Appendix A

Notation and Useful Identities

The purpose of this appendix is to introduce (or remind) the reader to some of the important results and notation of differential geometry that we make use of in this thesis. Wherever possible, we will be making full advantage of normal coordinates (due to the point-wise nature of most of our estimates), making our calculations far neater.

Unless otherwise stated, $\{\boldsymbol{\tau}_i\}_{i=1}^n$ will always denote an orthonormal basis of T_xM, while $\{\mathbf{e}_i\}_{i=1}^{n+1}$ will always be the standard basis on $\mathbb{R}^{n+1}$. Also, we will be following the Einstein summation convention over repeated indices, e.g.

$$\mathbf{X} = X_i\boldsymbol{\tau}_i = \sum_{i=1}^{n} X_i\boldsymbol{\tau}_i$$

A.1 Geometry in Coordinates

While normal coordinates are immensely useful, it is sometimes useful to work in a particular coordinate system. To that end, we introduce this notation, closely following that used by Huisken [14].

Suppose that M is given as the image of a smooth immersion $\mathbf{F} : U \to \mathbb{R}^{n+1}$, $U \subset \mathbb{R}^n$ open, i.e. $M = \mathbf{F}(U)$. We will denote by $\mathbf{x}$ the position vector on M, i.e. $\mathbf{x} = \mathbf{F}(\boldsymbol{p}), \boldsymbol{p} \in U$.

We have a basis for the tangent space $T_{\mathbf{x}}M$ at $\mathbf{x} \in M$ given by the coordinate tangent vectors, i.e.

$$T_xM = \operatorname*{span}_{1\leqslant i\leqslant n} \left\{ \left.\frac{\partial \mathbf{F}}{\partial p^i}\right|_{\boldsymbol{p}} \right\}$$

The canonical metric on $\mathbb{R}^{n+1}$ will be denoted by $\langle\cdot,\cdot\rangle$, with which we define the components of the first fundamental form g_{ij} (induced metric) on

$T_{\mathbf{x}}M$ by

$$g_{ij}(\mathbf{x}) = \left\langle \left. \frac{\partial \mathbf{F}}{\partial p^i} \right|_{\boldsymbol{p}}, \left. \frac{\partial \mathbf{F}}{\partial p^j} \right|_{\boldsymbol{p}} \right\rangle$$

The inverse metric g^{ij} is defined by $\left(g^{ij}\right) = (g_{ij})^{-1}$.

Thus, we define the inner product for tensors T, S of rank (p,q) on M by

$$\langle T, S\rangle = g_{j_1k_1}\cdots g_{j_pk_p} g^{l_1m_1}\cdots g^{l_qm_q} T^{j_1\ldots j_p}_{l_1\ldots l_q} S^{k_1\ldots k_p}_{m_1\ldots m_q}$$

The second fundamental form h_{ij} is computed by

$$h_{ij}(\mathbf{x}) = \left\langle \left. \frac{\partial \boldsymbol{\nu}}{\partial p^i} \right|_{\boldsymbol{p}}, \left. \frac{\partial \mathbf{F}}{\partial p^j} \right|_{\boldsymbol{p}} \right\rangle$$

The Christoffel symbols are given by

$$\Gamma^k_{ij}(\mathbf{x}) = \frac{1}{2} g^{kl} \left(\frac{\partial}{\partial p^i} g_{lj} + \frac{\partial}{\partial p^j} g_{il} - \frac{\partial}{\partial p^k} g_{ij} \right)$$

giving us the unique Levi-Civita connection on M of a tensor field $T = \left\{ T^{i_1\ldots i_p}_{j_1\ldots j_q} \right\}^n_{i_1,j_1\ldots i_p,j_q=1}$ of rank (p,q)

$$\begin{aligned} \nabla_k T^{i_1\ldots i_p}_{j_1\ldots j_q} = \frac{\partial}{\partial p^k} T^{i_1\ldots i_p}_{j_1\ldots j_q} &+ \Gamma^{i_1}_{km} T^{m\ldots i_p}_{j_1\ldots j_q} + \cdots + \Gamma^{i_p}_{km} T^{i_1\ldots m}_{j_1\ldots j_q} \\ &- \Gamma^m_{kj_1} T^{i_1\ldots i_p}_{m\ldots j_q} - \cdots - \Gamma^m_{kj_q} T^{i_1\ldots i_p}_{j_1\ldots m} \end{aligned}$$

We will often use the fact that $\nabla_k g_{ij} = 0, \forall i,j,k$.

