Appendix B

Geometric Flows

B.1 General Flows

We are going to describe a general geometric flow of a hyper-surface in $\mathbb{R}^{n+1}$. To that end, we define the family of immersions $\mathbf{F} : M^n \times [0, T) \to \mathbb{R}^{n+1}$ where M^n is an n-dimensional manifold. We denote the evolving hyper-surface by $M_t = \mathbf{F}(\cdot, t)(M^n)$ and by $\mathbf{x} = \mathbf{F}(\boldsymbol{p}, t) \in M_t$ we denote the position vector on M_t. The presented method of deriving the following geometric evolution equations was made known to the author through [8].

Now, we stipulate that M_t satisfies the evolution equation

$$\frac{d\mathbf{x}}{dt} = \eta\boldsymbol{\nu}, \quad \mathbf{x} \in M_t, t \in [0, T) \tag{B.1}$$

where $\eta = \eta(\mathbf{x}, t)$ is a smooth function on M_t and $\boldsymbol{\nu} = \boldsymbol{\nu}(\mathbf{x}, t)$ is a (consistent) choice of unit normal to M_t at $\mathbf{x}$. Thus in this system, we have the surface M_t evolving in the direction of $\boldsymbol{\nu}$, with velocity equal to η.

It is up to us to decide what we would like η to be. Our choice will depend upon the properties that we want the flow to have. In mean curvature flow, we choose $\eta = -H$, a choice we shall motivate later.

In studying the properties of the flow (B.1) we need to derive the evolution equations of some geometric quantities on M_t, such as the metric, the second fundamental form and the measure. One way to do this is to consider the way that these vary when the surface is *deformed* along the path of the flow.

We will be deforming M_t in a neighbourhood of a point $\mathbf{x}_0 \in M_t$. To that end, we define a local deformation of the hyper-surface M_t in $\mathbb{R}^{n+1}$ in an open neighbourhood $\Omega \subset \mathbb{R}^{n+1}$ of $\mathbf{x}_0$ as a family of diffeomorphisms

$$\boldsymbol{\varphi} : \Omega \times (\varepsilon, \varepsilon) \to \mathbb{R}^{n+1}$$

which satisfy

(i) $\boldsymbol{\varphi}(\mathbf{z}, 0) = \mathbf{z}, \quad \forall \mathbf{z} \in \Omega$

(ii) $\frac{d}{ds}\big|_{s=0} \boldsymbol{\varphi}(\mathbf{z}, s) = \mathbf{X}(\mathbf{z}), \quad \forall \mathbf{z} \in \Omega$

where $\mathbf{X} \in C_0^\infty\left(\Omega, \mathbb{R}^{n+1}\right)$.

We shall be choosing $\mathbf{X} = \eta\bar{\boldsymbol{\nu}}$ where $\bar{\boldsymbol{\nu}} \in C_0^\infty\left(\Omega, \mathbb{R}^{n+1}\right)$ is a smooth extension of $\boldsymbol{\nu}$ into Ω. Thus, in this setting the deformation $\boldsymbol{\varphi}$ locally has the same 'action' as the flow.

Now, as always, we will be using normal coordinates $\{\boldsymbol{\tau}_i\}$ at $\mathbf{x}_0$. Defining $\boldsymbol{\tau}_i(s) = d\boldsymbol{\varphi}_s(\boldsymbol{\tau}_i)$, we obtain a local coordinate system $\{\boldsymbol{\tau}_i(s), \boldsymbol{\nu}\}$ in Ω, which has the desirable property

$$\overline{\nabla}_{\boldsymbol{\tau}_i}\mathbf{X} - \overline{\nabla}_{\mathbf{X}}\boldsymbol{\tau}_i = [\mathbf{X}, \boldsymbol{\tau}_i], \qquad \text{in } \Omega \times (-\varepsilon, \varepsilon)$$

With the technical details out of the way, we are now prepared to calculate our evolution equations.

