# Appendix E

# Maximum and Comparison Principles

One of the, if not the, most important tools employed in the geometric analysis of (MCF) and indeed geometric evolution equations in general, is the maximum principle, or to be more precise, a collection of maximum principles, each tuned to the particular application.

In this section, we will present and derive a number of maximum principles that are used extensively throughout the results contained in this thesis. Apart from the maximum principles, the main result derived here is an application of the non-compact maximum principle of Ecker and Huisken [11]; a comparison principle for graphs over surfaces, which exist in the convex tubular neighborhood of these base surfaces.

## E.1 Global Maximum Principle

This 'quick-and-dirty' maximum principle utilises Huisken's monotonicity formula [15] to derive a global maximum principle, useful for deriving global bounds and convergence results. The following maximum principle is as in [10].

**Theorem E.1.** *Suppose $(M_t)_{t\in[0,T)}$ for $T > 0$ is a solution of (MCF) and $f : M_t \times [0,T) \to \mathbb{R}$ is a function satisfying the evolution equation*

$$\left(\frac{d}{dt} - \Delta\right) f \leqslant \langle \mathbf{a}, \nabla f \rangle\,, \quad t \in [0,T)$$

*for some vector field $\mathbf{a} = \mathbf{a}(\mathbf{x},t)$ such that $\mathbf{a}$ is well-defined in an open neighbourhood of all maximum points of $f$ on $[0,T)$, then we have the estimate*

$$\sup_{M_t} f(\cdot,t) \leqslant \sup_{M_0} f(\cdot,0)$$

*for all $t \in [0,T]$.*

*Proof.* First, we define the function $f_k$, where $k = \sup_{M_0} f$, as

$$f_k(\mathbf{x}, t) = \max(f(\mathbf{x}, t) - k, 0)$$

So, if $f(\mathbf{x}, t)$ is less than $k$, then $f_k$ is zero. Now, we calculate

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) f_k^2 &= 2f_k \left(\frac{d}{dt} - \Delta\right) f_k - 2|\nabla f_k|^2 \\ &\leqslant 2f_k \mathbf{a} \nabla f_k - 2|\nabla f_k|^2 \quad \text{by assumption} \\ &\leqslant \frac{1}{2} a_0^2 f_k^2 \quad \text{by Young's inequality}\end{aligned}$$

Now, using the Monotonicity Formula (see below) with $\varphi = f_k^2$, we have that

$$\begin{aligned}\frac{d}{dt}\int_{M_t} f_k^2 \rho d\mu_t &= \int_{M_t} \rho \left(\frac{d}{dt} - \Delta\right) f_k^2 d\mu_t - \int_{M_t} f_k^2 \left|\mathbf{H} + \frac{1}{2}\left(\frac{\mathbf{x} - \mathbf{x}_0}{T - t}\right)^{\perp}\right|^2 \rho d\mu_t \\ &\leqslant \frac{1}{2} a_0^2 \int_{M_t} f_k^2 \rho d\mu_t\end{aligned}$$

Which implies that

$$\int_{M_t} f_k^2 \rho d\mu_t \leqslant e^{\frac{1}{2}a_0^2 t} \int_{M_0} f_k^2 \rho d\mu_0$$

Thus, we see that

$$\int_{M_0} f_k^2 \rho d\mu_0 = 0$$

and so,

$$\int_{M_t} f_k^2 \rho d\mu_t = 0$$

Therefore, $f_k(\cdot, t) \equiv 0$ for $t \in [0, T)$ and thus $f$ is bounded by its initial data on $M_0$. □

To prove this useful maximum principle, we used Huisken's monotonicity formula, which helps us deal with the fact that $M$ is not compact. The monotonicity formula is, in effect, an evolution equation for the measure of the surface area of an evolving surface, weighted by the backwards heat kernel. Furthermore, it gives as the name suggests, monotonicity for this measure.

