Deformations of Rational Varieties with Codimension-One Torus Action

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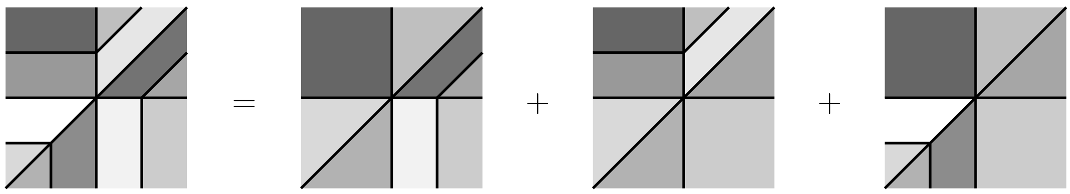

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Eidesstattliche Erklärung

Gemäß §7 (4) der Promotionsordnung des Fachbereichs Mathematik und Informatik der Freien Universität Berlin versichere ich hiermit, diese Arbeit selbstständig verfasst zu haben. Ich habe alle bei der Erstellung dieser Arbeit benutzten Hilfsmittel und Hilfen angegeben.

Abstract

We construct so-called T-deformations of rational, normal varieties with effective codimension-one torus action by gluing together deformations of affine varieties. We then analyze the properties of these T-deformations. Considering toric varieties with a subtorus action, we show that T-deformations span the space of infinitesimal deformations for smooth, complete toric varieties. For a T-deformation of any complete variety, we show that there is a natural isomorphism between the Picard group of its special fiber and a naturally defined subgroup of the Picard group of the general fiber. We show that rational $\mathbb{C}^*$-surfaces of fixed Picard number larger than two are deformation connected via T-deformations. We prove that T-deformations of a projective variety can always be embedded. Furthermore, we also provide sufficient criteria for the existence of partial smoothings of certain toric varieties. Finally, we use the techniques we have developed to provide a purely combinatorial proof that the Hilbert polynomials for geometric models of binary symmetric trivalent trees with equal numbers of leaves are equal.

Contents

List of Figures

Remarks on Notation

In this thesis, a number of mathematical objects appear whose symbols can take on a number of sub- and superscripts: polyhedral divisors (definition 1.2.2), divisorial fans (definition 1.2.8), divisorial support functions (definition 1.3.1), marked fansy divisors (definition 1.4.1), and divisorial polytopes (definition 1.4.4). The utilization of sub- and superscripts is essentially uniform for all these objects. We try to clarify it on the example of a polyhedral divisor $\mathcal{D}$ on a variety Y:

- Subscripts: A symbol appearing as a subscript represents a divisor or (not necessarily closed) point on Y. For a divisor P, the scripted symbol $\mathcal{D}_P$ represents the coefficient of $\mathcal{D}$ at P, whereas for a point y, $D_y := \sum_{y \in P} \mathcal{D}_P$.
- Superscript indices: A superscript index denotes a combinatorial decomposition of the corresponding coefficient. For example, $\mathcal{D}_P = \mathcal{D}_P^0 + \mathcal{D}_P^1$ is an equation stating that the polyhedron $\mathcal{D}_P$ is the Minkowski sum of the polyhedra $\mathcal{D}_P^0$ and $\mathcal{D}_P^1$.
- Superscripts with parentheses: A superscript appearing with parenthesis is used to denote elements of a family coming from a combinatorial decomposition of the scripted object. For example, given a Minkowski decomposition of $\mathcal{D}$, we have a family of polyhedral divisors $\mathcal{D}^{(s)}$, see section 2.2. For any fixed s, the coefficients of the polyhedral divisor $\mathcal{D}^{(s)}$ can be recovered as above via subscript, i.e. $\mathcal{D}_P^{(s)}$.
- Other superscripts: Other possible superscripts include index sets e.g. $\mathcal{I}$ or $\mathcal{J}$, or the designation "tot", or a combination of both. For example, the notation $\mathcal{D}^{\mathcal{I}}$ represents a polyhedral divisor attained via intersection over the elements of $\mathcal{I}$. On the other hand, $\mathcal{D}^{\mathrm{tot}}$ represents a polyhedral divisor for the total space of a deformation coming from a Minkowski decomposition, and $\mathcal{D}^{\mathcal{I},\mathrm{tot}}$ represents intersections of these polyhedral divisors for the total space, see section 3.2

Introduction

A fascinating subject of study in the field of algebraic geometry is deformation theory. The goal of this theory is, given some algebraic scheme, to systematically determine all ways in which this scheme can be deformed, i.e. how the defining algebraic data can be perturbed in a way such that the corresponding geometric objects vary 'continuously'. The applications of this theory are widespread, be it from the classification of singularities to the study of moduli spaces.

In general, the aforementioned goal is intractable. Thus, it is often necessary to restrict to the study of special cases, for example, complete intersection singularities, normal surface singularities, or Fano or Calabi-Yau varieties. Another special case for which much progress has been made is that of toric varieties. In a series of papers including [Alt95], [Alt97], and [Alt00], K. Altmann has succeeded in describing many aspects of the deformation theory of toric singularities. More recently, deformations of nonaffine toric varieties have been studied by A. Mavlyutov in [Mav04] and [Mav05]. The author of this dissertation has also studied deformations of nonaffine toric varieties in [Ilt09b] for the special case of partial resolutions of toric surface singularities.

A natural generalization of toric varieties are T-varieties, that is, normal varieties admitting an effective action by an algebraic torus T. K. Altmann and J. Hausen began the systematic study of such varieties in [AH06], joined by H. Süß in [AHS08]. Their main result is that, similar to the case of toric varieties, a T-variety X can be described in terms of convex combinatorial data with dimension equal to the dimension of T, together with a k-dimensional variety, where $k = \dim X - \dim T$. We call k the *complexity* of X. The combinatorial data appearing in this description bears the name *polyhedral divisor* in the affine case and *divisorial fan* in the nonaffine case.

The topic of this dissertation is the deformation theory of normal rational varieties admitting an effective codimension-one torus action, that is, rational complexity-one T-varieties. For the affine case, R. Vollmert has constructed so-called T-deformations, which arise from combinatorial decompositions of polyhedral divisors, see [IV09] and [Vol10]. We however focus on the case of nonaffine rational complexity-one T-varieties. It is important to keep in mind that any toric variety can also be considered as a rational complexity-one T-variety. Thus, our results also provide new results for deformations of nonaffine toric varieties.

We now provide an overview of the structure of this thesis, as well as stating our main results. The first two chapters present necessary background information and previous results. While this thesis is designed to be as self-contained as possible, we expect the reader to have some background in algebraic and toric geometry, say on the level of [Har77] and [Ful93]. In chapter 1 we introduce T-varieties. In addition to presenting the general construction of T-varieties from polyhedral divisors and divisorial fans, we specialize to the complexity-one case, where we recall results regarding divisors on T-varieties, complete and projective T-varieties, and singularities of T-varieties. In chapter

2, we then introduce the necessary notions from deformation theory, as well as presenting Vollmert's construction of T-deformations of affine rational complexity-one T-varieties.

Chapter 3 contains the most essential construction of this thesis. Here, we provide a combinatorial criterion which allows one to glue affine T-deformations together to get a deformation of an arbitrary rational complexity-one T-variety. We also call the resulting deformations T-deformations. As in the affine case, the fibers of such a T-deformation are rational complexity-one T-varieties, and we can describe them combinatorially via a family of divisorial fans. In this chapter, we also begin the first investigations of this construction. In particular, we state criteria for T-deformations to be separated and proper, show how the construction simplifies for complete T-varieties, provide necessary and sufficient criteria for one-parameter T-deformations to be locally trivial, and compute the image under the Kodaira-Spencer map for certain one-parameter T-deformations.

In chapter 4, we present the first application of our T-deformations by considering T-deformations of toric varieties. We focus especially on the case of smooth toric varieties coming from a full-dimensional fan with convex support, for which we provide an explicit description for the vector space of infinitesimal deformations. Our theorem 4.3.2 then implies the following:

Main Result 1. *Let X_0 be a smooth complete toric variety. Then T-deformations of X_0 span the vector space of infinitesimal deformations.*

We move on to consider the behaviour of T-invariant Cartier divisors under T-deformation in chapter 5. We do this by lifting invariant divisors on the fibers of a T-deformation to an invariant divisor on the total space. It turns out that for any fiber of a T-deformation, there is a naturally defined subgroup Pic' of the Picard group which maps injectively to the Picard group of the special fiber. Combining theorem 5.1.8 and theorem 5.2.2 gives us our second main result:

Main Result 2. *Let X_0 be a complete rational complexity-one T-variety with some T-deformation π. For any fiber X_s of π, there is a natural isomorphism $\bar{\pi}_{s,0} : \mathrm{Pic}'(X_s) \xrightarrow{\sim} \mathrm{Pic}(X_0)$ preserving Euler characteristic and intersection numbers.*

In chapter 6, we turn to the study of rational $\mathbb{C}^*$-surfaces, that is, smooth, complete, rational two-dimensional complexity-one T-varieties. We present a simplified description of T-deformations, and discuss how T-deformations behave with respect to blowing up and blowing down. In theorem 6.3.2 we then show that rational $\mathbb{C}^*$-surfaces with equal Picard number are connected by T-deformations:

Main Result 3. *Let X and X' be rational $\mathbb{C}^*$-surfaces with equal Picard number larger than two. Then there is a sequence of rational $\mathbb{C}^*$-surfaces $X = X^0, X^1, \ldots, X^j = X'$ such that for each $i \geq 0$, X^i and X^{i+1} are fibers of some common T-deformation.*

Chapter 7 concerns itself with the study of T-deformation of projective T-varieties. After proving some basic results concerning quotients of T-varieties by $\mathbb{C}^*$ actions, we show in theorem 7.2.1 the following:

Main Result 4. *Let X_0 be a T-variety with some fixed projectively normal embedding. Then any T-deformation of X_0 can be realized as an embedded deformation with respect to the embedding of X_0.*

We then introduce another combinatorial gadget to more handily describe embedded deformations. As an application, we consider deformations of geometric models for binary symmetric phylogenetic trivalent trees as studied by W. Buczyńska and J. Wiśniewski in [BW07]. We then use the combinatorial nature of our T-deformations to provide a purely combinatorial proof of the following result, which was previously proven using deformation theory:

Main Result 5. *The geometric models for binary symmetric phylogenetic trivalent trees with equal numbers of leaves have equal Hilbert polynomials.*

In chapter 8, we show how the construction of affine T-deformations can be modified to create deformations which are no longer homogeneous of degree zero. We then apply this technique to construct multidegree deformations of certain toric varieties. While we no longer have exact descriptions of all the fibers of such deformations in combinatorial terms, we show that these multidegree deformations are isomorphic to T-deformations when we restrict to certain strata of the base space. Thus, we do retain information on some of the fibers of the deformation, and this is often enough to make certain claims about the general fiber. One special case is our corollary 8.3.4, which shows the existence of certain partial smoothings for toric Fano varieties:

Main Result 6. *Let Δ be a Gorenstein reflexive polytope and let X_0 be the toric Fano variety associated to the face fan of Δ. Suppose X_0 is smooth in codimension d, and that $\Delta = \Delta^0 + \ldots + \Delta^r$ for lattice polytopes Δ^i whose faces of dimension less than or equal to d generate smooth cones in height one. Then X_0 admits a smoothing in codimension $d+1$.*

Several portions of this dissertation have appeared in a number of preprints authored or coauthored by the present author. Chapter 3 contains some material from [IV09], and 4 contains material from [Ilt09a] and [IV09]. Furthermore, chapters 5 and 6 contain a significant amount of material from [HI09]. However, any constructions or results presented in this dissertation without further attribution are solely due to the current author. A complementary approach to the study of deformations of nonaffine toric varieties has been begun by A. Mavlyutov in [Mav09]. His independently developed approach constructs deformations from similar combinatorial ingredients by using the homogeneous coordinate ring of a toric variety.

Chapter 1

T-Varieties

In this chapter, we shall recall a number of basic notions which will be utilized throughout the remainder of this dissertation. In section 1.1, we set some general notation. Essential is the combinatorial description of T-varieties via polyhedral divisors and divisorial fans; we cover this in section 1.2. After this, we will concentrate on complexity-one T-varieties. We give descriptions of invariant Weil and Cartier divisors in section 1.3. In section 1.4 we recall a simplified description of complete complexity-one T-varieties in terms of marked fansy divisors. Finally, we discuss singularities of complexity-one T-varieties in section 1.5.

1.1 General Notation

We will be working over an arbitrary algebraically closed field of characteristic zero which we will denote by $\mathbb{C}$. For any variety X, let $\mathbb{C}(X)$ denote its field of rational functions. For $f \in \mathbb{C}(X)$, let $V(f)$ denote the effective part of $\operatorname{div}(f)$. We call a variety X *semiprojective* if $A := \Gamma(X, \mathcal{O}_X)$ is a finitely generated $\mathbb{C}$-algebra and X is projective over $\operatorname{Spec} A$. For any $\mathbb{Q}$-Cartier divisor D on a scheme X, we will write $H^i(X, D)$ for the cohomology group $H^i(X, \mathcal{O}(D))$. Consider any module B graded by a group G. For any $g \in G$, we denote the homogeneous piece of B of degree g by $B(g)$.

Let N denote a lattice with dual M, and let $N_\mathbb{Q}$ and $M_\mathbb{Q}$ be the associated $\mathbb{Q}$ vector spaces. Let $T^N := N \otimes \mathbb{C}^* = \operatorname{Spec} \mathbb{C}[M]$ be the torus with cocharacter group N and character group M. For any point $v \in N_\mathbb{Q}$, let $\mu(v)$ denote the smallest positive integer such that $\mu(v) \cdot v$ is in N. For any cone $\sigma \subset N_\mathbb{Q}$, we denote its dual cone by $\sigma^\vee$. For any subset $S \subset N_\mathbb{Q}$, we denote the closure of the cone spanned by positive linear combinations of S by $\operatorname{cone}(S)$.

For any polyhedron $\Delta \subset N_\mathbb{Q}$, let $\operatorname{tail}(\Delta)$ denote its *tailcone*, that is, the cone of unbounded directions in Δ. Thus, Δ can be written as the Minkowski sum of some bounded polyhedron and its tailcone. Now for $u \in \operatorname{tail}(\Delta)^\vee \cap M$, denote by $\operatorname{face}(\Delta, u)$ the face of Δ upon which u achieves its minimum. We denote the relative interior of Δ by $\operatorname{relint}(\Delta)$.

Recall that a polyhedral *subdivision* C in some vector space V is a set of polyhedra in V closed under intersections, such that all inclusions are face relations. We furthermore impose the convention that all elements of C have trivial lineality space. If C is any polyhedral subdivision in $N_\mathbb{Q}$, we denote by $\operatorname{tail}(C)$ the *tailfan* of C, that is, the set of all cones $\operatorname{tail}(\Delta)$ for $\Delta \in C$, which is easily seen to form a fan. For any polyhedral subdivision C, the *support* of the subdivision, denoted by $|C|$, is the union of all elements of C. A subdivision is *complete* if its support is $N_{\mathbb{Q}}$. For any integer k, $C(k)$ denotes the set of all elements of C of dimension k.

We introduce several abuses of notation. Where convenient, we identify a polyhedron Δ with the polyhedral subdivision containing all its faces. Likewise, for any ray $\rho \subset N_{\mathbb{Q}}$, we also use ρ to denote its minimal lattice generator as long as no confusion can arise.

1.2 Polyhedral Divisors and T-Varieties

We begin by defining our main objects of study.

Definition 1.2.1. A *T-variety* is a normal variety X together with an effective torus action $T \times X \to X$. The *complexity* of a T-variety X is the dimension of X less the dimension of T.

Note that complexity zero T-varieties are simply toric varieties, which are in one to one correspondence with polyhedral fans. This correspondence was generalized to arbitrary complexity by K. Altmann and J. Hausen in [AH06] for the affine case, and together with H. Süß in [AHS08] for the general case. In the following, we will recall this generalization.

Consider a smooth semiprojective variety Y over $\mathbb{C}$ and let $\sigma \subset N_{\mathbb{Q}}$ be a pointed polyhedral cone.

Definition 1.2.2. A *polyhedral divisor* on Y with tail cone σ is a formal sum

$$\mathcal{D} = \sum_{P} \mathcal{D}_P \otimes P,$$

where P runs over all prime divisors on Y such that

(i) For all prime divisors P, $\mathcal{D}_P$ is either the empty set or a polyhedron with tailcone σ;

(ii) $\mathcal{D}_P = \sigma$ for all but finitely many P.

We can evaluate a polyhedral divisor for every element $u \in \sigma^\vee \cap M$ via

$$\mathcal{D}(u) := \sum_{\substack{P \\ \mathcal{D}_P \neq \emptyset}} \min_{v \in \mathcal{D}_P} \langle v, u \rangle P$$

in order to obtain an ordinary divisor on $\operatorname{Loc}(\mathcal{D})$, where $\operatorname{Loc}(\mathcal{D}) := Y \setminus \left(\bigcup_{\mathcal{D}_P = \emptyset} P\right)$. We will use these evaluations to restrict our interest to a subclass of polyhedral divisors.

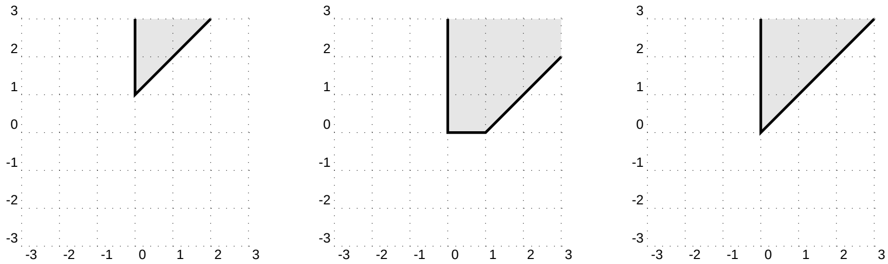
Definition 1.2.3. A polyhedral divisor $\mathcal{D}$ is called *proper* if for all $u \in \sigma^\vee \cap M$, $\mathcal{D}(u)$ is Cartier and semiample and if for all $u \in (\operatorname{relint} \sigma^\vee) \cap M$, $\mathcal{D}(u)$ is big. Recall that a divisor is semiample if a multiple is globally generated, and that a divisor is big if some multiple has a section with affine complement.

Remark 1.2.4. If $\mathcal{D}$ is a polyhedral divisor on a curve Y, we define the degree of $\mathcal{D}$ to be $\deg(\mathcal{D}) = \sum \mathcal{D}_P$. Then $\mathcal{D}$ is proper if and only if $\deg(\mathcal{D})$ is strictly contained in σ and for all $u \in \sigma^\vee$ with $\min_{v \in \deg(\mathcal{D})} \langle v, u \rangle = 0$ it follows that a multiple of $\mathcal{D}(u)$ is principal. Note that if $\operatorname{Loc}(\mathcal{D})$ is affine, then $\deg(\mathcal{D}) = \emptyset$, and $\mathcal{D}$ is automatically proper.

To a proper polyhedral divisor we associate an M-graded $\mathbb{C}$-algebra and consequently an affine scheme admitting a T^N-action:

$$X(\mathcal{D}) := \operatorname{Spec} \bigoplus_{u \in \sigma^\vee \cap M} H^0(Y, \mathcal{D}(u)) \cdot \chi^u.$$

Theorem 1.2.5 ([AH06] Theorems 3.1 and 3.4)**.** *$X(\mathcal{D})$ is a T^N-variety of complexity equal to the dimension of Y. Furthermore, any affine T^N-variety can be constructed in this manner. The construction of $X(\mathcal{D})$ comes with a surjective rational quotient map $p_{\mathcal{D}} : X(\mathcal{D}) \dashrightarrow \operatorname{Loc}(\mathcal{D})$ which is a morphism if $\operatorname{Loc}(\mathcal{D})$ is affine.*



(a) $\mathcal{D}_0$ (b) $\mathcal{D}_1$ (c) $\mathcal{D}_\infty$

Figure 1.1: A proper polyhedral divisor on $\mathbb{P}^1$

Example 1.2.6 (A proper polyhedral divisor on $\mathbb{P}^1$)**.** In figure 1.1, we picture three two-dimensional polytopes $\mathcal{D}_0$, $\mathcal{D}_1$, and $\mathcal{D}_\infty$, all with the same tailcone. Thus, $\mathcal{D} = \mathcal{D}_0 \otimes \{0\} + \mathcal{D}_1 \otimes \{1\} + \mathcal{D}_\infty \otimes \{\infty\}$ is a polyhedral divisor on $\mathbb{P}^1$. Furthermore, from remark 1.2.4, it follows that $\mathcal{D}$ is in fact proper. The corresponding T-variety $X(\mathcal{D})$ is in fact just $\mathbb{A}^3$ with a $(\mathbb{C}^*)^2$-action. We can see this by an explicit calculation. $\operatorname{tail}(\mathcal{D})^\vee$ is generated by $[1,0]$ and $[-1,1]$. For $u = [u_1, u_2] \in \operatorname{tail}(\mathcal{D})^\vee$ we have

$$\begin{aligned} \mathcal{D}(u) &= u_2 \cdot \{0\} \qquad \text{if} \qquad u_1 \geq 0; \\ \mathcal{D}(u) &= u_1 \cdot \{1\} + u_2 \cdot \{0\} \qquad \text{if} \qquad u_1 \leq 0. \end{aligned}$$

One then easily checks that the coordinate ring of $X(\mathcal{D})$ is generated by $\chi^{[1,0]}$, $\chi^{[0,1]}$, and $y \cdot \chi^{[-1,1]}$, where $y \in \mathbb{C}(\mathbb{P}^1)$ is such that $\operatorname{div}(y) = \{1\} - \{0\}$.

We now wish to glue these affine varieties together; this requires some further definitions.

Definition 1.2.7. Let $\mathcal{D} = \sum_P \mathcal{D}_P \otimes P$, $\mathcal{D}' = \sum_P \mathcal{D}'_P \otimes P$ be two proper polyhedral divisors on Y with tail cones σ and σ'.

- We define their *intersection* by

$$\mathcal{D} \cap \mathcal{D}' := \sum_P (\mathcal{D}_P \cap \mathcal{D}'_P) \otimes P.$$

- We say $\mathcal{D}' \subset \mathcal{D}$ if $\mathcal{D}'_P \subset \mathcal{D}_P$ for every prime divisor $P \in Y$.

- For $y \in Y$ a not necessarily closed point, set $\mathcal{D}_y := \sum_{y \in P} \mathcal{D}_P$, where summation is via Minkowski addition.

- $\mathcal{D}'$ is a *face* of $\mathcal{D}$ i.e. $\mathcal{D}' \prec \mathcal{D}$ if $\mathcal{D}' \subset \mathcal{D}$ and for each $y \in \operatorname{Loc}(\mathcal{D}')$ there is a pair $(w_y, D_y) \in (\sigma^\vee \cap M) \times |\mathcal{D}(w_y)|$ such that $y \notin \operatorname{supp}(D_y)$, $\mathcal{D}'_y = \operatorname{face}(\mathcal{D}_y, w_y)$, and $\operatorname{face}(\mathcal{D}'_v, w_y) = \operatorname{face}(\mathcal{D}_v, w_y)$ for all $v \in \operatorname{Loc}(\mathcal{D}) \setminus \operatorname{supp}(D_y)$.

If $\mathcal{D}' \subset \mathcal{D}$ then we have an inclusion

$$\bigoplus_{u \in \sigma'^\vee \cap M} H^0(Y, \mathcal{D}'(u)) \supset \bigoplus_{u \in \sigma^\vee \cap M} H^0(Y, \mathcal{D}(u))$$

which corresponds to a dominant morphism $X(\mathcal{D}') \to X(\mathcal{D})$. This is an open embedding exactly when $\mathcal{D}' \prec \mathcal{D}$, see proposition 3.4 of [AHS08].

Definition 1.2.8. A *divisorial fan* is a finite set $\mathcal{S}$ of proper polyhedral divisors such that for $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$ we have $\mathcal{D} \succ (\mathcal{D}' \cap \mathcal{D}) \prec \mathcal{D}'$ with $\mathcal{D}' \cap \mathcal{D}$ also in $\mathcal{S}$. For a not necessarily closed point $y \in Y$, the set of all $\mathcal{S}_y$ defined by the polyhedra $\mathcal{D}_y$, $\mathcal{D} \in \mathcal{S}$ is called a *slice* of $\mathcal{S}$. The *tailfan* of $\mathcal{S}$ is the fan formed by the tailcones $\operatorname{tail}(\mathcal{D})$ for $\mathcal{D} \in \mathcal{S}$. We say that a subset $\mathcal{I} \subset \mathcal{S}$ *induces* $\mathcal{S}$ if any $\mathcal{D} \in \mathcal{S}$ is an intersection of elements of $\mathcal{I}$.

Remark 1.2.9. In general $\mathcal{S}_y$ need not be a polyhedral subdivision. However, for points y corresponding to some prime divisor P, $\mathcal{S}_P := \mathcal{S}_y$ is indeed a subdivision.

For polyhedral divisors $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$, we may glue the affine varieties $X(\mathcal{D})$ and $X(\mathcal{D}')$ via

$$X(\mathcal{D}) \leftarrow X(\mathcal{D} \cap \mathcal{D}') \to X(\mathcal{D}')$$

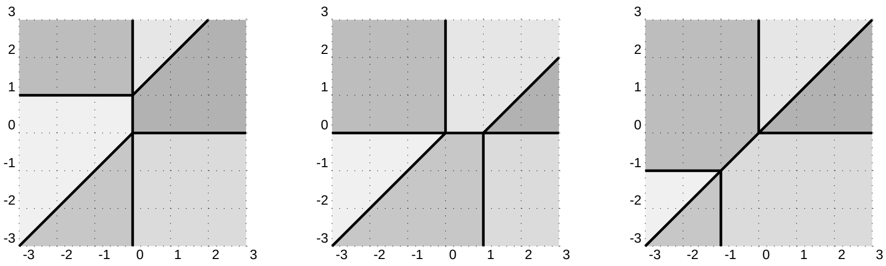
to get a scheme $X(\mathcal{S})$.

Theorem 1.2.10 ([AHS08] Theorems 5.3 and 5.6)**.** *$X(\mathcal{S})$ is a normal scheme of dimension* $\dim Y + \dim N_{\mathbb{Q}}$ *with an effective torus action by T^N. Furthermore, all T^N-varieties can be constructed in this manner.*

Remark 1.2.11. A divisorial fan $\mathcal{S}$ on a curve Y is called *complete* if all slices $\mathcal{S}_y$ are complete subdivisions of $N_{\mathbb{Q}}$ and Y is complete. For any divisorial fan $\mathcal{S}$ on a curve, $X(\mathcal{S})$ is complete if and only if $\mathcal{S}$ is complete. Completeness of T-varieties can be characterized in higher complexity cases as well, but this is slightly more complicated. Furthermore, note that $X(\mathcal{S})$ is always separated if $\mathcal{S}$ is a divisorial fan on a curve.

Remark 1.2.12. Throughout this thesis, we will especially be interested in rational complexity-one T-varieties. Since a T-variety coming from a divisorial fan on Y is birational to the product of Y and a torus, these varieties are exactly those arising from the case $Y = \mathbb{P}^1$.

Example 1.2.13 (The projectivized cotangent bundle of $\mathbb{P}^2$)**.** We consider now an example of a divisorial fan $\mathcal{S}$ on $\mathbb{P}^1$ which is induced by six proper polyhedral divisors. The only nontrivial slices of $\mathcal{S}$ are at 0, 1, and ∞, and are pictured in figure 1.2. The coefficients of the six inducing polyhedral divisors are never $\emptyset$; the coefficients belonging to a common polyhedral divisor share a color (and can be recognized by the fact that their tailcones are equal). Note that the polyhedral divisor $\mathcal{D}$ from example 1.2.6 is one of the six inducing polyhedral divisors. The corresponding variety $X(\mathcal{S})$ is in fact the projectivized cotangent bundle of $\mathbb{P}^2$, see example 8.5 in [AHS08].



(a) $\mathcal{S}_0$ (b) $\mathcal{S}_1$ (c) $\mathcal{S}_\infty$

Figure 1.2: Divisorial fan slices for $\mathbb{P}(\Omega_{\mathbb{P}^2})$

Remark 1.2.14. An important class of examples of T-varieties arises by considering a subtorus action on a toric variety. This can be made explicit as follows. Consider some lattice N' with Σ' a polyhedral fan in $N'_{\mathbb{Q}}$. Let $\mathrm{TV}(\Sigma')$ denote the corresponding toric variety; see [Ful93] for details on toric varieties. As usual, let M' be the dual lattice of N'. Now, consider some surjection of lattices $\deg : M' \twoheadrightarrow M$; this corresponds to a monomorphism of tori $T^N \hookrightarrow T^{N'}$. If $\deg^*$ is the dual homomorphism and N'' the cokernel of $\deg^*$, we then have the following exact sequence of lattices

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\deg^*"] & N' \arrow[l, bend left, "s"] \arrow[r, "p"] & N'' \arrow[r] & 0 \end{tikzcd}$$

where we have chosen some cosection s.

Now let Σ'' be the coarsest common refinement of the images of the cones of Σ' in $N''_{\mathbb{Q}}$. Then $Y = \mathrm{TV}(\Sigma'')$ is the Chow quotient of $\mathrm{TV}(\Sigma')$ by the action of T^N. Note that each ray ρ of Σ'' corresponds to an invariant prime divisor D_ρ on $\mathrm{TV}(\Sigma'')$. Now, for any cone $\sigma' \in \Sigma'$, define the polyhedral divisor

$$\mathcal{D}^{\sigma'} = \sum_{\rho \in \Sigma''(1)} s(p^{-1}(\rho) \cap \sigma') \otimes D_\rho.$$

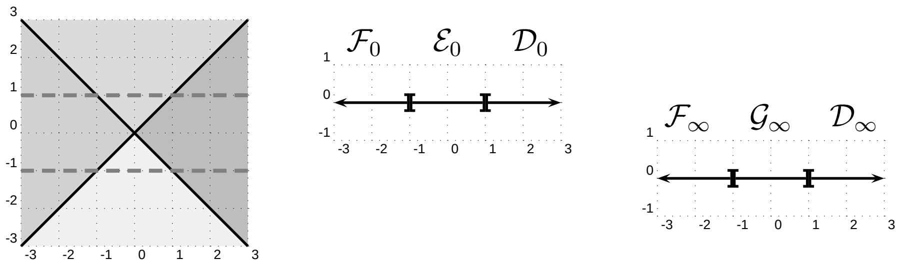
Then $\mathcal{D}^{\sigma'}$ is in fact proper, and these polyhedral divisors fit together to a divisorial fan $\mathcal{S}^{\Sigma'}$. Furthermore, $X(\mathcal{S}^{\Sigma'})$ is exactly the variety $X(\Sigma')$ endowed with the action of the subtorus T^N.

We will especially be interested in the above situation where the action of the subtorus has complexity one. This arises by choosing some primitive degree $R \in M'$ and setting $M = M'/\langle R \rangle$. Then $N = N' \cap R^\perp \subset N'$. In this case, $Y = \mathbb{P}^1$ and the divisorial fan $\mathcal{S}^{\Sigma'}$ consists of polyhedral divisors $\mathcal{D}^{\sigma'}$ for each $\sigma' \in \Sigma'$, where

$$\mathcal{D}^{\sigma'} = s(\sigma' \cap [R = 1]) \otimes \{0\} + s(\sigma' \cap [R = -1]) \otimes \{\infty\}.$$

Note that we define the set $[R = a]$ to be $\{v \in N_{\mathbb{Q}} | \langle v, R \rangle = a\}$, that is, the set of points in $N_{\mathbb{Q}}$ for which R takes the value a.

Example 1.2.15 (A toric Fano surface)**.** We consider a downgrade of the unique toric Fano surface with four A_1 singularities. The fan Σ' corresponding to this surface, generated by four two-dimensional cones, is pictured in figure 1.3(a). We consider the subtorus T generated by the basis vector $(1, 0)$. The quotient of $\mathrm{TV}(\Sigma')$ by T is just $\mathbb{P}^1$. The preimages of the two primitive generators of the corresponding fan are pictured in the previous figure as dark gray dashed lines. We take the cosection s just to be the natural projection.



(a) Σ' (b) $\mathcal{S}_0^{\Sigma'}$ (c) $\mathcal{S}_\infty^{\Sigma'}$

Figure 1.3: Fan and divisorial fan for a toric Fano surface

The slices of the divisorial fan $\mathcal{S}^{\Sigma'}$, with labeling, are shown in parts (b) and (c) of figure 1.3. The four two-dimensional cones of Σ' correspond to four proper polyhedral divisors $\mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}$ which induce $\mathcal{S}^{\Sigma'}$. Note that $\mathcal{D}$ and $\mathcal{F}$ have complete locus, whereas $\mathcal{E}$ and $\mathcal{G}$ both have affine locus.

1.3 Divisors on Complexity-One T-Varieties

We will be dealing extensively with invariant divisors on complete T-varieties with codimension one torus action; these divisors have been described in [PS08] in combinatorial terms. We will mainly be interested in Cartier divisors, but Weil divisors will also be of interest. In this section, we recall the results of [PS08].

Let C be any polyhedral subdivision in $N_\mathbb{Q}$ with tailfan Σ. Consider some continuous function $f : |C| \to \mathbb{Q}$ which is affine on the elements of C. We call f a $\mathbb{Q}$-*support function.* For such a $\mathbb{Q}$-support function, we define the *linear part* of f to be the function

$$\operatorname{tail}(f) : |\operatorname{tail}(C)| \to \mathbb{Q}$$

where for any $\Delta \in C$, if $f_{|\Delta} = \langle \cdot, u \rangle + a$, then $\operatorname{tail}(f)_{|\operatorname{tail}\Delta} = \langle \cdot, u \rangle$. We say that f is *integral* or just a *support function* if for any $v \in |C|$ and $k \in \mathbb{N}$ with $k \cdot v \in N$, $k \cdot f(v) \in \mathbb{Z}$.

Let Y be a smooth projective curve and $\mathcal{S}$ a divisorial fan on Y; set $\Sigma = \operatorname{tail}(\mathcal{S})$. Note that now the prime divisors of Y are just its closed points.

Definition 1.3.1. A *(divisorial) support function* on $\mathcal{S}$ is a formal sum of the form

$$h = \sum_{P \in Y} h_P \otimes P$$

where $h_P : |\mathcal{S}_P| \to \mathbb{Q}$ are support functions such that:

(i) $\operatorname{tail}(h_P)$ is the same for all P. We call $\operatorname{tail}(h_P)$ the linear part of h and denote it by $\operatorname{tail}(h)$;

(ii) $h_P \neq \operatorname{tail}(h)$ for only finitely many P.

We say that h is *principal* if there exists $u \in M$ and $f \in \mathbb{C}(Y)$ such that for all $P \in Y$ and $v \in |\mathcal{S}_P|$, $h_P(v) = -\langle u, v \rangle - \nu_P(f)$, where $\nu_P(f)$ is the order of f in P. We say that h is *Cartier* if for every $\mathcal{D} \in \mathcal{S}$ with complete locus, the restriction $h_{|\mathcal{D}}$ of h to $\mathcal{D}$ is principal. By $\operatorname{SF}(\mathcal{S})$ and $\operatorname{CaSF}(\mathcal{S})$ we respectively denote the groups of all support functions and Cartier support functions on $\mathcal{S}$, where we take the addition to be the natural addition of functions.

Let T-CDiv$(X(\mathcal{S}))$ denote the group of T-invariant Cartier divisors on $X(\mathcal{S})$. Now, for any element $h \in \mathrm{CaSF}(\mathcal{S})$, we can associate an invariant divisor $D_h \in \text{T-CDiv}(X(\mathcal{S}))$ as follows. We first will define an invariant open covering $\mathfrak{U}(\mathcal{S})$ of $X(\mathcal{S})$. Let $\mathcal{P}$ be the set of all $P \in Y$ such that $\mathcal{S}_P$ is nontrivial. For $P \in Y$ and $\mathcal{D} \in \mathcal{S}$ with noncomplete locus, set

$$\mathcal{D}(P) = \mathcal{D} + \sum_{\substack{Q \in \mathcal{P} \\ Q \neq P}} \emptyset \otimes Q.$$

We take $\mathfrak{U}(\mathcal{S})$ to consist of the open sets $X(\mathcal{D}(P))$ for $\mathcal{D}$ with noncomplete locus together with the sets $X(\mathcal{D})$ for $\mathcal{D}$ with complete locus.

We now define D_h locally with respect to the covering $\mathfrak{U}(\mathcal{S})$. Indeed, for $P \in Y$ and $\mathcal{D} \in \mathcal{S}$ with noncomplete locus, the restriction of h to $\mathcal{D}(P)$ is principal; let $u_{\mathcal{D},P}$ and $f_{\mathcal{D},P}$ denote corresponding elements of M and $\mathbb{C}(Y)$. Likewise, for $\mathcal{D}$ with complete locus, the restriction of h to $\mathcal{D}$ is also principal; let $u_{\mathcal{D}}$ and $f_{\mathcal{D}}$ once again denote corresponding elements. Restricted to an open set $X(\mathcal{D}(P))$ for $\mathcal{D}$ with noncomplete locus, we take D_h to be $\mathrm{div}(f_{\mathcal{D},P} \cdot \chi^{u_{\mathcal{D},P}})$. Likewise, restricted to an open set $X(\mathcal{D})$ for $\mathcal{D}$ with complete locus, we take D_h to be $\mathrm{div}(f_{\mathcal{D}} \cdot \chi^{u_{\mathcal{D}}})$.

Theorem 1.3.2 ([PS08] Proposition 3.10)**.** *The above construction gives a well-defined divisor $D_h \in \text{T-CDiv}(X(\mathcal{S}))$. The corresponding map $\mathrm{CaSF}(\mathcal{S}) \to \text{T-CDiv}(X(\mathcal{S}))$ is an isomorphism of abelian groups taking principal support functions to principal divisors.*

We now briefly turn our attention to Weil divisors. Let $\mathcal{D}$ be a proper polyhedral divisor on a curve Y with tailcone σ. There are exactly two types of invariant prime Weil divisors on $X(\mathcal{D})$:

(i) Closures of codimension-one orbits in $X(\mathcal{D})$;

(ii) Families of closures of codimension-two orbits in $X(\mathcal{D})$.

Proposition 1.3.3 ([PS08] Propositions 3.13 and 3.14)**.** *There are one-to-one correspondences*

(i) Between divisors of type (i) and pairs (P, v), $P \in Y$ and v a vertex of $\mathcal{D}_P$;

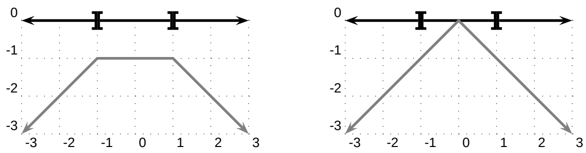
(ii) Between divisors of type (ii) and rays ρ of $\mathrm{tail}(\mathcal{D})$ such that $\rho \cap \deg(\mathcal{D}) = \emptyset$.

Denote such divisors by respectively $D_{(P,v)}$ and D_ρ. For any $h \in \mathrm{CaSF}(\mathcal{S})$, we have

$$D_h = -\sum_{\rho} \mathrm{tail}(h)(\rho) D_\rho - \sum_{(P,v)} \mu(v) h_P(v) D_{(P,v)}.$$

Proposition 1.3.4 ([PS08] Corollary 3.28)**.** *Consider $h \in \mathrm{CaSF}(\mathcal{S})$ for some complete divisorial fan $\mathcal{S}$. Then D_h is ample if and only if h is strictly concave, and if for all $\mathcal{D} \in \mathcal{S}$ with noncomplete locus and $\mathrm{tail}(\mathcal{D})$ full-dimensional, the linear extension of $h_{|\mathcal{D}}$ has strictly negative degree when evaluated at 0.*

Example 1.3.5 (An anticanonical divisor on a toric Fano surface)**.** We now consider the toric Fano surface $X(\mathcal{S})$ from example 1.2.15 with $\mathcal{S} := \mathcal{S}^{\Sigma'}$. We define a divisorial support function $h \in \mathrm{CaSF}(\mathcal{S})$ by $h = h_0 \otimes \{0\} + h_\infty \otimes \{\infty\}$, where $h_0 = h_\infty$ are depicted in figure 1.4(a). The linear part of h is then pictured in figure 1.4(b). Now, using proposition 1.3.3, we have that $D_h = D_{(0,-1)} + D_{(0,1)} + D_{(\infty,-1)} + D_{(\infty,1)}$. Furthermore, from proposition 1.3.4, we have that D_h is ample. In fact, one can easily check that D_h is an anticanonical divisor for $X(\mathcal{S})$.



(a) $h_0 = h_\infty$ (b) $\text{tail}(h)$

Figure 1.4: A divisorial support function for a toric Fano surface

1.4 Marked Fansy Divisors and Divisorial Polytopes

We now recall some results from [IS09] on complete and projective complexity-one T-varieties. We first show how the combinatorial data describing a complete complexity-one T-variety can be simplified to a so-called *marked fansy divisor*. Then we recall a correspondence between polarized complexity-one T-varieties and so-called *divisorial polytopes*.

