ is an integer, this is enough to show that $m - \frac{d}{2} + \frac{d}{b} - 1 \ge 0$. Thus, the estimate follows. □

We can now state the following optimal discretization error bound for harmonic maps.

Theorem 5.9. *Let $d < 4$, $m \ge 1$ with $2(m+1) > d$, and $m = 1$ if $d = 1$. Assume that G is a conforming grid of width h and order m, such that $F_h^{-1} : T \to T_h$ scales with order 2 if $d > 1$.*

Let u be a local minimizer of the harmonic energy $\mathfrak{J}$ on $W_\phi^{1,2}(\Omega, M)$, where M has either nonpositive sectional curvature, or (5.3) holds. If M does not have nonpositive *sectional curvature, we additionally assume that either* $d \in \{1,2\}$*, or that* $\phi(\Omega)$ *is contained in a geodesically convex ball.*

If h *is small enough, there exists a local minimizer* u_h *in* S_h *such that*

$$d_{L^2}(u,u_h) \leq C\, h^{m+1}\, \theta^2_{m+1,2,\Omega}(u) \tag{5.8}$$

holds.

Proof. Set $k = m+1$, and choose $q \geq 4$ admissible in the proof of Theorem 3.8. Note that for $d = 1,2$, this simply implies $q < \infty$. Only in the case $d = 3$, $m = 1$, q has to fulfill a stronger upper bound, namely $q < 6$.

Theorem 5.4, Lemmas 5.3, 5.6, 5.7, and Proposition 5.8 then show that we can apply Theorem 3.15. □

Remark 5.10. The numerical experiments in [San13] confirm the result of Theorem 5.9 for a test case with $M = S^2$, $d = 2$, and $m \in 1,2,3$. In [San12], the result (5.8) has also been observed for $M = S^2$, $d = 3$, and $m = 1$.

5.3 Heat Flow

The L^2-gradient flow of the harmonic energy (5.2), also called harmonic map heat flow

$$\langle u'(t), V\rangle_{L^2(\Omega,u(t)^{-1}TM)} = -\frac{d}{ds}_{|s=0}\mathfrak{J}(\exp_{u(t)}(sV)), \tag{5.9}$$

has been extensively studied in the literature. It is well-known that for a general target manifold M and $d \geq 2$ solutions to (5.9) may blow up in finite time (see, e.g., [CG89, CDY92, Gro93]).

For manifolds of non-positive curvature, however, we have the following result [Ham75].

Lemma 5.11. *Let* M *have non-positive curvature,* ϕ *denote smooth boundary data on* $\partial\Omega$*, and let* u_0 *be smooth initial data assuming the boundary data on* $\partial\Omega$*.*
Then there exists a unique smooth solution $u \in C^\infty(\Omega\times[0,\infty),M)\cap C(\overline{\Omega}\times[0,\infty),M)$ *to* (5.9) *assuming the boundary and intial data and fulfilling uniform bounds.*

We discretize (5.9) by the methods of layers we introduced in Chapter 4.

Remark 5.12. In order to apply Theorem 4.13, we need the existence and the H^2-regularity of minimizers of $\mathfrak{J}_{\tau,W} = \mathfrak{J} + \frac{1}{2\tau}d^2_{L^2}(W,\cdot)\ \mathfrak{J}$ at $W \in H$. Furthermore, the H^2-bound has to be independent of τ.

Typically, standard theory for the harmonic map heat-flow uses heat-kernels instead of the outlined procedure (see, e.g., [Ham75, CS89]). A detailed regularity analysis for minimizers of $\mathfrak{J}_{\tau,W}$ is outside of the scope of this work. We are, however, confident that future work will indeed close this gap.

We obtain the following discretization error bound.

Theorem 5.13. *Assume that M is complete and has nonpositive curvature which is bounded from below. For $d < 4$, $\mathfrak{J}$ defined by* (5.2)*, and $u_0 \in C^\infty(\Omega, M)$ let u be a solution* (5.9).

Consider a uniform partition $P_\tau = \{0 = t_0 < t_1 < \ldots < t_M < T\}$ of $(0,T)$ with $\tau = t_n - t_{n-1}$. Assume that for any $W \in H$ there exist a minimizer of $\mathfrak{J}_{\tau,W} = \mathfrak{J} + \frac{1}{2\tau} d^2_{L^2}(W, \cdot)$, which is bounded in $H^{2,2}(\Omega, M)$ independent of τ.

