Chapter 3

Estimates on Surfaces of Revolution

3.1 Height Estimates

The main aim in this chapter is to obtain a range of local and global estimates for the graph quantities u and v, and local and global estimates for the curvature $|A|^2$. To do this we make use of the evolution equations of Chapter 2, coupled with specially crafted test- and cutoff-functions, allowing us to obtain suitable evolution equations to which we may apply the maximum principle.

The global estimates in this section make use of a global maximum principle (Theorem E.1, [10]) while the local estimates require only the use of the standard maximum principle on compact subsets of $\mathbb{R}^{n+1}$ or a special localised version. For a discussion of the various forms of the maximum principle that we make use of in this thesis, see Chapter E.

The global estimates in this chapter are inspired by the corresponding estimates for graphs over planes, as seen in [10], while the local estimates are motivated by similar constructions to those presented in [11].

3.1.1 Local Estimates

A particularly useful cutoff-function for local height, gradient and curvature estimates on cylindrical graphs is the 'band function'. Note that the function $\langle \mathbf{x}, \boldsymbol{\vartheta} \rangle$ measures the distance along the axis of the reference cylinders.

Lemma 3.1 (Band Function). *The function $\xi = \xi(\mathbf{x}, t)$ defined by*

$$\xi(\mathbf{x}, t) = (R^2 - \langle \mathbf{x} - \mathbf{x}_0, \boldsymbol{\vartheta} \rangle^2 - 2t)_+^{2(p+1)}$$

for $R, p > 0$ and some $\mathbf{x}_0 \in \mathbb{R}^{n+1}$, satisfies

$$\left(\frac{d}{dt} - \Delta \right) \xi \leqslant -\frac{1}{2} \left(\frac{2p+1}{p+1} \right) \xi^{-1} |\nabla \xi|^2$$

for $\mathbf{x} \in M_t \cap \operatorname{spt} \xi$.

Proof. Compute

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right)\xi &= 2(p+1)\xi^{\frac{1}{2}\left(\frac{2p+1}{p+1}\right)}\left(2|\nabla\langle \mathbf{x}, \boldsymbol{\vartheta}\rangle|^2 - 2\right)\\ &\quad - 2(p+1)(2p+1)\xi^{\frac{p}{p+1}}|\nabla(R^2 - \langle \mathbf{x}, \boldsymbol{\vartheta}\rangle^2 - 2t)|^2\\ &= -4(p+1)\xi^{\frac{1}{2}\left(\frac{2p+1}{p+1}\right)}|\nabla u|^2 - \frac{1}{2}\left(\frac{2p+1}{p+1}\right)\xi^{-1}|\nabla\xi|^2\end{aligned}$$

since $|\nabla u|^2 = 1 - |\nabla\langle \mathbf{x}, \boldsymbol{\vartheta}\rangle|^2$. □

If we consider the set $M_t \cap \operatorname{spt}\xi$, we have selected out a 'band' of width $2\sqrt{R^2 - 2t}$ on the evolving surface, centered at $\mathbf{x}_0$. Without loss of generality, we will set $\mathbf{x}_0 = 0$.

Using this cutoff-function we may derive a local estimate for the height function. Set $g = u^2\xi$ and compute

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right)g &= -2(n-1)\xi - 2|\nabla u|^2\xi - 2\left\langle \nabla u^2, \nabla\xi\right\rangle\\ &\quad - 4(p+1)|\nabla u|^2 u^2 \xi^{\frac{1}{2}\left(\frac{2p+1}{p+1}\right)} - \frac{1}{2}\left(\frac{2p+1}{p+1}\right)\xi^{-1}|\nabla\xi|^2 u^2\end{aligned}$$

Estimating the cross-term using Young's inequality

$$-2\left\langle \nabla u^2, \nabla\xi\right\rangle \leqslant \frac{1}{2}\left(\frac{2p+1}{p+1}\right)\xi^{-1}|\nabla\xi|^2u^2 + 8\left(\frac{p+1}{2p+1}\right)|\nabla u|^2\xi$$

we obtain the evolution equation

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right)g &\leqslant -2(n-1)\xi\\ &\quad - 2\left[1 + 2\left(\frac{p+1}{2p+1}\right)\left((2p+1)u^2\xi^{-\frac{1}{2}\left(\frac{1}{p+1}\right)} - 2\right)\right]|\nabla u|^2\xi \qquad (3.1)\end{aligned}$$

the last term of which is negative for p sufficiently large, depending on $u_0 = \inf\{u(\mathbf{x}, t) : \mathbf{x} \in M_t, t \in [0, T)\}$. Note that on compact subsets of M_t, we may use the comparison principle to bound u_0 below, depending only on M_0, by comparing with the exact solution of the co-axial homothetically shrinking cylinders on the interval $\left[0, \bar{T}\right)$ where $\bar{u}_0 = \inf_{M_0} u(\cdot, 0)$ and $\bar{T} = \frac{1}{2}\left(\frac{\bar{u}_0^2}{n-1}\right)$, the existence time for the cylinder.

Using Equation (3.1) we obtain the following local estimate for the height

Theorem 3.2. *Set* $S_R(t) = \left\{\mathbf{x} \in \mathbb{R}^{n+1} : \langle \mathbf{x} - \mathbf{x}_0, \boldsymbol{\vartheta}\rangle^2 \leqslant R^2 - 2t\right\}, R > 0$ *then for* $p \geqslant 0$ *sufficiently high depending only on* M_0*, we have*

$$\sup_{M_t \cap S_R(t)} \left(\left(R^2 - \langle \mathbf{x} - \mathbf{x}_0, \boldsymbol{\vartheta}\rangle^2 - 2t\right)_+^{2(p+1)} u^2\right)(\cdot, t) \leqslant \sup_{M_0 \cap S_R(0)} \left(\left(R^2 - \langle \mathbf{x} - \mathbf{x}_0, \boldsymbol{\vartheta}\rangle^2\right)_+^{2(p+1)} u^2\right)(\cdot, 0) \quad (3.2)$$

for $t \in \left[0, \min\left\{T, \bar{T}, \frac{R^2}{2}\right\}\right)$ *where* $\bar{T} = \frac{1}{2}\left(\frac{\bar{u}_0^2}{n-1}\right)$.

Proof. This is a simple application of the maximum principle for compactly supported functions applied to the evolution equation (3.1) for the function $g = u^2\xi$ on the compact (shrinking) sets $S_R(t)$. □

Corollary 3.3 (Local Height Estimate)**.** *Suppose that on the set* $S_R = \{\mathbf{x} \in \mathbb{R}^{n+1} : \langle \mathbf{x}, \boldsymbol{\vartheta}\rangle^2 \leqslant R^2\}$ *for some* $R > 0$ *we have the estimate*

$$\sup_{M_0 \cap S_R} u(\cdot, 0) \leqslant C_0$$

for some $C_0 < \infty$*, then if* $u \geqslant u_0 > 0$ *we have the estimate*

$$\sup_{M_t \cap S_{\frac{R}{2}}} u(\cdot, t) \leqslant C$$

for $t \in \left[0, \min\left\{T, \bar{T}, \left(\frac{R}{2}\right)^2\right\}\right)$ *and some constant* C *depending only on* C_0, R *and* u_0*.*

Proof. Since, by going to a smaller set we have

$$\sup_{M_t \cap S_{\frac{R}{2}}(t)} \left(u^2\xi\right)(\cdot, t) \leqslant \sup_{M_t \cap S_R(t)} \left(u^2\xi\right)(\cdot, t)$$

Theorem 3.2 gives us

$$\sup_{M_t \cap S_R(t)} \left(u^2\xi\right)(\cdot, t) \leqslant \sup_{M_0 \cap S_R(0)} \left(u^2\xi\right)(\cdot, 0) \leqslant C_0$$

which is bounded by assumption, and since on $S_{\frac{R}{2}} \times \left[0, \left(\frac{R}{2}\right)^2\right)$ we have $\xi \geqslant C(R)$, thus the result follows. □

3.1.2 Global Estimates

First we derive an evolution equation for a useful cutoff function which allows us to bound quantities which initially grow as a function of axial distance.

