Chapter 5

Self-Similar Solutions

In this chapter, we shall investigate the possible self-similarly shrinking and expanding solutions to (MCF).

5.1 Homothetic Solutions

Suppose that we have a self-similar solution to (MCF). That is, our surface is evolving homothetically, i.e. for some $\lambda : [0,T) \to [0,\infty)$ we have

$$M_t = \lambda(t) M_0 \tag{5.1}$$

Equivalently, up to tangential diffeomorphisms (see Appendix D) we consider the family of immersions, as in [9]

$$\tilde{\mathbf{F}}(\boldsymbol{q},t) = \lambda(t)\tilde{\mathbf{F}}(\boldsymbol{q},0) \tag{5.2}$$

evolving by the equation

$$\left(\frac{d\tilde{\mathbf{F}}}{dt}(\boldsymbol{q},t)\right)^{\perp} = \mathbf{H}\left(\tilde{\mathbf{F}}(\boldsymbol{q},t)\right), \quad (\boldsymbol{q},t) \in M_r^n \times [0,T) \tag{5.3}$$

with $M_t = \tilde{\mathbf{F}}(\cdot,t)(M_r^n)$.

Immediately, using (5.2) with (5.3) we obtain the equation

$$\frac{d\lambda}{dt}(t)\left(\tilde{\mathbf{F}}(\boldsymbol{q},0)\right)^{\perp} = \frac{1}{\lambda(t)}\mathbf{H}\left(\tilde{\mathbf{F}}(\boldsymbol{q},0)\right) \tag{5.4}$$

using the fact that the mean curvature vector scales homothetically like $\frac{1}{\lambda}$ for the immersions (5.2). From this equation, we may infer that the quantity

$$\alpha = \frac{d\lambda^2}{dt}(t) = 2\frac{d\lambda}{dt}(t)\lambda(t)$$

is independent of time. So, integrating in time from t_0 to t and assuming that $\lambda(t_0) = 1$ we obtain

$$\lambda(t) = \sqrt{1 + \alpha(t - t_0)} \tag{5.5}$$

We can see that if $\alpha > 0$ we would have a self-similarly expanding solution to (MCF), defined for all positive times. The self-similarly shrinking solutions are those with $\alpha < 0$, which only exist on a finite time interval.

Combining (5.2), (5.4) and (5.5) we obtain, setting $\mathbf{x} = \tilde{\mathbf{F}}(\boldsymbol{q}, t)$, the equation

$$\mathbf{H}(\mathbf{x}) = \frac{1}{2}\left(\frac{\alpha}{\lambda^2}\right)\mathbf{x}^{\perp} \tag{5.6}$$

Now suppose we were to normalise our solution using a similar rescaling to that introduced in Chapter 2. Let us define new space and time variables,

$$\tilde{\mathbf{x}}(s) = \psi(t)\mathbf{x}(t), \quad t \in [t_0, T)$$

and

$$s = \xi \log \lambda(t), \quad t \in [t_0, T)$$

where $\psi(t) = \sqrt{\frac{\xi\alpha}{2}}\lambda^{-1}(t)$, $\xi = \operatorname{sign}\alpha$ and

$$T = \begin{cases} t_0 + \frac{\xi}{\alpha}, & \alpha < 0 \\ \infty, & \alpha > 0 \end{cases}$$

The normalisation has the effect of scaling out the homothety while extending the (finite, in the case $\alpha < 0$) time interval to be defined for all positive times s.

This gives us a normalised flow $(\tilde{M}_s)_{s\in[0,\infty)}$ defined by

$$\frac{d\tilde{\mathbf{x}}}{ds} = \tilde{\mathbf{H}} - \xi\tilde{\mathbf{x}} \tag{5.7}$$

Since the homothety has been scaled out, we obtain $\tilde{M} = \sqrt{\frac{\xi\alpha}{2}}M_0$ as a stationary solution, satisfying the equation

$$\tilde{\mathbf{H}}(\tilde{\mathbf{x}}) = \xi\tilde{\mathbf{x}}^{\perp}, \quad \tilde{\mathbf{x}} \in \tilde{M} \tag{5.8}$$

holds.

Proposition 5.1. *Let M be a rotationally symmetric cylindrical graph upon which the equations*

$$H(\mathbf{x}) = -\langle \xi\mathbf{x}, \boldsymbol{\nu}\rangle, \quad \mathbf{x} \in M$$

holds, where $\xi^2 = 1$, then we have the following equations on M

(i) $\Delta v = |A|^2 v + 2v^{-1}|\nabla v|^2 - \left(\frac{n-1}{u^2}\right) v - \rho^{-1}\langle \nabla\rho, \nabla v\rangle$

(ii) $\Delta u = \xi u + \frac{n-1}{u} - \rho^{-1}\langle \nabla\rho, \nabla u\rangle$

(iii) $\Delta H = -\xi H - H|A|^2 - \rho^{-1}\langle \nabla\rho, \nabla H\rangle$

(iv) $\Delta |A|^2 = -2\xi|A|^2 - 2|A|^4 + 2|\nabla A|^2 - \rho^{-1}\left\langle \nabla\rho, \nabla |A|^2\right\rangle$

where $\rho(\mathbf{x}) = e^{\frac{\xi}{2}|\mathbf{x}|^2}$.

Proof. First, we begin by computing

$$\nabla H = -\langle \xi\mathbf{x}, \boldsymbol{\tau}_i\rangle\, \overline{\nabla}_{\boldsymbol{\tau}_i}\boldsymbol{\nu} \tag{5.9}$$

To prove (i), refer to the computation (C.14) in Chapter C. For a rotationally symmetric cylindrical graph, this gives us

$$\Delta v = |A|^2 v + 2v^{-1}|\nabla v|^2 - \left(\frac{n-1}{u^2}\right) v - v^2\langle \nabla H, \boldsymbol{\omega}\rangle$$

Also, we calculate

$$\begin{aligned}
-v^2\langle \nabla H, \boldsymbol{\omega}\rangle &= v^2\langle \xi\mathbf{x}, \boldsymbol{\tau}_i\rangle \left\langle \overline{\nabla}_{\boldsymbol{\tau}_i}\boldsymbol{\nu}, \boldsymbol{\omega}\right\rangle \\
&= -\langle \xi\mathbf{x}, \nabla v\rangle \\
&= -\rho^{-1}\langle \nabla\rho, \nabla v\rangle
\end{aligned}$$

since $\nabla\rho = \xi\rho\mathbf{x}^\top$. Equation (i) then follows.

For (ii), again refer to Chapter C, this time to calculation (C.6), which gives us

$$\Delta u = \frac{n-1}{u} + \langle \mathbf{H}, \boldsymbol{\omega}\rangle$$

and attacking the last term, we find

$$\begin{aligned}
\langle \mathbf{H}, \boldsymbol{\omega}\rangle &= -Hv^{-1} \\
&= \langle \xi\mathbf{x}, \langle \boldsymbol{\omega}, \boldsymbol{\nu}\rangle \boldsymbol{\nu}\rangle \\
&= \left\langle \xi\mathbf{x}, \boldsymbol{\omega} - \boldsymbol{\omega}^\top\right\rangle \\
&= \xi u - \rho^{-1}\langle \nabla\rho, \nabla u\rangle
\end{aligned}$$

since $\nabla u = \boldsymbol{\omega}^\top$. The second equation then follow immediately.

Equation (iii) we obtain by differentiating equation (5.9), i.e.

$$\begin{aligned}
\Delta H &= \operatorname{div}_M \nabla H \\
&= -\left\langle \overline{\nabla}_{\boldsymbol{\tau}_j}\left(\langle \xi\mathbf{x}, \boldsymbol{\tau}_i\rangle \overline{\nabla}_{\boldsymbol{\tau}_i}\boldsymbol{\nu}\right), \boldsymbol{\tau}_j\right\rangle \\
&= -\left(\xi\langle \boldsymbol{\tau}_i, \boldsymbol{\tau}_j\rangle + \left\langle \xi\mathbf{x}, \overline{\nabla}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_j\right\rangle\right)\left\langle \boldsymbol{\tau}_j, \overline{\nabla}_{\boldsymbol{\tau}_i}\boldsymbol{\nu}\right\rangle - \langle \xi\mathbf{x}, \boldsymbol{\tau}_i\rangle \left\langle \overline{\nabla}_{\boldsymbol{\tau}_i}\overline{\nabla}_{\boldsymbol{\tau}_j}\boldsymbol{\nu}, \boldsymbol{\tau}_j\right\rangle \\
&= -\xi H - H|A|^2 - \rho^{-1}\langle \nabla\rho, \nabla H\rangle
\end{aligned}$$

where we have used the Gauss-Weingarten relations and the Codazzi equations.