Let $m = 1$, G a conforming grid of width $h \leq \sqrt{\kappa\tau}$, and set $V_h := W^{1,q}_{K;\phi}(\Omega, M) \cap S_h$, with K and q as in the proof of Theorem 3.8. Consider $U^n_{\tau,h}$ defined by (4.29).

If h is small enough, then we have for $t_n \in P_\tau$

$$d_{L^2}(u(t_n), U^n_{\tau,h}) \leq C_{29}\ (\tau + h^2) \tag{5.10}$$

$$D_{1,2}(u(t_n), U^n_{\tau,h}) \leq C_{29}(\sqrt{\tau} + h). \tag{5.11}$$

Proof. In view of Lemmas 5.11 and 5.1, we can directly apply Theorem 4.13 to obtain the assertion. □

Remark 5.14. No numerical studies using geodesic finite elements for gradient flows have been conducted so far. An experimental validation of Theorem 5.13 is planned.

Appendices

Appendix A
Estimates for the Exponential Map

In this appendix we will introduce some notation and estimates for the exponential map that are used throughout the main chapters of this work.

The exponential map is defined by

$$\exp_p : T_pM \to M, \qquad \exp_p(V) = \gamma_V(1),$$

where γ_V is a geodesic with $\gamma_V(0) = p$, $\dot{\gamma}_V(0) = V$. We denote the inverse of the exponential map by

$$\log : M^2 \to TM, \qquad \log(p,q) = \log_q p = \exp_q^{-1} p.$$

The differential of the exponential map is defined by

$$d\exp_p V \ : \ T_pM \to T_{\exp_p V}M, \qquad d\exp_p V(W) = \frac{d}{dt}_{t=0} \exp_p(V + tW).$$

For the differential with respect to the base point of exp we write $d_2 \exp$, i.e., for $V, W \in T_pM$

$$d_2 \exp_p V(W) = \frac{d}{dt}_{t=0} \exp_{\gamma_W(t)} \pi^{\gamma_W}_{\gamma_W(0) \mapsto \gamma_W(t)} V,$$

where π^{γ_W} denotes the parallel transport along γ_W.

For the bivariate logarithm $\log : (p,q) \mapsto \log_q p$ we denote the covariant derivative with respect to the first and second component by d_1 and d_2, respectively.

It is well known (see, e.g., [Jos08]) that

$$J(s) := d\exp_p(sV)(sW).$$

defines for $V, W \in T_pM$ the Jacobi field along the geodesic $\gamma(s) := \exp_p(sV)$ with $J(0) = 0$ and $\dot{J}(0) = W$. Jacobi field theory can be used to compare the derivative of log to parallel transport along γ and the identity map between tangent spaces. For the reader's convenience we shall here prove these elementary estimates. A similar proof can be found in the appendix of [Kar77].

Proposition A.1. *Let $p,q \in B_\rho \subset M$ with ρ small enough. Let* Rm *denote the Riemannian curvature tensor of M, and assume* Rm *to be bounded. Then*

$$\|d_2 \log_p q + Id\| \leq \frac{1}{2}|R|_\infty\, d^2(p,q) \tag{A.1}$$

$$\|d \log_p q - \pi_{q\mapsto p}\| \leq \frac{1}{2}|R|_\infty\, d^2(p,q), \tag{A.2}$$

where $\pi_{q\mapsto p} : T_qM \to T_pM$ denotes parallel transport along a geodesic.

Proof. We only prove (A.1) as (A.2) can be proved analogously.

Let $W \in T_pM$. Then

$$W + d_2 \log_p q(W) = J(1) - \dot{J}(1),$$

where J is the Jacobi field along the geodesic γ connecting q to p with $J(0) = 0$ and $J(1) = W$. Define $f : I \to T_pM$ by $f(t) = \pi_p^\gamma(J(t) - t\dot{J}(t))$. Then

$$\begin{aligned} W + d_2 \log_p q(W) &= f(1) - f(0) \\ &= \int_0^1 \frac{d}{dt} f(t)\, dt \\ &= \int_0^1 \pi_p^\gamma(-t\ddot{J}(t))\, dt \\ &= \int_0^1 t\pi_p^\gamma R(J(t),\dot{\gamma}(t))\dot{\gamma}(t)\, dt. \end{aligned}$$