**Proposition E.2** (Monotonicity Formula)**.** *The backward heat kernel, $\rho$*

$$\rho(\mathbf{x},t) = (4\pi(T-t))^{-\frac{n}{2}} e^{-\frac{1}{4}\left(\frac{|\mathbf{x}_0-\mathbf{x}|^2}{T-t}\right)}, \quad t < T$$

*on a manifold $M_t$ evolving by (MCF) satisfies*

$$\frac{d}{dt}\int_{M_t} \rho d\mu_t = -\int_{M_t} \left|\mathbf{H} + \frac{1}{2}\left(\frac{\mathbf{x}-\mathbf{x}_0}{T-t}\right)^{\perp}\right|^2 \rho d\mu_t$$

*Proof.* We begin by calculating (without loss of generality setting $\mathbf{x}_0 = 0$ and $T = 0$)

$$\begin{aligned}
\left(\frac{d}{dt} + \Delta\right)\rho - |\mathbf{H}|^2\rho &= \frac{\partial}{\partial t}\rho + \left\langle \overline{\nabla}\rho, \mathbf{H}\right\rangle + \operatorname{div}_{M_t} \overline{\nabla}\rho + \left\langle \overline{\nabla}\rho, \mathbf{H}\right\rangle - |\mathbf{H}|^2\rho \\
&= \frac{\partial\rho}{\partial t} + \operatorname{div}_{M_t} \overline{\nabla}\rho + 2\left\langle \nabla\rho + \nabla^{\perp}\rho, \mathbf{H}\right\rangle - |\mathbf{H}|^2\rho \\
&= \frac{\partial\rho}{\partial t} + \operatorname{div}_{M_t} \overline{\nabla}\rho + 2\left\langle \nabla^{\perp}\rho, \mathbf{H}\right\rangle \\
&\quad - |\mathbf{H}|^2\rho + \frac{|\nabla^{\perp}\rho|^2}{\rho} - \frac{|\nabla^{\perp}\rho|^2}{\rho} \\
&= \frac{\partial\rho}{\partial t} + \operatorname{div}_{M_t} \overline{\nabla}\rho \\
&\quad - \rho\left(|\mathbf{H}|^2 - \frac{2\left\langle \nabla^{\perp}\rho, \mathbf{H}\right\rangle}{\rho} + \frac{|\nabla^{\perp}\rho|^2}{\rho^2}\right) + \frac{|\nabla^{\perp}\rho|^2}{\rho} \\
&= \frac{\partial\rho}{\partial t} + \operatorname{div}_{M_t} \overline{\nabla}\rho + \frac{|\nabla^{\perp}\rho|^2}{\rho} - \left|\mathbf{H} - \frac{\nabla^{\perp}\rho}{\rho}\right|^2\rho
\end{aligned}$$

Now, lets look at the first three terms

$$\begin{aligned}
\frac{\partial\rho}{\partial t} &= -\frac{n}{2}(-4\pi t)^{-\frac{n}{2}-1}(-4\pi)e^{\frac{|\mathbf{x}|^2}{4t}} + \rho\left(-\frac{|\mathbf{x}|^2}{4t^2}\right) \\
&= -\rho\left(\frac{n}{2t} + \frac{|\mathbf{x}|^2}{4t^2}\right)
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{div}_{M_t} \overline{\nabla}\rho &= \operatorname{div}_{M_t}\left(\left(\frac{\mathbf{x}}{2t}\right)\rho\right) \\
&= \left(\frac{n}{2t}\right)\rho + \left\langle \frac{\mathbf{x}}{2t}, \nabla\rho\right\rangle \\
&= \left(\frac{n}{2t}\right)\rho + \left\langle \frac{\mathbf{x}}{2t}, \rho\left(\frac{\mathbf{x}^{\top}}{2t}\right)\right\rangle \\
&= \left(\frac{n}{2t} + \frac{|\mathbf{x}^{\top}|^2}{4t^2}\right)\rho
\end{aligned}$$

