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Specialization map between stratified bundles and the pro-étale fundamental group

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Zusammenfassung

Zu einer projektiven Familie semi-stabiler Kurven über einem vollständigen diskreten Bewertungsring in Charakteristik $p > 0$ konstruieren wir einen Spezialisierungsfunktor zwischen der Kategorie der stetigen Darstellungen der pro-étalen Fundamentalgruppe der speziellen Faser und der Kategorie der stratifizierten Bündel auf der geometrischen generischen Faser. Dieser Funktor induziert einen Morphismus zwischen den via Tannaka Dualität korrespondierenden affinen Gruppenschemata. Wir zeigen, dass dieser Morphismus ein Lift von Grothendiecks Spezialisierungsabbildung zwischen den entsprechenden étalen Fundamentalgruppen ist, die in [SGA 1] konstruiert wurde. Darüber hinaus ergeben unsere Methoden ein allgemeines Framework um Giesekers Konstruktion von stabilen Kurven mit degenerierter spezieller Faser aus [Gie73] zu verstehen.

Summary

Given a projective family of semi-stable curves over a complete discrete valuation ring of characteristic $p > 0$ with algebraically closed residue field, we construct a specialization functor between the category of continuous representations of the pro-étale fundamental group of the closed fibre and the category of stratified bundles on the geometric generic fibre. By Tannakian duality, this functor induces a morphism between the corresponding affine group schemes. We show that this morphism is a lifting of the specialization map between the étale fundamental groups constructed by Grothendieck in [SGA 1]. Moreover, the setting in which we work provides a general framework to understand Gieseker's construction for stable curves with degenerate closed fibre explained in [Gie73].

Contents

Introduction

In [SGA 1, Exp. V] Grothendieck stated the axiomatic conditions on a category $\mathcal{C}$ endowed with a functor $F\colon \mathcal{C} \to \mathrm{FSets}$, which ensure that the category $\mathcal{C}$ is equivalent to the category of finite sets with a continuous action of the pro-finite group $\mathrm{Aut}(F)$. The pairs $(\mathcal{C}, F)$ that satisfy the axioms given by Grothendieck are called Galois categories. Grothendieck's motivation to introduce this framework was to find an algebraic notion of fundamental group for schemes. For a connected locally Noetherian scheme X, he defined the category $\mathrm{Fét}_X$ of finite étale covers $\pi\colon Y \to X$. Given a geometric point x of X, he defined a functor $F_x(Y) = \pi^{-1}(x)$ from $\mathrm{Fét}_X$ to FSets and he proved that the pair $(\mathrm{Fét}_X, F_x)$ is a Galois category. The corresponding pro-finite group $\mathrm{Aut}(F_x)$ is called the étale fundamental group of X and it is denoted by $\pi_1^{\text{ét}}(X, x)$. If X is a smooth connected complete scheme over $\mathbb{C}$, then $\pi_1^{\text{ét}}(X, x)$ is just the pro-finite completion of $\pi_1^{\text{top}}(X^{\text{an}}, x)$, the topological fundamental group of the analytification of X.

One of the interesting features of the étale fundamental group is the existence of a specialization morphism. In [SGA 1, Exp. X] Grothendieck proved that, given Y a locally Noetherian scheme, $f\colon X \to Y$ a proper morphism with geometrically connected fibres and y_0, y_1 two points of Y such that y_0 is a specialization of y_1, if X_0 and X_1 are the geometric fibres over y_0 and y_1 with two geometric points x_0, x_1, then a morphism of specialization can be defined in a natural way

$$\mathrm{sp}_{\text{ét}}\colon \pi_1^{\text{ét}}(X_1, x_1) \to \pi_1^{\text{ét}}(X_0, x_0).$$

Moreover, when the morphism f is smooth, he showed that if $k(y_0)$ has characteristic zero then the specialization morphism is an isomorphism, whereas if $k(y_0)$ is a field of positive characterestic $p > 0$ then the specialization morphism induces an isomorphism between the maximal prime to p quotients.

Throughout the years there have been many attempts to generalize the construction of the étale fundamental group and to produce a topological group that encodes the information of a larger class of coverings.

The first generalization is due to Grothendieck himself and it is explained in [SGA 3, Exp. X, §6]. Given a connected locally Noetherian scheme X and a geometric point x of X, he proved that the functor, which associates with every abstract group G the set of isomorphic classes of pointed torsors for the constant group scheme associated with G over X, is representable. He defined the enlarged fundamental group of X, which we denote by $\pi_1^{\text{SGA3}}(X, x)$, as its representative. Furthermore, he showed that the functor F_x, defined as above, induces an equivalence between the category of étale locally constant schemes over X and the category of sets with a continuous $\pi_1^{\text{SGA3}}(X, x)$-action.

This fundamental group was implicitly used by Mumford in [Mum72]. In this article, given a complete discrete valuation ring A of characteristic $p > 0$ with fraction field K and residue field k, Mumford associated with a flat Schottky group $G \subset \mathrm{PGL}_2(K)$ a tree Λ_G on which G acts freely, deducing that G is a free group. Then he constructed a stable curve X over A with k-split degenerate closed fibre X_0 and non-singular generic fibre X_K such that G is the group of covering transformations of the universal covering Y_0 of X_0. Moreover, he also showed that the dual graph of Y_0 is Λ_G. In the same article Mumford proved that every stable curve X over A with k-split degenerate closed fibre and non-singular generic fibre can be constructed in this way for a unique flat Schottky group G. As we will remark in the third chapter of the thesis, the group G turns out to be isomorphic to the enlarged fundamental group of the closed fibre.

The setting introduced by Mumford was later used by Gieseker in [Gie73] to prove that, for any prime $p > 0$ and every integer $g > 1$, there exists a stable curve of arithmetic genus g in characteristic p that admits a semi-stable bundle of rank two whose Frobenius pull-back is not semi-stable. He first proved that, given an algebraically closed field k of characteristic $p > 0$ and a stable curve X of genus g over $k[[t]]$ with smooth generic fibre and degenerate closed fibre, there exists a semi-stable bundle of rank two on the geometric generic fibre $X_{\overline{K}}$ whose Frobenius pull-back is not semi-stable. In order to do so, he introduced the notion of coherent sheaves with meromorphic descent data on the universal covering $\mathcal{Y}$ of the completion $\widehat{X}$ of X along its degenerate closed fibre and he proved that the category they form is equivalent to the category of coherent sheaves on the generic fibre X_K. This construction allowed him to associate with a representation of the group of covering transformations of $\mathcal{Y}$ a stratified bundle on the geometric generic fibre. Then he concluded showing that such a curve X exists for every p and g.

The main goal of the thesis is to generalize the construction above and to make it applicable to every projective semi-stable curve over a complete discrete valuation ring, removing the assumption that the closed fibre is degenerate. To achieve this goal, the last main ingredient is the pro-étale fundamental group defined by Bhatt and Scholze, which is a generalization of both the étale and the enlarged fundamental group.

In [BS15] the authors generalized the construction of Galois categories to infinite Galois categories. These were first defined by Noohi in [Noo08] but the conditions he imposed were too weak. In particular, Bhatt and Scholze introduced the notion of tame infinite Galois categories and proved that every such category is equivalent to the category of sets with a continuous action of a Noohi complete topological group. Then, for a connected locally topologically Noetherian scheme X and a geometric point x, they showed that the pair (Cov_X, F_x) is a tame infinite Galois category, where Cov_X denotes the category of étale schemes over X that satisfy the valuative criterion of properness and F_x is the fibre functor over x. They defined the group associated with (Cov_X, F_x) as the pro-étale fundamental group of X, which we denote by $\pi_1^{\text{proét}}(X, x)$. The pro-finite completion of the pro-étale fundamental group is the étale fundamental group, while its pro-discrete completion is the enlarged fundamental group, hence we recover from the pro-étale fundamental group both the groups constructed by Grothendieck. Another important result is that $\mathbb{Q}_l$-local systems on X are equivalent to continuous representations of $\pi_1^{\text{proét}}(X, x)$ on finite-dimensional $\mathbb{Q}_l$-vector spaces, but this will not be the focus of the thesis.

Our aim will be to extend Gieseker's results to connected projective semi-stable curves over a complete discrete valuation ring and to define a specialization functor between the category of continuous representations of the proétale fundamental group of the closed fibre and the category of stratified bundles on the geometric generic fibre.

Leitfaden

In the first section of Chapter 1 we first give an introductory overview on infinite Galois categories. We recall the definition of Noohi groups and state some results of [BS15], among which the fact that a tame infinite Galois category is equivalent to the category G-Sets for a Noohi group G. In the second section we present the definition of the pro-étale site and the pro-étale fundamental group. In the third section, after defining normal crossing curves (see Def. 1.3.4), we produce a concrete computation of the pro-étale fundamental group of connected projective normal crossing curves defined over an algebraically closed field. In particular we prove the following theorem.

Theorem (Prop. 1.3.23)**.** *Let X be a connected projective normal crossing curve defined over an algebraically closed field and let ξ be a geometric point of X. Denote by C_j the irreducible components of X, by $\overline{C_j}$ their normalization and fix a geometric point ξ_j for every $\overline{C_j}$, then*

$$\pi_1^{\text{proét}}(X,\xi) \simeq \mathbb{Z}^{\star|I|-N+1} \star_N \pi_1^{\text{ét}}(\overline{C_1},\xi_1) \star_N \cdots \star_N \pi_1^{\text{ét}}(\overline{C_N},\xi_N),$$

where I is the set of singular points of X and $\mathbb{Z}^{\star|I|-N+1}$ is the free product of $|I|-N+1$ copies of $\mathbb{Z}$.

We begin Chapter 2 by presenting a brief overview on Tannakian categories, which provides us with the formalism that is needed in order to construct the specialization functor. We focus in particular on the category of F-linear representations, with F any field, of a given abstract group G. We define the algebraic hull of G to be the affine group scheme over F associated with this category (see Def. 2.2.1 for details). We prove that, if the field F is perfect, then the algebraic hull over F of any abstract group is reduced. Then we generalize this result to the category of F-linear continuous representations of a fixed topological group H, whose associated group scheme is called topological algebraic hull of H over F (see Def. 2.3.1 and Def. 2.3.2). Moreover, we describe explicitly the topological algebraic hull of a complete pro-finite group.

Lemma (Lemma 2.3.5)**.** *Let F be a field and $\pi = \varprojlim_i \pi_i$ be a complete pro-finite group with surjective transition maps, then the topological algebraic hull of π over F is isomorphic to the group scheme*

$$\varprojlim_i (\pi_i)_F,$$

where $(\pi_i)_F$ are the constant group schemes over F associated with the finite groups π_i.

These results will allow us to describe the topological algebraic hull of the pro-étale fundamental group of a connected projective normal crossing curve over an algebraically closed field. In particular, a required tool for this purpose is the notion of free product of affine group schemes, which we introduce in the last part of this chapter (see Def. 2.4.4). We also prove in Lemma 2.4.5 that this notion is compatible with the free product of abstract groups.

In Chapter 3 and Chapter 4 we work with the following setting: we denote by k a given algebraically closed field of characteristic $p > 0$, we fix A a complete discrete valuation ring of residue field k and we denote its fraction field by K, then we set $S = \mathrm{Spec}(A)$ and we fix X a projective semi-stable curve X over S with smooth generic fibre X_K.

In Chapter 3, given such a curve X, we construct a functor from the category of K-linear continuous representation of the pro-étale fundamental group of the closed fibre, denoted by $\mathrm{Rep}_K^{\mathrm{cts}}(\pi_1^{\mathrm{proét}}(X_0, \xi))$, to $\mathrm{Coh}(X_K)$, the category of coherent sheaves on X_K, and hence we generalize Gieseker's construction.

In the first section we recall Gieseker's results presented in [Gie73], which are based on the notion of coherent sheaves with meromorphic descent data (see Def. 3.1.10). The first step to generalize these results, in particular [Gie73, Prop. 1], is to construct a geometric covering of the closed fibre X_0 associated with a given K- linear continuous representation of $\pi_1^{\mathrm{proét}}(X_0, \xi)$. This step is explained in the second section, where we denote by Y_0^ρ the covering associated with the representation ρ and we define $\mathcal{Y}_\rho$ to be the corresponding covering of $\widehat{X}$, the completion of X along its closed fibre. In the third section we prove that coherent sheaves on $\mathcal{Y}_\rho$ with meromorphic descend data descend to coherent sheaves on a finite étale cover of X_K, denoted by Z_K^ρ.

Theorem (Thm. 3.3.8)**.** *Let X be a projective semi-stable curve over S with smooth generic fibre. Fix $(K^n, \rho) \in \mathrm{Rep}_K^{\mathrm{cts}}(\pi_1^{\mathrm{proét}}(X_0, \xi))$ and let $\mathcal{Y}_\rho$ and $\mathcal{Z}_\rho$ be the formal geometric coverings of $\widehat{X}$ defined in Section 3.2. Let Z_ρ be the finite étale covering of X corresponding to $\mathcal{Z}_\rho$ and $Z_K^\rho = Z_\rho \times_S K$ its generic fibre, then the category of coherent sheaves on $\mathcal{Y}_\rho$ with meromorphic descend data relative to $\mathcal{Z}_\rho$ is equivalent to the category of coherent sheaves on Z_K^ρ.*

In Def. 3.4.1 we construct meromorphic descent data associated with the continuous representation ρ for the trivial vector bundle $\mathcal{O}^n_{\mathcal{Y}_\rho}$ on $\mathcal{Y}_\rho$. Then, in the fourth section, we prove that this sheaf not only descends to a coherent sheaf on Z_K^ρ, as proved in the third section, but it also descends to a coherent sheaf on X_K, as stated in the following theorem.

Theorem (Thm. 3.4.5)**.** *Let X, (K^n, ρ), $\mathcal{Y}_\rho$ as above, then the sheaf with meromorphic descent data $\{\mathcal{O}^n_{\mathcal{Y}_\rho}, h^\rho_w\}_{w \in Aut(\mathcal{Y}_\rho|\widehat{X})}$ associated with ρ (see Def. 3.4.1) descends to a coherent sheaf on X_K.*

In particular these theorems lead us to the construction of the following functor, which is explained at the end of the third chapter.

Theorem (Thm. 3.4.8)**.** *Given X a projective semi-stable curve over S with smooth generic fibre, associating with $(K^n, \rho) \in \mathrm{Rep}_K^{cts}(\pi_1^{proét}(X_0, \xi))$ the coherent sheaf with meromorphic descent data $\{\mathcal{O}^n_{\mathcal{Y}_\rho}, h^\rho_w\}_{w \in Aut(\mathcal{Y}^\rho_\rho|\widehat{X})}$ induces a functor*

$$F\colon \mathrm{Rep}_K^{\mathrm{cts}}(\pi_1^{\mathrm{proét}}(X_0, \xi)) \to \mathrm{Coh}(X_K).$$

In Chapter 4 we recall the definition of stratified bundles, we introduce the notion of stratified bundles with meromorphic descent data and we show that we can extend the results of the previous chapter to all the Frobenius twists of X, which proves the following theorem.

Theorem (Prop. 4.2.4). *Given X a projective semi-stable curve over S with smooth generic fibre and $(K^n, \rho) \in \mathrm{Rep}_K^{\mathrm{cts}}(\pi_1^{\mathrm{pro\acute{e}t}}(X_0))$, let $\{\mathcal{O}^n_{\mathcal{Y}_\rho^{(i)}}, h_w^{\rho,i}\}$ be the stratified bundle on the i-th Frobenius twist of $\mathcal{Y}_\rho$ with meromorphic descent data induced by ρ, then $\{\mathcal{O}^n_{\mathcal{Y}_\rho^{(i)}}, h_w^{\rho,i}\}$ descends to a stratified bundle on X_K.*

Using the Tannakian formalism, we can finally prove the main result of the thesis.

Theorem (Thm. 4.2.7). *For a given projective semi-stable curve X over S with smooth generic fibre, the descent of stratified bundles with meromorphic descent data induces a functor*

$$sp_K\colon \mathrm{Rep}_K^{\mathrm{cts}}(\pi_1^{\mathrm{pro\acute{e}t}}(X_0,\xi)) \to \mathrm{Strat}(X_K).$$

Moreover, if $\overline{K}$ is a fixed algebraic closure of K, sp_K can be extended to

$$sp_{\overline{K}}\colon \mathrm{Rep}_{\overline{K}}^{\mathrm{cts}}(\pi_1^{\mathrm{pro\acute{e}t}}(X_0,\xi)) \to \mathrm{Strat}(X_{\overline{K}}),$$

and it induces a morphism of group schemes over $\overline{K}$

$$sp\colon \pi^{\mathrm{strat}}(X_{\overline{K}}) \to (\pi_1^{\mathrm{pro\acute{e}t}}(X_0,\xi))^{\mathrm{cts}}.$$

We conclude the fourth chapter by comparing this morphism of group schemes with the specialization map between the étale fundamental groups of $X_{\overline{K}}$ and X_0 constructed by Grothendieck in [SGA 1]. In particular, we show in Proposition 4.3.3 that the following diagram is commutative

$$\begin{array}{ccc}
\mathrm{Rep}_{\overline{K}}^{\mathrm{cts}}(\pi_1^{\mathrm{\acute{e}t}}(X_0,\xi)) & \xrightarrow{\mathrm{sp}_{SGA1}} & \mathrm{Rep}_{\overline{K}}^{\mathrm{cts}}(\pi_1^{\mathrm{\acute{e}t}}(X_{\overline{K}},\epsilon)) \\
\downarrow & & \downarrow \\
\mathrm{Rep}_{\overline{K}}^{\mathrm{cts}}(\pi_1^{\mathrm{pro\acute{e}t}}(X_0,\xi)) & \xrightarrow{\mathrm{sp}_{\overline{K}}} & \mathrm{Strat}(X_{\overline{K}})\ ,
\end{array}$$

where the left vertical arrow corresponds to composition with the pro-finite completion $\pi_1^{\mathrm{pro\acute{e}t}}(X_0,\xi) \to \pi_1^{\mathrm{\acute{e}t}}(X_0,\xi)$, the right vertical arrow is defined as descent along finite étale covering of $X_{\overline{K}}$ and the upper horizontal arrow is defined as the composition via the specialization map defined by Grothendieck.

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Conventions and notations

(a) The letter k will denote an algebraically closed field of characteristic $p > 0$.

(b) The letter A will denote a complete discrete valuation ring of characteristic $p > 0$ with residue field k.

(c) The letter K will denote the fraction field of A.

(d) The letter S will denote the scheme given by spectrum of A, i.e., $S = \mathrm{Spec}(A)$.

(e) Given a scheme X over S, we denote by X_0 its closed fibre and by X_K its generic fibre.

(f) Given a scheme X over S, we denote by $\widehat{X}$ the formal completion of X along its closed fibre X_0.

(g) All topological groups are assumed to be Hausdorff.

(h) Given G a topological group, we denote by G-Sets the category of sets S with a left G-action that is continuous with respect to the discrete topology on S.

(i) If G is a topological group we denote by G-FSets the category of finite sets with a continuous left G-action.

(j) Given a field F we denote by Vec_F the category of finite dimensional F-vector spaces.

(k) Given G and H two abstract groups, we denote the *free product of G and H* by $G \star H$. The elements of the underlying set are alternating sequences of non-trivial elements of G and H and the word with no letter, which is called the *empty word*. The group law on $G \star H$ is given by concatenation followed by reduction. We recall that reduction is the map that associates with any word an alternating sequence of elements in H and G by removing any instance of the identity element of either G or H, replacing any pair of the form $g_1 g_2$ by its product in G and any pair $h_1 h_2$ by its product in H. By abuse of notation we will call this group law *concatenation*. The neutral element for this group law is the empty word.

(l) If G and H are topological (Hausdorff) groups we call *free topological product of H and G* the co-product of G and H in the category of topological group, which was constructed in [Gra48]. Its underlying abstract group is free product $G \star H$, by abuse of notation, we will denote the free topological product of H and G by $G \star H$ too.

Chapter 1

Fundamental groups of normal crossing curves

In this chapter we describe the pro-étale fundamental group of of a normal crossing curve defined over an algebraically closed field F. In the first two sections we recall the definition of the pro-étale fundamental group and the main results of [BS15]. In the third section we compute explicitly the pro-étale fundamental of a normal crossing model via a descent argument.

1.1 Infinite Galois categories

In this section we present the definition of infinite Galois categories. This notion was first introduced in [Noo08] and later improved in [BS15]. Infinite Galois theory studies the conditions that force a category $\mathcal{C}$ to be equivalent to the category G-Sets for some topological group G.

Definition 1.1.1. Given a category $\mathcal{C}$ that admits colimits and finite limits, we say that an object $X \in \mathcal{C}$ is *connected* if it is not empty (i.e., not initial) and every subobject $Y \subset X$ (i.e., $Y \simeq Y \times_X Y$) is either empty or it coincides with X.

Definition 1.1.2. An *infinite Galois category* is a pair $(\mathcal{C}, F)$ consisting of a category $\mathcal{C}$ and a functor $F: \mathcal{C} \to \text{Sets}$ that satisfy the following conditions:

1. $\mathcal{C}$ admits colimits and finite limits,
2. each object $X \in \mathcal{C}$ is a disjoint union of connected objects,
3. $\mathcal{C}$ is generated under colimits by a set of connected objects,
4. F is faithful, conservative and commutes with colimits and finite limits.

Definition 1.1.3. Given $(\mathcal{C}, F)$ an infinite Galois category, we define its *fundamental group* to be the group $\pi_1(\mathcal{C}, F) := \text{Aut}(F)$ endowed with the topology induced by the compact-open topology on $\text{Aut}(S)$ for all $S \in \text{Sets}$.

We recall that a basis of open neighborhoods of $1 \in \text{Aut}(S)$ for the compact-open topology is given by the stabilizers $U_F \subset \text{Aut}(S)$ of finite subsets $F \subset S$.

In order to present some examples we need to define a specific class of topological groups.

Definition 1.1.4. Let G be a topological group and $F_G \colon G\text{-Sets} \to \text{Sets}$ be the forgetful functor. We say that G is a *Noohi group* if the natural map $G \to \text{Aut}(F_G)$ is an isomorphism of topological groups, where $\text{Aut}(F_G)$ is topologized by the compact-open topology on $\text{Aut}(S)$ for all $S \in \text{Sets}$.

Example 1.1.5 ([BS15], Ex. 7.1.2)**.** For every set S, the group $\text{Aut}(S)$ endowed with the compact-open topology is a Noohi group.

Definition 1.1.6. Given G a topological group, we define the *Raîkov completion of* G as its completion with respect to its two-sided uniformity (see [AT08]). We denote the Raîkov completion of G by $\widehat{G}^R$.

Definition 1.1.7. We say that a topological group G is *Raîkov complete* if the natural morphism $\sigma \colon G \to \widehat{G}^N$, constructed in [AT08, Thm. 3.6.10], is an isomorphism.

Remark 1.1.8. By [AT08, Thm. 3.6.10], given G a topological group there exists a continuous morphism $\sigma \colon G \to \widehat{G}^R$, whose image is dense in $\widehat{G}^R$. Let $(S, \rho) \in G$-Sets, then, by [BS15, Lemma 7.1.4], the group $\text{Aut}(S)$ endowed with the compact-open topology is Raîkov complete. Hence, by [AT08, Prop. 3.6.12], the action ρ of G on S admits an extension to $\hat{\rho}$ such that the following diagram commutes

$$\begin{array}{ccc} & \widehat{G}^R & \\ {\scriptstyle\sigma}\nearrow & & \searrow{\scriptstyle\hat{\rho}} \\ G & \xrightarrow{\quad\rho\quad} & \text{Aut}(S)\ . \end{array}$$

This induces an equivalence of categories between

$$(G\text{-Sets}, F_G) \simeq (\widehat{G}^R\text{-Sets}, F_{\widehat{G}^N}).$$

Proposition 1.1.9 ([BS15], Prop. 7.1.5)**.** *Let G be a topological group with a basis of open neighborhoods of $1 \in G$ given by open subgroups, then there is a natural isomorphism $\widehat{G}^R \simeq \text{Aut}(F_G)$. In particular, G is a Noohi group if and only if it is Raîkov complete.*

Example 1.1.10. By [AT08, Thm. 3.6.24], any locally compact group G is Raîkov complete. Hence, the previous proposition implies that locally compact groups G with a a basis of open neighborhoods of $1 \in G$ are Noohi groups. In particular, pro-finite groups and discrete groups are Noohi groups.

Example 1.1.11 ([BS15], Ex. 7.2.2)**.** Let G be a Noohi group and let F_G be the forgetful functor $F_G \colon G\text{-Sets} \to \text{Sets}$. The pair $(G\text{-Sets}, F_G)$ is an infinite Galois category. Moreover $\pi_1(G\text{-Sets}, F_G) = G$.

Remark 1.1.12. Not all the infinite Galois categories are of the type G-Sets for some topological group G. A counterexample is presented in [BS15, Ex. 7.2.3].

In order to describe the infinite Galois categories $(\mathcal{C}, F)$ that are equivalent to the category G-Sets, for a topological group G, we need to assume further conditions on the pair $(\mathcal{C}, F)$.

Definition 1.1.13. An infinite Galois category $(\mathcal{C}, F)$ is called *tame* if, for every connected object $X \in \mathcal{C}$, the group $\pi_1(\mathcal{C}, F)$ acts transitively on $F(X)$.

This definition is crucial and leads us to the following result.

Proposition 1.1.14 ([BS15], Thm 7.2.5)**.** *If $(\mathcal{C}, F)$ is an infinite Galois category its fundamental group $\pi_1(\mathcal{C}, F)$ is a Noohi group.*
If $(\mathcal{C}, F)$ is also tame the functor F induces an equivalence of categories

$$\mathcal{C} \simeq \pi_1(\mathcal{C}, F)\text{-Sets}.$$

1.2 Pro-étale fundamental group

In this section we give an overview of the definition of the pro-étale fundamental group, introduced in [BS15]. We start by defining the pro-étale site.

Definition 1.2.1. A morphism of schemes $f\colon Y \to X$ is called *weakly étale* if f and the diagonal morphism $\Delta_f\colon Y \to Y \times_X Y$ are both flat.

Lemma 1.2.2 ([BS15], Prop. 2.3.3)**.** *The composition and base change of weakly étale morphisms are weakly étale.*

Definition 1.2.3. We set $X_{\text{proét}}$ to be the category of weakly étale X-schemes, which we give the structure of a site by endowing it with the fpqc topology. We call $X_{\text{proét}}$ the *pro-étale site.*

Remark 1.2.4. The name pro-étale is justified by [BS15, Thm. 2.3.4], which implies that every weakly étale map $f\colon Y \to X$ is Zariski locally on X and locally in $Y_{\text{proét}}$ a pro-étale morphism, i.e., an inverse limit of étale morphisms. However, since pro-étale morphisms are not local on the target (as shown in [BS15, Ex. 4.1.12]), it is preferable to work with weakly étale morphisms.

The main ingredient needed for the definition of the pro-étale fundamental group is the category of locally constant sheaves on the pro-étale site.

Lemma 1.2.5 ([BS15], Lem. 4.2.12)**.** *Let T be a topological space, then the presheaf $\mathcal{F}_T$ that associates to every $U \in X_{\text{proét}}$ the set of continuous maps from U to T, i.e.,*

$$\mathcal{F}_T(U) = \text{Map}_{\text{cts}}(U, T),$$

is a sheaf on the pro-étale site. Moreover, if T is discrete, then $\mathcal{F}_T$ is the constant sheaf associated with T, i.e.,

$$\mathcal{F}_T(U) = \text{Map}(\pi_0(U), T),$$

where $\pi_0(U)$ is the set of connected components of $U \in X_{\text{proét}}$.

Definition 1.2.6. We say that a sheaf $\mathcal{F}$ on $X_{proét}$ is *constant* if there exists a topological space T such that $\mathcal{F} \simeq \mathcal{F}_T$.

Note that by Lemma 1.2.5 this notion coincides with the usual definition of constant sheaves only if the topological space T is discrete.

Definition 1.2.7. Let $\mathcal{F} \in \text{Shv}(X_{\text{proét}})$, then $\mathcal{F}$ is called *locally constant* if there exists a covering $\{Y_i \to X\}_i$ in $X_{\text{proét}}$ such that $\mathcal{F}|_{Y_i}$ is constant.

We denote by Loc_X the full subcategory of $\text{Shv}(X_{\text{proét}})$ spanned by locally constant sheaves.

In [BS15] the authors prove that the category of locally constant sheaves is equivalent to the category of geometric coverings, which is defined as follows.

Definition 1.2.8. A sheaf $\mathcal{F} \in \mathrm{Shv}(X_{\text{proét}})$ is called *geometric covering* if it represented by an étale $Y \to X$ that satisfies the valuative criterion of properness.

We denote by Cov_X the full subcategory of $\mathrm{Shv}(X_{\text{proét}})$ spanned by geometric coverings.

Proposition 1.2.9 ([BS15], Lemma 7.3.9)**.** *Let X be a locally topologically Noetherian connected scheme, then*

$$\mathrm{Loc}_X = \mathrm{Cov}_X \subset \mathit{Shv}(X_{\text{proét}}).$$

Definition 1.2.10. Let X be a locally topologically Noetherian connected scheme. Let Ω be an algebraically closed field and $\xi \colon \mathrm{Spec}(\Omega) \to X$ be a geometric point of X. A *pro-étale neighbourhood of* ξ is defined as a pair (U, u) of a scheme U that is weakly étale over X and a geometric point $u \in U(\Omega)$ such that the following diagram is commutative,

$$\begin{array}{ccc} & U & \\ {\scriptstyle u}\nearrow & & \searrow \\ \mathrm{Spec}(\Omega) & \xrightarrow{\quad \xi \quad} & X\,. \end{array}$$

Definition 1.2.11. Let X and $\xi \colon \mathrm{Spec}(\Omega) \to X$ be as before, we define a *morphism of pro-étale neighbourhood* from (U, u) to (U', u') to be an morphism $f \colon U \to U'$ over X such that

$$u' = f \circ u.$$

Lemma 1.2.12. *Given X a locally topologically Noetherian connected scheme and ξ a geometric point of X, the category of pro-étale neighbourhood of ξ is cofiltered.*

Proof. This is proven following the same argument of [Stacks, Tag 04JW]. For more details see also [Stacks, Tag 0991]. □

It follows from the previous lemma that the opposite category of pro-étale neighbourhood of ξ is filtered. Hence, we can define the stalk of a sheaf on $X_{\text{proét}}$ at a given geometric point of X.

Definition 1.2.13. Let X be a locally topologically Noetherian connected scheme and ξ a geometric point of X, then for every $\mathcal{F} \in \mathrm{Shv}(X_{\text{proét}})$ we define *the stalk of $\mathcal{F}$ at ξ* as

$$\mathcal{F}_\xi := \mathrm{colim}_{(U,u)} \mathcal{F}(U),$$

where the colimit runs over the opposite category of pro-étale neighbourhoods of the geometric point ξ.

Proposition 1.2.14 ([BS15], Lemma 7.4.1)**.** *Let X be a locally topologically Noetherian connected scheme, ξ a geometric point of X and set ev_ξ to be the following functor:*

$$ev_\xi \colon \mathrm{Loc}_X \to \mathrm{Sets},\ ev_\xi(\mathcal{F}) = \mathcal{F}_\xi,$$

then the pair (Loc_X, ev_ξ) is an infinite tame Galois category.