Lemma 3.4. *The function* $\eta = \eta(\mathbf{x}, t)$ *defined by*

$$\eta(\mathbf{x}, t) = \left(u_0^2 + \langle \beta\mathbf{x}, \boldsymbol{\vartheta}\rangle^2 + 2\beta^2(2\gamma + 1)t\right)^{-p}$$

for $u_0 > 0$ *and* $\beta, \gamma, p > 0$ *satisfies the evolution equation*

$$\left(\frac{d}{dt} - \Delta\right)\eta \leqslant -\left(\frac{p + \gamma + 1}{p}\right)\eta^{-1}|\nabla\eta|^2$$

Proof. This is a direct consequence of the fact that

$$\left(\frac{d}{dt} - \Delta\right)\langle \mathbf{x}, \boldsymbol{\vartheta}\rangle = 0$$

and $|\nabla \langle \mathbf{x}, \boldsymbol{\vartheta}\rangle|^2 \leqslant 1$, along with the computation

$$\eta^{-1}|\nabla\eta|^2 = p^2\eta^{\frac{p+2}{p}}|\nabla \langle \beta\mathbf{x}, \boldsymbol{\vartheta}\rangle^2|^2$$

which we estimate to obtain

$$\begin{aligned}\eta^{-1}|\nabla\eta|^2 &\leqslant 4p^2\beta^2\eta^{\frac{p+2}{p}}\langle \beta\mathbf{x}, \boldsymbol{\vartheta}\rangle^2 \\ &\leqslant 4p^2\beta^2\eta^{\frac{p+1}{p}}\end{aligned}$$

since $\langle \beta\mathbf{x}, \boldsymbol{\vartheta}\rangle^2 \leqslant \eta^{-\frac{1}{p}}$. □

We now derive an *a priori* estimate for the height function which shows an initial polynomial height bound is preserved by the flow. Note that this estimate also applies to non-rotationally symmetric solutions.

Proposition 3.5. *Suppose that on* M_0 *we have the estimate*

$$u^2 \leqslant \left(u_0^2 + \langle \beta\mathbf{x}, \boldsymbol{\vartheta}\rangle^2\right)^p$$

for some $u_0 > 0$ *and* $\beta, p \geqslant 0$*, then on* M_t *we have the estimate*

$$u^2 \leqslant c_1\left(u_0^2 + \langle \beta\mathbf{x}, \boldsymbol{\vartheta}\rangle^2 + 2\beta^2(2p - 1)t\right)^p$$

for some $c_1 \leqslant \infty$*, for all* $t > 0$*.*

Proof. Define the function $\eta = \eta(\mathbf{x}, t)$ as in the above lemma with some fixed $\gamma > 0$ to be chosen later. Now, using Lemma 3.4, we compute

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right)u^2\eta &\leqslant -2\eta|\nabla u|^2 - 2(n - 1 - \kappa)\eta \\ &\qquad - \left(\frac{p + \gamma + 1}{p}\right)u^2\eta^{-1}|\nabla\eta|^2 - 2\left\langle \nabla u^2, \nabla\eta\right\rangle\end{aligned}$$

and estimating the last term using Young's inequality

$$-2\left\langle \nabla u^2, \nabla \eta \right\rangle \leqslant 2\eta|\nabla u|^2 + 2u^2|\nabla\eta|^2$$

and thus, one finds

$$\left(\frac{d}{dt} - \Delta\right) u^2\eta \leqslant -\left(\frac{\gamma + 1 - p}{p}\right) u^2\eta^{-1}|\nabla\eta|^2$$

so, choosing $\gamma \geqslant p - 1$, we have

$$\left(\frac{d}{dt} - \Delta\right) u^2\eta \leqslant 0$$

and thus by the weak maximum principle (Theorem E.1) the function $u^2\eta$ is bounded by its initial data and the result follows. □

Next, we have another estimate which applies in the rotationally symmetric case where the evolving surface is developing a neck that pinches off. The argument presented here is similar to that in [1]. The argument also works in the non-rotationally symmetric case, however we only present here the rotational case for simplicity.

Proposition 3.6. *Suppose that M_t has a neck with minimum height at $z = \zeta(t)$ that pinches at time T, then we have the estimate*

$$\rho(\zeta(t), t) \leqslant \sqrt{2(n-1)(T-t)}, \quad t \in [0, T]$$

Proof. Recall that $\rho = \rho(z, t)$ satisfies the equivalent (up to tangential diffeomorphism) equation

$$\frac{\partial\rho}{\partial t}(z, t) = -\frac{n-1}{\rho} + \frac{\rho''}{1 + \rho'^2}, \quad (z, t) \in \mathbb{R} \times [0, T)$$

Set $\rho(t) = \rho(\zeta(t), t)$, then since $\zeta(t)$ defines a minimum point, we have $\rho'(\zeta(t), t) = 0$ and $\rho''(\zeta(t), t) \geqslant 0$ for $t \in [0, T)$.

Thus, we have the inequality

$$\frac{d\rho}{dt}(t) \geqslant -\frac{n-1}{\rho(t)}, \quad t \in [0, T) \tag{3.3}$$

or

$$\frac{d}{dt}\rho^2(t) \geqslant -2(n-1), \quad t \in [0, t)$$

Integrating (3.3) over $[t, T)$ for $t \leqslant T$, we have

$$\rho^2(T) - \rho^2(t) \geqslant -2(n-1)(T-t)$$

and since $\rho(T) = 0$, we are finished. □

3.2 Gradient Estimates

3.2.1 Local Estimates

In this section, we derive estimates on the gradient function, local in space and time, under the assumption that our surface is rotationally symmetric.

Suppose that on the compact subset $M_t \cap S_R(t)$, where $S_R(t)$ is the support of some as yet to be chosen cutoff function, we have that

$$u_0 = \inf_{t\in[0,T)} \inf_{M_t \cap S_R(t)} u(\cdot, t)$$

is bounded away from zero. From Corollary 3.3 we also have that

$$u_1 = \sup_{t\in[0,T)} \sup_{M_t \cap S_R(t)} u(\cdot, t)$$

is bounded from above by a constant which only depends on $\sup u(\cdot, 0)$, so long as we choose R sufficiently large. Having $u_0 > 0$ will be essential to our analysis, since it ensures that our setup is well-defined (i.e. $\boldsymbol{\omega}$ may be defined).

Recall the evolution equation for v,

$$\left(\frac{d}{dt} - \Delta\right) v = -|A|^2 v - 2v^{-1}|\nabla v|^2 + \left(\frac{n-1}{u^2}\right) v$$

For planar graphs, the evolution equation for v is identical, except for the last term which is not present. Somehow, we must handle that term as it could be the source of blowup in the gradient function.