Finally, equation (iv) is proved using the Simons identity (see Proposition A.4), which gives us

$$\Delta|A|^2 = -2|A|^4 + 2|\nabla A|^2 + 2Hh_{ik}h_{kj}h_{ij} + 2h_{ij}\boldsymbol{\tau}_i\boldsymbol{\tau}_j(H)$$

The last term, we may compute as

$$\begin{aligned} 2h_{ij}\boldsymbol{\tau}_i\boldsymbol{\tau}_j(H) &= -2h_{ij}\boldsymbol{\tau}_i\left(\langle \xi\mathbf{x}, \boldsymbol{\tau}_k\rangle h_{kj}\right) \\ &= -2\xi h_{ij}h_{ij} - 2h_{ij}\langle \xi\mathbf{x}, -h_{ik}\boldsymbol{\nu}\rangle h_{kj} - \left\langle \xi\mathbf{x}, \nabla|A|^2\right\rangle \\ &= -2\xi|A|^2 - 2Hh_{ij}h_{jk}h_{ik} - \rho^{-1}\left\langle \nabla\rho, \nabla|A|^2\right\rangle \end{aligned}$$

and identity (iv) follows. □

5.2 Self-Similarly Shrinking Solutions

Suppose that time T is the cut-off time for the flow, i.e. $\lambda(T) = 0$. Thus we have $\alpha = -\frac{1}{T}$ and we obtain

$$\lambda(t) = \sqrt{\frac{T-t}{T}}, \quad t \in [0, T) \tag{5.10}$$

so, we have a self-similarly shrinking solution to (MCF), vanishing at time T.

5.3 Self-Similar Expanding Solutions

Suppose that a homothetic surface M_t is self-similarly expanding, i.e. $\alpha > 0$, then we have

$$\lambda(t) = \sqrt{1+\alpha t}, \quad t \in [0, \infty) \tag{5.11}$$

5.3.1 Properties of Rotationally Symmetric Self-Similar Expanders

Consider the initial-value problem

$$\begin{cases} \rho''(z) = (1 + (\rho'(z))^2)\left(\frac{n-1}{\rho(z)} - (z\rho'(z) - \rho(z))\right), z \in \mathbb{R} \\ \rho(0) = \rho_0 > 0 \\ \rho'(0) = 0 \end{cases} \tag{5.12}$$

This equation may be obtained by expressing $\tilde{M}$ as a rotationally symmetric graph, i.e. $\tilde{\mathbf{x}} = \rho(z)\boldsymbol{\omega} + z\boldsymbol{\vartheta}$ where $z = \langle \mathbf{x}, \boldsymbol{\vartheta}\rangle$, and computing the terms in (5.8) under this assumption.

Standard theory gives us local existence and uniqueness of solutions to this problem. What we want to show are some properties of solutions to this problem. We shall also show that solutions to this problem are entire.

To make things clearer, we will introduce some notation and group some fundamental terms. Set $f(z) = z\rho'(z) - \rho(z)$, and $\varphi(z) = (\rho'(z))^2$, then the equation becomes

$$\rho'' = (1+\varphi)\left(\frac{n-1}{\rho} - f\right) \tag{5.13}$$

It will suffice to consider this equation for $z \geqslant 0$, since it is invariant under reflections about the origin.

Now, at the origin, $f(0) = -\rho_0 < 0$, so by continuity, we have that $f(z) < 0, z \in [0,\delta)$ for some $\delta > 0$. Thus, by (5.13), we have $\rho''(z) > 0, z \in [0,\delta)$, and integrating this, we obtain $\rho'(z) > 0, z \in (0,\delta)$.

To show that ρ is monotonically increasing for all $z > 0$, assume that there exists a point $z^* < \infty$ such that $\rho'(z^*) = 0$ and $\rho'(z) > 0, z \in (0, z^*)$. Thus, by the mean value theorem, there exists a $\delta > 0$ such that $\rho''(z) < 0, z \in (z^* - \delta, z^*)$. However, continuity of ρ' and Equation (5.12) implies the existence of an $\varepsilon > 0$ such that for $z \in (z^* - \varepsilon, z^*)$ we have

$$\rho''(z) = (1 + (\rho'(z))^2)\left(\frac{n-1}{\rho(z)} - (z\rho'(z) - \rho(z))\right) > 0$$

which is a contradiction. Hence, we have that $\rho'(z) > 0, z > 0$.

Multiplying (5.12) by ρ' and we obtain the inequality

$$\frac{\varphi'}{1+\varphi} \leqslant \left(\log \rho^{2(n-1)} + \rho^2\right)', \quad z > 0$$

from which we obtain the estimate

$$\rho'(z) \leqslant \left(\frac{\rho}{\rho_0}\right)^{n-1} e^{\frac{1}{2}(\rho^2 - \rho_0^2)} - 1, \quad z \geqslant 0$$

Now, we want to derive some bounds upon the behaviour of f. Notice that $f'(z) = z\rho''(z)$, giving us the equation

$$f' = z(1+\varphi)\left(\frac{n-1}{\rho} - f\right) \tag{5.14}$$

Clearly, since $f(0) = -\rho_0 < 0$ we have that there exists a maximal finite $z_0 > 0$ such that

$$f'(z) > 0, \quad z \in (0, z_0]$$

with $f(z_0) = 0$.

At z_0, we see from (5.14) that $f'(z_0) > 0$, so f is still increasing. We also see on the (bounded, since ρ is increasing) interval $[z_0, z_1)$ upon which

$f(z) < \frac{n-1}{\rho(z)}$, we still have $f'(z) > 0$. At z_1, we have $f(z_1) = \frac{n-1}{\rho(z_1)}$. What happens next?

Let us compute

$$\frac{d}{dz}\bigg|_{z=z_1} \left(\frac{n-1}{\rho} - f\right)(z) = -\left(\frac{n-1}{\rho^2(z_1)}\right)\rho'(z_1)$$
$$< 0$$

so we have

$$\left(\frac{n-1}{\rho} - f\right)(z) < \left(\frac{n-1}{\rho} - f\right)(z_1) = 0$$

for $z \in (z_1, z_1 + \delta)$ for some $\delta > 0$.

Extending this result, we see that for all $z > z_1$, $f(z) > \frac{n-1}{\rho(z)}$. We also see that on this interval $f' < 0$, so we conclude that

$$\lim_{z\to\infty} f(z_1 + z) = 0^+$$

Thus, since asymptotically we have $\rho'' < 0, \rho' > 0$ for $z > z_1$ and in some sense,

$$\rho(z) \to z\rho'(z)$$

More precisely, integrating $f(z) = z\rho'(z) - \rho(z)$, we have

$$\rho(z) = C\left(1 + \int_{z_1}^{z} \frac{f(y)}{y^2} dy\right) z, \quad z > z_1$$

for some $C > 0$. Since $f' < 0$ and thus f is bounded, it is clear that the solution becomes asymptotically linear at infinity, since the integral converges. An unsolved problem is getting an estimate on asymptotic slope of ρ; interestingly, numerical results suggest that smallest possible asymptotic slope is bounded below by some constant depending only on n (see Remark 5.4).

We haven't yet solved the general problem where $\rho'(0) \neq 0$. This case can be proven in much the same way as above, with a few modifications. The case where $\rho'(0) > 0$ is easy, and is covered by the above method, since as soon as $\rho'(z) > 0$ for any $z \geqslant 0$ we must have that it remains so for all $z > 0$. So, all we need to do is cover the case when $\rho'(0) < 0$.

We want to show that $\delta = \sup\{z : \rho(z) > 0, \rho'(z) < 0\}$ exists and is finite and that at this point we have $\rho(\delta) > 0$ and $\rho'(\delta) = 0$ so that we may proceed as above. That $\delta > 0$ is guaranteed by the initial data and continuity.

Setting $h(z) = \rho'(z)$ and $g(z) = \frac{n-1}{\rho(z)} + \rho(z)$, we have from the ODE,

$$h'(z) > g(z) - zh(z), \quad z \in [0, \delta)$$

or, setting $q(z) = e^{\frac{z^2}{2}} h(z)$ we have

$$q'(z) > e^{\frac{z^2}{2}} g(z), \quad z \in [0, \delta)$$

and integrating this, we obtain

$$h(z) > e^{-\frac{z^2}{2}} \left(h(0) + \int_0^z g(y) e^{\frac{y^2}{2}} dy \right), \quad z \in [0, \delta)$$

or

$$\rho'(z) > e^{-\frac{z^2}{2}} \left(\rho'(0) + \int_0^z \left(\frac{n-1}{\rho(y)} + \rho(y) \right) e^{\frac{y^2}{2}} dy \right), \quad z \in [0, \delta)$$

from which we easily see that before ρ may reach zero, we must have $\rho' = 0$, and that the point at which this occurs is δ. It is also easily seen that this δ is finite. Now once more, we may proceed as above and show that the solution is asymptotically linear.