Thus, we obtain

$$|W + d_2 \log_p q(W)| \leq |R|_\infty d^2(p,q) \int_0^1 t|J(t)|\, dt.$$

We assume that the sectional curvature of M is bounded by $K \geq 0$, and if $K > 0$, we further assume $d(p,q) \leq \frac{\pi}{2K}$. Set

$$s_K(t) = \begin{cases} \frac{1}{\sqrt{K}} \sin(\sqrt{K}t) & \text{if } K > 0 \\ t & \text{if } K = 0. \end{cases}$$

Then $s_K(t\|\dot{\gamma}(0)\|) > 0$ for all $t \in (0,1]$. Thus by Rauch comparison (see, e.g., [Jos08]), we have

$$|J(t)| \leq \frac{s_K(t\|\dot{\gamma}(0)\|)}{s_K(\|\dot{\gamma}(0)\|)}|W| \leq |W|,$$

and thus

$$|W + d_2 \log_p q(W)| \leq \frac{1}{2}|R|_\infty d^2(p,q)|W|.$$

□

Second derivatives of the logarithm fulfill similar bounds.

Proposition A.2. *Let $p, q \in B_\rho \subset M$ with ρ small enough. Then*

$$\|d_2 d \log_p q\| = C_{30}\, d(p,q) \tag{A.3}$$

$$\|d_2^2 \log_p q\| = C_{30}\, d(p,q), \tag{A.4}$$

where C_{30} depends on Rm *and* ∇Rm.

Proof.

(A.3): Let $V \in T_pM$ and $W \in T_qM$, and let $\gamma(t) = \exp_p(t \log_p q)$. We consider the family of curves given by

$$c(t,s,r) := \exp_{\exp_p(sV)}\left(t \log_{\exp_p(sV)} \exp_q(rW)\right).$$

Then $c(t,0,0) = \gamma(t)$, c is a geodesic in t for all r and s, and the vector fields

$$J_s(t) := \nabla_s c(t,0,0)$$
$$J_r(t) := \nabla_r c(t,0,0)$$

are Jacobi fields along γ with

$$J_s(0) = V, \qquad J_s(1) = 0$$
$$J_r(0) = 0, \qquad J_r(1) = W.$$

These Jacobi fields further fulfill

$$|\dot{J}_s(0)| = |d_2 \log_p q(V)| \leq \left(1 + \frac{1}{2}|R|_\infty d^2(p,q)\right)|V|$$
$$|\dot{J}_r(1)| = |d_2 \log_q p(W)| \leq \left(1 + \frac{1}{2}|R|_\infty d^2(p,q)\right)|W|.$$

Assuming $d(p,q) \leq \frac{\pi}{2K}$, we have (as in the proof of (A.1))

$$|J_s(t)| \leq |V|$$
$$|J_r(t)| \leq |W|,$$

and hence

$$|\dot{J}_s(t)| \leq |\dot{J}_s(0)| + \int_0^t |\ddot{J}_s(\tau)|\, d\tau \leq \frac{1}{2}\left(2 + (1+2t)|R|_\infty d^2(p,q)\right)|V|$$
$$|\dot{J}_r(t)| \leq |\dot{J}_r(1)| + \int_t^1 |\ddot{J}_r(\tau)|\, d\tau \leq \frac{1}{2}\left(2 + (1+2(1-t))|R|_\infty d^2(p,q)\right)|W|.$$

Set $X(t) := \nabla_s \nabla_r c(t,0,0)$. Then

$$\begin{aligned} d_2 d \log_p q(W,V) &= \nabla_s \nabla_r \nabla_t c(0,0,0) \\ &= \nabla_t \nabla_s \nabla_r c(0,0,0) + R(\nabla_t c(0,0,0), \nabla_s c(0,0,0)) \nabla_r c(0,0,0) \\ &= \dot{X}(0) + R(\dot{\gamma}(0), J_s(0)) J_r(0) \\ &= \dot{X}(0). \end{aligned}$$