The evolution equation for u contains an almost identical term

$$\left(\frac{d}{dt} - \Delta\right) u = -\frac{n-1}{u}$$

so, some function φ of u might do the trick. This technique has also been used by M. Simon, and Athenassenas ([19], [3]) to obtain similar gradient estimates on cylindrical graphs for (MCF) and volume preserving (MCF) respectively.

Lemma 3.7. *Let $\varphi = \varphi(u)$ be a non-negative function on $\mathbb{R}$, then*

$$\left(\frac{d}{dt} - \Delta\right) \varphi(u) = -u\varphi' \left(\frac{n-1}{u^2}\right) - \left(\frac{\varphi''}{\varphi'^2}\right) |\nabla\varphi|^2$$

Proof. This is a simple application of the product and chain rules. □

Setting $f = \varphi v$ (where φ is yet to be chosen) we have

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) f = &- \left[u\left(\frac{\varphi'}{\varphi}\right) - 1\right]\left(\frac{n-1}{u^2}\right) f - |A|^2 f \\ &- \left(\frac{\varphi''}{\varphi'^2}\right) f\varphi^{-1}|\nabla\varphi|^2 - 2f^{-1}\varphi^2|\nabla v|^2 \\ &- 2\langle\nabla\varphi, \nabla v\rangle\end{aligned}$$

and we also compute

$$f^{-1}|\nabla f|^2 = f^{-1}\varphi^2|\nabla v|^2 + 2\langle\nabla\varphi, \nabla v\rangle + f\varphi^{-2}|\nabla\varphi|^2$$

and using that

$$2\langle\nabla\varphi, \nabla v\rangle = 2\varphi^{-1}\langle\nabla\varphi, \nabla f\rangle - 2f\varphi^{-2}|\nabla\varphi|^2$$

we obtain

Lemma 3.8. *The function $f = \varphi(u)v$ satisfies the evolution equation*

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) f = &- \left[u\left(\frac{\varphi'}{\varphi}\right) - 1\right]\left(\frac{n-1}{u^2}\right) f - |A|^2 f \\ &- \left(\frac{\varphi''}{\varphi'^2}\right) f\varphi^{-1}|\nabla\varphi|^2 - 2f^{-1}|\nabla f|^2 + 2\varphi^{-1}\langle\nabla\varphi, \nabla f\rangle \end{aligned} \tag{3.4}$$

Now, introduce a general cut-off function ξ (to be chosen later) and, setting $g = f\xi$, we have the evolution equation

Proposition 3.9. *The function g satisfies the evolution equation*

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) g = &-|A|^2 g - 2g^{-1}|\nabla g|^2 + 2(\varphi\xi)^{-1}\langle\nabla(\varphi\xi), \nabla g\rangle \\ &- \left[u\left(\frac{\varphi'}{\varphi}\right) - 1\right]\left(\frac{n-1}{u^2}\right) g - \left(\frac{\varphi''}{\varphi'^2}\right)\varphi^{-1}|\nabla\varphi|^2 g \\ &+ g\xi^{-1}\left(\frac{d}{dt} - \Delta\right)\xi - 2(\varphi\xi)^{-1} g\langle\nabla\varphi, \nabla\xi\rangle\end{aligned} \tag{3.5}$$

Proof. Using the evolution equation for f, we find

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) g = &-|A|^2 g - 2f^{-1}|\nabla f|^2\xi + 2\varphi^{-1}\langle\nabla\varphi, \xi\nabla f\rangle \\ &- \left[u\left(\frac{\varphi'}{\varphi}\right) - 1\right]\left(\frac{n-1}{u^2}\right) g - \left(\frac{\varphi''}{\varphi'^2}\right)\varphi^{-1}|\nabla\varphi|^2 g \\ &+ f\left(\frac{d}{dt} - \Delta\right)\xi - 2\langle\nabla f, \nabla\xi\rangle\end{aligned}$$

We calculate

$$g^{-1}|\nabla g|^2 = f^{-1}|\nabla f|^2\xi + 2\langle\nabla f, \nabla\xi\rangle + f\xi^{-1}|\nabla\xi|^2$$

to obtain

$$\begin{aligned}-2f^{-1}|\nabla f|^2\xi - 2\langle\nabla f, \nabla\xi\rangle &= -2g^{-1}|\nabla g|^2 + 2\langle\nabla\xi, \nabla f\rangle + 2f\xi^{-1}|\nabla\xi|^2\\ &= -2g^{-1}|\nabla g|^2 + 2\xi^{-1}\langle\nabla\xi, \nabla g\rangle\end{aligned}$$

Finally, noting that

$$2\varphi^{-1}\langle\nabla\varphi, \xi\nabla f\rangle = 2\varphi^{-1}\langle\nabla\varphi, \nabla g\rangle - 2f\varphi^{-1}\langle\nabla\varphi, \nabla\xi\rangle$$

we have the result by combining these two identities. □

What remains, is to choose test- and cutoff-functions appropriately so that all terms on the right hand side of equation (3.5) remain non-positive, apart from terms involving the gradient of g, which are, for the most part, irrelevant.

Estimating the last term of (3.5) using Young's inequality, for $\mu > 0$ we obtain the equation

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) g \leqslant &-|A|^2 g - 2g^{-1}|\nabla g|^2 + 2(\varphi\xi)^{-1}\langle\nabla(\varphi\xi), \nabla g\rangle\\ &- \left[u\left(\frac{\varphi'}{\varphi}\right) - 1\right]\left(\frac{n-1}{u^2}\right) g - \left[\left(\frac{\varphi''\varphi}{\varphi'^2}\right) - \mu^{-1}\right]\varphi^{-2}|\nabla\varphi|^2 g\\ &+ \xi^{-1}\left[\left(\frac{d}{dt} - \Delta\right)\xi + \mu\xi^{-1}|\nabla\xi|^2\right] g \end{aligned} \tag{3.6}$$

Let us choose the cut-off function ξ. Let

$$\xi(\mathbf{x}, t) = (R^2 - \langle\mathbf{x} - \mathbf{x}_0, \boldsymbol{\vartheta}\rangle^2 - 2t)_+^{2(p+1)}$$

for $R, p > 0$ and some $\mathbf{x}_0$, that is, the 'band function' as defined in Lemma 3.1.

Inserting the result from Lemma 3.1 into (3.6), and setting $\mu = \frac{1}{2}\left(\frac{2p+1}{p+1}\right)$ we obtain

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) g \leqslant &-|A|^2 g - 2g^{-1}|\nabla g|^2 + 2(\varphi\xi)^{-1}\langle\nabla(\varphi\xi), \nabla g\rangle\\ &- \left[u\left(\frac{\varphi'}{\varphi}\right) - 1\right]\left(\frac{n-1}{u^2}\right) g\\ &- \left[\left(\frac{\varphi''\varphi}{\varphi'^2}\right) - 2\left(\frac{p+1}{2p+1}\right)\right]\varphi^{-2}|\nabla\varphi|^2 g \end{aligned} \tag{3.7}$$

Finally, we choose the test-function φ. Let

$$\varphi(u) = e^{\frac{1}{\lambda+1}\left(\frac{u}{u_0}\right)^{\lambda+1}}$$

for $\lambda \geqslant 0$ and we find that

$$u\left(\frac{\varphi'}{\varphi}\right) - 1 = \left(\frac{u}{u_0}\right)^{\lambda+1} - 1 \geqslant 0$$

and

$$\begin{aligned}\left(\frac{\varphi''\varphi}{\varphi'^2}\right) - 2\left(\frac{p+1}{2p+1}\right) &= 1 + (\lambda+1)\left(\frac{u}{u_0}\right)^{-(\lambda+1)} - 2\left(\frac{p+1}{2p+1}\right) \\ &= (\lambda+1)\left(\frac{u}{u_0}\right)^{-(\lambda+1)} - \frac{1}{2p+1} \\ &\geqslant 0\end{aligned}$$

for $p \geqslant 0$ sufficiently high.