As usual, let Y be a curve. Different divisorial fans $\mathcal{S}, \mathcal{S}'$ on Y can in fact yield the same T-variety $X(\mathcal{S}) = X(\mathcal{S}')$. The differing divisorial fans simply correspond to different open affine coverings. On the other hand, divisorial fans with identical slices might yield differing T-varieties, even in the complexity-one case. For example, one could blow up the toric Fano surface from example 1.2.15 by adding a ray through $(1, 0)$ to the fan Σ' of figure 1.3 without changing the slices of the corresponding divisorial fan. However, for complete complexity-one T-varieties, we can save the situation by noting that the entire information needed to reconstruct our T-variety can be recovered from the (unlabeled) slices of the divisorial fan together with the additional information of those polyhedral divisors with complete locus. This motivates the following definition:

Definition 1.4.1. A *marked fansy divisor* on a curve Y is a formal sum $\Xi = \sum_{P \in Y} \Xi_P \otimes P$ together with a fan Σ and some subset $\mathfrak{M} \subset \Sigma$, such that

(i) Ξ_P is a complete polyhedral subdivision of $N_{\mathbb{Q}}$, $\text{tail}(\Xi_P) = \Sigma$ for all $P \in Y$, and $\Xi_P = \Sigma$ for all but finitely many P;

(ii) For full-dimensional $\sigma \in \mathfrak{M}$ the polyhedral divisor $\mathcal{D}(\sigma) = \sum \Delta_P^\sigma \otimes P$ is proper, where Δ_P^σ is the unique element of Ξ_P with $\text{tail}(\Delta_P^\sigma) = \sigma$;

(iii) For $\sigma \in \mathfrak{M}$ of full dimension and $\tau \prec \sigma$, we have $\tau \in \mathfrak{M}$ if and only if
$$\deg(\mathcal{D}(\sigma)) \cap \tau \neq \emptyset;$$

(iv) If $\tau \prec \sigma$ and $\tau \in \mathfrak{M}$, then $\sigma \in \mathfrak{M}$.

We say that the elements of $\mathfrak{M}$ are *marked*. The support of a fansy divisor is the set of points $P \in Y$, where Ξ_P differs from the tailfan Σ.

Now, given any complete divisorial fan $\mathcal{S}$ on Y, we can associate a marked fansy divisor, by setting $\Xi = \sum \mathcal{S}_P \otimes P$ and adding marks to the tailcones of all $\mathcal{D} \in \mathcal{S}$ with complete locus. We call this marked fansy divisor $\Xi(\mathcal{S})$. The fact that all polyhedral divisors in $\mathcal{S}$ with the same tailcone either all have complete or noncomplete locus follows from a simplified face relation for complexity-one T-varieties, see [IS09] proposition 1.1.

Example 1.4.2 (Two marked fansy divisors)**.** We can associate a marked fansy divisor to the divisorial fan $\mathcal{S}$ of example 1.2.13 for $\mathbb{P}(\Omega_{\mathbb{P}^2})$. Indeed,

$$\Xi(\mathcal{S}) = \mathcal{S}_0 \otimes \{0\} + \mathcal{S}_1 \otimes \{1\} + \mathcal{S}_\infty \otimes \{\infty\}$$

with marks on all nonzero cones of $\mathrm{tail}(\mathcal{S})$. Likewise, we can associate a marked fansy divisor to the divisorial fan $\mathcal{S}^{\Sigma'}$ of example 1.2.15 for the toric Fano surface. Indeed, $\Xi(\mathcal{S}^{\Sigma'}) = \mathcal{S}_0^{\Sigma'} \otimes \{0\} + \mathcal{S}_\infty^{\Sigma'} \otimes \{\infty\}$ with marks on the cones $[0, \infty)$ and $(-\infty, 0]$.

Proposition 1.4.3 ([IS09] Proposition 1.6)**.** *For any marked fansy divisor Ξ, there exists a complete divisorial fan $\mathcal{S}$ with $\Xi = \Xi(\mathcal{S})$. If for two divisorial fans $\mathcal{S}, \mathcal{S}'$ we have that $\Xi(\mathcal{S}) = \Xi(\mathcal{S}')$, then it follows that $X(\mathcal{S}) = X(\mathcal{S}')$.*

We can make the first claim of the above proposition explicit. Let $\mathcal{P}$ be the set of points in $\mathbb{P}^1$ for which Ξ_P isn't trivial. For each $P \in \mathcal{P}$ and $\Delta \in \Xi_P$ with $\mathrm{tail}(\Delta) \notin \mathfrak{M}$, we have a polyhedral divisor $\mathcal{D}(P, \Delta) = \Delta \otimes P + \sum_{Q \in \mathcal{Q} \cup \mathcal{P} \setminus P} \emptyset \otimes Q$. Likewise, for each $\sigma \in \mathfrak{M}$ of full dimension, we have a polyhedral divisor $\mathcal{D}(\sigma)$, see definition 1.4.1. These polyhedral divisors induce a divisorial fan $\mathcal{S}$ via intersection, see [IS09], proposition 1.6, and $\Xi(\mathcal{S}) = \Xi$.

By the above proposition, we can thus define $X(\Xi)$ to be $X(\mathcal{S})$ for any $\mathcal{S}$ with $\Xi = \Xi(\mathcal{S})$. We can similarly define $\mathrm{CaSF}(\Xi)$ to be $\mathrm{CaSF}(\mathcal{S})$ for any $\mathcal{S}$ with $\Xi = \Xi(\mathcal{S})$. Furthermore, every complete complexity-one T-variety can be described via a marked fansy divisor. We thus can avoid divisorial fans and work instead with the somewhat more handy notion of marked fansy divisors. If we are only interested in describing projective complexity-one T-varieties together with a polarization, there is an even more handy description, namely that of *divisorial polytopes*:

Definition 1.4.4. A divisorial polytope $(\Psi, \Box)$ consists of a lattice polytope $\Box \subset M_{\mathbb{Q}}$ and a piecewise affine concave function

$$\Psi = \sum \Psi_P \otimes P : \Box \to \mathrm{Div}_{\mathbb{Q}} Y,$$

such that

(i) $\deg \Psi(u) > 0$ for u in the interior of $\Box$;

(ii) For u a vertex of $\Box$, $\deg \Psi(u) > 0$ or $\lambda \Psi(u) \sim 0$ for some $\lambda \in \mathbb{N}$;

(iii) For all $P \in Y$, the graph of Ψ_P is integral, i.e. has its vertices in $M \times \mathbb{Z}$.

We often will call the pair $(\Psi, \Box)$ simply Ψ.

We now show how to associate a marked fansy divisor and support function to a divisorial polytope $(\Psi, \Box)$. We begin by setting $\Psi_P^*(v) = \min_{u \in \Box}(\langle v, u \rangle - \Psi_P(u))$, which is a piecewise affine concave function on $N_{\mathbb{Q}}$. Now let Ξ_P be the polyhedral subdivision of $N_{\mathbb{Q}}$ induced by Ψ_P^* and take $\Xi(\Psi) = \sum \Xi_P \otimes P$. Furthermore, we add a mark to an element $\sigma \in \mathrm{tail}(\Xi)$ if $(\deg \circ \Psi)|_{F_\sigma} \equiv 0$, where $F_\sigma \prec \Box$ is the face where $\langle \cdot, v \rangle$ takes its minimum for all $v \in \sigma$.

Theorem 1.4.5 ([IS09] Theorem 3.2)**.** *There is a one-to-one correspondence between divisorial polytopes and pairs $(X, \mathcal{L})$ of complexity-one varieties with an equivariant ample line bundle gotten by sending*

$$\Psi \mapsto (\Xi(\Psi), \mathcal{O}(D_{\Psi^*})).$$

Furthermore, the global sections of D_{Ψ^*} are

$$\bigoplus_{u \in \square \cap M} H^0(Y, \Psi(u)).$$

Remark 1.4.6. Let Ψ be a divisorial polytope. Then the corresponding T-variety is the Proj of

$$\bigoplus_{k \in \mathbb{Z}_{\geq 0}} \bigoplus_{u \in k \cdot \square \cap M} H^0(Y, k \cdot \Psi(u/k)).$$

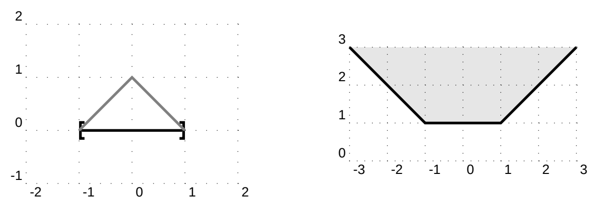
However, this $\mathbb{Z}$-graded algebra is generated in degree one if and only if D_{Ψ^*} is very ample and gives a projectively normal embedding.

Let Ξ be a marked fansy divisor on a curve Y, and $h \in \mathrm{CaSF}(\Xi)$ such that D_h is globally generated and ample. Then the sections of D_h determine a map $f : X(\Xi) \to \mathbb{P}^n$; we denote the image of f by X. Note that X also comes with a natural complexity-one T-action, but in general X need not be normal. By $C(X)$ we denote the affine cone over X with respect to this embedding; let $\widetilde{C(X)}$ be the normalization of $C(X)$. The following proposition tells us how to describe $\widetilde{C(X)}$ in terms of a polyhedral divisor:

Proposition 1.4.7 ([IS09] Proposition 4.1)**.** *With h as above, set*

$$\mathcal{D} = \sum_P \mathrm{conv}(\Gamma_{-h_P}) \otimes P$$

where Γ_{-h_P} is the graph of $-h_P$. Then $\mathcal{D}$ is a proper polyhedral divisor on Y and $\widetilde{C(X)} = X(\mathcal{D})$.



(a) $\Psi_0 = \Psi_\infty$ (b) $D_0 = \mathcal{D}_\infty$

Figure 1.5: A divisorial polytope and affine cone for a toric Fano surface

Example 1.4.8 (A toric Fano surface)**.** We now consider a divisorial polytope and an affine cone for the toric Fano surface X presented in example 1.2.15. Consider first the divisorial polytope $\Psi = \Psi_0 \otimes \{0\} + \Psi_\infty \otimes \{\infty\}$ on $\mathbb{P}^1$, with Ψ_0 and Ψ_∞ as pictured in figure 1.5(a) and $\square = [-1, 1]$. Then one easily checks that $\Xi(\Psi)$ is the marked fansy divisor for the aforementioned Fano surface, and Ψ^* is the support function h from example 1.3.5. We now consider the embedding of X given by D_h, i.e. the anticanonical embedding. This embedding is in fact projectively normal, and the cone over X is equal to $X(\mathcal{D})$, where $\mathcal{D} = \mathcal{D}_0 \otimes \{0\} + \mathcal{D}_\infty \otimes \{\infty\}$ is a polyhedral divisor on $\mathbb{P}^1$ and $\mathcal{D}_0$ and $\mathcal{D}_\infty$ are as pictured in figure 1.5(b).

Remark 1.4.9. Just as in remark 1.2.14 where we consider a toric variety with a subtorus action, we can consider a polarized toric variety with some subtorus action. Let $\Delta \subset M'_{\mathbb{Q}}$ be a polytope in some lattice M', and consider an exact sequence of lattices

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "F"] & M' \arrow[r, "\deg"] & M \arrow[l, bend left, "s^*"] \arrow[r] & 0 \end{tikzcd}$$

where we have chosen some section s^*. Consider the map $\Psi_\Delta : \deg(\Delta) \to \mathrm{Div}(\mathbb{P}^1)$ given by

$$(\Psi_\Delta)_0(u) = \max\{a \in \mathbb{Q} \mid F_{\mathbb{Q}}(a) + s^*(u) \in \Delta \cap \deg_{\mathbb{Q}}^{-1}(u)\},$$
$$(\Psi_\Delta)_\infty(u) = -\min\{a \in \mathbb{Q} \mid F_{\mathbb{Q}}(a) + s^*(u) \in \Delta \cap \deg_{\mathbb{Q}}^{-1}(u)\}.$$

Then $(\Psi_\Delta, \deg(\Delta))$ is a divisorial polytope. Moreover, Ψ_Δ corresponds exactly to the toric variety and the ample divisor coming from Δ but with the restricted torus action of T^N.

1.5 Singularities

We will be interested in criteria for smoothness and mildness of singularities for rational complexity-one T-varieties. We briefly recall several results in this direction from [Süß08].

Definition 1.5.1. Let Δ be a polyhedron in $N_{\mathbb{Q}}$. We say that Δ is *conically smooth/terminal/canonical* if $\mathrm{cone}(\Delta \times 1) \subset (N \times \mathbb{Z})_{\mathbb{Q}}$ is smooth/terminal/canonical.[1] We say that Δ is *conically smooth in dimension d* if all faces of Δ with dimension less than or equal to d are conically smooth.

Consider now a polyhedral divisor $\mathcal{D}$ on $Y = \mathbb{P}^1$.

Proposition 1.5.2 ([Süß08] Theorem 3.1)**.** *Suppose* $\mathrm{Loc}(\mathcal{D}) = \mathbb{P}^1$*. Then* $X(\mathcal{D})$ *is smooth if and only if there exist two special points* $0, \infty \in Y$ *along with lattice vectors* $v_P \in N$ *for all* $P \in Y$ *such that*

(i) Only finitely many $v_P \neq 0$*;*

(ii) $\sum_{P \in Y} v_P = 0$*;*

(iii) $\mathcal{D}_P + v_P = \mathrm{tail}(\mathcal{D})$ *for* $P \neq 0, \infty$*;*

(iv) The cone

$$\mathrm{cone}\Big(((\mathcal{D}_0 + v_0) \times 1) \cup ((\mathcal{D}_\infty + v_\infty) \times -1)\Big)$$

is smooth.

Proposition 1.5.3 (cf. [Süß08] Theorem 3.3)**.** *Suppose that* $\mathrm{Loc}(\mathcal{D}) \subsetneq \mathbb{P}^1$*. Then* $X(\mathcal{D})$ *is smooth/has terminal singularities/has canonical singularities if and only if for all* $P \in \mathrm{Loc}(\mathcal{D})$*,* $\mathcal{D}_P$ *is conically smooth/conically terminal/conically canonical. Likewise,* $X(\mathcal{D})$ *is smooth in codimension* $d+1$ *if and only if for all* $P \in \mathrm{Loc}(\mathcal{D})$*,* $\mathcal{D}_P$ *is conically smooth in dimension* d*. Furthermore, the singularity in any fiber* $p_{\mathcal{D}}^{-1}(P)$ *is algebraically isomorphic to the toric singularity*

$$\mathrm{TV}(\mathrm{cone}(\mathcal{D}_P \times 1)).$$

[1] Note that a cone is smooth/terminal/canonical if the corresponding affine toric variety has the same property. These properties have easy combinatorial characterizations, see for example [Dai02].

Proof. Since $Y = \mathbb{P}^1$, one easily checks that we can replace the analytic isomorphism from [Süß08], theorem 3.3 with an algebraic one. The smoothness/terminal/canonical properties then follow from this isomorphism. $\square$

Chapter 2

Preliminaries on Deformations

In this chapter, we recall necessary results on deformations of algebraic varieties. Section 2.1 contains some basic definitions and standard results. We then describe the construction of homogeneous deformations of affine, rational, complexity-one T-varieties in section 2.2.

2.1 General Deformation Theory

For general facts on deformation theory, we refer to [Ser06].

Definition 2.1.1. A *deformation* of a scheme X_0 is a flat family of schemes $\pi : X \to B$ with $0 \in B$ such that $\pi^{-1}(0) = X_0$. In other words, we have a cartesian diagram

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X \arrow[d, "\pi"] \\ 0 \arrow[r, hook] & B. \end{tikzcd}$$

X is called the *total space* of π and B is called the base space. For any $s \in B$, we denote $\pi^{-1}(s)$ by X_s.

Given deformations $\pi : X \to B$ and $\pi' : X' \to B$ of X_0, an *isomorphism* of π' with π is a map $\phi : X \to X'$ over B inducing the identity on X_0. For any scheme X_0, we actually have a contravariant functor Def_{X_0} where $\mathrm{Def}_{X_0}(B)$ is the set of deformations of X_0 over B modulo isomorphism, and morphisms are mapped to the natural pullbacks.

We will also be interested in deformations of line bundles:

Definition 2.1.2. Let X_0 be a scheme and $\mathcal{L}_0$ a line bundle on X_0. A *deformation* of the pair $(X_0, \mathcal{L}_0)$ consists of a deformation $\pi : X \to B$ of X_0 together with a line bundle $\mathcal{L}$ on X such that $\mathcal{L}_{|X_0} = \mathcal{L}_0$. For any $s \in B$, we denote $\mathcal{L}_{|X_s}$ by $\mathcal{L}_s$.

Given deformations $(\pi, \mathcal{L})$ and $(\pi', \mathcal{L}')$ of $(X_0, \mathcal{L}_0)$, an isomorphism between these pairs is an isomorphism ϕ of the deformations π and π' with $\mathcal{L} = \phi^*(\mathcal{L}')$. As above, we then have a contravariant functor $\mathrm{Def}_{(X_0,\mathcal{L}_0)}$, where $\mathrm{Def}_{(X_0,\mathcal{L}_0)}(B)$ is the set of deformations of $(X_0, \mathcal{L}_0)$ over B modulo isomorphism, and morphisms are mapped to the natural pullbacks.

We will also concern ourselves with embedded deformations:

Definition 2.1.3. Let $X_0 \hookrightarrow \mathbb{P}^n$ be a projectively embedded variety. An *embedded deformation* π of X_0 consists of a projective variety $X \hookrightarrow \mathbb{P}^N_B$ such that the projection π to B is a deformation of X_0, and the embedding of X restricts to the embedding of X_0. In other words, we have the following diagram:

$$\begin{tikzcd}
& \mathbb{P}^n \arrow[r, hook] & \mathbb{P}^n_B \arrow[ddl, bend left] \\
X_0 \arrow[ur, hook] \arrow[r, hook] \arrow[d] & X \arrow[ur, hook] \arrow[d, "\pi"] & \\
0 \arrow[r, hook] & B. &
\end{tikzcd}$$

An isomorphism of embedded deformations is an isomorphism of the ambient projective spaces along with an isomorphism of deformations which is compatible with the embeddings of the total spaces. As above, we then have a contravariant functor $\mathrm{Def}_{X_0 \hookrightarrow \mathbb{P}^n}$, where $\mathrm{Def}_{X_0 \hookrightarrow \mathbb{P}^n}(B)$ is the set of embedded deformations of $X_0 \hookrightarrow \mathbb{P}^n$ over B modulo isomorphism, and morphisms are mapped to the natural pullbacks.

For any projective variety $X_0 \hookrightarrow \mathbb{P}^n$, we then have a natural transformation

$$\mathrm{Def}_{X_0 \hookrightarrow \mathbb{P}^n} \to \mathrm{Def}_{(X_0, \mathcal{O}_{X_0}(1))}.$$

Likewise, for any scheme X_0 and line bundle $\mathcal{L}_0$ we have a natural transformation

$$\mathrm{Def}_{(X_0, \mathcal{L}_0)} \to \mathrm{Def}_{X_0}$$

which simply forgets the line bundle.

We introduce some more terms and notation:

Definition 2.1.4. Let $\pi : X \to B$ be a deformation of X_0.

(i) We call π a *k-parameter* deformation if B is an open subset of $\mathbb{A}^k$.

(ii) Let $\mathbb{C}[\epsilon]$ be the $\mathbb{C}$-algebra in one variable with the relation $\epsilon^2 = 0$. We call π a *first-order* deformation if $B = \operatorname{Spec} \mathbb{C}[\epsilon]$.

(iii) Let a torus $T = T^N$ act on X_0. We call π *homogeneous* with respect to this action if T acts on X and B and the maps π and $X_0 \hookrightarrow X$ are T-equivariant.

(iv) We say that π is *homogeneous of degree zero* if π is homogeneous and the action of T on B is trivial.

Remark 2.1.5. Consider a homogeneous deformation π with respect to some torus T. The action of T preserves fibers of π if and only if π is homogeneous of degree zero.

Now, given any schemes X_0 and B, one always has a deformation of X_0 with base space B, namely, the product family $\pi : X_0 \times B \to B$. We say that any deformation is *trivial* if it is isomorphic to the product family. We say that a deformation $\pi : X \to B$ is *locally trivial* if X admits an open cover $\mathfrak{U}$ such that for each $U \in \mathfrak{U}$, $\pi_{|U}$ is a trivial deformation. We will need the following theorems later:

Theorem 2.1.6 ([Ser06] Theorem 1.2.4). *Let X_0 be a smooth variety, and π a first-order deformation of X_0. Then π is locally trivial.*

Theorem 2.1.7 ([Ser06] Proposition 1.2.9)**.** *Let X_0 be a variety. There is a one-to-one correspondence*

$$\kappa : \left\{ \begin{array}{c} \textit{isomorphism classes of first-order} \\ \textit{locally trivial deformations of } X_0 \end{array} \right\} \to H^1(X_0, \mathcal{T}_{X_0})$$

called the Kodaira-Spencer correspondence *with $\kappa(\xi) = 0$ if and only if ξ is trivial.*

Let $\pi : X \to B$ now be a one-parameter deformation of some scheme X_0, where we have fixed some isomorphism $B \cong \operatorname{Spec} C$, where C is some localization of $\mathbb{C}[t]$ with $0 = V(t)$. Then we have a map $\operatorname{Spec} \mathbb{C}[\epsilon] \to B$ (canonical up to scaling) given by mapping t to ϵ. This gives us a first-order deformation

$$\pi^\epsilon : X \times_B \operatorname{Spec} \mathbb{C}[\epsilon] \to \operatorname{Spec} \mathbb{C}[\epsilon]$$

by pulling back π. We say that π is *infinitesimally (locally) trivial* if π^ϵ is (locally) trivial. We can extend κ to map infinitesimally locally trivial one-parameter deformations to elements of $H^1(X_0, \mathcal{T}_{X_0})$ by setting $\kappa(\pi) = \kappa(\pi^\epsilon)$; we call this the *Kodaira-Spencer map*. This map gives us an important invariant of any infinitesimally locally trivial one-parameter deformation.

For any variety X_0, we set $T^1_{X_0} = \operatorname{Def}_{X_0}(\operatorname{Spec} \mathbb{C}[\epsilon])$. This carries the structure of a $\mathbb{C}$-vector space. Note that if X_0 is smooth, then we in fact have $T^1_{X_0} = H^1(X_0, \mathcal{T}_{X_0})$. We say that X_0 is *rigid* if $T^1_{X_0} = 0$. In this case, all first-order deformations are trivial.

2.2 Deformations of Rational, Affine, Complexity-One T-Varieties

In this thesis, we will be studying deformations of nonaffine, rational, complexity-one T-varieties. In order to do this, we first need to understand the affine case. This has been studied in section 3 of [IV09] as well as [Vol10]. We summarize the essential results here.

Definition 2.2.1. Let $\alpha = (\alpha^1, \ldots, \alpha^r)$ be an r-tuple of natural numbers. An *α-term Minkowski decomposition* of a polyhedron Δ consists of polyhedra Δ^i, $i = 0, \ldots, r$ all with tailcone $\operatorname{tail}(\Delta)$ such that

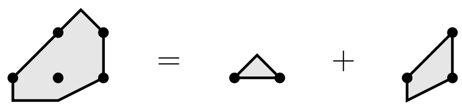
(i) $\Delta = \Delta^0 + \alpha^1 \cdot \Delta^1 + \ldots + \alpha^r \cdot \Delta^r$.

Furthermore, we call an α-term Minkowski decomposition *α-admissible* if:

(ii) For any $u \in \operatorname{tail}(\Delta)^\vee \cap M$, at most one of the evaluations $\min\langle \Delta^i, u \rangle$ is non-integral.

(iii) For each $1 \leq i \leq r$ with $\alpha^i > 1$, Δ^i is a lattice polyhedron.

For example,



is an α-admissible Minkowski decomposition of a non-lattice polyhedron with tailcone 0 for $\alpha = (1)$.

Remark 2.2.2. In the above definition, we also allow $\Delta = \emptyset$, in which case we replace $\mathrm{tail}(\Delta)$ with some cone σ of our choice. Note that if $\Delta \neq \emptyset$, the second condition above is equivalent to the following: for all vertices v of Δ, at most one of the corresponding vertices v_i of Δ^i, $i = 0, \dots, r$ is not a lattice point.

Now let $\mathcal{D}$ be a proper polyhedral divisor on $Y = \mathbb{P}^1$. As usual, let $\mathcal{P}$ be the set of points P in Y such that $\mathcal{D}_P$ is nontrivial. Consider a finite set of points $\mathcal{Q}$ of Y along with an α_Q-admissible Minkowski decomposition

$$\mathcal{D}_Q = \mathcal{D}_Q^0 + \sum_{i=1}^{r_Q} \alpha_Q^i \mathcal{D}_Q^i.$$

for each $Q \in \mathcal{Q}$, where if $\mathcal{D}_Q = \emptyset$ we take $\sigma = \mathrm{tail}(\mathcal{D})$. We call this data a *Minkowski decomposition* of $\mathcal{D}$. As we shall see, this data encodes an r-parameter deformation of $X(\mathcal{D})$, where $r = \sum_{Q\in\mathcal{Q}} r_Q$.

For each point $P \in \mathbb{P}^1$, let $y_P \in \mathbb{C}(Y)$ be a rational function with its sole zero at P. Let $t_{Q,1}, \dots, t_{Q,r_Q}$ be coordinates on $\mathbb{A}^{r_Q}$ for $Q \in \mathcal{Q}$, and set $t_{Q,0} = 0$ and $\alpha_Q^0 = 1$. Let B be any open affine neighborhood of the origin in $\prod_{Q\in\mathcal{Q}} \mathbb{A}^{r_Q}$ such that a divisor on $\mathbb{P}^1 \times B$ of the form $V(y_Q^{\alpha_Q^i} - t_{Q,i})$ doesn't intersect any divisor of the form $V(y_P)$ or $V(y_P^{\alpha_P^j} - t_{P,j})$ for $P \neq Q$ and respectively $P \in \mathcal{P}$ or $P \in \mathcal{Q}$. We then set $Y^{\mathrm{tot}} = \mathbb{P}^1 \times B$, and consider the polyhedral divisor

$$\mathcal{D}^{\mathrm{tot}} = \sum_{P\in\mathcal{P}\setminus\mathcal{Q}} \mathcal{D}_P \otimes V(y_P) + \sum_{\substack{Q\in\mathcal{Q}\\ 0\leq i\leq r_Q}} \mathcal{D}_Q^i \otimes V(y_Q^{\alpha_Q^i} - t_{Q,i})$$

on Y^{tot}. This is in fact a proper polyhedral divisor. We set $X^{\mathrm{tot}} = X(\mathcal{D}^{\mathrm{tot}})$. Consider now the composition of the quotient map $X(\mathcal{D}^{\mathrm{tot}}) \dashrightarrow Y^{\mathrm{tot}}$ with the projection to B. This can be extended to a morphism $\pi : X^{\mathrm{tot}} \to B$. For any $s = (s_{Q,i}) \in B$, we define a polyhedral divisor on $\mathbb{P}^1$:

$$\mathcal{D}^{(s)} = \sum_{P\in\mathcal{P}\setminus\mathcal{Q}} \mathcal{D}_P \otimes V(y_P) + \sum_{\substack{Q\in\mathcal{Q}\\ 0\leq i\leq r_Q}} \mathcal{D}_Q^i \otimes V(y_Q^{\alpha_Q^i} - s_{Q,i}).$$

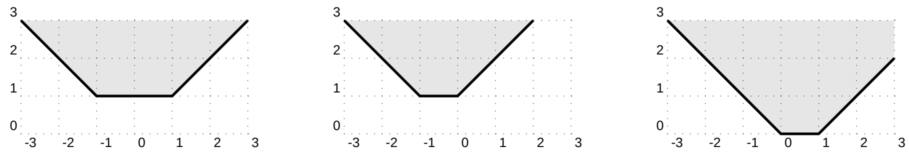
$\mathcal{D}^{(s)}$ is in fact also proper, and we have $\mathcal{D}^{(0)} = \mathcal{D}$. These polyhedral divisors encode the desired deformation:

Theorem 2.2.3 ([IV09])**.** *$\pi : X^{\mathrm{tot}} \to B$ is an r-parameter deformation of $X(\mathcal{D})$, and for any $s \in B$, $\pi^{-1}(s) = X(\mathcal{D}^{(s)})$.*

We call deformations of the above form *T-deformations*. Note that a T-deformation is always homogeneous of degree 0. Before finishing this section, we note the following fact:

Lemma 2.2.4. *If $\mathcal{D}$ has affine locus, then* $\mathrm{Loc}(\mathcal{D}^{\mathrm{tot}})$ *is affine.*

Proof. Since $\mathcal{D}$ has affine locus, $\mathrm{Loc}(\mathcal{D}^{\mathrm{tot}}) = Y^{\mathrm{tot}} \setminus V(f)$ for some nontrivial $f \in \mathbb{C}(Y^{\mathrm{tot}})$. Now because $Y^{\mathrm{tot}} = \mathbb{P}^1_B$ and B is affine, the claim follows from [Har77], proposition II.2.5(c). □



(a) $\mathcal{D}_0$ (b) $\mathcal{D}_0^0$ (c) $\mathcal{D}_0^1$

Figure 2.1: A Minkowski decomposition for the cone over a toric Fano surface

Example 2.2.5 (Cone over a toric Fano surface)**.** Consider the proper polyhedral divisor $\mathcal{D}$ from example 1.4.8; $X(\mathcal{D})$ is the cone over a toric Fano surface. We can decompose $\mathcal{D}_0$ as $\mathcal{D}_0 = \mathcal{D}_0^0 + \mathcal{D}_0^1$ as pictured in figure 2.1. This gives us a deformation π of $X(\mathcal{D})$. Note that the fiber of π at some $s \neq 0$ can't be toric, since $\mathcal{D}^{(s)}$ has nontrivial coefficients at 0, s, and ∞.

Chapter 3

Homogeneous Deformations of Nonaffine T-Varieties

This chapter contains the essential construction introduced in this dissertation, and forms a basis for all following chapters. In section 3.1, we introduce Minkowski decompositions of polyhedral subdivisions, and show how they behave especially nicely for polyhedral subdivisions with convex support. In section 3.2 we show how these decompositions can be used to construct deformations of nonaffine rational complexity-one T-varieties. In section 3.3 we provide criteria for these deformations to be separated and proper, as well as specializing to deformations of complete T-varieties. In section 3.4 we discuss the special case of locally trivial deformations. Finally, in section 3.5 we compute the image of the Kodaira-Spencer map for certain infinitesimally locally trivial one-parameter deformations.

3.1 Minkowski Decompositions of Polyhedral Subdivisions

As we saw in section 2.2, the essential ingredient for the construction of T-deformations of affine T-varieties was Minkowski decompositions of polyhedra. For T-deformations of nonaffine T-varieties, this role will be played by Minkowski decompositions of polyhedral subdivisions. In this section, we introduce these decompositions, and prove some preliminary results concerning them.

Definition 3.1.1. Let C be any polyhedral subdivision in $N_{\mathbb{Q}}$ and $\alpha \in \mathbb{N}^r$. An *α-term Minkowski predecomposition* of C consists of α-term Minkowski decompositions

$$\Delta = \Delta^0 + \alpha^1 \cdot \Delta^1 + \ldots + \alpha^r \cdot \Delta^r$$

for all $\Delta \in C$ such that

(i) If $\Delta \cap \nabla \neq \emptyset$ with $\Delta, \nabla \in C$, then $(\Delta \cap \nabla)^i = \Delta^i \cap \nabla^i$ for any $i \in \{0, \ldots, r\}$.

Such a predecomposition is called a *decomposition* if additionally

(ii) We have
$$\sum_{i \in I} \bigcap_{\Delta \in \mathcal{I}} \Delta^i \prec \sum_{i \in I} \bigcap_{\Delta \in \mathcal{J}} \Delta^i$$
for any $\mathcal{J} \subset \mathcal{I} \subset C$ and $I \subset \{0, \ldots, r\}$.

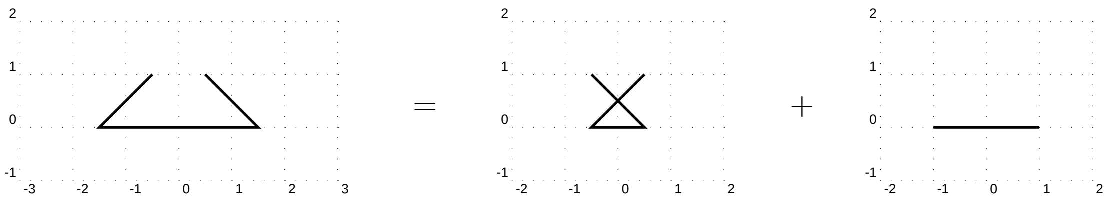


Figure 3.1: A Minkowski predecomposition

Finally, any α-term Minkowski (pre)decomposition of C is *α-admissible* if for each $\Delta \in C$, the corresponding decomposition of Δ is α-admissible, see definition 2.2.1.

Remark 3.1.2. Definitions quite similar to the above have appeared for example in [AHS08] (see definition 7.3) and [Mav09] (see section 3). Likewise, the fan decompositions in section 5 of [IV09] are a special case of Minkowski decompositions.

Condition (ii) in the above definition appears somewhat cumbersome. However, if C has convex support, it follows automatically from condition (i):

Proposition 3.1.3. *Let C be a polyhedral subdivision with convex support. Then any α-term Minkowski predecomposition of C is a decomposition.*

Remark 3.1.4. If $|C|$ is not convex, the above proposition need not hold. Consider for example the polyhedral subdivision C pictured in the left of figure 3.1. Then the following gives a (1)-term Minkowski predecomposition of C:

$$\begin{array}{ccccc}
\overline{(-1.5,0)(-.5,1)} & = & \overline{(-.5,0)(.5,1)} & + & (-1,0) \\
\overline{(1.5,0)(.5,1)} & = & \overline{(.5,0)(-.5,1)} & + & (1,0) \\
\overline{(-1.5,0)(1.5,0)} & = & \overline{(-.5,0)(.5,0)} & + & \overline{(-1,0)(1,0)} \\
(-1.5,0) & = & (-.5,0) & + & (-1,0) \\
(1.5,0) & = & (.5,0) & + & (1,0) \\
(-.5,1) & = & (.5,1) & + & (-1,0) \\
(.5,1) & = & (-.5,1) & + & (1,0)
\end{array}$$

However, it is clearly not a decomposition, since $\overline{(-.5,0)(.5,1)}$ and $\overline{(.5,0)(-.5,1)}$ do not intersect in a common face.

In order to prove the above proposition, we introduce something quite interesting in its own right: the *cone of Minkowski summands*. This was first introduced by K. Altmann in [Alt97] for compact polyhedra. We now generalize it to polyhedral subdivisions with convex support as follows:

Let C be a polyhedral subdivision of $N_{\mathbb{Q}}$ with convex support. Let $E \subset C(1)$ be the set of compact edges in C. Set $V := \mathbb{Q}^E$; for any $v \in V$ and $e \in E$, denote the e-component of v by v_e. Now for each edge $e \in E$, fix some orientation; we can thus identify e with a vector $\vec{e}$ of $N_{\mathbb{Q}}$. For any compact two-face $\tau \in C(2)$, define its sign vector

$$\varepsilon^\tau = (\varepsilon^\tau_e)_{e \in E} \in V$$

by

$$\varepsilon_e^\tau := \begin{cases} \pm 1 & \text{if } e \prec \tau \\ 0 & \text{otherwise} \end{cases}$$

such that $\sum_{e \in E} \varepsilon_e^\tau \vec{e} = 0$. This determines ε^τ up to sign, and we choose one of both possibilities.

Definition 3.1.5. For any polyhedral subdivision C with convex support, its *cone of Minkowski summands* is

$$\mathrm{cone}_{\mathrm{MS}}(C) := \left\{ v \in V \;\middle|\; \begin{array}{ll} v_e \geq 0 & \text{for all } e \in E \\ \sum v_e \varepsilon_e^\tau \vec{e} = 0 & \text{for all compact } \tau \in C(2) \end{array} \right\}.$$

One easily checks that $\mathrm{cone}_{\mathrm{MS}}(C)$ is a pointed polyhedral cone. Furthermore, it is at least one-dimensional, containing the vector $\underline{1}$, where $\underline{1}_e = 1$.

We now show how, given $v \in \mathrm{cone}_{\mathrm{MS}}(C)$, to construct a set of polyhedra C_v. Consider any sequence γ of edges $e \in E$ defining a noncyclic path from vertices w_1 to w_2 in $C(0)$. For $e \in \gamma$, set ε_e^γ to 1 if e was oriented from w_1 to w_2 and -1 if not. Now for any $\Delta \in C$, fix some vertex $w_\Delta \in \Delta(0)$. We set

$$\tilde{\Delta}_v := \mathrm{conv} \left\{ \sum_{e \in \gamma} v_e \varepsilon_e^\gamma \vec{e} \;\middle|\; \gamma \subset \Delta(1) \text{ noncyclic path} \right\} + \mathrm{tail}(\Delta).$$

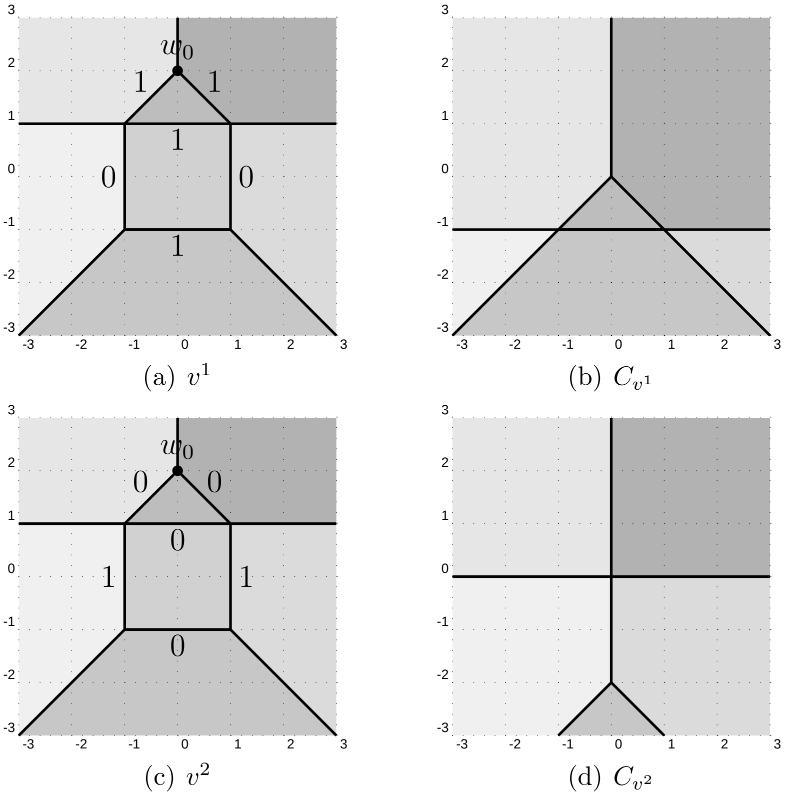
In other words, we get the polyhedron $\tilde{\Delta}_v$ by scaling the compact edges of Δ according to v and translating w_Δ to the origin. Now fix some $w_0 \in C(0)$. Then set

$$\Delta_v := \tilde{\Delta}_v + \sum_{e \in \gamma} v_e \varepsilon_e^\gamma \vec{e}$$

where γ is a path from w_0 to w_Δ. Finally, set $C_v := \{\Delta_v \mid \Delta \in C\}$. One easily checks that the whole construction is independent of all choices except that of w_0; different choices of w_0 correspond to uniform translation of the elements of C_v. Note that $C_{\underline{1}}$ is just some translation of C.

Consider now $\Delta, \nabla \in C$ such that $\Delta \cap \nabla \neq \emptyset$. Then it follows immediately from definition 3.1.5 and the above construction that $\Delta_v \cap \nabla_v = (\Delta \cap \nabla)_v$. Furthermore, consider $v^0, \ldots, v^r \in \mathrm{cone}_{\mathrm{MS}}(C)$ and some $\alpha \in \mathbb{N}^r$. Then $\Delta_{v^0} + \ldots + \Delta_{\alpha^r v^r} = \Delta_{v^1 + \ldots + \alpha^r v^r}$. In particular, if $v^0 + \sum_{i \geq 1} \alpha^i v^i = \underline{1}$, we get an α-term Minkowski predecomposition of $C_{\underline{1}}$. On the other hand, one easily checks that (modulo uniform translations) all α-term Minkowski predecompositions arise in this way. Thus, $\mathrm{cone}_{\mathrm{MS}}(C)$ parametrizes possible "summands" in predecompositions of C. From proposition 3.1.3 it will follow that $\mathrm{cone}_{\mathrm{MS}}(C)$ in fact parametrizes possible summands in *decompositions* of C.

Example 3.1.6 (A two-dimensional subdivision)**.** Consider the two-dimensional polyhedral subdivision C pictured in figures 3.2(a) and (c). Then $\#E = 6$, and there are two compact two-faces which give rise to four linearly independent equations; $\mathrm{cone}_{\mathrm{MS}}(C)$ is consequently two dimensional. By attaching numbers to edges in figures 3.2(a) and (c), we represent two elements $v^1, v^2 \in \mathrm{cone}_{\mathrm{MS}}(C)$ which in fact generate the cone. The corresponding sets of polyhedra C_{v^i} are pictured in figures 3.2(b) and (d), where in this case we have chosen $w_0 = (0, 2)$.