The Laplacian of a tensor is defined by

$$\Delta T^{i_1\ldots i_p}_{j_1\ldots j_q} = g^{mn} \nabla_m \nabla_n T^{i_1\ldots i_p}_{j_1\ldots j_q}$$

We have the matrix of the Weingarten map given by

$$\left(h^i_j\right) = \left(g^{ij}\right)(h_{ij})$$

the eigenvalues of which are the principle curvatures. We will denote the principle curvatures by $\{\kappa_i\}$.

The following notation will be used to denote the mean curvature H and the norm $|A|^2$ of the second fundamental form.

$$H = g^{ij} h_{ij} \qquad\qquad |A|^2 = g^{ij} g^{kl} h_{ik} h_{jl}$$

The measure on the surface, $d\mu$ is given by

$$d\mu = \mu(\mathbf{x}) d\boldsymbol{p}$$

where μ is given by

$$\mu(\mathbf{x}) = \sqrt{\det(g_{ij})}$$

and $d\boldsymbol{p}$ is the n-dimensional volume measure on $\mathbb{R}^n$.

This gives the definition of the n-dimensional Hausdorff measure on M, allowing us to measure the area of the surface

$$\mathcal{H}^n(M) = |M| = \int_{M_t} d\mu$$

A.2 Geometry Coordinate-Free

Now, geometry in a coordinate system is fun (to be sure), but it gets a little tedious working with difficult to calculate Christoffel symbols. To vastly simplify our calculations we shall introduce normal coordinates.

Proposition A.1 (Normal Coordinates)**.** *There exists a smooth frame field* $\{\boldsymbol{\tau}_i\}$ *on* M *such that at a point* $\mathbf{x} \in M$

(i) $g_{ij}(\mathbf{x}) = \delta_{ij}, \quad 1 \leqslant i, j \leqslant n$

(ii) $\left\langle \overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\tau}_j, \boldsymbol{\tau}_k \right\rangle(\mathbf{x}) = 0, \quad 1 \leqslant i, j, k \leqslant n$

Note that (ii) implies that $\overline{\nabla}_{\boldsymbol{\tau}_k} g_{ij} = 0$ at $\mathbf{x}$.

The thing to remember, is that this coordinate system is valid only at a single point; not even valid in a neighbourhood.

In this basis, we define the second fundamental form on M by

$$h_{ij} = \left\langle \overline{\nabla}_{\tau_i} \boldsymbol{\nu}, \boldsymbol{\tau}_j \right\rangle = -\left\langle \boldsymbol{\nu}, \overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\tau}_j \right\rangle$$

Now, since in normal coordinates, the Weingarten map is given by the diagonal of the second fundamental form, so the eigenvalues of the Weingarten map and hence the principle curvatures are given by the diagonal of the second fundamental form.

Thus, we calculate H, the mean curvature of M at $\mathbf{x}$ by

$$H = g^{ij} h_{ij} = h_{ii} = \sum_{i=1}^{n} \kappa_i$$

and the norm of the second fundamental form is given by

$$|A|^2 = g^{ik} g^{jl} h_{ij} h_{kl} = h_{ik} h_{ik} = \sum_{i=1}^{n} \kappa_i^2$$

The tangential gradient on M will be denoted ∇, defined as follows (where $^\top$ denotes projection onto the tangent space)

$$\begin{aligned}\nabla f &= \left(\overline{\nabla} f\right)^\top \\ &= \overline{\nabla} f - \left\langle \overline{\nabla} f, \boldsymbol{\nu} \right\rangle \boldsymbol{\nu} \\ &= \left\langle \overline{\nabla} f, \boldsymbol{\tau}_i \right\rangle \boldsymbol{\tau}_i \\ &= \boldsymbol{\tau}_i(f)\boldsymbol{\tau}_i\end{aligned}$$

Note that ∇ is the unique Levi-Civita connection on M.