So, we obtain the equation

$$\left(\frac{d}{dt} - \Delta\right) g \leqslant -|A|^2 g - 2g^{-1}|\nabla g|^2 + 2(\varphi\xi)^{-1}\left\langle \nabla(\varphi\xi), \nabla g\right\rangle \tag{3.8}$$

This equation is now in a form to which the maximum principle may be applied and we obtain the local estimate

Theorem 3.10. *Set* $S_R(t) = \left\{\mathbf{x} \in \mathbb{R}^{n+1} : \langle \mathbf{x} - \mathbf{x}_0, \boldsymbol{\vartheta}\rangle^2 \leqslant R^2 - 2t\right\}, R > 0$ *and suppose that on* M_t *we have* $u_0 > 0$*, then for all* $\lambda > 0$ *and* $p \geqslant 0$ *sufficiently high, we have*

$$\begin{aligned}\sup_{M_t \cap S_R(t)} &\left(e^{\frac{1}{\lambda+1}\left(\frac{u}{u_0}\right)^{\lambda+1}} \left(R^2 - \langle \mathbf{x} - \mathbf{x}_0, \boldsymbol{\vartheta}\rangle^2 - 2t\right)_+^{2(p+1)} v\right)(\cdot, t) \leqslant \\ &\sup_{M_0 \cap S_R(0)} \left(e^{\frac{1}{\lambda+1}\left(\frac{u}{u_0}\right)^{\lambda+1}} \left(R^2 - \langle \mathbf{x} - \mathbf{x}_0, \boldsymbol{\vartheta}\rangle^2\right)_+^{2(p+1)} v\right)(\cdot, 0)\end{aligned} \tag{3.9}$$

for $t \in \left[0, \min\left\{T, \frac{R^2}{2}\right\}\right)$.

Proof. This is a simple application of the maximum principle for compactly supported functions applied to the evolution equation (3.8) for the function $g = f\xi$ on the compact (shrinking) sets $S_R(t)$. □

Remark 3.11. Since $v \geqslant 1$, the local gradient estimate Theorem 3.10 implies an alternative (but less sharp) local height estimate to Theorem 3.2 of the form

$$\sup_{M_t \cap S_{\frac{R}{2}}} e^{\frac{1}{\lambda+1}\left(\frac{u}{u_0}\right)^{\lambda+1}}(\cdot, t) \leqslant C(R)$$

for $t \in \left[0, \min\left\{T, \frac{R^2}{2}\right\}\right)$.

Corollary 3.12 (Local Gradient Estimate). *Suppose that on the set $S_k = \{\mathbf{x} \in \mathbb{R}^{n+1} : \langle \mathbf{x}, \boldsymbol{\vartheta} \rangle^2 \leqslant k^2\}$ for some $k > 0$ we have the estimate*

$$\sup_{M_0 \cap S_k} v(\cdot, 0) \leqslant C_0$$

for some $C_0 < \infty$, then if $u \geqslant u_0 > 0$ we have the estimate

$$\sup_{M_t \cap S_{\frac{k}{2}}} v(\cdot, t) \leqslant C$$

for $t \in \left[0, \left(\frac{k}{2}\right)^2\right)$ and some constant C depending only on C_0, k and u_0.

There are many other alternatives to using the band cut-off function in order to obtain compact subsets on which to derive local estimates. One such example is the spot cut-off function.

Consider the function $k = k(\mathbf{x}, t)$ defined by

$$k = \frac{1}{2}\left(1 - \langle \mathbf{Z}, \boldsymbol{\omega} \rangle\right)$$

where $\mathbf{Z} = \boldsymbol{\omega}(\mathbf{x}_0)$ is the normal to the cylinder at some $\mathbf{x}_0$ in M_0. This function can be used to measure angular deviation from some given direction. This function can be used to select out a spot on the evolving surface M_t.

Lemma 3.13 (Spot Function). *The function $\psi = \psi(\mathbf{x}, t)$ defined by*

$$\psi(\mathbf{x}, t) = \left(R^2 - \langle \mathbf{x}, \boldsymbol{\vartheta} \rangle^2 - \left(\frac{R}{\theta}\right)^2 k^2 - \left(\left(\frac{R}{\theta}\right)^2 \left(\frac{n+2}{u_0^2}\right) + 2\right) t\right)_+^{2(p+1)} \tag{3.10}$$

for some given $R, \theta > 0$, $p > 0$ satisfies

$$\left(\frac{d}{dt} - \Delta\right) \psi \leqslant -\frac{1}{2}\left(\frac{2p+1}{p+1}\right) \psi^{-1} |\nabla \psi|^2$$

for $\mathbf{x} \in M_t \cap \operatorname{spt} \psi$ so long as $u_0 > 0$.

Proof. First we compute

$$\begin{aligned} \frac{d}{dt} \langle \mathbf{Z}, \boldsymbol{\omega} \rangle &= \left\langle \mathbf{Z}, \overline{\nabla}_{\mathbf{H}} \boldsymbol{\omega} \right\rangle \\ &= \frac{1}{u} \langle \mathbf{Z}, \mathbf{H}_\tau \rangle \\ &= \frac{1}{u} \langle \mathbf{Z}_\tau, \mathbf{H} \rangle \end{aligned}$$

and also

$$\begin{aligned}\Delta\langle\mathbf{Z},\boldsymbol{\omega}\rangle &= \boldsymbol{\tau}_i\boldsymbol{\tau}_i\langle\mathbf{Z},\boldsymbol{\omega}\rangle\\ &= \boldsymbol{\tau}_i\left(\frac{1}{u}\langle\mathbf{Z}_\tau,\boldsymbol{\tau}_i\rangle\right)\\ &= -\frac{1}{u^2}\langle\mathbf{Z}_\tau,\nabla u\rangle+\frac{1}{u}\left\langle\overline{\nabla}_{\boldsymbol{\tau}_i}\mathbf{Z}_\tau,\boldsymbol{\tau}_i\right\rangle+\frac{1}{u}\langle\mathbf{Z}_\tau,\mathbf{H}\rangle\\ &= -\frac{2}{u^2}\langle\mathbf{Z}_\tau,\nabla u\rangle-\left(\frac{n-1-\kappa}{u^2}\right)\langle\mathbf{Z},\boldsymbol{\omega}\rangle+\frac{1}{u}\langle\mathbf{Z}_\tau,\mathbf{H}\rangle\end{aligned}$$

which gives us

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)k &= -\frac{1}{2}\left(\frac{n-1-\kappa}{u^2}\right)\langle\mathbf{Z},\boldsymbol{\omega}\rangle-\frac{1}{u^2}\langle\mathbf{Z}_\tau,\nabla u\rangle\\ &\geqslant -\frac{1}{2}\left(\frac{n-1-\kappa}{u^2}\right)-\frac{1}{u^2}\\ &\geqslant -\frac{1}{2}\left(\frac{n+1}{u_0^2}\right)\end{aligned}$$