Definition 1.2.15. Let X be a locally topologically Noetherian connected scheme and ξ a geometric point of X, then the *pro-étale fundamental group* of X at ξ is defined as the group $\pi_1^{\text{proét}}(X, \xi) := \text{Aut}(\text{ev}_\xi)$.

Corollary 1.2.16. *If X is a locally topologically Noetherian connected scheme and ξ is a geometric point of X, then the group $\pi_1^{\text{proét}}(X, \xi)$ is a Noohi group. Moreover, the functor ev_ξ induces an equivalence of categories*

$$ev_\xi \colon \text{Loc}_X \simeq \pi_1^{\text{proét}}(X, \xi)\text{-Sets}.$$

Remark 1.2.17. Let Y be a connected object in the category Cov_X, as in Definition 1.1.1, then clearly Y is a connected scheme. In the proof of [BS15, Lemma 7.4.1], the authors show also that if $Y \in \text{Cov}_X$ is a connected scheme, then it is a connected object in the category Cov_X. This implies, by the previous corollary, that connected geometric coverings of X correspond to sets with a continuous transitive $\pi_1^{\text{proét}}(X, \xi)$-action.

Remark 1.2.18. Let ϵ be another geometric point of X then it follows from Proposition 1.2.16 that the categories $\pi_1^{\text{proét}}(X, \xi)$-Sets and $\pi_1^{\text{proét}}(X, \epsilon)$-Sets are equivalent. Moreover, since both groups are Noohi groups, by [BS15, Thm. 7.2.5.(2)],

$$\pi_1^{\text{proét}}(X, \xi) \simeq \pi_1^{\text{proét}}(X, \epsilon).$$

Note that the condition of being Noohi groups is necessary to conclude that the groups are isomorphic. Indeed, if G is a topological group with a basis of open neighbourhoods of $1 \in G$ given by open subgroups and G is not a Noohi group, then Proposition 1.1.9 implies that G is not Raîkov complete. Moreover, it shows that the categories G-Sets and $\widehat{G}^R$-Sets are equivalent even if G and $\widehat{G}^R$ are not isomorphic.

From the pro-étale fundamental group, we can retrieve both the enlarged fundamental and the étale fundamental group, defined by Grothendieck in [SGA 3] and [SGA 1] respectively.

Proposition 1.2.19 ([BS15], Lemma 7.4.3 and Lemma 7.4.6)**.** *Let X be a locally topological Noetherian connected scheme and ξ a geometric point, then*

- *the pro-discrete completion of $\pi_1^{\text{proét}}(X, \xi)$ is isomorphic to the enlarged fundamental group $\pi_1^{\text{SGA3}}(X, \xi)$,*
- *the pro-finite completion of $\pi_1^{\text{proét}}(X, \xi)$ is isomorphic to the étale fundamental group $\pi_1^{\text{ét}}(X, \xi)$.*

Proposition 1.2.20 ([BS15], Lemma 7.4.10)**.** *If X is geometrically unibranch, then*

$$\pi_1^{\text{proét}}(X, \xi) \simeq \pi_1^{\text{ét}}(X, \xi).$$

1.3 Descent for étale and pro-étale fundamental group

The aim of this section is to generalize [SGA 1, Exp. IX Cor. 5.4] in terms of the pro-étale fundamental group and to present a concrete computation of the pro-étale fundamental group of a connected projective normal crossing curve.

Before we proceed, we state the definitions of normal crossing curves and stable curves.

Definition 1.3.1. Let C be a scheme of dimension 1 of finite type over an algebraically closed field F, then C is a *semi-stable curve* if it is reduced and its singular points are ordinary double points.

Definition 1.3.2. We say that a scheme C of dimension 1 of finite type over an algebraically closed field F is a *stable curve* if it is a semi-stable curve and the following conditions are satisfied

- C is a connected projective curve,
- C has arithmetic genus $p_a(C) = \dim_F \mathrm{H}^1(C, \mathcal{O}_C) \geq 2$,
- the non-singular rational components of C, if they exist, intersect the other irreducible components in at least 3 points.

Definition 1.3.3. Let F be an algebraically closed field and let C be a scheme of dimension 1 of finite type over F, then C is a *normal crossing curve* if the associated reduced scheme C_{red} is a semi-stable curve.

Definition 1.3.4. Let F be a field and let $\overline{F}$ be a fixed algebraic closure of F. A curve C over F is a *(semi-)stable curve* if $C_{\overline{F}} = C \times_F \mathrm{Spec}(\overline{F})$ is a (semi-)stable curve over $\overline{F}$. Similarly we say that C is a *normal crossing curve* if $C_{\overline{F}}$ is a normal crossing curve over $\overline{F}$.

Definition 1.3.5. Given a scheme S, we define a *semi-stable curve over S* to be a flat scheme X over S whose fibres are geometrically connected semi-stable curves.

Remark 1.3.6. Note that in the literature the condition that the fibres are geometrically connected is usually not assumed. However, in the thesis we need this assumption in order to define the fundamental groups of the geometric fibres and hence to construct the specialization functor.

Remark 1.3.7. If S is a Dedekind scheme and X is a semi-stable curve over S with smooth generic fibre, by [Liu02, Prop. 10.3.15.(c)], the scheme X is normal.

Definition 1.3.8. Given a scheme S, a *stable curve over S of genus g* is a proper flat scheme over S, whose fibres are stable curves of arithmetic genus g.

Definition 1.3.9. A stable curve C over an algebraically closed field F is called *degenerate* if the normalization of every irreducible component of C is isomorphic to $\mathbb{P}^1_F$.

There is a well-known structure theorem for the étale fundamental group of such curves ([SGA 1, Exp. IX Cor. 5.4]), but as we are not aware of a reference for a complete proof, we explain it in next paragraphs. In order to do this, we need the notion of the coproduct in the category of pro-finite groups.

Definition 1.3.10. Given two pro-finite groups G and H we define their *coproduct in the category of pro-finite groups* to be the pro-finite completion of their free topological product $G \star H$. We denote the coproduct of G and H in the category of pro-finite groups by $G \star_F H$.

Lemma 1.3.11 ([SGA 1], Exp. IX Cor. 5.4)**.** *Given X a connected projective semi-stable curve over an algebraically closed field F and ξ a geometric point of X, for $j = 1, \ldots, N$ let C_j be the irreducible components of X, $\overline{C_j}$ their normalizations and fix a geometric point ξ_j for every $\overline{C_j}$, then*

$$\pi_1^{\text{ét}}(X, \xi) \simeq \widehat{\mathbb{Z}}^{\star_F r} \star_F \pi_1^{\text{ét}}(\overline{C_1}, \xi_1) \star_F \cdots \star_F \pi_1^{\text{ét}}(\overline{C_N}, \xi_N),$$

where $r = p_a(X) - \sum_i p_a(\overline{C_i})$ is the difference of the arithmetic genera and $\widehat{\mathbb{Z}}^{\star_F r}$ is the coproduct in the category of pro-finite groups of r copies of $\widehat{\mathbb{Z}}$.

We prove the statement by induction on N, the number of irreducible components of X.

Base step: $N = 1$.

If I is the set of singular points of X, by [Liu02, Prop. 10.3.18] we have that

$$r = p_a(X) - p_a(\overline{X}) = |I|.$$

Since by assumption $X = C_1$ is irreducible, the normalization $\overline{X} = \overline{C_1}$ is connected. Moreover, the normalization $g \colon \overline{X} \to X$ is finite and surjective, thus by [SGA 1, Exp. IX, Thm. 4.12] it is a morphism of effective descent for finite étale coverings.

In this simple setting, the descent data can be described explicitly. We denote by (a_i, b_i) the pair of points of $\overline{X}$ that are identified to $x_i \in I$ in X and we set F_{a_i} and F_{b_i} to be the functors associating to a finite étale cover Y of $\overline{X}$ its fibers over a_i and b_i respectively. Then giving descent data for a finite étale scheme Y with respect to g is equivalent to giving a collection of bijections $\{\alpha_i \colon F_{a_i}(Y) \to F_{b_i}(Y)\}_{x_i \in I}$.

Let $\mathcal{C}$ be the category whose objects are given by the datum $(Y, \alpha_1, \ldots, \alpha_r)$, with Y a finite étale cover of $\overline{X}$ and $\alpha_i \colon F_{a_i}(Y) \to F_{b_i}(Y)$ isomorphisms of sets, and whose morphisms from (Y, α_i) to (Z, β_i) are given by $\overline{X}$-scheme morphisms $\varphi \colon Y \to Z$ such that, for every $i \in I$, the following diagram commutes

$$\begin{array}{ccc} F_{a_i}(Y) & \xrightarrow{\alpha_i} & F_{b_i}(Y) \\ {\scriptstyle F_{a_i}(\varphi)}\downarrow & & \downarrow{\scriptstyle F_{b_i}(\varphi)} \\ F_{a_i}(Z) & \xrightarrow{\beta_i} & F_{b_i}(Z) \end{array} .$$

By construction, the category $\mathcal{C}$ is equivalent to the category of finite étale coverings of $\overline{X}$ with descent data with respect to the map g, and hence it is equivalent to the category of finite étale coverings of X. In particular, if we set $\widetilde{F}_{\xi_1}$ to be the functor

$$\widetilde{F}_{\xi_1}(Y, \alpha_i) = F_{\xi_1}(Y),$$

then the pair $(\mathcal{C}, \widetilde{F}_{\xi_1})$ defines a Galois category. Hence, to prove the lemma it suffices to show that the pro-finite group associated with $(\mathcal{C}, \widetilde{F}_{\xi_1})$ is the coproduct in the category of pro-finite groups of the étale fundamental group of $\overline{X}$ and r copies of $\hat{\mathbb{Z}}$.

We claim that $(\mathcal{C}, \widetilde{F}_{\xi_1})$ is equivalent to $(\mathbb{Z}^{\star r} \star \pi_1^{\text{ét}}(\overline{X}, \xi_1)\text{-FSets}, \text{forg})$, the category of finite sets with a continuous $\mathbb{Z}^{\star r} \star \pi_1^{\text{ét}}(\overline{X}, \xi_1)$-action, where forg is the forgetful functor. If the claim is true, then it follows that $\mathcal{C}$ is equivalent to the category of finite sets with a continuous action of the pro-finite completion of $\mathbb{Z}^{\star r} \star \pi_1^{\text{ét}}(\overline{X}, \xi_1)$, i. e., $\widehat{\mathbb{Z}}^{\star_F r} \star_F \pi_1^{\text{ét}}(\overline{X}, \xi_1)$, and the base step is proved.

To prove the claim we first show that the functor $\widetilde{F}_{\xi_1}$ factors through the category of finite sets with a continuous $\mathbb{Z}^{\star r} \star \pi_1^{\text{ét}}(\overline{X}, \xi_1)$-action.

By definition, $\pi_1^{\text{ét}}(\overline{X}, \xi_1) = \text{Aut}(F_{\xi_1})$ acts on $\widetilde{F}_{\xi_1}(Y, \alpha_i) = F_{\xi_1}(Y)$ for every $(Y, \alpha_i) \in \mathcal{C}$. Since $\overline{X}$ is connected, we can choose, for every i, a path τ_i from a_i to b_i and a path σ_i from a_i to ξ_1, that are natural isomorphisms of functors

$$\tau_i \colon F_{a_i} \to F_{b_i} \text{ and } \sigma_i \colon F_{a_i} \to F_{\xi_1}.$$

We notice that $\alpha_i \in \text{Hom}(F_{a_i}(Y), F_{b_i}(Y))$ can be written as

$$\alpha_i = \tau_i \circ g_i \text{ for some } g_i \in \text{Aut}(F_{a_i}(Y)).$$

Hence, we can define the action ρ_i of i-th copy of $\mathbb{Z}$ on $F_{\xi_1}(Y)$ as

$$\rho_i(1) = \sigma_i \circ g_i \circ \sigma_i^{-1}.$$

To prove that $\widetilde{F}_{\xi_1}$ induces an equivalence of categories, we construct a quasi-inverse functor. Given an object $(S, \rho_1, \ldots, \rho_r, \rho_{\xi_1}) \in \mathbb{Z}^{\star r} \star \pi_1^{\text{ét}}(\overline{X}, \xi_1)$-FSets, there exists a finite étale cover Y of $\overline{X}$ such that

$$F_{\xi_1}(Y) \simeq (S, \rho_{\xi_1}).$$

Thus, we can define the following functor:

$$G_{\xi_1}(S, \rho_i, \rho_{\xi_1}) = (Y, \tau_i \circ \sigma_i^{-1} \circ \rho_i(1) \circ \sigma_i).$$

From the construction it is easy to see that G_{ξ_1} and $\widetilde{F}_{\xi_1}$ are quasi-inverse functors.

Inductive step: $N - 1 \implies N$.

We fix C_1 an irreducible component of X such that the geometric point ξ does not lie in C_1 and such that $X \setminus C_1$ is connected. We denote by (a_i^1, b_i^1) the pairs of points of $\overline{C_1}$ identified to a singular point x_i^1 of C_1 and we denote by I_1 the set of these pairs. We set $r_1 = |I_1| \geq 0$, then by the base case we conclude that

$$\pi_1^{\text{ét}}(C_1, \xi_1) \simeq \widehat{\mathbb{Z}}^{\star_F r_1} \star_F \pi_1^{\text{ét}}(\overline{C_1}, \xi_1).$$

Let $\overline{X}_{N-1}$ be the complement of $\overline{C_1}$ in $\overline{X}$. We denote by (a_i^{N-1}, b_i^{N-1}) the pairs of points of $\overline{X}_{N-1}$ identified to a singular point x_i^{N-1} of X, and we denote by I_{N-1} the set of these pairs. Moreover, we set X_{N-1} to be the curve obtained from $\overline{X}_{N-1}$ identifying the pairs in I_{N-1}. By construction, X_{N-1} is a connected projective semi-stable curve with $N-1$ irreducible components. Hence, by the inductive hypothesis,

$$\pi_1^{\text{ét}}(X_{N-1}, \xi) \simeq \widehat{\mathbb{Z}}^{\star_F r_{N-1}} \star_F \pi_1^{\text{ét}}(\overline{C_2}, \xi_2) \star_F \cdots \star_F \pi_1^{\text{ét}}(\overline{C_N}, \xi_N),$$

where $r_{N-1} = p_a(X) - \sum_{i=2}^{N} p_a(\overline{C_i})$. Note that by [Liu02, Prop. 10.3.18] we have

$$r_{N-1} = |I_{N-1}| - (N-1) + 1 = |I_{N-1}| - N + 2.$$

Finally, we denote by $I_{1,N-1}$ the set of pairs (a_i^1, b_i^{N-1}), with a_i^1 a point of $\overline{C_1}$ and b_i^{N-1} a point of $\overline{X_{N-1}}$, that are identified in the remaining singular points of X. We fix a pair $(a_0^1, b_0^{N-1}) \in I_{1,N-1}$. Note that $I_{1,N-1} \neq \emptyset$ because X is connected. Let X' be the curve obtained from gluing C_1 and X_{N-1} along the pair $(a_0^1, b_0^{N-1}) \in I_{1,N-1}$. We define $\mathcal{C}_0$ to be the category whose objects are of the form

$$(Y_1, Y_{N-1}, \alpha_0) \in \mathcal{C}', \text{ with}$$

- Y_1 a finite étale cover of C_1,
- Y_{N-1} a finite étale cover of X_{N-1},
- $\alpha_0\colon F_{a_0^1}(Y_1) \to F_{b_0^{N-1}}(Y_{N-1})$ isomorphism of sets,

and whose morphisms

$$(Y_1, Y_{N-1}, \alpha_0) \to (Z_1, Z_{N-1}, \beta_0)$$

are given by a pair $(\varphi_1, \varphi_{N-1})$ with

- $\varphi_1\colon Y_1 \to Z_1$ a morphism of C_1-schemes,
- $\varphi_{N-1}\colon Y_{N_1} \to Z_{N-1}$ a morphisms X_{N-1}-schemes,

such that the following diagram commutes

$$\begin{array}{ccc} F_{a_0^1}(Y_1) & \xrightarrow{\alpha_0} & F_{b_0^{N-1}}(Y_{N-1}) \\ {\scriptstyle F_{a_0}(\varphi_1)}\downarrow & & \downarrow{\scriptstyle F_{b_0}(\varphi_{N-1})} \\ F_{a_0^1}(Z_1) & \xrightarrow{\beta_0} & F_{b_0^{N-1}}(Z_{N-1}) . \end{array}$$

Clearly $\mathcal{C}_0$ is equivalent to the category of finite étale coverings of X'.

We claim that $\mathcal{C}_0$ is equivalent to $\pi_1^{\text{ét}}(C_1, \xi_1) \star \pi_1^{\text{ét}}(X_{N-1}, \xi)$-FSets, the category of finite sets with a continuous $\pi_1^{\text{ét}}(C_1, \xi_1) \star \pi_1^{\text{ét}}(X_{N-1}, \xi)$-action. If the claim is true, then we can conclude that

$$\pi_1^{\text{ét}}(X', \xi) \simeq \widehat{\mathbb{Z}}^{\star_F r_1 + r_{N-1}} \star_F \pi_1^{\text{ét}}(\overline{C_1}, \xi_1) \star_F \cdots \star_F \pi_1^{\text{ét}}(\overline{C_N}, \xi_N).$$

To prove the claim, we show that the functor $\widetilde{F}_{\xi_1}(Y_1, Y_{N-1}, \alpha_0) := F_{\xi_1}(Y_1)$ induces the wanted equivalence of categories.

As in the base step, we first define the actions of $\pi_1^{\text{ét}}(C_1, \xi_1)$ and $\pi_1^{\text{ét}}(X_{N-1}, \xi)$ on $F_{\xi_1}(Y_1)$. Since $\pi_1^{\text{ét}}(C_1, \xi_1) = \mathrm{Aut}(F_{\xi_1})$, clearly $\pi_1^{\text{ét}}(C_1, \xi_1)$ acts on $F_{\xi_1}(Y_1)$. Furthermore, the schemes C_1 and X_{N-1} are connected, so we can choose the paths

$$\sigma_1\colon F_{a_0^1} \to F_{\xi_1} \text{ and } \sigma_{N-1}\colon F_{b_0^{N-1}} \to F_\xi.$$

We call ρ the action of $\pi_1^{\text{ét}}(X_{N-1}, \xi) \simeq \mathrm{Aut}(F_\xi)$ on $F_\xi(Y_{N-1})$ and we define, for every $g \in \mathrm{Aut}(F_\xi)$,

$$\tau(g) = (\sigma_{N-1} \circ \alpha_0 \circ \sigma_1^{-1})^{-1} \circ \rho(g) \circ (\sigma_{N-1} \circ \alpha_0 \circ \sigma_1^{-1}).$$

Then τ is an action of $\mathrm{Aut}(F_\xi)$ on $F_{\xi_1}(Y_1)$.

To prove that $\widetilde{F}_{\xi_1}$ induces an equivalence of categories we construct a quasi-inverse functor. Given $(S, \rho_1, \rho_{N-1}) \in \pi_1^{\text{ét}}(C_1, \xi_1) \star \pi_1^{\text{ét}}(X_{N-1}, \xi)$-FSets, there exists a finite étale cover Y_1 of C_1 such that

$$F_{\xi_1}(Y_1) \simeq (S, \rho_1),$$

and a finite étale cover Y_{N-1} of X_{N-1} such that

$$F_\xi(Y_{N-1}) \simeq (S, \rho_{N-1}).$$

Thus, we can define the functor

$$G_{\xi_1}(S, \rho_1, \rho_{N-1}) = (Y_1, Y_{N-1}, \sigma_{N-1}^{-1} \circ \mathrm{Id}_S \circ \sigma_1).$$

From the construction it is easy to see that G_{ξ_1} and $\widetilde{F}_{\xi_1}$ are quasi-inverse functors.

So far we have computed the étale fundamental group of X', now we finally compute the étale fundamental group of X.

We observe that a finite étale covering of X corresponds to the datum of a finite étale covering Y of X' and the isomorphisms $\alpha_i \colon F_{a_i^1}(Y) \to F_{b_i^{N-1}}(Y)$ for every remaining pair of points $\{a_i^1, b_i^{N-1}\} \in I_{1,N-1}$. Using the same argument of the base step, where we replace $\overline{X}$ with X', we conclude that

$$\pi_1^{\text{ét}}(X, \xi) \simeq \widehat{\mathbb{Z}}^{\star_F |I_{1,N-1}|-1} \star_F \pi_1^{\text{ét}}(X', \xi).$$

Hence, we obtain that

$$\pi_1^{\text{ét}}(X, \xi) \simeq \widehat{\mathbb{Z}}^{\star_F r_1 + r_{N-1} + |I_{1,N-1}|-1} \star_F \pi_1^{\text{ét}}(\overline{C_1}, \xi_1) \star_F \cdots \star_F \pi_1^{\text{ét}}(\overline{C_N}, .\xi_N).$$

Moreover, by [Liu02, Prop. 10.3.18],

$$r = r_1 + r_{n-1} + |I_{1,n-1}| - 1 = |I| - N + 1 = p_a(X) - \sum_{i=1}^{N} p_a(\overline{C_j}).$$

Remark 1.3.12. The statement of the proposition does not depend on the choice of ξ_j, only the construction of the isomorphism does. Indeeed, since $\overline{C_j}$ are connected, if ϵ_j is another geometric point of $\overline{C_j}$ then we have

$$\pi_1^{\text{ét}}(\overline{C_j}, \xi_j) \simeq \pi_1^{\text{ét}}(\overline{C_j}, \epsilon_j).$$

In the next chapters the construction of the isomorphism will not matter, thus we will not specify the choice of geometric points ξ_j and we will often write instead

$$\pi_1^{\text{ét}}(X, \xi) \simeq \widehat{\mathbb{Z}}^{\star_F r} \star_F \pi_1^{\text{ét}}(\overline{C_1}) \star_F \cdots \star_F \pi_1^{\text{ét}}(\overline{C_N}).$$

Lemma 1.3.13 ([SGA 1], Exp I, Cor. 8.4)**.** *Let X be a locally Noetherian connected scheme, X_{red} its associated reduced subscheme and ξ a geometric point of X then*

$$\pi_1^{\text{ét}}(X_{\text{red}}, \xi) \simeq \pi_1^{\text{ét}}(X, \xi)$$

The previous lemma let us generalize Lemma 1.3.11 to normal crossing curves.

Corollary 1.3.14. *Given X a connected projective normal crossing curve over an algebraically closed field F and ξ a geometric point of X, for $j = 1, \ldots, N$ let C_j be the irreducible components of X, $\overline{C_j}$ their normalization and fix a geometric point ξ_j for every $\overline{C_j}$, then*

$$\pi_1^{\text{ét}}(X, \xi) \simeq \widehat{\mathbb{Z}}^{\star_F |I|-N+1} \star_F \pi_1^{\text{ét}}(\overline{C_1}, \xi_1) \star_F \cdots \star_F \pi_1^{\text{ét}}(\overline{C_N}, \xi_N),$$

where I is the set of singular points of X and $\widehat{\mathbb{Z}}^{\star_F |I|-N+1}$ is the coproduct in the category of pro-finite groups of $|I| - N + 1$ copies of $\widehat{\mathbb{Z}}$.

Proof. By Lemma 1.3.13 we have

$$\pi_1^{\text{ét}}(X,\xi) \simeq \pi_1^{\text{ét}}(X_{\text{red}},\xi).$$

Since, by definition of normal crossing curve, X_{red} is a semi-stable curve and it is projective and connected by assumption, we can apply Lemma 1.3.11 and we get

$$\pi_1^{\text{ét}}(X,\xi) \simeq \pi_1^{\text{ét}}(X_{\text{red}},\xi) \simeq \widehat{\mathbb{Z}}^{\star_F r} \star_F \pi_1^{\text{ét}}(\overline{C_1},\xi_1) \star_F \cdots \star_F \pi_1^{\text{ét}}(\overline{C_N},\xi_N),$$

where $r = p_a(X_{\text{red}}) - \sum_{i=1}^{N} p_a((\overline{C_j})_{\text{red}})$ is the difference of the arithmetic genera. Hence, we conclude by [Liu02, Prop. 10.3.18], which implies that

$$|I| - N + 1 = p_a(X_{\text{red}}) - \sum_{i=1}^{N} p_a((\overline{C_j})_{\text{red}}).$$

□

The pro-étale fundamental group is associated with the category of geometric coverings, described in Definition 1.2.8. Hence, to prove an analogous result for the pro-étale fundamental group we need first to show that finite surjective morphisms are morphisms of effective descent for geometric coverings.

Proposition 1.3.15 ([Ryd10], Thm. 5.19). *An universally closed surjective morphism of finite presentation $g\colon X' \to X$ is a morphism of effective descent for étale algebraic spaces.*

Corollary 1.3.16. *Let $g\colon X' \to X$ be a proper surjective morphism of finite presentation, then g is a morphism of effective descent for étale separated schemes.*

Proof. This follows from the previous proposition and [Ryd10, Thm. 5.4]. □

Remark 1.3.17. Geometric coverings of X are étale X-schemes that satisfy the valuative criterion of properness (see Definition 1.2.8), so they are, in particular, separated étale schemes over X. Let g be as in Corollary 1.3.16, then a geometric covering Y' of X' with descent data relative to g descends to a separated étale X-scheme Y. Moreover, since g is proper, Y' satisfies the valuative criterion of properness if and only if Y does. Hence, g is a morphism of effective descent for geometric coverings.

Furthermore, we need a notion of coproduct of Noohi groups.

Definition 1.3.18. Given two Noohi groups G and H, we define $\mathcal{C}_{G,H}$ to be the category of triples (S,ρ_G,ρ_H), where $S \in$ Sets and ρ_G and ρ_H are continuous actions of G and H on S, and we set $F_{G,H}(S,\rho_G,\rho_H) = S$ to be the forgetful functor. By [BS15, Example 7.2.6], $(\mathcal{C}_{G,H}, F_{G,H})$ is an infinite tame galois cateogory. We define the *coproduct of G and H in the category of Noohi groups* as the Noohi group $G \star_N H := \text{Aut}(F_{G,H})$.

Remark 1.3.19. The topological free product of two discrete groups is their abstract free product endowed with the discrete topology. Since by Example 1.1.10 discrete groups are Noohi groups, using the universal property of the coproduct it is easy to see that the coproduct of two discrete groups in the category of Noohi groups coincides with their topological free product.

We give now an alternative description of the coproduct in the category of Noohi groups. In what follows, given two topological groups G and H, we denote by $G \star H$ their topological free product constructed in [Gra48].

Lemma 1.3.20. *For two Noohi groups G and H with a basis of open neighborhoods of 1 given by open subgroups, we set $\mathcal{B}$ to be the collection of open subsets of $G \star H$ of the form*

$$x_1\Gamma_1 y_1 \cap \cdots \cap x_n \Gamma_n y_n,$$

with $n \in \mathbb{N}$, $x_i, y_i \in G \star H$ and $\Gamma_i \subseteq G \star H$ open subgroups of $G \star H$. If we restrict the topology on $G \star H$ to the topology induced by $\mathcal{B}$, we obtain a topological group $G \star_{\mathcal{B}} H$ with a basis of open neighborhoods of $1 \in G \star H$ given by open subgroups.

Proof. Given $x, y \in G \star H$ and $\Gamma \subset G \star H$ an open subgroup, $(z_1, z_2) \in m^{-1}(x\Gamma y)$ implies that

$$y z_2^{-1} z_1^{-1} x = (x^{-1} z_1 z_2 y^{-1})^{-1} \in \Gamma.$$

Hence, the multiplication is continuous because we have, for every x, y and Γ,

$$(z_1, z_2) \in x\Gamma y z_2^{-1} \times z_1^{-1} x \Gamma y \subset m^{-1}(x\Gamma y).$$

Let i be the inverse morphism, then $y^{-1}\Gamma x^{-1} \subset i^{-1}(x\Gamma y)$, for every x, y and every Γ, thus $G \star_{\mathcal{B}} H$ is a topological group.

To conclude, it suffices to show that every set $x\Gamma y \in \mathcal{B}$ such that $1 \in x\Gamma y$ contains an open subgroup of $G \star_{\mathcal{B}} H$. The condition $1 \in x\Gamma y$ implies that $x^{-1}y^{-1} \in \Gamma$. The set $y^{-1}\Gamma y$ is, by definition, an open subgroup of $G \star_{\mathcal{B}} H$. Moreover, we see that $y^{-1}\Gamma y \subset x\Gamma y$ because, given $\delta \in y^{-1}\Gamma y$, we have, for some $\gamma \in \Gamma$,

$$\delta = y^{-1}\gamma y = x(x^{-1}y^{-1})\gamma y \in x\Gamma y.$$

□

Corollary 1.3.21. *Let G and H be two Noohi groups with a basis of open neighborhoods of 1 given by open subgroups, then the coproduct in the category of Noohi groups $G \star_N H$ is isomorphic to the Raîkov completion of the topological group $G \star_{\mathcal{B}} H$, defined above.*

Proof. By Lemma 1.3.20, $G \star_{\mathcal{B}} H$ has a basis of open neighbourhoods of 1 given by open subgroups. Hence, by Proposition 1.1.9, it suffices to prove that the categories $G \star_N H$-Sets and $G \star_{\mathcal{B}} H$-Sets are equivalent.

By the universal property of the free topological product, $G \star_N H$ - Sets and $G \star H$-Sets are equivalent categories. Furthermore, the identity induces a continuous morphism $G \star H \to G \star_{\mathcal{B}} H$, which corresponds to a fully faithful functor $G \star_{\mathcal{B}} H$-Sets $\to G \star H$-Sets. As we will see in Lemma 2.3.4, an action ρ of $G \star_T H$ on a set S is continuous with respect to the discrete topology on S if and only if the corresponding map $\rho\colon G \star_T H \to \mathrm{Aut}(S)$ is continuous with respect to the compact-open topology on $\mathrm{Aut}(S)$. By definition, a basis of open neighborhoods of $1 \in \mathrm{Aut}(S)$ is given by stabilizers of finite subsets of S, hence every inverse image of an open neighborhoods of $1 \in \mathrm{Aut}(S)$ contains an open subgroup. This implies that also the map $\rho\colon G \star_{T'} H \to \mathrm{Aut}(S)$ is continuous, and hence that the functor induced by the identity is an equivalence of category. □

Before we generalize Lemma 1.3.14 for the pro-étale fundamental of normal crossing curves we prove the analogue of Lemma 1.3.13.

Lemma 1.3.22. *Let X be a locally Noetherian connected scheme, X_{red} its associated reduced subscheme and ξ a geometric point of X, then*

$$\pi_1^{\text{proét}}(X_{red}, \xi) \simeq \pi_1^{\text{proét}}(X, \xi).$$

Proof. By [SGA 1, Exp. I, Thm. 8.3] the category of schemes that are étale over X is equivalent to the category of schemes that are étale over X_{red}. Thus, it suffices to prove that an étale scheme Y over X satisfies the valuative criterion of properness if and only if $Y \times_X X_{\text{red}} = Y_{\text{red}}$ does.

