The Cauchy Problem for Membranes

Dissertation

The Cauchy Problem for Membranes

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Zusammenfassung in deutscher Sprache

Diese Dissertation untersucht das Anfangswertproblem für Membranen. Eine Membran ist eine raumartige Untermannigfaltigkeit Σ_0 in einer umgebenden Mannigfaltigkeit ausgestattet mit einer Lorentz-Metrik. Die Bewegungsgleichung für eine Membran ist gegeben durch die folgende Bedingung für das Weltvolumen Σ der Membran, das eine zeitartige Untermannigfaltigkeit darstellt, welches durch Bewegung aus der Membran entsteht. Die Bewegung soll so erfolgen, dass das induzierte Volumen des Weltvolumen extremal wird. Die Euler-Lagrange Gleichung für das Volumenfunktional ergibt sich zu

$$H(\Sigma) \equiv 0,$$

wobei $H(\Sigma)$ als der mittlere Krümmungsvektor von Σ definiert ist. In der String-Theorie, ein Kanditat für die Vereinheitlichung der Allgemeinenen Relativitätstheorie und der Quantenfeldtheorie, werden solche Objekte p-Branen genannt. Diese sind höher-dimensionale Analoga zu strings mit den gleichen dynamischen Eigenschaften.
Wir betrachten folgendes Anfangswertproblem.

Sei eine raumartige Untermannigfaltigkeit Σ_0 gegeben. Sei ν ein zeitartiges zukunftsgerichtetes Einheitsvektorfeld auf Σ_0.
Existenz: Finde offene zeitartige Lösung Σ des Problems

$$H(\Sigma) \equiv 0,\ \Sigma_0 \subset \Sigma \text{ und } \nu \text{ ist tangential zu } \Sigma.$$

Eindeutigkeit: Seien Σ_1 und Σ_2 zwei Lösungen des Problems. Zeige, dass eine offene Umgebung $\Sigma_0 \subset U$ von Σ_0 existiert mit

$$U \cap \Sigma_1 = U \cap \Sigma_2.$$

Die Membrangleichung führt zu einem quasilinearen System hyperbolischer Differentialgleichungen. Um jedoch Existenz und Eindeutigkeit für solche Systeme benutzen zu können, wird die Diffeomorphismeninvarianz der Gleichung durch die Wahl einer geeigneten Eichung gebrochen. In dieser Arbeit wird eine Eichung benutzt, die auch bei der Lösung der Einstein-Gleichung der Allgemeinen Relativitätstheorie zum Einsatz kommt.

Es wird eine positive Antwort auf das Problem der Existenz und Eindeutigkeit gegeben. Das Ergebnis gilt für jede Dimension der umgebenden Mannigfaltigkeit und jede Codimension von Σ_0. Ebenso sind nicht-kompakte Anfangswerte zulässig und Anfangswerte, die beliebig nahe am Lichtkegel sind. Im Falle einer global hyperbolischen Umgebungsmannigfaltigkeit und uniformen Schranken an Ableitungen der zweiten Fundamentalform und Ableitungen des Anfangswertes ν sowie quantitative Kontrolle des Abstandes der Anfangswerte zum Lichtkegel wird eine Existenzzeit der Lösung Σ in geometrischen Größen gegeben.

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Contents

0 Introduction

0.1 Geometric differential equations

Nonlinear geometric partial differential equations play a fundamental role in mathematical physics. The requirement that the physical laws be formulated in a coordinate invariant way leads naturally to geometric equations concerning curvature. Recently, developments in the field of nonlinear equations make it possible to justify models of theoretical physics.

This can be done by showing existence and uniqueness of solutions under circumstances where the physics suggests that existence and uniqueness should hold. The main result of this thesis is existence and uniqueness for an equation originating in higher-dimensional extensions of String Theory.