Lemma B.1. *The g_{ij} and g^{ij} of the metric satisfy the equation*

$$\left.\frac{d}{ds}\right|_{s=0} g_{ij} = 2\eta h_{ij} \tag{B.2}$$

$$\left.\frac{d}{ds}\right|_{s=0} g^{ij} = -2\eta h_{ij} \tag{B.3}$$

Proof.

$$\begin{aligned}\left.\frac{d}{ds}\right|_{s=0} g_{ij} &= \mathbf{X}\left\langle \boldsymbol{\tau}_i, \boldsymbol{\tau}_j\right\rangle \\ &= 2\left\langle \overline{\nabla}_{\mathbf{X}}\boldsymbol{\tau}_i, \boldsymbol{\tau}_j\right\rangle \\ &= 2\left\langle \overline{\nabla}_{\boldsymbol{\tau}_i}\mathbf{X}, \boldsymbol{\tau}_j\right\rangle \\ &= 2\eta h_{ij}\end{aligned}$$

The evolution equation for the g^{ij} is a simple corollary of the fact

$$\mathbf{X}(g^{ik}g_{kj}) = 0$$

□

Lemma B.2. *The unit normal, $\boldsymbol{\nu}$ evolves according to the equation*

$$\left.\frac{d\boldsymbol{\nu}}{ds}\right|_{s=0} = -\nabla\eta$$

Proof.

$$\begin{aligned}\left.\frac{d\boldsymbol{\nu}}{ds}\right|_{s=0} &= -\overline{\nabla}_{\mathbf{X}}\boldsymbol{\nu} \\ &= \left\langle \overline{\nabla}_{\mathbf{X}}\boldsymbol{\nu}, \boldsymbol{\tau}_k\right\rangle \boldsymbol{\tau}_k \quad \text{since } \overline{\nabla}_{\mathbf{X}}\boldsymbol{\nu} \in T_{\mathbf{x}}M_t \\ &= -\left\langle \boldsymbol{\nu}, \overline{\nabla}_{\mathbf{X}}\boldsymbol{\tau}_k\right\rangle \boldsymbol{\tau}_k \\ &= -\left\langle \boldsymbol{\nu}, \overline{\nabla}_{\boldsymbol{\tau}_k}\mathbf{X}\right\rangle \boldsymbol{\tau}_k \\ &= -\left\langle \boldsymbol{\nu}, \boldsymbol{\tau}_k(\eta)\boldsymbol{\nu} + \eta\overline{\nabla}_{\boldsymbol{\tau}_k}\boldsymbol{\nu}\right\rangle \boldsymbol{\tau}_k \\ &= -\nabla\eta\end{aligned}$$

$\square$

Proposition B.3. *The measure μ satisfies*

$$\frac{d\mu}{dt} = \eta H \mu$$

Proof.

$$\begin{aligned}
\frac{d\mu}{dt} &= \mu \frac{d}{dt} \log \mu \\
&= \mu \frac{d}{dt} \log \sqrt{\det g_{ij}} \\
&= \frac{\mu}{2} \frac{\frac{d}{dt} \det g_{ij}}{\det g_{ij}} \\
&= \frac{\mu}{2} \frac{\det g_{ij} g^{ab} \frac{d}{dt} g_{ab}}{\det g_{ij}} \\
&= \frac{\mu}{2} g^{ij} \frac{d}{dt} g_{ij} \\
&= \frac{\mu}{2} g^{ij} \left(2\eta h_{ij}\right) \\
&= \eta H \mu
\end{aligned}$$

$\square$

Proposition B.4. *The second fundamental form h_{ij} evolves by the equation*

$$\left.\frac{d}{ds}\right|_{s=0} h_{ij} = -\boldsymbol{\tau}_i \boldsymbol{\tau}_j(\eta) + \eta h_{ik} h_{jk}$$

Proof.