Now we compute

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)\psi &= 2(p+1)\psi^{\frac{1}{2}\left(\frac{2p+1}{p+1}\right)}\left[-\left(\frac{d}{dt}-\Delta\right)\langle\mathbf{x},\boldsymbol{\vartheta}\rangle^2-\left(\frac{R}{\theta}\right)^2\left(\frac{d}{dt}-\Delta\right)k^2-\beta\right]\\ &\quad -\frac{1}{2}\left(\frac{2p+1}{p+1}\right)\psi^{-1}|\nabla\psi|^2\\ &\leqslant 2(p+1)\psi^{\frac{1}{2}\left(\frac{2p+1}{p+1}\right)}\left[2|\nabla\langle\mathbf{x},\boldsymbol{\vartheta}\rangle|^2+\left(\frac{R}{\theta}\right)^2\left(\left(\frac{n+1}{u_0^2}\right)k+2|\nabla k|^2\right)-\beta\right]\\ &\quad -\frac{1}{2}\left(\frac{2p+1}{p+1}\right)\psi^{-1}|\nabla\psi|^2\end{aligned}$$

and we check that

$$\begin{aligned}|\nabla k|^2 &= \frac{1}{4u^2}\left[|\mathbf{Z}_\tau|^2-\langle\mathbf{Z}_{\boldsymbol{\tau}},\boldsymbol{\nu}\rangle\right]\\ &\leqslant \frac{1}{4u_0^2}\end{aligned}$$

So, recalling that $|\nabla\langle\mathbf{x},\boldsymbol{\vartheta}\rangle|^2\leqslant 1$ and setting

$$\beta=\left(\left(\frac{R}{\theta}\right)^2\left(\frac{n+2}{u_0^2}\right)+2\right)$$

we have the result. □

Note that for $\theta\in(0,1)$, the set $M_t\cap\operatorname{spt}\psi$ is a subset of M_t which does not wrap around the axis completely, however if we let $\theta\to\infty$ we obtain the same sets to those given by $M_t\cap\operatorname{spt}\xi$, the band function.

3.2.2 Global Estimates

Without some extra assumptions on the growth of the height of the evolving surface, it is quite difficult to obtain estimates for the gradient that do not grow strongly in time (non-polynomially). This will be remedied in Chapter 5, however, in the meantime a quick global estimate for the gradient is presented here.

Lemma 3.14. *The function $\hat{v}$ defined by*

$$\hat{v} = e^{-\left(\frac{n-1}{u_0^2}\right)t} v$$

satisfies the evolution equation

$$\left(\frac{d}{dt} - \Delta\right) \hat{v} \leqslant -|A|^2 \hat{v} - 2\hat{v}^{-1} |\nabla \hat{v}|^2 \tag{3.11}$$

Proposition 3.15. *Suppose that $u(\mathbf{x}, t) \geqslant u_0 > 0$ for all $\mathbf{x} \in M_t, t \in [0, T)$, and suppose that on M_0 we have the estimate*

$$v \leqslant \left(\varepsilon^2 + \langle \beta \mathbf{x}, \boldsymbol{\vartheta} \rangle^2\right)^p$$

for some $\varepsilon > 0, \beta, p \geqslant 0$ then on M_t we have the estimate

$$v \leqslant c_2 e^{\left(\frac{n-1}{u_0^2}\right)t} \left(\varepsilon^2 + \langle \beta \mathbf{x}, \boldsymbol{\vartheta} \rangle^2 + 2\beta^2 t\right)^p$$

for some $c_2 < \infty$ for all $t \in [0, T)$

Proof. Let η be defined as in Lemma 3.4, with $\gamma = 0$, then we compute

$$\left(\frac{d}{dt} - \Delta\right) \hat{v}\eta \leqslant -|A|^2 \hat{v}\eta - 2\eta \hat{v}^{-1} |\nabla \hat{v}|^2 - \left(\frac{p+1}{p}\right) \hat{v}\eta^{-1} |\nabla \eta|^2 - 2 \langle \nabla \hat{v}, \nabla \eta \rangle$$

Estimating the cross term using Young's inequality,

$$-2 \langle \nabla \hat{v}, \nabla \eta \rangle \leqslant 2\eta \hat{v} |\nabla \hat{v}|^2 + \frac{1}{2} \hat{v} \eta^{-1} |\nabla \eta|^2$$

and, since $p \geqslant 0$, we obtain

$$\left(\frac{d}{dt} - \Delta\right) \hat{v}\eta \leqslant -|A|^2 \hat{v}\eta - \frac{1}{2}\left(\frac{p+2}{p}\right) \hat{v}\eta^{-1} |\nabla \eta|^2$$

and thus, applying the maximum principle to $\hat{v}\eta$ we obtain the result. □

While this estimate is sufficient to prove longtime existence, in some applications it won't be very useful because it still grows very strongly in time. Since we haven't used the fact that the minimum height may in fact be increasing, we can still improve on this estimate. We shall attend to these details (Chapter 5) once we have derived some barriers for the height.

3.3 Curvature Estimate

3.3.1 Local Estimates

In this section we will derive local estimates on the curvature and covariant derivatives of the curvature of all orders. We will do this using a version ([9]) of the Ecker and Huisken ([11]) localised weak maximum principle (see Appendix E for more details).

Without some extra work, we will not be able to derive local curvature estimates which do not grow strongly in time. The technical reason is that the order v term in the evolution equation for v needs to be taken care of.

While we can control this term using the evolution equation for u, in doing so we would lose some of the 'good' gradient of v term. If we are very careful we can retain just enough of the gradient term and obtain a local estimate for the curvature, independent of time.

The following proposition is motivated by the local curvature estimates derived in [9] and [11], adapted to cylindrical graphs.

Proposition 3.16. *Let $(M_t)_{t\in(t_0-R^2,t_0)}$, $R>0$ be a smooth, properly embedded solution in the set $Q_R(\mathbf{x}_0)=\{\mathbf{x}\in\mathbb{R}^{n+1}:\langle\mathbf{x}-\mathbf{x}_0,\boldsymbol{\vartheta}\rangle^2\leqslant R^2\}$ for some $\mathbf{x}_0\in\mathbb{R}^{n+1}$. Suppose that in this set $u_0>0$ and for $f=\varphi v$ (as above) the estimate*

$$f_1=\sup_{M_t\cap Q_R(\mathbf{x}_0)} f(\cdot,t)\leqslant 2\inf_{M_t\cap Q_R(\mathbf{x}_0)} f(\cdot,t)=f_0$$

holds for $t\in(t_0-R^2,t_0)$, then there exists a constant $c_0=c_0(n,u_0,f_1)$ such that

$$\sup_{M_t\cap Q_{\frac{R}{2}}(\mathbf{x}_0)}|A|^2(\cdot,t)\leqslant\frac{c_0}{R^2}$$

for $t\in\left(t_0-\left(\frac{R}{2}\right)^2,t_0\right)$.

Proof. First, we shall scale the solution such that

$$(M_t)_{t\in(t_0-R^2,t_0)}\mapsto(M_s^R)_{s\in(0,1)}$$

Set for any $R>0$

$$M_s^R=\frac{1}{R}\left(M_{t_0-R^2(1-s)}-\mathbf{x}_0\right)$$

for $s\in(0,1)$.

This is effected by the mapping

$$t\mapsto 1-\left(\frac{t_0-t}{R^2}\right)$$
$$\mathbf{x}\mapsto\frac{\mathbf{x}-\mathbf{x}_0}{R}$$

for $R > 0$ which preserves (MCF), that is $(M_s^R)_{s\in(0,1)}$ also solves (MCF).