and

$$\begin{aligned}\frac{|\nabla^\perp\rho|^2}{\rho} &= \frac{1}{\rho}\left\langle \nabla^\perp\rho, \nabla^\perp\rho\right\rangle \\ &= \frac{\rho}{4t^2}\left\langle \mathbf{x}^\perp, \mathbf{x}^\perp\right\rangle \\ &= \frac{|\mathbf{x}^\perp|^2}{4t^2}\rho\end{aligned}$$

so we have that

$$\frac{\partial\rho}{\partial t} + \operatorname{div}_{M_t}\overline{\nabla}\rho + \frac{|\nabla^\perp\rho|^2}{\rho} = 0$$

which implies the result

$$\begin{aligned}\frac{d}{dt}\int_{M_t}\rho d\mu_t &= \int_{M_t}\left(\frac{d\rho}{dt} - |\mathbf{H}|^2\rho\right)d\mu_t \quad \text{since } \frac{d\mu_t}{dt} = -|\mathbf{H}|^2\mu_t \\ &= \int_{M_t}\left(\left(\frac{d}{dt}+\Delta\right)\rho - |\mathbf{H}|^2\rho\right)d\mu_t \quad \text{by divergence theorem} \\ &= -\int_{M_t}\left|\mathbf{H} - \frac{\nabla^\perp\rho}{\rho}\right|^2\rho d\mu_t \\ &= -\int_{M_t}\left|\mathbf{H} + \frac{\mathbf{x}^\perp}{2t}\right|^2\rho d\mu_t\end{aligned}$$

□

**Remark E.3.** More generally, for a test function $\varphi = \varphi(\mathbf{x}, t)$ on $M_t$ we have the formula

$$\frac{d}{dt}\int_{M_t}\varphi\rho d\mu_t = \int_{M_t}\rho\left(\frac{d}{dt} - \Delta\right)\varphi d\mu_t - \int_{M_t}\varphi\rho\left|\mathbf{H} + \frac{1}{2}\left(\frac{\mathbf{x}-\mathbf{x}_0}{T-t}\right)^\perp\right|^2 d\mu_t$$

## E.2 Extending the Maximum Principle

Sometimes we wish to apply the maximum principle to a non-compact manifold (such as an entire graph), however we don't wish to impose any boundary conditions or growth restraints on the manifold. A convenient method to obtaining estimates on compact subsets of a non-compact manifold is to use localisation functions.

**Definition E.4** (Localisation Function)**.** A (sufficiently) smooth non-negative function $\varphi : \mathbb{R}^{n+1} \times [0, T] \to \mathbb{R}$ is called a localisation function if there exists a constant $c_\varphi$ such that

$$\varphi + \left|\frac{\partial\varphi}{\partial t}\right| + |\overline{\nabla}\varphi| + |\overline{\nabla}^2\varphi| + \frac{1}{\varphi}|\overline{\nabla}\varphi| \leqslant c_\varphi$$

Importantly, all localisation functions satisfy the estimate

$$\left|\left(\frac{d}{dt} - \Delta\right)\varphi\right| \leqslant c(n, c_\varphi)$$

Using this estimate, we obtain the following local maximum principle as derived in [9].

**Proposition E.5** (Localised Maximum Principle)**.** *Let* $(M_t)_{t\in[0,T)}, T > 0$ *be a solution of (MCF) and let* $\varphi : \mathbb{R}^{n+1} \times [0, T) \to \mathbb{R}$ *be a localisation function such that for all* $t \in [0, T)$*,* $M_t \cap \operatorname{spt}\varphi(\cdot, t)$ *is compact. Now, let* $f : M_t \times [0, T), \to \mathbb{R}$ *be a non-negative twice continuously differentiable function satisfying*

$$\left(\frac{d}{dt} - \Delta\right) f \leqslant \langle \mathbf{a}, \nabla f\rangle - \delta f^2 - df + K$$

