CHAPTER 1

Quaternionic Analysis

In this chapter we present some basic concepts of the algebraic structure properties of complex quaternions. We also recall some important results on Banach spaces as well as Sobolev spaces which arise in studying the boundary value problems for the Helmholtz equation in the next chapters. Finally, we introduce the quaternionic Stokes formulas and the Cauchy - Pompeiu integral representation formulas of first order which are strong powerful tools in the strategy of this work.

1. Algebra of complex quaternions

The most natural and close generalization of complex analysis is quaternionic analysis. It is known that many problems from different engineering areas can be formulated as quaternion optimization problems. Thus, by using quaternions we have an elegant mathematical method to solve many complicated problems in different areas of engineering. We refer to [**47, 48, 59, 60**] for wider applications of quaternionic analysis. On the other hand, it is seen that an overwhelming majority of physically meaningful problems can not be reduced to two - dimensional models. Therefore, it is necessary to develop an analogous theory to complex analysis for higher dimensions. We begin with the definition of the algebra of real quaternions (see [**46, 47**], [**60**, Chap. 1]).

Let $\{e_0, e_1, e_2, e_3\}$ be an orthonormal basis of $\mathbb{R}^4$ such that $x \in \mathbb{R}^4$ is represented as

$$x = \sum_{k=0}^{3} x_k e_k, \qquad x_k \in \mathbb{R}, \quad 0 \leq k \leq 3. \tag{1.1}$$

The part $x_0 e_0 =: \mathrm{Sc}(x)$ is called the scalar part of x and $\vec{x} = \sum\limits_{k=1}^{3} x_k e_k =: \mathrm{Vec}(x)$ the vector part of x.

For $x,\ y \in \mathbb{R}^4$, it is said $x = y$ if and only if they have exactly the same components i.e., $x = y$ iff $x_k = y_k, \quad k = 0, 1, 2, 3$. The sum $x+y$ is defined by adding the corresponding components

$$x + y = \sum_{k=0}^{3} (x_k + y_k) e_k.$$

And a product is defined in $\mathbb{R}^4$ which satisfies the conditions

(i) $e_1^2 = e_2^2 = e_3^2 = -1$,

(ii) $e_1 e_2 = -e_2 e_1 = e_3;\ e_2 e_3 = -e_3 e_2 = e_1;\ e_3 e_1 = -e_1 e_3 = e_2$.

The element e_0 is regarded as the usual unit, that is, $e_0 = 1$. For $x, y \in \mathbb{R}^4$ we define

$$< \vec{x}, \vec{y} > \ = \ x_1 y_1 + x_2 y_2 + x_3 y_3,$$

$$[\vec{x} \times \vec{y}] \quad = \quad \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

Then, the algebraic rules (i), (ii) yield the quaternionic product xy

$$xy = x_0y_0 - < \vec{x}, \vec{y} > + x_0\vec{y} + \vec{x}y_0 + [\vec{x} \times \vec{y}]. \tag{1.2}$$

We are now prepared to give the definition of the algebra of real quaternions.

DEFINITION 1.1. The tuple $(\mathbb{R}^4, \cdot)$ is called *the algebra of real quaternions.* We signify $(\mathbb{R}^4, \cdot)$ by $\mathbb{H}(\mathbb{R})$.

The quaternion $\bar{x} = x_0 - \vec{x}$ is called the conjugate to x. The number $|x|$ defined by

$$|x|^2 := x\bar{x} \tag{1.3}$$

is named the absolute value of x.

REMARK 1.2.

1. xy is an $\mathbb{R}-$bilinear and associative product, but it is *not commutative* due to equation (1.2).
2.

$$\begin{array}{rcl} \overline{x+y} & = & \bar{x} + \bar{y} \\ \overline{\mu x} & = & \mu\bar{x} \quad \text{for all } \mu \in \mathbb{R} \\ \bar{\bar{x}} & = & x \\ \overline{xy} & = & \bar{y}\bar{x} \\ \vec{x}^2 & = & -|\vec{x}|^2. \end{array}$$

If now in the definition of a quaternion (see (1.1)) we suppose that all components can be complex (instead of real) numbers we arrive at the definition of complex quaternions (biquaternions).

DEFINITION 1.3. *A complex quaternion (biquaternions)* x is an object of the form

$$x = \sum_{k=0}^{3} x_k e_k, \qquad x_k \in \mathbb{C}, \quad 0 \leq k \leq 3$$

with the commutation rule for the usual complex imaginary unit i with the quaternionic imaginary unit e_k, $k = 1, 2, 3$, $ie_k = e_k i$. The algebra of complex quaternions will be denoted by $\mathbb{H}(\mathbb{C})$.