Setting $s = 1 - \left(\frac{t_0 - t}{R^2}\right)$ and $\mathbf{y} = \frac{\mathbf{x}-\mathbf{x}_0}{R}$, we deduce that A^R, the second fundamental form of (M_s^R), satisfies

$$|\nabla^m A^R|^2(\mathbf{y}, s) = R^{2(m+1)}|\nabla^m A|^2(\mathbf{x}, t) \tag{3.12}$$

for $\mathbf{y} \in M_s$ and all $m \geqslant 0$.

Thus, to prove the proposition, we need only show that

$$\sup_{M_t^R \cap Q_{\frac{1}{2}}(0)} |A|^2(\cdot, t) \leqslant c_0$$

for $t \in (0, 1)$.

As shown in Appendix B, the curvature $|A|^2$ satisfies the evolution equation

$$\left(\frac{d}{dt} - \Delta\right)|A|^2 = 2|A|^4 - 2|\nabla A|^2$$

We must construct a suitable test function to control the first term of the evolution equation of $|A|^2$, since it might cause the curvature to blow up. We try a test function of $f = \varphi v$.

As in Theorem 3.10, we set $\varphi = \varphi(u)$ to be

$$\varphi = e^{\frac{1}{\lambda+1}\left(\frac{u}{u_o}\right)^{\lambda+1}}$$

for $\lambda > 0$, and analogously from earlier calculations, we have

$$\left(\frac{d}{dt} - \Delta\right) f \leqslant -|A|^2 f - (1+\delta) f^{-1}|\nabla f|^2 \tag{3.13}$$

where $\delta = \left(\frac{\bar{\delta}}{1+\bar{\delta}}\right) > 0$ where $\bar{\delta} = \lambda\left(\frac{u}{u_0}\right)^{-(\lambda+1)}$.

Remember that we have an *a priori* bound on f on compact subsets. Thus we may assume that f is uniformly bounded on the set $Q_R(\mathbf{x}_0)$ for all times $t \in [t_0 - R^2, t_0]$.

Now, let $h = h(f^2)$ be a test-function to be chosen later, and setting $g = |A|^2 h$, we compute

$$\left(\frac{d}{dt} - \Delta\right) g \leqslant -\frac{2}{h^2}\left[f^2 h' - h\right] g^2 - 2h|\nabla A|^2 \\ - \frac{2}{h^2}\left[(2+\delta)hh' + 2f^2 hh''\right]|\nabla f|^2 g - 2\left\langle \nabla h, \nabla |A|^2 \right\rangle$$

Using the identity

$$\frac{1}{2} g^{-1}|\nabla g|^2 = 2h|\nabla|A||^2 + \left\langle \nabla h, \nabla |A|^2 \right\rangle + 2f^2\left(\frac{h'}{h}\right)^2 |\nabla f|^2 g$$

and Kato's inequality, we have

$$\begin{aligned}
-2h|\nabla A|^2 - 2\left\langle \nabla h, \nabla |A|^2\right\rangle &\leqslant -2h|\nabla |A||^2 - 2\left\langle \nabla h, \nabla |A|^2\right\rangle \\
&= -\frac{1}{2}g^{-1}|\nabla g|^2 - \left\langle \nabla h, \nabla |A|^2\right\rangle \\
&\quad + 2f^2\left(\frac{h'}{h}\right)^2 |\nabla f|^2 g \\
&= -\frac{1}{2}g^{-1}|\nabla g|^2 - h^{-1}\left\langle \nabla h, \nabla g\right\rangle \\
&\quad + 6f^2\left(\frac{h'}{h}\right)^2 |\nabla f|^2 g
\end{aligned}$$

from which we obtain

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta\right) g \leqslant -\frac{1}{2}g^{-1}|\nabla g|^2 - h^{-1}\left\langle \nabla h, \nabla g\right\rangle - \frac{2}{h^2}\left[f^2h' - h\right] g^2 \\
- \frac{2}{h^2}\left[(2+\delta)hh' + f^2(2hh'' - 3h'^2)\right]|\nabla f|^2 g \quad (3.14)
\end{aligned}$$

Now the task is to find a function h such that all the terms are bounded in such a way as to satisfy the hypotheses of Theorem E.5.

Let us begin by letting $h = h(q)$ be defined by

$$h(q) = \left(\frac{\left(\frac{q}{q_1}\right)}{\left(1 - L\left(\frac{q}{q_1}\right)^{\varepsilon}\right)^{\frac{1}{\varepsilon}}}\right)^{\gamma}, \qquad q \in [q_0, q_1]$$

for $\varepsilon, \gamma > 0$ and $0 < L < 1$.

Consider the coefficient of the order g^2 term, then we have

$$\frac{2}{h^2}\left[f^2h' - h\right] = \frac{2}{h}\left(1 - L\left(\frac{q}{q_1}\right)^{\varepsilon}\right)^{-1}\left[L\left(\frac{q}{q_1}\right)^{\varepsilon} + \gamma - 1\right]$$

and similarly, the coefficient of the order g term is

$$\frac{2}{h^2}\left[(2+\delta)hh' + q(2hh'' - 3h'^2)\right]|\nabla f|^2 = 2\gamma q^{-1}\left[(2\varepsilon - \delta)L\left(\frac{q}{q_1}\right)^{\varepsilon} + \delta - \gamma\right]|\nabla f|^2$$

So, in order to satisfy Theorem E.5, we require the inequalities

$$L\left(\frac{q}{q_1}\right)^{\varepsilon} + \gamma - 1 > 0 \quad \text{and} \quad (2\varepsilon - \delta)L\left(\frac{q}{q_1}\right)^{\varepsilon} + \delta - \gamma \geqslant 0$$

to hold, for any $\delta > 0$. Let $\varepsilon = \delta$, then we require the inequality

$$q_1 < L^{\frac{1}{\delta}}\left(\frac{1+\delta}{1-\delta}\right)^{\frac{1}{\delta}} q, \quad q \in [q_0, q_1]$$

to hold to be able to choose γ to satisfy the two inequalities for $\delta > 0$.

Choose $L = 2^{-\delta}$ and in noting that $\left(\frac{1+\delta}{1-\delta}\right)^{\frac{1}{\delta}} > e^2$ it is sufficient to require

$$q_1 < \frac{1}{2}e^2 q_0$$

which is satisfied, since by assumption

$$f_1 \leqslant 2f_0$$

So, with this, and choosing a suitable γ, we have

$$\left(\frac{d}{dt} - \Delta\right) g \leqslant \langle \mathbf{a}, \nabla g\rangle - kg^2 - dg \tag{3.15}$$

with $k > 0$, $d \geqslant 0$ and $|\mathbf{a}|^2 \leqslant a_0^2(\delta)(1+d)$.

Let ψ be the function defined by

$$\psi(\mathbf{x}, t) = t(1 - \langle \mathbf{x}, \boldsymbol{\vartheta}\rangle)_+^3$$

We can easily verify that ψ is a localisation function. Note that

$$\operatorname{spt}\psi(\cdot, t) = \begin{cases} \emptyset, & t = 0 \\ Q_1(0), & t \in (0, 1] \end{cases}$$

Since the surfaces $(M_t^R)_{t\in(0,1)}$ are properly embedded, the set $M_t^R \cap Q_1(0)$ is compact for all $t \in (0, 1)$.

Using ψ, we are now able to apply Theorem E.5 to g in the set $Q_1(0)$, since all the hypotheses are satisfied. Thus, we arrive at the estimate

$$\sup_{M_t^R \cap Q_1(\mathbf{x}_0)} (g\psi)(\cdot, t) \leqslant C(n, k, a_0, c_\psi)$$

for $t \in (0, 1)$ since $\psi(\cdot, 0) = 0$.