Let R be any discrete valuation with fraction field F, then any morphism $\text{Spec}(F) \to Y$ factors through Y_{red} and similarly any morphism $\text{Spec}(R) \to X$ factors through X_{red}. Hence, it is clear that, for any diagram of the form

$$\begin{array}{ccc} \text{Spec}(F) & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \text{Spec}(R) & \longrightarrow & X, \end{array}$$

there exists a unique map $\text{Spec}(A) \to Y$ that makes the diagram commutative if and only if there exist a unique map $\text{Spec}(A) \to Y_{\text{red}}$ that makes the diagram between the associated reduced schemes commutative. □

Proposition 1.3.23. *Given X a connected projective normal crossing curve over an algebraically closed field F and ξ be a geometric point of X, for $j = 1, \ldots, N$ let C_j be the irreducible components of X, $\overline{C_j}$ their normalization and fix a geometric point ξ_j for every $\overline{C_j}$, then*

$$\pi_1^{\text{proét}}(X, \xi) \simeq \mathbb{Z}^{\star|I|-N+1} \star_N \pi_1^{\text{ét}}(\overline{C_1}, \xi_1) \star_N \cdots \star_N \pi_1^{\text{ét}}(\overline{C_N}, \xi_N),$$

where I is the set of singular points of X and $\mathbb{Z}^{\star|I|-N+1}$ is the free product of $|I| - N + 1$ copies of $\mathbb{Z}$.

Proof. By Lemma 1.3.22, we can assume that X is reduced and hence that it is a connected projective semi-stable curve. By Remark 1.3.16, the normalization is a morphism of effective descent for geometric coverings. Thus, using the same reasoning of Lemma 1.3.14, we can conclude that

$$\pi_1^{\text{proét}}(X, \xi) \simeq \mathbb{Z}^{\star r} \star_N \pi_1^{\text{proét}}(\overline{C_1}, \xi_1) \star_N \cdots \star_N \pi_1^{\text{proét}}(\overline{C_N}, \xi_N),$$

where $r = |I| - N + 1$. Morevoer, since $\overline{C_j}$ are normal, by Proposition 1.2.20 we have

$$\pi_1^{\text{proét}}(\overline{C_j}, \xi_j) \simeq \pi_1^{\text{ét}}(\overline{C_j}, \xi_j).$$

□

Remark 1.3.24. As for the étale case, we will not specify the choice of the geometric points of $\overline{C_j}$ and we will simply write

$$\pi_1^{\text{proét}}(X, \xi) \simeq \mathbb{Z}^{\star|I|-N+1} \star_N \pi_1^{\text{ét}}(\overline{C_1}) \star_N \cdots \star_N \pi_1^{\text{ét}}(\overline{C_N}).$$

Remark 1.3.25. With the same argument of Proposition 1.3.23, one can prove the analogous result for the enlarged fundamental group, defined by Grothendieck in [SGA 3, Exp.X]. Given G and H two pro-discrete groups, we set their coproduct in the category of pro-discrete group to be the pro-discrete completion of $G \star H$ and we denote it by $G \star_D H$. Then, under the assumptions of the previous proposition, we find that

$$\pi_1^{\mathrm{SGA3}}(X, \xi) \simeq \mathbb{Z}^{\star |I|-N+1} \star_D \pi_1^{\text{ét}}(\overline{C_1}) \star_D \cdots \star_D \pi_1^{\text{ét}}(\overline{C_N}).$$

Corollary 1.3.26. *Let F be an algebraically closed field and X be a degenerate stable curve over F, then*

$$\pi_1^{\text{proét}}(X) \simeq \mathbb{Z}^{\star r},$$

where $r = p_a(X)$.

Remark 1.3.27. If X is a degenerate stable curve over an algebraically closed field F, then by Proposition 1.2.19 and the previous corollary we have

$$\pi_1^{\text{proét}}(X) \simeq \pi_1^{\mathrm{SGA3}}(X).$$

As shown in Deligne's counterexample (see [BS15, Example 7.4.9]), if X is not degenerate this is no longer true. Remark 1.3.25 emphasizes that this is because the Noohi completion is, in general, not pro-discrete.

Chapter 2

Algebraic hulls

The goal of this chapter is to define, using Tannakian duality, the topological algebraic hull of a topological group and to describe its properties. This notion will be used in the next chapters to define the specialization homomorphism.

2.1 Tannakian categories

In this section we give a brief overview on Tannakian categories.

Definition 2.1.1. Given $\mathcal{T}$ a category and $\otimes\colon \mathcal{T}\times\mathcal{T}\to\mathcal{T}$ a functor, let $\mathbb{I}$ be an object of $\mathcal{T}$ and e an isomorphism $e\colon \mathbb{I}\to\mathbb{I}\otimes\mathbb{I}$, then we say that the pair $(\mathbb{I},e)$ is an *identity object* for $\otimes$ if the functors $X\mapsto X\otimes\mathbb{I}$ and $X\mapsto\mathbb{I}\otimes X$ are equivalences of categories.

Definition 2.1.2. A *tensor category* is a pair $(\mathcal{T},\otimes)$ given by a category $\mathcal{T}$ and an associative and commutative functor $\otimes\colon \mathcal{T}\times\mathcal{T}\to\mathcal{T}$, for which there exists an identity object.

Definition 2.1.3. Given $(\mathcal{T},\otimes)$ and $(\mathcal{T}',\otimes')$ two tensor categories, a *tensor functor* between them is a pair (F,φ), given by a functor $F\colon\mathcal{T}\to\mathcal{T}'$ and a functorial isomorphism $\varphi_{X,Y}\colon F(X)\otimes' F(Y)\to F(X\otimes Y)$ that is compatible with associative and commutative constraints and such that if $(\mathbb{I},e)$ is an identity object for $\otimes$ then $(F(\mathbb{I}),F(e))$ is an identity object for $\otimes'$.

Definition 2.1.4. Let $(\mathcal{T},\otimes)$ and $(\mathcal{T}',\otimes')$ be two tensor categories and F, G two tensor functors from $(\mathcal{T},\otimes)$ to $(\mathcal{T}',\otimes')$, then a morphism of functors $\lambda\colon F\to G$ is a *morphism of tensor functors* if, for all finite families $(X_i)_{i\in I}$ of objects of $\mathcal{T}$, the following diagram is commutative

$$\begin{array}{ccc} \bigotimes_{i\in I} F(X_i) & \longrightarrow & F(\bigotimes_{i\in I} X_i) \\ \big\downarrow{\scriptstyle \otimes_i\lambda_{X_i}} & & \big\downarrow{\scriptstyle \lambda_{\otimes_i X_i}} \\ \bigotimes_{i\in I} G(X_i) & \longrightarrow & G(\bigotimes_{i\in I} X_i)\ . \end{array}$$

Definition 2.1.5. A tensor category $(\mathcal{T},\otimes)$ is *rigid* if for every object X of $\mathcal{T}$ there exists an object $X^\vee$ of $\mathcal{T}$ and morphisms $\mathrm{ev}\colon X\otimes X^\vee\to\mathbb{I}$, $\delta\colon\mathbb{I}\to X\otimes X^\vee$, such that the compositions

$$X \xrightarrow{X\otimes\delta} X \otimes X^\vee \otimes X \xrightarrow{\mathrm{ev}\otimes X} X.$$

$$X^\vee \xrightarrow{\delta\otimes X^\vee} X^\vee \otimes X \otimes X^\vee \xrightarrow{X^\vee\otimes \mathrm{ev}} X^\vee.$$

are the identity.

Definition 2.1.6. A tensor category $(\mathcal{T}, \otimes)$ is *abelian* if $\mathcal{T}$ is an abelian category and $\otimes$ is a bi-additive functor.

Definition 2.1.7. Given a field F, a *neutral Tannakian category over* F is a rigid abelian tensor category $(\mathcal{T}, \otimes)$ such that $\mathrm{End}(\mathbb{I}) = F$ and such that it admits an exact F-linear tensor functor $\omega\colon \mathcal{T} \to \mathrm{Vec}_F$, which is called *fibre functor.*

Since we will deal only with neutral Tannakian categories, we will sometimes drop the adjective neutral.

Example 2.1.8. Let G be an affine group scheme over a field F, $\mathrm{Rep}_F(G)$ the category of its finite dimensional representations and $\omega_G\colon \mathrm{Rep}_F(G) \to \mathrm{Vec}_F$ the forgetful functor, then $(\mathrm{Rep}_F(G), \omega_G)$ is a neutral Tannakian category.

In fact, by the theorem of Tannakian duality, all neutral Tannakian categories will be of this form. Moreover, the tensor functors between them will be characterized by the corresponding group morphisms.

Definition 2.1.9. Let $\mathcal{T}$ be a neutral Tannakian category over a field F and ω a fibre functor of $\mathcal{T}$, then we define the functor $\underline{\mathrm{Aut}}^{\otimes}(\omega)\colon F\text{-Alg} \to \mathrm{Grp}$ on an F-algebra R to be the set $\mathrm{Aut}^{\otimes}(\phi_R \circ \omega)$ of automorphisms of the tensor functor $\phi_R \circ \omega$, where $\phi_R\colon \mathrm{Vec}_F \to \mathrm{Mod}_R$ is the functor $V \mapsto V \otimes_F R$.

Theorem 2.1.10 ([DM82], Thm. 2.11)**.** *If $\mathcal{T}$ is a neutral Tannakian category over a field F and ω is a fibre functor of $\mathcal{T}$, then*

- *the functor $\underline{Aut}^{\otimes}(\omega)$ is represented by an F-Hopf algebra B, i.e., by an affine group scheme $G = \mathrm{Spec}(B)$ over F,*
- *the functor ω defines an equivalence of category between $\mathcal{T}$ and $\mathrm{Rep}_F(G)$, the category of representations of G.*

Definition 2.1.11. Let $\mathcal{T}$ be a neutral Tannakian category over a field F and ω a fibre functor, then the group scheme over F that represents $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is called *Tannakian fundamental group of* $(\mathcal{T}, \omega)$ and it is denoted by $\pi(\mathcal{T}, \omega)$.

Proposition 2.1.12 ([DM82], Cor. 2.9)**.** *Let G and G' be two affine F-group schemes, $\mathrm{Rep}_F(G)$ and $\mathrm{Rep}_F(G')$ the neutral Tannakian categories of their representations, ω and ω' the forgetful functors, and $F\colon \mathrm{Rep}_F(G) \to \mathrm{Rep}_F(G')$ a tensor functor such that $\omega' \circ F = \omega$, then there exists a unique homomorphism $f\colon G' \to G$ such that*

$$F(V, \rho) = (V, \rho \circ f) \text{ for every } (V, \rho) \in \mathrm{Rep}_k(\pi(\mathcal{T}, \omega)) \simeq \mathcal{T}.$$

Proposition 2.1.13 ([DM82], Prop. 2.21)**.** *Let $\mathcal{T}$ and $\mathcal{T}'$ be two neutral Tannakian categories over a field F, ω and ω' fibre functors for $\mathcal{T}$ and $\mathcal{T}'$, $G\colon \mathcal{T} \to \mathcal{T}'$ a tensor functor such that $\omega' \circ G = \omega$ and $g\colon \pi(\mathcal{T}', \omega') \to \pi(\mathcal{T}, \omega)$ the corresponding morphism of group schemes over F, then*

- *g is faithfully flat if and only if G is fully faithful and, for every $X \in \mathcal{T}$, every subobject of $G(X)$ is isomorphic to the image of a subobject of X,*
- *g is a closed immersion if and only if every object of $\mathcal{T}'$ is isomorphic to a subquotient of an object of the form $G(X)$, for some $X \in \mathcal{T}$.*

2.2 Representations of abstract groups

We define, in this section, the algebraic hull of an abstract group and we present some examples that are relevant for the next chapters.

Definition 2.2.1. Given F a field and G an abstract group, let $\mathrm{Rep}_F(G)$ be the category of finite dimensional F-linear representations of G and ω_G the forgetful functor, then $\mathrm{Rep}_F(G)$ is a neutral Tannakian category over F and ω_G is a fibre functor. We define the *algebraic hull of G over F* as the affine group scheme $G^{\mathrm{alg}} := \pi(\mathrm{Rep}_F(G), \omega_G)$.

An explicit example is given by the following well known result.

Lemma 2.2.2. *Let G be a finite group, then its algebraic hull G^{alg} over a given field F is isomorphic to the constant group scheme over F associated with G.*

Proof. Since G is finite, the category of finite dimensional F-linear representations $\mathrm{Rep}_F(G)$ is equivalent to the category of finitely generated $F[G]$-modules, where $F[G]$ is the F-Hopf algebra generated by the elements of G. Let F^G be the dual F-Hopf algebra of $F[G]$, then $\mathrm{Rep}_F(G)$ is equivalent to the category of finitely generated F^G-comodules. This implies that $G^{\mathrm{alg}} = \mathrm{Spec}(F^G)$ and hence, by [Wat79, §2.3], G^{alg} is the constant group scheme associated with G. □

Remark 2.2.3. If G is an infinite abstract group, the previous statement does not hold. Indeed, if G is infinite, the constant group scheme over F associated with G is not quasi-compact, as it is an infinite disjoint union of $\mathrm{Spec}(k)$. Thus, it is not isomorphic to the affine group scheme G^{alg}.

In general, it is difficult to compute explicitly the algebraic hull of an abstract group. The next result shows that if F is a perfect field of characteristic $p > 0$, then the algebraic hull over F of any abstract group is reduced.

Definition 2.2.4. Let A be a ring of characteristic $p > 0$. We say that $\mathrm{Spec}(A)$ is a *perfect scheme* if A is a perfect ring, i.e., if the absolute Frobenius $F_A \colon A \to A$ is an isomorphism.

Note that if $\mathrm{Spec}(A)$ is a perfect scheme then it is, in particular, reduced.

Lemma 2.2.5. *If F is a perfect field of characteristic $p > 0$ and G is an abstract group, then the algebraic hull of G over F is a perfect scheme.*

Proof. The argument we will use is similar to the one in [San07, Thm. 11], but our case is much simpler.

Let $G^{\mathrm{alg}} = \mathrm{Spec}(B)$, we denote by $G^{\mathrm{alg}^{(1)}} = \mathrm{Spec}(B^{(1)})$ the Frobenius twist of G^{alg} and by $F^{(1)} \colon G^{\mathrm{alg}} \to G^{\mathrm{alg}^{(1)}}$ the relative Frobenius. Since F is perfect, proving that G^{alg} is perfect is equivalent to proving that $F^{(1)}$ is an isomorphism.

Let V be a finite dimensional F-vector space, we define $V^{(1)}$ to be the F-vector space whose underlying additive group is V and whose scalar multiplication is given by $\lambda \cdot v = \mu v$, for the unique $\mu \in F$ such that $\mu^p = \lambda$.

By Theorem 2.1.10, a G-representation on V corresponds to a B-comodule structure $\rho\colon V \to V \otimes_F B$. With a G-representation ρ on V we can associate also a $B^{(1)}$-comodule structure on $V^{(1)}$. Namely, we can define

$$\rho^{(1)}\colon V^{(1)} \to (V \otimes_F B)^{(1)} \simeq V^{(1)} \otimes_F B^{(1)},$$

as the map ρ between the underlying additive groups, which is F-linear because

$$\rho^{(1)}(\lambda \cdot v) = \rho(\mu v) = \mu\rho(v) = \lambda \cdot \rho^{(1)}(v).$$

We set $(-)^{(1)}\colon \mathrm{Rep}_F(G) \to \mathrm{Rep}_F(G^{\mathrm{alg}^{(1)}})$ to be the functor that associates to every representation ρ the corresponding $\rho^{(1)}$, which clearly is an equivalence of categories.

Given $(V, \rho) \in \mathrm{Rep}_F(G)$, we define its Frobenius twist as

$$\mathrm{Ft}(V, \rho) = (V^{(1)}, \mathrm{Ft}(\rho)),$$

where $\mathrm{Ft}(\rho)$ is the B-comodule structure defined as the composition

$$V^{(1)} \xrightarrow{\rho^{(1)}} V^{(1)} \otimes_F B^{(1)} \xrightarrow{\mathrm{id}\otimes F^{(1)}} V^{(1)} \otimes_F B.$$

In terms of representations, if we fix a base of V such that $\rho\colon G \to \mathrm{Aut}(V)$ is defined by $\rho(g) = (a_{ij}(g))$, then $\mathrm{Ft}(\rho)(g)$ is defined by the matrix whose coefficients are given by

$$\mathrm{Ft}(\rho)(g) = (a_{ij}^p(g)) \in \mathrm{Aut}(V^{(1)}).$$

With this second description it is easy to see that, since F is perfect, the Frobenius twist is an equivalence of categories.

Furthermore, by construction, the following diagram is commutative

$$\begin{array}{ccccc} & & \mathrm{Rep}_F(G^{\mathrm{alg}^{(1)}}) & & \\ & \overset{(-)^{(1)}}{\nearrow} & & \overset{F^{(1)}}{\searrow} & \\ \mathrm{Rep}_F(G) & & \xrightarrow{\qquad\mathrm{Ft}\qquad} & & \mathrm{Rep}_F(G) \end{array}.$$

Hence, the functor induced by the relative Frobenius $F^{(1)}$ is also an equivalence of categories. Since the relative Frobenius is F-linear, this implies, by Prop. 2.1.13, that the corresponding F-group schemes G^{alg} and $G^{\mathrm{alg}^{(1)}}$ are isomorphic.

Note that, instead, the functors $(-)^{(1)}$ and Ft are not F-linear. □

2.3 Representations of topological groups

Another example of Tannakian categories is the category of continuous representations of a topological group. In this section we study the properties of the affine group scheme associated with this category.

Definition 2.3.1. Given a field F and a topological group G, we call *continuous F-linear representation of G* a pair (V,ρ) given by a finite dimensional F-vector space V and an F-linear action $\rho\colon G\times V\to V$ that is continuous with respect to the discrete topology on V. We denote by $\mathrm{Rep}_F^{\mathrm{cts}}(G)$ the category of continuous F-linear representations of G.

It is easy to see that the category $\mathrm{Rep}_F^{\mathrm{cts}}(G)$ is a neutral Tannakian category and that the the forgetful functor, denoted by ω_G, is a fibre functor.

Definition 2.3.2. Let F be a field and G a topological group, then the *topological algebraic hull of G over F* is defined as the affine group scheme $G^{\mathrm{cts}} := \pi(\mathrm{Rep}_F^{\mathrm{cts}}(G),\omega_G)$.

Remark 2.3.3. Let F be a field and G be an abstract group endowed with the discrete topology, then it is clear that $G^{\mathrm{cts}} = G^{\mathrm{alg}}$.

The following lemma is a well-known elementary result in topology theory. We recall it here as it will be the starting point for the construction of the specialization functor.

Lemma 2.3.4. *Given F a field, V a finite dimensional F-vector space and $\rho\colon G\times V\to V$ an F-linear G-action on V, the following are equivalent:*

(a) *ρ is continuous with respect to the discrete topology on V,*

(b) *the group morphism $\tilde\rho\colon G\to \mathrm{Aut}(V)$, induced by ρ, is continuous with respect to the compact-open topology on* $\mathrm{Aut}(V)$*,*

(c) *the group morphism $\tilde\rho\colon G\to \mathrm{Aut}(V)$, induced by ρ, is continuous with respect to the discrete topology on* $\mathrm{Aut}(V)$*.*

Proof. We recall that a basis of open neighbourhoods of $\mathrm{Id}\in\mathrm{Aut}(V)$ for the compact-open topology is given by the stabilizers U_F of finite subsets $F\subset V$.

Condition (a) is satisfied if and only if for every $v\in V$ the set

$$A_{\{v\}} := \{(g,w)\in G\times V|\ \rho(g,w)=v\}\subset G\times V$$

is open in $G\times V$. For every $v,w\in V$, let $U_{\{w\}}\subset\mathrm{Aut}(V)$ be the stabilizer of w, then we have

$$B_{v,w} := \{g\in G|\ \rho(g,w)=v\} = \begin{cases}\emptyset \text{ if } v \text{ is not in the orbit of } w\\ g\cdot\tilde\rho^{-1}(U_{\{w\}}) \text{ for } g\in B_{v,w}, \text{ otherwise}\end{cases}.$$

Hence, if condition (b) is satisfied, the set $B_{v,w}$ is open in G for every $v,w\in V$. Since

$$A_{\{v\}} = \bigcup_{w\in V}(B_{v,w}\times\{w\})\subset G\times V,$$

it is clear that (b) implies (a).

On the other hand (b) is satisfied if and only if for every finite subset $F\subset V$ $\tilde\rho^{-1}(U_F)\subset G$ is open. By condition (a), the set

$$A_F := \rho^{-1}(F) = \{(g,w)\in G\times V|\ \rho(g,w)\in F\}$$

is open. Hence, in particular, for every $v \in F$ the set $A_F \cap (G \times \{v\})$ is open. Let $p_1 \colon G \times V \to G$ be the first projection, then

$$\tilde{\rho}^{-1}(U_F) = \{g \in G | \rho(g, v) \in F, \text{ for every } v \in F\} = \bigcap_{v \in F} p_1(A_F \cap G \times \{v\}).$$

Since p_1 is an open map, it follows that (a) and (b) are equivalent.

It remains to prove that (b) and (c) are equivalent. Let us choose a basis $\{e_1, \ldots, e_n\}$ of V, then the singleton

$$\{\mathrm{Id}\} = U_{\{e_1\}} \cap \cdots \cap U_{\{e_n\}} \subset \mathrm{Aut}(V)$$

is open for the the compact-open topology on $\mathrm{Aut}(V)$. Hence, for every automorphism $\varphi \in \mathrm{Aut}(V)$ the singleton $\{\varphi\}$ is open for the the compact-open topology on $\mathrm{Aut}(V)$. This implies that the compact-open topology on $\mathrm{Aut}(V)$ coincides with the discrete topology, so, clearly, (b) and (c) are equivalent. □

In the next statement we describe the topological algebraic hull of a pro-finite group. This will be later applied to compute the topological algebraic hull of the étale fundamental group of a scheme.

Lemma 2.3.5. *Let F be a field and $\pi = \varprojlim_i \pi_i$ a complete pro-finite group with surjective transition maps, then π^{cts}, the topological algebraic hull of π over F, is isomorphic to F- group scheme*

$$\varprojlim_i (\pi_i)_F,$$

where $(\pi_i)_F$ are the constant group schemes over F associated with the finite quotients π_i.

Proof. Since π_i is finite, $\pi_i^{\mathrm{cts}} = \pi_i^{\mathrm{alg}}$ and hence, by Lemma 2.2.2, π_i^{cts} is the constant group scheme over F associated with π_i, which we denote by $(\pi_i)_F$.

The natural map $\mathrm{pr}_i \colon \pi \to \pi_i$ induces a tensor functor between the categories of continuous representations

$$F_{\varphi_i} \colon \mathrm{Rep}_F^{\mathrm{cts}}(\pi_i) \to \mathrm{Rep}_F^{\mathrm{cts}}(\pi),\ F_{\varphi_i}(V, \rho) := (V, \rho \circ \mathrm{pr}_i)$$

and, by Proposition 2.1.12, it induces, for each i, a morphism of F-group schemes

$$\varphi_i \colon \pi^{\mathrm{cts}} \to (\pi_i)_F.$$

Hence, there exists a natural morphism of F-group schemes

$$\varphi \colon \pi^{\mathrm{cts}} \to \varprojlim_i (\pi_i)_F,$$

which corresponds to a functor

$$F_\varphi \colon \mathrm{Rep}_F(\varprojlim_i (\pi_i)_F) \to \mathrm{Rep}_F(\pi^{\mathrm{cts}}) \simeq \mathrm{Rep}_F^{\mathrm{cts}}(\pi).$$

By hypothesis, the maps pr_i are surjective and this implies that the functor F_{φ_i} is fully faithful. Furthermore, it is easy to show that F_{φ_i} satisfies the criterion of Proposition 2.1.13.(i). Thus, the corresponding morphism of F-group scheme φ_i is faithfully flat and, in particular, it is surjective.

Let us set $\pi^{\mathrm{cts}} = \mathrm{Spec}(A)$ and $(\pi_i)_F = \mathrm{Spec}(B_i)$. Since $(\pi_i)_F$ is reduced, the affine morphism φ_i corresponds to an injective morphism of F-Hopf algebras $\varphi_i \colon B_i \subset A$. Then the induced map $\varinjlim_i B_i \to A$, which corresponds to the morphism φ, is injective because filtered colimits of rings are left exact. By [Mil12, VI,Thm 11.1], we can conclude that φ is faithfully flat, and hence by Proposition 2.1.13.(i), F_φ is fully faithful.

It remains to show that F_φ is essentially surjective. By Lemma 2.3.4, given an object $(V, \rho) \in \mathrm{Rep}_F^{\mathrm{cts}}(\pi)$, the map $\tilde{\rho} \colon \pi \to \mathrm{Aut}(V)$, induced by ρ, is continuous with respect to the discrete topology on $\mathrm{Aut}(V)$. Thus, $\tilde{\rho}$ factors through a finite quotient of π, say π_i. This means that there exists a group morphism

$$\widetilde{\rho_i} \colon \pi_i \to \mathrm{Aut}(V) \text{ such that } \widetilde{\rho_i} \circ pr_i = \tilde{\rho}.$$

In particular, this implies that the π^{cts}-action on V corresponding to ρ, which, by abuse of notation, we denote again by ρ, factors through the $(\pi_i)_F$-action ρ_i induced by $\widetilde{\rho_i}$.

Let $p_i \colon \varprojlim_i \pi_i^{\mathrm{cts}} \to \pi_i^{\mathrm{cts}}$ be the natural morphism of F- group schemes, then, by construction, the following diagram commutes.

$$\begin{array}{ccccc} & & \varprojlim_i \pi_i^{\mathrm{cts}} & & \\ & \overset{\varphi}{\nearrow} & & \overset{p_i}{\searrow} & \\ \pi^{\mathrm{cts}} & & \xrightarrow{\quad \varphi_i \quad} & & \pi_i^{\mathrm{cts}} \end{array}.$$

Hence, we have

$$F_\varphi(V, \rho_i \circ p_i) = (V, \rho_i \circ p_i \circ \varphi) = (V, \rho_i \circ \varphi_i) = (V, \rho).$$

□

Remark 2.3.6. The group scheme $\varprojlim_i (\pi_i)_F$ is often denoted in the literature as π_F, even though it is not the constant group associated with π over F. We will also follow this notation.

Remark 2.3.7. We can generalize the previous lemma to pro-discrete groups. Namely, we can prove that, if $\pi = \varprojlim_i \pi_i$ is a complete pro-discrete group with surjective transition maps, then its topological algebraic hull over a given field F is isomorphic to the F-group scheme

$$\varprojlim_i \pi_i^{\mathrm{alg}}.$$

Indeed, by Lemma 2.2.5, the group schemes π_i^{alg} are reduced and hence we can apply the same argument of the previous lemma.

The argument of Lemma 2.2.5, can be easily translated in terms of topological algebraic hulls and we find the following result.

Lemma 2.3.8. *Let F be a perfect field of characteristic $p > 0$ and G a topological group, then the topological algebraic hull of G over F is a perfect scheme. In particular, G^{cts} is reduced.*

2.4 Free product of algebraic hulls

We are interested in studying the topological algebraic hull of the pro-étale fundamental group of a projective semi-stable curve. As saw in the previous chapter, this essentially consists in studying the topological algebraic hull of a free product of topological groups. In this section we introduce the notion of free product of affine group schemes and we show that the free product of topological algebraic hulls is compatible with the free product of topological groups.

Definition 2.4.1. Given $\mathcal{T}_1$ and $\mathcal{T}_2$ two neutral Tannakian categories over a field F and ω_1, ω_2 respectively fixed fibre functors, we set $\mathcal{T}_1 \times_F \mathcal{T}_2$ to be the category whose objects are given by triples (V, ρ_1, ρ_2), with

- $V \in \mathrm{Vec}_F$,
- $\rho_1 \colon \pi(\mathcal{T}_1, \omega_1) \to \mathrm{GL}_V$ an F-linear $\pi(\mathcal{T}_1, \omega_1)$-action,
- $\rho_2 \colon \pi(\mathcal{T}_2, \omega_2) \to \mathrm{GL}_V$ an F-linear $\pi(\mathcal{T}_2, \omega_2)$-action,

and whose morphisms from (V, ρ_1, ρ_2) to (W, τ_1, τ_2) are given by F-linear morphisms $\varphi \colon V \to W$ that are $\pi(\mathcal{T}_1, \omega_1)$-equivariant and $\pi(\mathcal{T}_2, \omega_2)$-equivariant.

We call $\mathcal{T}_1 \times_F \mathcal{T}_2$ the *Tannakian product category of* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *over* F.

If for $i = 1, 2$ $\widetilde{\omega}_i \colon \mathcal{T}_i \to \mathrm{Rep}_F(\pi(\mathcal{T}_i, \omega_i))$ is the equivalence of categories induced by ω_i, we call *projections* the functors

$$p_i \colon \mathcal{T}_1 \times_F \mathcal{T}_2 \to \mathcal{T}_i,\ p_i(V, \rho_1, \rho_2) = \widetilde{\omega}_i^{-1}(V, \rho_i),$$

where $\widetilde{\omega}_i^{-1}$ is the quasi-inverse of $\widetilde{\omega}_i$.

Remark 2.4.2. Given $(\mathcal{T}_1, \omega_1)$ and $(\mathcal{T}_2, \omega_2)$ two neutral Tannakian categories over a field F, it is easy to check that their Tannakian product category over F, $\mathcal{T}_1 \times_F \mathcal{T}_2$, is a rigid abelian tensor category. Moreover, the forgetful functor $\mathrm{forg}(V, \rho_1, \rho_2) = V$ is a fibre functor, hence $\mathcal{T}_1 \times_F \mathcal{T}_2$, is a neutral Tannakian category.

The following lemma explains the universal property that the Tannakian category $(\mathcal{T}_1 \times_F \mathcal{T}_2, \mathrm{forg})$ satisfies.