The above arguments give us the following result for solutions to 5.12

Theorem 5.2. *Let ρ be a solution to the ODE problem (5.12), then ρ is asymptotically linear.*

5.3.2 Solutions out of Cones

Recall that (MCF) is equivalent (up to tangential diffeomorphisms) to the scalar evolution equation

$$\frac{\partial \Phi_\rho}{\partial t}(\mathbf{x}, t) = |\overline{\nabla} \Phi_\rho(\mathbf{x}, t)| \operatorname{div}_{\mathbb{R}^{n+1}} \left(\frac{\overline{\nabla} \Phi_\rho(\mathbf{x}, t)}{|\overline{\nabla} \Phi_\rho(\mathbf{x}, t)|} \right), \quad \mathbf{x} \in M_t, t \in [0, T) \quad (5.15)$$

where the zero level sets of Φ_ρ define the evolving surface, i.e.

$$M_t = \Phi_\rho^{-1}(\cdot, t)\{0\}$$

Using a similar argument to that in [20] we shall show that for solutions with initial data equal to a ‘skewed cone’ we obtain self-similar solutions to mean curvature flow. These solutions are important because they provide useful barriers with which to derive further estimates.

If we let Φ_ρ^λ be defined by

$$\Phi_\rho^\lambda(\mathbf{x}, t) = \frac{1}{\lambda} \Phi_\rho(\lambda \mathbf{x} . \lambda^2 t) \quad (5.16)$$

for any $\lambda > 0$, then we easily see that Φ_ρ^λ still satisfies the level set equation.

To consider graphs over a given surface, we set

$$\Phi_\rho(\mathbf{x}, t) = \Lambda(\mathbf{x}) - \rho(\mathbf{S}(\mathbf{x}), t)$$

where Λ and $\mathbf{S}$ are the signed distance function and closest point projection of M_r^n respectively. This allows us to describe graphs over M_r^n in terms of a height function ρ.

Remark 5.3. For a cylindrical graph (i.e. $M_r^n = C_r^n$), we have

$$\Lambda(\mathbf{x}) = |\mathbf{x}_{\perp\boldsymbol{\vartheta}}| - r$$

and

$$\mathbf{S}(\mathbf{x}) = r\left(\frac{\mathbf{x}_{\perp\boldsymbol{\vartheta}}}{|\mathbf{x}_{\perp\boldsymbol{\vartheta}}|}\right) + \langle \mathbf{x}, \boldsymbol{\vartheta}\rangle\, \boldsymbol{\vartheta}$$

Suppose that we have initial data for equation (5.15) satisfying

$$\Phi_{\rho_0}(\lambda\mathbf{x}) = \lambda\Phi_{\rho_0}(\mathbf{x}) \tag{5.17}$$

for all $\lambda > 0$ and all $\mathbf{x} \in M_0$, i.e. M_0 is a cone 'over M_r^n'.

It is clear that graphs over an arbitrary surface can not necessarily satisfy this equation. The required property is that the signed distance function Λ for M_r^n satisfies

$$\Lambda(\lambda\mathbf{x}) = \lambda\Lambda(\mathbf{x})$$

for all $\lambda > 0$. Examples of surfaces that do satisfy this property are spheres, cylinders and planes (centered about the origin).

For a cylindrical graph, Equation (5.16) requires that ρ_0 is linear in the distance $z = \langle \boldsymbol{q}, \boldsymbol{\vartheta}\rangle$ along the axis, i.e. for $\boldsymbol{q} = \boldsymbol{q}_{\perp\boldsymbol{\vartheta}} + \langle \boldsymbol{q}, \boldsymbol{\vartheta}\rangle\, \boldsymbol{\vartheta}$

$$\rho_0(\boldsymbol{q}_{\perp\boldsymbol{\vartheta}} + \lambda\,\langle \boldsymbol{q}, \boldsymbol{\vartheta}\rangle\, \boldsymbol{\vartheta}) = \lambda\rho_0(\boldsymbol{q})$$

for all $\boldsymbol{q} \in C_r^n$.

For $\lambda > 0$, set

$$\Phi_{\rho_0}^{\lambda}(\mathbf{x}) = \frac{1}{\lambda}\Phi_{\rho_0}(\lambda\mathbf{x})$$

then

$$\Phi_{\rho}^{\lambda}(\mathbf{x}, t) = \frac{1}{\lambda}\Phi_{\rho}(\lambda\mathbf{x}, \lambda^2 t)$$

satisfies equation (5.15) with initial data $\Phi_{\rho_0}^{\lambda}$.

Notice however that $\Phi_{\rho_0}^{\lambda} = \Phi_{\rho_0}$ for all $\lambda > 0$, due to the linearity of ρ_0 and M_r^n. Furthermore, assuming solutions to Equation (5.15) are unique, an issue we shall return to, then this implies that

$$\Phi_{\rho}^{\lambda}(\mathbf{x}, t) = \Phi_{\rho}(\mathbf{x}, t)$$

Moreover, in view of the linearity of the signed distance function Λ we have

$$\rho(\mathbf{S}(\mathbf{x}), t) = \frac{1}{\lambda}\rho(\mathbf{S}(\lambda\mathbf{x}), \lambda^2 t)$$

Setting $\lambda = \frac{1}{\sqrt{2t}}$ we have

$$\rho(\mathbf{S}(\mathbf{x}), t) = \sqrt{2t}\rho\left(\mathbf{S}\left(\frac{\mathbf{x}}{\sqrt{2t}}\right), \frac{1}{2}\right), \quad t > 0$$

and more specifically, for cylindrical graphs, setting $z = \langle \mathbf{x}, \boldsymbol{\vartheta} \rangle$ we have

$$\rho(z,t) = \sqrt{2t}\rho\left(\frac{z}{\sqrt{2t}}, \frac{1}{2}\right), \quad t > 0$$

which gives us a self-similarly expanding solution M_t, with $M_{\frac{1}{2}}$ satisfying Equation (5.8) with $\xi = 1$.

Now, let us return to the issue of uniqueness of solutions of (5.15). In the case of planar graphs, the equation is uniformly parabolic over all of $\mathbb{R}^{n+1}$ so existence and uniqueness is not such an issue. However, in the cylindrical graph case, the equation is non-uniformly parabolic, and conical initial data in the cylindrical case has an isolated point through the region in which (5.15) is non-parabolic (i.e. directly through the cylinder axis).

We shall not properly address the problem of existence and uniqueness of solutions to this problem, however we do give some heuristic arguments for existence by way of approximating the initial data with hyperboloids.

Consider M^n_ε, the family of 'skewed hyperboloids', defined by

$$M^n_\varepsilon = \varphi^{-1}(\cdot)\{\varepsilon\}, \quad \varepsilon > 0$$

where

$$\varphi(\mathbf{x}) = \sqrt{|\mathbf{x}_{\perp\boldsymbol{\vartheta}}|^2 - \left\langle \mathbf{x}, r^{\gamma,\beta}_\delta(\mathbf{x})\boldsymbol{\vartheta}\right\rangle^2}$$

and $r^{\gamma,\beta}_\delta(\mathbf{x}) = r^{\gamma,\beta}_\delta(\langle \mathbf{x}, \boldsymbol{\vartheta}\rangle)$ is the smooth function defined by

$$r^{\gamma,\beta}_\delta(z) = (\beta - \gamma)\int_{-\delta}^{z} \eta_\delta(y)dy + \gamma, \quad z \in \mathbb{R}$$

with $\beta \geqslant \gamma > 0$, $\eta_\delta(z) = \frac{1}{\delta}\eta\left(\frac{z}{\delta}\right)$ for $\delta > 0$ and η is the smooth function defined by

$$\eta(z) = \begin{cases} Ce^{-\frac{1}{1-z^2}}, & |z| < 1 \\ 0, & |z| \geqslant 1 \end{cases}$$

with $C = \left(\int_{-1}^{1} e^{-\frac{1}{1-y^2}} dy\right)^{-1}$.

The surfaces M^n_ε are called skewed since whenever $\beta \neq \gamma$ the angle of the cone to which the hyperboloids are asymptotic is different on either side of the origin. Whenever $\beta = \gamma$, then we recover the usual hyperboloids.

It is clear that as we let ε (which controls the neck width) and δ (which controls the rapidity of the skew) tend to zero, the skewed hyperboloids converge to skewed cones, that is, two (opposing) cones with vertex at the origin and axis in the direction of $\boldsymbol{\vartheta}$.