Upon $Q_{\frac{1}{2}}(0) \times \left(\frac{3}{4}, 1\right)$ we have ψ uniformly bounded away from zero. Furthermore, on the same set we also have h uniformly bounded above and below. Thus, since $\sup_{M_t^R \cap Q_{\frac{1}{2}}(\mathbf{x}_0)}(g\psi) \leqslant \sup_{M_t^R \cap Q_1(\mathbf{x}_0)}(g\psi)$, we obtain the estimate

$$\sup_{M_t^R \cap Q_{\frac{1}{2}}(\mathbf{x}_0)} |A|^2 \leqslant C$$

for $t \in [\frac{3}{4}, 1)$ where $C = C(n, k, a_0, c_\psi, f_1)$

Scaling back to the original flow yields the estimate in

$$Q_{\frac{R}{2}}(\mathbf{x}_0) \times \left(t_0 - \left(\frac{R}{2}\right)^2, t_0\right)$$

□

Proposition 3.16 gives an interior estimate on the curvature, however, it requires a rather restrictive technical assumption, which renders it somewhat limited when applied to proving long-time existence. If we sacrifice some of the sharpness of this estimate, then we may remove this technical assumption, obtaining an estimate suitable for proving long-time existence results.

Proposition 3.17. *Let $(M_t)_{t\in(t_0-R^2,t_0)}$, $R>0$ be a smooth, properly embedded solution in the set $Q_R(\mathbf{x}_0)=\{\mathbf{x}\in\mathbb{R}^{n+1}:\langle \mathbf{x}-\mathbf{x}_0,\boldsymbol{\vartheta}\rangle^2\leqslant R^2\}$ for some $\mathbf{x}_0\in\mathbb{R}^{n+1}$. Suppose that in this set $u_0>0$ and the estimate*

$$\sup_{M_t\cap Q_R(\mathbf{x}_0)} v(\cdot,t)\leqslant v_0$$

holds for $t\in(t_0-R^2,t_0)$, then there exist constants $c_0=c_0(n,v_0)$ and $\zeta_0=\zeta_0(n,u_0)$ such that

$$\sup_{M_t\cap Q_{\frac{R}{2}}(\mathbf{x}_0)}|A|^2(\cdot,t)\leqslant\left(\frac{c_0}{R^2}\right)e^{\zeta_0\left(t-(t_0-R^2)\right)}$$

for $t\in\left(t_0-\left(\frac{R}{2}\right)^2,t_0\right)$.

Proof. The proof of this proposition is almost identical to that in Proposition 3.16, except that instead of working with the evolution equation of f, we work directly with the evolution equation of $\hat{v}=e^{-\zeta_0\left(t-(t_0-R^2)\right)}v$

$$\left(\frac{d}{dt}-\Delta\right)\hat{v}\leqslant-|A|^2\hat{v}-2\hat{v}^{-1}|\nabla\hat{v}|^2$$

for $\zeta_0\geqslant\left(\frac{n-1}{u_0^2}\right)$.

Working on the scaled solution $(M_t^R)_{t\in(0,1)}$, analogously to Equation (3.14) for the function $g=|A|^2h(\hat{v}^2)$, where h is an arbitrary positive twice-differentiable function, we obtain the evolution equation

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)g\leqslant-\frac{1}{2}g^{-1}|\nabla g|^2-h^{-1}\langle\nabla h,\nabla g\rangle-\frac{2}{h^2}\left[\hat{v}^2h'-h\right]g^2\\-\frac{2}{h^2}\left[3hh'+\hat{v}^2(2hh''-3h'^2)\right]|\nabla\hat{v}|^2g\end{aligned}\quad(3.16)$$

Choosing $h=h(q)$ to be defined by

$$h(q)=\left(\frac{\left(\frac{q}{q_1}\right)}{1-\frac{1}{2}\left(\frac{q}{q_1}\right)}\right),\quad q\in[1,q_1]$$

we have

$$\frac{2}{h^2}\left[\hat{v}^2h'-h\right]=1$$

and
$$\frac{2}{h^2}\left[3hh' + \hat{v}^2(2hh'' - 3h'^2)\right] = \frac{1}{v_0^2\left(1 - \frac{1}{2}\left(\frac{\hat{v}}{v_0}\right)^2\right)^2}$$

As in Proposition 3.16, we are able to apply Theorem E.5 with the localisation function Ψ defined by
$$\psi(\mathbf{x}, t) = t(1 - \langle \mathbf{x}, \boldsymbol{\vartheta}\rangle)_+^3$$
to obtain the estimate
$$\sup_{M_t^R \cap Q_1(\mathbf{x}_0)} (g\psi)(\cdot, t) \leqslant C(n, k, a_0, c_\psi)$$
for $t \in (0, 1)$ since $\psi(\cdot, 0) = 0$.

Upon $Q_{\frac{1}{2}}(0) \times \left(\frac{3}{4}, 1\right)$ we have ψ uniformly bounded away from zero. Furthermore, on the same set we also have h uniformly bounded above and below, since
$$\left(\frac{\hat{v}}{v_0}\right)^2 \leqslant h(\hat{v}^2) \leqslant 2\left(\frac{\hat{v}}{v_0}\right)^2$$

Thus, since $\sup_{M_t^R \cap Q_{\frac{1}{2}}(\mathbf{x}_0)}(g\psi) \leqslant \sup_{M_t^R \cap Q_1(\mathbf{x}_0)}(g\psi)$, we obtain the estimate
$$\sup_{M_t^R \cap Q_{\frac{1}{2}}(\mathbf{x}_0)} |A|^2 \leqslant Ce^{\zeta t}$$
for $t \in [\frac{3}{4}, 1)$ where $C = C(n, k, a_0, c_\psi, v_0)$.

Scaling back to the original flow yields the estimate in
$$Q_{\frac{R}{2}}(\mathbf{x}_0) \times \left(t_0 - \left(\frac{R}{2}\right)^2, t_0\right)$$
□

Either of the local curvature estimates can be extended to a local estimate on all derivatives of the curvature.

Theorem 3.18 (Smoothness Estimate). *Let $(M_t)_{t\in(t_0-R^2,t_0)}$, $R > 0$ be a smooth, properly embedded solution in the set $Q_R(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^{n+1} : \langle \mathbf{x} - \mathbf{x}_0, \boldsymbol{\vartheta}\rangle^2 \leqslant R^2\}$ for some $\mathbf{x}_0 \in \mathbb{R}^{n+1}$, satisfying the estimate*
$$\sup_{M_t \cap Q_R(\mathbf{x}_0)} |A|^2(\cdot, t) \leqslant \frac{c_0}{R^2}$$
for $t \in \left(t_0 - R^2, t_0\right)$ then for every $m \geqslant 1$ there exists a constant $c_m = c_m(n, m)$ such that
$$\sup_{M_t \cap Q_{\frac{R}{2}}(\mathbf{x}_0)} |\nabla^m A|^2(\cdot, t) \leqslant \frac{c_m}{R^{2(m_1)}}$$
for $t \in \left(t_0 - \left(\frac{R}{2}\right)^2, t_0\right)$.