Lemma 2.4.3. *Let $(\mathcal{T}_1, \omega_1)$ and $(\mathcal{T}_2, \omega_2)$ be two neutral Tannakian categories over a field F, then, for every neutral Tannakian category $(\mathcal{T}, \omega)$ over F and every pair of tensor functors*

$$G_1 \colon \mathcal{T} \to \mathcal{T}_1 \text{ and } G_2 \colon \mathcal{T} \to \mathcal{T}_2$$

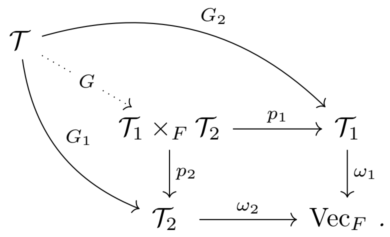
such that

$$\omega = \omega_1 \circ G_1 = \omega_2 \circ G_2,$$

there exists a tensor functor, unique up to natural isomorphisms,

$$G \colon \mathcal{T} \to \mathcal{T}_1 \times_k \mathcal{T}_2$$

such that, for $i = 1, 2$ $p_i \circ G \simeq G_i$ and, up to these natural isomorphisms, the following diagram commutes,



Proof. For $i = 1, 2$ and $(V, \rho_1, \rho_2) \in \mathcal{T}_1 \times_F \mathcal{T}_2$, we set

$$q_i(V, \rho_1, \rho_2) = (V, \rho_i) \in \mathrm{Rep}_F(\pi(\mathcal{T}_i, \omega_i)) \text{ and } \mathrm{forg}_i(V, \rho_i) = V \in \mathrm{Vec}_F.$$

By Theorem 2.1.10 and Proposition 2.1.12, to prove the statement it is equivalent to check that, for every pair of tensor functors (G_1, G_2)

$$G_1 \colon \mathrm{Rep}_F(\pi(\mathcal{T}, \omega)) \to \mathrm{Rep}_F(\pi(\mathcal{T}_1, \omega_1))$$

$$G_2 \colon \mathrm{Rep}_F(\pi(\mathcal{T}, \omega)) \to \mathrm{Rep}_F(\pi(\mathcal{T}_2, \omega_2))$$

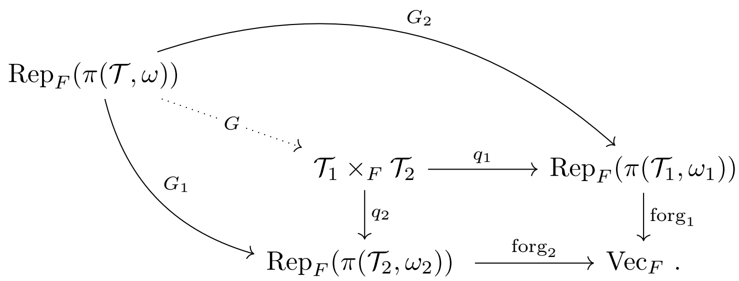
such that for $i = 1, 2$

$$\mathrm{forg} = \mathrm{forg}_i \circ G_i,$$

there exists a unique tensor functor

$$G \colon \mathrm{Rep}_F(\pi(\mathcal{T}, \omega)) \to \mathcal{T}_1 \times_F \mathcal{T}_2$$

such that the following diagram commutes



If we set, for $(V, \rho) \in \mathrm{Rep}_F(\pi(\mathcal{T}, \omega))$,

$$(V_1, \rho_1) := G_1(V, \rho) \text{ and } (V_2, \rho_2) := G_2(V, \rho),$$

then we get for $i = 1, 2$

$$V = \mathrm{forg}(V, \rho) = \mathrm{forg} \circ \widetilde{G_i}(V, \rho) = V_i.$$

Thus, the functor $G \colon \mathrm{Rep}_F(\pi(\mathcal{T}, \omega)) \to \mathcal{T}_1 \times_F \mathcal{T}_2$ has to be

$$G(V, \rho) := (V, \rho_1, \rho_2).$$

□

Definition 2.4.4. Given G_1 and G_2 two affine group schemes over a field F, we define the *free algebraic product of* G_1 *and* G_2 *over* F as the affine group associated with the Tannakian product category $\mathrm{Rep}_F(G_1) \times_F \mathrm{Rep}_F(G_2)$. We denote the free algebraic product of G_1 and G_2 over F by $G_1 \star_F G_2$.

Lemma 2.4.5. *Let F be a field and G_1, G_2 two abstract groups, then the algebraic hull over F of their free product $G_1 \star G_2$ is isomorphic to $G_1^{\mathrm{alg}} \star_F G_2^{\mathrm{alg}}$, the free algebraic product of the algebraic hulls of G_1 and G_2 over F.*

Proof. By the properties of the free product of abstract groups, the category $\mathrm{Rep}_F(G_1 \star G_2)$ is equivalent to the category $\mathcal{T}$ whose objects are given by triples (V, ρ_1, ρ_2) with

- $V \in \mathrm{Vec}_F$,
- $\rho_1 \colon G_1 \to \mathrm{GL}(V)$ a G_1-action,
- $\rho_2 \colon G_2 \to \mathrm{GL}(V)$ a G_2-action,

and whose morphisms from (V, ρ_1, ρ_2) to (W, τ_1, τ_2) are given by F-linear maps $\varphi \colon V \to W$ that are both G_1-equiviariant and G_2-equiviariant.

By Definition 2.2.1, for $i = 1, 2$, $\mathrm{Rep}_F(G_i)$ is equivalent to $\mathrm{Rep}_F(G_i^{\mathrm{alg}})$. Thus, it is clear that $\mathcal{T}$ is equivalent to the category $\mathrm{Rep}_F(G_1^{\mathrm{alg}}) \times_F \mathrm{Rep}_F(G_2^{\mathrm{alg}})$ and the conclusion follows by Theorem 2.1.10. □

Lemma 2.4.6. *Let F a field and G_1, G_2 two topological groups, then the topological algebraic hull over F of their topological free product is isomorphic to $G_1^{\mathrm{cts}} \star_F G_2^{\mathrm{cts}}$, the free algebraic product of the topological algebraic hulls of G_1 and G_2 over F.*

Proof. The statement follow from Definition 2.3.2 and the same argument of Lemma 2.4.5. □

Chapter 3

Descent of sheaves with meromorphic data

Notation. The following notation will be used throughout all these last two chapters.

- We fix k an algebraically closed of characteristic $p > 0$,
- we set A to be a complete discrete valuation ring of characteristic p with residue field k,
- we denote by K the fraction field of A and we set $S = \mathrm{Spec}(A)$.

The goal of this chapter is to construct, given a projective semi-stable curve X over S, a functor from $\mathrm{Rep}_K^{\mathrm{cts}}(\pi_1^{\text{proét}}(X_0))$, the category of continuous K-linear representations of the pro-étale fundamental group of X_0, to $\mathrm{Coh}(X_K)$, the category of coherent sheaves on the generic fibre X_K of X. The main idea is to associate with a continuous representation of $\pi_1^{\text{proét}}(X_0, \xi)$ a coherent sheaf with meromorphic descent data, which will be defined following [Gie73], and then prove that this sheaf descends to a coherent sheaf on X_K.

3.1 Meromorphic descent data

In this section we introduce the notion of coherent sheaves with meromorphic descent data and we state [Gie73, Prop. 1], which holds for a stable curve with degenerate closed fibre. In the next sections we will prove analogous results for projective semi-stable curves with smooth generic fibre.

The setting of [Gie73] is based on the construction illustrated in [Mum72, pg. 41], where given S as above and X a stable curve over S with degenerate closed fibre X_0, the author constructs the universal covering Y_0 of X_0. For our purposes the computational description of Y_0 in [Mum72, §3] and [Mum72, pg. 41] is not relevant, its geometrical description will instead play a very important role in the thesis. After illustrating the main properties of Y_0, we will prove a lemma that justifies the adjective universal used by Mumford.

Proposition 3.1.1 ([Mum72], Step I-V). *Given X a stable curve over S with degenerate closed fibre X_0, there exists an X_0-scheme Y_0 such that*

1. *$Y_0 \to X_0$ is an étale surjective morphism;*
2. *Y_0 is a connected infinite union of $\mathbb{P}^1_k$;*
3. *$p_a(Y_0) = 0$;*
4. *the group $\mathrm{Aut}(Y_0|X_0)$ is a free group with $r = p_a(X_0)$ generators, i.e.*

$$\mathrm{Aut}(Y_0|X_0) \simeq \mathbb{Z}^{\star r}.$$

Remark 3.1.2. If X_0 is a degenerate stable curve over the field k, by Corollary 1.3.26 and Remark 1.3.27, we have that

$$\pi_1^{\text{proét}}(X_0) \simeq \pi_1^{\text{SGA3}}(X_0) \simeq \mathbb{Z}^{\star r}.$$

Since $\pi_1^{\text{proét}}(X_0)$ is a discrete group, the left regular $\pi_1^{\text{proét}}(X_0, \xi)$-action on the set $S = \pi_1^{\text{proét}}(X_0, \xi)$ is continuous with respect to the discrete topology on S. Thus, the set $\pi_1^{\text{proét}}(X_0, \xi)$, endowed with the left regular action, is an object of the category $\pi_1^{\text{proét}}(X_0, \xi)$-Sets.

The following lemma gives us a geometrical interpretation of Y_0.

Lemma 3.1.3. *If X is a stable curve over S with degenerate closed fibre X_0 and ξ is a geometric point of X_0, then the scheme Y_0, given by the previous proposition, is a geometric covering of X_0. Moreover, it corresponds, via the equivalence*

$$\mathrm{Cov}_X \simeq \pi_1^{\text{proét}}(X_0, \xi)\text{-Sets},$$

to the set $\pi_1^{\text{proét}}(X_0, \xi)$ endowed with the left regular action.

Proof. From Proposition 3.1.1.(1)-(2) it follows that Y_0 satisfies the conditions of Definition 1.2.8 and thus it is a geometric covering of X_0. Since Y_0 is connected, it corresponds to a set S with a transitive action ρ of $\pi_1^{\text{proét}}(X_0, \xi)$.

As explained in Remark 3.1.2, the set $\pi_1^{\text{proét}}(X_0, \xi)$ endowed the regular representation is an object of the category $\pi_1^{\text{proét}}(X_0, \xi)$-Sets. Fixing an element $s \in S$, we can define the surjective $\pi_1^{\text{proét}}(X_0, \xi)$-equivariant map

$$\varphi_s \colon \pi_1^{\text{proét}}(X_0, \xi) \to S,\ \varphi_s(g) = \rho(g)(s).$$

By Corollary 1.2.16, the set $\pi_1^{\text{proét}}(X_0, \xi)$ endowed with the regular action corresponds to a geometric cover Z_0 of X_0 and φ_s corresponds to a map of X_0-scheme $\widetilde{\varphi}_s \colon Z_0 \to Y_0$. By construction, Z_0 is étale over Y_0 and $\widetilde{\varphi}_s$ satisfies the valuative criterion of properness, hence Z_0 is a geometric covering of Y_0.

On the other hand, by Proposition 3.1.1.(3) and Proposition 1.3.23, we can conclude that

$$\pi_1^{\text{proét}}(Y_0) = 1.$$

Thus, every geometric covering covering of Y_0 is isomorphic to a disjoint union of copies of Y_0. Since by construction Z_0 is connected, it follows that $\widetilde{\varphi}_s$ is an isomorphism of X_0-schemes, and hence φ_s is an isomorphism of $\pi_1^{\text{proét}}(X_0, \xi)$-Sets. □

Definition 3.1.4. If X is a stable curve over S with degenerate closed fibre X_0, we call *the universal cover of* X_0 the geometric covering Y_0 of X_0 that corresponds to the set $\pi_1^{\text{proét}}(X_0, \xi)$ endowed with the regular action.

By the previous lemma, the universal cover of X_0 is isomorphic to scheme constructed by Mumford, hence the notation is not ambiguous.

Remark 3.1.5. The left regular action on $S = \pi_1^{\text{proét}}(X_0, \xi)$ is continuous with respect to the discrete topology on S if and only if the group $\pi_1^{\text{proét}}(X_0, \xi)$ is discrete. By Proposition 1.3.23, this condition is satisfied if and only if all the normalizations of the irreducible components of X_0 have trivial étale fundamental group, that is, if and only if X_0 is degenerate. Thus, if X_0 is not degenerate $\pi_1^{\text{proét}}(X_0, \xi)$ is not an object of the category $\pi_1^{\text{proét}}(X_0, \xi)$-Sets, and hence we can not define Y_0 as in Lemma 3.1.3.

Before defining coherent sheaves with meromorphic descent data we recall how to associate with a geometric covering of X_0 a formal étale scheme over $\widehat{X}$, the completion of X along X_0.

Proposition 3.1.6 ([SGA 1], Exp. IX Prop 1.7)**.** *Let X be an S-scheme and $\widehat{X}$ the completion of X along its closed fibre, then category of étale schemes over the closed fibre X_0 and the category of étale formal schemes over $\widehat{X}$ are equivalent.*

Note that [SGA 1, Exp. IX, Prop 1.7] does not require the étale morphisms to be finite. In particular, for any geometric covering Y_0 of X_0 there exists an étale formal scheme $\mathcal{Y}$, unique up to $\widehat{X}$-isomorphisms, which reduces to Y_0.

Definition 3.1.7. Let X be an S-scheme, X_0 its closed fibre and $\widehat{X}$ the completion of X along X_0, then we denote by $\mathrm{Et}_{\widehat{X}}$ the category of formal schemes that are étale over $\widehat{X}$. We define $\mathrm{Cov}_{\widehat{X}}$ to be the full subcategory $\mathrm{Et}_{\widehat{X}}$ given by the essential image of Cov_{X_0} via the equivalence in Proposition 3.1.6. We call the objects of $\mathrm{Cov}_{\widehat{X}}$ *formal geometric coverings of* $\widehat{X}$.

Remark 3.1.8. It is important to note that, while the category of finite étale X_0-schemes is equivalent to the category of finite étale X-schemes, the categories Cov_{X_0} and Cov_X are not equivalent.

A counterexample is given by stable curves over S with smooth generic fibre and degenerate closed fibre. Indeed, if X is such a curve, then, by [Liu02, Prop.10.3.15 (c)], X is a normal scheme. Hence, by Proposition 1.2.20,

$$\pi_1^{\text{proét}}(X) \simeq \pi_1^{\text{ét}}(X)$$

and, in particular, $\pi_1^{\text{proét}}(X)$ is profinite. On the other hand, by Proposition 1.3.23, $\pi_1^{\text{proét}}(X_0) \simeq \mathbb{Z}^{\star r}$, with $r = p_a(X_0) \geq 2$. Thus, the groups $\pi_1^{\text{proét}}(X)$ and $\pi_1^{\text{proét}}(X_0)$ are non isomorphic Noohi groups and, by Proposition 1.2.14, this implies that the categories Cov_{X_0} and Cov_X are not equivalent.

Remark 3.1.9. Let $\mathcal{Y}$ be a formal geometric covering of $\hat{X}$, then $\mathcal{Y}$ is a S-scheme via the map $\mathcal{Y} \to \widehat{X} \to X \to S$, and the composition $\mathcal{Y} \to S = \mathrm{Spec}(A)$ corresponds to a morphism $A \to \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$. Hence a coherent $\mathcal{O}_{\mathcal{Y}}$-module is a sheaf of A-modules.

Definition 3.1.10. Let X be an S-scheme, $\mathcal{Y}$ a formal geometric covering of $\widehat{X}$ and $\mathcal{F}$ a coherent sheaf on $\mathcal{Y}$, then *meromorphic descent data on* $\mathcal{F}$ are given by a collection of elements

$$h_g \in \mathrm{H}^0(\mathcal{Y}, \mathcal{H}om_{\mathcal{O}_\mathcal{Y}}(\mathcal{F}, g^\star \mathcal{F}) \otimes_A K),\ g \in \mathrm{Aut}(\mathcal{Y}|\widehat{X})$$

that satisfy:

- $g^\star h_{g'} \circ h_g = h_{g' \circ g}$ for every $g, g' \in \mathrm{Aut}(\mathcal{Y}|\widehat{X})$,
- $h_{\mathrm{Id}} = \mathrm{Id}_{\mathcal{F} \otimes_A K}$.

Definition 3.1.11. Let X be an S-scheme and $\mathcal{Y}$ a formal geometric covering of $\widehat{X}$ then, given $\{\mathcal{F}, h_g\}$ and $\{\mathcal{G}, k_g\}$ two coherent sheaves on $\mathcal{Y}$ with meromorphic descent data, a *morphism of meromorphic descent data* from $\{\mathcal{F}, h_g\}$ to $\{\mathcal{G}, k_g\}$ is given by an element

$$f \in \mathrm{H}^0(\mathcal{Y}, \mathcal{H}om_{\mathcal{O}_\mathcal{Y}}(\mathcal{F}, \mathcal{G}) \otimes_A K)$$

such that for every $g \in \mathrm{Aut}(\mathcal{Y}|\widehat{X})$

$$k_g \circ f = g^\star(f) \circ h_g.$$

We denote by $\mathrm{Coh}^m(\mathcal{Y})$ the category of coherent sheaves on $\mathcal{Y}$ with meromorphic descent data.

Proposition 3.1.12 ([Gie73], Prop. 1)**.** *Let X be a stable curve over S with degenerate closed fibre and smooth generic fibre, Y_0 the universal covering of X_0 and $\mathcal{Y}$ the formal geometric covering of $\widehat{X}$ corresponding to Y_0, then* $\mathrm{Coh}^m(\mathcal{Y})$ *is equivalent to the category* $\mathrm{Coh}(X_K)$ *of coherent sheaves on X_K.*

Theorem 3.1.13. *If X is a stable curve over S with degenerate closed fibre and smooth generic fibre, then there exists a functor*

$$\mathrm{Rep}_K(\pi_1^{\text{proét}}(X_0, \xi)) \to \mathrm{Coh}(X_K).$$

Proof. Let Y_0 be the universal covering of X_0, $\mathcal{Y}$ the formal geometric covering of $\widehat{X}$ corresponding to Y_0 and $\rho\colon \mathrm{Aut}(\mathcal{Y}|\widehat{X}) \to \mathrm{GL}_n(K)$ be a representation of $\mathrm{Aut}(\mathcal{Y}|\widehat{X})$. Since, for every $g \in \mathrm{Aut}(\mathcal{Y}|\widehat{X})$, we have $g^\star \mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{Y}$, we can define meromorphic descent data $\{h_g^\rho\}$ on $\mathcal{O}_\mathcal{Y}^n$ as follows

$$\begin{array}{ccc} \mathcal{O}_\mathcal{Y}^n(U) \otimes_A K \cong \mathcal{O}_\mathcal{Y}(U) \otimes_A K^n & \xrightarrow{h_g^\rho} & \mathcal{O}_\mathcal{Y}(U) \otimes_A K^n \cong \mathcal{O}_\mathcal{Y}^n(U) \otimes_A K \\ f \otimes v & \longrightarrow & f \otimes \rho(g)(v)\ . \end{array}$$

Clearly, we have that $g^\star h_{g'}^\rho \circ h_g^\rho = h_{g' \circ g}^\rho$ and $h_{\mathrm{Id}} = \mathrm{Id}_{\mathcal{O}_\mathcal{Y}^n \otimes_A K}$. Thus, we can set

$$F'(\rho) := \{\mathcal{O}_\mathcal{Y}^n, h_g^\rho\} \in \mathrm{Coh}^m(\mathcal{Y}).$$

Let $\varphi\colon (K^n, \rho) \to (K^m, \tau)$ a morphism of representations. By construction $\mathrm{F}'(\rho) = \{\mathcal{O}_\mathcal{Y}^n, h_g^\rho\}$ and $\mathrm{F}'(\tau) = \{\mathcal{O}_\mathcal{Y}^m, h_g^\tau\}$, then we define $\mathrm{F}'(\varphi)$ as follows:

$$\begin{array}{ccc} \mathcal{O}_\mathcal{Y}(U) \otimes_A K^n & \xrightarrow{\mathrm{F}'(\varphi)} & \mathcal{O}_\mathcal{Y}(U) \otimes_A K^m \\ f \otimes v & \longrightarrow & f \otimes \varphi(v)\ . \end{array}$$

Since φ is a morphism of representations, we get

$$h_g^\tau \circ \mathrm{F}'(\varphi) = g^\star \mathrm{F}'(\varphi) \circ h_g^\rho.$$

Hence, $\mathrm{F}'(\varphi)$ is a morphism of meromorphic data. This implies that F' defines a functor

$$\mathrm{F}' \colon \mathrm{Rep}_K(\mathrm{Aut}(\mathcal{Y}|\widehat{X})) \to \mathrm{Coh}^m(\mathcal{Y}).$$

Moreover, by Proposition 3.1.1, we have that

$$\mathrm{Aut}(\mathcal{Y}|\widehat{X}) \simeq \mathrm{Aut}(Y_0|X_0) \simeq \mathbb{Z}^{\star r} \simeq \pi_1^{\text{proét}}(X_0, \xi).$$

Let $\gamma \colon \pi_1^{\text{proét}}(X_0) \to \mathrm{Aut}(\mathcal{Y}|\widehat{X})$ be such an isomorphism, then pre-composing with γ induces an equivalence of categories

$$\widetilde{\gamma} \colon \mathrm{Rep}_K(\pi_1^{\text{proét}}(X_0, \xi)) \to \mathrm{Rep}_K(\mathrm{Aut}(\mathcal{Y}|\widehat{X})).$$

Hence, we can define the following functor

$$\mathrm{F}' \circ \widetilde{\gamma} \colon \mathrm{Rep}_K(\pi_1^{\text{proét}}(X_0, \xi)) \to \mathrm{Coh}^m(\mathcal{Y}).$$

Finally, by composing this functor with the equivalence of categories in Proposition 3.1.12 we obtain the desired functor. □

The aim of the next sections is to generalize this result to any projective semi-stable curve over S with smooth generic fibre.

3.2 Coverings associated with representations

From this section on, given $S = \mathrm{Spec}(A)$ as before, we set X to be a projective semi-stable curve over S and we denote by X_0 its closed fibre and by X_K its generic fibre.

The first obstacle we encounter while trying to generalize the argument of Theorem 3.1.13 to projective semi-stable curves is that if the closed fibre X_0 is not degenerate we can't construct its universal covering, as explained in Remark 3.1.5. We overcome this first issue by associating with each representation a specific geometric covering of X_0.

We start by analyzing the category $\mathrm{Rep}_K^{\text{cts}}(\pi_1^{\text{proét}}(X_0, \xi))$ of continuous representation of $\pi_1^{\text{proét}}(X_0, \xi)$, defined as in Definition 2.3.1.

Lemma 3.2.1. *Given G a topological group with a basis of open neighborhood of 1 given by subgroups and $\widehat{G}^R$ its Raîkov completion, there exists an equivalence of categories*

$$\mathrm{Rep}_K^{\text{cts}}(\widehat{G}^R) \to \mathrm{Rep}_K^{\text{cts}}(G).$$

Proof. As in Example 1.1.8, we see that there exists a continuous morphism $\sigma \colon G \to \widehat{G}^R$, whose image is dense in $\widehat{G}^R$ and that σ induces a fully faithful functor

$$\widetilde{\sigma} \colon \mathrm{Rep}_K^{\text{cts}}(\widehat{G}^R) \to \mathrm{Rep}_K^{\text{cts}}(G).$$

Let (V, ρ) be a representation of G, then, by Lemma 2.3.4, ρ induces a morphism $\rho \colon G \to \mathrm{Aut}(V)$ that is a continuous with respect to the discrete topology on $\mathrm{Aut}(V)$. As remarked in Example 1.1.10, groups with discrete topology are Noohi complete and hence, by [AT08, Prop. 3.6.12], ρ admits an extension to $\hat{\rho}\colon \widehat{G}^R \to \mathrm{Aut}(V)$ such that $\hat{\rho} \circ \sigma = \rho$. This implies that $\widetilde{\sigma}$ is also essentially surjective. □

Corollary 3.2.2. *Let ξ be a geometric point of X_0 and, for $i = 1, \dots, N$, let C_i be the irreducible components of X_0 and $\overline{C_i}$ their normalizations, then there exists an equivalence of categories*

$$\mathrm{Rep}_K^{\mathrm{cts}}(\pi_1^{\mathrm{pro\acute{e}t}}(X_0, \xi)) \simeq \mathrm{Rep}_K^{\mathrm{cts}}(\mathbb{Z}^{\star |I|-N+1} \star \pi_1^{\mathrm{\acute{e}t}}(\overline{C_1}) \star \dots \star \pi_1^{\mathrm{\acute{e}t}}(\overline{C_N})).$$

Proof. By Proposition 1.3.23, we have that, for $r = |I| - N + 1$,

$$\pi_1^{\mathrm{pro\acute{e}t}}(X_0, \xi) \simeq \mathbb{Z}^{\star r} \star_N \pi_1^{\mathrm{\acute{e}t}}(\overline{C_1}) \star_N \dots \star_N \pi_1^{\mathrm{\acute{e}t}}(\overline{C_N}).$$

Moreover, by Corollary 1.3.21, $\mathbb{Z}^{\star r} \star_N \pi_1^{\mathrm{\acute{e}t}}(\overline{C_1}) \star_N \dots \star_N \pi_1^{\mathrm{\acute{e}t}}(\overline{C_N})$ is isomorphic to the Raîkov completion of $\mathbb{Z}^{\star r} \star_{\mathcal{B}} \pi_1^{\mathrm{\acute{e}t}}(\overline{C_1}) \star_{\mathcal{B}} \dots \star_{\mathcal{B}} \pi_1^{\mathrm{\acute{e}t}}(\overline{C_N})$, defined as in Lemma 1.3.20. Thus, Lemma 3.2.1 implies that

$$\mathrm{Rep}_K^{\mathrm{cts}}(\pi_1^{\mathrm{pro\acute{e}t}}(X_0, \xi)) \simeq \mathrm{Rep}_K^{\mathrm{cts}}(\mathbb{Z}^{\star r} \star_{\mathcal{B}} \pi_1^{\mathrm{\acute{e}t}}(\overline{C_1}) \star_{\mathcal{B}} \dots \star_{\mathcal{B}} \pi_1^{\mathrm{\acute{e}t}}(\overline{C_N})).$$

Futhermore, as in Corollary 1.3.21, we see that the identity induces an equivalence of categories

$$\mathrm{Rep}_K^{\mathrm{cts}}(\pi_{\mathcal{B}}) \simeq \mathrm{Rep}_K^{\mathrm{cts}}(\mathbb{Z}^{\star r} \star \pi_1^{\mathrm{\acute{e}t}}(\overline{C_1}) \star \dots \star \pi_1^{\mathrm{\acute{e}t}}(\overline{C_N})).$$

□

Remark 3.2.3. With the same reasoning of Lemma 3.2.1, one can show that, if G is a given topological group and $\widehat{G}^D$ is its pro-discrete completion, then there exists an equivalence of categories

$$\mathrm{Rep}_K^{\mathrm{cts}}(\widehat{G}^D) \to \mathrm{Rep}_K^{\mathrm{cts}}(G).$$

By Proposition 1.2.19, the pro-discrete completion of $\pi_1^{\mathrm{pro\acute{e}t}}(X_0, \xi)$ is isomorphic to $\pi_1^{\mathrm{SGA3}}(X_0, \xi)$, hence it follows that

$$\mathrm{Rep}_K^{\mathrm{cts}}(\pi_1^{\mathrm{pro\acute{e}t}}(X_0, \xi)) \simeq \mathrm{Rep}_K^{\mathrm{cts}}(\pi_1^{\mathrm{SGA3}}(X_0, \xi)).$$

Note that this equivalence of categories holds even in the cases, presented for example in [BS15, Example 7.4.9], where $\pi_1^{\mathrm{pro\acute{e}t}}(X_0, \xi)$ and $\pi_1^{\mathrm{SGA3}}(X_0, \xi)$ are not isomorphic as topological groups.

We proceed by associating with a continuous K-linear representation of $\pi_1^{\mathrm{pro\acute{e}t}}(X_0, \xi)$ a geometric covering of X_0. The notation that we introduce will be used repeatedly in this chapter and in the following one.

Let us fix an element $(K^n, \rho) \in \mathrm{Rep}_K^{\mathrm{cts}}(\pi_1^{\mathrm{proet}}(X_0, \xi))$. By Corollary 3.2.2 and Lemma 2.3.4, ρ corresponds to a K-linear representation

$$\rho\colon \mathbb{Z}^{\star r} \star \pi_1^{\mathrm{\acute{e}t}}(\overline{C_1}) \star \dots \star \pi_1^{\mathrm{\acute{e}t}}(\overline{C_N}) \to \mathrm{GL}_n(K),$$

which is continuous with respect to the discrete topology on $\mathrm{GL}_n(K)$.

Thus, by the universal property of the free product, (K^n, ρ) corresponds to the data

- $\rho_i^{\text{dis}} \colon \mathbb{Z} \to \mathrm{GL}_n(K)$ for $i = 1, \ldots, r$,
- $\rho_j^{\text{ét}} \colon \pi_1^{\text{ét}}(\overline{C_j}) \to \mathrm{GL}_n(K)$ for $j = 1, \ldots, N$,

where ρ_i^{dis} and $\rho_j^{\text{ét}}$ are group morphisms that are continuous with respect to the discrete topology on $\mathrm{GL}_n(K)$.

By construction, each morphism $\rho_j^{\text{ét}}$ factors through the finite quotient of $\pi_1^{\text{ét}}(\overline{C_j})$ given by $G_j = \pi_1^{\text{ét}}(\overline{C_j})/U$, where $U = {\rho_j^{\text{ét}}}^{-1}(\mathrm{Id})$. Hence, we have the following commutative diagram

$$\begin{array}{ccccc} & & G_j & & \\ & {}^{q_j}\nearrow & & \searrow^{\widetilde{\rho_j}^{\text{ét}}} & \\ \pi_1^{\text{ét}}(\overline{C_j}) & & \xrightarrow{\quad \rho_j^{\text{ét}} \quad} & & \mathrm{Gl}_n(K)\ . \end{array}$$

In particular, (K^n, ρ) induces to the following data

- $\rho_i^{\text{dis}} \colon \mathbb{Z} \to \mathrm{GL}_n(K)$ for $i = 1, \ldots, r$,
- $\widetilde{\rho_j}^{\text{ét}} \colon G_j \to \mathrm{GL}_n(K)$ for $j = 1, \ldots, N$,

which correspond, by the universal property of the free product, to a K-linear representation

$$\rho \colon \mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N \to \mathrm{GL}_n(K),$$

which is continuous with respect to the discrete topology on $\mathrm{GL}_n(K)$.

Clearly, $\mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N$ is a quotient of $\mathbb{Z}^{\star r} \star \pi_1^{\text{ét}}(\overline{C_1}) \star \cdots \star \pi_1^{\text{ét}}(\overline{C_N})$. Since it is a discrete group, by [AT08, Prop. 3.6.12] it is also a quotient of $\pi_1^{\text{proét}}(X_0, \xi)$ and we denote the quotient map by

$$q \colon \pi_1^{\text{proét}}(X_0, \xi) \to \mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N.$$

By Proposition 1.2.14, the set $\mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N$, endowed with the action given by q, corresponds to a connected geometric covering of X_0, which we denote by Y_0^{ρ}.

Definition 3.2.4. We set $\mathcal{Y}_\rho$ to be the geometric covering of $\widehat{X}$ that corresponds to the geometric covering Y_0^{ρ} of X_0 defined above.

By construction, we see that

$$\mathrm{Aut}(\mathcal{Y}_\rho | \widehat{X}) \simeq \mathrm{Aut}(Y_0^{\rho} | X_0) \simeq (\mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N)^{\mathrm{op}}. \tag{3.2.5}$$

Similarly, we can endow the set $G_1 \times \cdots \times G_N$ with a $\pi_1^{\text{proét}}(X_0)$-action, by composing the map q with the quotient map

$$\alpha \colon \mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N \to G_1 \times \cdots \times G_N.$$

Hence, we can associate with $G_1 \times \cdots \times G_N$ a finite étale cover Z_0^{ρ} of X_0.

Definition 3.2.6. We set $\mathcal{Z}_\rho$ to be the finite étale covering of $\widehat{X}$ that corresponds to the finite étale covering Z_0^{ρ} of X_0 defined above.