(a) v^1 (b) C_{v^1} (c) v^2 (d) C_{v^2}

Figure 3.2: Generators for $\mathrm{cone}_{\mathrm{MS}}(C)$

Proposition 3.1.7. *As constructed above, C_v is a polyhedral subdivision with convex support for any $v \in \mathrm{cone}_{MS}(C)$. Furthermore, if C is complete, then C_v is complete.*

Proof. In this proof, Δ, ∇, and $\Box$ will always be elements of C. For $0 \leq t \leq 1$, set $v(t) := t \cdot \underline{1} + (1-t) \cdot v$. Now if for all t, $\nabla \cap \Box = \emptyset$ implies $\nabla_{v(t)} \cap \Box_{v(t)} = \emptyset$, then C_v is clearly a polyhedral subdivision. Thus, for the moment we will be assuming that this isn't the case. Then the set

$$\{0 \leq t \leq 1 \mid \text{There exist } \nabla, \Box \in C \text{ with } \nabla \cap \Box = \emptyset,\ \nabla_{v(t)} \cap \Box_{v(t)} \neq \emptyset\}$$

isn't empty, and we take t_0 to be the maximal element. Note that we can actually assume that there are $\nabla, \Box \in C$ such that $\nabla \cap \Box = \emptyset$ but $\mathrm{relint}(\nabla_{v(t_0)}) \cap \mathrm{relint}(\Box_{v(t_0)}) \neq \emptyset$, and that the codimension of both ∇ and $\Box$ is larger than zero.

We claim that $t_0 = 0$; suppose not. If both ∇ and $\Box$ were contained in the boundary of C, then they would have to intersect, so we assume that ∇ isn't in the boundary. Then the polyhedra

$$\{\Delta_{v(t_0)} \mid \Delta \in C,\ \nabla \prec \Delta\}$$

surround the relative interior of $\nabla_{v(t_0)}$. Thus, there is some $\delta > 0$ and some $\Delta \in C$ such that $\nabla \prec \Delta$ and $\Delta_{v(t_0+\delta)} \cap \Box_{v(t_0+\delta)} \neq \emptyset$. But then $\Delta \cap \Box \neq \emptyset$, and for all t we have

$$\nabla_{v(t)} \cap \Box_{v(t)} = \nabla_{v(t)} \cap \Delta_{v(t)} \cap \Box_{v(t)} = \nabla_{v(t)} \cap (\Delta \cap \Box)_{v(t)}.$$

Both polyhedra in the intersection on the right hand side of the above equation are faces of $\Delta_{v(t)}$. Thus, if $\nabla_{v(t)} \cap \Box_{v(t)} \neq \emptyset$ for some $t > 0$, this must hold for all $t > 0$. It follows that t_0 must in fact be 0.

Now, if both ∇ and $\Box$ are contained in the boundary of C, it is clear from construction that $\nabla_0 \cap \Box_0$ is a face of both ∇_0 and $\Box_0$. Thus, we can again assume that ∇ isn't in the boundary of C. Then we note that the polyhedra

$$\{\Delta_v \mid \Delta \in C,\ \nabla_v \prec \Delta_v\}$$

surround the relative interior of ∇_v. Thus, there is some $\delta > 0$ and some $\Delta \in C$ such that $\nabla_v \prec \Delta_v$ and $\Delta_{v(\delta)} \cap \Box_{v(\delta)} \neq \emptyset$. Since $\delta > t_0 = 0$, it follows that $\Delta \cap \Box \neq \emptyset$. Then we have

$$\nabla_v \cap \Box_v = \nabla_v \cap \Delta_v \cap \Box_v = \nabla_v \cap (\Delta \cap \Box)_v.$$

Since both ∇_v and $(\Delta \cap \Box)_v$ are faces of Δ_v, their intersection must be a face of both polyhedra.

The convexity of $|C_v|$ follows easily from construction. If C is complete, then $|C_v|$ is unbounded in every direction, so since it is convex as well, C_v must be complete. $\Box$

We now return to the proof of proposition 3.1.3:

Proof of proposition 3.1.3. Let $|C|$ be convex, and consider any α-term predecomposition of C. Then let $v^0, v^1, \ldots, v^r \in \mathrm{cone}_{\mathrm{MS}}(C)$ correspond to this predecomposition. Now, fix some $I \subset \{0, \ldots, r\}$. Set $v := \sum_{i\in I} v^i$. One easily checks that $v^i \in \mathrm{cone}_{\mathrm{MS}}(C_v)$ for all $i \in I$.

We claim that for any $\mathcal{I} \subset C$,

$$\sum_{i\in I} \bigcap_{\Delta\in\mathcal{I}} \Delta^i = \bigcap_{\Delta\in\mathcal{I}} \sum_{i\in I} \Delta^i. \tag{3.1.1}$$

Indeed, if right hand side is equal to $\emptyset$, the equality is immediate. On the other hand, if this isn't the case, the equality follows from the fact that the v^i for $i \in I$ give a predecomposition of C_v, coupled with repeated used of property (i) from definition 3.1.1.

Property (ii) from definition 3.1.1 follows from equation (3.1.1) together with the fact that C_v is a polyhedral subdivision. $\Box$

3.2 T-Deformations of Nonaffine T-Varieties

Let $Y = \mathbb{P}^1$ and let $\mathcal{S}$ be a divisorial fan on Y. We now show how to construct homogeneous deformations of the rational nonaffine T-variety $X(\mathcal{S})$. This is done by gluing together T-deformations of affine T-varieties.

Definition 3.2.1. Fix some finite set of points $\mathcal{Q} \subset \mathbb{P}^1$. A *Minkowski (pre)decomposition* of $\mathcal{S}$ consists of α_Q-admissible Minkowski (pre)decompositions of the polyhedral subdivisions $\mathcal{S}_Q$ for $Q \in \mathcal{Q}$.

Consider now some Minkowski decomposition of $\mathcal{S}$. We shall show how this can be used to construct a deformation of $X(\mathcal{S})$. Now, for any $\mathcal{D} \in \mathcal{S}$, $Q \in \mathcal{Q}$, and $0 \leq i \leq r_Q$, let $\mathcal{D}_Q^i$ be the corresponding summand in the Minkowski decomposition of the polyhedral subdivision $\mathcal{S}_Q$ if $\mathcal{D}_Q \neq \emptyset$, and take $\mathcal{D}_Q^i = \emptyset$ otherwise. Thus, for each $\mathcal{D} \in \mathcal{S}$, we have a Minkowski decomposition giving rise to polyhedral divisors $\mathcal{D}^{\mathrm{tot}}$ and $\mathcal{D}^{(s)}$ on respectively Y^{tot} and $\mathbb{P}^1$, where $Y^{\mathrm{tot}} = \mathbb{P}^1 \times B$ is defined as in section 2.2. For each such polyhedral divisor $\mathcal{D}$, we also have a T-deformation $\pi_{\mathcal{D}} : X(\mathcal{D}^{\mathrm{tot}}) \to B$. To get a deformation of $X(\mathcal{S})$, we will glue the deformations of $\pi_{\mathcal{D}}$ together.

For any $\mathcal{I} \subset \mathcal{S}$, we set

$$\mathcal{D}^{\mathcal{I}} = \bigcap_{\mathcal{D}\in\mathcal{I}} \mathcal{D}; \qquad \mathcal{D}^{\mathcal{I},\mathrm{tot}} = \bigcap_{\mathcal{D}\in\mathcal{I}} \mathcal{D}^{\mathrm{tot}}; \qquad \mathcal{D}^{\mathcal{I},(s)} = \bigcap_{\mathcal{D}\in\mathcal{I}} \mathcal{D}^{(s)}.$$

Note that $\mathcal{D}^{\mathcal{I}} \in \mathcal{S}$. We then set

$$\mathcal{S}^{\mathrm{tot}} = \left\{\mathcal{D}^{\mathcal{I},\mathrm{tot}}\right\}_{\mathcal{I}\subset\mathcal{S}}; \qquad \mathcal{S}^{(s)} = \left\{\mathcal{D}^{\mathcal{I},(s)}\right\}_{\mathcal{I}\subset\mathcal{S}}.$$

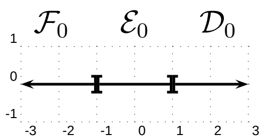
Proposition 3.2.2. *$\mathcal{S}^{\mathrm{tot}}$ is a divisorial fan on Y^{tot}. Likewise, each $\mathcal{S}^{(s)}$ is a divisorial fan on $\mathbb{P}^1$.*

We will prove this proposition shortly, but first we wish to state the main theorem of this section:

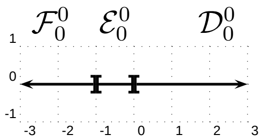
Theorem 3.2.3. *The maps $\pi_{\mathcal{D}} : X(\mathcal{D}^{\mathrm{tot}}) \to B$ glue together to a flat family $\pi : X(\mathcal{S}^{\mathrm{tot}}) \to B$ such that for $s \in B$, $\pi^{-1}(s) = X(\mathcal{S}^{(s)})$. Thus, π is an r-parameter deformation of $X(\mathcal{S})$.*

We call a deformation π of the above form a T-*deformation* as well.

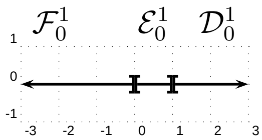
Remark 3.2.4. T-deformations of non-affine varieties are also homogeneous of degree zero. This follows from the fact that they are locally homogeneous of degree zero, and all gluing respects the T-action.



(a) $\mathcal{S}_0$



(b) $\mathcal{S}^{\mathrm{tot}}_{V(y_0)}$



(c) $\mathcal{S}^{\mathrm{tot}}_{V(y_0-t)}$

Figure 3.3: A T-deformation of a toric Fano surface

Example 3.2.5 (A T-deformation of a toric Fano surface)**.** We construct a T-deformation of the toric Fano surface from example 1.2.15. Let $\mathcal{S}$ be the divisorial fan $\mathcal{S}^{\Sigma'}$ from this example. It is induced by four polyhedral divisors $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, and contains five further polyhedral divisors gotten via intersections. Now, we define a Minkowski decomposition of $\mathcal{S}_0$ as follows:

$$\begin{array}{ccccc} [1,\infty) & = & [0,\infty) & + & [1,\infty) \\ [-1,1] & = & [-1,0] & + & [0,1] \\ (-\infty,-1] & = & (-\infty,-1] & + & (-\infty,0] \\ \{1\} & = & \{0\} & + & \{1\} \\ \{-1\} & = & \{-1\} & + & \{0\} \end{array}$$

This leads to the decompositions

$$\begin{array}{ccccc} \mathcal{D}_0 & = & [0,\infty) & + & [1,\infty) \\ \mathcal{E}_0 & = & [-1,0] & + & [0,1] \\ \mathcal{F}_0 & = & (-\infty,-1] & + & (-\infty,0] \\ (\mathcal{D}\cap\mathcal{E})_0 & = & \{0\} & + & \{1\} \\ (\mathcal{F}\cap\mathcal{G})_0 & = & \{-1\} & + & \{0\} \end{array}$$

with the decomposition of the 0-coefficient for the four remaining elements of $\mathcal{S}$ simply given by $\emptyset + \emptyset$. We thus get a deformation π of $X(\mathcal{S})$. In figure 3.3, we show the slice $\mathcal{S}_0$, along with the slices in $V(y_0)$ and $V(y_0 - t)$ of $\mathcal{S}^{\text{tot}}$. By looking at the divisorial fans $\mathcal{S}^{(s)}$ for $s \neq 0$ and using proposition 1.5.3, one sees that π smoothes the A_1 singularity in the chart $X(\mathcal{E})$. However, the singularities in the charts $X(\mathcal{D})$, $X(\mathcal{F})$, and $X(\mathcal{G})$ are not smoothed by this deformation.

Remark 3.2.6. If we are only given a predecomposition of $\mathcal{S}$, we can still construct a deformation of $X(\mathcal{S})$ as follows. Indeed, we still get a Minkowski decomposition of every $\mathcal{D} \in \mathcal{S}$ as described above. One can show that if $\mathcal{D} \prec \mathcal{E}$, then $\mathcal{D}^{\text{tot}} \prec \mathcal{E}^{\text{tot}}$. Thus, we can construct a scheme $\tilde{X}^{\text{tot}}$ by gluing $X(\mathcal{D}^{\text{tot}})$ and $X(\mathcal{E}^{\text{tot}})$ along $X((\mathcal{D} \cap \mathcal{E})^{\text{tot}})$ for any $\mathcal{D}, \mathcal{E} \in \mathcal{S}$. We can then naturally glue the deformations $\pi_{\mathcal{D}}$ together to get a map $\tilde{\pi} : \tilde{X}^{\text{tot}} \to B$. Then the special fiber of $\tilde{\pi}$ is still $X(\mathcal{S})$, and this deformation is also homogeneous of degree zero. However, it has the distinct disadvantage that $\tilde{X}^{\text{tot}}$ and the general fibers of $\tilde{\pi}$ may not be describable via divisorial fans. Furthermore, it can be shown that if $\tilde{X}^{\text{tot}}$ is separated, then the predecomposition must have in fact been a decomposition, and $\tilde{X}^{\text{tot}} = X(\mathcal{S}^{\text{tot}})$. Thus, in all further discussion we will only be considering true decompositions.

The remainder of the section is dedicated to proving proposition 3.2.2 and theorem 3.2.3. We split up the proofs into several smaller lemmata. Note that we will only be proving the claim of 3.2.2 for $\mathcal{S}^{\text{tot}}$. The proof for $\mathcal{S}^{(s)}$ is similar, and left to the reader.

Lemma 3.2.7. *(i) Consider polyhedra Δ^0 and Δ^1, and set $\Delta = \Delta^0 + \Delta^1$. For any $w \in \operatorname{tail}(\Delta)^\vee$ we have $\operatorname{face}(\Delta, w) = \operatorname{face}(\Delta^0, w) + \operatorname{face}(\Delta^1, w)$.*

(ii) For any polyhedra Δ and $w \in \operatorname{tail}(\Delta)^\vee$, $\operatorname{face}(\operatorname{tail}(\Delta), w) = \operatorname{tail}(\operatorname{face}(\Delta, w))$.

Proof. The proof of (i) is straightforward and left to the reader. The second claim follows from the first by writing $\Delta = \Delta^c + \operatorname{tail}(\Delta)$ for some compact polyhedron Δ^c. ☐

Lemma 3.2.8. *Consider polytopes $\nabla^i \subset \Delta^i$ for $1 \leq i \leq n$. Set $\Delta := \sum_{i=1}^n \Delta^i$ and $\nabla := \sum_{i=1}^n \nabla^i$ and let I be any subset of $\{1, \dots, n\}$.*

(i) For any $w \in (\operatorname{tail}\Delta)^\vee$ with $\operatorname{face}(\Delta, w) = \nabla$, we have $\operatorname{face}(\sum_{i\in I} \Delta^i, w) = \sum_{i \in I} \nabla^i$.

(ii) For any $w \in (\operatorname{tail}\Delta)^\vee$ with $\operatorname{face}(\Delta, w) = \operatorname{face}(\nabla, w)$, we have $\operatorname{face}(\sum_{i\in I} \Delta^i, w) = \operatorname{face}(\sum_{i\in I} \nabla^i, w)$.

Proof. Note that the first claim follows from the second. Indeed, if $\nabla = \operatorname{face}(\Delta, w)$, then w is constant on ∇, and must also be constant on ∇^i. Thus, $\nabla^i = \operatorname{face}(\nabla^i, w)$.

For part (ii), observe that $\nabla^i \subset \Delta^i$ implies $\min\langle \nabla^i, w\rangle \geq \min\langle \Delta^i, w\rangle$ for $1 \leq i \leq n$. But

$$\sum \min\langle \nabla^i, w\rangle = \min\langle \nabla, w\rangle = \min\langle \Delta, w\rangle = \sum \min\langle \Delta^i, w\rangle$$

so we in fact have $\min\langle \nabla^i, w\rangle = \min\langle \Delta^i, w\rangle$. Coupled with $\nabla^i \subset \Delta^i$ we then get that $\operatorname{face}(\nabla^i, w) \subset \operatorname{face}(\Delta^i, w)$. Applying lemma 3.2.7, we then have the following diagram:

$$\begin{array}{ccccccc}
\operatorname{face}(\Delta^1, w) & + & \dots & + & \operatorname{face}(\Delta^n, w) & = & \operatorname{face}(\Delta, w) \\
\cup & & & & \cup & & \| \\
\operatorname{face}(\nabla^1, w) & + & \dots & + & \operatorname{face}(\nabla^n, w) & = & \operatorname{face}(\nabla, w).
\end{array}$$

We can conclude that the inclusions must be equalities and again apply lemma 3.2.7. ☐

Lemma 3.2.9. *Let $\mathcal{D}', \mathcal{D}$ be proper polyhedral divisors on some curve C with $\mathcal{D}' \prec \mathcal{D}$, $\mathrm{Loc}(\mathcal{D})$ complete, and $\mathrm{Loc}(\mathcal{D}')$ not complete. Then for any $w \in \mathrm{tail}(\mathcal{D})^\vee$ with*

$$\mathrm{face}(\mathrm{tail}(\mathcal{D}), w) = \mathrm{tail}(\mathcal{D}'),$$

we have $\deg(\mathcal{D}(w)) > 0$.

Proof. Let y be the general point of C. Then from $\mathcal{D}' \prec \mathcal{D}$ and the conditions of definition 1.2.7, we find w_y such that $\mathrm{face}(\mathrm{tail}(\mathcal{D}), w_y) = \mathrm{tail}(\mathcal{D}')$ and $\deg(\mathcal{D}(w_y)) > 0$, since the fact that some coefficients of $\mathcal{D}'$ are the empty set implies the existence of a divisor $D_y \in |\mathcal{D}(w_y)|$ with nontrivial support. From this it follows that $\deg(\mathcal{D}) \cap \mathrm{tail}(\mathcal{D}') = \emptyset$.

Now let w be as in the statement of the lemma. The hyperplane determined by $\langle \cdot, w\rangle = 0$ intersects $\mathrm{tail}(\mathcal{D})$ in exactly $\mathrm{tail}(\mathcal{D}')$. Thus, it cannot intersect $\deg(\mathcal{D})$, since $\deg(\mathcal{D}) \subset \mathrm{tail}(\mathcal{D})$. It follows that $\deg(\mathcal{D}(w)) = (\deg(\mathcal{D}))(w) \neq 0$, so it must be strictly positive. ☐

Lemma 3.2.10. *For any $\mathcal{I} \subset \mathcal{S}$ with $\mathrm{Loc}(\mathcal{D}^{\mathcal{I}}) = \mathbb{P}^1$, we have $(\mathcal{D}^{\mathcal{I}})^{\mathrm{tot}} = \mathcal{D}^{\mathcal{I},\mathrm{tot}}$.*

Proof. From point (i) of definition 3.1.1 we have

$$\begin{aligned}
(\mathcal{D}^{\mathcal{I}})^{\mathrm{tot}} &= \sum_{P \in \mathcal{P}\setminus\mathcal{Q}} (\bigcap_{\mathcal{D}\in\mathcal{I}} \mathcal{D})_P \otimes V(y_P) + \sum_{\substack{Q\in\mathcal{Q}\\ 0\le i\le r_Q}} (\bigcap_{\mathcal{D}\in\mathcal{I}} \mathcal{D})_Q^i \otimes V(y_Q^{\alpha_Q^i} - t_{Q,i}) \\
&= \sum_{P \in \mathcal{P}\setminus\mathcal{Q}} (\bigcap_{\mathcal{D}\in\mathcal{I}} \mathcal{D}_P) \otimes V(y_P) + \sum_{\substack{Q\in\mathcal{Q}\\ 0\le i\le r_Q}} (\bigcap_{\mathcal{D}\in\mathcal{I}} \mathcal{D}_Q^i) \otimes V(y_Q^{\alpha_Q^i} - t_{Q,i}) \\
&= \mathcal{D}^{\mathcal{I},\mathrm{tot}}.
\end{aligned}$$

☐

Lemma 3.2.11. *For $\mathcal{I} \subset \mathcal{S}$, $\mathcal{D}^{\mathcal{I},\mathrm{tot}}$ arises from a Minkowski decomposition of $\mathcal{D}^{\mathcal{I}}$.*

Proof. Let $\mathcal{D}_Q^{\mathcal{I},i} = \mathcal{D}_x^{\mathcal{I},\mathrm{tot}}$ for $x = V(y_Q^{\alpha_Q^i} - t_{Q,i})$. Then clearly $\mathcal{D}_Q^{\mathcal{I}} = \sum \alpha_Q^i \mathcal{D}_Q^{\mathcal{I},i}$, and all $\mathcal{D}_Q^{\mathcal{I},i}$ have the correct tail cone. Thus, we just need to check the admissibility of the decomposition. But from definition 3.1.1(ii) coupled with 3.2.8(i) we have that for any $\mathcal{D} \in \mathcal{I}$ there exists $w \in \mathrm{tail}(\mathcal{D})^\vee$ such that for $0 \le i \le r_Q$ either $\mathcal{D}_Q^{\mathcal{I},i} = \emptyset$ or $\mathcal{D}_Q^{\mathcal{I},i} = \mathrm{face}(\mathcal{D}_Q^i, w)$. Thus, since the $\mathcal{D}_Q^i$ form an admissible decomposition, the $\mathcal{D}_Q^{\mathcal{I},i}$ must as well. ☐

Lemma 3.2.12. *For any $\mathcal{I} \subset \mathcal{S}$, $\mathcal{D}^{\mathcal{I},\mathrm{tot}}$ is proper.*

Proof. This follows directly from lemma 3.2.11 and the fact that the affine construction of T-deformations always gives proper polyhedral divisors. ☐

The next lemma is the essential point in the proof of proposition 3.2.2. It is rather technical in that we must consider a number of different cases, but each case just requires an application of some of the above lemmata.

Lemma 3.2.13. *For any $\mathcal{J} \subset \mathcal{I} \subset \mathcal{S}$, $\mathcal{D}^{\mathcal{I},\mathrm{tot}} \prec \mathcal{D}^{\mathcal{J},\mathrm{tot}}$.*

Proof. For any $x \in Y^{\mathrm{tot}}$ not necessarily closed, we define a point $\hat{x} \in Y$ as follows. If x is contained in some $V(y_Q^{\alpha_Q^i} - t_{Q,i})$ for $Q \in \mathcal{Q}$ and $0 \leq i \leq r_Q$, then let $\hat{x}$ be equal to the point Q in $\mathbb{P}^1$. Note that such a Q, if it exists, is unique due to the way B was constructed. In this case, we say x is *special.* Otherwise, if x is contained in any divisor $V(y_P)$ for $P \in Y$, let $\hat{x} = P$. Finally, for any other point x, take $\hat{x}$ to be the general point in $\mathbb{P}^1$. We now describe $\mathcal{D}_x^{\mathcal{I},\mathrm{tot}}$. If x is contained in some $V(y_Q^{\alpha_Q^i} - t_{Q,i})$, let I be the set of all i for which this holds. Then one easily sees that $\mathcal{D}_x^{\mathcal{I},\mathrm{tot}} = \sum_{i\in I} \bigcap_{\mathcal{D}\in\mathcal{I}} \mathcal{D}_Q^i$. Otherwise, $\mathcal{D}_x^{\mathcal{I},\mathrm{tot}} = \mathcal{D}_{\hat{x}}^{\mathcal{I}}$. One can describe $\mathcal{D}_x^{\mathcal{J},\mathrm{tot}}$ similarly.

Fix now any point $y \in \mathrm{Loc}(\mathcal{D}^{\mathcal{I},\mathrm{tot}})$, not necessarily closed. We must show that the requirements of definition 1.2.7 hold for this point y. Now, since $\mathcal{D}^{\mathcal{I}} \prec \mathcal{D}^{\mathcal{J}}$, there exists $(w_{\hat{y}}, D_{\hat{y}}) \in M \times |\mathcal{D}^{\mathcal{J}}(w_{\hat{y}})|$ fulfilling the face relation of definition 1.2.7 for $\mathcal{D}^{\mathcal{I}} \prec \mathcal{D}^{\mathcal{J}}$ and the point $\hat{y}$. In the remainder of the proof, we will consider several cases:

(a) $\mathrm{Loc}(\mathcal{D}^{\mathcal{I},\mathrm{tot}}) = Y^{\mathrm{tot}}$ and $\deg(D_{\hat{y}}) = 0$;

(b) $\mathrm{Loc}(\mathcal{D}^{\mathcal{I},\mathrm{tot}}) = Y^{\mathrm{tot}}$ and $\deg(D_{\hat{y}}) > 0$;

(c) $\mathrm{Loc}(\mathcal{D}^{\mathcal{I},\mathrm{tot}}) \neq Y^{\mathrm{tot}}$ and $\mathrm{Loc}(\mathcal{D}^{\mathcal{J},\mathrm{tot}}) = Y^{\mathrm{tot}}$;

(d) $\mathrm{Loc}(\mathcal{D}^{\mathcal{J},\mathrm{tot}}) \neq Y^{\mathrm{tot}}$.

Starting with case (a), set $w_y = w_{\hat{y}}$. We take D_y to be the trivial divisor on Y^{tot}. Note that we have $D_y \in |\mathcal{D}^{\mathcal{J},\mathrm{tot}}(w_y)|$. Clearly $y \notin \operatorname{supp} D_y$. Furthermore, we claim

$$\mathrm{face}(\mathcal{D}_y^{\mathcal{J},\mathrm{tot}}, w_y) = \mathcal{D}_y^{\mathcal{I},\mathrm{tot}}. \tag{3.2.1}$$

Indeed, if y isn't special, then this follows from $\mathcal{D}_y^{\mathcal{I},\mathrm{tot}} = \mathcal{D}_{\hat{y}}^{\mathcal{I}}$ and $\mathcal{D}_y^{\mathcal{J},\mathrm{tot}} = \mathcal{D}_{\hat{y}}^{\mathcal{J}}$. For y special, point (i) of definition 3.1.1 gives us

$$\mathcal{D}_Q^{\mathcal{I}} = \sum_{i=0}^{r_Q} \sum_{j=1}^{\alpha_Q^i} \bigcap_{\mathcal{D}\in\mathcal{I}} (\mathcal{D}_Q^i)$$

$$\mathcal{D}_Q^{\mathcal{J}} = \sum_{i=0}^{r_Q} \sum_{j=1}^{\alpha_Q^i} \bigcap_{\mathcal{D}\in\mathcal{J}} (\mathcal{D}_Q^i)$$

whereas we automatically have

$$\bigcap_{\mathcal{D}\in\mathcal{I}} \mathcal{D}_Q^i \subset \bigcap_{\mathcal{D}\in\mathcal{J}} \mathcal{D}_Q^i$$

for all $0 \leq i \leq r_Q$. Since $\mathrm{face}(\mathcal{D}_Q^{\mathcal{J}}, w_y) = \mathcal{D}_Q^{\mathcal{I}}$, we can thus apply 3.2.8(i) to show equation (3.2.1). Now finally, for all $v \in Y^{\mathrm{tot}}$, we claim that $\mathrm{face}(\mathcal{D}_v^{\mathcal{J},\mathrm{tot}}, w_y) = \mathrm{face}(\mathcal{D}_v^{\mathcal{I},\mathrm{tot}}, w_y)$. Indeed, for all v we have $\mathrm{face}(\mathcal{D}_v^{\mathcal{J}}, w_y) = \mathrm{face}(\mathcal{D}_v^{\mathcal{I}}, w_y)$. For v not special the claim is then immediate. On the other hand, for v special we use lemma 3.2.8(ii), where the hypothesis of the lemma is once again satisfied due to point (i) of definition 3.1.1. Thus, the pair (w_y, D_y) satisfies the requirements of definition 1.2.7.

We now move to case (b). Since $\deg(D_{\hat{y}}) > 0$, clearly we can find some $k \in \mathbb{N}$ such that $|\mathcal{D}^{\mathcal{J},\mathrm{tot}}(k \cdot w_{\hat{y}})|$ contains a divisor D such that $Y^{\mathrm{tot}} \setminus \operatorname{supp} D$ contains only those special points x with $\hat{x} = \hat{y}$, and none of the points lying in $V(y_P)$ for $P \in \mathcal{P}$ with $P \neq \hat{y}$. We then set $w_y = k \cdot w_{\hat{y}}$ and take $D_y = D$. Now, we have $y \notin \operatorname{supp} D_y$, and $\operatorname{face}(\mathcal{D}_y^{\mathcal{J},\mathrm{tot}}, w_y) = \mathcal{D}_y^{\mathcal{I},\mathrm{tot}}$ exactly as in case (a). We claim that we also have $\operatorname{face}(\mathcal{D}_v^{\mathcal{J},\mathrm{tot}}, w_y) = \operatorname{face}(\mathcal{D}_v^{\mathcal{I},\mathrm{tot}}, w_y)$ for all $v \in Y^{\mathrm{tot}} \setminus \operatorname{supp} D_y$. If $\hat{v} \neq \hat{y}$, then $\mathcal{D}_v^{\mathcal{I},\mathrm{tot}}$ and $\mathcal{D}_v^{\mathcal{J},\mathrm{tot}}$ are both trivial and the claim follows from the properties of $w_{\hat{y}}$. Likewise, if $\hat{v} = \hat{y}$ and y is trivial, then $\mathcal{D}_v^{\mathcal{I},\mathrm{tot}} = \mathcal{D}_y^{\mathcal{I}}$, $\mathcal{D}_v^{\mathcal{J},\mathrm{tot}} = \mathcal{D}_y^{\mathcal{J}}$ and the claim again follows from the properties of $w_{\hat{y}}$. Finally, if $\hat{v} = \hat{y}$ and y is not trivial, we can apply lemma 3.2.8(ii) as in part (a). Thus, we again have that the pair (w_y, D_y) satisfies the requirements of definition 1.2.7.

We now consider case (c). Suppose first that $\hat{y} \in \operatorname{Loc}(\mathcal{D}^{\mathcal{I}})$. Then one easily sees that $\deg(D_{\hat{y}}) > 0$ and one can proceed as in case (b). Thus, we have reduced to the case that $\hat{y} \notin \operatorname{Loc}(\mathcal{D}^{\mathcal{I}})$, from which it follows that y must be special. Let Q and I be the corresponding point and index set. Now, we can find $w \in \operatorname{tail}(\mathcal{D}^{\mathcal{J}})$ with $\operatorname{face}(\mathcal{D}_y^{\mathcal{J},\mathrm{tot}}, w) = \mathcal{D}_y^{\mathcal{I},\mathrm{tot}}$ by point (ii) of definition 3.1.1. Furthermore, by lemma 3.2.9 we have $\deg(\mathcal{D}^{\mathcal{J}}(w)) > 0$. Similar to in case (b), we can find some $k \in \mathbb{N}$ such that $|\mathcal{D}^{\mathcal{J},\mathrm{tot}}(k \cdot w)|$ contains a divisor D such that $Y^{\mathrm{tot}} \setminus \operatorname{supp} D$ contains only those special points x with $\hat{x} = \hat{y}$ and $\mathcal{D}_x^{\mathcal{I},\mathrm{tot}} \neq \emptyset$, and none of the points lying in $V(y_P)$ for $P \in \mathcal{P}$ with $P \neq \hat{y}$. We then set $w_y = k \cdot w_{\hat{y}}$ and take $D_y = D$. The claim of $\operatorname{face}(\mathcal{D}_y^{\mathcal{J},\mathrm{tot}}, w_y) = \mathcal{D}_y^{\mathcal{I},\mathrm{tot}}$ is satisfied automatically by our choice of w_y. The claim that $\operatorname{face}(\mathcal{D}_v^{\mathcal{J},\mathrm{tot}}, w_y) = \operatorname{face}(\mathcal{D}_v^{\mathcal{I},\mathrm{tot}}, w_y)$ for all $v \in Y^{\mathrm{tot}} \setminus \operatorname{supp} D_y$ is immediate for v nonspecial, and follows from lemma 3.2.8(ii) for v special. Thus, we again have that the pair (w_y, D_y) satisfies the requirements of definition 1.2.7.

The final case (d) is essentially identical to the case of (c), with the simplification that since $\operatorname{Loc}(\mathcal{D}^{\mathcal{J}})$ is affine, we needn't worry about the degrees of evaluations of $\mathcal{D}^{\mathcal{J}}$. $\square$

We are now ready to prove proposition 3.2.2:

Proof of proposition 3.2.2. First, all elements of $\mathcal{S}^{\mathrm{tot}}$ are proper polyhedral divisors due to lemma 3.2.12. Secondly, we claim that intersections of elements of $\mathcal{S}^{\mathrm{tot}}$ are themselves elements of $\mathcal{S}^{\mathrm{tot}}$. Indeed, for $\mathcal{I}, \mathcal{J} \subset \mathcal{S}$, we have $\mathcal{D}^{\mathcal{I},\mathrm{tot}} \cap \mathcal{D}^{\mathcal{J},\mathrm{tot}} = \mathcal{D}^{\mathcal{I}\cup\mathcal{J},\mathrm{tot}}$. Finally, from lemma 3.2.13 we have the necessary face relations:

$$\mathcal{D}^{\mathcal{I},\mathrm{tot}} \succ \mathcal{D}^{\mathcal{I}\cup\mathcal{J},\mathrm{tot}} \prec \mathcal{D}^{\mathcal{J},\mathrm{tot}}$$

$\square$

We conclude the section with the proof of theorem 3.2.3:

Proof of theorem 3.2.3. Since the maps $\pi_{\mathcal{D}}$ arise from a projection of the quotient map, they agree along intersections of polyhedral divisors and we can clearly glue them together to a map π. Flatness of π can be checked locally on each $X(\mathcal{D}^{\mathcal{I},\mathrm{tot}})$ for $\mathcal{I} \subset \mathcal{S}$; this follows then directly from lemma 3.2.11 and theorem 2.2.3. From this theorem, we also know $\pi^{-1}_{|X(\mathcal{D}^{\mathcal{I},\mathrm{tot}})}(s) = X(\mathcal{D}^{\mathcal{I},(s)})$, so we just need to check that everything glues properly. But for $\mathcal{I}, \mathcal{J} \subset \mathcal{S}$,

$$\pi^{-1}_{|X(\mathcal{D}^{\mathcal{I}\cup\mathcal{J},\mathrm{tot}})}(s) = X(\mathcal{D}^{\mathcal{I}\cup\mathcal{J},(s)}) = X(\mathcal{D}^{\mathcal{I},(s)}) \cap X(\mathcal{D}^{\mathcal{J},(s)}) = \pi^{-1}_{|X(\mathcal{D}^{\mathcal{I},\mathrm{tot}})}(s) \cap \pi^{-1}_{|X(\mathcal{D}^{\mathcal{J},\mathrm{tot}})}(s).$$

Thus, the gluing on $X(\mathcal{S}^{\mathrm{tot}})$ induces the gluing on $X(\mathcal{S}^{(s)})$. $\square$

Remark 3.2.14. The total space X^{tot} of a T-deformation need not be separated. For example, take $N = \mathbb{Z}$ and consider the divisorial fan consisting of polyhedral divisors

$$\begin{aligned} &[0,1] \otimes \{0\} + \emptyset \otimes \{\infty\} \\ &[2,3] \otimes \{0\} + \emptyset \otimes \{\infty\} \\ &\emptyset \otimes \{0\} + \emptyset \otimes \{\infty\} \end{aligned}$$

on $\mathbb{P}^1$. Then we can decompose $\mathcal{S}_0$ as $[0,1] = [0,1] + \{0\}$ and $[2,3] = [3,4] + \{-1\}$. This gives a T-deformation with X^{tot} not separated; this can be easily checked via theorem 7.5 of [AHS08] by considering the valuation ν on $\mathbb{C}(Y^{\text{tot}})$ with $\nu(y_0) = 1$ and $\nu(y_0 - t) = 2$. However, we shall see in the following section that this kind of pathology is avoided for T-deformations of complete T-varieties, or more generally, T-varieties whose divisorial fans only have slices with convex support.

3.3 Separated and Proper T-Deformations

We now provide criteria for T-deformations to be separated and proper:

Theorem 3.3.1. *Let $\mathcal{S}$ be a divisorial fan together with some Minkowski decomposition, and $\pi : X(\mathcal{S}^{\text{tot}}) \to B$ the corresponding T-deformation. If $|\mathcal{S}_Q|$ is convex for all $Q \in \mathcal{Q}$, then $X(\mathcal{S}^{\text{tot}})$ is separated. Likewise, π is proper if and only if $\mathcal{S}$ is complete.*

Proof. Let $\nu : \mathbb{C}(Y^{\text{tot}}) \to \mathbb{Q}$ be a valuation with center $y \in Y^{\text{tot}}$. Then ν defines a set of polyhedra $\mathcal{S}_\nu^{\text{tot}}$ called a weighted slice, see [AHS08], section 7. Now assume that $|\mathcal{S}_Q|$ is convex for all $Q \in Q$. We first claim that any such weighted slice $\mathcal{S}_\nu^{\text{tot}}$ is a polyhedral subdivision. Indeed, using the notation of the proof of lemma 3.2.13, if y isn't special, then $\mathcal{S}_\nu^{\text{tot}}$ is simply a dilation of a slice $\mathcal{S}_P$ for some $P \in Y$. On the other hand, if $y \in V(y_Q^{\alpha_Q^i} - t_{Q,i})$, then elements of $\mathcal{S}_\nu^{\text{tot}}$ are of the form

$$\sum_{i=0}^{r_Q} \lambda_i \bigcap_{\Delta \in \mathcal{I}} \Delta^i$$

for some $\lambda_i \in \mathbb{Q}_{\geq 0}$ and $\mathcal{I} \subset \mathcal{S}_Q$. Now let $v^0, \ldots, v^{r_Q} \in \text{cone}_{\text{MS}}(\mathcal{S}_Q)$ correspond to the Minkowski decomposition of $C := \mathcal{S}_Q$; take $v := \sum \lambda_i v^i$. Then one easily checks that $C_v = \mathcal{S}_\nu^{\text{tot}}$, up to translation, so the claim follows by proposition 3.1.7. Furthermore, we have that

$$\nu(\mathcal{D}^{\mathcal{I},\text{tot}}) \cap \nu(\mathcal{D}^{\mathcal{J},\text{tot}}) = \nu(\mathcal{D}^{\mathcal{I}\cup\mathcal{J},\text{tot}}) \tag{3.3.1}$$

for all $\mathcal{I}, \mathcal{J} \subset \mathcal{S}$. Indeed, this follows from an adapted version of equation (3.1.1). An application of the evaluation criterion of [AHS08], section 7 then shows that $X(\mathcal{S}^{\text{tot}})$ is separated.

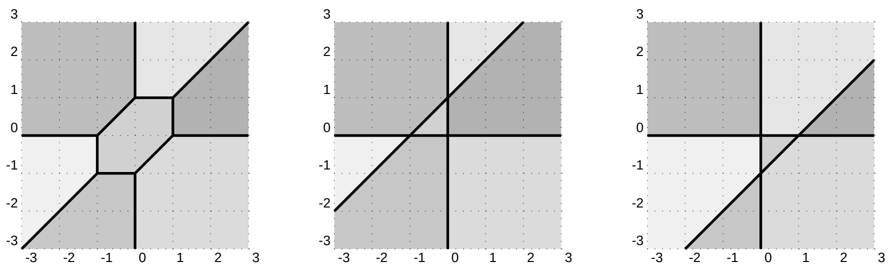
The claim regarding properness uses a relative version of the evaluation criterion of [AHS08], section 7 for completeness; this is described in [Süß09], theorem 7.1. Using the notation from [Süß09], the map π is in fact a torus equivariant morphism corresponding to the triple $(\text{pr}, F, 0)$, where $\text{pr} : Y \times B \to B$ is the projection and $F : N \to 0$ is the zero map. Since pr is proper, π is proper if and only if equation (3.3.1) holds and each weighted slice of $\mathcal{S}^{\text{tot}}$ is complete by theorem 7.1 of [Süß09]. If $\mathcal{S}$ is complete, this follows from the above discussion together with the last claim of proposition 3.1.7. Conversely, if all weighted slices of $\mathcal{S}^{\text{tot}}$ are complete, clearly $\mathcal{S}$ must be complete as well. ☐

An application of the above theorem is that T-deformations of complete T-varieties don't depend on the divisorial fan being used, just the marked fansy divisor. Let us make this precise. Consider any marked fansy divisor Ξ on $\mathbb{P}^1$ and fix some finite set of points $\mathcal{Q} \subset \mathbb{P}^1$. A *Minkowski decomposition* of Ξ consists of α_Q-admissible decompositions for the polyhedral subdivisions Ξ_Q for $Q \in \mathcal{Q}$. Similar to the case of a decomposition of a divisorial fan, we can associate a deformation of $X(\Xi)$ to the decomposition of Ξ. Indeed, let $\mathcal{S}$ be any divisorial fan with $\Xi(\mathcal{S}) = \Xi$. Then the decomposition of Ξ automatically defines a decomposition of $\mathcal{S}$, since the slices are the same, thus giving us a deformation of $X(\mathcal{S}) = X(\Xi)$. The following proposition shows us that this construction doesn't depend on the choice of $\mathcal{S}$:

Proposition 3.3.2. *Let $\mathcal{S}, \mathcal{S}'$ be divisorial fans on $\mathbb{P}^1$ with $\Xi(\mathcal{S}) = \Xi(\mathcal{S}')$ and consider Minkowski decompositions of $\mathcal{S}$ and $\mathcal{S}'$ coming from identical decompositions of slices $\mathcal{S}_Q = \mathcal{S}'_Q$. Then the corresponding deformations π and π' are equal.*

Proof. Without loss of generality we can assume that $\mathcal{S}$ is the divisorial fan we described after proposition 1.4.3. We will then show that there is an open embedding $\iota : X(\mathcal{S}^{\mathrm{tot}}) \hookrightarrow X(\mathcal{S}'^{\mathrm{tot}})$ which commutes with π and π'. Both π and π' are proper by theorem 3.3.1, so ι must be proper as well, and is thus an isomorphism, from which the theorem follows.