*where* $\delta > 0$*,* $K < \infty$*,* $d(\mathbf{x}, t) \geqslant 0$ *and* $|\mathbf{a}(\mathbf{x}, t)|^2 \leqslant a_0^2(1 + d(\mathbf{x}, t))$ *for all* $(\mathbf{x}, t) \in \operatorname{spt} f$*, then we have the estimate*

$$\sup_{M_t}(f\varphi)(\cdot, t) \leqslant \sup_{M_0}(f\varphi)(\cdot, 0) + C(n, \delta, a_0, K, c_\varphi)$$

*for all* $t \in [0, T)$*.*

**Remark E.6.** The constant $C = C(n, \delta, a_0, K, c_\varphi)$ in Theorem E.5 is of the form

$$C(n, \delta, a_0, K, c_\varphi) = c(n, c_\varphi)(1 + K)(1 + a_0^2)(1 + 1/\delta)$$

## E.3 Comparison Principle

One major application of a maximum principle is the derivation of a comparison principle. Simply, a comparison principle for (MCF) says that two initially disjoint solutions to (MCF) will remain disjoint.

This is useful in the case where an explicit solution to (MCF) is known, allowing us to create impassable barrier surfaces, perhaps protecting other solutions from developing singularities, or from moving to regions where certain equations are not well defined, for example a cylindrical graph moving towards the axis.

The following comparison principle has been adapted from a more general version given by Barles et al. in [4].

**Theorem E.7** (Comparison Principle)**.** *Let* $\rho_1 : \mathbb{R} \times [0, T) \to \mathbb{R}$ *and* $\rho_2 : \mathbb{R} \times [0, T) \to \mathbb{R}$*,* $T > 0$ *be two rotationally symmetric graphical solutions to (MCF) with at most polynomial growth. If*

$$\rho_1(z, 0) \leqslant \rho_0(z) \leqslant \rho_2(z, 0), \quad z \in \mathbb{R}$$

*for some function $\rho_0$ satisfying*

$$|\rho_0(y) - \rho_0(z)| \leqslant m\left((1 + |y| + |z|)^\nu |y - z|\right), \quad y, z \in \mathbb{R}$$

*where $m$ is a modulus of continuity and $0 \leqslant \nu < (1 + \sqrt{5})/2$, then*

$$\rho_1(z, t) \leqslant \rho_2(z, t) \quad z \in \mathbb{R}, t \in [0, T)$$

*Proof.* Let $\rho_i, i = 1, 2$ be a solution to the equation

$$\frac{\partial \rho_i}{\partial t}(z, t) = b(\rho_i')\rho_i'' + H(\rho_i), \quad z \in \mathbb{R}, t \in [0, T)$$

where

$$b(p) = \frac{1}{1 + p^2}, \quad p \in \mathbb{R}$$

and

$$H(u) = -\frac{n - 1}{u}, \quad u \in \mathbb{R}$$

In this simplified 1-dimensional case, for Theorem 2.1 of [4] to be applicable, we must have that $H$ is a non-decreasing function of $u$ and that there exists a function $\sigma : \mathbb{R} \to \mathbb{R}$ such that

$$b(p) = \sigma^2(p)$$

and

$$|\sigma(p) - \sigma(q)| \leqslant \frac{C|p - q|}{1 + |p| + |q|}$$

The function $H$ is clearly non-decreasing, and by Lemma 3.1 of [4], $\sigma$ satisfies the required inequality. Thus, we may apply Theorem 2.1 of [4] to the solutions $\rho_1$ and $\rho_2$ satisfying the initial condition

$$\rho_1(z, 0) \leqslant \rho_0(z) \leqslant \rho_2(z, 0), \quad z \in \mathbb{R}$$

for some function $\rho_0$ satisfying

$$|\rho_0(y) - \rho_0(z)| \leqslant m\left((1 + |y| + |z|)^\nu |y - z|\right), \quad y, z \in \mathbb{R}$$

where $m$ is a modulus of continuity (i.e. $\lim_{s \to 0_+} m(s) = 0$ and $m(t + s) \leqslant m(t) + m(s)$) and $0 \leqslant \nu < (1 + \sqrt{5})/2$, and obtain the result. □