We can observe that

$$\mathrm{Aut}(\mathcal{Z}_\rho|\widehat{X}) \simeq \mathrm{Aut}(Z_0^\rho|X_0) \simeq (G_1 \times \cdots \times G_N)^{\mathrm{op}}. \tag{3.2.7}$$

Moreover, $\mathcal{Y}_\rho \to \widehat{X}$ factors through $q\colon \mathcal{Y}_\rho \to \mathcal{Z}_\rho$ and we have

$$\mathrm{Aut}(\mathcal{Y}_\rho|\mathcal{Z}_\rho) \simeq \mathrm{Aut}(Y_0^\rho|Z_0^\rho) \simeq \ker(\alpha)^{\mathrm{op}}. \tag{3.2.8}$$

Remark 3.2.9. Since this thesis focuses on the categories of continuous representations of the above groups, the isomorphisms in Equation 3.2.5, Equation 3.2.7 and Equation 3.2.8 will not play a role, hence we will treat them as equalities.

It is instead important to distinguish the left and right multiplication in the groups, so we use two different notations. We denote the multiplication in $\ker(\alpha)^{\mathrm{op}}$ by $w \circ w'$, while when we do not write any symbol (i.e., ww'), we refer to the concatenation in $\ker(\alpha)$. We will use the analogous notation for $(\mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N \to G_1 \times \cdots \times G_N)^{\mathrm{op}}$ and $(G_1 \times \cdots \times G_N)^{\mathrm{op}}$.

Remark 3.2.10. The group $\ker(\alpha)$ is torsion free. Indeed, by the properties of the free product, if $w \in \ker(\alpha)$ is a torsion element, then it is of the form

$$w = v g_j v^{-1} \text{ for some } v \in \mathbb{Z}^{\star r} \star \pi_1^{\text{ét}}(\overline{C_1}) \star \cdots \star \pi_1^{\text{ét}}(\overline{C_N}),\ g_j \in G_j.$$

Since $\alpha(w) = (1, \ldots, 1, , g_j, 1, \ldots, 1)$ and $w \in \ker(\alpha)$, we conclude that $g_j = 1$ and hence w is the neutral element of $\mathbb{Z}^{\star r} \star \pi_1^{\text{ét}}(\overline{C_1}) \star \cdots \star \pi_1^{\text{ét}}(\overline{C_N})$, which is the empty word.

3.3 Infinite descent

In the previous section we have constructed, for a projective semi-stable curve X over S, the formal geometric coverings $\mathcal{Y}_\rho$ and $\mathcal{Z}_\rho$ of $\widehat{X}$ associated with a given representation $(K^n, \rho) \in \mathrm{Rep}_K^{\mathrm{cts}}(\pi_1^{\text{proét}}(X_0, \xi))$. The aim of this section is to prove that coherent sheaves on $\mathcal{Y}_\rho$ with meromorphic descend data descend to coherent sheaves on $\mathcal{Z}_\rho$.

Definition 3.3.1. Given X an S-scheme, $\mathcal{Y}$ and $\mathcal{Z}$ two formal geometric coverings of $\widehat{X}$, $q_{Y/Z}\colon \mathcal{Y} \to \mathcal{Z}$ an $\widehat{X}$-morphism and $\mathcal{F}$ a coherent sheaf on $\mathcal{Y}$, we call *meromorphic descent data relative to* $\mathcal{Z}$ *on* $\mathcal{F}$ a collection of elements

$$h_g \in \mathrm{H}^0(\mathcal{Y}, \mathcal{H}om_{\mathcal{O}_\mathcal{Y}}(\mathcal{F}, g^\star\mathcal{F}) \otimes_A K),\ g \in \mathrm{Aut}(\mathcal{Y}|\mathcal{Z})$$

that satisfy:

- $g^\star h_{g'} \circ h_g = h_{g' \circ g}$ for every $g, g' \in \mathrm{Aut}(\mathcal{Y}|\mathcal{Z})$;
- $h_{\mathrm{Id}} = \mathrm{Id}_{\mathcal{F}\otimes_A K}$.

We define morphisms of meromorphic descent data relative to $\mathcal{Z}$ as in Definition 3.1.11 and we denote by $\mathrm{Coh}^m(\mathcal{Y}|\mathcal{Z})$ the category of coherent sheaves on $\mathcal{Y}$ with meromorphic descent data relative to $\mathcal{Z}$.

Definition 3.3.2. Given X an S-scheme, $\mathcal{Y}$, $\mathcal{Z}$ and $q_{Y/Z}\colon \mathcal{Y} \to \mathcal{Z}$ as above, let $\{\mathcal{F}, h_w\}_{w\in \mathrm{Aut}(\mathcal{Y}|\mathcal{Z})}$ be a coherent sheaf on $\mathcal{Y}$ with meromorphic descent data relative to $\mathcal{Z}$. We say that $\{\mathcal{F}, h_w\}_{w\in \mathrm{Aut}(\mathcal{Y}|\mathcal{Z})}$ *descends to a coherent sheaf on* $\mathcal{Z}$ if there exists $\mathcal{G} \in \mathrm{Coh}(\mathcal{Z})$ such that

$$\{\mathcal{F}, h_w\}_{w\in \mathrm{Aut}(\mathcal{Y}|\mathcal{Z})} \simeq \{q^\star_{Y/Z}\mathcal{G}, h^q_w\}_{w\in \mathrm{Aut}(\mathcal{Y}|\mathcal{Z})},$$

where $h^q_w\colon q^\star_{Y/Z}\mathcal{G} \to w^\star q^\star_{Y/Z}\mathcal{G}$ are the natural isomorphisms.

In this section we work mainly with meromorphic descent data on $\mathcal{Y}_\rho$ relative to $\mathcal{Z}_\rho$, where $\mathcal{Y}_\rho$ and $\mathcal{Z}_\rho$ are the coverings of $\widehat{X}$ associated with ρ constructed in Definition 3.2.4 and Definition 3.2.6.

The following proposition is a generalization of [Gie73, Prop.1].

Proposition 3.3.3. *Given X a projective semi-stable curve over S and a coherent sheaf $\{\mathcal{F}, h_w\}_{w\in Aut(\mathcal{Y}_\rho|\mathcal{Z}_\rho)}$ with meromorphic descent data relative to $\mathcal{Z}_\rho$, there exists a coherent sheaf $\{\mathcal{F}', k_w\}_{\mathrm{Aut}(\mathcal{Y}_\rho|\mathcal{Z}_\rho)}$ with meromorphic descent data relative to $\mathcal{Z}_\rho$ that is isomorphic to $\{\mathcal{F}, h_w\}_{w\in Aut(\mathcal{Y}_\rho|\mathcal{Z}_\rho)}$ and such that*

$$k_w \in \mathrm{H}^0(X, \mathcal{H}om_{\mathcal{O}_{\mathcal{Y}_\rho}}(\mathcal{F}', w^\star\mathcal{F}')).$$

Proof. As in [Gie73, Prop.1], it suffices to show that for any $\mathrm{Aut}(\mathcal{Y}_\rho|\mathcal{Z}_\rho)$-invariant open $U \subset \mathcal{Y}_\rho$ there exists a quasi-compact open V of $\mathcal{Y}_\rho$ such that

- V is not contained in U,
- $V \cap wV \subseteq U$ for all $w \in \mathrm{Aut}(\mathcal{Y}_\rho|\mathcal{Z}_\rho)$, $w \neq Id_{\mathcal{Y}_\rho}$.

By Proposition 1.3.23, an irreducible component of $\mathcal{Y}_\rho$ corresponds to an orbit of the left G_j-action on $\mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N$, for some $j \in \{1, \dots, N\}$. Given a word $s \in \mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N$, we denote $\mathcal{Y}^j_s$ the irreducible component of $\mathcal{Y}_\rho$ corresponding to the G_j-orbit of s. Since the action of $\ker(\alpha)^{\mathrm{op}}$ on $\mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N$ is defined by right concatenation, given $\mathcal{Y}^j_s$ an irreducible component of $\mathcal{Y}_\rho$ and $w \in \ker(\alpha)^{\mathrm{op}}$, $w \neq Id_{\mathcal{Y}_\rho}$, we have

$$w(G_js) = G_jsw \neq G_js \text{ and } w(\mathcal{Y}^j_s) = \mathcal{Y}^j_{sw} \neq \mathcal{Y}^j_s.$$

Hence, the action of $\ker(\alpha)^{\mathrm{op}}$ on the set of irreducible components of $\mathcal{Y}_\rho$ is free.

Let us suppose that we are given an open $\mathrm{Aut}(Y_\rho|Z_\rho)$-invariant set $U \subset \mathcal{Y}_\rho$, then for the construction of V there are two possible cases.

First case: there exists $x \in \mathcal{Y}_\rho \setminus U$ that is a non-singular point.

We set $\mathcal{Y}_x$ to be the irreducible component of $\mathcal{Y}_\rho$ containing x, I_x to be the set of the singular points of $\mathcal{Y}_x$ and we define $V = \mathcal{Y}_x \setminus I_x$. By construction, V is not contained in U. Since $\ker(\alpha)^{\mathrm{op}}$ acts freely on the set of irreducible components, for all $w \in \ker(\alpha)^{\mathrm{op}}$, $w \neq Id_{\mathcal{Y}_\rho}$, we have

$$V \cap wV = \emptyset \subset U.$$

Second case: $\mathcal{Y}_\rho \setminus U \subset I$, where I is the set of singular points of $\mathcal{Y}_\rho$.

Let $x \in \mathcal{Y}_\rho \setminus U$, then x belongs to exactly two irreducible components of $\mathcal{Y}_\rho$, say $\mathcal{Y}^i_s$ and $\mathcal{Y}^l_t$. Let I_x be the set of singular points of $\mathcal{Y}^i_s \cup Y^l_t$ different from x, then we set $V = (\mathcal{Y}^i_s \cup \mathcal{Y}^l_t) \setminus I_x$. Clearly, V is not contained in U. Since $\ker(\alpha)^{\mathrm{op}}$

acts freely on the set of irreducible components, for any non-trivial $w \in \ker(\alpha)^{\mathrm{op}}$ we have

$$V \cap wV = ((\mathcal{Y}_s^i \cap \mathcal{Y}_{tw}^l) \cup (\mathcal{Y}_t^l \cap \mathcal{Y}_{sw}^i)) \setminus \{\text{sing. pts}\}.$$

Thus, there are three possibilities:

- $\mathcal{Y}_{tw}^l \neq \mathcal{Y}_s^i$ and $\mathcal{Y}_{sw}^i \neq \mathcal{Y}_t^l$, that implies $V \cap wV = \emptyset \subset U$,
- $\mathcal{Y}_{tw}^l = \mathcal{Y}_s^i$ and $\mathcal{Y}_{sw}^i \neq \mathcal{Y}_t^l$, that implies $V \cap wV = \mathcal{Y}_s^i \setminus \{\text{sing. pts of } \mathcal{Y}_s^i\} \subset U$,
- $\mathcal{Y}_{tw}^l \neq \mathcal{Y}_s^i$ and $\mathcal{Y}_{sw}^i = \mathcal{Y}_t^l$, that implies $V \cap wV = \mathcal{Y}_t^l \setminus \{\text{sing. pts of } \mathcal{Y}_t^l\} \subset U$.

Note that the case where $\mathcal{Y}_{tw}^l = \mathcal{Y}_s^i$ and $\mathcal{Y}_{sw}^i = \mathcal{Y}_t^l$ does not occur because it would imply that $w^2 = Id_{\mathcal{Y}_\rho}$, which is not possible because by Remark 3.2.10 $\ker(\alpha)^{\mathrm{op}}$ is torsion free. □

Remark 3.3.4. The action of $(\mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N)^{\mathrm{op}}$ on the set of irreducible components of $\mathcal{Y}_\rho$ is not free. Indeed, if $\emptyset$ is the empty word and $\mathcal{Y}_\emptyset^j$ is the irreducible component of $\mathcal{Y}_\rho$ that corresponds to $G_j \subset F_\xi(\mathcal{Y}_\rho)$, then for every $g_j \in G_j$,

$$g_j(\mathcal{Y}_\emptyset^j) = \mathcal{Y}_{g_j}^j = \mathcal{Y}_\emptyset^j.$$

The following theorem generalizes [Gie73, Prop.2].

Theorem 3.3.5. *Any coherent sheaf $\{\mathcal{F}, h_w\}_{w \in Aut(\mathcal{Y}_\rho|\mathcal{Z}_\rho)}$ on $\mathcal{Y}_\rho$ with meromorphic descent data relative to $\mathcal{Z}_\rho$ descends to a coherent sheaf on $\mathcal{Z}_\rho$.*

Proof. As in [Gie73, Prop.2], it suffices to prove that there exists a quasi-compact open subscheme T of $\mathcal{Y}_\rho$ such that its $\mathrm{Aut}(\mathcal{Y}_\rho|\mathcal{Z}_\rho)$-translates cover $\mathcal{Y}_\rho$.

We fix a non-trivial word $w \in \ker(\alpha)^{\mathrm{op}}$. Note that the irreducible components of the form $\mathcal{Y}_{s'}^j$ with $\alpha(s') = \alpha(s)$, defined as in the previous thereom's proof, are $\ker(\alpha)^{\mathrm{op}}$-translates of $\mathcal{Y}_{ws}^j$. Indeed, the word $t = s^{-1}w^{-1}s' \in \ker(\alpha)^{\mathrm{op}}$ satifies

$$t(\mathcal{Y}_{ws}^j) = \mathcal{Y}_{wst}^j = \mathcal{Y}_{s'}^j.$$

Given an element $g = (g_1, \ldots, g_N) \in G_1 \times \cdots \times G_N$, we denote by $\sigma(g)$ the word $g_1 \cdots g_N \in \mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N$ with letters $g_i \in G_i$ and we define the map

$$\sigma \colon G_1 \times \cdots \times G_N \to \mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N,\ \sigma(g_1, \ldots, g_N) = g_1 \cdots g_N.$$

We denote by 1_i the word whose only letter is the element $1 \in \mathbb{Z}$ belonging to the i-th copy of $\mathbb{Z}$. Then we set

$$T_G = \bigcup_{j=1}^{N} \bigcup_{g \in G_1 \times \cdots \times G_N} \left(\mathcal{Y}_{w\sigma(g)}^j \cup \bigcup_{i=1}^{r} \mathcal{Y}_{1_i w\sigma(g)}^j \right).$$

and we define I_G to be set of points of T_G that are intersection points with irreducible components of $\mathcal{Y}_\rho$ not contained in T_G. Finally, we set

$$T = T_G \setminus I_G.$$

By construction, T is an open quasi-compact sub-scheme of $\mathcal{Y}_\rho$, so it suffices to prove that its $\ker(\alpha)^{\mathrm{op}}$-translates cover $\mathcal{Y}_\rho$.

Given $s \in \mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N$, we set $g_s := \alpha(s)$. Since $\alpha(s) = \alpha(w\sigma(g_s))$, there exists $t \in \ker(\alpha)^{\mathrm{op}}$ such that

$$t(\mathcal{Y}^j_{w\sigma(g_s)}) = \mathcal{Y}^j_s.$$

This implies that

$$\mathcal{Y}_\rho = \bigcup_{t \in \ker(\alpha)^{\mathrm{op}}} t(T_G).$$

To conclude, it suffices to prove that for every $x \in I_G$ there exist $y \in T$ and $t \in \ker(\alpha)^{\mathrm{op}}$ such that $t(y) = x$. A point $x \in I_G$ is, by definition, an intersection point of an irreducible component of T_G and an irreducible component of $\mathcal{Y}_\rho$ not contained in T_G. We assume that $x \in \mathcal{Y}^j_{w\sigma(g)} \cap \mathcal{Y}^k_{s'}$ for some $j, k = 1, \ldots, N$, $s' \in \mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N$ and $g \in G_1 \times \cdots \times G_N$. We recall that $\mathcal{Y}^j_{w\sigma(g)}$ corresponds to the G_j-orbit

$$G_j w\sigma(g) \subset F_\xi(\mathcal{Y}_\rho) = \mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N,$$

while $\mathcal{Y}^k_{s'}$ corresponds to the G_k-orbit

$$G_k s' \subset F_\xi(\mathcal{Y}_\rho) = \mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N.$$

Hence, the point x corresponds to the identification of two words of the form

$$w_{x_j} = h_j w\sigma(g) \in G_j w\sigma(g) \text{ and } w_{x_k} = h_k s' \in G_k s'.$$

However, by Proposition 1.3.23, the identification of the points in the different orbits is given either by the identity or by the action of $\mathbb{Z}^{\star r}$ on $\mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N$. Hence, there are only two possibilities:

1. $w_{x_k} = w_{x_j}$, which implies $h_k s' = h_j w\sigma(g)$,
2. there exists i such that $w_{x_k} = 1_i w_{x_j}$, which implies $h_k s' = 1_i h_j w\sigma(g)$.

Let $a \in \widehat{X}$ be the singular point over which x lies and set $g' = \alpha(h_j w\sigma(g))$, then we define $y \in F_a(\mathcal{Y}_\rho)$ to be the point corresponding, via a chosen path from ξ to a, to the word

$$w_y = w\sigma(g') \in G_j w\sigma(g') \subset F_\xi(\mathcal{Y}_\rho).$$

Since $\alpha(w_y) = \alpha(w_{x_j})$, there exists $t \in \ker(\alpha)^{\mathrm{op}}$ such that $t(w_y) = w_{x_j}$, which implies $t(y) = x$. It remains to prove that $y \in T$.

If 1. is satisfied, the word w_y is identified to the word $w\sigma(g') \in G_k w\sigma(g')$ and we have

$$y \in \mathcal{Y}^j_{w\sigma(g')} \cap \mathcal{Y}^k_{w\sigma(g')} \subset T.$$

If 2. is satisfied, the word w_y is identified to the word $1_i w\sigma(g') \in G_k w\sigma(g')$ and we have

$$y \in \mathcal{Y}^j_{w\sigma(g')} \cap \mathcal{Y}^k_{1_i w\sigma(g')} \subset T.$$

The case where x is an intersection point of two irreducible components of the form $\mathcal{Y}^j_{1_i w\sigma(g)} \cap \mathcal{Y}^k_{s'}$ follows by the same argument. □

Lemma 3.3.6. *Let $\{\mathcal{F}, h_w\}_{w\in Aut(\mathcal{Y}_\rho|\mathcal{Z}_\rho)}$ and $\{\mathcal{F}', h'_w\}_{w\in Aut(\mathcal{Y}_\rho|\mathcal{Z}_\rho)}$ be two coherent sheaves on $\mathcal{Y}_\rho$ with meromorphic descent data relative to $\mathcal{Z}_\rho$ and assume that they descent to the coherent sheaves $\mathcal{G}$ and $\mathcal{G}'$ on $\mathcal{Z}_\rho$, then we have the following group isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{Z}_\rho}}(\mathcal{G}, \mathcal{G}') \otimes_A K \simeq \mathrm{Hom}(\{\mathcal{F}, h_w\}, \{\mathcal{F}', h'_w\}).$$

Proof. The result follows by the same reasoning of [Gie73, Prop.2]. □

Proposition 3.3.7. *Given W a finite étale covering of X and $\mathcal{W} \to \widehat{X}$ the corresponding finite étale covering of $\widehat{X}$, let $\mathrm{Coh}^K(\mathcal{W})$ be the category whose objects are coherent sheaves on $\mathcal{W}$ and whose morphisms are defined by*

$$\mathrm{Hom}_{Coh^K(\mathcal{W})}(\mathcal{F}, \mathcal{G}) := \mathrm{Hom}_{\mathcal{O}_\mathcal{W}}(\mathcal{F}, \mathcal{G}) \otimes_A K.$$

Let $W_K = W \times_S S_K$ be the generic fibre of W, then $\mathrm{Coh}^K(\mathcal{W})$ is equivalent to the category $\mathrm{Coh}(W_K)$.

Proof. By Grothendieck's existence theorem [EGA III, Cor.5.1.6], the category $\mathrm{Coh}^K(\mathcal{W})$ is equivalent to the category $\mathrm{Coh}^K(W)$, whose objects are coherent sheaves on W and whose maps are given by

$$\mathrm{Hom}_{\mathrm{Coh}^K(W)}(\mathcal{F}, \mathcal{G}) := \mathrm{Hom}_{\mathcal{O}_W}(\mathcal{F}, \mathcal{G}) \otimes_A K.$$

Denoting $j\colon W_K \to W$ the open immersion, it suffices to show that the functor

$$j^\star\colon \mathrm{Coh}^K(W) \to \mathrm{Coh}(W_K)$$

is an equivalence of categories.

By flat base change [Liu02, p. 5.2.27], for every coherent sheaf $\mathcal{F}$ on W and for any $p \geq 0$,

$$\mathrm{H}^p(W, \mathcal{F}) \otimes_A K \cong \mathrm{H}^p(W_K, j^\star\mathcal{F}).$$

Applying this for $p = 0$ to the sheaf $\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{G})$, for every $\mathcal{F}$ and $\mathcal{G}$ coherent sheaves, we get that $j^\star$ is a fully faithful functor.

Moreover, since W is proper over S, we can apply [EGA I, Thm. 9.4.8] and deduce that the functor $j^\star$ is essentially surjective. □

Theorem 3.3.8. *Given $(K^n, \rho) \in \mathrm{Rep}_K^{\mathrm{cts}}(\pi_1^{\text{proét}}(X_0, \xi))$, let Z_ρ be the finite étale covering of X corresponding to $\mathcal{Z}_\rho$ and $Z_K^\rho = Z_\rho \times_S K$ its generic fibre, then the category $Coh^m(\mathcal{Y}_\rho|\mathcal{Z}_\rho)$ of coherent sheaves on $\mathcal{Y}_\rho$ with meromorphic descent relative to $\mathcal{Z}_\rho$ is equivalent to the category $\mathrm{Coh}(Z_K^\rho)$ of coherent sheaves on Z_K^ρ.*

Proof. By Theorem 3.3.5 and Lemma 3.3.6, it follows that $\mathrm{Coh}^m(\mathcal{Y}_\rho|\mathcal{Z}_\rho)$ is equivalent to the category $\mathrm{Coh}^K(\mathcal{Z}_\rho)$, whose objects are coherent sheaves on $\mathcal{Z}_\rho$ and whose morphisms defined by

$$\mathrm{Hom}_{\mathrm{Coh}^K(\mathcal{Z}_\rho)}(\mathcal{F}, \mathcal{G}) := \mathrm{Hom}_{\mathcal{O}_{\mathcal{Z}_\rho}}(\mathcal{F}, \mathcal{G}) \otimes_A K.$$

Moreover, by Proposition 3.3.7, the category $\mathrm{Coh}^K(\mathcal{Z}_\rho)$ is equivalent to the category $\mathrm{Coh}(Z_K^\rho)$. □

3.4 Finite descent

In this section, given (K^n, ρ) a continuous $\pi_1^{\text{proét}}(X_0, \xi)$-representation, we construct meromorphic descent data on the trivial sheaf $\mathcal{O}^n_{\mathcal{Y}_\rho}$. Assuming that the generic fibre of X is smooth, we prove that the coherent sheaf on Z^ρ_K that corresponds to this descend data, via the equivalence in Theorem 3.3.8, further descends to a coherent sheaf on X_K.

Note that, as explained in Remark 1.3.7, if the generic fibre is smooth then X is normal. This property will be crucial in Theorem 3.4.5.

Definition 3.4.1. Given $(K^n, \rho) \in \mathrm{Rep}_K^{\text{cts}}(\pi_1^{\text{proét}}(X_0, \xi))$, set

$$\gamma \colon \mathrm{Aut}(\mathcal{Y}_\rho|\widehat{X}) \to \mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N$$

to be the composition of the isomorphism in Equation 3.2.5 and the inversion and set $\tilde{\rho} := \rho \circ \gamma$, then we define the *meromorphic descent data on $\mathcal{O}^n_{\mathcal{Y}_\rho}$ induced by ρ* as the collection $\{h^\rho_w\}_{w \in \mathrm{Aut}(\mathcal{Y}_\rho|\widehat{X})}$, where the morphisms h^ρ_w are given by

$$\begin{aligned} \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n &\xrightarrow{h^\rho_w} \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n \\ f \otimes v &\longrightarrow f \otimes \tilde{\rho}(w)(v) \ . \end{aligned}$$

Definition 3.4.2. Let (K^n, ρ) a continuous $\pi_1^{\text{proét}}(X_0, \xi)$)-representation that factors through $\mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N$, then for every $j \in \{1, \dots, N\}$ we define the group morphism

$$s_j \colon G_j^{\text{op}} \to (\mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N)^{\text{op}},$$

which sends an element $g_j \in G_j$ to the word that consists of the letter g_j.

Lemma 3.4.3. *Given $(K^n, \rho) \in \mathrm{Rep}_K^{\text{cts}}(\pi_1^{\text{proét}}(X_0, \xi))$, $j \in \{1, \dots, N\}$, $g_j \in G_j$, let $\{\mathcal{O}^n_{\mathcal{Y}_\rho}, k^{g_j}_w\}_{w \in Aut(\mathcal{Y}_\rho|\mathcal{Z}_\rho)}$ be the sheaf with meromorphic descend data relative to $\mathcal{Z}_\rho$ given by*

$$\begin{aligned} \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n &\xrightarrow{k^{g_j}_w} \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n \\ f \otimes v &\longrightarrow f \otimes \tilde{\rho}(s_j(g_j) \circ w \circ s_j(g_j)^{-1})(v) \ , \end{aligned}$$

where s_j is defined as above. If the sheaf $\{\mathcal{O}^n_{\mathcal{Y}_\rho}, h^\rho_w\}_{w \in Aut(\mathcal{Y}_\rho|\mathcal{Z}_\rho)}$ with meromorphic descent data relative to $\mathcal{Z}_\rho$ induced by ρ descends to the coherent sheaf $\mathcal{F}$ on $\mathcal{Z}_\rho$, then $\{\mathcal{O}^n_{\mathcal{Y}_\rho}, k^{g_j}_w\}_{w \in Aut(\mathcal{Y}_\rho|\mathcal{Z}_\rho)}$ descends to $g_j{}^\star\mathcal{F}$.

Proof. If $\{\mathcal{O}^n_{\mathcal{Y}_\rho}, h^\rho_w\}_{w \in \mathrm{Aut}(\mathcal{Y}_\rho|\mathcal{Z}_\rho)}$ descends to the coherent sheaf $\mathcal{F}$on $\mathcal{Z}_\rho$, there exists an isomorphism

$$\psi \colon q^\star_{Y/Z}\mathcal{F} \otimes_A K \to \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n$$

such that for every $w' \in \ker(\alpha)^{\text{op}}$ the following diagram commutes

$$\begin{array}{ccc} q^\star_{Y/Z}\mathcal{F} \otimes_A K & \xrightarrow{\psi} & \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle h^\rho_{w'}} \\ w'^\star q^\star_{Y/Z}\mathcal{F} \otimes_A K = q^\star\mathcal{F} \otimes_A K & \xrightarrow{w'^\star\psi} & \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n \ . \end{array} \tag{3.4.4}$$

Moreover, we observe that, for every $g_j \in G_j^{\mathrm{op}}$,

$$g_j \circ q_{Y/Z} = q_{Y/Z} \circ s_j(g_j).$$

The proof of this identity is obvious when one translates it in terms of the corresponding maps of sets between $\mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N$ and $G_1 \times \cdots \times G_N$.

Thus, we can define the isomorphism of sheaves

$$s_j(g_j)^\star q_{Y/Z}^\star \mathcal{F} = q_{Y/Z}^\star g_j{}^\star \mathcal{F} \otimes_A K \xrightarrow{s_j(g_j)^\star \psi} \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n.$$

The Equation 3.4.4 applied to $w' = s_j(g_j) \circ w \circ s_j(g_j)^{-1}$ for every $w \in \ker(\alpha)$ tells us that the following diagram commutes

$$\begin{array}{ccc} q_{Y/Z}^\star \mathcal{F} \otimes_A K & \xrightarrow{\psi} & \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle k_w^{g_j}} \\ q_{Y/Z}^\star \mathcal{F} \otimes_A K & \xrightarrow{(s_j(g_j) \circ w \circ s_j(g_j)^{-1})^\star \psi} & \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n . \end{array}$$

Then applying $s_j(g_j)^\star$ to the previous diagram, we get that the following diagram commutes

$$\begin{array}{ccc} s_j(g_j)^\star q_{Y/Z}^\star \mathcal{F} \otimes_A K & \xrightarrow{s_j(g_j)^\star \psi} & \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle k_w^{g_j}} \\ s_j(g_j)^\star q_{Y/Z}^\star \mathcal{F} \otimes_A K & \xrightarrow{w^\star s_j(g_j)^\star \psi} & \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n . \end{array}$$

Hence, $s_j(g_j)^\star \psi$ induces an isomorphism of meromorphic descent data between $\{q_{Y/Z}^\star g_j{}^\star \mathcal{F}\}_{w \in \mathrm{Aut}(\mathcal{Y}_\rho | \mathcal{Z}_\rho)}$ and $\{\mathcal{O}_{\mathcal{Y}_\rho}^n, k_w^{g_j}\}_{w \in \mathrm{Aut}(\mathcal{Y}_\rho | \mathcal{Z}_\rho)}$ and this proves that $\{\mathcal{O}_{\mathcal{Y}_\rho}^n, k_w^{g_j}\}_{w \in \mathrm{Aut}(\mathcal{Y}_\rho | \mathcal{Z}_\rho)}$ descends to $g_j{}^\star \mathcal{F}$ on $\mathcal{Z}_\rho$. □

Let $\mathcal{Z}_\rho$ be the finite étale cover of $\widehat{X}$ associated with (K^n, ρ) that we constructed in Section 3.2. We denote by Z_ρ the corresponding finite étale covering of X and by Z_K^ρ the generic fibre of Z_ρ.

Theorem 3.4.5. *Given X a projective semi-stable curve over S with smooth generic fibre and $(K^n, \rho) \in \mathrm{Rep}_K^{\mathrm{cts}}(\pi_1^{\mathrm{proét}}(X_0, \xi))$, let $\mathcal{F}_K$ be the coherent sheaf on Z_K^ρ that corresponds, via the equivalence in Theorem 3.3.8, to the sheaf $\{\mathcal{O}_{\mathcal{Y}_\rho}^n, h_w^\rho\}_{w \in Aut(\mathcal{Y}_\rho | \mathcal{Z}_\rho)}$ with meromorphic descent data induced by ρ, then $\mathcal{F}_K$ descends to a coherent sheaf on X_K.*

Proof. The idea of the proof is to break the argument in N steps and proceed by induction. First let us define the intermediate steps.

We set $\mathcal{Z}_N = \mathcal{Z}_\rho$ and $Z_K^N = Z_K^\rho$. By definition,

$$\mathrm{Aut}(Z_\rho | X) = \mathrm{Aut}(\mathcal{Z}_N | \widehat{X}) = (G_1 \times \cdots G_N)^{\mathrm{op}}.$$

For $i = 1, \ldots, N-1$, we define q_{N-i} to be the projection

$$q_{N-i} \colon G_i \times G_{i+1} \times \cdots \times G_N \to G_{i+1} \times \cdots \times G_N,$$

while

$$q_0 \colon G_N \to 1 \text{ and } q_N := \alpha.$$

For $i = 0, \ldots, N-1$, the set $G_{i+1} \times \cdots \times G_N$, is endowed with a $\pi_1^{\text{proét}}(X_0, \xi)$-action, via the map

$$\alpha_{N-i} = q_{N-i} \circ \cdots \circ q_{N-1} \circ q_N \colon \mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N \to G_{i+1} \times \cdots \times G_N.$$

Thus, it corresponds to a finite étale coverings $Z_0^{N-i} \to X_0$. We define $\mathcal{Z}_{N-i}$ to be the corresponding finite étale covering of $\widehat{X}$.