Note that any $x \in \mathbb{H}(\mathbb{C})$ can be represented as $x = \mathrm{Re}x + i\mathrm{Im}x$, where $\mathrm{Re}x = \sum\limits_{k=0}^{3} \mathrm{Re}x_k e_k$ and $\mathrm{Im}x = \sum\limits_{k=0}^{3} \mathrm{Im}x_k e_k$ belong to $\mathbb{H}(\mathbb{R})$. Then the conjugate to x also belong to $\mathbb{H}(\mathbb{C})$. It can be written as $\bar{x} = \overline{\mathrm{Re}x} + i\overline{\mathrm{Im}x}$.

We now consider the complex quaternion $x = 1 + ie_1$ and its conjugate $\bar{x} = 1 - ie_1$. Their product is

$$x\bar{x} = (1 + ie_1)(1 - ie_1) = 1 - 1 = 0 \tag{1.4}$$

This means that the algebra of complex quaternions contains zero divisors. For more information on the characterizations of the set of all zerodivisors as well as its structure we refer the readers to [**60**, Lemma 1],(see also [**47, 64**]).

As we can observe already, the modulus introduced by the equality (1.3) in the case of the complex quaternions in general does not give information about the absolute values of their components (see (1.4)). This is why another kind of modulus is used frequently. The norm $||x||_{\mathbb{H}(\mathbb{C})}$, where $x \in \mathbb{H}(\mathbb{C})$, is defined by

$$||x||_{\mathbb{H}(\mathbb{C})} := \sqrt{|x_0|^2 + |x_1|^2 + |x_2|^2 + |x_3|^2}, \tag{1.5}$$

where $x_k \in \mathbb{C}$, $|x_k|^2 = x_k \bar{x}_k$, $\bar{x}_k$ stands for the usual complex conjugation. It is easily seen that (1.5) represents a natural Euclidean metric in $\mathbb{R}^8$ and can be expressed as $||x||^2_{\mathbb{H}(\mathbb{C})} = |\mathrm{Re} x|^2 + |\mathrm{Im} x|^2$.

Looking at the following example $x = y = 1 + ie_1$. Then

$$||xy||_{\mathbb{H}(\mathbb{C})} = 2||1 + ie_1||_{\mathbb{H}(\mathbb{C})} = 2\sqrt{2},$$

but $||x||_{\mathbb{H}(\mathbb{C})}||y||_{\mathbb{H}(\mathbb{C})} = 2$. It means that, in general, $||xy||_{\mathbb{H}(\mathbb{C})} \neq ||x||_{\mathbb{H}(\mathbb{C})}||y||_{\mathbb{H}(\mathbb{C})}$ and even $||xy||_{\mathbb{H}(\mathbb{C})}$ can be greater than the product $||x||_{\mathbb{H}(\mathbb{C})}||y||_{\mathbb{H}(\mathbb{C})}$.

LEMMA 1.4. *Let x and y be complex quaternions. Then* $||xy||_{\mathbb{H}(\mathbb{C})} \leq \sqrt{2}||x||_{\mathbb{H}(\mathbb{C})}||y||_{\mathbb{H}(\mathbb{C})}$, *see* [**60**, Chap. 1, Lemma 2].

We now introduce the *isomorphic embedding* in $\mathbb{H}(\mathbb{C})$. An arbitrary element $x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ corresponds to the quaternion $x = \sum_{k=0}^{3} x_k e_k$. In this way $\mathbb{R}$ and $\mathbb{R}^3$ are embedded in $\mathbb{H}(\mathbb{R})$ so that we can say they are embedded in $\mathbb{H}(\mathbb{C})$. The corresponding embedding mappings are defined by

$$\begin{array}{rcl} \cdot : \mathbb{R} & \to & \mathbb{H}(\mathbb{C}) \\ x_0 & \mapsto & x_0 e_0 \end{array}$$

and

$$\begin{array}{rcl} \cdot : \mathbb{R}^3 & \to & \mathbb{H}(\mathbb{C}) \\ (x_1, x_2, x_3) & \mapsto & \displaystyle\sum_{k=1}^{3} x_k e_k \end{array}$$

respectively.

2. The Moisil - Teodorescu differential operator

Our purpose is treating the boundary value problems of the Helmholtz equation. To that purpose throughout this thesis Ω assume is to be an open bounded domain in $\mathbb{R}^3$ with a sufficiently smooth Liapunov surface.

We recall that a closed bounded surface Γ is called a Liapunov surface if in each point $x \in \Gamma$ there exists a normal $\vec{n}(x)$ satisfying a Hölder condition on Γ, i.e., there exists numbers $C > 0$ and $0 < \varepsilon < 1$ such that

$$|\vec{n}(x) - \vec{n}(y)| \leq C|x - y|^{\varepsilon} \text{ for arbitrary } x, y \in \Gamma.$$

We refer the readers to [**88**] for more information on Liapunov surfaces.