The Laplace-Beltrami operator of a scalar function $f : M_t \to \mathbb{R}$ is defined and calculated as follows

$$\begin{aligned}\Delta f &= \operatorname{div}_{M_t} \nabla f \\ &= \operatorname{div}_{M_t} \left(\overline{\nabla} f - \left\langle \overline{\nabla} f, \boldsymbol{\nu} \right\rangle \boldsymbol{\nu}\right) \\ &= \operatorname{div}_{M_t} \overline{\nabla} f - \operatorname{div}_{M_t} \left(\left\langle \overline{\nabla} f, \boldsymbol{\nu} \right\rangle \boldsymbol{\nu}\right) \\ &= \operatorname{div}_{M_t} \overline{\nabla} f - \left\langle \overline{\nabla} f, \boldsymbol{\nu} \right\rangle \operatorname{div}_{M_t} \boldsymbol{\nu} - \left\langle \nabla \left\langle \overline{\nabla} f, \boldsymbol{\nu} \right\rangle, \boldsymbol{\nu} \right\rangle \\ &= \operatorname{div}_{M_t} \overline{\nabla} f + \left\langle \mathbf{H}, \overline{\nabla} f \right\rangle\end{aligned}$$

Note that $\{\boldsymbol{\tau}_i, \boldsymbol{\nu}\}$ is an orthonormal basis of $\mathbb{R}^{n+1}$ and thus we have

$$\begin{aligned}\operatorname{div}_{\mathbb{R}^{n+1}} X &= \left\langle \overline{\nabla}_{\boldsymbol{\tau}_i} X, \boldsymbol{\tau}_i \right\rangle + \left\langle \overline{\nabla}_{\boldsymbol{\nu}} X, \boldsymbol{\nu} \right\rangle \\ &= \operatorname{div}_{M_t} X + \left\langle \overline{\nabla}_{\boldsymbol{\nu}} X, \boldsymbol{\nu} \right\rangle\end{aligned}$$

Which leads to this formula for the Laplacian

$$\begin{aligned}\Delta f &= \operatorname{div}_{\mathbb{R}^{n+1}} \overline{\nabla} f - \left\langle \overline{\nabla}_{\boldsymbol{\nu}} \left(\overline{\nabla} f\right), \boldsymbol{\nu} \right\rangle + \left\langle \mathbf{H}, \overline{\nabla} f \right\rangle \\ &= \Delta_{\mathbb{R}^{n+1}} f - \overline{\nabla}^2 f(\boldsymbol{\nu}, \boldsymbol{\nu}) + \left\langle \mathbf{H}, \overline{\nabla} f \right\rangle\end{aligned}$$

Equivalently, due to our expedient choice of basis, we may also calculate the Laplacian as

$$\Delta f = \boldsymbol{\tau}_i \boldsymbol{\tau}_i(f)$$

The total derivative of a function $f : M_t \to \mathbb{R}$ is defined as

$$\begin{aligned}\frac{df}{dt}(\mathbf{x}, t) &= \frac{\partial}{\partial t} f(F(p,t), t) \\ &= \frac{\partial f}{\partial t}(\mathbf{x}, t) + \left\langle \overline{\nabla} f(\mathbf{x}, t), \mathbf{H} \right\rangle\end{aligned}$$

thus

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \left\langle \mathbf{H}, \overline{\nabla} f \right\rangle$$