Examples of solutions of a geometric equation are minimal surfaces and surfaces of prescribed mean curvature. Additionally, physical situations involving surface tension such as capillary surfaces and soap surfaces lead to curvature equations. In General Relativity the fundamental equation consists of the vanishing of the Ricci curvature, which leads to a quasilinear hyperbolic equation. Solving the initial value problem is a central issue in General Relativity. In contrast to solutions of linear equations, solutions of nonlinear equations can have singular behaviour. To cope with this behaviour new analytical methods need to be applied. In this work we consider existence and uniqueness of solutions to the initial value problem for a hyperbolic minimal surface, whose structure is similar to the initial value problem of General Relativity.

0.2 Unified Theories

In theoretical physics the search for a unified description of the universe is a central issue. Two successful theories exist at this time representing different viewpoints. Quantum Field Theory deals with very small length scales. Forces between entities are transferred by particles — a central part of this theory. The other viewpoint, General Relativity, acts at great length scales and is the theoretical groundwork for the movement of planets and stars. The difference becomes more obvious when considering the role of space and time in both theories. Whereas in Quantum Field Theory they are assumed to be fixed, in General Relativity both quantities are glued together in a manifold called *spacetime*. This manifold carries a Lorentzian metric, i. e. the metric has one negative eigenvalue and the others are positive. The metric of the spacetime has to satisfy the Einstein equations relating the Ricci curvature to the stress-energy tensor. The latter tensor models information about the distribution of matter and energy within the universe. Some attempts exist to overcome the inconsistency in the description of nature. One of them is String Theory. In String Theory one-dimensional objects called *strings* are studied in place of zero-dimensional particles of Quantum Field Theory. Considered as classical objects, i. e. without connection to Quantum Field Theory, strings are submanifolds of a Lorentzian manifold. Recently, even higher dimensional objects called p-*branes* have been introduced as part of an extension of String Theory ($p = 1$). Branes also arise as Dirichlet boundary values for strings if one is not dealing with free strings and are called *D-branes* in this case. The next section describes the equation of motion for p-branes.

0.3 Membranes

To phrase the problem we consider two notions from Lorentzian geometry. A submanifold of a Lorentzian ambient manifold is called *spacelike* if the induced metric is Riemannian and *timelike* if the induced metric is Lorentzian.

A spacelike submanifold Σ_0 of a Lorentzian ambient manifold is called a *membrane* in this thesis. During a time evolution a membrane sweeps out a timelike submanifold Σ called *world volume*. If $\dim \Sigma_0 = p$, then $\dim \Sigma = p+1$; in physics literature the term membrane is reserved for two-dimensional objects ($p = 2$), and the spacelike submanifold Σ_0 is called a p-brane for higher-dimensional objects. If $p = 1$, then Σ_0 is called string and Σ is called the worldsheet of the string.

The equation of motion of a membrane is determined by the condition that the world volume of the membrane is a critical point for the volume functional induced by the ambient manifold. The Euler-Lagrange equation of the volume functional is the vanishing of the mean curvature of the world volume Σ. A discussion of this fact in the context of Riemannian manifolds can be found in [GHL04]. The central system of partial differential equations considered in this thesis is therefore

$$H(\Sigma) \equiv 0, \tag{0.3.1}$$

where $H(\Sigma)$ denotes the mean curvature vector of the submanifold Σ. Equation (0.3.1) will be called the *membrane equation* in the sequel.

Surfaces in Euclidean space satisfying this equation are called minimal surfaces. Many aspects of minimal surfaces are well understood, a treatment in Euclidean space was done by J. Douglas in [Dou39] and in general Riemannian manifolds by C. B. Morrey in [Mor48].

A powerful tool for investigating such geometric equations is the theory of partial differential equations. Here, a distinction arises between minimal submanifolds in Riemannian geometry and timelike world volumes in Lorentzian geometry. Whereas the equation describing minimal submanifolds leads to elliptic equations, the Lorentzian metric induced on the world volume causes the resulting equations for membranes to be hyperbolic. Due to the dependency of the induced volume on the induced metric of the submanifold the derived equations in both cases are nonlinear.

Static examples of membranes can be obtained from minimal surfaces, which are solutions in Minkowski space if a timelike real line is attached. Exact solutions corresponding to pulsating and rotating objects have been studied by H. Nicolai and J. Hoppe in [HN87].