Further, $X(0) = 0 \in T_pM$, and $X(1) = 0 \in T_qM$. By deriving the previous equality by T again, we obtain an ODE for X

$$\begin{aligned} \ddot{X}(t) &= \nabla_t \nabla_t \nabla_s \nabla_r c(t,0,0) \\ &= \nabla_t \nabla_s \nabla_t \nabla_r c(t,0,0) + \nabla_t (R(\dot{\gamma(t)}, J_s(t)) J_r(t)) \\ &= \nabla_s \nabla_t \nabla_t \nabla_r c(t,0,0) + R(J_s(t), \dot{\gamma}(t)) \dot{J}_r(t) + \nabla_t (R(\dot{\gamma(t)}, J_s(t)) J_r(t)) \\ &= \nabla_{J_s} R(\dot{\gamma}(t), J_r(t))) \dot{\gamma}(t) + \nabla_{\dot{\gamma}} R(\dot{\gamma(t)}, J_s(t)) J_r(t)) + 2R(\dot{\gamma(t)}, J_s(t)) \dot{J}_r(t)) \\ &\quad + 2R(\dot{\gamma(t)}, J_r(t)) \dot{J}_s(t)) + R(\dot{\gamma}(t), X(t))) \dot{\gamma}(t). \end{aligned}$$

Assuming $d^2(p,q) \leq \frac{1}{2|R|_\infty}$, we can estimate

$$\begin{aligned} \|\ddot{X}\|_\infty &\leq 2|\nabla R||\dot{\gamma}|^2 |J_s||J_r| + 2|R||\dot{\gamma}| \left(|\dot{J}_s||J_r| + |J_s||\dot{J}_r| \right) + \frac{1}{2}|X| \\ &\leq 2|\nabla R||\dot{\gamma}|^2 |J_s||J_r| + \frac{1}{2}\|\ddot{X}\|_\infty, \end{aligned}$$

from which follows

$$\|\ddot{X}\|_\infty \leq 4\nabla R||\dot{\gamma}|^2 |J_s||J_r| + 4|R||\dot{\gamma}| \left(|\dot{J}_s||J_r| + |J_s||\dot{J}_r| \right).$$

Thus, we obtain

$$\|\ddot{X}\|_\infty \leq C(|R|, |\nabla R|) d(p,q)\ |V|\ |W|.$$

Set now $f(t) := \pi_p^\gamma (X(t) + (1-t)\dot{X}(t))$. Then

$$d_2 d \log_p q(W,V) = \dot{X}(0) = -(f(1) - f(0)) = -\int_0^1 (1-t) \pi_p^\gamma \ddot{X}(t)\, dt,$$

and thus

$$|d_2 d \log_p q(W,V)| \leq \frac{1}{2}\|\ddot{X}\|_\infty \leq C(|R|, |\nabla R|) d(p,q) |V|\ |W|.$$

(A.4): The proof is analogous to the previous one: We set for $V, W \in T_pM$

$$c(t,s,r) := \exp_q \left((1-t) \log_q \exp_q (sV + rW) \right).$$

The Jacobi fields J_s and J_r then fulfill

$$J_s(0) = V, \qquad J_s(1) = 0$$
$$J_r(0) = W, \qquad J_r(1) = 0.$$

Setting $X(t) := \nabla_s \nabla_r c(t,0,0)$, we obtain the same differential equation for X as above, and $X(0) = d^2 \exp_p(0)(W,V) = 0$ and $X(1) = 0$. Thus, we obtain

$$\|\ddot{X}\|_\infty \le C(|R|, |\nabla R|) d(p,q)\ |V|\ |W|,$$

and hence

$$\|\dot{X}(0)\| \le C(|R|, |\nabla R|) d(p,q)\ |V|\ |W|.$$

We can calculate

$$\begin{aligned} d_2 d \log_p q(W,V) &= \nabla_s \nabla_r \nabla_t c(0,0,0) \\ &= \dot{X}(0) + R(J_s(0), \dot{\gamma}(0)) J_r(0) \\ &= \dot{X}(0) + R(V, \dot{\gamma}(0)) W, \end{aligned}$$

which implies (A.4). □

Appendix B
A Linear System of Elliptic Equations

Let M be a smooth manifold, $\Omega \subset \mathbb{R}^d$ a domain with smooth boundary. In the context of Section 5.2, we consider problems essentially of the form

$$W \in W_0^{1,2}(\Omega, u^{-1}TM) : \int_\Omega \langle \nabla W, \nabla V \rangle \, dx = -\int_\Omega \langle V, F \rangle \, dx \quad \forall V \in W_0^{1,2}(\Omega, u^{-1}TM), \tag{B.1}$$

where $F \in L^2(\Omega, u^{-1}TM)$, and $u \in W_K^{1,q}(\Omega, M)$ with $q > \max\{2, d\}$.