2.1. Spaces of complex quaternions valued functions. By the *isomorphic embedding* we can identify $(x_1, x_2, x_3) = \vec{x} \in \mathbb{R}^3$ with $x = \sum\limits_{k=1}^{3} x_k e_k \in \mathbb{H}(\mathbb{R}) \subset \mathbb{H}(\mathbb{C})$.
Now we consider functions f defined in Ω with values in $\mathbb{H}(\mathbb{C})$. Those functions may be written as

$$f(x) = \sum_{k=0}^{3} f_k(x) e_k, \quad f_k(x) \in \mathbb{C},\ x \in \Omega. \tag{1.6}$$

Properties such as continuity, differentiability, integralbility, and so on, which are described to f have to be possessed by all components $f_k(x)$ which are complex-valued functions defined on Ω.

Let $\mathcal{B}(\Omega)$ be a function space of complex functions defined on Ω. For example, $\mathcal{B}$ may be $C^k,\ C^{(k,\varepsilon)},\ L_p,\ W_p^k$ and so on. We then define a function space

$$\mathcal{B}(\Omega, \mathbb{H}(\mathbb{C})) := \{f : \Omega \to \mathbb{H}(\mathbb{C}) :\ \text{all the component of } f \text{ belong to } B(\Omega)\}.$$

If $\mathcal{B}(\Omega)$ is normed with norm $||\cdot||_{\mathcal{B}}$ then we can define a norm on $\mathcal{B}(\Omega, \mathbb{H}(\mathbb{C}))$ by

$$||f||_{\mathcal{B}} = \big(\sum_{k=0}^{3} ||f_k||^2_{\mathcal{B}}\big)^{\frac{1}{2}} \text{ for } f \in \mathcal{B}(\Omega, \mathbb{H}(\mathbb{C})).$$

If $\mathcal{B}(\Omega)$ is a Banach space then the space $\mathcal{B}(\Omega, \mathbb{H}(\mathbb{C}))$ defined in this manner is also a complex Banach space. In this way the usual Banach spaces of these functions are denoted by $C^k(\Omega, \mathbb{H}(\mathbb{C})),\ C^{(k,\varepsilon)}(\Omega, \mathbb{H}(\mathbb{C})),\ L_p(\Omega, \mathbb{H}(\mathbb{C})), W_p^k(\Omega, \mathbb{H}(\mathbb{C}))$ and so on.

$C^k(\Omega, \mathbb{H}(\mathbb{C}))$ is the space of k times continuously differentiable function in Ω.

$C^{(k,\varepsilon)}(\Omega, \mathbb{H}(\mathbb{C}))$ is the space of k times continuously differentiable function, whose $k\!-\!th$ derivative is Hölder continuous with the exponent ε.

$C_c^\infty(\Omega, \mathbb{H}(\mathbb{C}))$ is defined as the vector space consisting of functions from Ω to $\mathbb{H}(\mathbb{C})$ with compact support which have continuous derivatives of all orders.

$L_p(\Omega, \mathbb{H}(\mathbb{C}))$ is the space of all functions, whose $p - th$ power is Lebesgue intergrable in Ω. Notice that $L_p(\Omega, \mathbb{H}(\mathbb{C}))$ is an $\mathbb{H}(\mathbb{C})$-bimodul. Moreover, it can be proved to be a Banach bimodul. We refer to [**72, 77, 91**] for more informations about right (left)-$\mathbb{H}(\mathbb{C})-$module as well as bi - $\mathbb{H}(\mathbb{C})-$module of $\mathbb{H}(\mathbb{C})-$valued continuous (differentiable, integrable) functions.

Let us consider $L_2(\Omega, \mathbb{H}(\mathbb{C}))$ as a right module. Then, it can be easilyverified that the formula

$$(f, g) := \int_{\Omega} \overline{f(x)} g(x) dx,$$

defines a scalar product turning $L_2(\Omega, \mathbb{H}(\mathbb{C}))$ into a right Hilbert $\mathbb{H}(\mathbb{C})$-module.

The following spaces are just the $\mathbb{H}(\mathbb{C})$-valued analogue to Sobolev spaces. Let k be a non - negative integer. Then $W_p^k(\Omega, \mathbb{H}(\mathbb{C}))$ is the space of $k-$times differentiable functions in Sobolev's sense, whose $k - th$ order derivatives belong to $L_p(\Omega, \mathbb{H}(\mathbb{C}))$.

$\overset{0}{W}{}_p^k(\Omega, \mathbb{H}(\mathbb{C}))$ is the space of $k-$times differentiable functions in Sobolev's sense, whose $k - th$ order derivatives belong to $L_p(\Omega, \mathbb{H}(\mathbb{C}))$ and vanish on the boundary Γ.

$W_p^{k,loc}(\Omega, \mathbb{H}(\mathbb{C})) := \{f, f \in W_p^k(\Omega, \mathbb{H}(\mathbb{C}))$ for every compact $K \subset \Omega\}$. One can find comprehensive information on Banach spaces for instance in ([**1**], [**38**, Section II]).