$$\begin{aligned}
\left.\frac{d}{ds}\right|_{s=0} h_{ij} &= -\mathbf{X} \left\langle \boldsymbol{\nu}, \overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\tau}_j \right\rangle \\
&= -\left\langle \overline{\nabla}_{\mathbf{X}} \boldsymbol{\nu}, \overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\tau}_j \right\rangle - \left\langle \boldsymbol{\nu}, \overline{\nabla}_{\mathbf{X}} \overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\tau}_j \right\rangle \\
&= -\left\langle \boldsymbol{\nu}, \overline{\nabla}_{\boldsymbol{\tau}_i} \overline{\nabla}_{\boldsymbol{\tau}_j} \mathbf{X} \right\rangle \\
&= -\left\langle \boldsymbol{\nu}, \overline{\nabla}_{\boldsymbol{\tau}_i} \left(\boldsymbol{\tau}_i(\eta) \boldsymbol{\nu} + \eta \overline{\nabla}_{\boldsymbol{\tau}_j} \boldsymbol{\nu} \right) \right\rangle \\
&= -\left\langle \boldsymbol{\nu}, \boldsymbol{\tau}_i \boldsymbol{\tau}_j(\eta) \boldsymbol{\nu} + \boldsymbol{\tau}_j(\eta) \overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\nu} + \boldsymbol{\tau}_i(\eta) \overline{\nabla}_{\boldsymbol{\tau}_j} \boldsymbol{\nu} + \eta \overline{\nabla}_{\boldsymbol{\tau}_i} \overline{\nabla}_{\boldsymbol{\tau}_j} \boldsymbol{\nu} \right\rangle \\
&= -\left\langle \boldsymbol{\nu}, \boldsymbol{\tau}_i \boldsymbol{\tau}_j(\eta) \boldsymbol{\nu} + \eta \nabla h_{ij} + \eta h_{jk} h_{ik} \right\rangle \\
&= -\boldsymbol{\tau}_i \boldsymbol{\tau}_j(\eta) + \eta h_{ik} h_{jk}
\end{aligned}$$

$\square$

Collecting these equations together, and evaluating along the path given by $t \mapsto \mathbf{x}(t) = \boldsymbol{\varphi}(\mathbf{x}_0, t)$ we obtain

(i) $\left(\frac{d}{dt} - \Delta\right) g_{ij} = 2\eta h_{ij}$

(ii) $\left(\frac{d}{dt} - \Delta\right) \mu = \eta H \mu$

(iii) $\left(\frac{d}{dt} - \Delta\right) \boldsymbol{\nu} = |A|^2 \boldsymbol{\nu} - \nabla(\eta + H)$

(iv) $\left(\frac{d}{dt} - \Delta\right) h_{ij} = |A|^2 h_{ij} - \boldsymbol{\tau}_i \boldsymbol{\tau}_j (\eta + H) + (\eta - H) h_{ik} h_{kj}$

where we have used the Simons' inequality in the last equation.

B.2 Mean Curvature Flow

To see why (MCF) is an interesting flow to investigate, consider the following variation calculation of the 'energy' $|M_t|$ under the evolution

$$\frac{d\mathbf{x}}{dt} = \eta \boldsymbol{\nu}, \mathbf{x} \in M_t$$

and we find

$$\begin{aligned} \frac{d}{dt}|M_t| &= \int_{M_t} \frac{1}{\mu} \frac{d\mu}{dt} d\mu \\ &= \int_{M_t} \operatorname{div}_{M_t} \left(\frac{d\mathbf{x}}{dt} \right) d\mu \\ &= -\int_{M_t} \left\langle \mathbf{H}, \frac{d\mathbf{x}}{dt} \right\rangle d\mu \end{aligned}$$

by the divergence theorem. Thus, it is clear that in setting

$$\frac{d\mathbf{x}}{dt} = \mathbf{H}, \quad \mathbf{x} \in M_t$$

we have chosen the least energy flow for the area functional of M_t. This is the equivalent of setting $\eta = -H$ in the previous calculations. From this, it is clear that the area of a compact surface monotonically decreases under (MCF).