The mean curvature $H_{H^n_\varepsilon}$ of the skewed hyperboloid is given by

$$H_{M^n_\varepsilon}(z) = \frac{n-1}{(\varepsilon^2 + (1 + (r^{\gamma,\beta}_\delta + z r^{\gamma,\beta\prime}_\delta)^2)(r^{\gamma,\beta}_\delta z)^2)^{\frac{1}{2}}} - \frac{\varepsilon^2 (r^{\gamma,\beta}_\delta + z r^{\gamma,\beta\prime}_\delta)^2 + (z r^{\gamma,\beta}_\delta)(2 r^{\gamma,\beta\prime}_\delta + z r^{\gamma,\beta\prime\prime}_\delta)(\varepsilon^2 + (z r^{\gamma,\beta}_\delta)^2)}{(\varepsilon^2 + (1 + (r^{\gamma,\beta}_\delta + z r^{\gamma,\beta\prime}_\delta)^2)(z r^{\gamma,\beta}_\delta)^2)^{\frac{3}{2}}}$$

where we have denoted $z = \langle \mathbf{x}, \boldsymbol{\vartheta} \rangle$. It is clear that the global minimum height occurs at those points at which $z = 0$. Note that at those points at which $z = 0$, we have

$$H_{M^n_\varepsilon}(0) = \frac{1}{\varepsilon}\left(n - 1 - \left(\frac{\beta+\gamma}{2}\right)^2\right) \tag{5.18}$$

and thus, for fixed $\beta, \gamma > 0$ such that $\beta + \gamma > 2\sqrt{n-1}$, we have that for all positive δ and ε there exists a neighbourhood about those points, upon which $H_{M^n_\varepsilon} < 0$.

This means that under (MCF), for a short time, on any of the approximating hyperboloids the neck won't immediately move towards the axis; in fact, it will move outwards for a short time. This makes it possible for us to use the (sufficiently steep) skewed hyperboloids to approximate (sufficiently steep) arbitrary cones, and thus use them as initial data for (MCF).

The interior estimates of Chapter 3 imply short time existence on each of the approximating hyperboloids. However, as the hyperboloids are limited to the cone, the existence time could potentially dwindle to zero, though it is clear from (5.18) that as ε approaches zero the speed with which the neck tries (at least initially) to escape the origin is increasingly faster.

Further work needs to be done here to ensure that this limiting process as $\varepsilon \to 0$ is valid and converges to a unique solution of (5.15). This result would allow us to take as initial data arbitrarily steep cones, obtaining arbitrarily steep self-similarly expanding solutions. This property of arbitrary steepness for self-similarly expanding solutions will be important in the following section, and thus we will be forced to assume it. The following remark gives some evidence to support this assumption.

Remark 5.4. The steepness condition which ensures that the approximating skewed hyperboloids have a region of negative mean curvature around the cone vertex appears to be necessary. Numerical modelling of the ODE (5.12) implies that for the no-skew case ($\rho'(0) = 0$), on the space of initial data, there is a minimum opening angle for the solutions (see Figure 5.4).

While there is no maximum opening angle; the solutions can be made to be arbitrarily steep by choosing ρ_0 sufficiently small or sufficiently large, there does appear to be a minimum (depending only on n), and this does

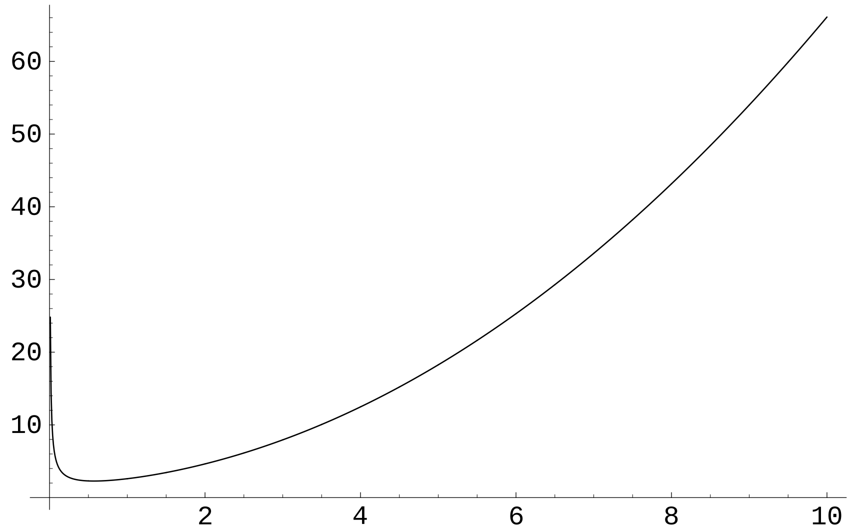

Figure 5.1: Neck Height vs Asymptotic Slope - $n = 2$

appear to be directly related to the steepness condition required to generate self-similarly expanding solutions from cones.

One further effect that is seen in the numerical results is the 'flattening' of the solution near the origin; the solution crosses the cone to which it is asymptotic. For an example of this, see Figure 5.4. This seems to be related to the steepness condition required to obtain a region of negative curvature around the neck, which lifts the solution near the origin, while the positive curvature on the 'arms' pushes the solution down, hence the flattening effect.

5.4 Convergence to Self-Similar Expanders

In [10], Ecker and Huisken obtain a result for planar graphs which are, in some way, growing linearly at infinity, giving convergence to self-similarly expanding solutions under a suitable normalisation. In this section, we intend to prove a similar result for cylindrical graphs.

In the cylindrical setting, the problem is compounded by the extra terms introduced by the curvature (which is not present in the planar case) of the base manifold which make it difficult to obtain suitable gradient bounds which, under rescaling, are uniform in time.

In Chapter 3, we developed local and global estimates for the gradient and curvature, assuming only that the height was bounded below uniformly. This unfortunately led to global gradient bounds which grew exponentially in time, whereas we require a bound on the gradient that is uniform in time.

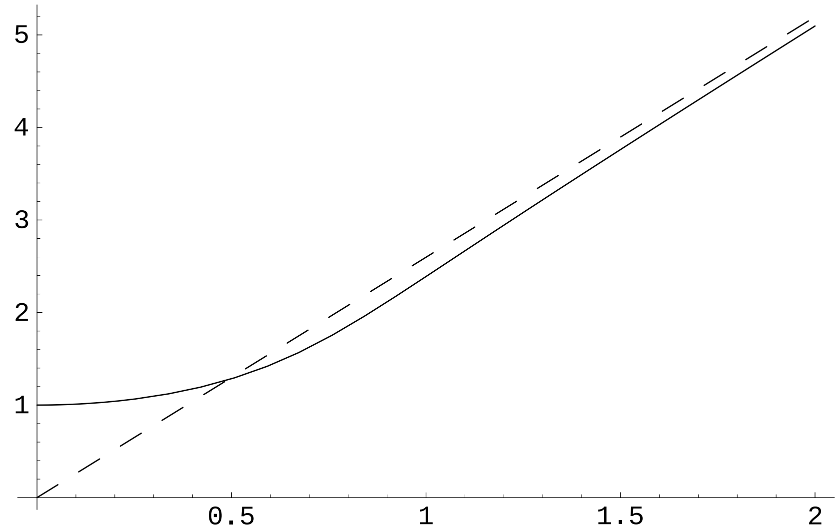

Figure 5.2: Numerical Solution - $n = 2, \rho_0 = 1$

5.4.1 Time Independent Estimates

The following definition gives the class of initial surfaces for which we shall be investigating long-time convergence.

Definition 5.5. Suppose for some $u_0, \beta_0 > 0$, that we have

$$u(\mathbf{x}, 0) \geqslant \sqrt{u_0^2 + \langle \beta_0 \mathbf{x}, \boldsymbol{\vartheta} \rangle^2}, \quad \mathbf{x} \in M_0$$

with β_0 large enough (see Remark 5.4) such that a self-similarly expanding cylindrical graph solution exists 'inside' M_0 then we say that M_0 is self-similarly bounded below.

The motivation for the above definition is clear when we consider the self-similar expanding solutions to Equation (5.12), $\rho(z, t) = \lambda(t)\rho(\lambda^{-1}(t)z)$ where $\lambda = \sqrt{1 + \alpha t}$, for some speed of expansion $\alpha > 0$.

The comparison principle (Theorem E.7) may be applied to the evolution of M_t and the self-similarly expanding solution, since the condition $\beta_0 \leqslant \beta$ allows a cone to be initially placed between the solutions, which satisfies the comparison principle hypothesis that the solutions be initially separated in a particular way (see Theorem E.7 for details). This yields the lower bound

$$u(\mathbf{x}, t) \geqslant \rho(\mathbf{x}, t), \quad \mathbf{x} \in M_t, t \in [0, T)$$

where we have set $z = \langle \mathbf{x}, \boldsymbol{\vartheta} \rangle$ and $\rho(\mathbf{x}, t) = \rho(z, t)$.