This system is closely related to the concept of linear systems of elliptic equations in divergence form. In particular, one can use coordinates to write the covariant derivatives in terms of ordinary derivatives and Christoffel symbols and obtain a linear system of elliptic equations of the form

$$\int_\Omega a_{ij}^{\alpha\beta} \, \partial_\alpha W^i \, \partial_\beta V^i + \tilde{b}_{ji}^{\beta} \, \partial_\beta V^j \, W^i + \tilde{b}_{ij}^{\alpha} \, \partial_\alpha W^i \, V^j + b_{ij} \, W^i \, V^j + f_j V^j \, dx = 0$$

with

$$\begin{aligned} a_{ij}^{\alpha\beta} &= \delta^{\alpha\beta} g_{ij} \\ \tilde{b}_{ij}^{\alpha} &= \delta^{\alpha\beta} g_{ik} \Gamma_{mj}^k d_\alpha u^m \\ b_{ij} &= \delta^{\alpha\beta} g_{kl} \Gamma_{im}^k \Gamma_{jn}^l d_\alpha u^m d_\beta u^n \\ f_j &= g_{kj} F^k, \end{aligned}$$

where greek indices denote coordinates on $\Omega \subset \mathbb{R}^d$, latin indices denote coordinates on M, and we sum over repeated indices. We can check that if $U \in L^2(\Omega, u^{-1}TM)$, and $u \in W_K^{1,q}$ with $q > \max\{2, d\}$, we have for the coefficients

$$\begin{aligned} \tilde{b}_{ij}^{\alpha} &\in L^q \\ b_{ij} &\in L^{\frac{q}{2}}, \end{aligned}$$

$$f_j \in L^2.$$

This means that one can apply standard theory for linear systems of elliptic equations (see, e.g., [LU68]) to obtain an estimate of the form

$$\|\hat{W}\|_{W^{2,2}(\Omega,\mathbb{R}^n)} \leq C\|F\|_{L^2},$$

where $\hat{W}$ is the coordinate vector for W. If $u \in W^{2,b}(\Omega, M)$, with b as in Lemma 3.14, we can then estimate the covariant norm by the coordinate one and obtain

$$\|W\|_{W^{2,2}(\Omega,u^{-1}TM)} \leq C\|F\|_{L^2}.$$

While this procedure is a straightforward application of standard theory it does not fit with the general spirit of this work to use intrinsic concepts. Thus we will include a direct proof for interior H^2-regularity for problems of the type (B.1).
We will follow the proof in [Eva98] for linear elliptic equations.

Theorem B.1. *Let $u \in W_K^{1,q}(\Omega, M)$ with $q \geq 2\max\{2,d\}$. Let $F \in L^2(\Omega, u^{-1}TM)$, and let $W \in W^{1,2}(\Omega, u^{-1}TM)$ solve* (B.1). *Then for each subset $\Omega' \Subset \Omega$ we have*

$$\|W\|_{W^{2,2}(\Omega',u^{-1}TM)} \leq C_{31}\left(\|F\|_{L^2(\Omega,u^{-1}TM)} + \|W\|_{L^2(\Omega,u^{-1}TM)}\right),$$

where C_{31} depends on M, Ω', Ω and K.

Proof. First note that by setting $V = W$ in (B.1), we can directly estimate

$$\|W\|^2_{W^{1,2}(\Omega,u^{-1}TM)} \leq \|F\|_{L^2(\Omega,u^{-1}TM)}\|W\|_{L^2(\Omega,u^{-1}TM)},$$

and thus

$$\|W\|_{W^{1,2}(\Omega',u^{-1}TM)} \leq \frac{1}{2}\left(\|F\|_{L^2(\Omega,u^{-1}TM)} + \|W\|_{L^2(\Omega,u^{-1}TM)}\right). \tag{B.2}$$