So, to summarise the earlier results when applied to (MCF), we have

(i) $\left(\frac{d}{dt} - \Delta\right) g_{ij} = -2H h_{ij}$

(ii) $\left(\frac{d}{dt} - \Delta\right) \mu = -H^2 \mu$

(iii) $\left(\frac{d}{dt} - \Delta\right) \boldsymbol{\nu} = |A|^2 \boldsymbol{\nu}$

(iv) $\left(\frac{d}{dt} - \Delta\right) h_{ij} = |A|^2 h_{ij} - 2H h_{ik} h_{kj}$

Corollary B.5. *The mean curvature and the norm of the second fundamental form satisfy the evolution equations* *(i)* $\left(\frac{d}{dt} - \Delta\right) H = |A|^2 H$

(ii) $\left(\frac{d}{dt} - \Delta\right) |A|^2 = -2|\nabla A|^2 + 2|A|^4$

Proof of (i). Using the evolution equations for h_{ij} and g^{ij} we calculate

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta\right) H &= \left(\frac{d}{dt} - \Delta\right) \left(g^{ij} h_{ij}\right) \\
&= h_{ij} \left(\frac{d}{dt} - \Delta\right) g^{ij} + g^{ij} \left(\frac{d}{dt} - \Delta\right) h_{ij} - 2 \left\langle \nabla g^{ij}, \nabla h_{ij} \right\rangle \\
&= h_{ij} \left(2H h_{kl} g^{ik} g^{jl}\right) + g^{ij} \left(-2H h_{il} g^{lm} g^{mj} + |A|^2 h_{ij}\right) \\
&= |A|^2 H
\end{aligned}$$

□

Proof of (ii). Again, using the equations for h_{ij} and g^{ij} we calculate

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta\right) |A|^2 &= \left(\frac{d}{dt} - \Delta\right) \left(g^{ik} g^{jl} h_{ij} h_{kl}\right) \\
&= 2 g^{ik} g^{im} h_{kl} \left(\frac{d}{dt} - \Delta\right) g^{ml} + 2 g^{ik} g^{ml} h_{im} \left(\frac{d}{dt} - \Delta\right) h_{kl} \\
&\quad - 2 \left\langle \nabla \left(g^{ik} g^{im} h_{kl}\right), \nabla h_{kl} \right\rangle \\
&= 2 g^{ik} g^{im} h_{kl} \left(2H h_{ab} g^{ma} g^{lb}\right) \\
&\quad + 2 g^{ik} g^{ml} h_{im} \left(-2H h_{ka} g^{ab} h_{al} + |A|^2 h_{kl}\right) \\
&\quad - 2 g^{ik} g^{ml} \left\langle \nabla h_{im}, \nabla h_{kl} \right\rangle \\
&= 2|A|^4 - 2|\nabla A|^2
\end{aligned}$$

□

B.3 Higher-order Evolution Equations

Suppose that some tensor T on a solution $(M_t)_{t \in [0,T)}$ of (MCF) satisfies the evolution equation

$$\left(\frac{d}{dt} - \Delta\right) T = Q \tag{B.4}$$

If we require estimates on higher derivatives of T, we will require evolution equations for these derivatives.

Proposition B.6. *Suppose a tensor T on a solution $(M_t)_{t \in [0,T)}$ to (MCF) satisfies the evolution equation*

$$\left(\frac{d}{dt} - \Delta\right) T = Q$$

then ∇T satisfies the evolution equation

$$\left(\frac{d}{dt} - \Delta\right) \nabla T = \nabla Q + A * (\nabla T * A + T * \nabla A)$$

where $*$ denotes any product with respect to the metric.