Next, we will introduce some basic notations of the Sobolev spaces (see [**1**], [**76**, Section VI]) used in our discussions in Chapter 2.

DEFINITION 1.5. Let X, Y be normed vector spaces.

(i) A linear mapping $L : X \longrightarrow Y$ is called bounded if there is a constant M such that $||Lx|| \leq M||x||$, for every $x \in X$.
(ii) We say that an operator $L : X \longrightarrow Y$ is continuous at an $x \in X$ if, whenever x_n is a sequence such that $x_n \longrightarrow x$, we have $L(x_n) \longrightarrow L(x)$.

REMARK 1.6. Let $L : X \longrightarrow Y$ be a linear mapping. L is continuous if and only if it is bounded.

Moreover, let $\mathcal{L}(X, Y)$ denote the set of all bounded linear mappings from X to Y. We denote by $C_b(\bar{\Omega})$ the set of all bounded continuous functions on $\bar{\Omega}$ with the norm $||u|| = \sup\limits_{x\in\bar{\Omega}} ||u(x)||_{\mathbb{H}(\mathbb{C})}$. Obviously, $C_b(\bar{\Omega})$ is a Banach space.

DEFINITION 1.7. Let $\tilde{L}_p(\Omega, \mathbb{H}(\mathbb{C}))$ be the set of all continuous functions $f : \Omega \longrightarrow \mathbb{H}(\mathbb{C})$ for which

$$||f||_{L_p} := \left(\int\limits_{\Omega} |f(x)|^p_{\mathbb{H}(\mathbb{C})} dx \right)^{\frac{1}{p}}$$

is finite with $p \in (1, +\infty)$.

THEOREM 1.8. *(Hölder's inequality)*
Let $f \in \tilde{L}_p(\Omega, \mathbb{H}(\mathbb{C}))$, $g \in \tilde{L}_q(\Omega, \mathbb{H}(\mathbb{C}))$, *where* $1 < p, q < +\infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* $fg \in \tilde{L}_1(\Omega, \mathbb{H}(\mathbb{C}))$ *and* $||fg||_{L_1} \leq ||f||_{L_p}||g||_{L_q}$.

REMARK 1.9. Let k be a non - negative integer and let $1 \leq p \leq +\infty$. Since Sobolev's imbedding theorem (see [**1, 84**], [**76**, Chap. 6, 6.4]), more general imbedding theorems for W_p^k spaces can be established. It is shown that

$$W_p^k(\Omega, \mathbb{H}(\mathbb{C})) \subset L_{\frac{3p}{3-kp}}(\Omega, \mathbb{H}(\mathbb{C})), \qquad \text{for} \qquad kp < 3, \tag{1.7}$$

$$W_p^k(\Omega, \mathbb{H}(\mathbb{C})) \subset C_b(\Omega, \mathbb{H}(\mathbb{C})), \qquad \text{for} \qquad kp > 3. \tag{1.8}$$

2.2. The Moisil - Teodorescu differential operator.

DEFINITION 1.10. Let $f \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$. The *Moisil-Teodorescu differential operator* is given by

$$Df := \sum_{k=1}^{3} e_k \partial_k f \text{ where } \partial_k := \frac{\partial}{\partial x_k}.$$

If we write $f = f_0 + \vec{f}$ then we get by a straightforward calculation

$$Df = -\text{div}\vec{f} + \text{grad} f_0 + \text{rot}\vec{f}. \tag{1.9}$$

The equation $Df = 0$ is equivalent to the *Moisil-Teodoresco system*

$$\begin{cases} \mathrm{div}\vec{f} &= 0, \\ \mathrm{grad}\ f_0 + \mathrm{rot}\vec{f} &= 0. \end{cases} \tag{1.10}$$

For detailed considerations of this system we refer the readers to [**25, 36, 46, 64, 75**].

Let us note that the *Moisil-Teodorescu operator* was introduced as acting from the left-hand side. The correponding operator acting from the right-hand side will be denoted by D_r

$$D_r f = \sum_{k=1}^{3} \partial_k f e_k$$

and in vertor form the application of D_r can be represented as

$$D_r f = -\mathrm{div}\vec{f} + \mathrm{grad} f_0 - \mathrm{rot}\vec{f}. \tag{1.11}$$

In [**40, 91**] a function f is called *left(right)-monogenic* in Ω if it fulfills $Df = 0,\ (D_r f = 0)$ in Ω. Note that $D^2 = D_r^2 = -\sum_{k=1}^{3} \partial_k^2 = -\Delta$. This property guarantees that each *left (right)-monogenic* is a harmonic function. An important example for *left (right)-monogenic* function is the so-called *generalized Cauchy kernel* given in [**23**]. For the detailed discussion on monogenic as well as harmonic functions we refer to [**5, 23, 40, 91**].