To obtain estimates for the second covariant derivatives, we introduce the covariant difference quotient for a vector field V along u

$$D^\varepsilon_\alpha V(x) = \frac{1}{\varepsilon}\left(\pi^u_{u(x+\varepsilon e_\alpha)\mapsto u(x)} V(x+\varepsilon e_\alpha) - V(x)\right),$$

where $\pi^u_{u(x)\mapsto u(y)}$ denotes the parallel transport along the curve $t \mapsto u(x+t(y-x))$, and x and ε are such that $x, x+\varepsilon e_\alpha \in \Omega$. The expression D^ε_α behaves like a difference quotient in the sense that

$$\lim_{\varepsilon\to 0} D^\varepsilon_\alpha V(x) = \nabla_\alpha V(x),$$

if the limit exists.

We can estimate for $V \in W^{1,2}(\Omega, u^{-1}TM)$

$$\sum_{\alpha=1}^{d} \int_{\Omega'} |D_\alpha^\varepsilon V|^2 \, dx \le C \|\nabla V\|^2_{L^2(\Omega, u^{-1}TM)} \tag{B.3}$$

for $\Omega' \Subset \Omega$, and $\varepsilon < \frac{1}{2}\operatorname{dist}(\Omega, \Omega')$.

Let $\Omega' \Subset \Omega'' \Subset \Omega$, and let ξ be a smooth cut-off function with

$$\xi = \begin{cases} 1 & \text{on } \Omega' \\ 0 & \text{on } \mathbb{R}^d \backslash \Omega'' \end{cases}, \qquad 0 \le \xi \le 1.$$

We set

$$V(x) = -D_\alpha^{-\varepsilon}(\xi^2 D_\alpha^\varepsilon W(x)).$$

Then $V \in W_0^{1,2}(\Omega, u^{-1}TM)$. Inserting V into (B.1), we obtain

$$-\sum_{\beta=1}^{d} \int_\Omega \langle \nabla_\beta W(x), \nabla_\beta D_\alpha^{-\varepsilon}(\xi^2 D_\alpha^\varepsilon W(x))\rangle \, dx = \int_\Omega \langle D_\alpha^{-\varepsilon}(\xi^2 D_\alpha^\varepsilon W(x)), F(x)\rangle \, dx. \tag{B.4}$$

We define an approximate Riemannian curvature tensor by

$$\mathrm{Rm}^\varepsilon(d^\alpha u, d^\beta u, X, Y) = \langle R^\varepsilon(d^\alpha u, d^\beta u)X, Y\rangle = \langle D_\alpha^\varepsilon \nabla_\beta X, Y\rangle - \langle \nabla_\beta D_\alpha^\varepsilon X, Y\rangle$$

and remark that

$$\int_\Omega \langle D_\alpha^\varepsilon X, Y\rangle \, dx = -\int_\Omega \langle X, D_\alpha^{-\varepsilon} Y\rangle \, dx$$

if either X or Y has compact support.

We can rewrite (B.4) to

$$\begin{aligned}
&-\sum_{\beta=1}^{d} \int_\Omega \left\langle \nabla_\beta W(x), \nabla_\beta D_\alpha^{-\varepsilon}(\xi^2 D_\alpha^\varepsilon W(x))\right\rangle \, dx \\
&= \sum_{\beta=1}^{d} \int_\Omega \left\langle \nabla_\beta D_\alpha^\varepsilon W(x), \xi^2 \nabla_\beta D_\alpha^\varepsilon W(x)\right\rangle \, dx + \sum_{\beta=1}^{d} \int_\Omega \left\langle D_\alpha^\varepsilon \nabla_\beta W(x), 2\xi \, d^\beta \xi \, D_\alpha^\varepsilon W(x)\right\rangle \, dx \\
&\quad + \sum_{\beta=1}^{d} \int_\Omega \mathrm{Rm}^{-\varepsilon}(d^\alpha u, d^\beta u, \xi^2 D_\alpha^\varepsilon W(x), \nabla_\beta W(x)) \, dx \\
&\quad + \sum_{\beta=1}^{d} \int_\Omega \mathrm{Rm}^\varepsilon(d^\alpha u, d^\beta u, W, \xi^2 \nabla_\beta D_\alpha^\varepsilon W) \, dx,
\end{aligned}$$