Furthermore, if Z_{N-i} is the finite étale cover of X corresponding to $\mathcal{Z}_{N-i}$, then we set Z_K^{N-i} to be its generic fibre. By construction,

$$\operatorname{Aut}(Z_{N-i}|X) = \operatorname{Aut}(\mathcal{Z}_{N-i}|\widehat{X}) = (G_{i+1} \times \cdots G_N)^{\mathrm{op}},$$

$$\operatorname{Aut}(Z_{N-i+1}|Z_{N-i}) = \operatorname{Aut}(\mathcal{Z}_{N-i+1}|\mathcal{Z}_{N-i}) = G_i^{\mathrm{op}},$$

$$\operatorname{Aut}(\mathcal{Y}_\rho|\mathcal{Z}_{N-i}) = \ker(\alpha_{N-i})^{\mathrm{op}}.$$

We claim that, for $i = 0, \ldots, N-1$,

$$\operatorname{Aut}(Z_{N-i}|X) \simeq \operatorname{Aut}(Z_K^{N-i}|X_K). \tag{3.4.6}$$

Applying the pullback along the open immersion $X_K \to X$ we get a natural morphism

$$\operatorname{Aut}(Z_{N-i}|X) \to \operatorname{Aut}(Z_K^{N-i}|X_K).$$

Since, if two automorphisms of Z_{N-i} coincide on the open Z_K^{N-i} they coincide on the all Z_{N-i}, then the above morphism is injective. It remains to show that it is surjective. As seen in Remark 1.3.7, X is a normal scheme. Hence Z_{N-i} is also normal and applying [LM99, Lem. 4.11] to the model Z_{N-i}, we see that an automorphism $f \in \operatorname{Aut}(Z_K^{N-i}|X_K)$ can be extended to the scheme Z_{N-i}. Thus, the morphism Equation 3.4.6 is an isomorphism.

Similarly, we see that

$$\operatorname{Aut}(Z_K^{N-i+1}|Z_K^{N-i}) \simeq \operatorname{Aut}(Z_{N-i+1}|Z_{N-i}) = G_i^{\mathrm{op}}.$$

Hence, to construct descent data for the sheaf $\mathcal{F}_K$ relative to the finite étale map $q_{N-1} \colon Z_K^N \to Z_K^{N-1}$, it suffices to construct, for every $g_1 \in G_1^{\mathrm{op}}$, isomorphisms

$$h_{g_1}^{N-1} \colon \mathcal{F}_K \to g_1{}^\star \mathcal{F}_K$$

satisfying the co-cycle condition.

The sheaf with meromorphic descent data $\{\mathcal{O}_{\mathcal{Y}_\rho}^n, k_w^{g_1}\}_{w \in \operatorname{Aut}(\mathcal{Y}_\rho|\mathcal{Z}_\rho)}$, defined in Lemma 3.4.3, descends to $g_1{}^\star\mathcal{F}$ on $\mathcal{Z}_N$ so, by Theorem 3.3.8, we need to construct

$$h_{g_1}^{N-1} \in \operatorname{Hom}_{\mathcal{O}_{Z_K^N}}(\mathcal{F}_K, g_1{}^\star \mathcal{F}_K) = \operatorname{Hom}(\{\mathcal{O}_{\mathcal{Y}_\rho}^n, h_w^\rho\}, \{\mathcal{O}_{\mathcal{Y}_\rho}^n, k_w^{g_1}\}).$$

Let s_1 be as in Definition 3.4.2, then we set, for every $g_1 \in G_1$,

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n & \xrightarrow{h_{g_1}^{N-1}} & \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n \\ f \otimes v & \longrightarrow & f \otimes \tilde{\rho}(s_1(g_1))(v). \end{array}$$

and we get that the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n & \xrightarrow{h^{N-1}_{g_1}} & \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n \\ {\scriptstyle h^\rho_w}\downarrow & & \downarrow{\scriptstyle k^{g_1}_w} \\ \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n & \xrightarrow[w^\star h^{N-1}_{g_1}]{} & \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n \end{array}$$

because

$$\tilde{\rho}(s_1(g_1)\circ w\circ s_1(g_1)^{-1})\circ\tilde{\rho}(s_1(g_1)) = \tilde{\rho}(s_1(g_1)\circ w\circ s_1(g_1)^{-1}\circ s_1(g_1)) = \tilde{\rho}(s_1(g_1)\circ w).$$

Hence, $\{h^{N-1}_{g_1}\}$ are isomorphisms of descent data. Since by construction they satisfy the co-cycle condition, they induce the isomorphisms we wanted. This shows that $\mathcal{F}_K$ descends to a coherent sheaf $\mathcal{F}^{N-1}_K$ on Z^{N-1}_K.

By construction, if $\mathcal{F}_{N-1}$ is the sheaf on $\mathcal{Z}_{N-1}$ corresponding to $\mathcal{F}^{N-1}_K$, then $\{\mathcal{F}, h^{N-1}_{g_1}\}_{g_1\in G_1^{\mathrm{op}}}$ descends to $\mathcal{F}_{N-1}$ as sheaf with meromorphic descent data, i.e., there exists an isomorphism ψ_{N-1} such that, for every $g_1 \in G_1^{\mathrm{op}}$, the following diagram commutes

$$\begin{array}{ccc} q^\star_{N-1}\mathcal{F}_{N-1} \otimes_A K & \xrightarrow{\psi_{N-1}} & \mathcal{F} \otimes_A K \\ \downarrow & & {\scriptstyle h^{N-1}_{g_1}}\downarrow \\ q^\star_{N-1}\mathcal{F}_{N-1} \otimes_A K & \xrightarrow{g_1^\star\psi_{N-1}} & s_1(g_1)^\star\mathcal{F} \otimes_A K \ . \end{array}$$

We want now to construct descent data for the sheaf $\mathcal{F}^{N-1}_K$ relative to the finite étale map $q_{N-2}\colon Z^{N-1}_K \to Z^{N-2}_K$. In order to do this, we need first to find a sheaf with meromorphic data on $\mathcal{Y}_\rho$ that descends to $\mathcal{F}_{N-1}$.

For $i = 1, \dots, N$, the following sequence of groups is exact

$$1 \to \ker(\alpha_{N-i+1})^{\mathrm{op}} \to \ker(\alpha_{N-i})^{\mathrm{op}} \to G_i^{\mathrm{op}} \to 1$$

and the map $s_i\colon G_i^{\mathrm{op}} \to \ker(\alpha_{N-i})^{\mathrm{op}}$, defined as in Definition 3.4.2, induces a section. Thus, $\ker(\alpha_{N-i})^{\mathrm{op}}$ is the semi-direct product of $\ker(\alpha_{N-i+1})^{\mathrm{op}}$ and G_i^{op}. In particular, any word $w \in \ker(\alpha_{N-1})^{\mathrm{op}}$ can be written as

$$w = w' \circ s_1(g_1) \text{ with } w' \in \ker(\alpha_N)^{\mathrm{op}} = \ker(\alpha)^{\mathrm{op}},\ g_1 \in G_1^{\mathrm{op}}.$$

Given $w = w' \circ s_1(g_1)$, we set, for $f \otimes v \in \mathcal{O}^n_{\mathcal{Y}_\rho} \otimes_A K$,

$$h^{N-2}_w(f \otimes v) := s_1(g_1)^\star h^\rho_{w'} \circ h^{N-1}_{g_1}(f \otimes v) = f \otimes \tilde{\rho}(w)v.$$

By construction, $\{\mathcal{O}^n_{\mathcal{Y}_\rho}, h^{N-2}_w\}_{w\in\ker(\alpha_{N-1})^{\mathrm{op}}}$ is a sheaf with meromorphic descent data on $\mathcal{Y}_\rho$ relative to $\mathcal{Z}_{N-1}$.

Moreover, the following diagram commutes:

$$\begin{array}{ccccc} q^\star q^\star_{N-1}\mathcal{F}_{N-1} \otimes_A K & \xrightarrow{q^\star\psi_{N-1}} & q^\star\mathcal{F} \otimes_A K & \xrightarrow{\psi} & \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n \\ \downarrow & & {\scriptstyle q^\star h^{N-1}_g}\downarrow & & {\scriptstyle h^{N-1}_g}\downarrow \\ q^\star q^\star_{N-1}\mathcal{F}_{N-1} \otimes_A K & \xrightarrow{q^\star g_1^\star\psi_{N-1}} & s_1(g_1)^\star q^\star\mathcal{F} \otimes_A K & \xrightarrow{s_1(g_1)^\star\psi} & \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n \\ {\scriptstyle \mathrm{Id}}\downarrow & & {\scriptstyle \mathrm{Id}}\downarrow & & {\scriptstyle s_1(g_1)^\star h_{w'}}\downarrow \\ q^\star q^\star_{N-1}\mathcal{F}_{N-1} \otimes_A K & \xrightarrow{s_1(g_1)^\star q^\star\psi_{N-1}} & s_1(g_1)^\star q^\star\mathcal{F} \otimes_A K & \xrightarrow{s_1(g_1)^\star w'^\star\psi} & \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n \end{array}$$

where for simplicity we wrote $q = q_{Y/Z}$. The upper left square commutes because $\mathcal{F}$ descends to $\mathcal{F}_{N-1}$ on Z_{N-1}; the upper right by construction of h_g^{N-1}; the lower left square commutes because $g_j \circ q = q \circ s_j(g_j)$; while the lower right commutes because $\{\mathcal{O}_{\mathcal{Y}_\rho}, h_w\}$ descends to $\mathcal{F}$.

In particular, also the external square commutes and this implies that the sheaf $\{\mathcal{O}^n_{\mathcal{Y}_\rho}, h_w^{N-2}\}_{w\in\ker(\alpha_{N-1})^{\mathrm{op}}}$ descends to $\mathcal{F}_{N-1}$ on $\mathcal{Z}_{N-1}$.

For every $g_2 \in G_2$, let $\{\mathcal{O}^n_{\mathcal{Y}_\rho}, k_w^{g_2}\}_{w\in\ker(\alpha_{N-1})^{\mathrm{op}}}$ be defined by

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n & \xrightarrow{k_w^{g_2}} & \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n \\ f \otimes v & \longrightarrow & f \otimes \tilde{\rho}(s_2(g_2) \circ w \circ s_2(g_2)^{-1})(v). \end{array}$$

As in the proof of Lemma 3.4.3, we find that $\{\mathcal{O}^n_{\mathcal{Y}_\rho}, k_w^{g_2}\}$ descends to the sheaf $g_2{}^\star\mathcal{F}_{N-1}$ on Z_{N-1}. Then, repeating the argument of the base step, we find that the maps

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n & \xrightarrow{h_{g_2}^{N-2}} & \mathcal{O}_{\mathcal{Y}_\rho} \otimes_A K^n \\ f \otimes v & \longrightarrow & f \otimes \tilde{\rho}(s_2(g_2))(v) \end{array}$$

induce isomorphisms of meromorphic descent data between $\{\mathcal{O}^n_{\mathcal{Y}_\rho}, h_w^{N-2}\}$ and $\{\mathcal{O}^n_{\mathcal{Y}_\rho}, k_w^{g_2}\}$. Moreover, by Theorem 3.3.8

$$h_{g_2}^{N-2} \in \mathrm{Hom}(\{\mathcal{O}^n_{\mathcal{Y}_\rho}, h_w^{N-2}\}, \{\mathcal{O}^n_{\mathcal{Y}_\rho}, k_w^{g_2}\}) = \mathrm{Hom}(\mathcal{F}_K^{N-1}, g_2{}^\star\mathcal{F}_K^{N-1}).$$

By construction $h_{g_2}^{N-2}$ satisfy the co-cycle condition, hence they induce descent data for $\mathcal{F}_K^{N-1}$ relative to the finite étale map $Z_K^{N-1} \to Z_K^{N-2}$.

By induction on N, the proof is complete. ☐

Remark 3.4.7. Let $\{\mathcal{O}^n_{\mathcal{Y}_\rho}, h_w^\rho\}_{w\in\mathrm{Aut}(\mathcal{Y}_\rho|\mathcal{Z}_\rho)}$ and $\mathcal{F}_K$ as above, then we showed that $\mathcal{F}_K$ descends to a coherent sheaf $\mathcal{G}_K^\rho$ on X_K. Let $\mathcal{G}_\rho$ be the coherent sheaf on $\widehat{X}$ that corresponds to $\mathcal{G}_K^\rho$ via the equivalence in Proposition 3.3.7, then it is clear from the construction that $\{\mathcal{O}^n_{\mathcal{Y}_\rho}, h_w^\rho\}_{w\in\mathrm{Aut}(\mathcal{Y}_\rho|\widehat{X})}$, the sheaf with meromorphic data induced by ρ, descends to $\mathcal{G}_\rho$ on $\widehat{X}$ in the sense of Definition 3.3.2, i.e., there exists a morphism of meromorphic descent data

$$\{\mathcal{O}^n_{\mathcal{Y}_\rho}, h_w^\rho\}_{w\in\mathrm{Aut}(\mathcal{Y}_\rho|\widehat{X})} \simeq \{q_Y^\star\mathcal{G}_\rho, h_w^q\}_{w\in\mathrm{Aut}(\mathcal{Y}_\rho|\widehat{X})},$$

where $q_Y : \mathcal{Y}_\rho \to \widehat{X}$ is the structure map of $\mathcal{Y}_\rho$ and $h_w^q : q_Y^\star\mathcal{G}_\rho \to w^\star q_Y^\star\mathcal{G}_\rho$ are the natural isomorphisms.

Theorem 3.4.8. *Given X a projective semi-stable curve over S with smooth generic fibre, associating with $(K^n, \rho) \in \mathrm{Rep}_K^{\mathrm{cts}}(\pi_1^{\mathrm{pro\acute{e}t}}(X_0, \xi))$ the coherent sheaf $\{\mathcal{O}^n_{\mathcal{Y}_\rho}, h_w^\rho\}_{w\in Aut(\mathcal{Y}_\rho|\widehat{X})}$ with meromorphic descent data induced by ρ defines a functor*

$$F\colon \mathrm{Rep}_K^{\mathrm{cts}}(\pi_1^{\mathrm{pro\acute{e}t}}(X_0, \xi)) \to \mathrm{Coh}(X_K).$$

Proof. Given $(K^n, \rho) \in \mathrm{Rep}_K^{\mathrm{cts}}(\pi_1^{\text{proét}}(X_0, \xi))$, we defined in Definition 3.4.1 $\{\mathcal{O}^n_{\mathcal{Y}_\rho}, h^\rho_w\}_{w\in \mathrm{Aut}(\mathcal{Y}_\rho|\widehat{X})}$ and proved in Theorem 3.3.5 and Theorem 3.4.5 that it corresponds to a coherent sheaf $\mathcal{F}^\rho_K$ on X_K. Then we can set

$$\mathrm{F}(K^n, \rho) = \mathcal{F}^\rho_K.$$

Hence, it suffices to define $\mathrm{F}(\varphi)$ for every morphism of continuous representations $\varphi\colon (K^n, \rho) \to (K^m, \tau)$. Assume that ρ factors through the quotient $\mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N$ and τ factors through $\mathbb{Z}^{\star r} \star H_1 \star \cdots \star H_N$, then we denote by $\mathcal{Y}_\rho$ and $\mathcal{Y}_\tau$ the formal geometric coverings of $\widehat{X}$ associated with ρ and τ, as in Definition 3.2.4.

We consider now the set

$$\mathbb{Z}^{\star r} \star (G_1 \times H_1) \star \cdots \star (G_N \times H_N).$$

Since, for every j, there exists a quotient map

$$\beta_j : \pi_1^{\text{ét}}(\overline{C_j}) \to G_j \times H_j,$$

the map

$$\beta\colon \pi_1^{\text{proét}}(X_0, \xi) \to \mathbb{Z}^{\star r} \star (G_1 \times H_1) \star \cdots \star (G_N \times H_N),$$

induced by β_j on the letters in $\pi_1^{\text{ét}}(\overline{C_j})$ and by the identity on the letters in $\mathbb{Z}^{\star r}$ is a continuous group morphism. Hence, it defines a $\pi_1^{\text{proét}}(X_0, \xi)$-action on $\mathbb{Z}^{\star r} \star (G_1 \times H_1) \star \cdots \star (G_N \times H_N)$ and we can associate with it a geometric covering of $\widehat{X}$, which we call $\mathcal{Y}_{\rho,\tau}$. By construction, there exist $\widehat{X}$-morphisms

$$p_\rho\colon \mathcal{Y}_{\rho,\tau} \to \mathcal{Y}_\rho \text{ and } p_\tau\colon \mathcal{Y}_{\rho,\tau} \to \mathcal{Y}_\tau.$$

We set

$$\rho'\colon \mathbb{Z}^{\star r} \star (G_1 \times H_1) \star \cdots \star (G_N \times H_N) \to \mathrm{GL}_n(K)$$

to be the unique group morphism such that

- $\rho'(w) = \rho(w)$ for every $w \in \mathbb{Z}^{\star r}$,
- $\rho'(g_i, h_i) = \rho(g_i)$ for every $(g_i, h_i) \in G_i \times H_i$, $i = 1, \ldots, N$.

Similarly, we define τ'. Then we have that

$$p_\rho^\star\{\mathcal{O}^n_{\mathcal{Y}_\rho}, h^\rho_w\} = \{\mathcal{O}^n_{\mathcal{Y}_{\rho,\tau}}, h^{\rho'}_w\} \text{ and } p_\tau^\star\{\mathcal{O}^m_{\mathcal{Y}_\tau}, h^\tau_w\} = \{\mathcal{O}^m_{\mathcal{Y}_{\rho,\tau}}, h^{\tau'}_w\}.$$

By construction, the sheaf $\{\mathcal{O}^n_{\mathcal{Y}_{\rho,\tau}}, h^{\rho'}_w\}_{w\in\mathrm{Aut}(\mathcal{Y}_{\rho,\tau}|\widehat{X})}$ descends to a coherent sheaf $\mathcal{F}_\rho$ on $\widehat{X}$, which corresponds via the equivalence in Proposition 3.3.7 to $F(\rho)$ on X_K and, similarly, $\{\mathcal{O}^m_{\mathcal{Y}_{\rho,\tau}}, h^{\tau'}_w\}$ descends to a coherent sheaf $\mathcal{F}_\tau$ on $\widehat{X}$, which corresponds to $F(\tau)$ on X_K.

By Proposition 3.3.7, in order to define $F(\varphi)$, it suffices to construct a morphism

$$\mathcal{F}_\rho \otimes_A K \to \mathcal{F}_\tau \otimes_A K.$$

However, by Theorem 3.3.5 and Theorem 3.4.5, such a morphism corresponds to a morphism of meromorphic descent data

$$\alpha_\varphi \colon \{\mathcal{O}^n_{\mathcal{Y}_{\rho,\tau}}, h^\rho_w\}_{w\in \mathrm{Aut}(\mathcal{Y}_{\rho,\tau}|\widehat{X})} \to \{\mathcal{O}^m_{\mathcal{Y}_{\rho,\tau}}, h^\tau_w\}_{w\in \mathrm{Aut}(\mathcal{Y}_{\rho,\tau}|\widehat{X})}.$$

We set

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{Y}_{\rho,\tau}} \otimes_A K^n & \xrightarrow{\alpha_\varphi} & \mathcal{O}_{\mathcal{Y}_{\rho,\tau}} \otimes_A K^m \\ f\otimes v & \longrightarrow & f \otimes \varphi(v)\ , \end{array}$$

then α_φ is the wanted morphism of meromorphic descent data. Indeed, since φ is a morphism of representations, for every $w \in \mathrm{Aut}(\mathcal{Y}_{\rho,\tau}|\widehat{X})$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{Y}_{\rho,\tau}} \otimes_A K^n & \xrightarrow{h^\rho_w} & w^\star \mathcal{O}_{\mathcal{Y}_{\rho,\tau}} \otimes_A K^n \\ {\scriptstyle \alpha_\varphi}\downarrow & & \downarrow {\scriptstyle w^\star \alpha_\varphi} \\ \mathcal{O}_{\mathcal{Y}_{\rho,\tau}} \otimes_A K^m & \xrightarrow[h^\tau_w]{} & w^\star \mathcal{O}_{\mathcal{Y}\rho,\tau} \otimes_A K^m\ . \end{array}$$

□

Remark 3.4.9. If X is a stable curve over S with smooth generic fibre and degenerate closed fibre, then the functor constructed in the previous lemma coincides with the functor constructed in Theorem 3.1.13.

Chapter 4

The specialization functor

Given S and K as in the previous chapter, $\overline{K}$ a fixed algebraic closure of K and X a projective semi-stable over S with smooth generic fibre, the goal of this chapter is to construct a tensor functor from the category $\mathrm{Rep}^{\mathrm{cts}}_{\overline{K}}(\pi_1^{\mathrm{proét}}(X_0, \xi))$ of continuous $\overline{K}$-linear $\pi_1^{\mathrm{proét}}(X_0, \xi)$-representations to the category $\mathrm{Strat}(X_{\overline{K}})$ of stratified bundles on the geometric generic fibre $X_{\overline{K}}$. By Tannakian duality, this functor will induce morphism of group schemes from $\pi^{\mathrm{strat}}(X_{\overline{K}})$ to $(\pi_1^{\mathrm{proét}}(X_0, \xi))^{\mathrm{cts}}$.

In analogy with the previous chapter, stratified bundles with meromorphic data will play a crucial role in the construction of the functor.

4.1 Stratified bundles with meromorphic descent data

In this section we recall the definition of stratified bundles and we state the properties of the group scheme associated with the Tannakian category that they form. These properties will be the starting point for the study of the correlation between the specialization functor that we construct and the usual étale specialization map. We conclude by introducing the notion of stratified bundles with meromorphic descent data.

Given F a field of positive characteristic $p > 0$ and T a smooth scheme of finite type over F, we denote by $T^{(i)}$ the i-th Frobenius twist of T and by $F^i_{T/F} \colon T^{(i)} \to T^{(i+1)}$ the i-th relative Frobenius over F.

For our purposes it would, in fact, be sufficient to consider $T = X_{\overline{K}}$, the geometric generic fibre of a projective semi-stable curve X.

Definition 4.1.1. Let T be a smooth scheme of finite type over a field F of positive characteristic, then an *F-divided sheaf on* T is given by a sequence $(\mathcal{E}_i, \sigma_i)_{i \geq 0}$, where $\mathcal{E}_i$ are bundles on $T^{(i)}$ and $\sigma_i \colon {F^i_{T/F}}^\star \mathcal{E}_{i+1} \to \mathcal{E}_i$ are $\mathcal{O}_{T^{(i)}}$-linear isomorphisms.

Definition 4.1.2. Given T as above and $(\mathcal{E}_i, \sigma_i)$,$(\mathcal{G}_i, \tau_i)$ two F-divided sheaves on T, a *morphism of F-divided sheaves* from $(\mathcal{E}_i, \sigma_i)$ to $(\mathcal{G}_i, \tau_i)$ is defined as a sequence of $\mathcal{O}_{T^{(i)}}$-linear maps, $\alpha = \{\alpha_i \colon \mathcal{E}_i \to \mathcal{G}_i\}$, such that the following diagram is commutative

$$\begin{array}{ccc} {F^i_{T/F}}^{\star}\mathcal{E}_{i+1} & \xrightarrow{{F^i_{T/F}}^{\star}\alpha_{i+1}} & {F^i_{T/F}}^{\star}\mathcal{G}_{i+1} \\ \sigma_i\big\downarrow & & \big\downarrow\tau_i \\ \mathcal{E}_i & \xrightarrow{\alpha_i} & \mathcal{G}_i \ . \end{array}$$

Definition 4.1.3. Let T be a smooth scheme of finite type over a field F and $\mathcal{D}_{T/F}$ the quasi coherent $\mathcal{O}_T$-module of differential operators defined in [EGA 4, Section 16], then a *stratified bundle on* T is a locally free $\mathcal{O}_T$-module of finite rank endowed with a $\mathcal{O}_T$-linear $\mathcal{D}_{T/F}$-action extending the $\mathcal{O}_T$-module structure via the inclusion $\mathcal{O}_T \subset \mathcal{D}_{T/F}$. A *morphism of stratified bundles* is a morphism of $\mathcal{D}_{T/F}$-modules.

Theorem 4.1.4 (**Katz's theorem**, [Gie75], Thm. 1.3)**.** *Let T be a smooth scheme of finite type over a field F of characteristic $p > 0$, then the category of stratified bundles on T and the category of F-divided sheaves on T are equivalent.*

Since they are equivalent, we identify these two categories and we use the term stratified bundles for both definitions. Moreover, we denote by $\mathrm{Strat}(T)$ the category of stratified bundles on T.

Proposition 4.1.5 ([SR72], Section. VI.1)**.** *Let T be a smooth scheme of finite type over a field F of characteristic $p > 0$, then the category* $\mathrm{Strat}(T)$ *of stratified bundles on T is a rigid abelian tensor category. Moreover, if T has a rational point $x \in T(F)$, the functor*

$$\omega_x \colon \mathrm{Strat}(T) \to \mathrm{Vec}_F,\ \omega_x(\mathcal{E}_i, \sigma_i) = x^*\mathcal{E}_0$$

is a fibre functor and $\mathrm{Strat}(T)$ *is a neutral Tannakian category over F.*

Definition 4.1.6. Let T a smooth scheme of finite type over a field F of characteristic $p > 0$ and suppose that T has a rational point $x \in T(F)$, then we denote by $\pi^{\mathrm{strat}}(T, x)$ the affine group scheme associated with $(\mathrm{Strat}(T), \omega_x)$ via Tannakian duality (Theorem 2.1.10).

Definition 4.1.7. Given $\mathcal{E} \in \mathrm{Strat}(T)$ we denote by $\langle \mathcal{E} \rangle_{\otimes}$ the full subcategory of $\mathrm{Strat}(T)$ whose objects are subquotients of objects of the form $P(\mathcal{E}, \mathcal{E}^{\vee})$, with $P(x, y) \in \mathbb{N}[x, y]$.

The category $\langle \mathcal{E} \rangle_{\otimes}$ is the smallest Tannakian subcategory of $\mathrm{Strat}(T)$ that is closed under subquotients and contains $\mathcal{E}$.

Definition 4.1.8. Suppose that T is a smooth scheme of finite type over a field F of characteristic $p > 0$ and that it has a rational point $x \in T(F)$, we define the *monodromy group of* $\mathcal{E} \in \mathrm{Strat}(T)$ to be the affine group scheme associated with the Tannakian category $(\langle \mathcal{E} \rangle_{\otimes}, \omega_x|_{\langle \mathcal{E} \rangle_{\otimes}})$ and we denote it by $G(\mathcal{E}, x)$.

Definition 4.1.9. Under the above assumptions, we say that $\mathcal{E} \in \mathrm{Strat}(T)$ *has finite monodromy* if the group scheme $G(\mathcal{E}, x)$ is finite over K.

To check if a stratified bundle has finite monodromy, we can use the following criterion.

Lemma 4.1.10 ([EL13], Lem. 1.1). *If F is an algebraically closed of characteristic $p > 0$ and T is a smooth scheme of finite type over F, then given $\mathcal{E} \in \mathrm{Strat}(T)$ the following are equivalent*

- *$\mathcal{E}$ has finite monodromy,*
- *there exists a finite étale cover $p\colon T' \to T$ such that $p^{\star}\mathcal{E}$ is trivial in $\mathrm{Strat}(T')$.*

Analysing the category $\mathrm{Strat}(T)$ one finds even more information about the group scheme $\pi^{\mathrm{strat}}(T, x)$.

Theorem 4.1.11 ([San07], Thm. 11). *If F is an algebraically closed of characteristic $p > 0$, T is a smooth scheme of finite type over F and $x \in T(F)$, then the group scheme $\pi^{\mathrm{strat}}(T, x)$ is perfect. In particular, it is reduced.*

Corollary 4.1.12 ([Kin14], Thm. 2.9). *Let F an algebraically closed field of characteristic $p > 0$, T a smooth scheme of finite type over F and $x \in T(F)$, then if a stratified bundle $\mathcal{E}$ on T has finite monodromy, its monodromy group $G(\mathcal{E}, x)$ is a constant group scheme over F.*

The following description of the pro-finite completion of the group scheme $\pi^{\mathrm{strat}}(T, x)$ will be very important for the next sections.

Proposition 4.1.13 ([Kin14], Prop. 2.15). *Given F an algebraically closed of characteristic $p > 0$, T a smooth scheme of finite type over F and $x \in T(F)$, let $\pi_1^{\text{ét}}(T, x) = \varprojlim_i \pi_i$ be the étale fundamental group of T, then*

(i) if a stratified bundle $\mathcal{E}$ on T has finite monodromy, its monodromy group $G(\mathcal{E}, x)$ is isomorphic to the constant group scheme over F associated with a finite quotient of $\pi_1^{\text{ét}}(T, x)$,

(ii) for every finite quotient π_i of $\pi_1^{\text{ét}}(T, x)$ there exists a stratified bundle $\mathcal{E}$ such that $G(\mathcal{E}, x) \simeq (\pi_i)_F$, where $(\pi_i)_F$ is the constant group scheme over F associated with π_i.

In particular, there exists a morphism of group schemes over F

$$\pi^{\mathrm{strat}}(T, x) \to \varprojlim_i (\pi_i)_F.$$

Remark 4.1.14. In analogy with Remark 2.3.6, the group scheme $\varprojlim_i (\pi_i)_F$ is denoted as $\pi_1^{\text{ét}}(T, x)_F$.

We return now to our original notation, where A, S and K are as in the previous chapter and we fix a projective semi-stable curve X over S with smooth generic fibre. We proceed defining stratified bundles with meromorphic descent data.

Definition 4.1.15. Given $\mathcal{Y}$ a formal geometric covering of $\widehat{X}$, a coherent sheaf $\mathcal{F}$ on $\mathcal{Y}$ is called *meromorphic bundle* if there exists a locally free sheaf $\mathcal{E}$ on $\mathcal{Y}$ such that $\mathcal{F} \otimes_A K \cong \mathcal{E} \otimes_A K$.

We denote by $\mathcal{Y}^{(i)}$ the i-th Frobenius twist of $\mathcal{Y}$ and by $F^i_{\mathcal{Y}/S}\colon \mathcal{Y}^{(i)} \to \mathcal{Y}^{(i+1)}$ the i-th relative Frobenius over S.

Remark 4.1.16. If X is a projective semi-stable curve over S with smooth generic fibre, then so are its Frobenius twists $X^{(i)}$.

Indeed, since X is projective over S, by base change also $X^{(i)}$ is projective over S. Moreover, by [Liu02, Prop. 10.3.15.(a)], $X^{(i)}$ is a semi-stable projective curve over $S^{(i)}$. The generic fibre of $X^{(i)}$ is given by $(X^{(i)})_K \cong (X_K)^{(i)}$, which is smooth by base change, and the closed fibre of $X^{(i)}$ is $(X^{(i)})_0 \cong (X_0)^{(i)}$. Hence, the fibres of $X^{(i)}$ are geometrically connected, as we required in Definition 1.3.5.