A natural question for hyperbolic equations is the *initial value problem* (IVP) or *Cauchy problem*. Solving such a Cauchy problem under appropriate assumptions only involving geometric quantities will be the central issue of this work. The Cauchy problem for closed strings, i. e. membranes diffeomorphic to the circle line, has been studied in Minkowski space and in globally hyperbolic ambient manifolds. T. Deck [Dec94] studied geometric evolution problems for a string using the work of C. H. Gu on the motion of strings (cf. [Gu83]). Solutions are obtained as timelike immersion in a neighborhood of the initial data.

In his Ph. D. thesis [Mül03], O. Müller also solved the Cauchy problem for a closed string adapting another work of C. H. Gu on the existence of harmonic maps from two-dimensional Minkowski space to a Riemannian manifold (cf. [Gu80]). In these works the question of global existence was investigated for globally hyperbolic ambient manifolds. The notion of global existence used involves the construction of a map which does not necessarily represent a regular submanifold.

The existence of global-in-time solutions and stability are further questions. T. Deck showed stability for a subclass of globally hyperbolic target manifolds and O. Müller showed global existence for general globally hyperbolic targets. In case the world volume in Minkowski space is represented as a graph, global existence for small initial data was shown in [Lin04]. This work uses techniques of D. Christodoulou (cf. [Chr86]) and S. Klainerman (cf. [Kla86]). These authors showed the stability of Minkowski space satisfying the Einstein equations of General Relativity (cf. [CK93]). This result was generalized to arbitrary codimensions in [AAI06].

Using techniques previously applied to the Einstein equation by D. Christodoulou and S. Klainerman, S. Brendle (cf. [Bre02]) investigated the stability of a hyperplane satisfying the membrane equation (0.3.1) in Minkowski space. In contrast to the work of H. Lindblad, S. Brendle requires less regularity of the initial data, comparable to that in [CK93].

0.4 Main results

We now introduce a formulation of an initial value problem for the membrane equation (0.3.1). The goal is to set up a problem independent of any choice of coordinates or parametrizations of the geometric quantities. Initial values for hyperbolic equations are composed of an initial position and an initial velocity. Here, one needs to find a geometric quantity which can fill these parts.

In the following we state the main problem of this thesis in non-technical terms.

Main Problem

Let (N^{n+1}, h) be an $(n+1)$-dimensional Lorentzian manifold called *ambient manifold* and let Σ_0 be an m-dimensional spacelike regularly immersed submanifold of N called *initial submanifold*. Assume ν to be a unit timelike future-directed vector field normal to Σ_0 which will be called *initial direction*.

Existence
Find an open $(m+1)$-dimensional regularly immersed submanifold Σ carrying a Lorentzian metric and satisfying

$$H(\Sigma) \equiv 0, \ \Sigma_0 \subset \Sigma, \text{ and } \nu \text{ is tangential to } \Sigma. \tag{0.4.1}$$

Uniqueness
Show that for Σ_1 and Σ_2 solving the IVP (0.4.1), there exists an open set V with $\Sigma_0 \subset V \subset N$ such that

$$V \cap \Sigma_1 = V \cap \Sigma_2. \tag{0.4.2}$$

The main result of this thesis is an affirmative answer to the problem above under suitable conditions. Since equation (0.3.1), as a geometric equation, is invariant under diffeomorphisms of the world volume Σ, the membrane equation is degenerate hyperbolic; this means that technically, the principal symbol of the differential operator has zero eigenvalues. As for the Einstein equations a choice of gauge is needed. From H. Friedrich and A. Rendall's treatise of the Cauchy problem for the Einstein equations in [FR00] we make use of the so-called harmonic map gauge, a generalized form of the harmonic (or wave-) coordinates used by Y. Choquet-Bruhat to solve the Cauchy problem for the Einstein equations (cf. [CB52]). This choice leads to a reduced equation which is quasilinear hyperbolic of second order; thus we need to investigate an existence theory for such equations.