Furthermore, since the self-similarly expanding solution becomes asymptotically linear, we have the estimate

$$\rho(\mathbf{x}, t) \geqslant \sqrt{\varepsilon_0^2 + \langle \beta_0 \mathbf{x}, \boldsymbol{\vartheta} \rangle^2 + 2\beta_0^2(2\gamma_0 + 1)t}, \quad \mathbf{x} \in M_t, t \in [0, T) \qquad (5.19)$$

for some $\varepsilon_0, \gamma_0 > 0$. Note that potentially, the slope parameter β_0 might have to be made just slightly smaller to fit this bound. The speed of expansion, α, of the self-similar surface has been arbitrarily set to $2\beta_0^2(2\gamma_0+1)$ purely for computational convenience.

Since the minimum height of the self-similar solution ρ is always increasing (it is self-similarly expanding, after all), using the extension theorem, Theorem 4.2, which basically states that the only thing preventing long-time existence is the formation of a singularity due to height going to zero, we actually obtain the estimate for all $t > 0$, that is

$$u(\mathbf{x},t) \geqslant \sqrt{\varepsilon_0^2 + \langle \beta_0 \mathbf{x}, \boldsymbol{\vartheta}\rangle^2 + 2\beta_0^2(2\gamma_0+1)t}, \quad \mathbf{x} \in M_t, t \in [0,\infty) \tag{5.20}$$

However, while we have extended the solution to exist for all time, we are moving towards another goal: convergence. That is, the derivation of initial conditions such solutions converge to a self-similarly expanding solution. A crucial ingredient of obtaining a convergence result will be a uniform estimate for the gradient function, something the local and global estimates of Chapter 3 failed to achieve.

Let us extend the cutoff function as defined in Lemma 3.4 using the fact that we're on a rotationally symmetric surface to squeeze a little more out of the right-hand side.

Lemma 5.6. *Let $\eta = \eta(\mathbf{x},t)$ be defined by*

$$\eta(\mathbf{x},t) = \left(\varepsilon^2 + \langle \beta \mathbf{x}, \boldsymbol{\vartheta}\rangle^2 + 2\beta^2(2\gamma+1)t\right)^{-p}$$

for $p > 0$, then η satisfies the evolution equation

$$\left(\frac{d}{dt} - \Delta\right)\eta \leqslant -\left(\frac{p+\gamma+1}{p}\right)\eta^{-1}|\nabla\eta|^2 - 2\beta^2 p\eta^{\frac{p+1}{p}}|\nabla u|^2$$

Proof. The proof of this is much the same as in Lemma 3.4, except that we use the fact that $\kappa = 0$ on our evolving surface, moreover, we use that

$$\langle \boldsymbol{\nu}, \boldsymbol{\vartheta}\rangle^2 = 1 - \langle \boldsymbol{\nu}, \boldsymbol{\omega}\rangle^2 = |\nabla u|^2$$

and the result follows. □

The following proposition illustrates how choosing initial surfaces which are sufficiently steep give control over important terms.

Proposition 5.7. *Suppose that M_0 is self-similarly bounded below, then for all $A, B > 0$ there exist constants $\varepsilon_0 \leqslant \varepsilon$, $\beta_0 \leqslant \beta$ and $\gamma_0 \leqslant \gamma$ such that the function $\varphi = \varphi(\mathbf{x},t)$ defined by $\varphi = u^2\eta$ (with $p = 1$) satisfies the evolution equation*

$$\left(\frac{d}{dt} - \Delta\right)\varphi \leqslant -2\left(\frac{n-1}{u^2}\right)\varphi - A\eta^{-1}|\nabla\eta|^2u^2 - B|\nabla u|^2\eta \tag{5.21}$$

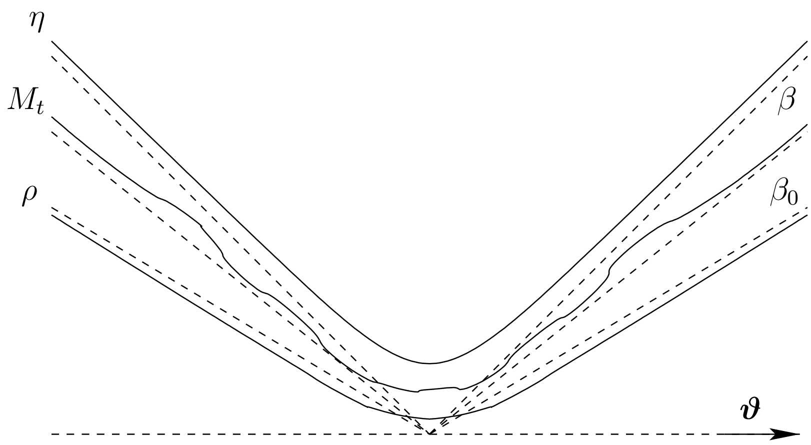

Figure 5.3: Barriers

Proof. First, using Lemma 5.6 with $p = 1$, we compute

$$\left(\frac{d}{dt} - \Delta\right)\varphi \leqslant -(2+\gamma)\eta^{-1}|\nabla\eta|^2u^2 - 2\beta^2\varphi|\nabla u|^2\eta \\ - 2\left(\frac{n-1}{u^2}\right)\varphi - 2|\nabla u|^2\eta - 2\left\langle\nabla\eta, \nabla u^2\right\rangle \quad (5.22)$$

and applying Young's inequality to the last term, we have

$$\left(\frac{d}{dt} - \Delta\right)\varphi \leqslant -2\left(\frac{n-1}{u^2}\right)\varphi - \gamma\eta^{-1}|\nabla\eta|^2u^2 - 2\beta^2\varphi|\nabla u|^2\eta$$

Now, consider the factors in the last term. Since M_0 is self-similarly bounded below we have the lower bound (5.20) on u, and therefore by the definition of η

$$2\beta^2\varphi \geqslant 2\beta^2\left(\frac{\varepsilon_0^2 + \langle\beta_0\mathbf{x}, \boldsymbol{\vartheta}\rangle^2 + 2\beta_0^2(2\gamma_0+1)t}{\varepsilon^2 + \langle\beta\mathbf{x}, \boldsymbol{\vartheta}\rangle^2 + 2\beta^2(2\gamma+1)t}\right) \quad (5.23)$$

Now, set $z = \langle\mathbf{x}, \boldsymbol{\vartheta}\rangle$ and

$$f(z,t) = 2\beta^2\left(\frac{\varepsilon_0^2 + (\beta_0 z)^2 + 2\beta_0^2(2\gamma_0+1)t}{\varepsilon^2 + (\beta z)^2 + 2\beta^2(2\gamma+1)t}\right), \quad z \in \mathbb{R}, t \geqslant 0$$

It is clear that

$$\frac{\partial f}{\partial z}(z,t) \geqslant 0 \quad \text{and} \quad \frac{\partial f}{\partial t}(z,t) \leqslant 0$$

for $\gamma_0 \leqslant \gamma$, and furthermore that

$$\lim_{t\to\infty} f(z,t) = 2\left(\frac{1+\gamma_0}{1+\gamma}\right)\beta_0^2$$

Thus, we may bound f below by a positive constant $B(\beta_0, \gamma_0)$ depending on β_0 and γ_0/γ. Moreover, this constant may be made arbitrarily large by choosing β_0 sufficiently large.

Thus, if we have $\varepsilon_0 \leqslant \varepsilon$, $\beta_0 \leqslant \beta$ and $\gamma_0 \leqslant \gamma$ then we have

$$\left(\frac{d}{dt} - \Delta\right)\varphi \leqslant -2\left(\frac{n-1}{u^2}\right)\varphi - A(\gamma_0)\eta^{-1}|\nabla\eta|^2u^2 - B(\beta_0,\gamma_0)|\nabla u|^2\eta$$

with $A, B > 0$ arbitrarily large, by choosing β_0 and γ_0 sufficiently large. $\square$

Corollary 5.8. *The function $\varphi = \varphi(\mathbf{x}, t)$ as defined above, satisfies the evolution equation*

$$\left(\frac{d}{dt} - \Delta\right)\varphi \leqslant -2\left(\frac{n-1}{u^2}\right)\varphi - C\varphi^{-1}|\nabla\varphi|^2$$

for any $C > 0$ if β_0 and γ_0 are sufficiently large.

Proof. First, we estimate

$$\begin{aligned}\varphi^{-1}|\nabla\varphi|^2 &= 4|\nabla u|^2\eta + 4u\langle\nabla u, \nabla\eta\rangle + \eta^{-1}|\nabla\eta|^2u^2 \\ &\leqslant 5\left(|\nabla u|^2\eta + \eta^{-1}|\nabla\eta|^2u^2\right)\end{aligned}$$

and then, choosing A and B sufficiently large in Equation (5.21), we have the result. $\square$

Corollary 5.9. *Let $f = \varphi v$, then f satisfies the evolution equation*

$$\left(\frac{d}{dt} - \Delta\right)f \leqslant -|A|^2f - (2-\delta)f^{-1}|\nabla f|^2 \tag{5.24}$$

with $\delta > 0$ arbitrarily small for sufficiently large β_0 and γ_0.