In complex analysis it is well-known that the product of two holomorphic functions is again a holomorphic function. However, this is not true in the quaternionic case. We can see this by the following example.

Example 1.11. Let be $f = x_1e_2 + x_2e_1$ then $Df = e_1e_2 + e_2e_1 = 0$, and $g = x_1e_1 - x_2e_2$ then $Dg = e_1^2 - e_2^2 = 0$. However

$$fg = (x_1e_2 + x_2e_1)(x_1e_1 - x_2e_2) = -(x_1^2 + x_2^2)e_3$$

and

$$D(fg) = -2(x_1e_1e_3 + x_2e_2e_3) = 2(x_1e_2 - x_2e_1).$$

The above equalities can be verified by using the following generalization of Leibniz's rule.

Theorem 1.12. *(Generalization of Leibniz's rule) Let $f, g \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$. Then*

$$D(fg) = (Df)g + \bar{f}(Dg) - \sum_{k=1}^{3} f_k \partial_k g.$$

Its proof can be done by direct calculation (see [**46, 60**]).

Remark 1.13. If $Vec(f) = 0$ then $D[f_0 g] = D[f_0]g + f_0 D[g]$.

To conclude this section, we briefly recall the definition of fractional order Sobolev spaces which can be found in [**1**, Chap. VII], [**76**, Section. VI]. First of all, we introduce weighted L_2- spaces.

Let Ω be a bounded smooth domain in $\mathbb{R}^3$ and $s \in \mathbb{R}^+$,

$$L_2^s(\Omega, \mathbb{H}(\mathbb{C})) := \left\{ f | (1 + |x|^2)^{\frac{s}{2}} f \in L_2(\Omega, \mathbb{H}(\mathbb{C})) \right\}.$$

These spaces are Hilbert spaces with the scalar product

$$< f, g >_{L_2^s} = \int\limits_{\Omega} (1 + |x|^2)^s \overline{f(x)} g(x) dx,$$

and the norm

$$||f||_{L_2^s} = \int\limits_{\Omega} (1 + |x|^2)^s \overline{f(x)} f(x) dx.$$

If $s = 0$ we have $||f||_{L_2^s} = ||f||_{L_2}$.

Further, in the next chapters we will use the weighted Sobolev spaces $W_2^s(\Omega, \mathbb{H}(\mathbb{C}))$ which are defined as

$$W_2^s(\Omega, \mathbb{H}(\mathbb{C})) = \left\{ f | (1 + |x|^2)^{\frac{s}{2}} f \in L_2(\Omega, \mathbb{H}(\mathbb{C})),\ (1 + |x|^2)^{\frac{s+k}{2}} D^k f \in L_2(\Omega, \mathbb{H}(\mathbb{C})) \right\},$$

where $k < [s]$ with $[s]$ denoting the largest integer less than or equal s and $D^k f$ is defined as its $k - th$ order derivatives in the distributional sense.

Analogously to the above definition about $W_2^s(\Omega, \mathbb{H}(\mathbb{C}))$ we have well-defined spaces $W_p^s(\Omega, \mathbb{H}(\mathbb{C}))$ for arbitrary values of s and $1 < p < \infty$ whose norm is written as

$$||f||_{W_p^s} = \left\{ ||f||_{W_p^{[s]}} + \sum_{|\alpha| = [s]} \int\limits_{\Omega} \int\limits_{\Omega} \frac{||D^\alpha f(x) - D^\alpha f(y)||^p}{||x - y||^{3 + p(s - [s])}} dx dy \right\}^{\frac{1}{p}}.$$

We now address the question if and in which sense functions in Sobolev spaces can be restricted to the boundary of the domain. Its answer is called trace theorem.

THEOREM 1.14. *Let $k \in \mathbb{N},\ k \neq 0$. Then there exists a continuous linear map*

$$tr_\Gamma T : \quad W_p^k(\Omega, \mathbb{H}(\mathbb{C})) \longrightarrow W_p^{k-1/p}(\Gamma, \mathbb{H}(\mathbb{C})),$$

called the trace operator.

A natural question is now which elements of $W_p^{k-1/p}(\Gamma, \mathbb{H}(\mathbb{C}))$ can be obtained as restrictions, or "traces" of functions in $W_p^k(\Omega, \mathbb{H}(\mathbb{C}))$. The answer is that all elements of $W_p^{k-1/p}(\Gamma, \mathbb{H}(\mathbb{C}))$ are obtained in this way. This means that there exists a linear mapping

$$T : \quad W_p^{k-1/p}(\Gamma, \mathbb{H}(\mathbb{C})) \longrightarrow W_p^k(\Omega, \mathbb{H}(\mathbb{C}))$$

such that $T tr_\Gamma T$ is the identity. We also refer to [**68**, Chap. 1, Section 11, 12] (see also [**69, 84**]) for complete information about $W_p^s(\Omega)$ with its norm and the trace theorem.