To construct ι, we simply show that every polyhedral divisor in $\mathcal{S}^{\mathrm{tot}}$ is a face of some polyhedral divisor in $\mathcal{S}'^{\mathrm{tot}}$, which locally guarantees an open embedding. Since the gluings coming from $\mathcal{S}'^{\mathrm{tot}}$ induce gluings for $\mathcal{S}^{\mathrm{tot}}$, this globally gives us an open embedding. Now, it is in fact sufficient to only consider the polyhedral divisors $\mathcal{D}(P, \Delta)^{\mathrm{tot}}$ and $\mathcal{D}(\sigma)^{\mathrm{tot}}$, since other polyhedral divisors in $\mathcal{S}^{\mathrm{tot}}$ are faces of these. One easily checks that the polyhedral divisor $\mathcal{D}(\sigma)^{\mathrm{tot}}$ must be in $\mathcal{S}'^{\mathrm{tot}}$ as well, since $\Xi(\mathcal{S}') = \Xi$ implies that $\mathcal{D}(\sigma)$ is in $\mathcal{S}'$ and the slices of $\mathcal{S}'$ and $\mathcal{S}$ have identical Minkowski decompositions. On the other hand, consider $P \in \mathcal{P}$ and $\Delta \in \Xi_P$ with $\mathrm{tail}(\Delta)$ not marked. Then there is a polyhedral divisor $\mathcal{D} \in \mathcal{S}'$ with affine locus and $\mathcal{D}_P = \Delta$. One easily confirms that $\mathcal{D}(P, \Delta)^{\mathrm{tot}}$ is a face of $\mathcal{D}^{\mathrm{tot}}$. $\square$



(a) Ξ_0 (b) Ξ_0^0 (c) Ξ_0^1

Figure 3.4: A T-deformation of $\overline{C(dP_6)}$

Example 3.3.3 (A compactified cone over the del Pezzo surface of degree six)**.** Let Ξ be the marked fansy divisor on $\mathbb{P}^1$ with sole nontrivial slice Ξ_0 as pictured in figure 3.4(a), and marks on all nonzero tail cones. Then $X_0 = X(\Xi)$ is a compactification of the (anticanonical) cone over the del Pezzo surface of degree six. Now, we can decompose the polyhedral subdivision Ξ_0 as pictured in figure 3.4 parts (b) and (c). Here, full-dimensional elements of Ξ_0 are decomposed into the sum of an element of Ξ_0^0 and Ξ_0^1 which have the same shade of gray (or the same tailcone). The decompositions of lower-dimensional elements of Ξ_0 are induced via intersection. This decomposition of Ξ_0 gives us a decomposition of the marked fansy divisor Ξ, and thus a T-deformation π of X_0. For $s \neq 0$, the fiber $\pi^{(-1)}(s)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Indeed, the fiber is described by a divisorial fan with exactly two nontrivial slices Ξ_0^0 and Ξ_0^1. The desired conclusion can be reached by reversing the downgrading procedure of remark 1.2.14.

We can construct a different deformation of X_0 as follows. Indeed, the polyhedral subdivisions $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{S}_\infty$ of figure 1.2 encode a $(1,1)$-admissible decomposition of Ξ_0

in a similar manner to above, where we note that the hexagon of Ξ_0 is decomposed into the sum of the three compact line segments present in figure 1.2. The generic fibers of this deformation are then all isomorphic to $\mathbb{P}(\Omega_{\mathbb{P}^2})$.

These two deformations were presented in [JR06] as examples of a Fano variety with canonical singularities admitting two different smoothings. A combinatorial description similar to the one presented here can be found in [Süß08].

3.4 Locally Trivial Deformations

In the following two sections, we will exclusively be considering one-parameter T-deformations. This is not really a limitation, since we can always restrict a k-parameter T-deformation naturally to k different one-parameter deformations. Our goal is now to combinatorially characterize T-deformations which are locally trivial.

Definition 3.4.1. Let Δ be a polyhedron in $N_{\mathbb{Q}}$, and $\alpha \in \mathbb{N}$. An α-admissible Minkowski decomposition $\Delta = \Delta^0 + \alpha \cdot \Delta^1$ is *essentially trivial* if either

(i) $\Delta = \emptyset$;

(ii) $\Delta^1 = \mathrm{tail}(\Delta) + v$ for some $v \in N$; or

(iii) $\alpha = 1$ and $\Delta^0 = \mathrm{tail}(\Delta) + v$ for some $v \in N$.

Fix a polyhedral divisor $\mathcal{D}$ on $Y = \mathbb{P}^1$, and consider a Minkowski decomposition of $\mathcal{D}$ leading to a one-parameter T-deformation of $X(\mathcal{D})$. We can assume that the decomposition of $\mathcal{D}$ comes from an α-admissible Minkowski decomposition $\mathcal{D}_0 = \mathcal{D}_0^0 + \mathcal{D}_0^1$. We say that the decomposition of $\mathcal{D}$ is *essentially trivial* if the decomposition of $\mathcal{D}_0$ is as well. If $\mathcal{S}$ is a divisorial fan on $\mathbb{P}^1$ with Minkowski decomposition leading to a one-parameter T-deformation of $X(\mathcal{S})$, we say the decomposition of $\mathcal{S}$ is *locally essentially trivial* if the decomposition of all elements of $\mathcal{S}_0$ are essentially trivial.

The first result of this section is the following, which gives one reason why essentially trivial decompositions are important:

Proposition 3.4.2. *Let $\mathcal{D}$ be a polyhedral divisor on $\mathbb{P}^1$ with Minkowski decomposition leading to an infinitesimally locally trivial one-parameter T-deformation. If $\mathrm{Loc}(\mathcal{D}) \neq \mathbb{P}^1$ or $X(\mathcal{D})$ is smooth, then the decomposition of $\mathcal{D}$ is essentially trivial.*

Proof. Suppose first that $X(\mathcal{D})$ is smooth.[1] Then by propositions 1.5.2 and 1.5.3, $X(\mathcal{D})$ is a localization of some toric variety. It follows by the downgrading procedure that $\mathcal{D}_0$ must be a hyperplane section in height one of some smooth cone. If $\mathcal{D}_0 = \emptyset$, then the Minkowski decomposition is clearly essentially trivial. Suppose thus that $\mathcal{D}_0 \neq \emptyset$. Without loss of generality, we can assume that $\mathcal{D}_0$ is compact, since in the current situation an α-admissible decomposition of $\mathcal{D}_0$ induces an α-admissible decomposition of the convex hull of the vertices of $\mathcal{D}_0$. Since $\mathcal{D}_0$ comes from a smooth cone, the only lattice points it can contain must be vertices. Thus, the decomposition of $\mathcal{D}_0$ must be essentially trivial.

Suppose instead that $X(\mathcal{D})$ has affine locus and isn't smooth. The singularity in the fiber $p_{\mathcal{D}}^{(-1)}(0)$ is isomorphic to the toric singularity $X_0 = \mathrm{TV}(\mathrm{cone}(\mathcal{D}_0 \times 1))$, and the deformation of $X(\mathcal{D})$ induces a deformation of this singularity, which must be infinitesimally

[1] In this case, every deformation is automatically infinitesimally locally trivial, see [Ser06] theorem 1.2.4.

locally trivial. From proposition 5.6 of [Alt95] it follows that the decomposition of $\mathcal{D}_0$ must be essentially trivial. $\square$

Remark 3.4.3. We believe that the conditions on $\mathcal{D}$ in the above proposition are unnecessary. However, at the moment there is not enough understanding of affine T-deformations to eliminate this.

The next theorem further demonstrates the importance of essentially trivial decompositions:

Theorem 3.4.4. *Let $\mathcal{D}$ be a polyhedral divisor on $\mathbb{P}^1$ with essentially trivial Minkowski decomposition. Suppose that either $\mathcal{D}$ has affine locus, or with at most three exceptions $\mathcal{D}_P$ is of the form $\mathcal{D}_P = \operatorname{tail}(\mathcal{D}) + v_P$ for some $v_P \in N$. Then the resulting one-parameter deformation is infinitesimally locally trivial.*

As we shall see, the reason that in the above theorem we only allow three exceptions in the complete case has to do with the fact that an automorphism of $\mathbb{P}^1$ is determined by its action on three points. The case of noncomplete locus is simpler, because we can localize to a cover where each corresponding polyhedral divisor has only one nontrivial coefficient which isn't the empty set. The above theorem leads to the following corollary:

Corollary 3.4.5. *Let $\mathcal{S}$ be a divisorial fan such that all polyhedral divisors $\mathcal{D} \in \mathcal{S}$ with $X(\mathcal{D})$ singular have affine locus. Then one-parameter infinitesimally locally trivial T-deformations of $X(\mathcal{S})$ correspond to locally essentially trivial Minkowski decompositions of $\mathcal{S}$.*

Proof. By proposition 3.4.2, infinitesimally locally trivial T-deformations must come from essentially trivial Minkowski decompositions. But by proposition 1.5.2, we have that all polyhedral divisors in $\mathcal{S}$ fulfill the conditions of theorem 3.4.4. Thus, the claim follows. $\square$

In the remainder of this section, we prove theorem 3.4.4. In doing so, we will construct a number of isomorphisms of deformations which will be useful in calculating the Kodaira-Spencer map in the following section. The following two lemmata are essentially special cases of proposition 8.6 in [AH06]:

Lemma 3.4.6. *Let Y be a normal semiprojective variety and $\mathcal{D}$ a proper polyhedral divisor on Y. For some $v \in N$ and $f \in \mathbb{C}(Y)$ let $\widetilde{\mathcal{D}} = \mathcal{D} + v \otimes \operatorname{div}(f)$. Then there is a canonical isomorphism $\phi_v \colon X(\mathcal{D}) \to X(\widetilde{\mathcal{D}})$ where $\phi_v^\#$ is defined by mapping χ^u to $f^{\langle v,u\rangle}\chi^u$ for $u \in M$.*

Proof. We have

$$\begin{aligned}\mathcal{O}_{X(\widetilde{D})} = \bigoplus_{u\in\delta^\vee\cap M} H^0\Big(Y, \widetilde{\mathcal{D}}(u)\Big)\cdot\chi^u &\cong \bigoplus_{u\in\delta^\vee\cap M} H^0\Big(Y, \mathcal{D}(u) + \langle v,u\rangle \operatorname{div}(f)\Big)\cdot\chi^u \\ &\cong \bigoplus_{u\in\delta^\vee\cap M} H^0\Big(Y, \mathcal{D}(u)\Big)\cdot f^{-\langle v,u\rangle}\chi^u\end{aligned}$$

and thus $\phi_v^\#$ induces an isomorphism $\mathcal{O}_{X(\widetilde{D})} \cong \mathcal{O}_{X(D)}$, since

$$\mathcal{O}_{X(D)} = \bigoplus_{u\in\delta^\vee\cap M} H^0\Big(Y, \mathcal{D}(u)\Big)\cdot\chi^u.$$

$\square$

Lemma 3.4.7. *Let Y be a normal semiprojective variety and $\bar{\gamma} \in \operatorname{Aut}(Y)$. For a proper polyhedral divisor $\mathcal{D}$ on Y, define $\bar{\gamma}(\mathcal{D}) = \sum_P \mathcal{D}_P \otimes \bar{\gamma}_*(P)$. Then there is a natural isomorphism $\gamma\colon X(\mathcal{D}) \to X(\bar{\gamma}(\mathcal{D}))$ induced by $\bar{\gamma}$.*

Proof. Similar to the proof of the above lemma, we have

$$\begin{aligned}\mathcal{O}_{X(\bar{\gamma}(\mathcal{D}))} &= \bigoplus_{u\in\delta^\vee\cap M} H^0\Big(Y, \bar{\gamma}(\mathcal{D})(u)\Big)\cdot\chi^u \\ &= \bigoplus_{u\in\delta^\vee\cap M} (\bar{\gamma}^\#)^{-1}\Big(H^0\big(Y,\mathcal{D}(u)\big)\Big)\cdot\chi^u.\end{aligned}$$

□

In the following, $\mathcal{D}$ will always be a proper polyhedral divisor on $Y = \mathbb{P}^1$, and we will consider a Minkowski decomposition of $\mathcal{D}_0$ giving rise to a one-parameter deformation π of $X(\mathcal{D})$. As usual, the total space of π is $X(\mathcal{D}^{\mathrm{tot}})$, with $\mathcal{D}^{\mathrm{tot}}$ a polyhedral divisor on Y^{tot}. By instead considering the Minkowski decomposition $\mathcal{D}_0 = \mathcal{D}_0 + \alpha\cdot\operatorname{tail}\mathcal{D}_0$, we get a possibly different polyhedral divisor $\mathcal{D}^{\mathrm{prod}}$ on Y^{tot} together with corresponding deformation $\pi_{\mathrm{prod}} : X(\mathcal{D}^{\mathrm{prod}}) \to B$, which is clearly just the product family.

We also need to fix some homogeneous coordinates on $\mathbb{P}^1$: let z_0, z_1 be homogeneous coordinates such that $y_0 = z_0/z_1$. Now, consider any points $P_1 = (1 : c_1)$ and $P_2 = (1 : c_2)$ with $P_i \in \mathcal{P}$. We define an automorphism $\bar{\gamma}_{P_1,P_2}$ of Y^{tot} by considering the following map $\bar{\gamma}^\#_{P_1,P_2}$ on the homogeneous coordinate ring of Y^{tot}:

$$\begin{aligned} z_0 &\mapsto \frac{(1-(c_1+c_2)t)z_0 + tz_1}{(1-c_1t)(1-c_2t)} \\ z_1 &\mapsto \frac{-c_1c_2tz_0 + z_1}{(1-c_1t)(1-c_2t)} \\ t &\mapsto t \end{aligned}$$

One easily checks that $\bar{\gamma}^\#_{P_1,P_2}$ is invertible with inverse

$$\begin{aligned} z_0 &\mapsto z_0 - tz_1 \\ z_1 &\mapsto c_1c_2tz_0 + (1-(c_1+c_2)t)z_1 \\ t &\mapsto t \end{aligned}$$

Lemma 3.4.8. *Assume that $\mathcal{D}_P = \operatorname{tail}(\mathcal{D})$ for $P \notin \{0, P_1, P_2\}$, and $\mathcal{D}_0^0 = \operatorname{tail}(\mathcal{D})$. Then $\bar{\gamma}_{P_1,P_2}(\mathcal{D}^{\mathrm{prod}}) = \mathcal{D}^{\mathrm{tot}}$, and the map γ_{P_1,P_2} induced by $\bar{\gamma}_{P_1,P_2}$ gives an isomorphism $\gamma_{P_1,P_2} : X(\mathcal{D}^{\mathrm{prod}}) \xrightarrow{\sim} X(\mathcal{D}^{\mathrm{tot}})$.*

Proof. Set $\bar{\gamma} = \bar{\gamma}_{P_1,P_2}$. A simple calculation shows

$$\begin{aligned} \bar{\gamma}_*(V(y_0)) &= \bar{\gamma}_*(V(z_0)) = V(z_0 - tz_1) = V(y_0 - t) \\ \bar{\gamma}_*(P_i\times B) &= \bar{\gamma}_*(V(c_iz_0 - z_1)) = V((1-c_jt)(c_iz_0 - z_1)) = P_i \times B \end{aligned}$$

where $i, j \in \{1,2\}$ with $i \neq j$. The claims then follow by definition of $\bar{\gamma}_{P_1,P_2}(\mathcal{D})$ and lemma 3.4.7. □

The next lemma is essential for the case where $\mathcal{D}$ has affine locus. It tells us that for any divisor D in Y^{tot} with $\mathcal{D}^{\mathrm{tot}}_D = \emptyset$, the only information pertinent for π^ϵ is the point of intersection of D with the fiber over $0 \in B$.

Lemma 3.4.9. *Consider two proper polyhedral divisors $\mathcal{E}^1, \mathcal{E}^2$ on Y^{tot} such that for all prime divisors $D \subset Y^{\text{tot}}$, either $\mathcal{E}^1_D = \mathcal{E}^2_D$ or $D = V(y_P^l - ct)$ with $P \in \mathbb{P}^1$, $l \geq 1$, $c \in \mathbb{C}$ such that there are l_i, c_i with $\mathcal{E}^i_{D_i} = \emptyset$ for $D_i = V(y_P^{l_i} - c_i t)$. Then $X(\mathcal{E}^1) \times_B \operatorname{Spec} \mathbb{C}[\epsilon] = X(\mathcal{E}^2) \times_B \operatorname{Spec} \mathbb{C}[\epsilon]$.*

Proof. Suppose that the $\mathcal{E}^i$ have locus Y^{tot}. Then the statement is trivial. If not, then $\operatorname{Loc}(\mathcal{E}^i)$ is affine similar to in lemma 2.2.4. Without loss of generality we can assume $\mathcal{E}^1_D \prec \mathcal{E}^2_D$ for all prime divisors D. Thus, $\operatorname{Loc}(\mathcal{E}^1)$ is a localization of $\operatorname{Loc}(\mathcal{E}^2)$ along certain divisors of the form $D = V(y_P^{l_1} - c_1 t)$. We have by assumption l_2 and c_2 with $\frac{y_P^k}{y_P^{l_2} - c_2 t}$ regular on $\operatorname{Loc}(\mathcal{E}^2)$ for $0 \leq k \leq l^1$. Now,

$$f := \left(\frac{y_P^{l_2}}{y_P^{l_2} - c_2 t} \right)^{-1} = \frac{y_P^{l_2} - c_2 t}{y_P^{l_2}} = 1 - t \frac{c_2}{y^{l_2}}$$

and

$$1 - t\frac{c_2}{y^{l_2}} \equiv 1 - t \frac{c_2}{y^{l_2} - c_2 t} \quad (t^2)$$

so f is regular on $\operatorname{Loc}(\mathcal{E}^2) \times_B \operatorname{Spec} \mathbb{C}[\epsilon]$. Thus, without loss of generality, we can actually assume that $\operatorname{Loc}(\mathcal{E}^2)$ has been localized along $V(y_P)$. Now,

$$g := \left(\frac{y_P^{l_1} - c_1 t}{y_P^{l_1}} \right)^{-1} \equiv 1 + t \frac{c_1}{y^{l_1}} \quad (t^2)$$

so g is regular on $\operatorname{Loc}(\mathcal{E}^2) \times_B \operatorname{Spec} \mathbb{C}[\epsilon]$. Thus, the regular functions on $\operatorname{Loc}(\mathcal{E}^1) \times_B \operatorname{Spec} \mathbb{C}[\epsilon]$ and $\operatorname{Loc}(\mathcal{E}^2) \times_B \operatorname{Spec} \mathbb{C}[\epsilon]$ are equal, and the claim follows. $\square$

Proof of theorem 3.4.4. First, suppose that $\mathcal{D}_0 = \emptyset$. Then it follows directly from lemma 3.4.9 that $\pi^\epsilon = \pi^\epsilon_{\text{prod}}$. Suppose instead that $\mathcal{D}_0 \neq \emptyset$, but $\operatorname{Loc}(\mathcal{D})$ is still affine. Then we can cover $X(\mathcal{D}^{\text{tot}})$ by two open sets $X(\mathcal{E}^1)$ and $X(\mathcal{E}^2)$, where we take $\mathcal{E}^1_D = \emptyset$ if $D = V(y_P)$ for $P \in \mathcal{P} \setminus \{0\}$, $\mathcal{E}^2_D = \emptyset$ for $D = V(y_0)$ or $D = V(y_0 - t)$, and $\mathcal{E}^i_D = \mathcal{D}_D$ otherwise. By lemma 3.4.9, $X(\mathcal{E}^2)$ is isomorphic to a product family modulo t^2. On the other hand, by first applying an isomorphism ϕ_v from lemma 3.4.6 and then possibly the isomorphism $\gamma^{-1}_{\infty,\infty}$, we get again by lemma 3.4.9 something isomorphic to a product family modulo t^2. Thus, in this case as well π is infinitesimally locally trivial.

Finally, we suppose that $\mathcal{D}$ has complete locus, in which case the preconditions of the theorem guarantee at most three points P_0, P_1, P_2 such that $\mathcal{D}_{P_i}$ isn't a lattice translated tailcone. We first apply isomorphisms ϕ_v so that we can assume without loss of generality that $\mathcal{D}_P = \operatorname{tail}(\mathcal{D})$ for $P \neq P_i$ and $\mathcal{D}^j_0 = \operatorname{tail}(\mathcal{D})$ for either $j = 0$ or $j = 1$. Now, if $\mathcal{D}^1_0 = \operatorname{tail}(\mathcal{D})$, then $X(\mathcal{D})$ is the trivial family. Otherwise, it follows that $P_i = 0$ for some i, without loss of generality $i = 0$. Then γ_{P_1,P_2} gives an isomorphism with the trivial family by lemma 3.4.8. $\square$

3.5 The Kodaira-Spencer Map

Let $\mathcal{S}$ be a divisorial fan on $Y = \mathbb{P}^1$ such that for any $\mathcal{D} \in \mathcal{S}$ with complete locus, for all but up to three slices of $\mathcal{D}$ we can write $\mathcal{D}_P = \operatorname{tail}(\mathcal{D}) + v_P$ with $v_P \in N$. Consider any locally essentially trivial Minkowski decomposition of $\mathcal{S}$ with resulting one-parameter deformation π. From theorem 3.4.4, we know that π is in fact infinitesimally locally trivial. Thus, we can compute $\kappa(\pi) \in H^1(X(\mathcal{S}), \mathcal{T}_{X(\mathcal{S})})$, the image of the Kodaira-Spencer map.

To describe $\kappa(\pi)$, we need to introduce some notation and conventions. First of all, we need a cover of $X(\mathcal{S})$ on which π^ϵ is trivial. To achieve this, we can actually assume that for all $\mathcal{D} \in \mathcal{S}$ with affine locus, either $\mathcal{D}_0 = \emptyset$ or $\mathcal{D}_P = \mathrm{tail}(\mathcal{D})$ for all $P \in \mathrm{Loc}(\mathcal{D}) \setminus 0$. Indeed, for any $\mathcal{D}$ with affine locus, one easily checks that we can cover $X(\mathcal{D}^{\mathrm{tot}})$ by $X(\mathcal{D}^{\mathrm{tot}} + \emptyset \otimes (V(y_0) + V(y_0^\alpha - t)))$ and $X(\mathcal{D}^{\mathrm{tot}} + \emptyset \otimes \sum_{P \in \mathcal{P}\setminus 0} V(y_P))$. Furthermore, it follows from the previous section that this cover has the desired property.

We associate the following data to the locally essentially trivial decomposition of $\mathcal{S}$. For each $\mathcal{D} \in \mathcal{S}$, we set $a_{\mathcal{D}} = 0$ if either $\mathcal{D}_0 = \emptyset$ or $\mathcal{D}_0^0 = \mathcal{D}_0 + \widetilde{v}_{\mathcal{D}}^0$ for some $\widetilde{v}_{\mathcal{D}}^0 \in N$. Otherwise, we set $a_{\mathcal{D}} = 1$ and define $\widetilde{v}_{\mathcal{D}}^0$ via $\mathcal{D}_0^1 = \mathcal{D}_0 + \widetilde{v}_{\mathcal{D}}^0$. If $a_{\mathcal{D}} = 0$, we set $c_{\mathcal{D}}^1 = c_{\mathcal{D}}^2 = v_{\mathcal{D}}^P = 0$ for all $P \in \mathcal{P}$.

For any point $P \in \mathbb{P}^1$, $P \neq 0$, we define $c_P \in \mathbb{C}$ via $P = V({y_0}^{-1} - c_P)$. Assume now instead that $a_{\mathcal{D}} = 1$. For all $P \in \mathcal{P} \setminus 0$ with at most two exceptions $P_{\mathcal{D}}^1$ and $P_{\mathcal{D}}^2$, we define $v_{\mathcal{D}}^P \in N$ via $\mathcal{D}_P = \mathrm{tail}(\mathcal{D}) + v_{\mathcal{D}}^P$. We then set $c_{\mathcal{D}}^i = c_{P_{\mathcal{D}}^i}$, where if the previous equation didn't have any exceptions, we can just take $c_{\mathcal{D}}^i = 0$. We can describe $\kappa(\pi)$ in terms of this data as follows:

Theorem 3.5.1. *Consider the open covering of $X(\mathcal{S})$ given by the open sets $X(\mathcal{D})$, $\mathcal{D} \in \mathcal{S}$. The image of π by the Kodaira-Spencer map in $H^1(X(\mathcal{S}), \mathcal{T}_{X(\mathcal{S})})$ is given by the Čech cocycle $d_{\mathcal{D},\mathcal{E}}$ defined by*

$$\begin{aligned}
d_{\mathcal{D},\mathcal{E}}(y_0) &= a_{\mathcal{E}}(c_{\mathcal{E}}^1 \cdot y_0 - 1)(c_{\mathcal{E}}^2 \cdot y_0 - 1) - a_{\mathcal{D}}(c_{\mathcal{D}}^1 \cdot y_0 - 1)(c_{\mathcal{D}}^2 \cdot y_0 - 1) \\
d_{\mathcal{D},\mathcal{E}}(\chi^u) &= \Bigg[(-1)^{a_{\mathcal{E}}} \left(\langle \widetilde{v}_{\mathcal{E}}^0, u\rangle - a_{\mathcal{E}} \sum \langle v_{\mathcal{E}}^P, u\rangle \left(1 + \frac{(c_{\mathcal{E}}^1 - y_0^{-1})(c_{\mathcal{E}}^2 - y_0^{-1})}{c_P - y_0^{-1}}\right)\right) \\
&\quad - (-1)^{a_{\mathcal{D}}} \left(\langle \widetilde{v}_{\mathcal{D}}^0, u\rangle - a_{\mathcal{D}} \sum \langle v_{\mathcal{D}}^P, u\rangle \left(1 + \frac{(c_{\mathcal{D}}^1 - y_0^{-1})(c_{\mathcal{D}}^2 - y_0^{-1})}{c_P - y_0^{-1}}\right)\right)\Bigg] \cdot y_0^{-\alpha}\chi^u.
\end{aligned}$$

Proof. We calculate the Kodaira-Spencer map as described in [Ser06]. First, fix some $\mathcal{D} \in \mathcal{S}$. Then as in the proof of theorem 3.4.4 we have isomorphisms

$$\theta_{\mathcal{D}} : X(\mathcal{D}) \times \operatorname{Spec} \mathbb{C}[\epsilon] \xrightarrow{\sim} X(\mathcal{D}^{\mathrm{tot}}) \times_B \operatorname{Spec} \mathbb{C}[\epsilon].$$

Using the explicit descriptions of the maps ϕ_v and γ_{P_1,P_2}, a routine calculation shows that $\theta_{\mathcal{D}}^{\#}$ maps

$$\begin{aligned}
y_0 &\mapsto y_0 + a_{\mathcal{D}} t (c_{\mathcal{D}}^1 \cdot y_0 - 1)(c_{\mathcal{D}}^2 \cdot y_0 - 1) \\
\chi^u &\mapsto \chi^u + (-1)^{a_{\mathcal{D}}} t y_0^{-\alpha} \left(\langle \widetilde{v}_{\mathcal{D}}^0, u\rangle - a_{\mathcal{D}} \sum \langle v_{\mathcal{D}}^P, u\rangle \left(1 + \frac{(c_{\mathcal{D}}^1 - y_0^{-1})(c_{\mathcal{D}}^2 - y_0^{-1})}{c_P - y_0^{-1}}\right)\right) \cdot \chi^u
\end{aligned}$$

modulo t^2. Likewise, we have that $(\theta_{\mathcal{D}}^{\#})^{-1}$ maps

$$\begin{aligned}
y_0 &\mapsto y_0 - a_{\mathcal{D}} t (c_{\mathcal{D}}^1 \cdot y_0 - 1)(c_{\mathcal{D}}^2 \cdot y_0 - 1) \\
\chi^u &\mapsto \chi^u - (-1)^{a_{\mathcal{D}}} t y_0^{-\alpha} \left(\langle \widetilde{v}_{\mathcal{D}}^0, u\rangle - a_{\mathcal{D}} \sum \langle v_{\mathcal{D}}^P, u\rangle \left(1 + \frac{(c_{\mathcal{D}}^1 - y_0^{-1})(c_{\mathcal{D}}^2 - y_0^{-1})}{c_P - y_0^{-1}}\right)\right) \cdot \chi^u
\end{aligned}$$

modulo t^2.

Now for $\mathcal{D}, \mathcal{E} \in \mathcal{S}$, set $\theta_{\mathcal{D},\mathcal{E}} = \theta_{\mathcal{D}}^{-1}\theta_{\mathcal{E}}$. We then define a derivation $d_{\mathcal{D},\mathcal{E}}$ via $\theta_{\mathcal{D},\mathcal{E}}^{\#} =$

$\mathrm{id} + t \cdot d_{\mathcal{D},\mathcal{E}}$. Since $\theta^{\#}_{\mathcal{D},\mathcal{E}} = \theta^{\#}_{\mathcal{E}}(\theta^{\#}_{\mathcal{D}})^{-1}$, we can calculate that $\theta^{\#}_{\mathcal{D},\mathcal{E}}$ maps

$$\begin{aligned}
&y_0 \mapsto y_0 - a_{\mathcal{D}} t(c^1_{\mathcal{D}} \cdot y_0 - 1)(c^2_{\mathcal{D}} \cdot y_0 - 1) + a_{\mathcal{E}} t(c^1_{\mathcal{E}} \cdot y_0 - 1)(c^2_{\mathcal{E}} \cdot y_0 - 1)\\
&\chi^u \mapsto \chi^u - (-1)^{a_{\mathcal{D}}} t y_0^{-\alpha} \left(\langle \widetilde{v}^0_{\mathcal{D}}, u\rangle - a_{\mathcal{D}} \sum \langle v^P_{\mathcal{D}}, u\rangle \left(1 + \frac{(c^1_{\mathcal{D}} - y_0^{-1})(c^2_{\mathcal{D}} - y_0^{-1})}{c_P - y_0^{-1}}\right)\right) \cdot \chi^u\\
&\qquad + (-1)^{a_{\mathcal{E}}} t y_0^{-\alpha} \left(\langle \widetilde{v}^0_{\mathcal{E}}, u\rangle - a_{\mathcal{E}} \sum \langle v^P_{\mathcal{E}}, u\rangle \left(1 + \frac{(c^1_{\mathcal{E}} - y_0^{-1})(c^2_{\mathcal{E}} - y_0^{-1})}{c_P - y_0^{-1}}\right)\right) \cdot \chi^u
\end{aligned}$$

once again modulo t^2. The claim of the theorem follows. ☐

The above formula is admittedly quite technical. This is the price we pay for the generality of divisorial fans we are considering. If we restrict to a special class of divisorial fans, the resulting formula is much more manageable:

Corollary 3.5.2. *Suppose that $\mathcal{S}$ is a divisorial fan such that for all $\mathcal{D} \in \mathcal{S}$ with complete locus, $\mathcal{D}_P = \mathrm{tail}(\mathcal{D})$ unless $P = 0$ or $P = \infty = V(y_0^{-1})$. Then as in theorem 3.5.1, $\kappa(\pi)$ is defined by the cocycle*

$$d_{\mathcal{D},\mathcal{E}} = (a_{\mathcal{E}} - a_{\mathcal{D}}) \frac{\partial}{\partial y_0} + y_0^{-\alpha} \sum_i \langle (-1)^{a_{\mathcal{E}}} \widetilde{v}^0_{\mathcal{E}} - (-1)^{a_{\mathcal{D}}} \widetilde{v}^0_{\mathcal{D}}, e^*_i \rangle \chi^{e^*_i} \frac{\partial}{\partial \chi^{e^*_i}}$$

where e^*_i is a basis of M.

Proof. Under the above assumptions, we always have $c^i_{\mathcal{D}} = 0$ as well as $v^P_{\mathcal{D}} = 0$. The statement then follows directly from theorem 3.5.1. ☐

Chapter 4

Homogeneous Deformations of Nonaffine Toric Varieties

As mentioned in the remark at the end of section 1.2, a toric variety can be viewed as a T-variety by considering the action of some subtorus of the big torus. Thus, we can use the divisorial fan decompositions of section 3.2 to construct deformations of an arbitrary toric variety. If in particular the deformation is locally trivial, we can use theorem 3.5.1 to calculate the image of the Kodaira-Spencer map. We briefly describe this situation in section 4.1. In section 4.2 we then give an explicit description of the vector space of infinitesimal deformations in terms of connected components of certain graphs for smooth toric varieties coming from a fan with convex full-dimensional support. We then use this description in 4.3 to show that T-deformations of these varieties span this vector space of infinitesimal deformations.

4.1 A Simplified Kodaira-Spencer Map

Let N' be an n-dimensional lattice with dual M', and let Σ be a fan on $N'_{\mathbb{Q}}$ with corresponding toric variety $X_0 = \mathrm{TV}(\Sigma)$. Fix some primitive vector $R \in M'$ and set $N = R^{\perp}$. As noted in remark 1.2.14, after choosing a section from N' to N we get a divisorial fan $\mathcal{S} = \mathcal{S}^{\Sigma}$, where $X(\mathcal{S}^{\Sigma})$ describes X_0 as a T^N-variety; the only nontrivial slices of $\mathcal{S}$ are of course at 0 and ∞.

Now, consider any Minkowski decomposition of $\mathcal{S}$; this gives rise to a T^N-deformation π of X_0. Without loss of generality, we can assume that $\mathcal{Q} \subset \{0, \infty\}$. Indeed, since for $P \neq 0, \infty$ $\mathcal{D}_P = \mathrm{tail}(\mathcal{D}_P)$ for all $\mathcal{D} \in \mathcal{S}$, one can use lemma 3.4.6 to construct an isomorphism with a deformation of the above type. We will furthermore assume that $\alpha = 1$.

Now suppose that the Minkowski decomposition of $\mathcal{S}$ is essentially locally trivial. Then by theorem 3.4.4, π is infinitesimally locally trivial, and we can calculate the Kodaira-Spencer map with theorem 3.5.1. To do this, let e_i be a basis of N' with dual basis e_i^* such that $R = e_n^*$. We define $a_\sigma := a_{\mathcal{D}^\sigma}$ and $v_\sigma := \tilde{v}^0_{\mathcal{D}^\sigma} - a_\sigma \cdot e_n$, where the terms on the right sides are as in section 3.5. Then we have the following:

Proposition 4.1.1. *The image of the Kodaira-Spender map $\kappa(\pi)$ is given by the Čech cocycle*

$$d_{\tau,\sigma} = \sum_i \langle (-1)^{a_\sigma} v_\sigma - (-1)^{a_\tau} v_\tau, e_i^* \rangle \chi^{e_i^* - e_n^*} \frac{\partial}{\partial \chi^{e_i^*}}$$

with respect to the open covering $\{\mathrm{TV}(\sigma)\}_{\sigma\in\Sigma}$ *of* X_0.

Proof. This follows directly from corollary 3.5.2 and the fact that $y_0 = \chi^{e_n^*}$. □

4.2 Infinitesimal Deformations of Smooth, Complete Toric Varieties

For the next two sections, X_0 will be a toric variety given by some fan Σ in the lattice N'. If Σ is full-dimensional, we say that X_0 has *no torus factors*. If $|\Sigma|$ is convex, we say that X_0 is *semicomplete*; X_0 is in fact semicomplete if and only if it admits an equivariant proper map to an affine toric variety. This class of varieties includes complete toric varieties as well as (partial) resolutions of toric singularities.

We are interested in the vector space of infinitesimal deformations $T^1_{X_0}$. When X_0 is smooth, we simply have $T^1_{X_0} = H^1(X_0, \mathcal{T}_{X_0})$. Now this vector spaces carries an M' grading, where M' is dual to N'. Thus, we can compute it by computing each homogeneous piece $T^1_Y(-R)$.

For any $\rho \in \Sigma(1)$ and $R \in M'$ with $\langle \rho, R\rangle = 1$, let $\Gamma_\rho(-R)$ be the graph embedded in $N'_{\mathbb{Q}}$ with vertices consisting of primitive lattice generators of rays $\tau \in \Sigma(1) \setminus \rho$ fulfilling $\langle \tau, R\rangle > 0$; two vertices τ_1 and τ_2 are connected by an edge if they are common faces of some cone in Σ. The main result of this section is the following:

Theorem 4.2.1. *For a smooth semicomplete toric variety* X_0 *with no torus factors,*

$$\dim T^1_{X_0}(-R) = \sum_{\langle \rho, R\rangle = 1} \max\{0, \dim H^0(\Gamma_\rho(-R), \mathbb{C}) - 1\}.$$

To prove the above theorem, we shall first need to calculate the cohomology of the boundary divisors of X_0; recall that the elements of $\Sigma(1)$ correspond exactly to the invariant prime divisors of $X_0 = \mathrm{TV}(\Sigma)$. We denote the divisor corresponding to ρ by D_ρ.

Proposition 4.2.2. *For any semicomplete toric variety* X_0 *we have*

$$H^1(X_0, D_\rho)(-R) = 0$$

if $\langle \rho, R\rangle \neq 1$. *Otherwise,*

$$\dim H^1(Y, D_\rho)(-R) = \max\{0, \dim H^0(\Gamma_\rho(-R), \mathbb{C}) - 1\}.$$

Proof. Let $U_\rho(-R) = \{v \in N_{\mathbb{Q}} | \langle v, -R\rangle < h(v)\}$, where h is the piecewise linear function on Σ given by $h(\rho) = -1$, $h(\rho') = 0$ for $\rho' \in \Sigma(1)$ with $\rho' \neq \rho$. Then by [Dem70], $H^i(X_0, D_\rho)(-R) \cong H^i(N_{\mathbb{Q}}, U_\rho(-R))$ for all $i \geq 0$. Thus, we have the following exact sequence coming from relative cohomology:

$$0 \to H^0(X_0, D_\rho)(-R) \to \mathbb{C} \to H^0(U_\rho(-R), \mathbb{C}) \to H^1(X_0, D_\rho)(-R) \to 0.$$

A standard calculation shows that if $U_\rho(-R) \neq \emptyset$, then $H^0(X_0, D_\rho)(-R) = 0$. On the other hand, if $U_\rho(-R) = \emptyset$, then clearly $H^1(X_0, D_\rho)(-R) = 0$.

Now suppose that $\langle \rho, R\rangle \neq 1$. If $R = 0$ then $U_\rho(-R) = \emptyset$ and thus $H^1(X_0, D_\rho)(-R) = 0$. Otherwise, one easily checks that $U_\rho(-R)$ deformation retracts to $|\Sigma| \cap [R = 1]$. Since

X_0 is semicomplete, this is convex, so in particular $U_\rho(-R)$ has at most a single connected component. Thus, $H^1(X_0, D_\rho)(-R) = 0$ by the above exact sequence.