and thus obtain

$$\sum_{\beta=1}^{d}\int_{\Omega}\left\langle \nabla_\beta D_\alpha^\varepsilon W(x), \xi^2 \nabla_\beta D_\alpha^\varepsilon W(x)\right\rangle\, dx$$
$$= \int_\Omega \left\langle D_\alpha^{-\varepsilon}(\xi^2 D_\alpha^\varepsilon W(x)), F(x)\right\rangle\, dx - \sum_{\beta=1}^{d}\int_\Omega \left\langle \nabla_\beta D_\alpha^\varepsilon W(x), 2\xi\; d^\beta \xi\; D_\alpha^\varepsilon W(x)\right\rangle\, dx$$
$$- \sum_{\beta=1}^{d}\int_\Omega \mathrm{Rm}^{-\varepsilon}(d^\alpha u, d^\beta u, \xi^2 D_\alpha^\varepsilon W(x), \nabla_\beta W(x))\, dx$$
$$- \sum_{\beta=1}^{d}\int_\Omega \mathrm{Rm}^{\varepsilon}(d^\alpha u, d^\beta u, W, \xi^2 \nabla_\beta D_\alpha^\varepsilon W)\, dx$$
$$- \sum_{\beta=1}^{d}\int_\Omega \mathrm{Rm}^{\varepsilon}(d^\alpha u, d^\beta u, W, 2\xi\; d^\beta \xi\; D_\alpha^\varepsilon W(x))\, dx.$$

Note that

$$D_\alpha^{-\varepsilon}(\xi^2 Y(x)) = \xi^2(x) D_\alpha^{-\varepsilon} Y(x) + (\xi(x-\varepsilon e_\alpha) + \xi(x)) D_\alpha^{-\varepsilon}\xi(x)\; \pi^u_{u(x-\varepsilon e_\alpha)\mapsto u(x)} Y(x).$$

Hence, we can estimate

$$\sum_{\beta=1}^{d}\int_\Omega \xi^2 \|\nabla_\beta D_\alpha^\varepsilon W(x)\|^2\, dx$$
$$\leq \int_\Omega \xi^2 \|D_\alpha^{-\varepsilon}(D_\alpha^\varepsilon W(x))\| \|F(x)\|\, dx + C\sum_{\beta=1}^{d}\int_\Omega \xi \|\nabla_\beta D_\alpha^\varepsilon W(x)\|\; \|D_\alpha^\varepsilon W(x)\|\, dx$$
$$+ \sum_{\beta=1}^{d}\int_\Omega \xi^2 \|\mathrm{Rm}^{-\varepsilon}\| \|d^\alpha u\|\; \|d^\beta u\|\; \|D_\alpha^\varepsilon W(x)\|\; \|\nabla_\beta W(x)\|\, dx$$
$$+ \sum_{\beta=1}^{d}\int_\Omega \xi^2 \|\mathrm{Rm}^{\varepsilon}\|\; \|d^\alpha u\|\; \|d^\beta u\|\; \|W\|\; \|\nabla_\beta D_\alpha^\varepsilon W\|\, dx$$
$$+ \sum_{\beta=1}^{d}\int_\Omega \xi \|\mathrm{Rm}^{\varepsilon}\| \|d^\alpha u\|\; \|d^\beta u\|\; \|W\|\; \|D_\alpha^\varepsilon W(x)\|\, dx.$$

Under the assumptions of the theorem, we can estimate use Hölder's and Young's inequalities to absorb the higher derivatives into the left hand side

$$\begin{aligned}
&\sum_{\beta=1}^{d} \int_{\Omega'} \|\nabla_\beta D_\alpha^\varepsilon W(x)\|^2 \, dx \\
&\qquad \le C \sum_{\beta=1}^{d} \left(\|F\|^2_{L^2(\Omega,u^{-1}TM)} + \|\nabla W\|^2_{L^2(\Omega,u^{-1}TM)} + \|W\|^2_{L^{\frac{2q}{q-4}}(\Omega,u^{-1}TM)} \right) \\
&\qquad \le C \sum_{\beta=1}^{d} \left(\|F\|^2_{L^2(\Omega,u^{-1}TM)} + \|W\|^2_{W^{1,2}(\Omega,u^{-1}TM)} \right).
\end{aligned}$$

Summing over α and using the $W^{1,2}$-estimate (B.2) for W yields the assertion. □

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