3. The Cauchy-Pompeiu integral representation

In the theory of complex variables formular the Cauchy-Pompeiu integral representation can be written as

$$w(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{w(\zeta)}{\zeta - z} d\zeta - \frac{1}{\pi} \int_{\Omega} \frac{f(\xi + i\eta)}{\xi + i\eta - z} d\xi d\eta$$

where $z = x + iy,\ \zeta = \xi + i\eta$. It represents the solution to the inhomogeneous Cauchy-Riemann equation, $\partial_{\bar{z}} w = f$ where $\partial_{\bar{z}} w = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. The area integral operator and the singular integral appearing leads to the generalized functions of a complex variable (see [**86**]). The results of this theory of functions of one complex variable are providing a wider and deeper impression on the investigation of some classes of differential equations, embracing many important equations of mathematical physics.

Moreover, the theory of the *holomorphic functions* of one complex variable is closely connected with the theory of *harmonic functions* which are the solution of the two-dimensional Laplace equation. This means that so any essential progress in each of these theories results in progress in the other. However, there is no such connection for $n-$dimentional complex analysis with $n > 1$. The paradoxical differences existing between the cases $n = 1$ and $n > 1$ can be explained. These paradoxies disappear if we consider quaternionic analysis (see [**77, 78, 79**]), more general Clifford analysis see [**75**].

On the other hand, the Helmholtz operator $(\Delta + \alpha^2)$ can be considered as the most simple and sufficiently natural generalization of the Laplace operator. The operator $(\Delta + \alpha^2)$, $\alpha \in \mathbb{C}$, plays an important role and it offen arises in applications of physics. That is why it seems quite natural to build an analogous theory to the theory of *harmonic functions* which is called theory of *metaharmonic functions.*

The theory of *metaharmonic functions* was investigated in [**41, 46**] for $\alpha \in \mathbb{R}_+$. Some of them were developed for Clifford-valued function, which can be found in [**91, 92, 93**]. Corresponding results have been done for $\alpha \in \mathbb{R}_-$ in [**54, 55, 75**]. A class of *hyperholomophic* quaternion-valued functions with *metaharmonic functions* and theorems about the connection between them are introduced in [**60, 65**] for complex α. All of these results can be obtained also for complex quaternionic parameter α, see [**60, 61, 65, 66**]. In this section we only briefly recall the results about the analogue of the Cauchy- Pompeiu formula in the case of complex α which represents the solution to the inhomogeneous equation $(D + \alpha)f = g$ in Ω. We also introduce the quaternionic Stokes formulas which is one of the main tools in latter discussions. To do this, we begin with the definition of the $D_\alpha-$ operator.

3.1. Factorization of the Helmholtz operator and fundamental solutions. As the Laplacial also the Helmholtz operator can be factorized in quaternionic analysis as

$$\Delta + \alpha^2 = -(D + \alpha)(D - \alpha). \tag{1.12}$$

Definition 1.15. Let the operator $D_\alpha = D + \alpha I$ be given where α is an arbitrary complex constant and I is the identity operator.

Hence, the equality (1.12) can be rewritten as

$$\Delta + \alpha^2 = -D_\alpha D_{-\alpha} = -D_{-\alpha} D_\alpha.$$

This means that any function satisfying the equation $D_\alpha f = 0$ or $D_{-\alpha} f = 0$ also satisfies the Helmholtz equation $(\Delta + \alpha^2)f = 0$.

We define the set of *poly-left(right)* $\alpha-$*hyperholomorphic functions* as

$$\mathcal{M}^k_p(\Omega, \mathbb{H}(\mathbb{C})) := \{f|\ f \in L^k_p(\Omega, \mathbb{H}(\mathbb{C})),\ D^k_\alpha f = 0\},$$

$$\mathcal{M}^k_{r,p}(\Omega, \mathbb{H}(\mathbb{C})) := \{f|\ f \in L^k_p(\Omega, \mathbb{H}(\mathbb{C})),\ D^k_{r,\alpha} f = 0\}.$$

In the case $p = 2,\ k = 1$ we write $\ker D_\alpha = \mathcal{M}^1_2(\Omega, \mathbb{H}(\mathbb{C}))$. Each element f belongs to $\ker D_\alpha$ is called *left* $\alpha-$*hyperholomorphic function.* Obviously, $\ker D_{r,\alpha}$ is the set of *right* $\alpha-$*hyperholomorphic functions.*