We can now assume that $\langle \rho, R \rangle = 1$. In this case, $H^0(U_\rho(-R), \mathbb{C}) = H^0(\Gamma_\rho(-R), \mathbb{C})$. Indeed, $U_\rho(-R)$ can be retracted to $\widetilde{U}_\rho(-R) = U_\rho(-R) \cap S$, where S is the unit sphere. Now for each cone σ of dimension larger than two in Σ, $\widetilde{U}_\rho(-R) \cap \sigma$ can be replaced in $\widetilde{U}_\rho(-R)$ with $\widetilde{U}_\rho(-R) \cap \partial\sigma$ without changing the connectivity of the set. Thus, we can replace $\widetilde{U}_\rho(-R)$ by its intersection with the union of all elements of $\Sigma(2)$. This is clearly homeomorphic to $\Gamma_\rho(-R)$. The desired formula then follows from above the exact sequence. $\square$

The next lemma connects the cohomology of the tangent bundle with that of the boundary divisors:

Lemma 4.2.3. *[Jac94] Let X_0 be a smooth toric variety with no torus factors. Then $H^i(X_0, \mathcal{T}_{X_0}) \cong \bigoplus_{\rho \in \Sigma(1)} H^i(X_0, D_\rho)$ as M-graded groups for $i \geq 1$.*

Proof. Taking the long exact cohomology sequence coming from the generalized Euler sequence[1]

$$0 \mapsto N_1(X_0) \otimes \mathcal{O}_{X_0} \to \bigoplus_{\rho \in \Sigma(1)} \mathcal{O}(D_\rho) \to \mathcal{T}_{X_0} \to 0\,, \tag{4.2.1}$$

and using the fact that $H^i(X_0, N_1 \otimes \mathcal{O}_{X_0})$ vanishes for $i \geq 1$ gives us

$$H^i(X_0, \mathcal{T}_{X_0}) \cong H^i(X_0, \bigoplus_{\rho \in \Sigma(1)} D_\rho) \cong \bigoplus_{\rho \in \Sigma(1)} H^i(X_0, D_\rho)$$

for $i \geq 1$. $\square$

Proof of theorem 4.2.1. Combine lemma 4.2.3 with proposition 4.2.2. $\square$

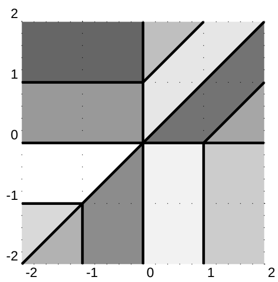


Figure 4.1: $\mathcal{S}_0$ for a smooth toric threefold

Example 4.2.4 (A smooth toric threefold)**.** Consider the complete fan Σ in $\mathbb{Z}^3_\mathbb{Q}$ with rays $\rho_1 = (1,0,1)$, $\rho_2 = (1,1,0)$, $\rho_3 = (0,1,1)$, $\rho_4 = (-1,0,0)$, $\rho_5 = (-1,-1,1)$, $\rho_6 = \rho_0 = (0,-1,0)$, $\rho_7 = (0,0,1)$, and $\rho_8 = -\rho_7$, and top-dimensional cones generated by $\rho_i, \rho_{i+1}, \rho_7$ or $\rho_i, \rho_{i+1}, \rho_8$ for $0 \leq i \leq 6$. Then $X_0 = \mathrm{TV}(\Sigma)$ is a smooth, complete toric threefold. As above, taking $R = [0,0,1]$ and $\rho = \rho_7$, we get a slice $\mathcal{S}_0$ as pictured in figure 4.1. The graph $\Gamma_\rho(-R)$ simply consists of the three nonzero vertices in the slice $\mathcal{S}_0$. Thus, $\dim T^1_{X_0}(-R) = 2$. Furthermore, one easily checks that this is the only degree contributing to $T^1_{X_0}$.

[1] See [CLS10] for the noncomplete case.

The above theorem also has a number of corollaries:

Corollary 4.2.5. *Let X_0 be a smooth semicomplete toric surface with no torus factors. Let $\widetilde{X}_0$ be an equivariant blow-up of X_0. Then $T^1_{X_0} \subset T^1_{\widetilde{X}_0}$.*

Proof. By adding rays to a two-dimensional fan Σ, the number of connected components of any $\Gamma_\rho(-R)$ can only increase. □

In [BB96] it was shown that a smooth, complete toric Fano variety is rigid. This has been generalized to Fano varieties with $\mathbb{Q}$-factorial terminal singularities in [dFH09]. It is possible to use our explicit cohomology calculations to generalize the first result in another direction. In many cases, it suffices to assume weakly Fano, that is, that the anticanonical divisor is nef:

Corollary 4.2.6. *Let X_0 be a complete, smooth, weakly Fano toric variety of dimension n, and assume that there is no equivariant embedding $\widetilde{A}_1 \times (\mathbb{C}^*)^{n-2} \hookrightarrow X_0$, where $\widetilde{A}_1$ is the minimal resolution of a toric A_1 singularity. Then X_0 is rigid.*

Proof. Let the roof of Σ be the support of the polyhedral subdivision consisting of simplices spanned by the primitive generators of any cone in Σ. Since X_0 is weakly Fano, then the roof of Σ is in fact concave.

Consider $R \in M$ and $\rho \in \Sigma(1)$ such that $\langle \rho, R \rangle = 1$ and take $\rho_1, \rho_2 \in \Gamma_\rho(-R)$ with $\rho_1 \neq \rho_2$. We shall show that ρ_1 and ρ_2 are in fact connected in $\Gamma_\rho(-R)$; the corollary then follows from theorem 4.2.1.

We first claim that it suffices to only consider ρ_i which share a common cone with ρ. Indeed, every connected component of $\Gamma_\rho(-R)$ clearly has a vertex sharing a cone with ρ. Now let γ be the line segment connecting ρ_1 and ρ_2 and let $\widetilde{\gamma}$ be the projection of γ to the roof of Σ. Now, since the roof of Σ is concave,

$$\langle v, R \rangle \geq \min\{\langle \rho_1, R \rangle, \langle \rho_2, R \rangle\} \geq 1$$

for all $v \in \widetilde{\gamma}$. Thus, if γ doesn't intersect ρ, $\widetilde{\gamma}$ is in $U_\rho(-R)$ and ρ_1 and ρ_2 are connected in $U_\rho(-R)$ and thus also in $\Gamma_\rho(-R)$, where $U_\rho(-R)$ is as in the proof of proposition 4.2.2. Suppose on the other hand that γ intersects ρ, that is, that $\rho \in \widetilde{\gamma}$. Then from the concavity of the roof and $\langle \rho, R \rangle = 1$, it follows that $\langle \rho_i, R \rangle = 1$ for $i = 1, 2$. Since however both ρ_i share common cones with ρ, one easily sees that the subfan of Σ with rays ρ_1, ρ_2, and ρ corresponds to the toric variety $\widetilde{A}_1 \times (\mathbb{C}^*)^{n-2}$, a contradiction. □

Now, any weakly Fano smooth, complete toric surface which isn't Fano does in fact admit an embedding of $\widetilde{A}_1$, so the above result doesn't provided anything new for $n = 2$. However, we can show that non-Fano surfaces are in fact never rigid:

Corollary 4.2.7. *A smooth, complete toric surface X_0 is rigid if and only if X_0 is Fano.*

Proof. If X_0 is Fano, it is rigid by [BB96]. On the other hand, suppose that X_0 isn't Fano, and let Σ be the corresponding fan. Then there must be some $\rho \in \Sigma(1)$ together with $R \in M$ such that $\langle \rho, R \rangle = 1$ and $\langle \rho', R \rangle \geq 1$ for rays $\rho' \in \Sigma(1)$ adjacent to ρ. Then $\Gamma_\rho(-R)$ has two connected components, so $T^1_{X_0}(-R) \neq 0$ by theorem 4.2.1. □

We end the section with another statement regarding toric surfaces:

Corollary 4.2.8. *Any smooth semicomplete toric surface X_0 with no torus factors is unobstructed, that is,*

$$H^2(X_0, \mathcal{T}_{X_0}) = 0.$$

Proof. Using Serre duality with lemma 4.2.3, we have

$$H^2(X_0, \mathcal{T}_{X_0}) \cong \bigoplus_{\rho \in \Sigma(1)} H^2(X_0, D_\rho) \cong \bigoplus_{\rho \in \Sigma(1)} H^0(X_0, -D_\rho - \sum_{\rho' \in \Sigma(1)} D_{\rho'}).$$

The terms on the right hand side are clearly zero. ☐

4.3 One-Parameter Deformations Spanning T^1

For this section we will be assuming that $X_0 = \mathrm{TV}(\Sigma)$ is smooth and semicomplete with no torus factors. Our goal is to construct one-parameter T-deformations spanning $T^1_{X_0}$. We do this by associating a Minkowski decomposition to each connected component of the graphs $\Gamma_\rho(-R)$ which appeared in theorem 4.2.1.

Choose some $R \in M'$ and let $\rho \in \Sigma(1)$ be some ray with $\langle \rho, R \rangle = 1$. We can choose a basis $e_1 \ldots, e_n$ of N' as above in proposition 4.1.1 such that $R = e_n^*$ and $\rho = e_n$. As above, we take $\mathcal{S} = \mathcal{S}^\Sigma$ to be the divisorial fan describing X_0 with a T^N-action, where $N = R^\perp$. We now consider the graph $\Gamma_\rho(-R)$ from the previous section; note that by rescaling with $\mathbb{Q}_{>0}$ we can consider $\Gamma_\rho(-R)$ to be embedded in the slice $\mathcal{S}_0$, with vertices of $\Gamma_\rho(-R)$ corresponding to non-zero vertices of $\mathcal{S}_0$ and with two vertices connected by an edge if they are in fact connected by a one-simplex in $\mathcal{S}_0$.

Now for $R \in M'$, define

$$\Omega(-R) = \{\rho \in \Sigma(1) \big| \ \langle \rho, R \rangle = 1 \text{ and } \Gamma_\rho(-R) \neq \emptyset\}.$$

Assume that $\rho \in \Omega(-R)$ and choose now some connected component C of $\Gamma_\rho(-R)$. This leads to a Minkowski decomposition of the polyhedral subdivision $\mathcal{S}_0$ as follows. Consider $\Delta \in \mathcal{S}_0$. If $\Delta \cap C = \emptyset$, then set $\Delta^0 = \Delta$ and $\Delta^1 = \mathrm{tail}(\Delta)$. If instead the intersection $\Delta \cap C$ is nonempty, set $\Delta^0 = \mathrm{tail}(\Delta)$ and $\Delta^1 = \Delta$.

Proposition 4.3.1. *The $\{\Delta^i\}$ form an admissible Minkowski decomposition of the polyhedral subdivision $\mathcal{S}_0$.*

Proof. We utilize the cone of Minkowski summands from section 3.1. Indeed, consider the vector $v \in \mathrm{cone}_{\mathrm{MS}}(\mathcal{S}_0)$ with $v_e = 1$ if $e \cap C = \emptyset$ and with $v_e = 0$ if $e \cap C \neq \emptyset$ for edges e in $\Gamma_\rho(-R)$; one easily confirms that this is indeed an element of the cone of Minkowski summands. Furthermore, $(\mathcal{S}_0)_v = \{\Delta^0\}$, and $(\mathcal{S}_0)_{\underline{1}-v} = \{\Delta^1\}$. Thus, it follows from proposition 3.1.7 that the $\{\Delta^i\}$ form a Minkowski decomposition of $\mathcal{S}_0$. The admissibility of this decomposition follows immediately from the construction. ☐

Now, consider the Minkowski decomposition of $\mathcal{S}$ coming from the above decomposition of $\mathcal{S}_0$. This gives us a one-parameter T-deformation of X_0 which we call $\pi(C, \rho, R)$. We can now formulate one of our main results:

Theorem 4.3.2. *Let X_0 be a smooth semicomplete toric variety with no torus factors. Then the one-parameter deformations $\pi(C, \rho, R)$ span $T^1_{X_0}(-R)$, where ρ ranges over all rays $\rho \in \Omega(-R)$ and C ranges over all connected components of the graphs $\Gamma_\rho(-R)$. Thus, $T^1_{X_0}$ is spanned by T-deformations.*

Proof. To prove the theorem, we simply calculate the Kodaira-Spencer map for the above deformations and then use the description of $T^1_{X_0}(-R)$ from theorem 4.2.1. For $\rho \in \Omega(-R)$, let $\partial(R,\rho)$ be the derivation taking $\chi^v \mapsto \langle \rho, v\rangle \chi^{v-R}$. If we choose the basis $e_1,\ldots,e_n$ such that $e_n = \rho$ and $R = e_n^*$, then $\partial(R,\rho) = \frac{\partial}{\partial \chi^{e_n^*}}$. Applying proposition 4.1.1 we then have that the image of $\pi(C,\rho,R)$ is given by

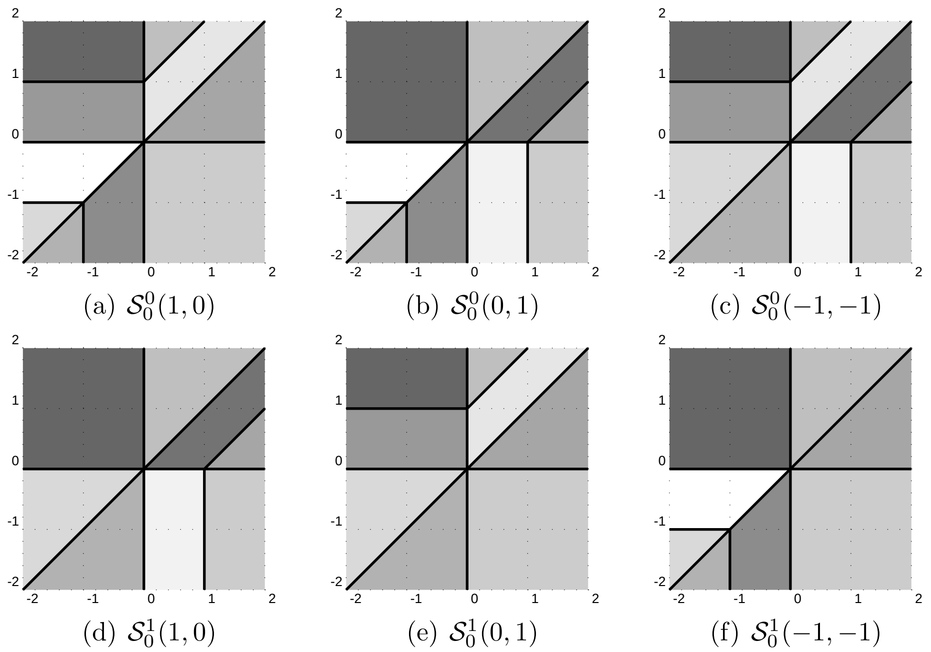
$$d_{\tau,\sigma} = (a'_\sigma - a'_\tau)\partial(R,\rho)$$

where $a'_\sigma = 0$ if $\mathcal{D}_0^i \cap C = \emptyset$ and $a'_\sigma = 1$ otherwise. Indeed, it follows from the above construction that $a_\sigma = a'_\sigma$. Furthermore, $\tilde{v}^0_\sigma = 0$ for all σ. Note that we will only have to consider cones $\sigma \in \Sigma(n)$, as the $\mathrm{TV}(\sigma)$ cover X_0.

Now let $f \in H^0(\Gamma_\rho(-R),\mathbb{C})$ and $\sigma \in \Sigma(n)$. If $\Gamma_\rho(-R)\cap\sigma = \emptyset$, set $f(\sigma) = 1$, otherwise set $f(\sigma) = f(v)$ for any $v \in \Gamma_\rho(-R)\cap\sigma$. Combining exact sequences from the proofs of proposition 4.2.2 and lemma 4.2.3 we then have the exact sequence

$$0 \longrightarrow \bigoplus_{\rho\in\Omega(-R)} \mathbb{C} \longrightarrow \bigoplus_{\rho\in\Omega(-R)} H^0(\Gamma_\rho(-R),\mathbb{C}) \xrightarrow{\Phi} H^1(X_0,\mathcal{T}_{X_0})(-R) \longrightarrow 0$$

where Φ maps $f \in H^0(\Gamma_\rho(-R),\mathbb{C})$ to the Čech cocycle $f_{\tau,\sigma} = \frac{1}{2}(f(\tau)-f(\sigma))\partial(R,\rho)$. Now, for $\rho \in \Omega(-R)$ and any connected component C in $\Gamma_\rho(-R)$ let $f(C,\rho,R) \in H^0(\Gamma_\rho(-R))$ be defined by $f(C,\rho,R)_{|C} \equiv -1$ and $f(C,\rho,R)_{|\Gamma_\rho(-R)\setminus C} \equiv 1$. Then we have that $\Phi(f(C,\rho,R))$ is equal to the image of $\pi(C,\rho,R)$ in $H^1(X_0,T^1_{X_0})$ by the above calculation. Furthermore, one easily sees that the $f(C,\rho,R)$ form a basis of $H^0(\Gamma_\rho(-R),\mathbb{C})/\mathbb{C}$, where C ranges over all connected components of $\Gamma_\rho(-R)$ except one. Thus, if we allow ρ to vary over the elements of $\Omega(-R)$ as well, the $\pi(C,\rho,R)$ span $T^1_{X_0}(-R)$. $\square$



(a) $\mathcal{S}^0_0(1,0)$ (b) $\mathcal{S}^0_0(0,1)$ (c) $\mathcal{S}^0_0(-1,-1)$

(d) $\mathcal{S}^1_0(1,0)$ (e) $\mathcal{S}^1_0(0,1)$ (f) $\mathcal{S}^1_0(-1,-1)$

Figure 4.2: Deformations of a smooth toric threefold

Example 4.3.3 (A smooth toric threefold)**.** We continue example 4.2.4 and construct Minkowski decompositions corresponding to deformations spanning $T^1_{X_0}$, where X_0 was our smooth toric threefold. As mentioned previously, the connected components of $\Gamma_\rho(-R)$ are simply the vertices $(1,0)$ $(0,1)$, and $(-1,-1)$. For each vertex, we thus get a decomposition of $\mathcal{S}_0$ with two summands. For the vertex v, we denote the resulting polyhedral subdivisions by $\mathcal{S}_0^0(v)$ and $\mathcal{S}_0^1(v)$; these are pictured in figure 4.2.

Note that we can find a decomposition of $\mathcal{S}_0$ with three summands, where the corresponding polyhedral subdivisions are $\mathcal{S}_0^1(1,0)$, $\mathcal{S}_0^1(0,1)$, $\mathcal{S}_0^1(-1,-1)$. Adding up any two of the summands gives us one of the three original decompositions. Thus, we have in fact found a two-parameter deformation of X_0 which spans $T^1_{X_0}$.

Chapter 5

Families of T-Invariant Divisors

Up until now, we have been primarily concerned with certain deformations of some T-variety X_0. In other words, we have been interested in properties of the functor Def_{X_0}. For this chapter, we switch our focus to the functor $\mathrm{Def}_{(X_0,\mathcal{L})}$ for some T-invariant line bundle $\mathcal{L}$. In section 5.1, we construct an explicit map between certain subgroups of invariant divisors of the fibers of a one-parameter T-deformation π. We then show that under certain conditions, this map preserves many nice properties, for example, Euler characteristic. In section 5.2 we then prove that this map is quite often surjective, in particular, for T-deformations of complete T-varieties.

5.1 A Map of Picard Groups

Throughout the next two sections, we will fix some rational complexity-one T-variety $X_0 = X(\mathcal{S})$ together with some one-parameter T-deformation $\pi : X^{\mathrm{tot}} \to B$. We will assume that π arises from a Minkowski decomposition of $\mathcal{S}$ occurring at the point $0 \in \mathbb{P}^1$; set $\alpha = \alpha_0$. For any $s \in B$, denote by $X_s = X(\mathcal{S}^{(s)})$ the fiber of π over s. We denote by $\alpha(s)$ the divisor $V(y_0^\alpha - s)$ on $\mathbb{P}^1$.

Our goal is now to compare the Picard groups $\mathrm{Pic}(X_s)$ as $s \in B$ varies. Our strategy is the following: for any fixed $b \in B \setminus \{0\}$ we will identify a subgroup $\text{T-CDiv}'(X_b)$ of $\text{T-CDiv}(X_b)$ such that any element of $\text{T-CDiv}'(X_b)$ naturally lifts to an invariant Cartier divisor on the total space X^{tot}. We can then restrict this divisor to any fiber X_s, giving us an element of $\text{T-CDiv}'(X_s)$. Thus, we will have a natural map $\pi_{b,s} : \text{T-CDiv}'(X_b) \to \text{T-CDiv}'(X_s)$. Since this map respects linear equivalence, we can then use it to compare subgroups of the Picard groups of the fibers.

Our first task is now to identify the special subgroups T-CDiv$'$ which allow for natural lifting of divisors to X^{tot}. This will be taken care of by the following definition:

Definition 5.1.1. For $s \in B \subset \mathbb{P}^1$, define $\mathrm{CaSF}'(\mathcal{S}^{(s)})$ to consist of those $h \in \mathrm{CaSF}(\mathcal{S}^{(s)})$ such that for all $\mathcal{D}^{(s)} \in \mathcal{S}^{(s)}$, we can find $u \in M$, and $a_0, a_s \in \mathbb{Z}$ satisfying

(i) $(h_{|\mathcal{D}^{(s)}})_0(v) = \langle v, u\rangle + a_0$;

(ii) $(h_{|\mathcal{D}^{(s)}})_\eta(v) = \langle v, u\rangle + a_s$ for all $\eta \in \operatorname{supp}\alpha(s)$.

We of course also always have $\mathrm{CaSF}'(\mathcal{S}^{(0)}) = \mathrm{CaSF}(\mathcal{S}^{(0)})$. Note that $\mathrm{CaSF}'(\mathcal{S}^{(s)}) = \mathrm{CaSF}(\mathcal{S}^{(s)})$ if the decomposition of $\mathcal{S}$ is essentially locally trivial and $\alpha = 1$. Finally, by T-CDiv$'$ we denote the image of CaSF$'$ under the natural map from section 1.3.

Fix now some $b \in B^* := B \setminus \{0\}$ and choose some support function $h^{(b)} \in \mathrm{CaSF}'(\mathcal{S}^{(b)})$; this corresponds to an invariant Cartier divisor $D_{h^{(b)}} \in \text{T-CDiv}'(X_b)$. We will be showing that this can be lifted to a Cartier divisor D_h^{tot} on X^{tot}. We first will need invariant open coverings of X_b and X^{tot}. For $P \in \mathcal{P} \setminus \{0\}$ and $\mathcal{D} \in \mathcal{S}$ with noncomplete locus, set

$$U_{\mathcal{D},P}^{(b)} = X\Big(\mathcal{D}^{(b)} + \emptyset \otimes \alpha(b) + \sum_{\substack{Q \in \mathcal{P} \\ Q \neq P}} \emptyset \otimes Q\Big)$$
$$U_{\mathcal{D},P}^{\mathrm{tot}} = X\Big(\mathcal{D}^{\mathrm{tot}} + \emptyset \otimes V(y_0^\alpha - t) + \sum_{\substack{Q \in \mathcal{P} \\ Q \neq P}} \emptyset \otimes V(y_Q)\Big)$$

and likewise set

$$U_{\mathcal{D},0}^{(b)} = X\Big(\mathcal{D}^{(b)} + \sum_{\substack{Q \in \mathcal{P} \\ Q \neq 0}} \emptyset \otimes Q\Big)$$
$$U_{\mathcal{D},0}^{\mathrm{tot}} = X\Big(\mathcal{D}^{\mathrm{tot}} + \sum_{\substack{Q \in \mathcal{P} \\ Q \neq 0}} \emptyset \otimes V(y_Q)\Big).$$

On the other hand, for $P \in \mathcal{P}$ and $\mathcal{D} \in \mathcal{S}$ with complete locus, set $U_{\mathcal{D},P}^{(b)} = X(\mathcal{D}^{(b)})$ and $U_{\mathcal{D},P}^{\mathrm{tot}} = X(\mathcal{D}^{\mathrm{tot}})$. One easily checks that $\{U_{\mathcal{D},P}^{(b)}\}$ and $\{U_{\mathcal{D},P}^{\mathrm{tot}}\}$ define invariant open coverings of respectively X_b and X^{tot}. These open coverings may in fact be finer than necessary for defining the desired Cartier divisor.

For each $P \in \mathcal{P}$ and $\mathcal{D} \in \mathcal{S}$, let $u_{\mathcal{D},P} \in M$, $f_{\mathcal{D},P}^{(b)} \in \mathbb{C}(Y)$ be such that $D_{h^{(b)}|U_{\mathcal{D},P}^{(b)}} = \mathrm{div}(f_{\mathcal{D},P}^{(b)} \cdot \chi^{u_{\mathcal{D},P}})$. Such $f_{\mathcal{D},P}^{(b)}, u_{\mathcal{D},P}$ exist since $h^{(b)} \in \mathrm{CaSF}'(\mathcal{S}^{(b)})$. Now set

$$f_{\mathcal{D},P}^{\mathrm{tot}} = f_{\mathcal{D},P}^{(b)} \cdot \left(\frac{y_0^\alpha - t}{y_0^\alpha - b}\right)^{\nu_\eta(f_{\mathcal{D},P}^{(b)})} \in \mathbb{C}(Y^{\mathrm{tot}}),$$

where ν_η is the valuation in the point η, and η is any point in the support of $\alpha(b)$.

Proposition 5.1.2. *With respect to the open covering $X^{\mathrm{tot}} = \bigcup U_{\mathcal{D},P}^{\mathrm{tot}}$, the functions $f_{\mathcal{D},P}^{\mathrm{tot}} \cdot \chi^{u_{\mathcal{D},P}} \in \mathbb{C}(X^{\mathrm{tot}})$ define an invariant Cartier divisor on X^{tot} which we denote by D_h^{tot}.*

Proof. Consider $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$ and $P, P' \in \mathcal{P}$. It is sufficient to show

$$\frac{f_{\mathcal{D},P}^{\mathrm{tot}}}{f_{\mathcal{D}',P'}^{\mathrm{tot}}} \cdot \chi^{u_{\mathcal{D},P} - u_{\mathcal{D}',P'}} \in H^0\Big(U_{\mathcal{D},P}^{\mathrm{tot}} \cap U_{\mathcal{D}',P'}^{\mathrm{tot}}, \mathcal{O}_{X^{\mathrm{tot}}}\Big).$$

Setting $\tilde{u} = u_{\mathcal{D},P} - u_{\mathcal{D}',P'}$ and choosing some $\eta \in \operatorname{supp} \alpha(b)$, this is equivalent to showing

$$g := \frac{f_{\mathcal{D},P}^{(b)}}{f_{\mathcal{D}',P'}^{(b)}} \cdot \left(\frac{y_0^\alpha - t}{y_0^\alpha - b}\right)^{\nu_\eta(f_{\mathcal{D},P}^{(b)}/f_{\mathcal{D}',P'}^{(b)})} \in H^0\Big(Y_{\mathcal{D},P}^{\mathrm{tot}} \cap Y_{\mathcal{D}',P'}^{\mathrm{tot}}, \mathcal{D}^{\mathrm{tot}} \cap \mathcal{D}'^{\mathrm{tot}}(\tilde{u})\Big),$$

where $Y_{D,P}^{\mathrm{tot}} = Y^{\mathrm{tot}}$ if $\mathrm{Loc}(\mathcal{D})$ is complete and $Y_{D,P}^{\mathrm{tot}}$ is the image of $U_{D,P}^{\mathrm{tot}}$ under the quotient map otherwise. This in turn is the same as showing that

$$\nu_D(g) \geq -(\mathcal{D}^{\mathrm{tot}} \cap \mathcal{D}'^{\mathrm{tot}})_D(\tilde{u}) \tag{5.1.1}$$

for all divisors D contained in $Y^{\mathrm{tot}}_{\mathcal{D},P} \cap Y^{\mathrm{tot}}_{\mathcal{D}',P'}$, where ν_D is the corresponding valuation. One immediately sees that this is automatically fulfilled unless D is of the form $V(y_0^\alpha - t)$ or $V(y_Q)$ for some $Q \in \mathbb{P}^1$, since both sides of the above inequality will be 0.

Now for $Q \in \mathbb{P}^1 \setminus \operatorname{supp}\alpha(b)$, $\nu_{V(y_Q)}(g) = \nu_Q(f^{(b)}_{\mathcal{D},P}/f^{(b)}_{\mathcal{D}',P'})$. Furthermore, for $\eta \in \operatorname{supp}\alpha(b)$, $\nu_{V(y_\eta)}(g) = 0$ and $\nu_{V(y_0^\alpha - t)}(g) = \nu_\eta(f^{(b)}_{\mathcal{D},P}/f^{(b)}_{\mathcal{D}',P'})$. On the other hand, we have

$$\begin{aligned}
(\mathcal{D}^{\mathrm{tot}} \cap \mathcal{D}'^{\mathrm{tot}})_{V(y_Q)}\big(\tilde{u}\big) &= (\mathcal{D}^{(b)} \cap \mathcal{D}'^{(b)})_Q\big(\tilde{u}\big);\\
(\mathcal{D}^{\mathrm{tot}} \cap \mathcal{D}'^{\mathrm{tot}})_{V(y_\eta)}\big(\tilde{u}\big) &= 0;\\
(\mathcal{D}^{\mathrm{tot}} \cap \mathcal{D}'^{\mathrm{tot}})_{V(y_0^\alpha - t)}\big(\tilde{u}\big) &= (\mathcal{D}^{(b)} \cap \mathcal{D}'^{(b)})_\eta\big(\tilde{u}\big).
\end{aligned}$$

Now, since the functions $f^{(b)}_{\mathcal{D},P}\chi^{u_{\mathcal{D},P}}$ define a Cartier divisor on X_b, we have

$$\frac{f^{(b)}_{\mathcal{D},P}}{f^{(b)}_{\mathcal{D}',P'}} \in H^0\Big(Y_{\mathcal{D},P} \cap Y_{\mathcal{D}',P'}, \mathcal{D}^{(b)} \cap \mathcal{D}'^{(b)}\big(\tilde{u}\big)\Big)$$

where $Y_{D,P}$ is defined similarly to $Y^{\mathrm{tot}}_{D,P}$. Consequently,

$$\nu_Q(g) \geq -(\mathcal{D}^{\mathrm{tot}} \cap \mathcal{D}'^{\mathrm{tot}})_Q\big(\tilde{u}\big) \tag{5.1.2}$$

for $Q \in Y_{D,P}$ and inequality (5.1.1) follows for the required divisors. □

Having checked that D^{tot}_h is indeed a Cartier divisor of X^{tot}, we now want to describe its restrictions to fibers X_s. This restriction $(D^{\mathrm{tot}}_h)_s$ will be T-invariant, and should thus correspond to some support function $h^{(s)}$. Indeed, we will be defining a support functions $h^{(s)} \in \mathrm{CaSF}(\mathcal{S}^{(s)})$ for any $s \in B$. For $s \neq 0$, the combinatorial data of $h^{(s)}$ will be essentially the same as that of $h^{(b)}$, just that the coefficients for prime divisors in $\alpha(b)$ have become coefficients for prime divisors in $\alpha(s)$. For $s = 0$, things are a little more tricky, and we have to somehow 'add' certain coefficients together:

Definition 5.1.3. For each $s \in B$, define $h^{(s)} \in \mathrm{CaSF}(\mathcal{S}^{(s)})$ as follows:

- For $P \in \mathbb{P}^1 \setminus (\operatorname{supp}\alpha(s) \cup \operatorname{supp}\alpha(b))$ set $h^{(s)}_P = h^{(b)}_P$;
- If $s \neq b$, set $h^{(s)}_P = (h^{(b)})^0$ for $P \in \operatorname{supp}\alpha(b)$;
- For $P \in \operatorname{supp}\alpha(s)$ and $s \neq 0$ set $h^{(s)}_P = h^{(b)}_\eta$ for any $\eta \in \operatorname{supp}\alpha(b)$;
- For $P = s = 0$, set
$$h^{(s)}_0(v) = h^{(b)}_0(v_0) + \alpha h^{(b)}_\eta(v_\eta)$$
where if $v \in \mathcal{D}_0$ for some $\mathcal{D} \in \mathcal{S}$, we take any $v_0 \in \mathcal{D}^{(b)}_0$, $v_\eta \in \mathcal{D}^{(b)}_\eta$ for any $\eta \in \alpha(b)$ such that $v_0 + \alpha v_\eta = v$.

Note that the requirement $h^{(b)} \in \mathrm{CaSF}'(\mathcal{S}^{(b)})$ ensures that $h^{(0)}_0(v)$ does not depend on the choice of such v_0 and v_η. One easily checks that $h^{(s)}$ is in fact an element of $\mathrm{CaSF}'(\mathcal{S}^{(s)})$.

Remark 5.1.4. For any support function f, let Γ_f denote the polyhedral subdivision of its graph. Then one easily checks that the elements of $\Gamma_{h^{(b)}_0}$ and $\Gamma_{h^{(b)}_\eta}$ give an (α)-term Minkowski decomposition of the polyhedral subdivision $\Gamma_{h^{(0)}_0}$.

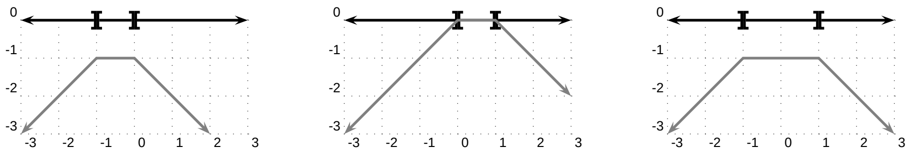
Proposition 5.1.5. *The restriction of D_h^{tot} to the fiber X_s for any $s \in S$ is equal to*

$$(D_h^{\mathrm{tot}})_s = D_{h^{(s)}}.$$

In particular, D_h^{tot} is a lifting of $D_{h^{(b)}}$.

Proof. A simple calculation shows that the restriction of the functions $f_{\mathcal{D},P}^{\mathrm{tot}} \cdot \chi^{u_{\mathcal{D},P}}$ to any fiber X_s are exactly those determined by $h^{(s)}$. □

In light of the two above propositions, we define a map $\pi_{b,s}$: T-CDiv$'(X_b) \to$ T-CDiv$'(X_s)$ by sending $D_{h^{(b)}}$ to $D_{h^{(s)}}$. For $s \in B^*$, it is obvious from the construction that $\pi_{b,s}$ is an isomorphism. For $s = 0$, the matter is slightly more delicate. It is clear from construction that $\pi_{b,0}$ is a group homomorphism sending principal divisors to principal divisors, with kernel contained in the set of principal divisors. Thus, $\pi_{b,s}$ always descends to an injective map $\bar{\pi}_{b,s} : \mathrm{Pic}'(X_b) \hookrightarrow \mathrm{Pic}'(X_s)$, where Pic$'$ is the image of CaSF$'$ modulo linear equivalence.



(a) $h_0^{(b)}$ (b) $h_b^{(b)}$ (c) $h_\infty^{(b)}$

Figure 5.1: A family of divisors for a toric Fano surface

Example 5.1.6 (A family of divisors for a toric Fano surface)**.** We consider the deformation π of the toric Fano surface from example 3.2.5, with corresponding family of divisorial fans $\mathcal{S}^{(s)}$. For some $b \neq 0$, we consider the support function $h^{(b)} = h_0^{(b)} \otimes \{0\} + h_b^{(b)} \otimes \{b\} + h_\infty^{(b)} \otimes \{\infty\}$, where $h_0^{(b)}$, $h_b^{(b)}$, and $h_\infty^{(b)}$ are as pictured in figure 5.1. Then $h^{(b)} \in \mathrm{CaSF}'(\mathcal{S}^{(b)})$. Furthermore, for this support function, we get that $h^{(0)}$ is the support function from example 1.3.5. Thus, we have constructed a family of divisors D_h^{tot} for π which restricts to the anticanonical divisor on the central fiber. One can actually check that all the divisors in the family are anticanonical divisors. Note that for this example, $\mathrm{CaSF}'(\mathcal{S}^{(b)})$ is not equal to $\mathrm{CaSF}(\mathcal{S}^{(b)})$.

Remark 5.1.7. The existence of an injection $\pi_{b,0} : \mathrm{Pic}'(X_b) \hookrightarrow \mathrm{Pic}'(X_0)$ for T-deformations is somewhat similar to the situation for smoothings of Fano varieties, see for example propositions 3.16 and 3.17 of [Gal07]. Indeed, if X_0 is an almost Fano variety with at most Gorenstein canonical singularities and $\pi : X^{\mathrm{tot}} \to B$ is some smoothing, then there is an injective map $\mathrm{Pic}(X^{\mathrm{tot}}) \to \mathrm{Pic}(X_s)$ for all $s \in B$ which is an isomorphism for $s = 0$.

Now if the special fiber X_0 is complete, the cohomology groups of coherent sheaves on all the fibers of π have finite vector space dimension. For invertible sheaves we then have the following theorem:

Theorem 5.1.8. *Let X_0 be complete. Consider some $b \in B^*$ and any $\mathcal{L} \in \mathrm{Pic}'(X_b)$. Then we have*

(i) $h^i(\bar{\pi}_{b,s}(\mathcal{L})) = h^i(\mathcal{L})$ for all $i \geq 0$ and any $s \in B^*$;

(ii) $h^i(\bar{\pi}_{b,0}(\mathcal{L})) \geq h^i(\mathcal{L})$ *for all* $i \geq 0$*;*

(iii) $\chi(\bar{\pi}_{b,s}(\mathcal{L})) = \chi(\mathcal{L})$ *for any* $s \in B$*.*

Proof. Consider $h^{(b)} \in \mathrm{CaSF}'(\mathcal{S}^{(b)})$ with $D_{h^{(b)}} \cong \mathcal{L}$. Then $\mathcal{O}(D_h^{\mathrm{tot}})$ is a line bundle on X^{tot} and thus flat over B, since π is flat. One easily checks that for $s \in B^*$, $h^i(\mathcal{O}(D_{h^{(s)}})) = h^i(\mathcal{L})$; this can be seen for example by comparing Čech cohomology. Now since X_0 is complete, we have that π is proper by proposition 3.3.1. The theorem then follows from the corollary in [Mum70] section II.5, since $\mathcal{O}(D_{h^{(s)}}) = \mathcal{O}(D_h^{\mathrm{tot}})_{|X_s}$ for all $s \in B$. $\square$

Similarly, if X_0 is complete, $\pi_{b,s}$ preserves intersection numbers:

Theorem 5.1.9. *Let* X_0 *be complete of dimension* n *and consider invariant divisors* $D^1, \dots, D^n \in \mathrm{T\text{-}CDiv}'(X_b)$ *for some* $b \in B^*$*. Then for all* $s \in B$*, the intersection numbers*

$$\pi_{s,b}(D^1). \cdots .\pi_{s,b}(D^n)$$

agree.

Proof. By proposition 5.1.2, we can lift the divisor D^i to a divisor $\tilde{D}^i$ on X^{tot}. Define γ to be the one-cycle class attained by intersecting the divisors $\tilde{D}^1, \dots, \tilde{D}^n$. Then γ_s, the restriction of γ to X_s, is the intersection of all $\pi_{b,s}(D^i)$. Thus, $\deg(\gamma_s)$ is the desired intersection number. The theorem then follows from a direct application of proposition 10.2 in [Ful98]. $\square$

Finally, $\pi_{b,s}$ maps canonical divisors to canonical divisors:

Theorem 5.1.10. *If for some* $b \in B^*$, $K \in \mathrm{T\text{-}CDiv}'(X_b)$ *is a canonical divisor on* X_s*, then* $\pi_{b,s}(K)$ *is a canonical divisor on* X_s *for all* $s \in B$*.*

Proof. If $K \in \mathrm{T\text{-}CDiv}'(X_b)$, we can assume (after possible modification with an invariant principal divisor) that it is of the form stated in theorem 3.19 of [PS08]. Coupled with proposition 3.16 of [PS08], we have that $K = D_{h^{(b)}}$, with $h^{(b)} \in \mathrm{CaSF}'(X_s)$ defined as follows:

(i) For $P \in Y \setminus \{0\}$ and v a vertex in $\mathcal{S}_P^{(b)}$, $h_P^{(b)}(v) = -1 + 1/\mu(v)$;

(ii) For $Q = 0$ and v a vertex in $\mathcal{S}_Q^{(b)}$, $h_Q^{(b)}(v) = 1 + 1/\mu(v)$;

(iii) $\mathrm{tail}(h^{(b)})$ has slope 1 along every ray of the tailfan of $\mathcal{S}^{(b)}$.

Indeed, this follows immediately by taking $K_Y = -2 \cdot \{0\}$ in theorem 3.19 of [PS08]. For $s \neq 0$, it immediately follows that $D_{h^{(s)}}$ is canonical. On the other hand, one easily checks then that $h^{(0)} \in \mathrm{CaSF}(X_0)$ is the support function defined by:

(i) For $P \in Y \setminus \{0\}$ and v a vertex in $\mathcal{S}_P^{(0)}$, $h_P^{(0)}(v) = -1 + 1/\mu(v)$;

(ii) For v a vertex in $\mathcal{S}_0^{(0)}$, $h_0^{(0)}(v) = 1 + 1/\mu(v)$;

(iii) $(h^{(0)})^0$ has slope 1 along every ray of the tailfan of $\mathcal{S}^{(0)}$.

Indeed, (i) and (iii) are immediate, and (ii) follows from the fact that any vertex $v \in \mathcal{S}_0^{(0)}$ is the sum of vertices of $\mathcal{S}_0^{(b)}$ and $\mathcal{S}_b^{(b)}$, one of which must be a lattice point. Taking again $K_Y = -2 \cdot \{0\}$, we see that $D_{h^{(0)}}$ is also canonical. $\square$

Example 5.1.11 (A compactification of $\overline{C(dP_6)}$)**.** We return to example 3.3.3 and consider the deformation π of $X_0 = \overline{C(dP_6)} = X(\Xi)$ to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathcal{S}$ be any divisorial fan with $\Xi(\mathcal{S}) = \Xi$ together with some Minkowski decomposition corresponding to π. We first observe that for $b \neq 0$, T-CDiv$'(X_b) \cong \mathbb{Z}^3 \times \mathrm{Div}^0(\mathbb{P}^1)$. Indeed, for $a_1, a_2, a_3 \in \mathbb{Z}$, let $h^{(b)}[a_1, a_2, a_3] \in \mathrm{CaSF}'(\mathcal{S}^{(b)})$ be the support function taking respective values $-a_1, -a_2, -a_3$ on the vertices $(0,0), (0,1), (-1,0)$ of $\mathcal{S}_0^{(b)}$, taking respective values $0, a_2 - a_1, a_3 - a_1$ on the vertices $(0,0), (0,-1), (1,0)$ of $\mathcal{S}_b^{(b)}$ and taking value 0 on all other vertices. Note that this completely determines $h^{(b)}[a_1, a_2, a_3]$. It is then obvious that any element of $\mathrm{CaSF}'(\mathcal{S}^{(b)})$ can be written uniquely as $h^{(b)}[a_1, a_2, a_3] + P$ for some $a_1, a_2, a_3 \in \mathbb{Z}$ and $P \in \mathrm{Div}^0(\mathbb{P}^1)$. This gives the above isomorphism.