Proof. As in Proposition 3.16, owing to the scaling behaviour of the second fundamental form and its derivatives, we may rescale appropriately and our assumption becomes
$$\sup_{M_t^R \cap Q_1(0)} |A|^2(\cdot,t) \leqslant c_0 \tag{3.17}$$
for $t \in (0,1)$. What is to be show is that that for $m \geqslant 0$ there exist constants c_m such that
$$\sup_{M_t^R \cap Q_{\frac{1}{2}}(0)} |\nabla^m A|^2(\cdot,t) \leqslant c_m$$
for $t \in \left(\frac{3}{4},1\right)$. This we shall show by induction. By the assumption (3.17), we trivially obtain the base case for $m=0$ since
$$\sup_{M_t^R \cap Q_{\frac{1}{2}}(0)} |A|^2(\cdot,t) \leqslant \sup_{M_t^R \cap Q_1(0)} |A|^2(\cdot,t) \leqslant c_0$$
for $t \in \left(\frac{3}{4},1\right) \subset (0,1)$.

Now, assume that for $k = 1, \ldots, m-1$ that we have established the inequalities
$$\sup_{M_t^R \cap Q_{\frac{1}{2}}(0)} |\nabla^k A|^2(\cdot,t) \leqslant c_k$$
for $t \in \left(\frac{3}{4},1\right)$.

Using these estimates, we aim to establish an estimate for $|\nabla^m A|^2$. From Corollary B.9, and Young's inequality, we have the evolution equation
$$\left(\frac{d}{dt} - \Delta\right) |\nabla^m A|^2 \leqslant -2|\nabla^{m+1}A|^2 + C_m(1+|\nabla^m A|^2) \tag{3.18}$$
where $C_m = C_m(c_0, \ldots, c_{m-1})$.

Let $g = |\nabla^m A|^2(|\nabla^{m-1}A|^2 + \Lambda)$ for some $\Lambda > 0$ and compute
$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) g \leqslant -2|\nabla^{m+1}A|^2(|\nabla^{m-1}A|^2+\Lambda)+C_m(1+|\nabla^m A|^2)(|\nabla^{m-1}A|^2+\Lambda)\\ - 2|\nabla^m A|^4 + C_{m-1}(1+|\nabla^{m-1}A|^2)|\nabla^m A|^2\\ - 2\left\langle \nabla|\nabla^{m-1}A|^2, \nabla|\nabla^m A|^2\right\rangle \end{aligned} \tag{3.19}$$
and estimate the cross-term using the Cauchy-Schwarz, Kato, and Young inequalities
$$\begin{aligned}-2\left\langle \nabla|\nabla^{m-1}A|^2, \nabla|\nabla^m A|^2\right\rangle &\leqslant 8|\nabla^{m-1}A||\nabla^m A|\,\big|\nabla|\nabla^{m-1}A|\big|\,\big|\nabla|\nabla^m A|\big|\\ &\leqslant 8|\nabla^{m-1}A||\nabla^m A||\nabla^m A||\nabla^{m+1}A|\\ &\leqslant 2|\nabla^{m+1}A|^2(|\nabla^{m-1}A|^2+\Lambda)\\ &\quad + 8\left(\frac{|\nabla^{m-1}A|^2}{|\nabla^{m-1}A|^2+\Lambda}\right)|\nabla^m A|^4\end{aligned}$$

which gives us

$$\left(\frac{d}{dt}-\Delta\right)g \leqslant -2\left[1-4\left(\frac{|\nabla^{m-1}A|^2}{|\nabla^{m-1}A|+\Lambda}\right)\right]|\nabla^m A|^2 + C|\nabla^m A|^2$$

where $C = C(\Lambda, C_m, C_{m-1})$, and again using Young's inequality, we obtain

$$\left(\frac{d}{dt}-\Delta\right)g \leqslant -\left[\frac{\Lambda - 7|\nabla^{m-1}A|^2}{(|\nabla^{m-1}A|+\Lambda)^3}\right]g^2 + K$$

Setting $\Lambda = 8c_{m-1}$ we succeed in obtaining an evolution equation of the form

$$\left(\frac{d}{dt}-\Delta\right)g \leqslant -kg^2 + K$$

where $k = \frac{1}{729c_0^2} > 0$ and $K = K(n, m, c_0, \ldots, c_{m-1})$ by the inductive assumption. As in Proposition 3.16, this equation is in a suitable form to apply Theorem E.5 with a suitable localisation function.

As in Proposition 3.16, we choose the localisation function ψ defined by

$$\psi(\mathbf{x}, t) = t(1 - \langle \mathbf{x}, \boldsymbol{\vartheta}\rangle)_+^3$$

and in applying Theorem E.5 we obtain the estimate

$$\sup_{M_t^R \cap Q_1(0)} (g\psi)(\cdot, t) \leqslant C(n, k, K, c_\psi)$$

for $t \in (0, 1)$ since $\psi(\cdot, 0) = 0$.

Upon $Q_{\frac{1}{2}}(0) \times \left(\frac{3}{4}\right)$ we have ψ bounded away from zero, and we obtain the estimate

$$\sup_{M_t^R \cap Q_{\frac{1}{2}}(0)} |\nabla^m A|^2 \leqslant c_m$$

for $t \in \left(\frac{3}{4}, 1\right)$ where $c_m = c_m(n, m, c_0, \ldots, c_{m-1})$.

Scaling back to the original flow yields the estimate in

$$Q_{\frac{R}{2}}(\mathbf{x}_0) \times \left(t_0 - \left(\frac{R}{2}\right)^2, t_0\right)$$

□

3.3.2 Global Estimates

Global curvature estimates that grow polynomially in time are quite difficult to obtain in the cylindrical graph setting, owing to the term linear in v in the evolution equation for v, a term that is not present in the planar graph setting. In absence of the more delicate barrier techniques developed in Chapter 5, we present a global estimate for the curvature, growing exponentially in time.

Proposition 3.19. *The function $g = |A|^2\hat{v}^2$ satisfies the evolution equation*

$$\left(\frac{d}{dt} - \Delta\right) g \leqslant -2\hat{v}^{-1} \langle \nabla\hat{v}, \nabla g\rangle$$

where $\hat{v} = e^{-\left(\frac{n-1}{u_0^2}\right)t} v$.

Proof. This follows easily from the estimate

$$-2\left\langle \nabla\hat{v}^2, \nabla|A|^2\right\rangle \leqslant -2\hat{v}^{-1}\langle\nabla\hat{v}, \nabla g\rangle + 6|A|^2|\nabla\hat{v}|^2 + 2\hat{v}^2|\nabla|A||^2$$

□

Corollary 3.20. *Let $(M_t)_{t\in[0,T)}, T > 0$ be a smooth solution to (MCF) with bounded curvature and gradient and $u \geqslant u_0 > 0$ on each M_t then we have the a priori estimate*

$$\sup_{M_t}(|A|^2v^2)(\cdot, t) \leqslant e^{\left(\frac{n-1}{u_0^2}\right)t} \sup_{M_0}(|A|^2v^2)(\cdot, 0)$$

for all $t \in [0, T)$.

Proof. This follows simply from Proposition 3.19 and the estimate

$$\begin{aligned}|\nabla v| &= v^2|\nabla\langle\boldsymbol{\nu}, \boldsymbol{\omega}\rangle| \\ &= v^2|h_{ij}\langle\boldsymbol{\omega}, \boldsymbol{\tau}_i\rangle\boldsymbol{\tau}_j| \\ &\leqslant |A|v^2\end{aligned}$$

which allows us to apply the non-compact maximum principle (Theorem E.1) with $\mathbf{a} = -2\hat{v}^{-1}\nabla\hat{v}$. □