By the definiton of the operators D_α and $D_{r,\alpha}$ as well as Remark 1.2 we have

$$\begin{aligned}
\overline{D_\alpha f(x)} &= \overline{(\sum_{k=1}^{3} e_k \frac{\partial}{\partial x_k} + \alpha)(\sum_{j=0}^{3} f_j(x) e_j)} \\
&= \overline{\sum_{k=1}^{3} e_k \frac{\partial}{\partial x_k}(\sum_{j=0}^{3} f_j(x) e_j)} + \overline{\alpha(\sum_{j=0}^{3} f_j(x) e_j)} \\
&= \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\overline{(\sum_{j=0}^{3} f_j(x) e_j)}\,\overline{e_k} + \alpha \overline{f(x)} \\
&= -\sum_{k=1}^{3} \frac{\partial}{\partial x_k}\overline{f(x)} e_k + \alpha \overline{f(x)} \\
&= -D_r \overline{f(x)} + \alpha \overline{f(x)}.
\end{aligned}$$

It means that

$$\overline{D_\alpha f(x)} = -D_{r,-\alpha}\overline{f(x)}, \text{ for any } \alpha \in \mathbb{C}. \tag{1.13}$$

Now, using the equality (1.12) and the fundamental solution of the Helmholtz equation, a fundamental solution can be constructed for the factors of the Helmholtz operator. Indeed, if we assume ϑ is a fundamental solution of the Helmholtz operator i.e a function satisfying $(\Delta + \alpha^2)\vartheta(x) = \delta(x)$, where $\delta(x)$ is Dirac delta distribution, then $K_\alpha(x) = -(D - \alpha)\vartheta(x)$ is a fundamental solution of D_α and $K_{-\alpha}(x) = -(D + \alpha)\vartheta(x)$ is a fundamental solution of $D_{-\alpha}$, i.e., $D_\alpha K_\alpha(x) = \delta(x)$, and $D_{-\alpha}K_{-\alpha}(x) = \delta(x)$.
As discussed in [**60**, p. 27] a unique fundamental solution to the Helmholtz operator related to its physical meaning is

$$\vartheta(x) = -\frac{e^{i\alpha|x|}}{4\pi|x|}.$$

Since $\vartheta(x)$ is a scalar function and using formulas (1.9), (1.11) we have $D_\alpha\vartheta(x) = D_{r,\alpha}\vartheta(x)$.

From formula (1.9) by a straightforward computation we get

$$K_\alpha(x) = -\mathrm{grad}\vartheta(x) + \alpha\vartheta(x) = (\alpha + \frac{x}{|x|^2} - i\alpha\frac{x}{|x|})\ (-\frac{e^{i\alpha|x|}}{4\pi|x|}) \tag{1.14}$$

$$K_{-\alpha}(x) = -\mathrm{grad}\vartheta(x) - \alpha\vartheta(x) = (-\alpha + \frac{x}{|x|^2} - i\alpha\frac{x}{|x|})\ (-\frac{e^{i\alpha|x|}}{4\pi|x|}) \tag{1.15}$$

These functions play a crucial role in many biological, chemical and physical systems, see [**60, 61, 64**]. Note that besides the equation $D_\alpha K_\alpha(x) = \delta(x)$, $K_\alpha(x)$ fulfils the radiation condition at infinity uniformly in all directions

$$(1 + \frac{ix}{|x|})K_\alpha(x) = 0(\frac{1}{|x|})$$

when $|x| \to \infty$ which is in agreement with the Silver- Müller radiation conditions in [**58**].

By direct caculation using formulas (1.9), (1.11) and Remark 1.13 again we obtain the following properties of $K_\alpha(x),$ and $K_{-\alpha}(x)$

$$\begin{cases} D_\alpha K_\alpha(x) & = D_{r,\alpha} K_\alpha(x), \\ D_\alpha K_{-\alpha}(x) & = D_{r,\alpha} K_{-\alpha}(x). \end{cases} \tag{1.16}$$

3.2. The quaternionic Cauchy-Pompeiu formulas. In classical real analysis integral representation formulas are deduced from the Gauss theorem. In complex analysis, a special case of the Cauchy-Pompeiu formula is the Cauchy representation of analytic functions which is also a consequence of the Gauss theorem. Analogously to this, we introduce here the Stokes formula and the Cauchy-Pompeiu integral representation in quaternionic analysis which is related to the factors of the Helmholtz operator.

THEOREM 1.16. *(Quaternionic Stokes formula)*
Let f and g belong to $C^1(\Omega, \mathbb{H}(\mathbb{C})) \cap C(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$ then

$$\int\limits_\Omega [(D_r f(y))g(y) + f(y)Dg(y)]dy = \int_\Gamma f(y)\vec{n}(y)g(y)d\Gamma_y, \tag{1.17}$$

where $\vec{n} := \sum\limits_{k=1}^{3} n_k e_k$ denotes the outward unitary normal vector on Γ.

The above theorem is taken from [**60**, Theorem 2], whose proof can be found in [**47**, Proposition 3.22], (see also [**64**, Chap. 4]).