Furthermore, given a geometric covering Y_0 of X_0 and $\mathcal{Y}$ the corresponding formal geometric covering of $\widehat{X}$, we see that, since k is perfect,

$$\operatorname{Aut}(Y_0|X_0) \simeq \operatorname{Aut}(Y_0^{(i)}|X_0^{(i)}) \simeq \operatorname{Aut}(\mathcal{Y}^{(i)}|\widehat{X^{(i)}}) \simeq \operatorname{Aut}(\mathcal{Y}|\widehat{X}).$$

We will identify the above groups, treating these isomorphisms as equalities.

Definition 4.1.17. Given $\mathcal{Y}$ a formal geometric covering of $\widehat{X}$, a *stratified bundle with meromorphic descent data on* $\mathcal{Y}$ is a sequence $\{\{\mathcal{E}_i, h_w^i\}_{w\in \operatorname{Aut}(\mathcal{Y}|\widehat{X})}, \sigma_i\}_{i\geq 0}$, with

- $\{\mathcal{E}_i, h_w^i\}_{w\in \operatorname{Aut}(\mathcal{Y}|\widehat{X})}$ meromorphic bundle on $\mathcal{Y}^{(i)}$ with meromorphic descent data
$$h_w^i \colon \mathcal{E}_i \otimes_A K \to w^\star \mathcal{E}_i \otimes_A K,$$
- σ_i isomorphisms of meromorphic descent data
$$\sigma_i \colon \{{F^i_{\mathcal{Y}/S}}^\star \mathcal{E}_{i+1}, {F^i_{\mathcal{Y}/S}}^\star h_w^{i+1}\}_{w\in \operatorname{Aut}(\mathcal{Y}|\widehat{X})} \to \{\mathcal{E}_i, h_w^i\}_{w\in \operatorname{Aut}(\mathcal{Y}|\widehat{X})}.$$

To simplify the notation, we will denote a stratified bundle with meromorphic descent data by $\{\mathcal{E}_i, h_w^i, \sigma_i\}$ or by $\{\mathcal{E}_i, h_w^i\}$, when the isomorphisms σ_i are clear.

Definition 4.1.18. Given $\mathcal{Y}$ a formal geometric covering of $\widehat{X}$, a *morphism of stratified bundles with meromorphic descent data* from $\{\mathcal{E}_i, h_w^i, \sigma_i\}$ to $\{\mathcal{G}_i, k_w^i, \tau_i\}$ is given by a sequence $\{\alpha_i\}_{i\geq 0}$ of morphisms of meromorphic descent data on $\mathcal{Y}^{(i)}$ such that the following diagram is commutative

$$\begin{array}{ccc} {F^i_{\mathcal{Y}/S}}^\star \{\mathcal{E}_{i+1}, h_w^{i+1}\}_{w\in \operatorname{Aut}(\mathcal{Y}|\widehat{X})} & \xrightarrow{{F^i_{\mathcal{Y}/S}}^\star \alpha_{i+1}} & {F^i_{\mathcal{Y}/S}}^\star \{\mathcal{G}_{i+1}, k_w^{i+1}\}_{w\in \operatorname{Aut}(\mathcal{Y}|\widehat{X})} \\ \downarrow{\sigma_i} & & \downarrow{\tau_i} \\ \{\mathcal{E}_i, h_w^i\}_{w\in \operatorname{Aut}(\mathcal{Y}|\widehat{X})} & \xrightarrow{\alpha_i} & \{\mathcal{G}_i, k_w^i\}_{w\in \operatorname{Aut}(\mathcal{Y}|\widehat{X})} \end{array}.$$

We denote by $\operatorname{Strat}^m(\mathcal{Y})$ the category of stratified bundle with meromorphic descent data on $\mathcal{Y}$.

4.2 Construction of the specialization functor

In this section we explain the construction of the specialization functor. The idea is to repeat the construction of Theorem 3.4.8 for all Frobenius twists of X and then prove that the obtained sequence of coherent sheaves is a stratified bundles.

Lemma 4.2.1. *Given X a projective semi-stable curve over S with smooth generic fibre, a representation $(K^n, \rho) \in \operatorname{Rep}_K^{\text{cts}}(\pi_1^{\text{proét}}(X_0, \xi))$ induces a stratified bundle with meromorphic descent data on $\mathcal{Y}_\rho$.*

Proof. Given a representation (K^n, ρ), we set $\{\mathcal{O}^n_{\mathcal{Y}_\rho}, h^\rho_w\}_{w \in \mathrm{Aut}(\mathcal{Y}_\rho|\widehat{X})}$ to be the sheaf with meromorphic descent data induced by ρ, as in Definition 3.4.1.

By Remark 4.1.16, we can repeat the construction on $X^{(i)}$. Namely we can define, on the sheaf $\mathcal{O}^n_{\mathcal{Y}^{(i)}_\rho}$, the meromorphic descent data $\{h^{\rho,i}_w\}_{w \in \mathrm{Aut}(\mathcal{Y}_\rho|\widehat{X})}$

$$\begin{aligned} \mathcal{O}_{\mathcal{Y}^{(i)}_\rho} \otimes_A K^n &\xrightarrow{h^{\rho,i}_w} \mathcal{O}_{\mathcal{Y}^{(i)}_\rho} \otimes_A K^n \\ f \otimes v &\longrightarrow f \otimes \tilde{\rho}(w)(v) \ , \end{aligned}$$

with $\tilde{\rho}$ as in Definition 3.4.1.

Hence, we get a sequence $\{\mathcal{O}^n_{\mathcal{Y}^{(i)}_\rho}, h^{\rho,i}_w\}_i$ of meromorphic bundles with meromorphic descent data, and, by construction of $h^{\rho,i}_w$, it is clear that

$$F^i_{\mathcal{Y}_\rho/S}{}^\star \{\mathcal{O}^n_{\mathcal{Y}^{(i+1)}_\rho}, h^{\rho,i+1}_w\} = \{\mathcal{O}^n_{\mathcal{Y}^{(i)}_\rho}, h^{\rho,i}_w\}.$$

Thus, $\{\mathcal{O}^n_{\mathcal{Y}^{(i)}_\rho}, h^{\rho,i}_w\}_i$ is a stratified bundle with meromorphic descent data. □

Definition 4.2.2. Given X a projective semi-stable curve over S with smooth generic fibre and $(K^n, \rho) \in \mathrm{Rep}^{\mathrm{cts}}_K(\pi_1^{pro\acute{e}t}(X_0, \xi))$, then a *stratified bundle with meromorphic descent data on $\mathcal{Y}_\rho$ relative to $\mathcal{Z}_\rho$* is given by the data of

- $\{\mathcal{E}_i, h^i_w\}_{w \in \mathrm{Aut}(\mathcal{Y}_\rho|\mathcal{Z}_\rho)}$, a meromorphic bundle on $\mathcal{Y}^{(i)}_\rho$ with meromorphic descent data
$$h^i_w \colon \mathcal{E}_i \otimes_A K \to w^\star \mathcal{E}_i \otimes_A K,$$
for each $i \geq 0$,
- σ_i, isomorphisms of meromorphic descent data
$$\sigma_i \colon \{F^i_{\mathcal{Y}_\rho/S}{}^\star \mathcal{E}_{i+1}, F^i_{\mathcal{Y}_\rho/S}{}^\star h^{i+1}_w\}_{w \in \mathrm{Aut}(\mathcal{Y}_\rho|\mathcal{Z}_\rho)} \to \{\mathcal{E}_i, h^i_w\}_{w \in \mathrm{Aut}(\mathcal{Y}_\rho|\mathcal{Z}_\rho)}.$$

The morphisms of stratified bundles with meromorphic descent data on $\mathcal{Y}_\rho$ relative to $\mathcal{Z}_\rho$ are defined as in Definition 4.1.18 and we denote by $\mathrm{Strat}^m(\mathcal{Y}_\rho|\mathcal{Z}_\rho)$ the category that these objects form.

The next proposition is the analogue of Theorem 3.3.8 for stratified bundles with meromorphic descent data.

Proposition 4.2.3. *Given X a projective semi-stable curve over S with smooth generic fibre and $(K^n, \rho) \in \mathrm{Rep}^{\mathrm{cts}}_K(\pi_1^{\mathrm{pro\acute{e}t}}(X_0))$, let Z_ρ be the finite étale covering of X corresponding to $\mathcal{Z}_\rho$ and $Z^\rho_K = Z_\rho \times_S K$ its generic fibre, then the categories $\mathrm{Strat}^m(\mathcal{Y}_\rho|\mathcal{Z}_\rho)$ and $\mathrm{Strat}(Z^\rho_K)$ are equivalent.*

Proof. By Remark 4.1.16, we can extend the results we found on X in the previous chapter to all its Frobenius twists $X^{(i)}$.

In particular, given an object $\{\mathcal{E}_i, h^i_w\}_i \in \mathrm{Strat}^m(\mathcal{Y}_\rho|\mathcal{Z}_\rho)$, by Theorem 3.3.5, the sheaf $\{\mathcal{E}_i, h^i_w\}_{w \in \mathrm{Aut}(\mathcal{Y}_\rho|\mathcal{Z}_\rho)}$ with meromorphic descent data relative to $\mathcal{Z}^{(i)}_\rho$ descends to a coherent sheaf $\mathcal{G}^i$ on $\mathcal{Z}^{(i)}_\rho$ for every i, and it corresponds via Groethendieck's existence theorem to a coherent sheaf $\widetilde{\mathcal{G}}^i$ on $Z^{(i)}_\rho$. Since by [Stacks, Tag 05B2] locally free sheaves of finite rank correspond via fpqc descent to locally free sheaves of finite rank, then, by construction, if $\mathcal{E}_i$ is a meromorphic bundle, also $\mathcal{G}^i$ is a meromorphic bundle on $\mathcal{Z}_\rho^{(i)}$. Moreover, as shown for example in [Her05, Cor. 1.15] locally free sheaves on $\mathcal{Z}_\rho^{(i)}$ correspond via Grothendieck's existence theorem to locally free sheaves on $Z_\rho^{(i)}$, hence also $\widetilde{\mathcal{G}}^i$ is a meromorphic bundle.

To simplify the notation we write $F_{\mathcal{Z}}^i = F_{\mathcal{Z}_\rho/S}^i$, $F_Z^i = F_{Z_\rho/S}^i$, and $F_{\mathcal{Y}}^i = F_{\mathcal{Y}_\rho/S}^i$. If $p_{\mathcal{Y}/Z}^i$ is the composition of $p_{\mathcal{Y}/\mathcal{Z}}^i \colon \mathcal{Y}_\rho^{(i)} \to \mathcal{Z}_\rho^{(i)}$ and the map $\mathcal{Z}_\rho^{(i)} \to Z_\rho^{(i)}$, then we have that

$$\operatorname{Hom}_{\mathcal{O}_{Z_\rho^{(i)}}}({F_Z^i}^\star \widetilde{\mathcal{G}}^{i+1}, \widetilde{\mathcal{G}}^i) \otimes_A K \simeq \operatorname{Hom}_{\mathcal{O}_{\mathcal{Y}_\rho^{(i)}}}({p_{\mathcal{Y}/Z}^i}^\star ({F_{Z_K}^i}^\star \widetilde{\mathcal{G}}^{i+1}), {p_{\mathcal{Y}/Z}^i}^\star \widetilde{\mathcal{G}}^i) \otimes_A K.$$

Since the relative Frobenius is functorial, i.e.,

$$F_Z^i \circ p_{\mathcal{Y}/Z}^i = p_{\mathcal{Y}/Z}^{i+1} \circ F_{\mathcal{Y}}^i,$$

and ${p_{\mathcal{Y}/Z}^i}^\star \widetilde{\mathcal{G}}^i \simeq \{\mathcal{E}_i, h_w^i\}$, we get

$$\operatorname{Hom}_{\mathcal{O}_{Z_\rho^{(i)}}}({F_Z^i}^\star \widetilde{\mathcal{G}}^{i+1}, \widetilde{\mathcal{G}}^i) \otimes_A K \simeq \operatorname{Hom}({F_{\mathcal{Y}}^i}^\star \{\mathcal{E}_{i+1}, h_w^{i+1}\}, \{\mathcal{E}_i, h_w^i\}).$$

Hence, the category $\operatorname{Strat}^m(\mathcal{Y}_\rho|\mathcal{Z}_\rho)$ is equivalent to the category $\operatorname{Strat}^K(Z_\rho)$, whose object are defined by sequences of meromorphic bundles $\{\widetilde{\mathcal{G}}_i, \sigma_i\}$ on $Z_\rho^{(i)}$ with isomorphisms

$$\sigma_i \colon {F_Z^i}^\star \widetilde{\mathcal{G}}_{i+1} \otimes_A K \to \widetilde{\mathcal{G}}_i \otimes_A K$$

and whose morphisms from $\{\widetilde{\mathcal{G}}_i, \sigma_i\}$ to $\{\widetilde{\mathcal{G}}'_i, \tau_i\}$ are sequences $\{\varphi_i\}$ of morphisms

$$\varphi_i \colon \widetilde{\mathcal{G}}_i \otimes_A K \to \widetilde{\mathcal{G}}'_i \otimes_A K$$

that are compatible with the isomorphisms σ_i and τ_i.

On the other hand by Proposition 3.3.7, $\widetilde{\mathcal{G}}_i$ corresponds to a coherent sheaf $\mathcal{G}_K^i$ on ${Z_K^\rho}^{(i)}$. Since $\widetilde{\mathcal{G}}_i$ are meromorphic bundles, the sheaves $\mathcal{G}_K^i$ are locally free. As above, by functoriality of the Frobenius, we see that

$$\operatorname{Hom}_{\mathcal{O}_{{Z_K^\rho}^{(i)}}}({F_{Z_K}^i}^\star \mathcal{G}_K^{i+1}, \mathcal{G}_K^i) \simeq \operatorname{Hom}_{\mathcal{O}_{Z_\rho^{(i)}}}({F_Z^i}^\star \widetilde{\mathcal{G}}^{i+1}, \widetilde{\mathcal{G}}^i) \otimes_A K.$$

Hence, the category $\operatorname{Strat}^K(Z_\rho)$ is equivalent to the category $\operatorname{Strat}(Z_K^\rho)$. □

The next proposition is the analogue of Theorem 3.4.5 for stratified bundles.

Proposition 4.2.4. *Given X a projective semi-stable curve over S with smooth generic fibre and $(K^n, \rho) \in \operatorname{Rep}_K^{\mathrm{cts}}(\pi_1^{\text{proét}}(X_0))$, let $\{\mathcal{O}^n_{\mathcal{Y}_\rho^{(i)}}, h_w^{\rho,i}\}_{i\geq 0}$ be the stratified bundle on with meromorphic descent data $\mathcal{Y}_\rho^{(i)}$ induced by ρ, then $\{\mathcal{O}^n_{\mathcal{Y}_\rho^{(i)}}, h_w^{\rho,i}\}$ descends to a stratified bundle on X_K.*

Proof. By Proposition 4.2.3, the stratified bundle with meromorphic descent data $\{\mathcal{O}^n_{\mathcal{Y}_\rho^{(i)}}, h_w^{\rho,i}\}_{i\geq 0}$ defined by ρ descends to a stratified bundle $\mathcal{G}_{\rho,K}^i$ on ${Z_K^\rho}^{(i)}$. Applying Theorem 3.4.5 to every Frobenius twists of X, we see that for every i the sheaf $\mathcal{G}_{\rho,K}^i$ further descends to a locally free sheaf $\mathcal{F}_{\rho,K}^i$ on $X_K^{(i)}$. Moreover, as in Remark 3.4.7, we can see that, if $\mathcal{F}_\rho^i$ is the meromorphic bundle on $\widehat{X^{(i)}}$ corresponding to $\mathcal{F}_{\rho,K}^i$ on $X_K^{(i)}$ via the equivalence in Proposition 3.3.7, then the meromorphic bundle $\{\mathcal{O}^n_{\mathcal{Y}^{(i)}_\rho}, h^{\rho,i}_w\}_{w\in \mathrm{Aut}(\mathcal{Y}_\rho|\widehat{X})}$ with meromorphic descent data induced by ρ descends to $\mathcal{F}^i_\rho$ for every $i \geq 0$.

To prove that $\{\mathcal{F}^i_{\rho,K}\}$ is a stratified bundle on X_K it suffices to construct the isomorphisms

$$\sigma_i \colon {F^i_{X_K/K}}^\star \mathcal{F}^{i+1}_{\rho,K} \to \mathcal{F}^i_{\rho,K}.$$

To simplify the notation, we write $F^i_{X_K} = F^i_{X_K/K}$, $F^i_{\widehat{X}} = F^i_{\widehat{X}/S}$ and also $F^i_{\mathcal{Y}} = F^i_{\mathcal{Y}_\rho/S}$. By functoriality of the Frobenius and Proposition 3.3.7, we have that

$$\mathrm{Hom}_{\mathcal{O}_{X_K}}({F^i_{X_K}}^\star \mathcal{F}^{i+1}_{\rho,K}, \mathcal{F}^i_{\rho,K}) \simeq \mathrm{Hom}_{\mathcal{O}_{\widehat{X}}}({F^i_{\widehat{X}}}^\star \mathcal{F}^{i+1}_{\rho}, \mathcal{F}^i_{\rho}) \otimes_A K.$$

Moreover, if $p^i_Y \colon \mathcal{Y}^{(i)}_\rho \to \widehat{X}^{(i)}$ is the structure map of $\mathcal{Y}^{(i)}_\rho$, since

$$F^i_{\widehat{X}} \circ p^i_Y = p^{i+1}_Y \circ F^i_{\mathcal{Y}}$$

and $\{\mathcal{O}^n_{\mathcal{Y}^{(i)}_\rho}, h^{\rho,i}_w\}_{w\in \mathrm{Aut}(\mathcal{Y}_\rho|\widehat{X})} \simeq {p^i_Y}^\star \mathcal{F}^i_\rho$, we get

$$\mathrm{Hom}_{\mathcal{O}_{X_K}}({F^i_{X_K}}^\star \mathcal{F}^{i+1}_{\rho,K}, \mathcal{F}^i_{\rho,K}) \simeq \mathrm{Hom}({F^i_{\mathcal{Y}}}^\star \{\mathcal{O}^n_{\mathcal{Y}^{(i+1)}_\rho}, h^{\rho,i+1}_w\}, \{\mathcal{O}^n_{\mathcal{Y}^{(i)}_\rho}, h^{\rho,i}_w\}).$$

Hence, the identities

$${F^i_{\mathcal{Y}_\rho/S}}^\star \{\mathcal{O}^n_{\mathcal{Y}^{(i+1)}_\rho}, h^{\rho,i+1}_w\} = \{\mathcal{O}^n_{\mathcal{Y}^{(i)}_\rho}, h^{\rho,i}_w\}$$

induce the isomorphisms σ_i that we wanted. □

Proposition 4.2.5. *Given X a projective semi-stable curve over S, the descent of stratified bundles with meromorphic descent data associated to continuous representations of $\pi_1^{\text{proét}}(X_0, \xi)$ induces a functor*

$$sp_K \colon \mathrm{Rep}^{\mathrm{cts}}_K(\pi_1^{\text{proét}}(X_0, \xi)) \to \mathrm{Strat}(X_K).$$

Proof. By Proposition 4.2.4, given $(K^n, \rho) \in \mathrm{Rep}^{\mathrm{cts}}_K(\pi_1^{\textit{proét}}(X_0))$, the stratified bundle with meromorphic descent data $\{\mathcal{O}^n_{\mathcal{Y}^{(i)}_\rho}, h^{\rho,i}_w\}_{i\geq 0}$ induced by ρ on $\mathcal{Y}_\rho$ descends to a stratified bundle $\{\mathcal{F}^i_\rho\}$ on X_K. Thus, we can define

$$\mathrm{sp}_K(K^n, \rho) := \{\mathcal{F}^i_\rho\}_i \in \mathrm{Strat}(X_K).$$

Let $\varphi \colon (K^n, \rho) \to (K^m, \tau)$ be a morphism of representations, then, using the same argument of Proposition 3.4.8 for every Frobenius twist of X, we can define the morphisms of sheaves $F_i(\varphi)$,

$$F_i(\varphi) \colon \{\mathcal{F}^i_\rho\} \to \{\mathcal{F}^i_\tau\}.$$

By construction, it is clear that the collection $\{F_i(\varphi)\}$ induces a morphism of stratified bundles from $\{\mathcal{F}^i_\rho\}$ to $\{\mathcal{F}^i_\tau\}$ and the statement follows. □

Lemma 4.2.6. *The functor sp_K constructed above is a tensor functor.*

Proof. Let $(K^n, \rho), (K^m, \tau) \in \mathrm{Rep}^{\mathrm{cts}}_K(\pi_1^{\text{proét}}(X_0, \xi))$ two continuous representation. As we saw in Section 3.2

- $\mathcal{Y}_\rho$ corresponds to a $\pi_1^{\text{proét}}(X_0, \xi)$-set of the form $\mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N$,

- $\mathcal{Y}_\tau$ corresponds to a $\pi_1^{\text{proét}}(X_0,\xi)$-set of the form $\mathbb{Z}^{\star r} \star H_1 \star \cdots \star H_N$.

As in Proposition 3.4.8, we define $\mathcal{Y}_{\rho,\tau}$ to be the geometric covering of $\widehat{X}$ corresponding to the $\pi_1^{\text{proét}}(X_0,\xi)$-set $\mathbb{Z}^{\star r} \star (G_1 \times H_1) \star \cdots \star (G_N \times H_N)$.

Moreover, we set

$$\rho'\colon \mathbb{Z}^{\star r} \star (G_1 \times H_1) \star \cdots \star (G_N \times H_N) \to \mathrm{GL}_n(K)$$

to be the unique group morphism such that

- $\rho'(w) = \rho(w)$ for every $w \in \mathbb{Z}^{\star r}$,
- $\rho'(g_i,h_i) = \rho(g_i)$ for every $(g_i,h_i) \in G_i \times H_i$ and every $i = 1,\dots,N$.

Similarly, we define τ'. Then we can consider

$$\rho' \otimes \tau'\colon \mathbb{Z}^{\star r} \star (G_1 \times H_1) \star \cdots \star (G_N \times H_N) \to \mathrm{GL}_{nm}(K)$$

and we can define the associated stratified bundle on $\mathcal{Y}_{\rho,\tau}$ with meromorphic descent data $\{\mathcal{O}^n_{\mathcal{Y}^{(i)}_{\rho,\tau}}, h_w^{\rho'\otimes\tau',i}\}$.

Let

$$p_\rho\colon \mathcal{Y}_{\rho,\tau} \to \mathcal{Y}_\rho \text{ and } p_\tau\colon \mathcal{Y}_{\rho,\tau} \to \mathcal{Y}_\tau,$$

then we have that

$$p_\rho^\star\{\mathcal{O}^n_{\mathcal{Y}^{(i)}_\rho}, h_w^{\rho,i}\} = \{\mathcal{O}^n_{\mathcal{Y}^{(i)}_{\rho,\tau}}, h_w^{\rho',i}\} \text{ and } p_\tau^\star\{\mathcal{O}^m_{\mathcal{Y}^{(i)}_\tau}, h_w^{\tau,i}\} = \{\mathcal{O}^m_{\mathcal{Y}^{(i)}_{\rho,\tau}}, h_w^{\tau',i}\}.$$

Hence, we can define

$$\{\mathcal{O}^n_{\mathcal{Y}^{(i)}_{\rho,\tau}}, h_w^{\rho',i}\} \otimes \{\mathcal{O}^m_{\mathcal{Y}^{(i)}_{\rho,\tau}}, h_w^{\tau',i}\} := \{\mathcal{O}^{nm}_{\mathcal{Y}^{(i)}_{\rho,\tau}}, h_w^{\rho',i} \otimes h_w^{\tau',i}\}.$$

We immediately find that

$$\{\mathcal{O}^n_{\mathcal{Y}^{(i)}_{\rho,\tau}}, h_w^{\rho'}\} \otimes \{\mathcal{O}^m_{\mathcal{Y}^{(i)}_{\rho,\tau}}, h_w^{\tau'}\} \simeq \{\mathcal{O}^{nm}_{\mathcal{Y}^{(i)}_{\rho\otimes\tau}}, h_w^{\rho'\otimes\tau'}\}.$$

Furthermore, by construction, we notice that $\{\mathcal{O}^n_{\mathcal{Y}^{(i)}_{\rho\otimes\tau}}, h_w^{\rho'}\}$ descends to the stratified bundle $\{\mathcal{F}^i_\rho\} = \mathrm{sp}_K(\rho)$, $\{\mathcal{O}^m_{\mathcal{Y}^{(i)}_{\rho,\tau}}, h_w^{\tau'}\}$ descends to the stratified bundle $\{\mathcal{F}^i_\tau\} = \mathrm{sp}_K(\tau)$ and also that $\{\mathcal{O}^{nm}_{\mathcal{Y}^{(i)}_{\rho\otimes\tau}}, h_w^{\rho'\otimes\tau'}\}$ descends to the stratified bundle $\{\mathcal{F}^i_{\rho\otimes\tau}\} = \mathrm{sp}_K(\rho\otimes\tau)$. Thus, by Theorem 3.3.8 and Theorem 3.4.5 it follows that

$$\mathrm{sp}_K(\rho) \otimes \mathrm{sp}_K(\tau) \simeq \mathrm{sp}_K(\rho\otimes\tau).$$

All the properties of tensor functor can be easily checked in a similar way. □

Theorem 4.2.7. *Given X a projective semi-stable curve over S with smooth generic fibre, the functor sp_K can be extended to a tensor functor*

$$sp_L\colon \mathrm{Rep}_L^{\mathrm{cts}}(\pi_1^{\text{proét}}(X_0,\xi)) \to \mathrm{Strat}(X_L),$$

where $X_L = X_K \times_K \mathrm{Spec}(L)$, for every finite extension L of K.

Moreover, fixing $x \in X_{\overline{K}}(\overline{K})$, it induces a morphism of group schemes

$$sp\colon \pi^{\mathrm{strat}}(X_{\overline{K}}) \to (\pi_1^{\text{proét}}(X_0,\xi))^{\mathrm{cts}}.$$

Proof. Let $(\overline{K}^n, \rho) \in \mathrm{Rep}_{\overline{K}}(\pi_1^{\text{proét}}(X_0))$, then there exists a finite field extension $K \subset L$ and $(L^n, \rho_L) \in \mathrm{Rep}_L(\pi_1^{\text{proét}}(X_0))$ such that

$$(L^n, \rho_L) \otimes_L \overline{K} = (\overline{K}^n, \rho).$$

We say that ρ is defined over L and we set

- A_L to be the integral closure of A in L,
- $S_L = \mathrm{Spec}(A_L)$,
- $X_{S_L} = X \times_S S_L$.

By definition, A_L is a complete discrete valuation ring, whose residue field is k and whose fraction field is L. By base change, X_{S_L} is proper over S_L. Moreover, we have that

$$(X_{S_L})_0 = X_{S_L} \times_{S_L} \mathrm{Spec}(k) = X \times_S \mathrm{Spec}(k) = X_0,$$

$$(X_{S_L})_{\overline{K}} = X_{S_L} \times_{S_L} \mathrm{Spec}(\overline{K}) = X \times_S \mathrm{Spec}(\overline{K}) = X_{\overline{K}}.$$

Hence, X_{S_L} is a projective semi-stable curve with smooth generic fibre.

We can apply Proposition 4.2.5 to X_{S_L} and we can define a functor sp_L that associates to (L^n, ρ_L) a stratified bundle on X_L. Let $\mathrm{bs}_L \colon X_{\overline{K}} \to X_L$ be the base change, we set

$$\mathrm{sp}(\overline{K}^n, \rho) := \mathrm{bs}_L^*(\mathrm{sp}_L(L^n, \rho_L)) \in \mathrm{Strat}(X_{\overline{K}}).$$

For every morphism $\varphi \colon (\overline{K}^n, \rho) \to (\overline{K}^m, \tau)$ there exists a finite extension L of K such that ρ and τ are defined over L and there exists $\varphi_L \colon (L^n, \rho) \to (L^m, \tau)$ such that $\varphi_L \otimes_L \overline{K} = \varphi$, then $\mathrm{sp}_L(\varphi_L)$ is a well defined morphism of stratified bundles on X_L and we can set

$$\mathrm{sp}(\varphi) := \mathrm{bs}_L^*(\mathrm{sp}_L(\varphi_L)).$$

Since, by Proposition 4.2.6 applied to X_{S_L}, sp_L is a tensor functor, for every finite extension $K \subset L$, then also sp is a tensor functor.

Let x be the fixed point of $X_{\overline{x}}$, ω_x be the associated fibre functor of $\mathrm{Strat}(X_{\overline{K}}$ and ω_π the fiber functor of $\mathrm{Rep}^{\mathrm{cts}}_{\overline{K}}(\pi_1^{\textit{proét}}(X_0, \xi))$ given by the forgetful functor. We claim that $\omega_x \circ \mathrm{sp} \simeq \omega_\pi$.