Proof. Using Corollary 5.8 and the evolution equation for v, we compute

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right)f \leqslant -|A|^2f - 2v^{-1}|\nabla v|^2\varphi + \left(\frac{n-1}{u^2}\right)f \\ - 2\left(\frac{n-1}{u^2}\right)f - C\varphi^{-2}|\nabla\varphi|^2f - 2\langle\nabla\varphi, \nabla v\rangle\end{aligned}$$

then using the identity

$$f^{-1}|\nabla f|^2 = v^{-1}|\nabla v|^2\varphi + 2\langle\nabla\varphi, \nabla v\rangle + \varphi^{-2}|\nabla\varphi|^2f$$

and

$$2\langle\nabla\varphi, \nabla v\rangle = 2\varphi^{-1}\langle\nabla\varphi, \nabla f\rangle - 2f\varphi^{-2}|\nabla\varphi|^2$$

we obtain

$$\left(\frac{d}{dt}-\Delta\right)f\leqslant -|A|^2f-2f^{-1}|\nabla f|^2+2\varphi^{-1}\left\langle\nabla\varphi,\nabla f\right\rangle-C\varphi^{-2}|\nabla\varphi|^2f$$

and after estimating the cross term with Young's inequality, for $\delta>0$ we have

$$\left(\frac{d}{dt}-\Delta\right)f\leqslant -|A|^2f-(2-\delta)f^{-1}|\nabla f|^2$$

so long as $C\geqslant\frac{1}{2\delta}$ which is effected by choosing γ_0 and β_0 sufficiently large. □

This evolution equation for $f=\varphi v$ implies a uniform estimate for the gradient function v.

Theorem 5.10. *Suppose that on M_0 the estimate*

$$\sup_{M_0} v(\cdot,0)\leqslant C$$

holds, and that M_0 is bounded below self-similarly then we have the estimate

$$\sup_{M_t} v(\cdot,t)\leqslant c_1 \tag{5.25}$$

on M_t for $t>0$ where c_1 is a bounded constant depending upon γ_0, β_0 and M_0.

Proof. This result follows from the non-compact maximum principle applied to Equation (5.24), since the self-similar lower bound on M_t gives the estimate

$$\varphi\geqslant C(\beta_0,\gamma_0)$$

for some constant $C(\beta_0,\gamma_0)>0$. □

Now that linear growth of cylindrical graphs has been shown to be conserved under (MCF), we can set about turning this result into a uniform global curvature bound.

Proposition 5.11. *Let $\psi=\psi(f^2)$ be the function defined for $\delta>0$ by*

$$\psi(q)=\frac{\left(\frac{q}{q_1}\right)^{1-\delta}}{\left(1-L\left(\frac{q}{q_1}\right)^{\frac{1}{2}(1-\delta)}\right)^2},\quad q\in[q_0,q_1]$$

with $f=vu^2\eta$ for $L\in(0,1)$ and $0<q_0\leqslant q_1<\infty$ then if M_0 is bounded below self-similarly, the function $g=|A|^2\psi(f^2)$ satisfies the evolution equation

$$\left(\frac{d}{dt}-\Delta\right)g\leqslant -\psi^{-1}\left\langle\nabla\psi,\nabla g\right\rangle-\frac{1}{2}g^{-1}|\nabla g|^2 \tag{5.26}$$

for δ sufficiently small.

Proof. We begin by computing the evolution equation for $g = |A|^2\psi(f^2)$ (similar to Proposition 3.16) for a general test-function $\psi = \psi(f^2)$

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)g &= \psi\left(\frac{d}{dt}-\Delta\right)|A|^2 + |A|^2\left(\frac{d}{dt}-\Delta\right)\psi - 2\left\langle\nabla\psi,\nabla|A|^2\right\rangle \\ &= \psi\left(\frac{d}{dt}-\Delta\right)|A|^2 + 2f\left(\frac{\psi'}{\psi}\right)g\left(\frac{d}{dt}-\Delta\right)f \\ &\quad - \frac{2}{\psi}\left[\psi' + 2f^2\psi''\right]|\nabla f|^2 g - 2\left\langle\nabla\psi,\nabla|A|^2\right\rangle\end{aligned}$$

and using the evolution equation for $|A|^2$ and (5.24) we have

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)g &\leqslant -2\psi|\nabla A|^2 - 2\left\langle\nabla\psi,\nabla|A|^2\right\rangle \\ &\quad - \frac{2}{\psi^2}\left[f^2\psi' - \psi\right]g^2 - \frac{2}{\psi}\left[(3-\delta)\psi' + 2f^2\psi''\right]|\nabla f|^2 g \end{aligned} \tag{5.27}$$

From the identity

$$\frac{1}{2}g^{-1}|\nabla g|^2 = 2\psi|\nabla|A||^2 + \left\langle\nabla\psi,\nabla|A|^2\right\rangle + 2f^2\left(\frac{\psi'}{\psi}\right)^2|\nabla f|^2 g$$

and Kato's inequality, we have

$$\begin{aligned}-2\psi|\nabla A|^2 - 2\left\langle\nabla\psi,\nabla|A|^2\right\rangle &\leqslant -2\psi|\nabla|A|^2 - 2\left\langle\nabla\psi,\nabla|A|^2\right\rangle \\ &= -\frac{1}{2}g^{-1}|\nabla g|^2 - \left\langle\nabla\psi,\nabla|A|^2\right\rangle \\ &\quad + 2f^2\left(\frac{\psi'}{\psi}\right)^2|\nabla f|^2 g \\ &= -\frac{1}{2}g^{-1}|\nabla g|^2 - \psi^{-1}\left\langle\nabla\psi,\nabla g\right\rangle \\ &\quad + 6f^2\left(\frac{\psi'}{\psi}\right)^2|\nabla f|^2 g\end{aligned}$$

and inserting this into (5.27) obtain

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)g &\leqslant -\psi^{-1}\left\langle\nabla\psi,\nabla g\right\rangle - \frac{1}{2}g^{-1}|\nabla g|^2 - \frac{2}{\psi^2}\left[f^2\psi' - \psi\right]g^2 \\ &\quad - \frac{2}{\psi^2}\left[(3-\delta)\psi\psi' + f^2(2\psi\psi'' - 3\psi'^2)\right]|\nabla f|^2 g \end{aligned} \tag{5.28}$$

Now, if we define ψ by

$$\psi(q) = \frac{\left(\frac{q}{q_1}\right)^{1-\delta}}{\left(1 - L\left(\frac{q}{q_1}\right)^{\frac{1}{2}(1-\delta)}\right)^2}, \quad q \in [q_0, q_1]$$

with $L \in (0,1)$ then we have

$$\frac{2}{\psi^2}\left[(3-\delta)\psi\psi' + q(2\psi\psi'' - 3\psi'^2)\right] = 0$$

and

$$\frac{2}{\psi^2}\left[q\psi' - \psi\right] = 2\psi^{-1}\left(\frac{L\left(\frac{q}{q_1}\right)^{\frac{1}{2}(1-\delta)} - \delta}{1 - L\left(\frac{q}{q_1}\right)^{\frac{1}{2}(1-\delta)}}\right)$$

Now, if we want this to be non-negative, then we require

$$\left(\frac{q_0}{q_1}\right)^{-\frac{1}{2}(1-\delta)}\delta \leqslant L < 1$$

We may bound $\frac{f_0}{f_1}$ below uniform in δ since $v \geqslant 1$ and $\varphi > 0$ on M_t due to M_0 being self-similarly bounded below, thus by choosing $\delta > 0$ sufficiently small we satisfy both of the inequalities and obtain the result. □

Using this evolution equation, we obtain a global curvature estimate.

Corollary 5.12 (Curvature Estimate)**.** *Let $(M_t)_{t\in[0,\infty)}$ be a smooth solution to (MCF), bounded below self-similarly, then we have the estimate*

$$\sup_{M_t} |A|^2(\cdot,t) \leqslant C \sup_{M_0} |A|^2(\cdot,0)$$

for $t \in [0,\infty)$, where $C = C(\beta_0, \gamma_0, c_1)$.