On the other hand, we also have that T-CDiv$(X_0) =$ T-CDiv$'(X_0) \cong \mathbb{Z}^3 \times \mathrm{Div}^0(\mathbb{P}^1)$. Indeed, for $a_1, a_2, a_3 \in \mathbb{Z}$, let $h^{(0)}[a_1, a_2, a_3] \in \mathrm{CaSF}'(\mathcal{S}^{(0)})$ be the support function taking respective values $-a_1, -a_2, -a_3$ at $(0,0), (0,1), (-1,0)$ of $\mathcal{S}_0^{(0)}$ and with value 0 on the vertex 0 of all other slices. As before, this completely determines $h^{(0)}[a_1, a_2, a_3]$ and as above, any element of $\mathrm{CaSF}'(\mathcal{S}^{(0)})$ can be written uniquely as $h^{(0)}[a_1, a_2, a_3] + P$. Now, if we set $h^{(b)} = h^{(b)}[a_1, a_2, a_3] + P$, then one easily checks that $h^{(0)} = h^{(0)}[a_1, a_2, a_3] + P$. Thus, in this case, the map $\pi_{b,0}$ is injective and thus an isomorphism. Factoring out by linear equivalence, we then have $\mathrm{Pic}(X_0) \cong \mathrm{Pic}'(X_b) \cong \mathbb{Z}$. Note that in this example $\mathrm{Pic}'(X_b) \neq \mathrm{Pic}(X_b)$, which is of course just $\mathbb{Z}^3$.

Now, $-K^{(b)} := D_{h^{(s)}[2,2,2]}$ is an anticanonical divisor for X_b. Then

$$-K^{(0)} := D_{h^{(0)}[2,2,2]} = \pi_{s,0}(-K^{(s)}),$$

and one easily checks that this is in fact an anticanonical divisor for X_0. Since both X_0 and X_b are toric Fano varieties, the higher cohomology groups of $-K^{(0)}$ and $-K^{(b)}$ vanish, so that in this case we actually have $h^i(-K^{(0)}) = h^i(-K^{(b)})$ for all $i \geq 0$, in particular for $i = 0$.

5.2 Surjectivity of $\pi_{b,0}$

The goal of this section is to establish the surjectivity of the map $\pi_{b,0}$ in certain cases. We first have the following proposition:

Proposition 5.2.1. *The map $\pi_{b,0}$ is surjective if* $\operatorname{rank} \mathrm{Pic}'(X_b) = \operatorname{rank} \mathrm{Pic}'(X_0)$ *and $|\mathcal{S}_0|$ is connected.*

Proof. We assume that $\alpha = 1$; the proof for $\alpha > 1$ is similar. Now, the map $\mathrm{Pic}'(X_b) \otimes \mathbb{Q} \to \mathrm{Pic}'(X_0) \otimes \mathbb{Q}$ induced by $\pi_{b,0}$ is an isomorphism. However, one easily checks that $\mathrm{Pic}'(X_0)$ is torsion free, since multiples of a non-constant support function are still non-constant, and $\mathrm{Pic}(\mathbb{P}^1)$ is torsion free. Thus, given any support function $f \in \mathrm{CaSF}'(\mathcal{S}^{(0)})$, we can find a not necessarily integral support function $\tilde{h}^{(b)} \in \mathrm{CaSF}'_{\mathbb{Q}}(\mathcal{S}^{(b)})$ with $\tilde{h}^{(0)} = f'$, where $\mathrm{CaSF}_{\mathbb{Q}}$ is defined as CaSF without the integrality condition, and $f - f'$ is a principal support function. We can correct $\tilde{h}^{(b)}$ by a principal support function to in fact assume that $f = f'$. It is clear from the construction of $\tilde{h}^{(0)}$ from $\tilde{h}^{(b)}$ that $\tilde{h}^{(b)}$ must have integral slopes. Now consider some vertex $e \in \mathcal{S}_0$ with decomposition $e = e^0 + e^1$, with (without loss of generality) e^0 a lattice point. Set $h_0^{(b)} = \tilde{h}_0^{(b)} - \tilde{h}_0^{(b)}(e^0)$, $h_b^{(b)} = \tilde{h}_b^{(b)} + \tilde{h}_0^{(b)}(e^0)$, and $h_P^{(b)} = \tilde{h}_P^{(b)}$ for $P \neq 0, b$.

We claim that $h_P^{(b)}$ is an integral support function for all P. Indeed, for $P \neq 0, b$ this is immediate. Furthermore, for any $\Delta \in \mathcal{S}_0$, there is $u \in M$ and $a \in \mathbb{Z}$ such that

f_0 restricted to Δ is given by $a + \langle \cdot, u\rangle$. By construction there are $a_0, a_b \in \mathbb{Q}$ such that $a = a_0 + a_b$ and $h_0^{(b)}$ (respectively $h_b^{(b)}$) restricted to Δ^0 (or Δ^1) is given by $a_0 + \langle \cdot, u\rangle$ (respectively $a_b + \langle \cdot, u\rangle$). Note that if $a_0 \in \mathbb{Z}$, then $a_b \in \mathbb{Z}$ and vice versa. Thus, for each Δ, we must show that either $a_0 \in \mathbb{Z}$ or $a_b \in \mathbb{Z}$. Note that if Δ^0 or Δ^1 contains a lattice point on which $h_0^{(b)}$ or $h_b^{(b)}$ takes an integral value, then a_0 respectively a_b is integral as well. Thus, $h_0^{(b)}$ and $h_b^{(b)}$ are integral on any Δ'^i, where $e_0 \in \Delta'^0$. Now, for any general $\Delta \in \mathcal{S}_0$ intersecting such a Δ', $\Delta^j \cap \Delta'^j$ must contain a lattice point for either $j = 0$ or $j = 1$, see remark 2.2.2. Thus, $h_0^{(b)}$ and $h_b^{(b)}$ are integral on such Δ^i as well. Proceeding by induction using the connectedness of $\mathcal{S}_0$ completes the claim.

We can thus conclude that $h^{(b)} := \sum h_P^{(b)} \otimes P$ is an element of $\mathrm{CaSF}'(\mathcal{S}^{(b)})$ with $h^{(0)} = f$, which completes the proof. $\square$

We will use proposition 5.2.1 to show the following:

Theorem 5.2.2. *Suppose that $|\mathcal{S}_0|$ is convex. Then the map $\pi_{b,0} : \text{T-CDiv}'(X_b) \to \text{T-CDiv}'(X_0)$ is surjective and thus induces an isomorphism $\bar{\pi}_{b,0} : \mathrm{Pic}'(X_b) \to \mathrm{Pic}'(X_0)$.*

Proof. Consider some $f \in \mathrm{CaSF}'(\mathcal{S}^{(0)}$. Although $|\Gamma_{f_0}|$ isn't convex, we can still define a cone of Minkowski summands as in definition 3.1.5 and one easily checks that all relevant results of section 3.1 still hold. Furthermore, the cones $\mathrm{cone}_{\mathrm{MS}}(\mathcal{S}_0)$ and $\mathrm{cone}_{\mathrm{MS}}(\Gamma_{f_0})$ are naturally isomorphic. In particular, the (α)-admissible decomposition of $\mathcal{S}_0$ induces an (α)-term decomposition of Γ_{f_0}, see the discussion following definition 3.1.5. But such a decomposition corresponds to an $h^{(b)} \in \mathrm{CaSF}'(\mathcal{S}^{(b)}) \otimes \mathbb{Q}$, that is, we have relaxed all integrality conditions; furthermore, $f = h^{(0)}$, see remark 5.1.4. But some multiple of $h^{(b)}$ will be in $\mathrm{CaSF}'(\mathcal{S}^{(b)})$, so $\mathrm{rank}\,\mathrm{Pic}'(X_b) = \mathrm{rank}\,\mathrm{Pic}'(X_0)$. The theorem then follows from proposition 5.2.1. $\square$

We can reformulate the theorem as follows:

Corollary 5.2.3. *Suppose that $|\mathcal{S}_0|$ is convex. Then for any line bundle $\mathcal{L}$ on $X(\mathcal{S})$, the fiber over π of the natural transformation $\mathrm{Def}_{(X(\mathcal{S}),\mathcal{L})} \to \mathrm{Def}_{X(\mathcal{S})}$ is nonempty.*

Remark 5.2.4. If $\mathcal{S}_0$ is not convex, there are simple examples such that $\pi_{b,0}$ is not surjective. One such example can be attained by a modification of example 5.1.11. Indeed, let X_0 be the open subset of $\overline{C(dP_6)}$ coming from the divisorial fan where we leave out any polyhedral divisors which give singular T-varieties. The single nontrivial slice of such a divisorial fan is as pictured in figure 3.4(a) with the omission of the hexagon in the middle. We then consider the deformation π of X_0 gotten by restricting our previous deformation of $\overline{C(dP_6)}$. Note that π is now locally trivial. One easily checks that with respect to this deformation, $\text{T-CDiv}(X_b) = \text{T-CDiv}'(X_b) \cong \mathbb{Z}^5 \times \mathrm{Div}^0(\mathbb{P}^1)$ whereas $\text{T-CDiv}(X_0) = \text{T-CDiv}'(X_0) \cong \mathbb{Z}^6 \times \mathrm{Div}^0(\mathbb{P}^1)$, and thus that $\pi_{b,0}$ isn't surjective.

Chapter 6

Homogeneous Deformations of Rational $\mathbb{C}^*$-Surfaces

We will now apply the techniques of chapters 3, 4, and 5 to study T-deformations of and families of invariant divisors on smooth, complete, rational surfaces with $\mathbb{C}^*$ action, henceforth referred to as rational $\mathbb{C}^*$-surfaces. In section 6.1 we recall some preliminary facts about toric and rational $\mathbb{C}^*$-surfaces. In section 6.2, we discuss T-deformations of these surfaces and how they can be blown up and down. We prove in section 6.3 that for a fixed Picard number larger than two, all rational $\mathbb{C}^*$-surfaces can be connected via T-deformations. Finally, in section 6.4, we give an explicit isomorphism of Picard groups preserving the intersection pairing for rational $\mathbb{C}^*$-surfaces with equal Picard number.

6.1 Rational $\mathbb{C}^*$-Surfaces

A rational $\mathbb{C}^*$-surface X is described by a complete marked fansy divisor Ξ on $\mathbb{P}^1$, with N a rank one lattice. For the rest of this chapter, we will thus fix an isomorphism $N \cong \mathbb{Z}$. In this case, $\mathrm{tail}(\Xi)$ is generated by the cones $\mathbb{Q}_{\geq 0}$ and $\mathbb{Q}_{\leq 0}$. Thus, to present the data of Ξ, we can simply draw the slices of Ξ and note which of the cones $\mathbb{Q}_{\geq 0}$, $\mathbb{Q}_{\leq 0}$ has a mark. In figures, we put a $\bullet$ on the right or left hand side to denote that $\mathbb{Q}_{\geq 0}$ or respectively $\mathbb{Q}_{\leq 0}$ is marked.

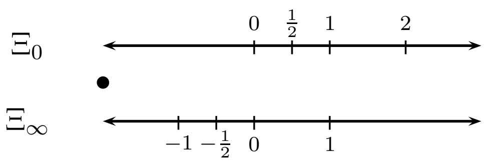


Figure 6.1: A possible marked fansy divisor on $\mathbb{P}^1$.

Example 6.1.1 (A blowup of $\mathbb{P}^2$)**.** Consider the marked fansy divisor pictured in figure 6.1. This corresponds to the rational $\mathbb{C}^*$-surface obtained by taking the toric variety $\mathbb{P}^2$, blowing up in two of the three toric fixpoints, and then blowing up in the four resulting fixpoints of the exceptional divisors. Since this marked fansy divisor only has two non-trivial slices, $X(\Xi)$ is in fact still toric.

In general, complete $\mathbb{C}^*$-surfaces can also be described in terms of weighted graphs, see [OW77]. If the surface is in fact a toric variety, then this graph is circular, see

[Ful93]. Up to isomorphism we can thus represent a smooth, complete toric surface X by some sequence $(b_0,\ldots,b_l)$, where the $-b_i$ are the self intersection numbers of the torus invariant prime divisors ordered in a suitable manner. In this case, we simply write $X = \mathrm{TV}(b_0,\ldots,b_l)$. Alternatively, if $\Sigma \subset \mathbb{Z}^2 \otimes \mathbb{Q}$ is some fan such such that X is the associated toric surface, we write $X = \mathrm{TV}(\Sigma)$. We then denote by ρ_i the ray of Σ corresponding to b_i. Note then that $b_i \cdot \rho_i = \rho_{i-1} + \rho_{i+1}$ and that the Picard number $\rho(X)$ of X is $l-1$. Furthermore, one has has the equality $\sum b_i = 3l - 9$.

Let $c_1, c_2, \ldots, c_k \in \mathbb{Z}$. The continued fraction $[c_1, c_2, \ldots, c_k]$ is inductively defined as follows if no division by 0 occurs: $[c_k] = c_k$, $[c_1, c_2, \ldots, c_k] = c_1 - 1/[c_2,\ldots,c_k]$. Now consider some $l+1$ tuple $(b_0,\ldots,b_l)$ defining a smooth toric surface, and suppose that $b_0 < 0$ and $l > 2$. Using induction on l, one can easily show that there exists a unique index ξ, $1 < \xi < l$ such that $[b_1,\ldots,b_{\xi-1}]$ is well defined and equals zero, or equivalently, that $\rho_\xi = -\rho_0$. If we are in this situation, we define

$$\gamma = \sum_{i=1}^{\xi-1}(3 - b_i) - 3. \tag{6.1.1}$$

We will use the following lemma in the next section:

Lemma 6.1.2. *We always have $\gamma \geq 0$. Likewise, $b_0 + b_\xi - \gamma \geq 0$. Finally, for $R \in (\mathbb{Z}^2)^*$ such that $\langle \rho_\xi, R\rangle = 1$, we have $\langle \rho_{\xi-1}, R\rangle - \langle \rho_1, R\rangle = \gamma$.*

Proof. All statements can be easily shown by induction on the number of rays in Σ. □

6.2 Degeneration Diagrams

Let $\pi : X^{\mathrm{tot}} \to B$ be a one-parameter T-deformation of the rational $\mathbb{C}^*$-surface X_0. For any $s \in B^*$, we say that X_0 *T-deforms* to $X_s = \pi^{-1}(s)$. Conversely, we say that X_s *T-degenerates* to X_0. By an abuse of terminology we will call any such X_s a general fiber. We now slightly reverse our perspective; instead of describing a way of deforming X_0, we describe degenerations of some X_s to X_0.

Definition 6.2.1. Let Ξ be a marked fansy divisor on $\mathbb{P}^1$ giving a rational $\mathbb{C}^*$-surface and choose some $s \in \mathbb{P}^1$, $s \neq 0$. A *degeneration diagram* for Ξ consists of the pair (Ξ, G), where G is a connected graph on the vertices of Ξ_0 and Ξ_s such that:

(i) G is bipartite with respect to the natural partition induced by Ξ_0 and Ξ_s;

(ii) G can be realized in the plane with all edges being line segments by embedding Ξ_0 and Ξ_s in parallel lines;

(iii) Every vertex of G with valence strictly larger than one is a lattice point.

To a degeneration diagram (Ξ, G) we can associate a deformation π as follows. Let $\Xi^{(0)}$ be the marked fansy divisor with marks identical to Ξ, $\Xi^{(0)}_P = \Xi_P$ for $P \in \mathbb{P}^1 \setminus \{0, s\}$, $\Xi^{(0)}_s$ trivial, and $\Xi^{(0)}_0$ the subdivision of $\mathbb{Q}$ with vertices of the form $v_0 + v_s$, with $\overline{v_0 v_s}$ an edge of G. Each polyhedron $[v, w]$ in the slice $\Xi^{(0)}_0$ comes with a natural decomposition $[v, w] = [v_0, w_0] + [v_s, w_s]$, where $v, w \in \Xi_P$ and $\overline{v_0 v_s}$, $\overline{w_0 w_s}$ are edges of G. This gives an admissible decomposition of $\Xi^{(0)}_0$ and thus a deformation $\pi : X^{\mathrm{tot}} \to B$ of $X(\Xi^{(0)})$, with $X_s = X(\Xi)$. Conversely, one easily checks that any one-parameter T-deformation of a rational $\mathbb{C}^*$-surface with fiber $X_s = X(\Xi)$ arises from a degeneration diagram of the form (Ξ, G) (up to isomorphism).

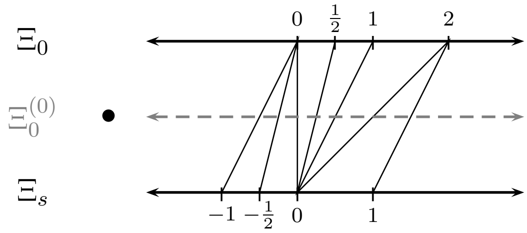


Figure 6.2: A possible degeneration diagram

Example 6.2.2 (A blowup of $\mathbb{P}^2$)**.** Figure 6.2 presents a degeneration diagram for the marked fansy divisor Ξ of figure 6.1 with $s = \infty$. The resulting slice $\Xi_0^{(0)}$ can then be seen quite easily as the induced subdivision on the dashed line in between Ξ_0 and Ξ_s scaled by a factor of two.

To distinguish between $\mathbb{C}^*$-invariant Weil divisors on X_0 and X_s, we write them with a superscript, i.e. $D^{(0)}_{(v,P)}$ and $D^{(s)}_{(v,P)}$, etc. We shall now see how such deformations are compatible with blowing up and blowing down. We will need the following lemma:

Lemma 6.2.3. *Let* (Ξ, G) *be a degeneration diagram. For any edge* $\overline{v_0 v_s}$ *of* G*,* $v_0 \in \Xi_0, v_s \in \Xi_s$*, with* $(D^{(0)}_{(v_0+v_s,0)})^2 = -1$*, one of the vertices* v_0, v_s *must have valence one. If both vertices have valence one, the deformation corresponding to* (Ξ, G) *is trivial.*

Proof. If $v_0 + v_s$ lies to the left or to the right of all other vertices of $\Xi_0^{(0)}$, then the edge $\overline{v_0 v_s}$ lies to the left or right of all other edges of G as well; it is then clear that either v_0 or v_s must have valence one. If on the other hand $v_0 + v_s$ has left and right neighboring vertices $v', v'' \in \Xi_0^{(0)}$, then $\mu(v_0 + v_s) = \mu(v') + \mu(v'') > 1$. If neither v_0 nor v_s has valence one, they must both be lattice points, in which case $\mu(v_0 + v_s) = 1$, a contradiction.

The second claim follows easily from the observation that if both vertices have valence one, G can only have one edge. □

Using this lemma, it is clear how to *blow down* any T-deformation. Indeed, let π correspond to the degeneration diagram (Ξ, G), and let $\phi : X_0 \to X_0'$ be the contraction of an invariant minus one curve.

Suppose first of all that this curve is of the form $D^{(0)}_{(v,P)}$ for $P \neq 0$ or of the form $\mathcal{D}_\rho$ for $\rho \in \{\mathbb{Q}_{\geq 0}, \mathbb{Q}_{\leq 0}\}$. Then we get a new marked fansy divisor Ξ' by respectively removing the vertex v from the subdivision Ξ_P or by adding a mark to ρ. Setting $G' = G$, we then have that (Ξ', G) is a degeneration diagram with $X(\Xi'^{(0)}) = X_0'$ and with $X_s' = X(\Xi')$ a blowdown of X_s.

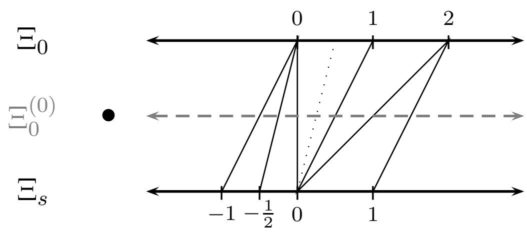


Figure 6.3: Blowing down a deformation

On the other hand, suppose that ϕ blows down a curve of the form $D^{(0)}_{(v,0)}$. Then v corresponds to an edge $\overline{v_0 v_s}$ of G and by the above lemma, either v_0 or v_s must have valence one; assume without loss of generality that this is v_0. We then get a new marked fansy divisor Ξ' by removing the vertex v_0 from the subdivision Ξ_0. Furthermore, we have a graph G' on Ξ' attained from G by removing the edge $\overline{v_0 v_s}$. Due to the fact that v_0 had valence one in G, one easily checks that (Ξ', G') is a degeneration diagram.

As in the other case, we have $X(\Xi'^{(0)}) = X'_0$ and $X'_s = X(\Xi')$ a blowdown of X_s. In this manner we define the *blowdown of* (Ξ, G) *by* ϕ to be (Ξ', G'). We call the deformation corresponding to (Ξ', G') the blowdown of π by ϕ.

Example 6.2.4 (Blowing down a deformation)**.** Consider the T-deformation π from example 6.2.2. The special fiber has a minus one curve corresponding to the point 1/2 in $\Xi^{(0)}_0$. The corresponding edge of G is pictured in figure 6.3 as a dotted line segment; by removing this edge and the vertex 1/2 of Ξ_0, we get the desired blowdown of π.

It is also possible to lift a T-deformation $\pi : X \to B$ by an invariant *blowup* ϕ of either the special fiber X_0 or the fiber X_s. Indeed, let (Ξ, G) be the corresponding degeneration diagram.

The first possible type of blowup of X_0 or X_s is by blowing up in an elliptic fixpoint of the $\mathbb{C}^*$ action, that is, by removing a mark from either $\mathbb{Q}_{\geq 0}, \mathbb{Q}_{\leq 0}$. If we define Ξ' to be equal to Ξ with the relevant modification of the marks, we get a degeneration diagram (Ξ', G') with either $X(\Xi'^{(0)})$ or $X(\Xi')$ the desired blowup of X_0 or respectively X_s.

Suppose instead that the blowup of X_0 or X_s corresponds to inserting a vertex v in the subdivision $\Xi^{(0)}_P = \Xi_P$ for $P \neq 0, s$. Then if we define Ξ' to come from Ξ by adding the vertex v to Ξ_P and setting $G = G'$, we get a degeneration diagram (Ξ', G') with the same property as in the previous case.

Suppose now that a blowup of X_0 corresponds to inserting a vertex v in the subdivision $\Xi^{(0)}_0$. This corresponds to the insertion of a vertex $\tilde{v}$ in either Ξ_0 or Ξ_s, which in turn corresponds to a blowup of X_s.[1] So assume that we have a blowup of X_s of this form. Then we can define a marked fansy divisor Ξ' from Ξ similar to the previous cases. Likewise, we can define a graph G' on the vertices of Ξ'_0, Ξ'_s by adding an edge between $\tilde{v}$ and the unique vertex connected to all neighboring vertices of $\tilde{v}$.

This defines a degeneration diagram (Ξ', G') with the same property as above. In such cases, we call (Ξ', G') a *blowup of* (Ξ, G) *by* ϕ.

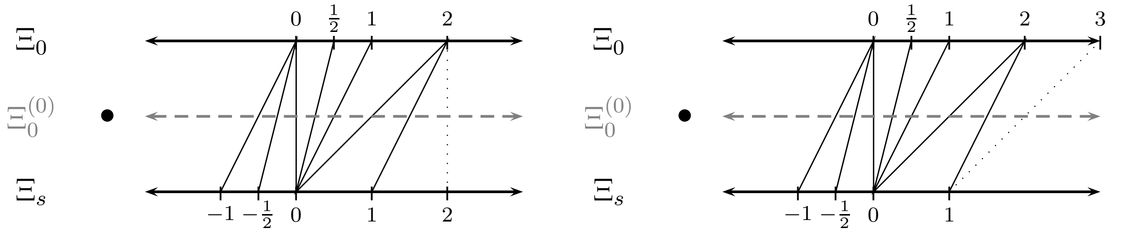


Figure 6.4: Blowing up a deformation

Example 6.2.5 (Blowing up a deformation)**.** Consider again the T-deformation π from example 6.2.2. We can blow up the special fiber by further subdividing $\Xi^{(0)}_0$ at the point

[1] Note that the placement of the vertex in either Ξ_0 or Ξ_s is uniquely determined if v isn't an extremal vertex of $\Xi^{(0)}_0$.

4. This leads to two possible blowups of π; the corresponding degeneration diagrams are pictured in figure 6.4, with the added edges being dotted.

We can sum up the preceding constructions in the following proposition:

Proposition 6.2.6. *Let (Ξ, G) be a degeneration diagram with corresponding special fiber X_0 and general fiber X_s.*

(i) If $\phi : X_0 \to X_0'$ is a blowdown of an invariant curve, there is a unique degeneration diagram (Ξ', G') called the blowdown of (Ξ, G) by ϕ such that $X(\Xi'^{(0)}) = X_0'$ and $X(\Xi')$ is an invariant blowdown of X_s.

(ii) if $\phi : X_0' \to X_0$ is an invariant blowup, there is a degeneration diagram (Ξ', G') called a blowup of (Ξ, G) by ϕ such that $X(\Xi'^{(0)}) = X_0'$ and $X(\Xi')$ is an invariant blowup of X_s.

(iii) if $\phi : X_s' \to X_s$ is an invariant blowup, there is a unique degeneration diagram (Ξ', G') called the blowup of (Ξ, G) by ϕ such that $X(\Xi') = X_s'$ and $X(\Xi'^{(0)})$ is an invariant blowup of X_0.

We will see in section 6.4 that these constructions commute with the corresponding maps of divisors. However, to end this section, we shortly turn our attention to T-deformations where all fibers are toric surfaces. The following theorem tells us how a number of these can be described nicely in terms of self-intersection numbers:

Theorem 6.2.7. *Consider a smooth complete toric surface $X_0 = \mathrm{TV}(b_0, \ldots, b_l)$ such that $b_0 < 0$ and $l > 2$. Then there exists a homogeneous deformation of X_0 with toric general fiber*

$$X_s = \mathrm{TV}(b_0 + \gamma + 2r, b_{\xi-1}, \ldots, b_1, b_\xi - \gamma - 2r, b_{\xi+1}, \ldots, b_l),$$

where ξ and γ are as defined in the previous section and $0 \leq r \leq -b_0$.

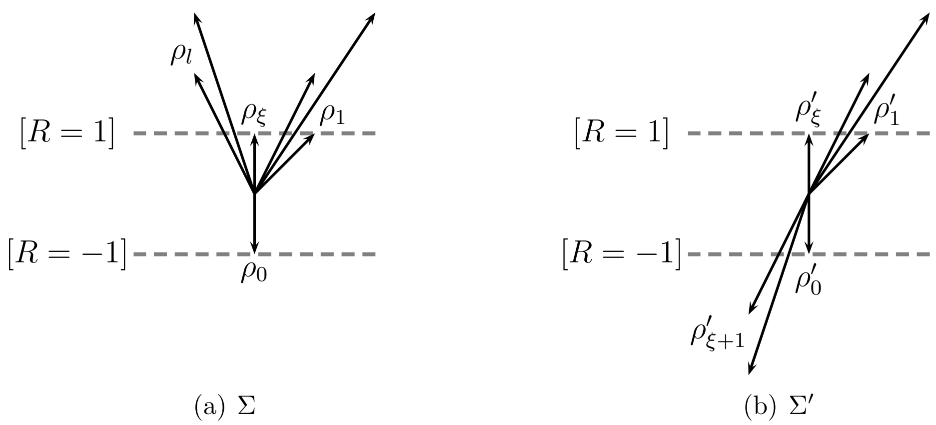


Figure 6.5: Possible fans from the proof of theorem 6.2.7

Proof. Let $X_0 = \mathrm{TV}(\Sigma)$ for some fan $\Sigma \subset \mathbb{Z}^2 \otimes \mathbb{Q}$ with rays ρ_i corresponding to the numbers b_i, see for example figure 6.5(a). Consider the unique $R \in (\mathbb{Z}^2)^*$ such that $\langle \rho_0, R \rangle = -1$ and $\langle \rho_1, R \rangle = r$. We will consider the $\mathbb{C}^*$-action of $T^{R^\perp}$. Corresponding to this action, we transform the fan Σ into a marked fansy divisor $\Xi^{(0)}$ by taking $\Xi_0^{(0)}$ to be the subdivision induced by Σ on the line $[R = 1]$, with ρ_ξ set as the origin, and by taking $\mathbb{Q}_{\leq 0}, \mathbb{Q}_{\geq 0}$ to be unmarked if and only if $r = 0$ or $r = -b_0$, respectively. Note that the condition on $0 \leq r \leq -b_0$ is exactly such that Σ subdivides the line $[R = -1]$ only with the ray ρ_0. Thus, we have $X_0 = X(\Xi^{(0)})$.

We now construct a degeneration diagram (Ξ, G) giving special fiber X_0. Indeed, let the vertices of Ξ_0 consist of those vertices $v \in \Xi_0^{(0)}$ with $v \geq 0$. Likewise, let the vertices of Ξ_s consist of those vertices $v \in \Xi_0^{(0)}$ with $v \leq 0$. Furthermore, take the marks of Ξ to be those of $\Xi^{(0)}$. We then define G to be the graph having edges $\overline{v_0 v_s}$ with $v_0 \in \Xi_0, v_s \in \Xi_s$ and either $v_0 = 0$ or $v_s = 0$. One easily confirms that (Ξ, G) is indeed a degeneration diagram with $\Xi_0^{(0)}$ being the marked fansy divisor for the corresponding special fiber.

Now, we see that the general fiber $X_s = X(\Xi)$ is toric, since Ξ only has two nontrivial slices. In fact, by embedding Ξ_0 and Ξ_s in height one and minus one respectively, we recover a fan Σ' with $X_s = X(\Xi) = X(\Sigma')$, see for example figure 6.5(b). Σ' then has rays $\rho'_0, \ldots, \rho'_l$ ordered cyclically with $\rho'_i = \rho_i$ for $\xi \leq i \leq l$ or $i = 0$ and ρ'_i a vertical reflection for $0 < i < \xi$. Let $-b'_i$ be the self-intersection number of the divisor corresponding to the ρ'_i; then X_s is represented by the chain $(b'_0, \ldots, b'_l)$. Now, it is immediate from this description that $b'_i = b_{\xi - i}$ for $1 \leq i < \xi$ and that $b'_i = b_i$ for $\xi < i \leq l$. Furthermore,

$$b'_\xi = \langle \rho'_{\xi-1}, R \rangle + \langle \rho'_{\xi+1}, R \rangle = -\langle \rho_1, R \rangle + \langle \rho_{\xi+1}, R \rangle. \tag{6.2.1}$$

We also have $\langle \rho_{\xi-1}, R \rangle = r + \gamma$ by lemma 6.1.2. We can then rewrite equation (6.2.1) as

$$b'_\xi = -r + b_\xi - \langle \rho_{\xi-1}, R \rangle = b_\xi - \gamma - 2r.$$

Since the sum of all the intersection numbers must remain constant, we also have $b'_0 = b_0 + \gamma + 2r$, completing the proof.

$\square$

6.3 Deformation Connectedness

Let X and X' be two rational $\mathbb{C}^*$-surfaces.

Definition 6.3.1. We say that X and X' are *T-deformation connected* if there is a finite sequence $X = X^0, X^1, \ldots, X^k = X'$ with X^i T-deforming to X^{i-1} or X^{i-1} T-deforming to X^i for each $1 \leq i \leq k$.

It is well-known that a Hirzebruch surface of even parity cannot be deformed to a Hirzebruch surface of odd parity and vice versa. An obstruction to such a deformation can be found by comparing the Chow rings. If we instead consider rational surfaces of fixed Picard number $\rho > 2$, it is an easy exercise to see that all the Chow rings are isomorphic. Thus, the obstruction to deformation we had for the case $\rho = 2$ no longer exists. In fact, for rational $\mathbb{C}^*$-surfaces it is sufficient to consider T-deformations:

Theorem 6.3.2. *Consider the set of all rational $\mathbb{C}^*$-surfaces with Picard number ρ for any integer $\rho > 2$. All elements of this set are homogeneously deformation connected.*

The proof of this theorem will constitute the remainder of this section. We first prove the following lemma:

Lemma 6.3.3. *Any rational $\mathbb{C}^*$-surface X can be degenerated to a toric surface via a finite number of T-degenerations.*

Proof. Let Ξ be a marked fansy divisor with $X = X(\Xi)$. Suppose that Ξ has more than three non-trivial slices. Then there are non-trivial slices Ξ_P, Ξ_Q with $P \neq Q$ such that the left-most vertex w_P of Ξ_P and the right-most vertex w_Q of Ξ_Q are lattice points; this follows from the smoothness criterion of 1.5.2 and 1.5.3. Setting $0 = P, s = Q$ and considering the graph G on the vertices of Ξ_P, Ξ_Q with edges of the form $\overline{v_0 w_Q}$ and $\overline{v_s w_P}$ for $v_0 \in \Xi_P$, $v_s \in \Xi_Q$ gives a degeneration diagram (Ξ, G). The corresponding special fiber has one less non-trivial slice than X.

We can apply the above procedure inductively, and can thus assume that Ξ has at most three non-trivial slices. If Ξ has less than three non-trivial slices, then $X(\Xi)$ is toric, and we are done. If as above there are non-trivial slices Ξ_P, Ξ_Q with $P \neq Q$ such that the left-most vertex v_P of Ξ_P and the right-most vertex v_Q of Ξ_Q are lattice points, then we can once again proceed as above and degenerate to something with only two non-trivial slices. We thus must only consider the remaining case, which is that where Ξ has three non-trivial slices Ξ_0, Ξ_1, Ξ_∞ and Ξ_0, Ξ_∞ have no extremal lattice vertices and both extremal vertices of Ξ_1 are lattice points. We show that this is actually impossible.

In this case, we can actually assume that the left-most vertex of Ξ_1 is 0, and that the right-most vertex is n. Let u_0^l/v_0^l, u_∞^l/v_∞^l be the left-most vertices of Ξ_0 and Ξ_∞ written in lowest terms and let u_0^r/v_0^r, u_∞^r/v_∞^r similarly be the right-most vertices. Due to smoothness we have

$$-u_0^l v_\infty^l - u_\infty^l v_0^l = 1; \tag{6.3.1}$$

$$u_0^r v_\infty^r + u_\infty^r v_0^r + v_0^r v_\infty^r \cdot n = 1. \tag{6.3.2}$$

Furthermore, we of course have

$$u_P^l v_P^r \leq u_P^r v_P^l \tag{6.3.3}$$

for $P = 0, \infty$. Solving equations (6.3.1) and (6.3.2) for v_0^l and v_0^r, substituting for these expressions in (6.3.3) for $P = 0$, and rearranging terms gives us

$$v_0^r v_\infty^r + v_0^l v_\infty^l + u_\infty^l v_\infty^r v_0^r v_0^l \geq u_\infty^r v_\infty^l v_0^l v_0^r + v_0^l v_\infty^l v_0^r v_\infty^r n.$$

Combining this with (6.3.3) for $P = \infty$ then gives us

$$v_0^r v_\infty^r + v_0^l v_\infty^l \geq v_0^l v_\infty^l v_0^r v_\infty^r n.$$

This however is a contradiction, since $n \geq 1$ and $v_0^l, v_\infty^l, v_0^r, v_\infty^r \geq 2$. Thus, this case never arises and we can always degenerate to a toric surface. □

In general, one can always construct a rational surface by iteratively blowing up a Hirzebruch surface in a number of points. This can in fact be done equivariantly for rational $\mathbb{C}^*$-surfaces. For marked fansy divisors Ξ with no marks, this is stated in [OW77]. However, we know of no proof of the general case and thus provide one here as an easy corollary of the above lemma:

Corollary 6.3.4. *Any rational $\mathbb{C}^*$-surface X with Picard number larger than two can be constructed from a Hirzebruch surface by a series of equivariant blowups.*

Proof. Suppose that $X = X_s$ isn't a Hirzebruch surface. By lemma 6.3.3, we know that X degenerates to some toric variety X_0 via T-deformations. But there is an invariant minus one curve on X_0 which can be blown down, since X_0 is toric, see [Ful93]. Blowing down the deformations from X_0 to X_s as in proposition 6.2.6 gives us a new general fiber X'_s which is an invariant blowdown of X_s. The proof then follows by induction on the Picard number. $\square$

We will collect several more lemmata we shall need:

Lemma 6.3.5. *Consider a smooth fan Σ with rays $\rho_0, \ldots, \rho_l$. Let Σ_1 and Σ_2 be the smooth fans attained by inserting a ray between ρ_0 and ρ_1 respectively ρ_1 and ρ_2. Then* $\mathrm{TV}(\Sigma_1)$ *is T-deformation connected to* $\mathrm{TV}(\Sigma_2)$.

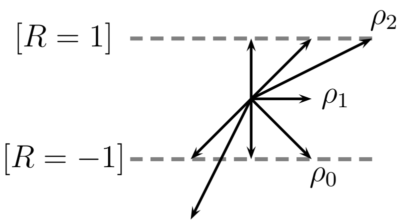


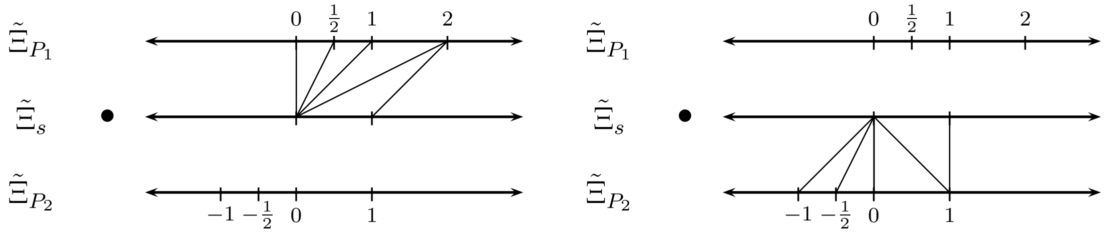
Figure 6.6: Example fan in proof of lemma 6.3.5

Proof. As in the proof of theorem 6.2.7, we transform Σ into a marked fansy divisor Ξ. For $P_1 \neq P_2 \in \mathbb{P}^1$, let Ξ_{P_1} and Ξ_{P_2} to be the subdivisions induced by Σ on the affine lines $[R = 1]$ and $[R = -1]$, where $R \in (\mathbb{Z}^2)^*$ is such that $\langle \rho_1, R \rangle = 0$ and $\langle \rho_0, R \rangle < 0$. We do not mark $\mathbb{Q}_{\geq 0}$, and mark $\mathbb{Q}_{\leq 0}$ unless there is $\xi \neq 1$ such that $\langle \rho_\xi, R \rangle = 0$. Then $X(\Xi) = X(\Sigma)$. See for example figure 6.6.

Now, for some $s \in \mathbb{P}^1 \setminus \{P_1, P_2\}$, let $\tilde{\Xi}$ be the marked fansy divisor with $\tilde{\Xi}_P = \Xi_P$ for $P \neq s$, and $\tilde{\Xi}_s$ the subdivision of $\mathbb{Q}$ with vertices 0 and 1. For $i = 1, 2$, let G_i be the graph on the vertices of $\tilde{\Xi}_s$ and $\tilde{\Xi}_{P_i}$ with edges $\overline{v_s w}$ for either vertices $v_s = 0 \in \tilde{\Xi}_s$ and $w \in \tilde{\Xi}_{P_i}$ or vertices $v_s = 1 \in \tilde{\Xi}_s$ and w the right-most vertex in Ξ_{P_i}. Setting $P_i = 0$, one easily checks that $(\tilde{\Xi}, G_i)$ is a degeneration diagram with general fiber $X(\tilde{\Xi})$ and special fiber $\mathrm{TV}(\Sigma_i)$, see for example figure 6.7. Thus, we have T-deformations from both $\mathrm{TV}(\Sigma_1)$ and $\mathrm{TV}(\Sigma_2)$ to some common rational $\mathbb{C}^*$-surface, making them T-deformation connected. $\square$

Remark 6.3.6. The two deformations constructed in the above proof can be naturally glued together to give a flat family X^{tot} over $\mathbb{P}^1$ with fibers $X_0 = \mathrm{TV}(\Sigma_1)$ and $X_\infty = \mathrm{TV}(\Sigma_2)$. In this family, the fiber over any point $s \in \mathbb{P}^1$ is simply the blowup of $\mathrm{TV}(\Sigma)$ in s, where we have identified the base space $\mathbb{P}^1$ with the divisor corresponding to the ray ρ_1.

Lemma 6.3.7. *The set* $\{\mathrm{TV}(b_0, 0, b_\xi, 1, 1) \mid b_0 + b_\xi = 3\}$ *is T-deformation connected.*



(a) $i = 1$ (b) $i = 2$

Figure 6.7: Example degeneration diagrams in proof of lemma 6.3.5

Proof. Consider b_0, b_ξ and b'_0, b'_ξ such that $b_0 + b_\xi = b'_0 + b'_\xi = 3$. Due to symmetry we can assume that b_0, b'_0 have the same parity. Let $b''_0 = -\max\{|b_0|, |b'_0|\}$. Then $\mathrm{TV}(b''_0, 0, n - b''_0, 1, 1)$ deforms to both $\mathrm{TV}(b_0, 0, b_\xi, 1, 1)$ and to $\mathrm{TV}(b'_0, 0, b'_\xi, 1, 1)$ by theorem 6.2.7, so the desired set is T-deformation connected. □

We now turn to the proof of the theorem:

Proof of theorem 6.3.2. We will prove the theorem by induction on ρ. Suppose that $\rho = 3$. From lemma 6.3.3 we have that any rational $\mathbb{C}^*$-surface can be degenerated to a toric surface via T-deformations. Furthermore, one easily checks that every toric surface with Picard number 3 is of the form $\mathrm{TV}(b_0, 0, b_\xi, 1, 1)$. Thus, for $\rho = 3$ the statement then follows from lemma 6.3.7.

