Chapter 4

Existence

In this section we will derive short- and long-time existence results for cylindrical graphs using similar methods to those used by Ecker and Huisken [10], [11] in the case of entire planar graphs. To do this we will use the local and global height, gradient and curvature estimates developed in Chapter 3

Theorem 4.1 (Local Short-Time Existence)**.** *Let $\Omega \subset C_r^n \subset \mathbb{R}^{n+1}$ be a compact domain with smooth boundary $\partial\Omega$ and suppose that $M_0 = \mathbf{F}_0(\cdot)(\Omega)$ is a smooth, cylindrical graph over the cylinder Ω then there exists a $T > 0$ depending only upon M_0 such that the initial/boundary-value problem*

$$\begin{cases} \dfrac{\partial \mathbf{F}}{\partial t}(\boldsymbol{p}, t) = \mathbf{H}(\mathbf{F}(\boldsymbol{p}, t)), & \boldsymbol{p} \in \Omega, t \in [0, T) \\ \mathbf{F}(\boldsymbol{p}, 0) = \mathbf{F}_0(\boldsymbol{p}), & \boldsymbol{p} \in \Omega \\ \mathbf{F}(\boldsymbol{p}, t) = \mathbf{F}_0(\boldsymbol{p}), & \boldsymbol{p} \in \partial\Omega, t \in [0, T) \end{cases} \tag{4.1}$$

has a unique, smooth, cylindrical graph solution over Ω for $t \in [0, T)$.

Proof. Using Chapter D we may express the initial/boundary-value problem (4.1) as an equivalent (up to tangential diffeomorphisms) quasi-linear scalar problem for some function $\rho : \Omega \times [0, T) \to \mathbb{R}$, i.e.

$$\begin{cases} \dfrac{\partial \rho}{\partial t}(\boldsymbol{q}, t) = a^{ij}(\rho, \overline{\nabla}_{\boldsymbol{q}}\rho)\rho_{ij} + b^i(\rho, \overline{\nabla}_{\boldsymbol{q}}\rho)\rho_i + f(\rho), & \boldsymbol{q} \in \Omega, t \in [0, T) \\ \rho(\boldsymbol{q}, 0) = \rho_0(\boldsymbol{q}), & \boldsymbol{q} \in \Omega \\ \rho(\boldsymbol{q}, t) = \rho_0(\boldsymbol{q}), & \boldsymbol{q} \in \partial\Omega, t \in [0, T) \end{cases} \tag{4.2}$$

Since M_0 is a cylindrical graph, we may assume that we have $\rho_0 > 0$. Comparison with the exact solution of the homothetically shrinking cylinder gives the existence of a T depending only on M_0 and n such that

$$\rho(\boldsymbol{q}, t) \geqslant \frac{\rho_0}{2}, \quad \boldsymbol{q} \in \Omega, t \in [0, T)$$

Using Proposition D.4 we have that (4.2) is a quasi-linear parabolic partial differential equation for solutions with bounded gradient.

It follows from the local gradient estimate, Theorem 3.10, and standard theory that (4.2) is uniformly parabolic on some time interval $[0, T)$ for some $T > 0$ depending only on M_0.

Using the local curvature estimate, Theorem 3.18, we may apply the results of Ladyženskaja et al. [16] to obtain a unique, smooth solution to (4.2) on $[0, T)$. Using an appropriate tangential diffeomorphism (see Chapter D for details), we construct the required solution to the original problem (4.1). □

Note that we haven't obtained an existence theorem that is global in time, which means that the solution can still have a finite existence time, unlike the equivalent theorems in the planar graph case. The reason for this is that solutions in the cylindrical graph case can easily develop singularities from neck-pinches.

In Chapter 5, we shall deal with the possibility of neck-pinches by deriving conditions under which they do not occur, however, in the meantime we have the following extension theorem (c.f. Theorem 3.24 [9]), allowing for the solution to be pushed a little further, so long as the solution is clear of the axis.

Theorem 4.2 (Extension Theorem)**.** *Let $(M_t)_{t\in[0,T)}$ be a smooth entire rotationally symmetric cylindrical graph moving by (MCF). Suppose that $u \geqslant u_0 > 0$ on $S_R(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^{n+1} : \langle \mathbf{x} - \mathbf{x}_0, \boldsymbol{\vartheta}\rangle^2 \leqslant R^2\} \times [0, T)$ for some $R > 0$, then the solution on $S_{\frac{R}{2}}(\mathbf{x}_0)$ may be extended smoothly up to time $T + \varepsilon$ for some $\varepsilon > 0$.*

Proof. See Theorem 3.24 in [9] for details. □