COROLLARY 1.17. *Let $g \in C^1(\Omega, \mathbb{H}(\mathbb{C})) \cap C(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$. Then*

$$\int\limits_\Omega Dg(y)dy = \int\limits_\Gamma \vec{n}(y)g(y)d\Gamma_y. \tag{1.18}$$

COROLLARY 1.18. *Let $g \in C^1(\Omega, \mathbb{H}(\mathbb{C})) \cap C(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$ and g be $\alpha-$left-hyperholomophic function. Then*

$$\int\limits_\Gamma \vec{n}(y)g(y)d\Gamma_y = -\alpha \int\limits_\Omega g(y)dy. \tag{1.19}$$

REMARK 1.19. From (1.17) with $\alpha \in \mathbb{C}$ we have

$$\int_\Omega [(D_{r,-\alpha} f(y))g(y) + f(y)(D_\alpha g(y))]dy = \int_\Gamma f(y)\vec{n}(y)g(y)d\Gamma_y, \tag{1.20}$$

$$\int_\Omega [(D_{r,\alpha} f(y))g(y) + f(y)(D_{-\alpha} g(y))]dy = \int_\Gamma f(y)\vec{n}(y)g(y)d\Gamma_y. \tag{1.21}$$

THEOREM 1.20. *(Quaternionic Cauchy-Pompeiu formulas)*
Let $f \in C^1(\Omega, \mathbb{H}(\mathbb{C})) \cap C(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$. Then

$$f(x) = -\int_\Gamma K_\alpha(x-y)\vec{n}(y)f(y)d\Gamma_y + \int_\Omega K_\alpha(x-y)D_\alpha f(y)dy. \tag{1.22}$$

The integral formula (1.22) can be rewritten as

$$f(x) = -\int_\Gamma K_\alpha(x-y)\vec{n}(y)f(y)d\Gamma_y + \int_\Omega K_\alpha(x-y)g(y)dy$$

It represents the solution to the equation $D_\alpha f = g$. The proof of this theorem can be found in [**60**, Theorem 4].
It is straighforward to see

$$f(x) = -\int_\Gamma K_{-\alpha}(x-y)\vec{n}(y)f(y)d\Gamma_y + \int_\Omega K_{-\alpha}(x-y)D_{-\alpha}f(y)dy. \tag{1.23}$$

The formulas (1.22), (1.23) are the quaternionic Cauchy-Pompeiu formulas for the operators $D_\alpha,\ D_{-\alpha}$. They express the any differentiable function through its boundary values and its first-order derivaties.
The quaternionic Cauchy-Pompeiu formulas immediately imply the following analogue of the Cauchy integral formula.

Theorem 1.21. *(Quaternionic Cauchy integral formula)*
Let $f \in C^1(\Omega, \mathbb{H}(\mathbb{C})) \cap C(\bar{\Omega}, \mathbb{H}(\mathbb{C}))$ and f be a left (right) α-hyperholomophic function. Then

$$f(x) = -\int_\Gamma K_\alpha(x-y)\vec{n}(y)f(y)d\Gamma_y,$$

for all $x \in \Omega$.

Let us now introduce the main integral operators whose properties are similar to their famous complex prototypes. The *Cauchy's integral operator,* the first order *Teodorescu transform* operator and the operator of *singular integration,* guarantee an efficient solution of different kinds of boundary value problems.

$$\begin{aligned} F_\alpha f(x) &= -\int_\Gamma K_\alpha(x-y)\vec{n}(y)f(y)d\Gamma_y, \quad x \in \mathbb{R}^3 - \Gamma, \\ T_{\alpha,1} f(x) &= \int_\Omega K_\alpha(x-y)f(y)dy, \quad x \in \mathbb{R}^3, \\ S_\alpha f(x) &= -2\int_\Gamma K_\alpha(x-y)\vec{n}(y)f(y)d\Gamma_y, \quad x \in \Gamma. \end{aligned}$$

Note that the integral in the definition of the operator S_α is taken in the sense of Cauchy's principal value. In the other words, it can be called a singular Cauchy operator.

As usual, the singular Cauchy operator generates two important operators P_α and Q_α. Which are defined by

$$\begin{cases} P_\alpha &= \frac{1}{2}(I + S_\alpha), \\ Q_\alpha &= \frac{1}{2}(I - S_\alpha). \end{cases} \tag{1.24}$$

The operator P_α denotes the projection onto the spaces of all $\mathbb{H}(\mathbb{C})-$valued functions which may be *left $\alpha-$hyperholomophic functions* extended into interior of the domain Ω. Q_α denotes the projection onto the spaces of all $\mathbb{H}(\mathbb{C})-$valued functions which may be *left $\alpha-$hyperholomophic functions* extended into exterior of the domain Ω. We refer to [**60**, Chap. 2] for more details.

We remark that the operators $F_\alpha,\ S_\alpha,\ P_\alpha,$ and Q_α allow an extension to $L_2(\Gamma, \mathbb{H}(\mathbb{C}))$. Their further properties will be introduced in Chapter 3 where some boudary value problems for the Helmholtz equation are investigated