Proof. Since f may be bounded above (see Theorem 5.10) and below, we have the estimate

$$\left(\frac{f_0}{f_1}\right)^{2(1-\delta)} \leqslant \psi(f^2) \leqslant (1-L)^{-2}, \quad f \in [f_0, f_1]$$

Thus, Proposition 5.11 gives the result since L can be chosen smaller than 1 for sufficiently small δ (which requires sufficiently large β_0 and γ_0). □

Similar to the local smoothness estimate Theorem 3.18, the curvature bound above can be extended to a global estimate on all derivatives of the curvature

Proposition 5.13 (Smoothness Estimate)**.** *Let $(M_t)_{t\in[0,\infty)}$ be a smooth solution to (MCF), bounded below self-similarly, and suppose that we have the estimate*

$$\sup_{M_t} v(\cdot,t) \leqslant c_1$$

then for any $m \geqslant 0$ we have the estimate

$$\sup_{M_t} |\nabla^m A|^2(\cdot,t) \leqslant C \sup_{M_0} |\nabla^m A|^2(\cdot,0)$$

for $t \in [0,\infty)$, where $C = C(\beta_o, \gamma_0, c_1, m, n, \sup_{M_0}|A|^2, \ldots, \sup_{M_0}|\nabla^{m-1}A|^2)$.

Proof. Using the above estimate in Proposition 5.11 as the base case ($m = 0$), we use an inductive argument very similar to Theorem 3.18 to obtain the global estimate for any $m \geqslant 1$. □

It turns out that a little more can be extracted from Equation (5.26), giving a curvature decay estimate.

Corollary 5.14. *Let $h = 2tg + \psi$ with $g = |A|^2\psi$ and $\psi = \psi(f^2)$ as defined above, then h satisfies the evolution equation*

$$\left(\frac{d}{dt} - \Delta\right) h \leqslant -\psi^{-1} \langle \psi, \nabla h \rangle \tag{5.29}$$

Proof. Using (5.26) and (5.24) we compute

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) h &\leqslant 2g + 2t \left(\frac{d}{dt} - \Delta\right) g + \left(\frac{d}{dt} - \Delta\right) \psi \\ &\leqslant 2g - \psi^{-1} \langle \nabla\psi, \nabla(2tg) \rangle \\ &\quad + \psi' \left[-2f^2|A|^2 - 2(3-\delta)|\nabla f|^2\right] - 4f^2\psi''|\nabla f|^2\end{aligned}$$

and since

$$-\psi^{-1} \langle \nabla\psi, \nabla(2tg) \rangle = -\psi^{-1} \langle \nabla\psi, \nabla h \rangle + 4f^2 \left(\frac{\psi'}{\psi}\right)^2 |\nabla f|^2$$

we have the equation

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) h \leqslant -\psi^{-1} \langle \psi, \nabla h \rangle &- \frac{2}{\psi} \left[f^2\psi' - \psi\right] g \\ &- \frac{2}{\psi} \left[(3-\delta)\psi\psi' + 2f^2 \left(\psi\psi'' - \psi'^2\right)\right] |\nabla f|^2\end{aligned}$$

Since we know from Proposition 5.11 that for sufficiently small $\delta > 0$ we have

$$\frac{2}{\psi} \left[f^2\psi' - \psi\right] > 0$$

and

$$\frac{2}{\psi} \left[(3-\delta)\psi\psi' + 2f^2 \left(\psi\psi'' - \psi'^2\right)\right] > 0$$

the result follows. □

Proposition 5.15 (Curvature Decay Estimate)**.** *Let $(M_t)_{t\in[0,\infty)}$ be a smooth solution to (MCF), bounded below self-similarly, and suppose that we have the estimate*

$$\sup_{M_t} v(\cdot, t) \leqslant c_1$$

then for any $m \geqslant 0$ we have the estimate

$$\sup_{M_t} |\nabla^m A|^2(\cdot, t) \leqslant \frac{C}{t^{m+1}}$$

for $t \in (0, \infty)$, where $C = C(\beta_0, \gamma_0, c_1, m, n)$.

Proof. Using Equation (5.29) we proceed iteratively as in [10] to prove estimates on the higher derivatives.

Using the non-compact maximum principle (Theorem E.1), Equation (5.29) gives the $m = 0$ decay estimate.

Next, from Corollary B.9 we have for any $l \geqslant 0$ the evolution equation

$$\left(\frac{d}{dt} - \Delta\right)(t^{l+1}|\nabla^l A|^2) \leqslant -2t^{l+1}|\nabla^{l+1}A|^2 + (l+1)t^l|\nabla^l A|^2 + t^{l+1}\sum_{i+j+k=l} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^l A \quad (5.30)$$

Now, assume that the decay estimate has been established up to $m-1$, then applying the Schwarz inequality and Young's inequality to the last term of Equation (5.30) we obtain

$$\left(\frac{d}{dt} - \Delta\right)(t^{l+1}|\nabla^l A|^2) \leqslant -2t^{l+1}|\nabla^{l+1}A|^2 + C(l)\sum_{k=1}^{l} t^k|\nabla^k A|^2 \quad (5.31)$$

where $C(l)$ is a constant depending only on $c_1, \beta_0, \gamma_0, l$ and n.

Now, set $h = \sum_{k=1}^m \zeta_k t^{k+1}|\nabla^k A|^2 + \zeta_0\psi$ and, using Equation (5.30) and (5.26), progressively choose each $\zeta_k, k = m, \ldots, l$ sufficiently large (depending upon $\zeta_{k+1}C(k+1)$), and we obtain

$$\left(\frac{d}{dt} - \Delta\right)h \leqslant 0$$

Applying Theorem E.1 to this equation and we obtain the result. □

Now, with Propositions 3.5, 5.10 and 5.15, we are ready to show convergence of solutions self-similarly bounded below to self-similar expanding solutions of (MCF).

5.4.2 The Convergence

Consider the rescaling introduced at the start of the chapter, and observe that on the rescaled surface $\tilde{M}_s$, Propositions 3.5, 5.10 and 5.15 yield the uniform (in s) estimates

$$\tilde{u}^2(\tilde{\mathbf{x}}, s) \leqslant c_0(1 + \langle\tilde{\mathbf{x}}, \boldsymbol{\vartheta}\rangle^2) \quad (5.32)$$

$$\tilde{v}(\tilde{\mathbf{x}}, s) \leqslant c_1 \quad (5.33)$$

$$|\tilde{A}|^2(\tilde{\mathbf{x}}, s) \leqslant c_2 \quad (5.34)$$

Armed with a suitable portfolio of height, gradient and curvature estimates which are, on the rescaled surface $\tilde{M}_s$, bounded uniformly in time, we may prove a convergence result for the rescaled flow. This result is in the spirit of the convergence result for planar graphs in [10].

The main result gives convergence of the rescaled flow to a stationary surface, satisfying the equation of a self-similar expanding solution to the original flow.

Theorem 5.16. *Suppose that M_0 is a rotationally symmetric entire cylindrical graph, which has at most linear growth, bounded curvature, is bounded below self-similarly and suppose also that M_0 satisfies the estimate*

$$\langle \mathbf{x}, \boldsymbol{\nu} \rangle^2 \leqslant c_3 \left(1 + \langle \mathbf{x}, \boldsymbol{\vartheta} \rangle^2\right)^{1-\delta}, \quad \mathbf{x} \in M_0 \tag{5.35}$$

for some $\delta > 0$ and $c_3 < \infty$ then the solution $\tilde{M}_s$ of normalised (MCF) converges as $s \to \infty$ to a limiting surface $\tilde{M}_\infty$ upon which the equation

$$\tilde{\mathbf{x}}^\perp = \tilde{\mathbf{H}}(\tilde{\mathbf{x}}), \quad \tilde{\mathbf{x}} \in \tilde{M}_\infty$$

is satisfied.

We need to ensure that our surface stays in the same spatial asymptotic class as it was in initially.

Lemma 5.17. *Suppose that M_0 satisfies the estimate (5.35), then we have*

$$\langle \tilde{\mathbf{x}}, \tilde{\boldsymbol{\nu}} \rangle^2 \leqslant C(s) \left(1 + \langle \tilde{\mathbf{x}}, \boldsymbol{\vartheta} \rangle^2\right)^{1-\delta}, \quad \tilde{\mathbf{x}} \in \tilde{M}_s$$

where C is a constant depending upon s and c_2.

Proof. Since we are allowing the constant in this estimate to depend on time, it will be sufficient to look at the un-normalised flow, and simply scale the result.