By adapting the results of T. Kato (cf. [Kat75]) for symmetric hyperbolic systems of first order to second-order equations, we obtain an existence theory with emphasis on a lower bound on the time of existence for a solution. The corresponding Theorem 2.9 shows existence of classical solutions of quasilinear wave equations using Sobolev space theory. To make use of the Sobolev embedding theorem, boundedness of higher-order derivatives is needed.

In Theorem 3.39 we consider general initial velocities with normal component pointing in direction ν and obtain parametrized immersions solving the Cauchy problem (0.4.1) in Minkowski space. The conditions imposed on the initial data are bounds for derivatives of the second fundamental form of the initial submanifold and the initial velocity up to order s, resp. $s+1$, where $s > \frac{m}{2} + 1$ denotes an integer. These tensors were measured with the Euclidean metric on Minkowski space. The value s, and therefore the dimension of the initial submanifold, enters the conditions through the use of Sobolev spaces in the existence theory of T. Kato (cf. [Kat75]). To bound the initial values away from the light cone we introduce a bound for the infimum of angles between unit timelike directions normal to the initial submanifold and the timelike direction of Minkowski space. Furthermore, a bound is imposed for the angle between the initial velocity and the timelike direction. These conditions give us a measure of how far the initial values are away from the light cone. A lower bound for the time parameter of the differential equation is obtained by uniformity of these conditions and in view of the bounds on the smoothness of the initial data.

By introducing a graph representation of the initial submanifold similar to graph representations for hypersurfaces of Euclidean space, but adapted to Minkowski space and valid for any codimension, we fulfill the conditions needed by Theorem 3.14 for the problem when localized in space.

A generalization to a globally hyperbolic Lorentzian ambient manifold is given in Theorem 4.32. To be able to adopt the conditions imposed on the initial values in Minkowski space, a substitute for the timelike direction is obtained in the following manner. The Lorentzian ambient manifold is assumed to admit a time function in a neighborhood of the initial submanifold, i.e. a function whose gradient is timelike everywhere. The unit normal to the level sets of this time function in that neighborhood is used to formulate the conditions on the angles between the lightcone and the timelike and spacelike directions of the initial data, replacing the timelike direction of Minkowski space.

The ambient manifold has to satisfy certain conditions. The foliation with levelsets of the time function gives a natural Riemannian metric by flipping the sign of the unit normal to the slices. The conditions imposed on the ambient manifold are boundedness of derivatives of the curvature up to order $s+1$, control over the norm of the gradient of the time function measured with the Lorentzian metric, and boundedness of derivatives of the gradient of the time function up to order $s+2$. The conditions on the initial values are the same as in Minkowski space if one uses the flipped Riemannian metric to measure the norm of the second fundamental form and the initial velocity.

Solutions obtained by the above procedure are non-geometric objects. As a next step towards the solution to problem (0.4.1) we show in Theorem 5.2 that we can solve the initial value problem for immersions if we allow the initial velocity to have a tangential part called *shift* and a factor — called *lapse* — scaling the initial direction. The uniqueness part of this theorem shows that the construction of geometric solutions to (0.4.1) in the preceding cases are independent of the choice of an immersion of the initial submanifold, the initial lapse and shift. This is done by constructing a reparametrization of two solutions in order to change one gauge into the other. The condition for such a change leads to the problem of solving a harmonic map equation from a Lorentzian manifold to another Lorentzian manifold. This equation is solved using the results for quasilinear hyperbolic equations previously developed for solving the membrane equation.

The existence part of the main problem is shown in Theorem 6.5. Uniform conditions on ambient manifold, initial submanifold, and initial direction lead to a lower bound on the time of existence of a solution in the sense that one obtains a lower bound on the length of timelike curves emanating from the initial submanifold. Local conditions lead to a solution in a neighborhood of the initial submanifold as is shown in Corollary 6.11. Uniqueness of solutions to the Cauchy problem (0.4.1) is established in Theorem 6.12. The strategy is to compare a solution to the initial value problem (0.4.1) with a solution constructed in Corollary 6.11. A parametrization of the given solution meeting the gauge condition used for the construction of a solution is obtained by solving a harmonic map equation similar to that in the above case. No conditions on the dimensions are necessary, nor is compactness. Therefore known examples for minimal surfaces, such as the plane and the catenoid, are permitted as initial values.