Assume that the theorem holds for Picard number ρ, and consider any two rational $\mathbb{C}^*$-surfaces X^1, X^2 with Picard number $\rho+1$. By again applying lemma 6.3.3, we can assume without loss of generality that X^1 and X^2 are toric. Let $\tilde{X}^i$ be an invariant blowdown of X^i. Then $\tilde{X}^1$ and $\tilde{X}^2$ are T-deformation connected by the induction hypothesis, and this series of deformations and degenerations can be blown up to connect $\hat{X}^1$ and $\hat{X}^2$, where $\hat{X}^i$ is an invariant blowup of $\tilde{X}^i$. Thus, we must only show that $\hat{X}^i$ and X^i are T-deformation connected, that is, any two invariant blowups in a point of a common toric surface are T-deformation connected. But this follows from repeated application of lemma 6.3.5, proving the theorem. □

6.4 An Isomorphism of Picard Groups

As in the previous sections, we will consider a one-parameter T-deformation $\pi : X^{\mathrm{tot}} \to B$ of a rational $\mathbb{C}^*$-surface X_0 with $X_s = X(\Xi)$ for some marked fansy divisor Ξ. The map $\pi_{s,0} : \text{T-CDiv}(X_s) \to \text{T-CDiv}(X_0)$ can be described quite nicely in terms of Weil divisors and the corresponding deformation diagram (Ξ, G).[2] Indeed, consider following proposition:

Proposition 6.4.1. *The map $\pi_{s,0}$ is defined by:*

$$D^{(s)}_\rho \mapsto D^{(0)}_\rho \qquad D^{(s)}_{(v_P,P)} \mapsto D^{(0)}_{(v_P,P)}$$
$$D^{(s)}_{(v_0,0)} \mapsto \sum_{\overline{v_0 v} \in E(G)} \mu(v) D^{(0)}_{(v_0+v,0)} \qquad D^{(s)}_{(v_s,s)} \mapsto \sum_{\overline{v_s v} \in E(G)} \mu(v) D^{(0)}_{(v_0+v,0)}$$

[2]Note that since everything here is smooth, we have T-CDiv = T-CDiv′, see the note at the end of definition 5.1.1.

for any $P \in S \setminus \{0, s\}$, $v_P \in \Xi_P$, $v_0 \in \Xi_0$, $v_s \in \Xi_s$, and $\rho \in \operatorname{tail}(\Xi)$, where $E(G)$ is the set of edges of the graph G.

Proof. This follows directly from the description of $h^{(0)}$ in section 5.1 and proposition 1.3.3. □

An important fact is that, in a sense, such a map of Cartier divisors is compatible with blowing up or down. More specifically, in the above setting, let $\phi_0 : X_0 \to X_0'$ be an invariant blowdown of a minus one curve with $E^{(0)}$ the corresponding exceptional divisor. Let π' be the blowdown of π by ϕ, with X_s' the general fiber of π'. From the description of the blowdown of a degeneration diagram, one easily confirms that we have an invariant blowdown $\phi_s : X_s \to X_s'$; let $E^{(s)}$ be the corresponding exceptional divisor.

Proposition 6.4.2. *In the above situation, $\pi_{s,0}(E^{(s)}) = E^{(0)}$. Furthermore, the following diagram commutes:*

$$\begin{tikzcd} \text{T-CDiv}(X_0) & \text{T-CDiv}(X_s) \arrow[l, "\pi_{s,0}"'] \\ \text{T-CDiv}(X_0') \arrow[u, "\phi_0^*"] & \text{T-CDiv}(X_s') \arrow[l, "\pi'_{s,0}"] \arrow[u, "\phi_s^*"'] \end{tikzcd}$$

Proof. The claim regarding the exceptional divisor follows from the description of the blowdown of a degeneration diagram and from proposition 6.4.1. Indeed, if (Ξ, G) is the degeneration diagram corresponding to π and $E^{(0)}$ corresponds to some edge e of G, then $E^{(s)}$ corresponds to the vertex of e with valence one, which then obviously maps to the desired divisor, since the other vertex of e must have height one. If on the other hand $E^{(0)}$ is some other divisor of X_0, the claim is immediate from proposition 6.4.1.

The commutativity of the diagram follows from the description of $h^{(0)}$ in section 5.1. Indeed, the pullback of an invariant Cartier divisor D_h on a T-variety $X(\Xi')$ to some blowup $X(\Xi)$ corresponds to the same piecewise affine function h. Furthermore, one easily sees from the description of $h^{(0)}$ that further refinement in a divisorial fan Ξ does not affect the construction of $h^{(0)}$. □

Example 6.4.3 (Hirzebruch surfaces)**.** We look at an explicit description of the map $\bar{\pi}_{s,0}$, where π is a deformation of a Hirzebruch surface. If $X(\Xi) = \mathcal{F}_r$, i.e. the rth Hirzebruch surface, and Ξ admits a non-trivial degeneration diagram, we can assume that Ξ_0 has vertices $-\frac{1}{r+\xi}$, 0 and that Ξ_s has vertices $0, \frac{1}{\xi}$ for some $\xi > 0$. We call this marked fansy divisor $\Xi(r, \xi)$. Note that with the exception of the case $r = 0, \xi = 1$, there is only one possible graph G making $(\Xi(r, \xi), G)$ into a degeneration diagram. Indeed, this is the bipartite graph where both 0 vertices have valence two and the other two vertices have valence one. For the case $r = 0, \xi = \pm 1$, there is also the possibility of the bipartite graph $\tilde{G}$ where both 0 vertices have valence one and the other two vertices, in this case lattice points, have valence two. In any case, the degeneration diagram $(\Xi(r, \xi), G)$ (or $(\Xi(r, \xi), \tilde{G})$) has corresponding special fiber $\mathcal{F}_{r+2\xi}$. The difference between G and $\tilde{G}$ corresponds to a flip on the total space of the deformation.

For any Hirzebruch surface $\mathcal{F}_r$ with $r > 0$, let η be the divisor class of the fiber of the ruling on $\mathcal{F}_r$, and let ζ be the unique class with $\zeta^2 = r$ and $\eta.\zeta = 1$. Now considering the isomorphism $X(\Xi(r, \xi)) \cong \mathcal{F}_r$, η and ζ can respectively be represented by $D^{(s)}_{(0,s)}$ and $D^{(s)}_{(1/\xi,s)}$. Consider now the deformation π from $\mathcal{F}_{r+2\xi}$ to $\mathcal{F}_r$ determined by the degeneration diagram $(\Xi(r, \xi), G)$ and assume $r > 0$. Then $\bar{\pi}_{s,0}(\eta)$ can be represented by

$D^{(0)}_{(1/\xi,0)}$ and $\bar{\pi}_{s,0}(\zeta)$ can be represented by $(r+\xi)D^{(0)}_{(-1/(r+\xi),0)} + D^{(0)}_{(0,0)}$. One easily checks that

$$\bar{\pi}_{s,0}(\eta) = \eta \tag{6.4.1}$$

$$\bar{\pi}_{s,0}(\zeta) = \zeta - \xi\eta \tag{6.4.2}$$

where by abuse of notation, the η, ζ on the right hand side of the equalities represent classes in $\mathrm{Pic}(\mathcal{F}_{r+2\xi})$.

The case of $r = 0$ requires slightly more care, since there are two possible rulings on $\mathcal{F}_0$. Fix an isomorphism $\mathcal{F}_0 \cong X(\Xi(0,\xi))$ and consider the ruling of $\mathcal{F}_0$ given by the quotient map of the $\mathbb{C}^*$-action on $X(\Xi(0,\xi))$; note that this doesn't depend on ξ. Then η and ζ can be represented exactly as above. For π corresponding to the degeneration diagram $(\Xi(0,\xi), G)$, we once again have equations (6.4.1) and (6.4.2). On the other hand, for π corresponding to the degeneration diagram $(\Xi(0,1), \tilde{G})$, we have $\bar{\pi}_{s,0}(\eta) = \zeta - \eta$ and $\bar{\pi}_{s,0}(\zeta) = \eta$. Thus, if in this case we instead consider the other possible ruling of $\mathcal{F}_0$ (and thus swap η and ζ), we once again have equations (6.4.1) and (6.4.2).

Chapter 7

Homogeneous Deformations of Projective T-Varieties

In this section we turn our attention exclusively to T-deformations of projective T-varieties. We first state some results regarding $\mathbb{C}^*$-quotients of complexity one T-varieties in section 7.1. In section 7.2 will show that one-parameter T-deformations of a projective variety can always be lifted to embedded deformations. In section 7.3 we approach deformations of projective T-varieties via decompositions of divisorial polytopes. This has the advantage of providing a completely combinatorial proof that Hilbert-Ehrhart polynomials remain constant under embedded T-deformation. Finally, in section 7.4, we use these techniques to provide a combinatorial proof that the Hilbert-Ehrhart polynomials of certain phylogenetic models are equal.

7.1 Quotients of T-Varieties by $\mathbb{C}^*$ Actions

Before considering deformations of projective T-varieties, we first need to gather some information concerning quotients of T-varieties by a $\mathbb{C}^*$-action. Our approach is similar to that of [AH08], but we are not only considering cones over projectively normal T-varieties. For details on quotients by tori in general, including good quotients, we refer to [BH06].

We first set some notation for the section. Let N be a lattice, and take $N' = N \oplus \mathbb{Z}$; we denote the dual lattices by M and M'. Let e_0 be a primitive generator of $\mathbb{Z}$ in N', and let $e_0^* \in M'$ be such that $\langle e_0, e_0^* \rangle = 1$ and $e_0^{*\perp} = N$. The inclusion $\mathbb{Z} \hookrightarrow N'$ induces a monomorphism of tori $\mathbb{C}^* \hookrightarrow T^{N'}$ with quotient T^N. Let pr denote the projection $N' \to N$.

For any semiprojective variety Y and proper polyhedral divisor $\mathcal{D}$ on Y with respect to the lattice N', we define $\mathrm{pr}(\mathcal{D})$ to be

$$\mathrm{pr}(\mathcal{D}) = \sum_P \mathrm{pr}(\mathcal{D}_P) \otimes P.$$

Note that $\mathrm{pr}(\mathcal{D})$ technically need not be a polyhedral divisor, since the corresponding tail cone may not be pointed. However, we can still associate an M-graded $\mathbb{C}$-algebra and thus an affine scheme $X(\mathrm{pr}(\mathcal{D})))$ just as in section 1.2.

Lemma 7.1.1. *The good quotient of $X(\mathcal{D})$ by the natural $\mathbb{C}^*$-action is $X(\mathrm{pr}(\mathcal{D}))$.*

Proof. Let $\sigma = \operatorname{tail}(\mathcal{D})$, $\tilde{\mathcal{D}} = \operatorname{pr}(\mathcal{D})$, and $\tilde{\sigma} = \operatorname{tail}(\tilde{\mathcal{D}})$. Then $\tilde{\sigma} = \operatorname{pr}(\sigma)$, and one easily checks that $\tilde{\sigma}^\vee = \sigma^\vee \cap M_\mathbb{Q}$.

Note that the good quotient of $X(\mathcal{D})$ is equal to

$$\operatorname{Spec}\left(\bigoplus_{u\in\sigma^\vee\cap M'} H^0(Y,\mathcal{D}(u))\right)^{\mathbb{C}^*}.$$

However, we have

$$\left(\bigoplus_{u\in\sigma^\vee\cap M'} H^0(Y,\mathcal{D}(u))\right)^{\mathbb{C}^*} = \bigoplus_{u\in\sigma^\vee\cap M} H^0(Y,\mathcal{D}(u)) = \bigoplus_{u\in\tilde{\sigma}^\vee\cap M} H^0(Y,\operatorname{pr}(\mathcal{D})(u)).$$

□

In many cases of most interest to us, $\operatorname{pr}(\mathcal{D})$ is in fact not a proper polyhedral divisor, and $X(\operatorname{pr}(\mathcal{D}))$ will have dimension less than $\operatorname{rank} N$. Thus, we are interested in other quotients of $X(\mathcal{D})$, where we have removed some closed $T^{N'}$-invariant subvariety Z. We first need to understand how to construct an open cover of $X(\mathcal{D}) \setminus Z$. From now on, we assume that $\operatorname{Pic} Y \cong \mathbb{Z}$. We choose the above isomorphism such that $\mathbb{Z}_{\geq 0}$ is the cone of ample divisors. Thus, all divisors on Y have a well-defined degree, and we can define the degree of any polyhedral divisor $\mathcal{D}$ on Y similar as in remark 1.2.4 to be

$$\deg(\mathcal{D}) = \sum_P \deg(P)\cdot\mathcal{D}_P.$$

Now, for any $u \in \operatorname{tail}(\mathcal{D})^\vee \cap M$, we define $\operatorname{face}(\mathcal{D},u) = \sum \operatorname{face}(\mathcal{D}_P,u)\otimes P$; this is a polyhedral divisor.

In this situation, we define the set $\mathcal{S}^\mathcal{D}$ to consist of those polyhedral divisors $\operatorname{face}(\mathcal{D},u)$ for $u \in \sigma^\vee \cap M' \setminus 0$ with $(\deg(\mathcal{D}))(u) = 0$, as well as $\operatorname{face}(\mathcal{D},u) + \emptyset \otimes D$ for $u \in M'$ with $(\deg(\mathcal{D}))(u) > 0$ and any effective $0 \neq D$ contained in the support of $\mathcal{D}$. Likewise, we define the set $\operatorname{pr}(\mathcal{S}^\mathcal{D})$ to consist of all $\operatorname{pr}(\mathcal{D}')$ for $\mathcal{D}' \in \mathcal{S}^\mathcal{D}$.

Theorem 7.1.2. *Consider $\mathcal{D}$, $\mathcal{S}^\mathcal{D}$, and $\operatorname{pr}(\mathcal{S}^\mathcal{D})$ as above and assume that $\sigma = \operatorname{tail}(\mathcal{D})$ is full dimensional and contains e_0 in its relative interior. Then $\mathcal{S} := \mathcal{S}^\mathcal{D}$ and $\operatorname{pr}(\mathcal{S}) := \operatorname{pr}(\mathcal{S}^\mathcal{D})$ are divisorial fans on Y. Furthermore, $X(\operatorname{pr}(\mathcal{S}))$ is a good quotient of $X(\mathcal{S})$ by the natural $\mathbb{C}^*$ action, and the Chow quotient $X(\mathcal{D})//^{\mathrm{ch}}\mathbb{C}^*$ is equal to $X(\operatorname{pr}(\mathcal{S}))$.*[1]

For the proof of theorem 7.1.2, we use the following lemma, which is similar to lemma 1.4 of [IS09].

Lemma 7.1.3. *Assume that $\operatorname{tail}(\mathcal{D})$ contains e_0 in its relative interior. Consider $u \in \sigma^\vee \cap M' \setminus 0$.*

(i) If $(\deg(\mathcal{D}))(u) = 0$, then $\operatorname{face}(\mathcal{D},u)$ is proper and a face of $\mathcal{D}$. Furthermore, $\operatorname{pr}(\operatorname{face}(\mathcal{D},u))$ is a proper polyhedral divisor.

(ii) If $(\deg(\mathcal{D}))(u) > 0$, then for any effective divisor D with $\deg D > 0$, $\operatorname{face}(\mathcal{D},u) + \emptyset \otimes D$ is proper and a face of $\mathcal{D}$. Furthermore, $\operatorname{pr}(\operatorname{face}(\mathcal{D},u) + \emptyset \otimes D)$ is a proper polyhedral divisor.

[1] Here we slightly abuse notation by defining the Chow quotient to be the inverse limit over all GIT quotients.

Proof. First, note that since e_0 is in the interior of $\mathrm{tail}(\mathcal{D})$, it is not contained in the tail of $\mathrm{face}(\mathcal{D}, u)$. Thus, $\mathrm{tail}(\mathrm{pr}(\mathrm{face}(\mathcal{D}, u)))$ must be pointed, so both $\mathrm{pr}(\mathrm{face}(\mathcal{D}, u))$ and $\mathrm{pr}(\mathrm{face}(\mathcal{D}, u) + \emptyset \otimes D)$ are polyhedral divisors. Furthermore, $\mathrm{tail}(pr(\mathrm{face}(\mathcal{D}, u)))^\vee$ is not a face of $\mathrm{tail}(\mathrm{face}(\mathcal{D}, u))^\vee$, so $\mathrm{pr}(\mathcal{D}')$ will be proper if $\mathcal{D}'$ is, for $\mathcal{D}' = \mathrm{face}(\mathcal{D}, u)$ or $\mathcal{D}' = \mathrm{face}(\mathcal{D}, u) + \emptyset \otimes D$.

Now, suppose that $(\deg(\mathcal{D}))(u) = 0$, and set $\mathcal{D}' = \mathrm{face}(\mathcal{D}, u)$. Then $\mathrm{tail}(\mathcal{D}') = \mathrm{tail}(\mathcal{D}) \cap u^\perp$, and $\mathrm{tail}(\mathcal{D}')^\vee = \mathrm{tail}(\mathcal{D})^\vee + \mathbb{Q} \cdot u$. Consider some $w' \in \mathrm{tail}(\mathcal{D}')^\vee$, which we write as $w + k \cdot u$ for some $w \in \mathrm{tail}(\mathcal{D})^\vee$. Then

$$\mathcal{D}'(w') = \sum_P \min\langle \mathcal{D}'_P, w + k \cdot u\rangle \otimes P = \mathcal{D}(w) + k\mathcal{D}(u),$$

since $\deg k\mathcal{D}(u) = 0$. Thus, $\mathcal{D}'(w')$ is big or semiample exactly when $\mathcal{D}(w)$ is. Furthermore, if w' is in the interior of $\mathrm{tail}(\mathcal{D}')$, then w is in the interior of $\mathrm{tail}(\mathcal{D})$. This shows the properness of $\mathcal{D}'$; the face relation $\mathcal{D}' \prec \mathcal{D}$ is immediate.

Suppose on the other hand that $(\deg(\mathcal{D}))(u) > 0$ and set $\mathcal{D}' = \mathrm{face}(\mathcal{D}, u) + \emptyset \otimes D$. Then any divisor on $\mathrm{Loc}(\mathcal{D}')$ is ample, so $\mathcal{D}'$ is proper. Furthermore, $\mathcal{D}' \prec \mathcal{D}$ by lemma 6.8 of [AHS08]. $\square$

Proof of theorem 7.1.2. The fact that $\mathcal{S}$ is a divisorial fan follows directly from lemma 7.1.3. To see that $\mathrm{pr}(\mathcal{S})$ is a divisorial fan, we must check that for $\mathcal{D}', \mathcal{D}'' \in \mathcal{S}$ with $\mathcal{D}' \prec \mathcal{D}''$, we have $\mathrm{pr}(\mathcal{D}') \prec \mathrm{pr}(\mathcal{D}'')$. This is done similar to in the proof of lemma 7.1.3 and is left to the reader.

It is immediate that $X(\mathrm{pr}(\mathcal{S}))$ is a good quotient of $X(\mathcal{S})$, since for each $\mathcal{D}' \in \mathcal{S}$, $X(\mathrm{pr}(\mathcal{D}'))$ is a good quotient of $X(\mathcal{D}')$ by lemma 7.1.1. To see that $X(\mathrm{pr}(\mathcal{S}))$ is the Chow quotient of $X(\mathcal{D})$, note that $X(\mathcal{S}) = X(\mathcal{D}) \setminus Z$, where Z is the union of all closed torus orbits of $X(\mathcal{D})$. Since the weight monoid of the $\mathbb{C}^*$ action on $X(\mathcal{D})$ is just $\mathbb{Z}_{\geq 0}$, the Chow quotient of $X(\mathcal{D})$ is equal to the GIT quotient of $X(\mathcal{D})$ corresponding to the ray $\mathbb{Z}_{\geq 0}$. The corresponding set of semistable points is the complement of those points whose orbit cone is simply 0; this is easily seen to be Z. $\square$

7.2 Embedded Deformations

The basic question we wish to consider here is the following: when can an arbitrary one-parameter T-deformation of a projective T-variety be realized as an embedded deformation? The following theorem says that this is always possible for projectively normal embeddings:

Theorem 7.2.1. *Let X_0 be a projective, rational, complexity-one T-variety together with some projectively normal embedding. Then any one-parameter T-deformation of X_0 can be realized as an embedded deformation with respect to the embedding of X_0.*

We now present a construction we will use to prove the theorem. The setup is the following: let $\mathcal{S}$ be a complete divisorial fan on $\mathbb{P}^1$ and consider any $h \in \mathrm{CaSF}(\mathcal{S})$ such that D_h is very ample and the embedding it determines is projectively normal. Now consider any α-admissible Minkowski decomposition of $\mathcal{S}_0$ giving a one-parameter T-deformation $\pi : X(\mathcal{S}^{\mathrm{tot}}) \to B$. For some $b \neq 0$, let $h^{(b)}$ be any support function in $\mathrm{CaSF}'(\mathcal{S}^{(b)})$ such that $D_{h^{(b)}} \in \pi_{b,0}^{-1}(D_h)$.

We now define the polyhedral divisor $\mathcal{D}$ on $\mathbb{P}^1$ by

$$\mathcal{D} = \sum_P \operatorname{conv}(\Gamma_{-h_P}) \otimes P$$

where $\Gamma_{-h_P} \subset N_{\mathbb{Q}} + \mathbb{Q}$ is the graph of $-h_P$. Likewise, set

$$\mathcal{D}_0^0 = \operatorname{conv}(\Gamma_{-h_0^{(b)}}); \qquad \mathcal{D}_0^1 = \operatorname{conv}(\Gamma_{-h_b^{(b)}}).$$

Then one easily checks that $\mathcal{D}_0 = \mathcal{D}_0^0 + \alpha \cdot \mathcal{D}_0^1$ is an α-admissible Minkowski decomposition of $\mathcal{D}_0$, which gives us a polyhedral divisor $\mathcal{D}^{\mathrm{tot}}$ on some $\tilde{Y}^{\mathrm{tot}}$, as well as polyhedral divisors $\mathcal{D}^{(s')}$ on $\mathbb{P}^1$. We now consider the following projective varieties:

$$\operatorname{Proj}(\mathcal{D}^{(s)}) := \operatorname{Proj} \bigoplus_u H^0(\mathbb{P}^1, \mathcal{D}^{(s)}(u))$$
$$\operatorname{Proj}(\mathcal{D}^{\mathrm{tot}}) := \operatorname{Proj} \bigoplus_u H^0(\tilde{Y}^{\mathrm{tot}}, \mathcal{D}^{\mathrm{tot}}(u))$$

where the $\mathbb{Z}$ grading of the algebras on the right hand side is given by the natural projection onto $N^{\perp}$. $\operatorname{Proj}(\mathcal{D}^{(s)})$ is projective over $\mathbb{C}$, and $\operatorname{Proj}(\mathcal{D}^{\mathrm{tot}})$ is projective over $\tilde{B} := \operatorname{Spec} H^0(\tilde{Y}^{\mathrm{tot}}, \mathcal{O}_{\tilde{Y}^{\mathrm{tot}}}) \hookrightarrow \mathbb{P}^1$. Let $\tilde{\pi} : \operatorname{Proj}(\mathcal{D}^{\mathrm{tot}}) \to \tilde{B}$ be the structure map.

Proposition 7.2.2. *$\tilde{\pi}$ is a flat family with fibers*

$$\tilde{\pi}^{(-1)}(s) = \operatorname{Proj}(\mathcal{D}^{(s)}) = X(\mathcal{S}^{(s)})$$

and with

$$\mathcal{O}(1)_{|\tilde{\pi}^{(-1)}(s)} = \mathcal{O}(D_{h^{(s)}}).$$

Furthermore, over $B \cap \tilde{B}$, $\pi \cong \tilde{\pi}$. In other words, $\tilde{\pi}$ realizes the deformation π as an embedded deformation with respect to the embedding given by D_h.

Proof. The Minkowski decomposition of $\mathcal{D}$ gives us a deformation $\hat{\pi} : X(\mathcal{D}^{\mathrm{tot}}) \to \tilde{B}$ of $X(\mathcal{D})$. Furthermore, $X(\mathcal{D}^{(s)})$ and $X(\mathcal{D}^{\mathrm{tot}})$ are just the affine cones over $\operatorname{Proj}(\mathcal{D}^{(s)})$ and $\operatorname{Proj}(\mathcal{D}^{\mathrm{tot}})$. Now, the equality $\tilde{\pi}^{(-1)}(s) = \operatorname{Proj}(\mathcal{D}^{(s)})$ is immediate; the equalities $X(\mathcal{S}^{(s)}) = \operatorname{Proj}(\mathcal{D}^{(s)})$ and $\mathcal{O}(1)_{|\tilde{\pi}^{(-1)}(s)} = \mathcal{O}(D_{h^{(s)}})$ follow directly from proposition 1.4.7.

We now must show that over $B \cap \tilde{B}$, $\pi \cong \tilde{\pi}$. This is equivalent to showing that

$$X(\mathcal{D}^{\mathrm{tot}})//^{\mathrm{ch}}\mathbb{C}^* \cong X(\mathcal{S}^{\mathrm{tot}})$$

where the $\mathbb{C}^*$ action corresponds to the above $\mathbb{Z}$ grading and we now consider both $\mathcal{D}^{\mathrm{tot}}$ and $\mathcal{S}^{\mathrm{tot}}$ as (polyhedral/fansy) divisors on $\mathbb{P}^1 \times (B \cap \tilde{B})$. Now from theorem 7.1.2, we have $X(\mathcal{D}^{\mathrm{tot}})//^{\mathrm{ch}}\mathbb{C}^* = X(\operatorname{pr}(\mathcal{S}^{\mathcal{D}^{\mathrm{tot}}}))$, where pr and $\mathcal{S}^{\mathcal{D}^{\mathrm{tot}}}$ are defined as in section 7.1. Using the strict concavity of h from proposition 1.3.4, it is then not difficult to see that the elements of $\operatorname{pr}(\mathcal{S}^{\mathcal{D}^{\mathrm{tot}}})$ with no $\emptyset$-coefficients are exactly the elements of $\mathcal{S}^{\mathrm{tot}}$ with no $\emptyset$-coefficients. One further sees that the remaining elements of $\mathcal{S}^{\mathrm{tot}}$ and $\operatorname{pr}(\mathcal{S}^{\mathcal{D}^{\mathrm{tot}}})$ can be localized to some common covering, so that indeed $X(\operatorname{pr}(\mathcal{S}^{\mathcal{D}^{\mathrm{tot}}})) = X(\mathcal{S}^{\mathrm{tot}})$. ☐

Proof of theorem 7.2.1. Let π be any T-deformation, and let h be a support function corresponding to the embedding of X_0. Then by theorem 5.2.2, we can find a support function $h^{(b)}$ as in the above construction with $D_{h^{(b)}} \in \pi_{b,0}^{-1}(D_h)$. The claim then follows from proposition 7.2.2. ☐

Example 7.2.3 (An embedded deformation of a toric Fano surface)**.** Consider the deformation of the toric Fano surface X_0 from example 3.2.5, and let $-K_{X_0} = D_h$ be the anticanonical divisor from example 1.3.5, with h the corresponding support function. Now, $-K_{X_0}$ gives a projectively normal embedding of X_0. Recall from example 5.1.6 that we have a support function $h^{(s)}$ such that $h^{(0)} = h$. Thus, we can apply proposition 7.2.2, and find that we can realize π as an embedded deformation with respect to the above embedding. In fact, this realization is just the projectivization of the deformation from example 2.2.5.

7.3 Decompositions of Divisorial Polytopes

We now change perspective from the previous section. Instead of trying to lift some T-deformation to an embedded one, we try to find a natural construction for any embedded T-deformation. Now, in toric geometry, the natural description of embedded toric varieties is via polytopes. Likewise, the natural description of embedded T-varieties is via divisorial polytopes. Thus, it is not surprising that our description of embedded T-deformations should involve decompositions of divisorial polytopes:

Definition 7.3.1. Let $\Psi : \Box \to \operatorname{div}_{\mathbb{Q}} \mathbb{P}^1$ be a divisorial polytope. An *α-admissible one-parameter decomposition* of Ψ consists of two piecewise affine functions $\Psi_0^0, \Psi_0^1 : \Box \to \mathbb{Q}$ such that:

(i) $\Gamma_{\Psi_0^i}$ has lattice vertices for $i = 0, 1$;

(ii) $\Psi_0(u) = \Psi_0^0(u) + \alpha \Psi_0^1(u)$ for all $u \in \Box$;

(iii) For any full-dimensional polyhedron $\nabla \subset \Box$ on which Ψ_0 is affine, Ψ_0^i has non-integral slope on ∇ for at most one $i \in \{0, 1\}$;

(iv) If $\alpha \neq 1$, then Ψ_0^1 always has integral slope.

For brevity, we call an α-admissible one-parameter decomposition of a divisorial polytope simply a decomposition. Now, consider any decomposition of some Ψ. Then we construct a T-deformation of $X(\Xi(\Psi))$ as follows: Set

$$\mathcal{D} = \sum_P \operatorname{conv}(\Gamma_{-\Psi_P^*}) \otimes P$$

and decompose $\mathcal{D}_0 = \mathcal{D}_0^0 + \alpha \cdot \mathcal{D}_0^1$, where

$$\mathcal{D}_0^i = \operatorname{conv}(\Gamma_{-(\Psi_0^i)^*}).$$

One easily confirms that this is an α-admissible Minkowski decomposition of $\mathcal{D}_0$. Thus, we get a polyhedral divisor $\mathcal{D}^{\mathrm{tot}}$ on Y^{tot} and deformation $\tilde{\pi} : X(\mathcal{D}^{\mathrm{tot}}) \to B$ of $X(\mathcal{D})$. Taking the natural $\mathbb{C}^*$ quotient gives us a deformation $\pi : X(\mathcal{D}^{\mathrm{tot}})//^{\mathrm{ch}}\mathbb{C}^* \to B$ of $X(\mathcal{D})//^{\mathrm{ch}}\mathbb{C}^*$. We call the total space of π X^{tot} and denote fibers $\pi^{-1}(s)$ by X_s.

Given the above decomposition of Ψ, define $\Psi^{\mathrm{tot}} : \Box \to \operatorname{Div}_{\mathbb{Q}} Y^{\mathrm{tot}}$ by

$$\Psi^{\mathrm{tot}}(u) = \sum_{P \neq 0} \Psi_P(u) \otimes V(y_P) + \Psi_0^0(u) \otimes V(y_0) + \Psi_0^1(u) \otimes V(y_0^\alpha - t).$$

Likewise, for any $s \in B$, define $\Psi^{(s)} : \Box \to \operatorname{Div}_{\mathbb{Q}} \mathbb{P}^1$ by

$$\Psi^{(s)}(u) = \sum_{P \neq 0} \Psi_P(u) \otimes V(y_P) + \Psi_0^0(u) \otimes V(y_0) + \Psi_0^1(u) \otimes V(y_0^\alpha - s).$$

One easily checks that $\Psi^{(s)}$ is in fact a divisorial polytope.

Theorem 7.3.2. *π is a T-deformation of the projective T-variety $X(\Xi(\Psi))$ with $X_s = X(\Xi(\Psi^{(s)}))$. Furthermore, if the divisor D_{Ψ^*} corresponding to Ψ is very ample and gives a projectively normal embedding, π can be realized as an embedded deformation with respect to the embedding given by D_{Ψ^*} with total space*

$$X^{\text{tot}} = \operatorname{Proj} \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \bigoplus_{u \in k \cdot \Box \cap M} H^0(Y^{\text{tot}}, k \cdot \Psi^{\text{tot}}(u/k)).$$

Proof. By theorem 7.1.2, the total space of π is the T-variety $X(\operatorname{pr}(\mathcal{S}^{\mathcal{D}^{\text{tot}}}))$. It is not difficult to check that $\operatorname{pr}(\mathcal{S}^{\mathcal{D}^{\text{tot}}})$ arises as $\mathcal{S}^{\text{tot}}$ for some divisorial fan $\mathcal{S}$ on $\mathbb{P}^1$ with Minkowski decomposition. The description of the fibers follows from

$$X_s = X(\mathcal{D}^{(s)}) /\!/^{\text{ch}} \mathbb{C}^* = X(\Xi(\Psi^{(s)}))$$

with the second equality coming from proposition 1.4.7.

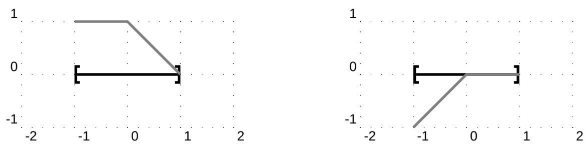
Now suppose that D_{Ψ^*} is very ample and gives a projectively normal embedding. Then the embedding of $X(\Xi(\Psi))$ is realized via

$$\operatorname{Proj} \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \bigoplus_{u \in k \cdot \Box \cap M} H^0(\mathbb{P}^1, k \cdot \Psi(u/k))$$

and the coordinate rings of $X(\mathcal{D})$ and $X(\mathcal{D}^{\text{tot}})$ are generated in degree 1 with respect to the natural $\mathbb{Z}$ grading. Thus,

$$\begin{aligned} X(\mathcal{D}^{\text{tot}}) /\!/^{\text{ch}} \mathbb{C}^* &= \operatorname{Proj} \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \bigoplus_{u \in k \cdot \Box \cap M} H^0(Y^{\text{tot}}, \mathcal{D}^{\text{tot}}(u, k)) \\ &= \operatorname{Proj} \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \bigoplus_{u \in k \cdot \Box \cap M} H^0(Y^{\text{tot}}, k \cdot \Psi^{\text{tot}}(u/k)). \end{aligned}$$

$\Box$



(a) Φ_0^0 (b) Φ_0^1

Figure 7.1: A decomposition of a divisorial polytope

Example 7.3.3 (A toric Fano surface)**.** We consider the divisorial polytope Ψ from example 1.4.8. We can construct a decomposition of Ψ with functions Ψ_0^0, Ψ_0^1 as pictured in figure 7.1. Since Ψ corresponds to a projectively normal embedding of the toric Fano surface X_0 from example 1.2.15, we thus get by theorem 7.3.2 that the above decomposition of Φ encodes an embedded deformation of X_0. In fact, this is exactly the embedded deformation we constructed in example 7.2.3.

Remark 7.3.4. It is straightforward to generalize the notion of one-parameter decompositions of divisorial polytopes to multi-parameter decompositions involving multiple coefficients. These can be then used to construct multi-parameter deformations similar to above. We leave the details for the reader to work out.

Now, for the remainder of the section, suppose that D_{Ψ^*} is very ample and gives a projectively normal embedding. The total space of π then comes with a twisting bundle $\mathcal{O}(1)$ which can be restricted to each fiber:

$$\mathcal{O}(1)_s := \mathcal{O}(1)_{|X_s}.$$

The Hilbert polynomial for a fiber $\pi^{-1}(s)$ is then defined to be the polynomial such that

$$\mathcal{H}_s(k) := \dim H^0(X_s, \mathcal{O}(k)_s)$$

for all $k \gg 0$. It is well known that in such a flat family, $\mathcal{H}_s$ is independent of s, see theorem 9.9 of [Har77]. However, in the case of the above T-deformations, we can easily see this combinatorially:

Corollary 7.3.5. *Suppose D_{Ψ^*} is very ample and gives a projectively normal embedding. Then $\mathcal{O}(1)_s = D_{(\Psi^{(s)})^*}$ for all $s \in B$, and $\mathcal{H}_s$ does not depend on s.*

Proof. The embedding of X_s is given by

$$\operatorname{Proj} \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \bigoplus_{\substack{u\in\square \\ k\cdot u\in M}} H^0(\mathbb{P}^1, k\cdot\Psi^{(s)}(u)).$$

Thus, by proposition 1.4.7, $\mathcal{O}(1)_s = D_{(\Psi^{(s)})^*}$.

We now show that $\mathcal{H}_s = \mathcal{H}_0$ for any $s \in B$. Indeed, for $k \gg 0$,

$$\begin{aligned}\mathcal{H}_s(k) = \sum_{\substack{u\in\square \\ k\cdot u\in M}} \dim H^0(\mathbb{P}^1, \Psi^{(s)}(u)) &= \sum_{\substack{u\in\square \\ k\cdot u\in M}} (\deg\lfloor\Psi^{(s)}(u)\rfloor + 1) \\ &= \sum_{\substack{u\in\square \\ k\cdot u\in M}} \left(\sum_{P\in\mathbb{P}^1} \lfloor\Psi_P^{(s)}(u)\rfloor + 1\right).\end{aligned}$$

Furthermore, we have

$$\lfloor\Psi_P^{(0)}(u)\rfloor = \begin{cases} \lfloor\Psi_P^{(s)}(u)\rfloor & P \neq 0, s \\ \lfloor\Psi_0^{(s)}(u)\rfloor + \lfloor\Psi_s^{(s)}(u)\rfloor & P = 0 \end{cases}$$

for any $u \in \square$ with $k\cdot u \in M$, with the case $P = 0$ following from definition 7.3.1. Combining this with the above equation completes the claim. $\square$

Example 7.3.6 (Some toric fivefolds)**.** This example will play an important role in the following section. Define $\square \subset \mathbb{Q}^4$ by

$$\square = \operatorname{conv}\Big\{(0,0,0,0),(1,1,1,1),(1,1,0,0),(0,0,1,1),\\ (1,0,1,0),(0,1,0,1),(1,0,0,1),(0,1,1,0)\Big\}$$

and consider it as a lattice polytope in the sublattice of $\mathbb{Z}^4$ generated by its vertices. $\square$ is in fact the convex hull of two squares intersecting in a point. Consider piecewise linear concave functions $f_i : \square \to \mathbb{Q}$ for $i = 1, 2, 3, 4$ where the values of f_i at the vertices of its graph are as in the following table:

u	$f_1(u)$	$f_2(u)$	$f_3(u)$	$f_4(u)$
$(0,0,0,0)$	0	0	0	0
$(1,1,1,1)$	0	0	0	0
$(1,1,0,0)$	1	-1	2	-2
$(0,0,1,1)$	-1	1	0	0
$(1,0,1,0)$	0	-1	1	0
$(0,1,0,1)$	0	-1	1	0
$(1,0,0,1)$	-1	0	1	0
$(0,1,1,0)$	-1	0	1	0

One easily checks that f_i has integral slopes for $i = 1, 2, 3, 4$.

Now, set $\Psi = (f_3 + f_4) \otimes \{0\} + (f_1 + f_2) \otimes \{\infty\}$ and $\Psi' = (f_2 + f_3) \otimes \{0\} + (f_1 + f_4) \otimes \{\infty\}$. One easily checks that both Ψ and Ψ' are divisorial polytopes which in fact give projectively normal embeddings. Furthermore, the coefficients of Ψ and Ψ' at 0 and ∞ each have a natural decomposition of the form $f_i + f_j$. These decompositions thus correspond to embedded deformations of $X(\Xi(\Psi))$ and $X(\Xi(\Psi'))$. One thus sees that it is possible to deform both $X(\Xi(\Psi))$ and $X(\Xi(\Psi'))$ to the T-variety $X(\Xi(\Psi''))$, where

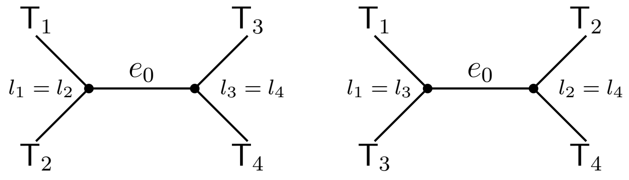
$$\Psi'' = \sum_{i=1}^{4} f_i \otimes P_i$$

for any four distinct points $P_i \in \mathbb{P}^1$.

7.4 Deformations of Phylogenetic Models

In [BW07], W. Buczyńska and J. Wiśniewski investigate projective toric varieties which are geometric models of binary symmetric phylogenetic trivalent trees. One main result is that geometric models of trees with the same number of leaves can be connected via embedded deformations; in particular their Hilbert-Ehrhart polynomials are equal. In this section, we provide alternate proofs of these statements using decompositions of divisorial polytopes. In fact, we show that the relevant geometric models are T-deformation connected (see definition 6.3.1). Furthermore, the proof of the equality of certain Hilbert-Ehrhart polynomials can be formulated in completely combinatorial terms without the use of algebraic geometry or deformation theory.

We adopt the following notation and definitions from [BW07] with slight modification; those further interested in phylogenetic trees may refer to this paper for more details.