First, using (MCF) and Lemma B.2 we find that $f = \langle \mathbf{x}, \boldsymbol{\nu} \rangle$ satisfies the evolution equation

$$\left(\frac{d}{dt} - \Delta\right) f = |A|^2 f - 2H$$

Now, we introduce the cutoff function $\eta = \eta(\mathbf{x}, t)$ defined by

$$\eta(\mathbf{x}, t) = \left(1 + \langle \mathbf{x}, \boldsymbol{\vartheta} \rangle^2 + 2t\right)^{\delta - 1}$$

and, since $0 < \delta < 1$ we have the evolution equation

$$\left(\frac{d}{dt} - \Delta\right) \eta \leqslant -\left(\frac{2-\delta}{1-\delta}\right) \eta^{-1} |\nabla \eta|^2$$

Combining these results, we have

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)f^2\eta &\leqslant 2|A|^2f^2\eta - 4Hf\eta - 2|\nabla f|^2\eta \\ &\quad - \left(\frac{2-\delta}{1-\delta}\right)\eta^{-1}|\nabla\eta|^2 f^2 - 2\left\langle\nabla\eta, \nabla f^2\right\rangle \\ &\leqslant C(f^2\eta+1)\end{aligned}$$

where we have used the uniform curvature estimate along with Young's inequality twice (on the cross term and the mean curvature term). Here, C is a constant depending upon c_2.

So, from this evolution equation, the maximum principle implies that $f^2\eta$ may grow at most exponentially in time, and thus after rescaling, the result follows if we allow C to depend on s. □

Proposition 5.18. *Suppose $\tilde{M}_s$ satisfies the requirements of Theorem 5.16, and that $\tilde{u} \geqslant \sqrt{\rho(n-1)} > 0$. Let $f = \left(\tilde{H} + \langle\tilde{\mathbf{x}}, \tilde{\boldsymbol{\nu}}\rangle\right)^2 \tilde{v}^2$ and $\tilde{\eta} = \eta(\tilde{\mathbf{x}}, s)$ be the cutoff function defined by*

$$\tilde{\eta}(\tilde{\mathbf{x}}, s) = \left(1 + \langle\beta\tilde{\mathbf{x}}, \boldsymbol{\vartheta}\rangle^2\right)^{\varepsilon-1} e^{2\gamma s}$$

with $\beta, \gamma > 0$ and $0 < \varepsilon < \delta \leqslant 1$, then the function $g = f\tilde{\eta}$ satisfies the evolution equation

$$\left(\frac{d}{ds} - \tilde{\Delta}\right) g \leqslant -2\tilde{v}^{-1}\langle\nabla\tilde{v}, \nabla g\rangle \tag{5.36}$$

so long as β and γ are sufficiently small and ρ is sufficiently large.

Proof. Using Lemma 2.10 we compute that

$$\left(\frac{d}{ds} - \tilde{\Delta}\right)(\tilde{H} + \langle\tilde{\mathbf{x}}, \tilde{\boldsymbol{\nu}}\rangle)^2 = 2(|\tilde{A}|^2 - 1)(\tilde{H} + \langle\tilde{\mathbf{x}}, \tilde{\boldsymbol{\nu}}\rangle)^2 - 2|\nabla(\tilde{H} + \langle\tilde{\mathbf{x}}, \tilde{\boldsymbol{\nu}}\rangle)|^2$$

Now, since $\tilde{u} \geqslant \sqrt{\rho(n-1)} > 0$ for $s > 0$, we have

$$\left(\frac{d}{ds} - \tilde{\Delta}\right)\tilde{v}^2 \leqslant -2|\tilde{A}|^2\tilde{v}^2 - 6|\nabla\tilde{v}|^2 + 2\rho^{-1}\tilde{v}^2$$

and combining these equations, and estimating, we obtain

$$\left(\frac{d}{ds} - \tilde{\Delta}\right) f \leqslant -2(1-\rho^{-1})f - \frac{1}{2}f^{-1}|\nabla f|^2 - 2\tilde{v}^{-1}\langle\nabla\tilde{v}, \nabla f\rangle \tag{5.37}$$

Now, we compute, since $\varepsilon < 1$

$$\begin{aligned}\left(\frac{d}{ds} - \tilde{\Delta}\right)\tilde{\eta} &\leqslant 2(1-\varepsilon)\left(\frac{\beta^2 + \langle\beta\tilde{\mathbf{x}}, \boldsymbol{\vartheta}\rangle^2}{1 + \langle\beta\tilde{\mathbf{x}}, \boldsymbol{\vartheta}\rangle^2}\right)\tilde{\eta} + 2\gamma\tilde{\eta} - \left(\frac{2-\varepsilon}{1-\varepsilon}\right)\tilde{\eta}^{-1}|\nabla\tilde{\eta}|^2 \\ &\leqslant 2\left[(1-\varepsilon)(1+\beta^2) + \gamma\right]\tilde{\eta} - \left(\frac{2-\varepsilon}{1-\varepsilon}\right)\tilde{\eta}^{-1}|\nabla\tilde{\eta}|^2\end{aligned}$$

where we have used that

$$\left(\frac{d}{ds} - \tilde{\Delta}\right)\langle \tilde{\mathbf{x}}, \boldsymbol{\vartheta}\rangle^2 \geqslant -2\left(\langle \tilde{\mathbf{x}}, \boldsymbol{\vartheta}\rangle^2 + 1\right)$$

Combining this evolution equation with that of equation (5.37) we have for $g = f\tilde{\eta}$

$$\begin{aligned}\left(\frac{d}{ds} - \tilde{\Delta}\right) g \leqslant -\frac{1}{2} f^{-1}|\nabla f|^2 \tilde{\eta} - \left(\frac{2-\varepsilon}{1-\varepsilon}\right)\tilde{\eta}^{-1}|\nabla\tilde{\eta}|^2 f - 2\langle \nabla \eta, \nabla f\rangle \\ - 2\left[\varepsilon(1+\beta^2) - (\beta^2 + \gamma + \rho^{-1})\right] g - 2\tilde{v}^{-1}\langle \nabla\tilde{v}, \tilde{\eta}\nabla f\rangle\end{aligned}$$

Now, we estimate

$$\begin{aligned}-2\tilde{v}^{-1}\langle \nabla\tilde{v}, \tilde{\eta}\nabla f\rangle &= -2\tilde{v}^{-1}\langle \nabla\tilde{v}, \nabla g\rangle + 2f\tilde{v}^{-1}\langle \nabla\tilde{v}, \nabla\tilde{\eta}\rangle \\ &\leqslant -2\tilde{v}^{-1}\langle \nabla\tilde{v}, \nabla g\rangle + 2c\beta g\end{aligned}$$

since $\tilde{v}^{-1}|\nabla\tilde{v}| \leqslant |\tilde{A}|\tilde{v} \leqslant c$ and $|\nabla\tilde{\eta}| \leqslant 2\beta\tilde{\eta}$, since by Proposition 5.15 we have $|\tilde{A}| \leqslant c_2$. Using this, and Young's inequality on the $\tilde{\eta}$-f cross term, since $\left(\frac{2-\varepsilon}{1-\varepsilon}\right) \geqslant 2$, we obtain

$$\left(\frac{d}{ds} - \tilde{\Delta}\right) g \leqslant -2\tilde{v}^{-1}\langle \nabla\tilde{v}, \nabla g\rangle - 2\left[\varepsilon(1+\beta^2) - (c\beta + \beta^2 + \gamma + \rho^{-1})\right] g$$

thus, in choosing β and γ sufficiently small, and as long as ρ is sufficiently large, we have the result. Note that β, γ and ρ depend only on ε, c_1 and c_2. □

Proof of 5.16. Using Lemma 5.17, we have that for all $s > 0$

$$g(\tilde{\mathbf{x}}, s) \to 0$$

for $|\tilde{\mathbf{x}}| \to \infty$. Thus, for $\varepsilon < \delta$, the function g takes its maximum on a bounded set, and we may apply the standard maximum principle to the evolution equation (5.36), which yields the estimate

$$\sup_{\tilde{M}_s} \frac{(\tilde{H} + \langle \tilde{\mathbf{x}}, \tilde{\boldsymbol{\nu}}\rangle)^2 \tilde{v}^2}{(1 + \langle \beta\tilde{\mathbf{x}}, \boldsymbol{\vartheta}\rangle^2)^{1-\varepsilon}} \leqslant e^{-2\gamma s} \sup_{M_0} \frac{(H + \langle \mathbf{x}, \boldsymbol{\nu}\rangle)^2 v^2}{(1 + \langle \beta\mathbf{x}, \boldsymbol{\vartheta}\rangle^2)^{1-\varepsilon}} \tag{5.38}$$

for $s \geqslant 0$ with β, γ and ρ chosen appropriately as above (depending only on ε, c_1 and c_2.

Thus it is clear by Lemma 5.17 and that $\tilde{v} \geqslant 1$ that in the limit as $s \to \infty$ we have

$$\tilde{H} + \langle \tilde{\mathbf{x}}, \tilde{\boldsymbol{\nu}}\rangle \to 0$$

i.e. there exists a limit surface $\tilde{M}_\infty$ such that

$$\tilde{\mathbf{x}}^\perp = \tilde{\mathbf{H}}(\tilde{\mathbf{x}}), \quad \tilde{\mathbf{x}} \in \tilde{M}_\infty$$

□