A.3 Identities

Proposition A.2 (Codazzi Equations)**.** *The second fundamental form is totally symmetric on M, i.e. it satisfies*

$$\boldsymbol{\tau}_i(h_{jk}) = \boldsymbol{\tau}_k(h_{ij}) = \boldsymbol{\tau}_j(h_{ki})$$

Proposition A.3 (Gauss Equations)**.** *On M, the Riemann tensor satisfies*

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}$$

Proposition A.4 (Simons' Identity)**.**

$$\Delta h_{ij} = \nabla_i \nabla_j H + H h_{il} g^{lm} h_{mj} - |A|^2 h_{ij} \tag{A.1}$$

Proof. Computing in normal coordinates, we find

$$\begin{aligned}
\Delta h_{ij} &= \boldsymbol{\tau}_k \boldsymbol{\tau}_k (h_{ij}) \\
&= \boldsymbol{\tau}_k \boldsymbol{\tau}_i (h_{kj}) && \text{by Codazzi} \\
&= \boldsymbol{\tau}_k \boldsymbol{\tau}_i (h_{jk}) && \text{by symmetry} \\
&= \boldsymbol{\tau}_i \boldsymbol{\tau}_k (h_{jk}) + R_{kijm} h_{mk} + R_{kikm} h_{jm} \\
&= \boldsymbol{\tau}_i \boldsymbol{\tau}_j (h_{kk}) + (h_{kj} h_{im} - h_{km} h_{ij}) h_{mk} \\
&\quad + (h_{kk} h_{im} - h_{km} h_{ik}) h_{jm} && \text{by Gauss equations} \\
&= \boldsymbol{\tau}_i \boldsymbol{\tau}_j (H) + H h_{ik} h_{kj} - |A|^2 h_{ij}
\end{aligned}$$

□

Lemma A.5 (Gauss-Weingarten Relations)**.** *Let $\{\boldsymbol{\tau}_i\}_{i=1}^n$ be a smooth frame field on M, then we have the relations*

(i) $\overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\nu} = h_{ik} g^{kl} \boldsymbol{\tau}_l$

(ii) $\overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\tau}_j = -h_{ij} \boldsymbol{\nu} + \left(\overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\tau}_j\right)^{\top}$

Proof. Relation (i) follows easily from the definition of h_{ij} and relation (ii) follows from resolving $\overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\tau}_j$ onto the tangent space, then using the definition of h_{ij}. Note that in normal coordinates, $g_{ij} = \delta_{ij}$ and $\left(\overline{\nabla}_{\boldsymbol{\tau}_i} \boldsymbol{\tau}_j\right)^{\top}$ vanishes at $\mathbf{x}$. □

We also have in normal coordinates

Proposition A.6.

$$\overline{\nabla}_{\boldsymbol{\tau}_i} \overline{\nabla}_{\boldsymbol{\tau}_j} \boldsymbol{\nu} = \nabla h_{ij} - h_{ik} h_{kj} \boldsymbol{\nu}$$

Proof. By GW(i), we compute

$$\begin{aligned}
\overline{\nabla}_{\boldsymbol{\tau}_i}\overline{\nabla}_{\boldsymbol{\tau}_j}\boldsymbol{\nu} &= \overline{\nabla}_{\boldsymbol{\tau}_i}\left(h_{jk}\boldsymbol{\tau}_k\right) \\
&= \boldsymbol{\tau}_k\overline{\nabla}_{\boldsymbol{\tau}_i}h_{jk} + h_{jk}\overline{\nabla}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_k \\
&= \boldsymbol{\tau}_k\overline{\nabla}_{\boldsymbol{\tau}_k}h_{ij} - h_{jk}h_{ik}\boldsymbol{\nu} \quad \text{by Codazzi and GW(ii)} \\
&= \nabla h_{ij} - h_{ik}h_{kj}\boldsymbol{\nu}
\end{aligned}$$

□

Note that Proposition A.6 implies

Proposition A.7. *The unit normal to M satisfies*

$$\Delta\boldsymbol{\nu} = \nabla H - |A|^2\boldsymbol{\nu} \tag{A.2}$$