Let $(L^n, \rho_L) \in \mathrm{Rep}_L(\pi_1^{\text{proét}}(X_0))$ for some finite extension L of K such that $(L^n, \rho_L) \otimes_L \overline{K} = (\overline{K}^n, \rho)$, then choosing a trivialization of $\mathrm{sp}(L^n, \rho_L))$ on a neighborhood of x induces an isomorphism

$$\omega_x(\mathrm{sp}(L^n, \rho_L)) \simeq \overline{K}^n = \omega_\pi(\overline{K}^n, \rho).$$

Given $\varphi_L \colon (L^n, \rho) \to (L^m, \tau)$ a morphism of representations, we denote by $\mathcal{F}^0_\rho$ and $\mathcal{G}^0_\tau$ the meromorphic bundles on X_{S_L} corresponding, via Theorem 3.3.8, to the first term of $\mathrm{sp}_L(L^n, \rho_L)$ and $\mathrm{sp}_L(L^m, \tau_L)$ respectively. We set $\{U_i\}$ to be a covering of X_{S_L} on which $\mathcal{F}_i$ and $\mathcal{G}_i$ are trivialized and we fix two trivializations $s_0 : \mathcal{F}_0|_{U_{j_0}} \simeq \mathcal{O}^n_{U_{j_0}}$, $t_0 : \mathcal{G}_0|_{U_{j_0}} \simeq \mathcal{O}^m_{U_{j_0}}$ on an affine open U_{j_0} that contains the point $x \in X_{S_L}(\overline{K})$. We notice that

$$\bigcup_{\{U_j \mid U_j \cap X_0 \neq \emptyset\}} U_j = X_{S_L},$$

because if it was a strict inclusion, then its complement in X_{S_L} would have non-trivial intersection with X_0, which is a contradiction. Therefore, we can assume that $U_{j_0} \cap X_0 \neq \emptyset$ and we can define $\widehat{U_{j_0}}$ to be the formal completion of U_{j_0} along $U_{j_0} \cap X_0$. By [EGA III, Cor. 5.1.3], the pullback along $\widehat{U_{j_0}} \to U_{j_0}$ is a fully faithful functor, hence the diagram

$$\begin{array}{ccc} \mathcal{F}^0_\rho|_{U_{j_0}} \otimes_A L & \xrightarrow{s_{j_0}} & \mathcal{O}_{U_{j_0}} \otimes_A L^n \\ \downarrow{\scriptstyle \mathrm{sp}_L(\varphi_L)|_{U_{j_0}}} & & \downarrow{\scriptstyle \mathrm{id}\otimes\varphi_L} \\ \mathcal{G}^0_\tau|_{U_{j_0}} \otimes_A L & \xrightarrow{t_{j_0}} & \mathcal{O}_{U_{j_0}} \otimes_A L^m \end{array}$$

commutes on X_{S_L} if and only if the corresponding diagram on $\widehat{X_{S_L}}$ commutes. Furthermore, by fpqc descent, the latter commutes if and only if its pullback on $\mathcal{Y}_{\rho,\tau} \times \widehat{U_{j_0}}$ commutes, but this follows by construction because the corresponding diagram on $\mathcal{Y}_{\rho,\tau} \times \widehat{U_{j_0}}$ is

$$\begin{array}{ccc} \mathcal{O}^n_{\mathcal{Y}_{\rho,\tau}\times\widehat{U_{j_0}}} \otimes_A L & \xrightarrow{\mathrm{Id}} & \mathcal{O}^n_{\mathcal{Y}_{\rho,\tau}\times\widehat{U_{j_0}}} \otimes_A L \\ \downarrow{\scriptstyle \mathrm{id}\otimes\varphi_L} & & \downarrow{\scriptstyle \mathrm{id}\otimes\varphi_L} \\ \mathcal{O}^m_{\mathcal{Y}_{\rho,\tau}\times\widehat{U_{j_0}}} \otimes_A L & \xrightarrow{\mathrm{Id}} & \mathcal{O}^m_{\mathcal{Y}_{\rho,\tau}\times\widehat{U_{j_0}}} \otimes_A L \end{array} .$$

Thus, we can conclude that the following diagram commutes

$$\begin{array}{ccc} x^\star(\mathcal{F}^0_\rho \otimes_A L) & \xrightarrow{x^\star s_0} & \overline{K}^n \\ {\scriptstyle x^\star \mathrm{sp}_L(\varphi_L)}\downarrow & & \downarrow{\scriptstyle \varphi_L\otimes_K \overline{K}} \\ x^\star(\mathcal{G}^0_\tau \otimes_A L) & \xrightarrow{x^\star t_0} & \overline{K}^m \end{array} ,$$

which implies that

$$\omega_x \simeq \omega_\pi(\varphi).$$

Let γ be a natural transformation between $\omega_x \circ \mathrm{sp}$ and ω_π constructed as above, then we set

$$\omega'_\pi(\overline{K}^n, \rho) = \gamma(\omega_\pi(\overline{K}^n, \rho)).$$

Hence, by [DM82, Cor. 2.9], the functor sp corresponds to a morphism of group schemes

$$\mathrm{sp}\colon \pi^{\mathrm{strat}}(X_{\overline{K}}) \to \pi(\mathrm{Rep}^{\mathrm{cts}}_{\overline{K}}(\pi_1\mathrm{pro\acute{e}t}(X_0)), \omega'_\pi).$$

Moreover, ω_π and ω'_π are naturally isomorphic, so we have that

$$\pi(\mathrm{Rep}^{\mathrm{cts}}_{\overline{K}}(\pi_1\mathrm{pro\acute{e}t}(X_0)), \omega'_\pi) \simeq (\pi_1^{\mathrm{pro\acute{e}t}}(X_0))^{\mathrm{cts}}$$

and, composing with this isomorphism, we get a morphism of group schemes

$$\mathrm{sp}\colon \pi^{\mathrm{strat}}(X_{\overline{K}}) \to (\pi_1^{\mathrm{pro\acute{e}t}}(X_0))^{\mathrm{cts}}.$$

□

Corollary 4.2.8. *For $j = 1, \dots, N$, let C_j be the irreducible components of X_0 and $\overline{C_j}$ their normalizations, then the functor sp induces a morphism of group schemes*

$$sp\colon \pi^{\mathrm{strat}}(X_{\overline{K}}) \to (\mathbb{Z}^{\mathrm{alg}})^{\star_{\overline{K}}|I|-N+1} \star_{\overline{K}} (\pi_1^{\text{ét}}(\overline{C_1})_{\overline{K}}) \star_{\overline{K}} \cdots \star_{\overline{K}} (\pi_1^{\text{ét}}(\overline{C_N})_{\overline{K}}),$$

where if I is the set of singular points of X, $(\mathbb{Z}^{\mathrm{alg}})^{\star_{\overline{K}} r}$ is the free algebraic product of r copies of $\mathbb{Z}^{\mathrm{alg}}$ and $\pi_1^{\text{ét}}(\overline{C_j})_{\overline{K}}$ is defined as in Remark 2.3.6.

Proof. By Corollary 3.2.2,

$$(\pi_1^{\text{proét}}(X_0, \xi))^{\mathrm{cts}} \simeq (\mathbb{Z}^{\star r} \star \pi_1^{\text{ét}}(\overline{C_1}) \star \cdots \star \pi_1^{\text{ét}}(\overline{C_N}))^{\mathrm{cts}},$$

thus by Lemma 2.4.6 we have that

$$(\pi_1^{\text{proét}}(X_0, \xi))^{\mathrm{cts}} \simeq (\mathbb{Z}^{\mathrm{cts}})^{\star_{\overline{K}} r} \star_{\overline{K}} \pi_1^{\text{ét}}(\overline{C_1})^{\mathrm{cts}} \star_{\overline{K}} \cdots \star_{\overline{K}} \pi_1^{\text{ét}}(\overline{C_N})^{\mathrm{cts}}.$$

Moreover, since $\mathbb{Z}$ is discrete, by Remark 2.3.3 we have that

$$\mathbb{Z}^{\mathrm{cts}} \simeq \mathbb{Z}^{\mathrm{alg}}$$

and by Lemma 2.3.5 we have that, for every j,

$$\pi_1^{\text{ét}}(\overline{C_j})^{\mathrm{cts}} \simeq \pi_1^{\text{ét}}(\overline{C_j})_{\overline{K}},$$

where $\pi_1^{\text{ét}}(\overline{C_j})_{\overline{K}}$ is defined as in Remark 2.3.6. ☐

4.3 Compatibility with the étale specialization map

In [SGA 1] Grothendieck constructed a specialization map for the étale fundamental group. In this section we compare the specialization functor that we defined with Grothendieck's construction.

Lemma 4.3.1. *Let X_0 be a connected noetherian scheme over k and ξ a geometric point of X_0, then the category $\mathrm{Rep}^{\mathrm{cts}}_{\overline{K}}(\pi_1^{\text{ét}}(X_0, \xi))$ is equivalent to a full sub-category of $\mathrm{Rep}^{\mathrm{cts}}_{\overline{K}}(\pi_1^{\text{proét}}(X_0, \xi))$.*

Proof. By Proposition 1.2.19, the pro-finite completion of $\pi_1^{\text{proét}}(X_0, \xi)$ is isomorphic to the étale fundamental group $\pi_1^{\text{ét}}(X_0, \xi)$. Hence, there exists a dense continuous group morphism

$$c\colon \pi_1^{\text{proét}}(X_0, \xi) \to \pi_1^{\text{ét}}(X_0, \xi).$$

This morphism induces a faithful functor

$$\mathrm{Rep}^{\mathrm{cts}}_{\overline{K}}(\pi_1^{\text{ét}}(X_0, \xi)) \to \mathrm{Rep}^{\mathrm{cts}}_{\overline{K}}(\pi_1^{\text{proét}}(X_0, \xi)).$$

Since the image of c is dense, this functor is also full. ☐

Given $x \in X_{\overline{K}}(\overline{K})$, we denote by sp_{SGA1} the specialization map constructed by Groethendieck in [SGA 1]

$$\mathrm{sp}_{SGA1} \colon \pi_1^{\text{ét}}(X_{\overline{K}}, x) \to \pi_1^{\text{ét}}(X_0, \xi).$$

By [SGA 1, Exp. X, Cor. 2.4], sp_{SGA1} is surjective. Hence, it induces a fully faithful functor

$$\mathrm{sp}_{SGA1} \colon \mathrm{Rep}^{\mathrm{cts}}_{\overline{K}}(\pi_1^{\text{ét}}(X_0, \xi)) \to \mathrm{Rep}^{\mathrm{cts}}_{\overline{K}}(\pi_1^{\text{ét}}(X_{\overline{K}}, x)).$$

We construct now a functor

$$F \colon \mathrm{Rep}^{\mathrm{cts}}_{\overline{K}}(\pi_1^{\text{ét}}(X_{\overline{K}}, x)) \to \mathrm{Strat}(X_{\overline{K}}).$$

Let $(\overline{K}^n, \rho) \in \mathrm{Rep}^{\mathrm{cts}}_{\overline{K}}(\pi_1^{\text{ét}}(X_{\overline{K}}, x))$, by continuity and Lemma 2.3.4, ρ factors through a finite quotient of $\pi_1^{\text{ét}}(X_{\overline{K}}, x)$ that we call G_ρ.

$$\begin{array}{ccc} & \pi_1^{\text{ét}}(X_{\overline{K}}, x) & \\ \swarrow & & \searrow^{\rho} \\ G_\rho & \xrightarrow{\bar{\rho}} & \mathrm{GL}_n(\overline{K}) \end{array}$$

In particular, there exists a finite Galois cover $W_{\overline{K}}$ of $X_{\overline{K}}$ such that

$$\mathrm{Aut}(W_{\overline{K}} | X_{\overline{K}}) = G_\rho^{\mathrm{op}}.$$

We can define descend data $\{h_g^\rho\}_{g \in G_\rho}$ for the sheaf $\mathcal{O}^n_{W_{\overline{K}}}$ on $W_{\overline{K}}$ as follows

$$\begin{array}{ccc} \mathcal{O}^n_{W_{\overline{K}}} & \xrightarrow{h_g^\rho} & \mathcal{O}^n_{W_{\overline{K}}} \\ (f_i) & \longrightarrow & \rho(g)(f_i)\ . \end{array}$$

Since $W_{\overline{K}} \to X_{\overline{K}}$ is a morphism of effective descent for coherent sheaves, $\{\mathcal{O}^n_{W_{\overline{K}}}, h_g^\rho\}$ descends to a coherent sheaf $\mathcal{E}$ on $X_{\overline{K}}$ that, by construction, is locally free. As in the proof of Proposition 4.2.5, if we repeat the argument for the Frobenius twists of $X_{\overline{K}}$ we can associate to ρ a stratified bundle.

It remains only to define the functor on the morphisms and this is done as in Proposition 3.4.8 and Proposition 4.2.5.

In the end, we get the following diagram

$$\begin{array}{ccc} \mathrm{Rep}^{\mathrm{cts}}_{\overline{K}}(\pi_1^{\text{ét}}(X_0, \xi)) & \xrightarrow{\mathrm{sp}_{SGA1}} & \mathrm{Rep}^{\mathrm{cts}}_{\overline{K}}(\pi_1^{\text{ét}}(X_{\overline{K}}, x)) \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle F} \\ \mathrm{Rep}^{\mathrm{cts}}_{\overline{K}}(\pi_1^{\text{proét}}(X_0, \xi)) & \xrightarrow{\mathrm{sp}} & \mathrm{Strat}(X_{\overline{K}})\ . \end{array} \tag{4.3.2}$$

Proposition 4.3.3. *If X is a projective semi-stable curve over S with smooth generic fibre, the diagram* (4.3.2) *is commutative up to a natural transformation.*

Proof. Let $(\overline{K}^n, \rho) \in \mathrm{Rep}^{\mathrm{cts}}_{\overline{K}}(\pi_1^{\text{ét}}(X_0, \xi))$, then there exists a finite field extension L of K and $(L^n, \rho_L) \in \mathrm{Rep}^{\mathrm{cts}}_L(\pi_1^{\text{ét}}(X_0, \xi))$ such that

$$(\overline{K}^n, \rho) = (L^n, \rho_L) \otimes_K \overline{K}.$$

For simplicity we call ρ also the representation with coefficients in L.

By continuity and Lemma 2.3.4, the morphism ρ factors through a representation $\bar{\rho}$ of a finite quotient G_ρ of $\pi_1^{\text{ét}}(X_0, \xi)$. Hence, we have the following commutative diagram

$$\begin{tikzcd} \pi_1^{\text{proét}}(X_0,\xi) \arrow[r, "c"] \arrow[d, "p"] & \pi_1^{\text{ét}}(X_0,\xi) \arrow[dl] \arrow[d, "\rho"] \\ G_\rho \arrow[r, "\bar{\rho}"] & \mathrm{GL}_n(L) \end{tikzcd},$$

where c is the morphism induced by the pro-finite completion.

Since G_ρ is endowed with the discrete topology, the continuous morphism p factors through the quotient $\mathbb{Z}^{\star r} \star H_1 \star \cdots \star H_N$, where

$$H_j = \pi_1^{\text{ét}}(\overline{C_j})/p_j^{-1}(1_{G_\rho}),$$

and $p_j = p \circ j$, with j is the natural morphism

$$j\colon \pi_1^{\text{ét}}(\overline{C_j}) \to \pi_1^{\text{proét}}(X_0, \xi).$$

We recall that in Section 3.2, given $\rho \circ c \in \mathrm{Rep}_{\overline{K}}^{\text{cts}}(\pi_1^{\text{proét}}(X_0,\xi))$ we defined

$$G_j = \pi_1^{\text{ét}}(\overline{C_j})/(\rho \circ c \circ j)^{-1}(\mathrm{Id}).$$

By commutativity of the diagram, $\rho \circ c \circ j = \bar{\rho} \circ p \circ j$. Since, by construction, $\bar{\rho}$ is injective, this implies that $H_j = G_j$. Thus, there exists a $\pi_1^{\text{proét}}(X_0,\xi)$-equivariant morphism

$$q\colon \mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N \to G_\rho,$$

such that the following diagram is commutative

$$\begin{tikzcd} & \pi_1^{\text{proét}}(X_0,\xi) \arrow[r, "c"] \arrow[dl] \arrow[d, "p"] & \pi_1^{\text{ét}}(X_0,\xi) \arrow[dl] \arrow[d, "\rho"] \\ \mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N \arrow[r, "q"] & G_\rho \arrow[r, "\bar{\rho}"] & \mathrm{GL}_n(L) \end{tikzcd}.$$

Let X_{S_L} be defined as in the proof of Theorem 4.2.7 and let $\widehat{X}_{S_L}$ be the completion of X_{S_L} along X_0. Then the $\pi_1^{\text{proét}}(X_0,\xi)$-sets $\mathbb{Z}^{\star r} \star G_1 \star \cdots \star G_N$ and G_ρ correspond, via the equivalence in Corollary 1.2.16, to two geometric coverings of $\widehat{X}_{S_L}$ that we call $\mathcal{Y}$ and $\mathcal{W}$ and q corresponds to a $\widehat{X}_{S_L}$-morphism

$$\begin{tikzcd} & \mathcal{W} \arrow[dr, "p_{\mathcal{W}}"] & \\ \mathcal{Y} \arrow[ur, "q"] \arrow[rr, "p_{\mathcal{Y}}"] & & \widehat{X}_{S_L} \end{tikzcd}$$

By construction, $\mathrm{sp}(L^n, \rho\circ c)$ corresponds to a sequence $\{\mathcal{F}^i_\rho\}$ of meromorphic bundles on $\widehat{X}_{S_L}$ such that

$$p^\star_{\mathcal{Y}}\{\mathcal{F}^i_\rho\} \simeq \{\mathcal{O}^n_{\mathcal{Y}^{(i)}}, h^{\bar\rho\circ q,i}_w\}_{w\in(\mathbb{Z}^{\star r}\star G_1\star\cdots\star G_N)^{\mathrm{op}}}.$$

We observe that, if W is the finite étale covering of X_{S_L} corresponding to $\mathcal{W}$ and $W_{\overline{K}}$ is its geometric generic fibre, then, using [LM99, Lem. 4.11] as in the proof of Theorem 3.4.5, we deduce that

$$\mathrm{Aut}(W_{\overline{K}}|X_{\overline{K}}) \simeq G^{\mathrm{op}}_\rho.$$

By construction of the functor F, if $p_{W_{\overline{K}}}\colon W_{\overline{K}} \to X_{\overline{K}}$ is the structure map of $W_{\overline{K}}$, then

$$p^\star_{W_{\overline{K}}} F(\mathrm{sp}_{SGA1}(L^n,\rho)) \simeq \{\mathcal{O}^n_{W^{(i)}_{\overline{K}}}, h^{\bar\rho,i}_g\}_{g\in G^{\mathrm{op}}_\rho}.$$

Hence, by Proposition 3.3.7, $F(\mathrm{sp}_{SGA1}(L^n,\rho))$ corresponds to a sequence of meromorphic bundles $\{\mathcal{G}^\rho_i\}$ on $\widehat{X}_{S_L}$ such that

$$p^\star_{\mathcal{W}}\mathcal{G}^\rho_i \simeq \{\mathcal{O}^n_{\mathcal{W}^{(i)}}, h^{\bar\rho,i}_g\}_{g\in G^{\mathrm{op}}_\rho}.$$

We claim that $\{\mathcal{O}^n_{\mathcal{Y}^{(i)}}, h^{\bar\rho\circ q,i}_w\}$ descends to $\{\mathcal{O}^n_{\mathcal{W}^{(i)}}, h^{\bar\rho,i}_g\}$ on $\mathcal{W}$.

By construction we have $\mathrm{Aut}(\mathcal{Y}|\mathcal{W}) = \ker(q)^{\mathrm{op}}$, so, if $w\in\ker(q)^{\mathrm{op}}$, then clearly $\bar\rho\circ q$ is trivial on w. Hence

$$\{\mathcal{O}^n_{\mathcal{Y}^{(i)}}, h^{\bar\rho\circ q,i}_w\}_{w\in\ker(q)^{\mathrm{op}}} = \{\mathcal{O}^n_{\mathcal{Y}^{(i)}}, \mathrm{Id}\}_{w\in\ker(q)^{\mathrm{op}}},$$

which implies that $\{\mathcal{O}^n_{\mathcal{Y}^{(i)}}, h^{\rho,i}_w\}_{w\in\ker(q)^{\mathrm{op}}}$ descends to the trivial stratified bundle $\{\mathcal{O}^n_{\mathcal{W}^{(i)}}\}$. Clearly, we can identify

$$\mathrm{Hom}(\{\mathcal{O}^n_{\mathcal{Y}^{(i)}},\mathrm{Id}\},\{\mathcal{O}^n_{\mathcal{Y}^{(i)}},\mathrm{Id}\})\otimes L = \mathrm{Hom}(\{\mathcal{O}^n_{\mathcal{W}^{(i)}}\},\{\mathcal{O}^n_{\mathcal{W}^{(i)}}\})\otimes L.$$

Then we see that for every $w,w'\in(\mathbb{Z}^{\star r}\star G_1\star\cdots\star G_N)^{\mathrm{op}}$ such that $q(w)=g$, $h^{q\circ\bar\rho,i}_w = h^{q\circ\bar\rho,i}_{w'}$ and it corresponds, via the identification above, to $h^{\bar\rho,i}_g$.

This implies that

$$q^\star\{\mathcal{O}^n_{\mathcal{W}^{(i)}}, h^{\bar\rho,i}_g\}_{g\in G^{\mathrm{op}}_\rho} = \{\mathcal{O}^n_{\mathcal{Y}^{(i)}}, h^{\bar\rho\circ q,i}_w\}_{w\in\mathbb{Z}^{\star r}\star G_1\star\cdots\star G_N}.$$

Hence, we find that there exists an isomorphism of stratified bundles with meromorphic descent data on $\mathcal{Y}$

$$\eta_\rho\colon \mathrm{sp}(c(L^n,\rho)) \simeq F(\mathrm{sp}_{SGA1}(L^n,\rho)).$$

It remains to show that the isomorphism η_ρ is functorial.

Given $\varphi\colon(\overline{K}^n,\rho)\to(\overline{K}^m,\tau)$ a morphism of representations, there exists a finite extension L of K, $(L^n,\rho),(\overline{K}^m,\tau)\in\mathrm{Rep}^{\mathrm{cts}}_L(\pi^{\text{ét}}_1(X_0,\xi))$ and a morphism $\varphi_L\colon(L^n,\rho)\to(L^m,\tau)$, such that $\varphi=\varphi_L\otimes_K\overline{K}$. As we did before, we drop the index L and we call both maps φ.

Let G_ρ and G_τ be the quotients of $\pi^{\text{ét}}_1(X_0,\xi)$ over which ρ and τ are defined and set $\mathcal{W}_\rho$ and $\mathcal{W}_\tau$ the corresponding geometric coverings of $\widehat{X}_{S_L}$. Then we set $\mathcal{W}_{\rho,\tau} := \mathcal{W}_\rho\times_{\widehat{X}_{S_L}}\mathcal{W}_\tau$, which corresponds to the $\pi^{\text{ét}}_1(X_0,\xi)$-set $G_\rho\times G_\tau$.

By Proposition 3.3.7 and Theorem 3.4.5, the diagram

$$\begin{array}{ccc} \mathrm{sp}(c(L^n,\rho)) & \xrightarrow{\eta_\rho} & F(\mathrm{sp}_{SGA1}(L^n,\rho)) \\ \Big\downarrow{\scriptstyle \mathrm{sp}(c(\varphi))} & & \Big\downarrow{\scriptstyle F(\mathrm{sp}_{SGA1}(\varphi))} \\ \mathrm{sp}(c(L^m,\tau)) & \xrightarrow{\eta_\tau} & F(\mathrm{sp}_{SGA1}(L^m,\tau)) \end{array}$$

commutes if the corresponding diagram on $\widehat{X}_{S_L}$ pulled back on $\mathcal{W}_{\rho,\tau}$ commutes, that is, if the following diagram commutes

$$\begin{array}{ccc} \{\mathcal{O}^n_{\mathcal{W}_{\rho,\tau}}, h_g^{\bar\rho\times\mathrm{Id},i}\} & \xrightarrow{\mathrm{Id}} & \{\mathcal{O}^n_{\mathcal{W}_{\rho,\tau}}, h_g^{\bar\rho\times\mathrm{Id},i}\} \\ \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle F\varphi} \\ \{\mathcal{O}^n_{\mathcal{W}_{\rho,\tau}}, h_g^{\bar\tau\times\mathrm{Id},i}\} & \xrightarrow{\mathrm{Id}} & \{\mathcal{O}^n_{\mathcal{W}_{\rho,\tau}}, h_g^{\bar\tau\times\mathrm{Id},i}\}\ . \end{array} \tag{4.3.4}$$

Since the diagram (4.3.4) clearly commutes, we conclude that the isomorphism is functorial and hence it induces a natural transformation. □

Lemma 4.3.5. *If X is a projective semi-stable curve over S with smooth generic fibre, the diagram* (4.3.2) *induces the following diagram of group schemes*

$$\begin{array}{ccc} \pi^{\mathrm{strat}}(X_{\overline{K}}) & \xrightarrow{sp} & \pi_1^{\mathrm{pro\acute{e}t}}(X_0,\xi)^{\mathrm{cts}} \\ \Big\downarrow{\scriptstyle F} & & \Big\downarrow{\scriptstyle c} \\ \pi_1^{\mathrm{\acute{e}t}}(X_{\overline{K}},x)_{\overline{K}} & \xrightarrow{sp_{SGA1}} & \pi_1^{\mathrm{\acute{e}t}}(X_0,\xi)_{\overline{K}}, \end{array} \tag{4.3.6}$$

where $\pi_1^{\acute{e}t}(X_{\overline{K}},x)_{\overline{K}}$ and $\pi_1^{\acute{e}t}(X_0,\xi)_{\overline{K}}$ are defined as in Remark 2.3.6.

Proof. We proved in Remark 4.2.7 that sp induces a morphism between the corresponding group schemes. With the analogous argument on F, one proves that also the functor F induces a morphism of group schemes between $\pi^{\mathrm{strat}}(X_{\overline{K}})$ and $\pi_1^{\acute{e}t}(X_{\overline{K}},x)^{\mathrm{cts}}$.

Clearly sp_{SGA1} and c are tensor functors and they commute with the forgetful functor, hence they induce a morphism between the corresponding group schemes.

Moreover, by Lemma 2.3.5, we have that

$$\pi_1^{\mathrm{\acute{e}t}}(X_{\overline{K}},x)^{\mathrm{cts}} = \pi_1^{\mathrm{\acute{e}t}}(X_{\overline{K}},x)_{\overline{K}} \text{ and } \pi_1^{\mathrm{\acute{e}t}}(X_0,\xi)^{\mathrm{cts}} = \pi_1^{\mathrm{\acute{e}t}}(X_0,\xi)_{\overline{K}}$$

in the sense of Remark 2.3.6. □

Proposition 4.3.7. *If X is a projective semi-stable curve over S with smooth generic fibre, the functor F of the diagram 4.3.2 is fully faithful and its essential image consists of stratified bundles with finite monodromy.*

Proof. Clearly the functor F is faithful, we prove now that it is also full.

Let $(\overline{K}^n,\rho),(\overline{K}^m,\tau)\in\mathrm{Rep}^{\mathrm{cts}}_{\overline{K}}(\pi_1^{\mathrm{\acute{e}t}}(X_{\overline{K}},x))$, then ρ factors through the finite quotient G_ρ, while τ factors through G_τ. Let W_ρ and W_τ be the corresponding finite étale coverings of $X_{\overline{K}}$ and $W_{\rho,\tau} := W_\rho\times_{X_{\overline{K}}} W_\tau$. Let $\varphi=\{\varphi_i\}_{i\geq 0}$ be a morphism of stratified bundles from $F(\overline{K}^n,\rho)$ to $F(\overline{K}^m,\tau)$. By construction and finite étale descent, we get

$$\mathrm{Hom}(F(\overline{K}^n,\rho),F(\overline{K}^m,\tau)) = \mathrm{Hom}(\{\mathcal{O}^n_{W^{(i)}_{\rho,\tau}}, h_g^{\rho,i}\}_{i\geq 0},\{\mathcal{O}^m_{W^{(i)}_{\rho,\tau}}, h_g^{\tau,i}\}_{i\geq 0}).$$

Moreover, since the pullbacks of $F(\overline{K}^n, \rho)$ and $F(\overline{K}^m, \tau)$ on $W_{\rho,\tau}$ are trivial stratified bundles and $\mathrm{Strat}(W_{\rho,\tau})$ is a neutral Tannakian category, we find that

$$\mathrm{Hom}(F(\overline{K}^n, \rho), F(\overline{K}^m, \tau)) \subset M_{mn}(\overline{K}).$$

Hence, the maps φ_i correspond to linear morphisms $\varphi_i \colon \overline{K}^n \to \overline{K}^m$. This implies that

$$F^i_{W^{(i)}_{\rho,\tau}|K}{}^{\star}\varphi_{i+1} = \varphi_i = \varphi_0 \in M_{mn}(\overline{K}).$$

To prove that F is full, it suffices to show that φ_0 is $\pi_1^{\text{ét}}(X_{\overline{K}}, x)$-equivariant. This follows by the fact that φ_0 commutes with the descent data, which are defined by ρ and τ.

It remains to analyse the essential image of F. By construction, $F(\overline{K}^n, \rho)$ is a stratified bundle on $X_{\overline{K}}$ trivialized on a finite étale cover of $X_{\overline{K}}$, then by Lemma 4.1.10, $F(\overline{K}^n, \rho)$ has finite monodromy. Thus, it suffices to show that all stratified bundles with finite monodromy are in the essential image.

Let $E = \{\mathcal{E}_i\} \in \mathrm{Strat}(X_{\overline{K}})$ with finite monodromy, then by Lemma 4.1.10, there exists a finite Galois cover $p\colon Y \to X_{\overline{K}}$ such that $p^\star E$ is trivial, i.e., $p^\star\{\mathcal{E}_i\} \simeq \{\mathcal{O}^n_{Y^{(i)}}\}$ for some n. We set $G = \mathrm{Aut}(Y|X_{\overline{K}})^{\mathrm{op}}$, which, by construction, is a finite quotient of the étale fundamental group $\pi_1^{\text{ét}}(X_{\overline{K}}, x)$. Then $\mathcal{O}^n_{Y^{(i)}}$ can be endowed with descent data $\{h^i_g\}_{g\in G^{\mathrm{op}}}$ such that $\{\mathcal{O}^n_{Y^{(i)}}, h^i_g\}$ descends to $\{\mathcal{E}_i\}$ on $X_{\overline{K}}$. Since $h^i_g \in M_{n^2}(\overline{K})$ by the above argument and they are isomorphisms, we can set for every $g \in G$

$$\rho(g) = h^0_{g^{-1}} \in \mathrm{GL}_n(\overline{K}).$$

As in Remark 3.2.9, we denote by $\circ$ the multiplication in G^{op} and we refer to the multiplication in G when we don't write a symbol. Then applying the co-cycle condition, we get

$$\rho(gg')(v) = h^0_{g'^{-1}g^{-1}}(v) = h^0_{g^{-1}\circ g'^{-1}}(v) = h^0_{g^{-1}} \circ h^0_{g'^{-1}}(v) = \rho(g)(\rho(g')(v)).$$

Hence, ρ defines a group morphism $\rho\colon G \to \mathrm{GL}_n(\overline{K})$.

By construction, it is clear that $F(\overline{K}^n, \rho) \simeq \{\mathcal{E}_i\}$, hence $\{\mathcal{E}_i\}$ is in the essential image of F. □

Remark 4.3.8. Note that in general a specialization morphism between the topological groups $\pi_1^{\text{proét}}(X_{\overline{K}}, x)$ and $\pi_1^{\text{proét}}(X_0, \xi)$ compatible with the étale specialization map does not exist.

Assume for example that X is a stable curve over S with smooth generic fibre and degenerate closed fibre. Since the generic fibre X_K is smooth, by [Liu02, Prop. 3.15] X is normal. Hence by Proposition 1.3.23 and Lemma 1.2.20,

$$\pi_1^{\text{proét}}(X_0, \xi) \simeq \mathbb{Z}^{\star r} \text{ and } \pi_1^{\text{proét}}(X_{\overline{K}}, x) \simeq \pi_1^{\text{ét}}(X_{\overline{K}}, x),$$

where $r = p_a(X_0)$ is the arithmetic genus of the closed fibre.

Any continuous morphism

$$\mathrm{sp}\colon \pi_1^{\text{ét}}(X_{\overline{K}}, x) \to \mathbb{Z}^{\star r}$$

factors through a finite quotient of $\pi_1^{\text{ét}}(X_{\overline{K}}, x)$, but since $\mathbb{Z}^{\star r}$ is a free group this implies that sp has to be the zero map.

Under the same assumptions, the étale specialization map is surjective. Hence, in particular, the following diagram does not commute

$$\begin{array}{ccc}
\pi_1^{\text{proét}}(X_{\overline{K}}, x) \simeq \pi_1^{\text{ét}}(X_{\overline{K}}, x) & \xrightarrow{\text{Id}} & \pi_1^{\text{ét}}(X_{\overline{K}}, x) \\
{\scriptstyle 0}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{sp}_{SGA1}} \\
\pi_1^{\text{proét}}(X_0, \xi) \simeq \mathbb{Z}^{\star r} & \xrightarrow{c} & \pi_1^{\text{ét}}(X_0, \xi) \ .
\end{array}$$

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- dass ich alle Hilfsmittel und Hilfen angegeben habe,
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Unterschrift: Elena Lavanda

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