(a) T (b) T'

Figure 7.2: Trees connected by an elementary mutation along e_0

Let T be a tree with edges E and vertices V. Let L be the subset of V consisting of vertices with valence one, and let N be the complement of L. We call elements of L and N respectively *leaves* and *nodes*. We call T *trivalent* if all elements of N have valence three. Now set

$$M(\mathsf{T}) := \bigoplus_{e \in E} \mathbb{Z} \cdot e$$

with $N(\mathsf{T})$ the associated dual lattice. There is a canonical inclusion of V in $N(\mathsf{T})$ where for some $v \in V$ we set $v(e) = 1$ if the edge e contains v and $v(e) = 0$ otherwise.

To a tree T as above, we can also associate a polytope $\Delta(\mathsf{T})$:

Definition 7.4.1. The *polytope model* of T is the convex hull in $M(\mathsf{T})_{\mathbb{Q}}$ of

$$\left\{ u = \sum a_i e_i \mid a_i \in \{0, 1\} \text{ and } v(u) \in 2\mathbb{Z} \text{ for all } v \in \mathsf{N} \right\}.$$

For any tree T, denote by $\widehat{M}(\mathsf{T})$ the sublattice of $M(\mathsf{T})$ generated by the vertices of $\Delta(\mathsf{T})$. Now recall that a polytope Δ in a lattice M is *normal* if the semigroup

$$\operatorname{cone}(\Delta \times \{1\}) \cap M \times \mathbb{Z}$$

is generated in height one.

Proposition 7.4.2 ([BW07] Proposition A.5)**.** *If T is trivalent then $\Delta(\mathsf{T})$ is a normal polytope in the lattice $\widehat{M}(\mathsf{T})$.*

For any trivalent tree T, we denote the projective toric variety associated to $\Delta(\mathsf{T})$ in the lattice $\widehat{M}(\mathsf{T})$ by $X(\mathsf{T})$ and call it the *geometric model* of T. In this case, the Hilbert polynomial of $X(\mathsf{T})$ is equal to the Ehrhart polynomial of $\Delta(\mathsf{T})$ in the lattice $\widehat{M}(\mathsf{T})$, that is, the function

$$k \mapsto k \cdot \Delta(\mathsf{T}) \cap \widehat{M}(\mathsf{T}).$$

Thus, we will speak of the Hilbert-Ehrhart polynomial associated to T.

Now consider four trees $\mathsf{T}_1, \ldots, \mathsf{T}_4$, each with some marked leaf l_i, and let τ be a permutation of the set $\{1, \ldots, 4\}$. Then we can create a new tree considering the union of the four trees after identifying the leaves $l_{\tau(i)}$ and $l_{\tau(i+1)}$ for $i = 1, 3$ and then joining $l_{\tau(1)}$ with $l_{\tau(2)}$ via a new edge e_0.

Definition 7.4.3. Suppose trivalent trees T and T' arise from the above construction by considering the same four trees T_i, see for example figure 7.2. Then we say that T and T' are *connected by an elementary mutation along e_0*. We say that two trees are mutation equivalent if there exists a sequence of elementary mutations from one to the other.

Lemma 7.4.4 ([BW07] Lemma 2.18)**.** *Any two trivalent trees with the same number of leaves are mutation equivalent.*

Now, in [BW07] lemma 2.23 it is shown that for any elementary mutation, a projective flat family can be found with the corresponding geometric models as fibers. Combining this with the above lemma yields that the geometric models of all trivalent trees with some fixed number of leaves are connected by embedded deformations, and that in particular the Hilbert-Ehrhart polynomials are equal. Here, we can use our theory of T-deformations to replace this lemma with the following:

Proposition 7.4.5. *Suppose two trivalent trees T and T' are connected by an elementary mutation. Then there are embedded T-deformations π of $X(\mathsf{T})$ and π' of $X(\mathsf{T}')$ which have an isomorphic embedded fiber. Thus, $X(\mathsf{T})$ and $X(\mathsf{T}')$ are connected via embedded deformations.*

Proof. Let T_i be trees such that T is created from T_i as above with respect to the permutation $\tau = \mathrm{id}$. Assume that T' is created from T_i with the permutation τ' exchanging 2 and 3; other cases can be dealt with similarly. For $i = 1, \ldots, 4$, let e_i be the sole edge containing l_i in T_i. Now let $M' = M(\mathsf{T}) = M(\mathsf{T}')$ and let deg be the projection from M' to the orthogonal complement of e_0 where we are using the standard scalar product on M'. Note that the images under deg of $\widehat{M}(\mathsf{T})$ and $\widehat{M}(\mathsf{T}')$ are equal; we denote this image lattice by M. Now, the restriction of deg to $\widehat{M}(\mathsf{T})$ and $\widehat{M}(\mathsf{T}')$ determines divisorial polytopes $\Phi : \nabla \to \mathrm{Div}_{\mathbb{Q}} \mathbb{P}^1$ and $\Phi' : \nabla' \to \mathrm{Div}_{\mathbb{Q}} \mathbb{P}^1$ in M coming from the polytopes $\Delta(\mathsf{T})$ and $\Delta(\mathsf{T}')$ as in remark 1.4.9; here we take the natural section coming from the natural splitting of M'. One easily confirms that in fact $\nabla = \nabla'$.

Consider now the lattice M'' generated by e_i, $i = 1, \ldots, 4$, and let $\phi : M' \to M''$ be the projection. Let $\square$, Ψ, Ψ', and f_i be as in example 7.3.6, where the lattice in which we consider $\square$ is simply equal to the image of M under ϕ. Furthermore, one easily checks that $\phi(\nabla) = \square$. For any function $f : \square \to \mathbb{Q}$, we can pull it back to a function on ∇:

$$\phi^* f(u) := f(\phi(u))$$

for all $u \in \nabla$. Likewise, we can pull back divisorial polytopes from $\square$ to ∇. One easily checks that $\phi^*\Psi = \Phi$ and $\phi^*\Psi' = \Phi'$. Furthermore, we can pull back the decomposition of Ψ and Ψ' to decompositions of Φ and Φ'. Since $\Delta(\mathsf{T})$ and $\Delta(\mathsf{T}')$ are normal in $\widehat{M}(\mathsf{T})$ and $\widehat{M}(\mathsf{T}')$, we get embedded deformations π and π' corresponding to these decompositions, both of which have a fiber corresponding to $\phi^*\Psi^{(1)}$. $\square$

Corollary 7.4.6. *For any trivalent trees T and T' with the same number of leaves, T and T' have the same Hilbert-Ehrhart polynomial.*

Remark 7.4.7. Our proof of the above corollary is completely combinatorial. Indeed, we can completely forget about T-deformations and just use our combinatorial proof of corollary 7.3.5.

Chapter 8

Non-Homogeneous Deformations

The T-deformations we have been considering have had the special property that they are homogeneous of degree zero. This has the advantage that we have been able to explicitly describe the fibers of our deformations. However, it is often beneficial to consider more general deformations, for example if one wants to construct partial smoothings. In this last chapter, we will consider certain deformations which are not homogeneous. In section 8.1, we introduce a general technique which can be used to 'perturb' a T-deformation of an affine T-variety so that it is no longer homogeneous. In section 8.2 we consider the case that the variety of interest is actually toric, and show that the restrictions of such a perturbed T-deformation to certain linear strata of the base space are in fact (homogeneous degree zero) T-deformations. Finally, in section 8.3, we show how this technique can be used to construct partial smoothings of nonaffine toric varieties.

8.1 A General Technique

The technique we use to perturb a homogeneous deformation is not original; see for example section 6.2 of [AvS09] and section 6.6 of [AvS00]. More recently, it was employed by A. Mavlyutov in [Mav09]. However, we present it for the first time here in the context of T-deformations.

Let $\mathcal{D}$ be a proper polyhedral divisor on $Y = \mathbb{P}^1$ along with some α-admissible Minkowski decomposition $\mathcal{D}_0 = \mathcal{D}_0^0 + \alpha^1 \cdot \mathcal{D}_0^1 \ldots + \alpha^r \cdot \mathcal{D}_0^r$. Let $\pi : X^{\mathrm{tot}} \to B$ be the corresponding T-deformation as constructed in section 2.2. Now for each $i = 1, \ldots, r$, consider regular functions $f_i^1, \ldots, f_i^{l_i}$ with $f_i^j \in H^0(X^{\mathrm{tot}}, \mathcal{O}_{X^{\mathrm{tot}}})$. We use these functions to turn π into a non-homogeneous deformation with $d = \sum_{i=1}^r l_i$ parameters.

This is done as follows. Let $\mathcal{X}$ be the subvariety of $X^{\mathrm{tot}} \times \prod_{i=1}^r \mathbb{A}^{l_i}$ given by equations

$$t_i = \sum_{j=1}^{l_i} t_{i,j} \cdot f_i^j \tag{8.1.1}$$

where $t_{i,j}$ are coordinates on the ith term of the product $\prod_{i=1}^r \mathbb{A}^{l_i}$, and t_i are coordinates on B. Let $\pi' : \mathcal{X} \to \prod_{i=1}^r \mathbb{A}^{l_i}$ be the map induced by the projection. By considering the embedding $X^{\mathrm{tot}} \hookrightarrow X^{\mathrm{tot}} \times \prod_{i=1}^r \mathbb{A}^{l_i}$ defined by $\mathrm{id} \times \{0\}$ we then have the following diagram:

$$
\begin{tikzcd}
X(\mathcal{D}) \arrow[r, hook] \arrow[d] & X^{\mathrm{tot}} \arrow[d, "\pi"] \arrow[rr, bend left] & \mathcal{X} \arrow[r, hook] \arrow[d, "\pi'"] & X^{\mathrm{tot}} \times \prod_{i=1}^{r} \mathbb{A}^{l_i} \arrow[dl] \\
0 \arrow[r, hook] & B & \prod_{i=1}^{r} \mathbb{A}^{l_i} &
\end{tikzcd}
$$

It is immediately clear that the image of $X(\mathcal{D})$ in $X^{\mathrm{tot}} \times \prod_{i=1}^{r} \mathbb{A}^{l_i}$ actually lies in $\mathcal{X}$. There it is cut out by the equations $t_{i,j} = 0$ and is thus a complete intersection, since $\dim \mathcal{X} = \dim X(\mathcal{D}) + d$. Thus we have the following:

Proposition 8.1.1. *After restricting the base space to some neighborhood of the origin, π' is a deformation of $X(\mathcal{D})$.*

Proof. Since $X(\mathcal{D})$ is a complete intersection in $\mathcal{X}$, arbitrary perturbations of the defining equations give a flat family around the origin. $\square$

Since we have essentially multiplied the deformation parameters with arbitrary regular functions, we can no longer expect that this deformation is homogeneous with respect to the T^N action.

8.2 Multidegree Deformations of Toric Varieties

We now show how the above non-homogeneous deformations can be used to combine certain T-deformations of toric varieties. Let Δ be a full-dimensional polyhedron in a lattice N with α-admissible Minkowski decomposition $\Delta = \Delta^0 + \alpha^1 \cdot \Delta^1 + \ldots + \alpha^r \cdot \Delta^r$. Let $\tau_1, \ldots, \tau_l$ be a collection of facets of Δ together with primitive weights $u_1, \ldots, u_l \in M$ such that $\mathrm{face}(\Delta, u_j) = \tau_j$ and

$$\min\langle \Delta, u_j \rangle = \langle \tau_j, u_j \rangle = -\alpha_j,$$

where $\alpha_j \in \mathbb{N}$ and α_j divides all α^i. For $1 \leq i \leq r$ and $1 \leq j \leq l$ set $\beta_j^i = \alpha^i/\alpha_j$. We call the Minkowski decomposition of Δ together with the collection of facets $\{\tau_j\}$ a *multidegree deformation datum*.

We use this data to construct a nonhomogeneous deformation of $X(\mathcal{D})$, where $\mathcal{D} = \Delta \otimes \{0\} + \emptyset \otimes \{\infty\}$ is a polyhedral divisor on $\mathbb{P}^1$. Let σ be the cone generated by $(\Delta, 1) \subset N \times \mathbb{Z}$. Note that $X(\mathcal{D})$ is in fact the toric variety $\mathrm{TV}(\sigma)$ determined by σ, see remark 1.2.14.

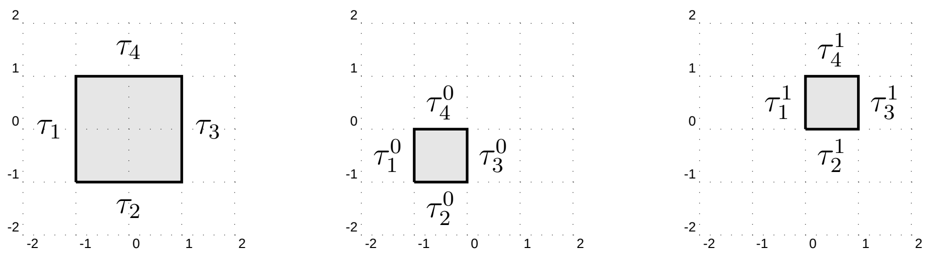
Let $\pi : X^{\mathrm{tot}} \to B$ be the deformation of $X(\mathcal{D})$ coming from the given α-admissible Minkowski decomposition of Δ. Note that the deformation parameter t_i has degree $[0, \ldots, 0, \alpha^i]$ in the $M \times \mathbb{Z}$ grading corresponding to the characters of the big torus acting on $X(\mathcal{D})$. The total space X^{tot} is actually also toric. Indeed, fix some basis $b_1, \ldots, b_n$ of N. Consider now the lattice $\widetilde{N} = N \times \mathbb{Z}^{r+1}$; here we take basis $\widetilde{b}_1, \ldots, \widetilde{b}_n, \widetilde{c}_0, \ldots, \widetilde{c}_r$, where $\widetilde{b}_i$ is the image of b_i under the natural inclusion and $\widetilde{c}_i$, $i = 0, \ldots, r$ is the image under inclusion of the standard $\mathbb{Z}^{r+1}$-basis. Now consider the cone

$$\widetilde{\sigma} = \mathrm{cone}(\mathrm{conv} \cup_{i=0}^{r} (\Delta_i + \widetilde{c}_i))$$

where we implicitly consider the inclusion of N in $\widetilde{N}$. Then $X^{\text{tot}} = \text{TV}(\widetilde{\sigma})$, and the inclusion $\text{TV}(\sigma) \hookrightarrow \text{TV}(\widetilde{\sigma})$ corresponds to the lattice map $N \times \mathbb{Z} \to \widetilde{N}$ sending (b_i, λ) to $\widetilde{b_i} + \lambda \cdot \left(\widetilde{c}_0 + \sum_{j=1}^{r} \alpha^j \cdot \widetilde{c}_j\right)$.

Now, the characters $\chi^{\beta_j^i[u_j,\alpha_j]}$ are regular functions on $X(\mathcal{D}) = \text{TV}(\sigma)$, since $[u_j, \alpha_j] \in \sigma^\vee$. Taking $\widetilde{M}$ to be the lattice dual to $\widetilde{N}$ with dual basis $\widetilde{b}_i^*, \widetilde{c}_i^*$, one easily checks that each $\beta_j^i[u_j, \alpha_j]$ lifts to a unique $\widetilde{u}_i^j \in \widetilde{\sigma}^\vee \cap \widetilde{M}$. We thus set $f_i^j = \chi^{\widetilde{u}_i^j}$ for $i = 1, \ldots r$ and $j = 1, \ldots, l$ and apply the construction of the previous section to get a $d = r \cdot l$-parameter deformation π' of $X(\mathcal{D}) = \text{TV}(\sigma)$. We call the deformation π' a *multidegree deformation.*

Remark 8.2.1. If Δ is a reflexive Gorenstein polytope, then the deformation π' we just constructed has already been constructed by A. Mavlyutov, see [Mav09], remark 6.3.



(a) Δ (b) Δ^0 (c) Δ^1

Figure 8.1: A multidegree deformation datum

Example 8.2.2 (A multidegree deformation of the cone over a toric Fano surface)**.** We consider the polytope Δ and its (1)-admissible Minkowski decomposition $\Delta = \Delta^0 + \Delta^1$ as pictured in figure 8.1. We consider all facets $\tau^1, \tau^2, \tau^3, \tau^4$ as shown in the figure. This is a multidegree deformation datum, with corresponding primitive weights $u_1 = [1, 0]$, $u_2 = [0, 1]$, $u_3 = [-1, 0]$, $u_4 = [0, -1]$, and $\alpha_j = \beta_j^i = 1$ for all i, j. If $\mathcal{D}$ is the corresponding polyhedral divisor, it turns out that $X(\mathcal{D})$ is in fact the same cone over the toric Fano surface from example 1.4.8, although with a different torus action. This can be seen by viewing both varieties as toric varieties. In this example, we then have

$$\widetilde{u}_1^1 = [1, 0, 1, 0] \qquad \widetilde{u}_1^2 = [0, 1, 1, 0]$$
$$\widetilde{u}_1^3 = [-1, 0, 0, 1] \qquad \widetilde{u}_1^4 = [0, -1, 0, 1]$$

Note that we have in fact embedded the total space of π' in a toric variety as well. Indeed, consider the lattice $\overline{N} = \widetilde{N} \times \mathbb{Z}^{r \cdot l}$ with basis $\overline{b}_i, \overline{c}_j, \overline{d}_{j,k}$, which are respectively the images under inclusion of b_i, $\widetilde{c}_j$, and the standard $\mathbb{Z}^{r \cdot l}$ basis. Then the total space of π' is embedded in the toric variety $\text{TV}(\overline{\sigma})$, where

$$\overline{\sigma} = \widetilde{\sigma} \times \mathbb{Q}_{\geq 0}^{r \cdot l}.$$

The embedding is given by the equations

$$\chi^{\overline{c}_i^*} - \chi^{\alpha^i \overline{c}_0^*} = \sum_{j=1}^{l} \chi^{\overline{d}_{i,j}^* + \overline{u}_i^j},$$

where $\overline{b}_i^*, \overline{c}_j^*, \overline{d}_{j,k}^*$ is the natural dual basis of $\overline{M} = \mathrm{Hom}(\overline{N}, \mathbb{Z})$, and $\overline{u}_i^j$ is the lift of $\widetilde{u}_i^j$ to $\overline{M}$ given by the natural splitting of $\widetilde{M}$.

We now wish to analyze certain special strata of the deformation π'. Fix some $1 \leq \nu \leq l$. Consider the sublattice $N_\nu := [u_\nu]^\perp \times \mathbb{Z} \subset N \times \mathbb{Z}$ together with some cosection s which respects the splitting of N_ν. Set

$$\begin{aligned} \sigma_\nu^+ &= s(\sigma \cap [[-u_\nu, 0] = 1]); \\ \sigma_\nu^- &= s(\sigma \cap [[-u_\nu, 0] = -1]). \end{aligned}$$

Then $\mathcal{D}^{\sigma_\nu} = \sigma_\nu^+ \otimes \{0\} + \sigma_\nu^- \otimes \{\infty\}$ is a proper polyhedral divisor on $\mathbb{P}^1$ whose corresponding T-variety is simply $\mathrm{TV}(\sigma)$ with action by the torus T^{N_ν}. The Minkowski decomposition of Δ induces an α-admissible Minkowski decomposition $\tau_\nu = \tau_\nu^0 + \alpha^1 \cdot \tau_\nu^1 \ldots + \alpha^r \cdot \tau_\nu^r$. Now, take

$$\begin{aligned} \sigma_\nu^0 &:= \frac{1}{\alpha_\nu} s(\tau_\nu^0 \times \{1\}) + \mathrm{tail}(\sigma_\nu^+) \\ \sigma_\nu^i &:= s(\tau_\nu^i \times \{0\}) + \mathrm{tail}(\sigma_\nu^+), \qquad 1 \leq i \leq r. \end{aligned}$$

Then $\sigma_\nu^+ = \sigma_\nu^0 + \beta_\nu^1 \cdot \sigma_\nu^1 + \ldots + \beta_\nu^r \cdot \sigma_\nu^r$ is a β_ν-admissible Minkowski decomposition, where $\beta_\nu := (\beta_\nu^1, \ldots, \beta_\nu^r)$. Denote the deformation corresponding to this decomposition by π_ν; its deformation parameters have degrees $[\beta_\nu^i \cdot u_\nu, 0]$ for $1 \leq i \leq r$ with respect to the $M \times \mathbb{Z}$-grading. In this setting, for $(s_1, \ldots, s_r) \in \mathbb{A}^r$, define the proper polyhedral divisor

$$\mathcal{D}^{\nu,(s)} = \sigma_\nu^0 \otimes \{0\} + \sum_{i=1}^r \sigma_\nu^i \otimes V(y_0^{\beta_\nu^i} - s_i) + \sigma_\nu^- \otimes \{\infty\}.$$

Note that the fibers of π_ν are just $X(\mathcal{D}^{\nu,(s)})$.

The content of the following proposition is that the deformation π' in a sense combines all the deformations π_ν:

Proposition 8.2.3. *The deformation π' restricted to $t_{i,j} = 0$ for $j \neq \nu$ is canonically isomorphic to the T-deformation π_ν.*

Proof. Restricting π' to $t_{i,j} = 0$ for $j \neq \nu$ has the following effect: We replace $\overline{N}$ by

$$\overline{N}_\nu = \overline{N} / \langle \overline{d}_{i,j} \rangle_{j \neq \nu}$$

and identify basis vectors of $\overline{N}$ with their images in $\overline{N}_\nu$. Taking $\overline{\sigma}_\nu$ to be the image of $\overline{\sigma}$ in $\overline{N}_\nu$, we have that the total space of the restriction of π' is embedded in $\mathrm{TV}(\overline{\sigma}_\nu)$ by requiring that the polynomials

$$g_i := \chi^{\overline{c}_i^*} - \chi^{\alpha^i \overline{c}_0^*} - \chi^{\overline{u}_i^\nu} \chi^{\overline{d}_{i,\nu}^*} \tag{8.2.1}$$

vanish for $i = 1, \ldots, r$.

We will use the techniques of [AHS08], proposition 5.5 to construct a polyhedral divisor giving the restriction of π'; this is a generalization of the downgrading we presented in remark 1.2.14. After possible basis change (and reordering of Minkowski summands if all $\alpha^i = 1$), we can assume that $\min\langle \alpha^i \cdot \Delta^i, u_\nu \rangle = 0$ for $i \neq 0$ and thus that $\min\langle \Delta^0, u_\nu \rangle = -\alpha_\nu$. Furthermore, we can assume that $b_2, \ldots, b_n$ is a basis for $[u_\nu]^\perp \subset N$ and that

$s(b_1) = 0$, where s is the cosection from above. Using the splitting of $\overline{N}_\nu$, we extend s to a cosection $\overline{s} : \overline{N}_\nu \to N_\nu$. Consider the exact sequence

$$0 \longrightarrow N_\nu \longrightarrow \overline{N}_\nu \xrightarrow{\ p\ } \widehat{N} \longrightarrow 0$$

where we have taken $\widehat{N}$ to be the quotient $\overline{N}_\nu / N_\nu$. We take a basis $\widehat{b}_1, \widehat{c}_1, \ldots \widehat{c}_r, \widehat{d}_1, \ldots, \widehat{d}_r$ of $\widehat{N}$ such that the map p sends

$$\begin{aligned}
\overline{b}_1 &\mapsto \widehat{b}_1 & \overline{b}_i &\mapsto 0, \qquad 2 \le i \le n \\
\overline{c}_0 &\mapsto \sum_{i=1}^{r} \alpha^i \widehat{c}_i & \overline{c}_j &\mapsto -\widehat{c}_j, \qquad 1 \le j \le r \\
\overline{d}_{k,\nu} &\mapsto \widehat{d}_k
\end{aligned}$$

Let $\widehat{M}$ be the dual lattice of $\widehat{N}$ with corresponding basis $\widehat{b}_1^*, \widehat{c}_i^*, \widehat{d}_j^*$.

Let Σ be the fan in $\widehat{N}_{\mathbb{Q}}$ induced by the images of all faces of $\overline{\sigma}_\nu$. Then the Chow quotient of $\mathrm{TV}(\overline{\sigma}_\nu)$ by the action of T^{N_ν} is $\mathrm{TV}(\Sigma)$. Let Y^{tot} be the image of the total space of the restriction of π' in $\mathrm{TV}(\Sigma)$. We wish to describe this more closely. To do so, we consider two special charts of $\mathrm{TV}(\Sigma)$. Let $\widehat{\sigma}_{\min}$ and $\widehat{\sigma}_{\max}$ be the images of the faces of $\overline{\sigma}$ on which $\overline{u}_i^\nu$ is respectively minimized and maximized for all i. It is immediate that both these cones must belong to Σ.

We first focus on $\widehat{\sigma}_{\min}$. Calculating the images of the relevant rays of $\overline{\sigma}$, we have

$$\widehat{\sigma}_{\min} = \langle v_0, v_1, \ldots, v_r, w_1, \ldots, w_r \rangle$$

with

$$v_0 = -\widehat{b}_1 + \sum_{i=1}^{r} \beta_\nu^i \widehat{c}_i, \qquad v_i = -\widehat{c}_i \qquad 1 \le i \le r, \qquad w_i = \widehat{d}_i \qquad 1 \le i \le r.$$

Thus, $\mathrm{TV}(\widehat{\sigma}_{\min}) = \mathbb{A}^{1+2r}$. We will take v_i^*, w_j^* to be a dual basis. Note that we also have

$$\overline{s}\Big(\overline{\sigma} \cap p^{-1}(v_0)\Big) = \sigma_\nu^0, \qquad \overline{s}\Big(\overline{\sigma} \cap p^{-1}(v_i)\Big) = \sigma_\nu^i, \qquad 1 \le i \le r \tag{8.2.2}$$

and

$$\overline{s}\Big(\overline{\sigma} \cap p^{-1}(w_j)\Big) = \mathrm{tail}(\sigma_\nu^+). \tag{8.2.3}$$

Now, on the torus $T^{\widehat{N}}$, the image of $V(g_i)$ is given by $V(\widehat{g}_i) \cap T^{\widehat{N}}$, where we take

$$\widehat{g}_i := 1 - \chi^{\widehat{c}_i^*} - \chi^{\beta_\nu^i \widehat{b}_1^* + \widehat{c}_i^*} \chi^{\widehat{d}_i^*},$$

see [Bir07]. On the other hand, the image of $V(g_i)$ in $\mathrm{TV}(\widehat{\sigma}_{\min})$ is given by setting

$$\widehat{g}_i \cdot \chi^{-\beta_\nu^i \widehat{b}_1^* - \widehat{c}_i^*} = 0,$$

since the left hand side is regular on $\mathrm{TV}(\widehat{\sigma}_{\min})$ and irreducible. Indeed, this can be seen by rewriting the above equation as

$$\chi^{v_i^*} - (\chi^{v_0^*})^{\beta_\nu^i} = \chi^{w_i^*}$$

which is just a hyperplane in $\mathbb{A}^{1+2r}$.

In general, Y^{tot} is contained in the intersections of the images of $V(g_i)$ in $\mathrm{TV}(\Sigma)$, but may actually be smaller if the intersection isn't irreducible. But on the chart $\mathrm{TV}(\widehat{\sigma}_{\min})$, the intersection of the images of $V(g_i)$ is just $\mathbb{A}^{r+1}$, so we in fact have that $Y^{\mathrm{tot}}_{|\mathrm{TV}(\widehat{\sigma}_{\min})} = \mathbb{A}^{r+1}$. For this chart, we can choose coordinates $y, t_1, \ldots, t_r$ on this space such that the restrictions of the toric divisors corresponding to v_i is given by the equation $y^{\beta^i_\nu} = t_i$ for $i \geq 1$ and $y = 0$ for $i = 0$.

We now turn shortly to the chart $\mathrm{TV}(\widehat{\sigma}_{\max})$. Let $\widehat{H}_{\max}$ denote the r-cycle corresponding to the face of $\widehat{\sigma}_{\max}$ where we omit rays generated by the $\widehat{d_i}$. This r-cycle must in fact lie in Y^{tot}. Indeed, consider the face $\overline{\sigma}_{\max}$ of $\overline{\sigma}_\nu$ which is spanned by all rays on which u_ν is maximal, omitting the rays $\overline{d}_{i,\nu}$; it defines an r-cycle $\overline{H}_{\max}$ on $\mathrm{TV}(\overline{\sigma}_\nu)$. Furthermore, since $p(\overline{\sigma}_{\max}) = \widehat{\sigma}_{\max}$, the quotient map restricted to $\mathrm{TV}(\overline{\sigma}_{\max})$ is in fact regular, and thus the image of $\overline{H}_{\max}$ under the quotient map is $\widehat{H}_{\max}$. We now just need to check that $V(g_i)$ contains $\overline{H}_{\max}$ for each i. Note that we have that neither $\overline{c}_i^*$, $\overline{c}_0^*$, nor $\overline{u}_\nu$ lie in $-\overline{\sigma}_\nu$. Thus, the monomials $\chi^{\overline{c}_i^*}$, $\chi^{\overline{c}_0^*}$, and $\chi^{\overline{u}_\nu}$ are not invertible in $\mathrm{TV}(\overline{\sigma}_\nu)$, and so $\overline{H}_{\max} \subset V(g_i)$. We can conclude that $\widehat{H}_{\max} \subset Y^{\mathrm{tot}}$ as desired.

We now have enough information to conclude that $Y^{\mathrm{tot}} = \mathbb{P}^1 \times \mathbb{A}^r$. Indeed, we have a natural map $Y^{\mathrm{tot}} \to \mathbb{A}^r$, whose fibers in the chart $\widehat{\sigma}_{\min}$ are affine lines and whose fibers in the chart $\widehat{\sigma}_{\max}$ contain a point not in the previous previous chart; thus, the fibers must all be $\mathbb{P}^1$. Extending our coordinates for Y^{tot} in the chart $\widehat{\sigma}_{\min}$ (where y is the coordinate on $\mathbb{P}^1$), we have that $Y^{\mathrm{tot}} \cap \widehat{H}_{\max}$ is given by the equation $y = \infty$. Thus, all the toric divisors on $\mathrm{TV}(\Sigma)$ corresponding to rays of $\widehat{\sigma}_{\min}$ not of the form $\widehat{d_i}$ restrict to $y^{-1} = 0$. We ignore toric divisors corresponding to rays $\widehat{d_i}$, since the associated polyhedral coefficient is trivial as we see from equation (8.2.3).

We must now check that the divisors given by the equations $y = 0, y = \infty, y^{\beta^i_\nu} = t_i$ have the correct polyhedral coefficients. But this follows from equation (8.2.2) for all divisors excluding $y = \infty$. But the coefficient for $y = \infty$ must be σ_ν^-, since we know that the special fibers of π' and π_ν are equal. ☐

Example 8.2.4 (A cone over a toric Fano surface)**.** We continue example 8.2.2, and restrict the corresponding deformation π' to the stratum $t_{1,j} = 0$, $j \neq \nu$ for $\nu = 1$. Choosing the natural cosection s, we get that $\mathcal{D}^{\sigma_\nu}$ is exactly the polyhedral divisor of example 1.4.8. Furthermore, the decomposition $\sigma_\nu^+ = \sigma_\nu^0 + \sigma_\nu^1$ is exactly the decomposition of example 2.2.5. By proposition 8.2.3, the restriction of π' is thus equal to the deformation π_1, where π_1 was the deformation from example 2.2.5.

8.3 Constructing Partial Smoothings

We now show how multidegree deformations can be used to construct partial smoothings of nonaffine toric varieties. Let N be a lattice, and $\Delta \subset N_{\mathbb{Q}}$ a polytope containing the origin in its interior. By $\Sigma(\Delta)$ we denote the *face fan* of Δ, that is the set of all cones of the form $\mathrm{cone}(\tau)$ for $\tau \prec \Delta$. Then $X_0 = \mathrm{TV}(\Sigma(\Delta))$ is a projective toric variety. We can also consider the cone $\sigma = \mathrm{cone}(\Delta \times \{1\}) \subset (N \times \mathbb{Z})_{\mathbb{Q}}$ and the associated toric variety $\mathrm{TV}(\sigma)$. Then the Chow quotient of $\mathrm{TV}(\sigma)$ by the subtorus $\mathbb{C}^* \subset T^{N\times\mathbb{Z}}$ corresponding to the one-parameter subgroup generated by $(\underline{0}, 1) \in N \times \mathbb{Z}$ is in fact X_0. Thus, we call $\mathrm{TV}(\sigma)$ the cone over X_0 and denote it by $C(X_0)$. Now any deformation of $C(X_0)$ which is homogeneous of degree 0 in the natural $\mathbb{Z}$ grading descends to a deformation of X_0. We shall thus consider degree 0 deformations of $C(X_0)$ to construct deformations of X_0.

Consider now a multidegree deformation datum $\Delta = \Delta^0 + \alpha^1 \cdot \Delta^1 + \ldots + \alpha^r \cdot \Delta^r$ with facets $\tau_1, \ldots, \tau_l$. This gives us a deformation π' of $C(X_0)$, which is homogeneous of degree 0 with respect to the $\mathbb{Z}$ grading, and we thus get an $r \cdot l$-parameter deformation $\widehat{\pi}$ of X_0.

Proposition 8.3.1. *Choose some $1 \leq \nu \leq l$. Along the r-dimensional stratum given by $t_{i,j} = 0$ for $j \neq \nu$, the fiber of $\widehat{\pi}$ over $t_{1,\nu} = s_1, \ldots, t_{r,\nu} = s_r$ equals $X(\mathrm{pr}(\mathcal{S}^{\mathcal{D}^{\nu,(s)}}))$.*

Proof. This is just a combination of proposition 8.2.3 and theorem 7.1.2. ☐

Now, since we know some of the fibers of $\widehat{\pi}$, we can reach some conclusions about its general fiber. For any $1 \leq \nu \leq l$, denote by Z_ν the (possibly empty) singular locus of $U_\nu := \mathrm{TV}(\mathrm{cone}(\tau_\nu))$, and set $d_\nu = \mathrm{codim}(Z_\nu, U_\nu)$.

Theorem 8.3.2. *For any $1 \leq \nu \leq l$, the deformed germ of Z_ν in the general fiber of $\widehat{\pi}$ is smooth in codimension $d_\nu + 1$ if $\frac{1}{\alpha_\nu}\tau_\nu^0, \tau_\nu^1, \ldots, \tau_\nu^r$ are conically smooth in dimension d_ν. Furthermore, if Z_ν is an isolated singularity, then the deformed germ of Z_ν in the general fiber of $\widehat{\pi}$ is smooth/terminal/canonical if $\frac{1}{\alpha_\nu}\tau_\nu^0, \tau_\nu^1, \ldots, \tau_\nu^r$ are conically smooth/terminal/canonical.*

Proof. For the first part, it is enough to show that there is some fiber of $\widehat{\pi}$ in which the general point of Z_ν has been smoothed. Thus, we restrict to the stratum $t_{i,j} = 0$ for $j \neq \nu$, where we know by proposition 8.3.1 that the fibers are of the form $X(\mathrm{pr}(\mathcal{S}^{\mathcal{D}^{\nu,(s)}})$. One easily checks that for each fiber away from zero, the polyhedral divisor for the deformation of the chart U_ν has affine locus, and polyhedral coefficients $\frac{1}{\alpha_\nu}\tau_\nu^0, \tau_\nu^1, \ldots, \tau_\nu^r$. Since these polytopes are conically smooth in dimension d_ν, the deformation of U_σ must be smooth in codimension $d_\nu + 1$ due to proposition 1.5.3. In particular, the general point of Z_ν has been smoothed.

The second claim is shown completely analogously, where we make use of the fact that the property of being canonical or terminal is preserved under deformation, see [Kaw99] and [Nak04], corollary 5.3. ☐

The following corollary is immediate:

Corollary 8.3.3. *Let $d = \min_{1 \leq \nu \leq l} d_\nu$ and suppose that $X_0 \setminus \bigcup Z_\nu$ is smooth in codimension $d + 1$. If for all $1 \leq \nu \leq l$ $\frac{1}{\alpha_\nu}\tau_\nu^0, \tau_\nu^1, \ldots, \tau_\nu^r$ are conically smooth in dimension d, then the general fiber of $\widehat{\pi}$ is smooth in codimension $d + 1$. Likewise, if X_0 only has isolated singularities which are all contained in the charts U_ν, and $\frac{1}{\alpha_\nu}\tau_\nu^0, \tau_\nu^1, \ldots, \tau_\nu^r$ are conically smooth/terminal/canonical for all $1 \leq \nu \leq l$, then the general fiber of $\widehat{\pi}$ is smooth/terminal/canonical.*

A special case of the above is the following:

Corollary 8.3.4. *Let Δ be a Gorenstein reflexive polytope and let $X_0 = \mathrm{TV}(\Sigma(\Delta))$ be smooth in codimension d. Suppose $\Delta = \Delta^0 + \ldots + \Delta^r$ for lattice polytopes Δ^i which are conically smooth in dimension d. Then X_0 admits a smoothing in codimension $d + 1$.*

Proof. The decomposition $\Delta = \Delta^0 + \ldots + \Delta^r$ together with all facets τ_i of Δ defines a multidegree deformation datum with $\alpha_j^i = 1$ for all i. Any τ_ν^i is then a face of Δ^i. ☐

Example 8.3.5 (A toric Fano surface)**.** Let Δ be the Gorenstein reflexive polytope from figure 8.1 with the multidegree deformation datum from example 8.2.2. Then $X_0 = \mathrm{TV}(\Sigma(\Delta))$ is just the singular toric Fano surface from example 1.2.15 with four

A_1 singularities. By corollary 8.3.4, X_0 admits a smoothing. If $\widehat{\pi}$ is the deformation of X_0 corresponding to the above datum, then restricting $\widehat{\pi}$ to $t_{1,\nu} = 0$ for some ν gives a smoothing of exactly one of the four A_1 singularities. Indeed, we see this for $\nu = 1$ by combining example 8.2.4 with example 7.2.3; for other values of ν this is similar. Looking at a fiber over a general point in the base space, we have then smoothed all four singularities.

Example 8.3.6 (A toric Fano threefold)**.** Let S be the convex hull of the origin and three standard basis vectors in $\mathbb{Z}^3$, and take $\Delta = 4 \cdot S + (-1, -1, -1)$. Then Δ is a Gorenstein reflexive polytope giving rise to a toric Fano threefold X_0; note that the singular locus of X_0 has dimension one. Now, Δ admits the decomposition $S+S+S+(S+(-1,-1,-1))$, where each summand is conically smooth. Thus, X_0 admits a smoothing in codimension two.

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Zusammenfassung

Diese Dissertation befasst sich mit Deformationen von rationalen, normalen Varietäten mit einer Toruswirkung der Komplexität eins. Solche Varietäten lassen sich durch *polyedrische Divisoren* und *divisorielle Fächer* kombinatorisch beschreiben, die von K. Altmann, J. Hausen und H. Süß entwickelt wurden. Ziel dieser Arbeit ist es, mithilfe dieser Kombinatorik die Deformationstheorie dieser Varietäten zu erforschen.

R. Vollmert konnte bereits zeigen, dass eine kombinatorische Zerlegung eines polyedrischen Divisors zu einer Deformation der zugehörigen affinen Varietät führt. Der Kern dieser Dissertation besteht darin, ein Verfahren anzugeben, wie diese Deformationen von affinen Varietäten verklebt werden können, um Deformationen von nicht zwangsweise affinen Varietäten zu konstruieren. Die daraus hervorgehenden Deformationen werden als *T-Deformationen* bezeichnet.

Nach der Einführung von T-Deformationen werden ihre grundlegenden Eigenschaften untersucht. Kriterien für die Separiertheit und Eigentlichkeit einer T-Deformation werden angegeben, sowie ein Kriterium für lokale Trivialität. Für den Fall von lokaler Trivialität wird das Bild der Kodaira-Spencer Abbildung berechnet.

Diese ersten Untersuchungsergebnisse lassen sich anwenden, um zu zeigen, dass die T-Deformationen einer glatten, kompletten, torischen Varietät den Vektorraum der infinitesimalen Deformationen dieser Varietät aufspannen. Das heißt, dass man allein durch T-Deformationen viele Informationen über die Deformationsmöglichkeiten einer solchen Varietät erhält. Als zweite Anwendung werden T-Deformationen von rationalen Flächen mit $\mathbb{C}^*$-Wirkung untersucht. Hier wird gezeigt, dass alle solche Flächen mit einer fixierten Picard-Zahl größer zwei durch T-Deformationen ineinander überführbar sind.

Die Theorie der T-Deformation wird weiterentwickelt, um Familien von Divisoren und projektiv eingebettete Deformationen zu untersuchen. Es stellt sich heraus, dass es bei T-Deformationen von glatten kompletten Varietäten eine natürliche Isomorphie zwischen den Picard-Gruppen der Fasern gibt. Diese Isomorphie verallgemeinert sich zu einer Abbildung zwischen Untergruppen der Picard-Gruppen im nicht glatten Fall. Diese Abbildung wird anschließend benutzt, um die Einbettbarkeit von T-Deformation zu untersuchen. Insbesondere wird gezeigt, dass alle T-Deformationen von projektiven Varietäten einbettbar sind.

Zudem werden noch zwei weitere Anwendungen besprochen. Zum einen liefert die zugrundeliegende Kombinatorik von T-Deformationen einen rein kombinatorischen Beweis dafür, dass die geometrischen Modelle von binären symmetrischen trivalenten phylogenetischen Bäumen mit gleicher Blattanzahl identisch sind. Zum anderen ermöglichen T-Deformationen auch die Konstruktion von partiellen Glättungen gewisser torischer Fanovarietäten.