Proof. Computing in normal coordinates,

$$\begin{aligned}\frac{d}{dt}\Gamma^k_{ij} &= \frac{1}{2}\left[\boldsymbol{\tau}_i\left(\frac{d}{dt}g_{kj}\right) + \boldsymbol{\tau}_j\left(\frac{d}{dt}g_{ik}\right) - \boldsymbol{\tau}_k\left(\frac{d}{dt}g_{ij}\right)\right] \\ &= -\left[\boldsymbol{\tau}_i(Hh_{kj}) + \boldsymbol{\tau}_j(Hh_{ik}) - \boldsymbol{\tau}_k(Hh_{ij})\right]\end{aligned}$$

thus, since

$$\begin{aligned}\nabla_k T^{i_1\ldots i_p}_{j_1\ldots j_q} = \frac{\partial}{\partial p^k} T^{i_1\ldots i_p}_{j_1\ldots j_q} &+ \Gamma^{i_1}_{km} T^{m\ldots i_p}_{j_1\ldots j_q} + \cdots + \Gamma^{i_p}_{km} T^{i_1\ldots m}_{j_1\ldots j_q} \\ &- \Gamma^m_{kj_1} T^{i_1\ldots i_p}_{m\ldots j_q} - \cdots - \Gamma^m_{kj_q} T^{i_1\ldots i_p}_{j_1\ldots m}\end{aligned}$$

we have that

$$\frac{d}{dt}(\nabla T) = \nabla\left(\frac{dT}{dt}\right) + T * A * \nabla A$$

Now, compute

$$\begin{aligned}\Delta(\nabla_k T) &= \nabla_i \nabla_i (\nabla_k T) \\ &= \nabla_i\left[\nabla_k(\nabla_i T) + (R * T)_{ik}\right] \\ &= \nabla_k(\Delta T) + R * \nabla_k T + \nabla_k R * T\end{aligned}$$

or, since $R = A * A$ by the Gauß relations, we have

$$\Delta(\nabla T) = \nabla(\Delta T) + A * (\nabla T * A + T * \nabla A)$$

and the result follows immediately. □

Corollary B.7. *The covariant derivative of the second fundamental form ∇A satisfies the evolution equation*

$$\left(\frac{d}{dt} - \Delta\right) \nabla A = \sum_{i+j+k=1} \nabla^i A * \nabla^j A * \nabla^k A \tag{B.5}$$

Proposition B.8. *The p^{th} covariant derivative of the second fundamental form $\nabla^p A$ satisfies the evolution equation*

$$\left(\frac{d}{dt} - \Delta\right) \nabla^p A = \sum_{i+j+k=p} \nabla^i A * \nabla^j A * \nabla^k A \tag{B.6}$$

for $p \geqslant 0$.

Proof. We proceed to prove this evolution equation by induction. The base case ($p = 0$) is already established by Corollary B.7. Now, suppose that for $p = m$ the evolution equation holds, then we compute

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) \nabla^{m+1} A &= \left(\frac{d}{dt} - \Delta\right) \nabla(\nabla^m A) \\ &= \nabla \left(\sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A \right) \\ &= \sum_{i+j+k=m} \nabla^{i+1} A * \nabla^j A * \nabla^k A \\ &= \sum_{i+j+k=m+1} \nabla^i A * \nabla^j A * \nabla^k A\end{aligned}$$

and the result follows by induction on p. □

Corollary B.9. *The length of the p^{th} covariant derivative $|\nabla^p A|^2$ satisfies the evolution equation*

$$\left(\frac{d}{dt} - \Delta\right) |\nabla^p A|^2 = -2|\nabla^{p+1} A|^2 + \sum_{i+j+k=p} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^p A \quad \text{(B.7)}$$

for $p \geqslant 0$.

Proof. First, we compute

$$\begin{aligned}\frac{d}{dt}|\nabla^p A|^2 &= \frac{d}{dt}(\nabla^p A) * \nabla^p A + A * A * \nabla^p A * \nabla^p A \\ &= \langle \Delta(\nabla^p A), \nabla^p A\rangle + \sum_{i+j+k=p} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^p A\end{aligned}$$

where the last term in the first line has been absorbed into the last term of the second line.

Now, using the Bochner formula

$$\Delta|\nabla^p A|^2 = \langle \Delta(\nabla^p A), \nabla^p A\rangle + 2|\nabla^{p+1} A|^2$$

the result is immediately clear. □