A natural question for scale-invariant equations such as the membrane equation is the behaviour of a solution under rescaling. The results presented here are scale-invariant, in the sense that we show that the time of existence equals the scale multiplied by a scale-invariant constant.

Solutions to the geometric Cauchy-problem (0.4.1) were obtained for various notions of the initial submanifold to be regularly immersed, locally embedded, and with locally finite intersections, in such a way that the solution has the same structure as the initial submanifold. Furthermore, uniqueness results in the sense of the main problem are established for these types of submanifolds.

The goal of this work was to provide short-time solutions to the Cauchy problem (0.4.1). Let us conclude this discussion by proposing directions for future research. An extension result and the analysis of the long-time behaviour of solutions of the membrane equation would be interesting to study. Another related goal would be the understanding of the structure of singularities occurring at finite time. From the viewpoint of the theory of partial differential equations it would also be desirable to weaken the condition needed on the differentiability of the initial data.

0.5 Thesis overview

We will now give a short overview of this work. *Chapter 1* provides notations to be used, various formulas, and the choice of the gauge to be used to reduce the membrane equation. The existence theory for hyperbolic equations obtained by the reduction process is the subject of *Chapter 2*. We obtain a lower bound on the time of existence of a solution and spatially-local uniqueness of solutions. More technical and non-enlightening lemmata are proven in *Chapter A* of the appendix. To provide an easier access to solutions of the membrane equation, *Chapter 3* is devoted to the membrane equation in the case where the ambient manifold is Minkowski space. By examining the Minkowski case one gains an understanding of what conditions should be imposed when generalizing to an arbitrary ambient manifold, which is done in *Chapter 4*. Up to this point all solutions were obtained as immersions; *Chapter 5* is then devoted to showing that each solution is independent of the choice of immersion of the initial submanifold, and of lapse and shift of the initial velocity. Finally, *Chapter 6* gives proofs of the main existence and uniqueness results providing an estimate on the time of existence.

1 Membrane equation

1.1 Notations

Let N^{n+1} be an $(n+1)$-dimensional smooth manifold endowed with a Lorentzian metric h. The Levi-Civita connection w.r.t. h and the corresponding Christoffel symbols will be denoted by $\mathbf{D}$ and $\mathbf{\Gamma}$, respectively. The $(1,3)$ or the $(0,4)$-version of the curvature of h will be denoted by $\mathbf{Rm}$ or $\mathbf{R}$.

Coordinates on N will carry two sets of indices as follows. Capital Latin letters as $A, B, C, \ldots$ will run from 0 to n and small underlined Latin letters as $\underline{a}, \underline{b}, \underline{c}, \ldots$ will run from 1 to n. Our convention for the signature of a Lorentzian metric will be $(-+\cdots+)$, so that the 0th component denotes the timelike direction. Partial derivatives in coordinates will be abbreviated by $\partial_A = \frac{\partial}{\partial y^A}$ and we abbreviate covariant derivatives by $\mathbf{D}_A = \mathbf{D}_{\partial_A}$. A covariant derivative of order ℓ will be denoted by $\mathbf{D}^\ell$. We will use the notation of contracted Christoffel symbols defined by $\mathbf{\Gamma}^A = h^{BC}\mathbf{\Gamma}^A_{BC}$. If E denotes a Riemannian metric on N, then we use the coordinate independent norm on tensors $T \in \mathcal{T}^\ell_k(N)$ induced by E as follows

$$|T|^2_E = E^{A_1C_1} \cdots E^{A_kC_k}\, E_{B_1D_1} \cdots E_{B_\ell D_\ell}\, T^{B_1,\ldots,B_\ell}_{A_1,\ldots,A_k}\, T^{D_1,\ldots,D_\ell}_{C_1,\ldots,C_k}.$$

Coordinate derivatives of order ℓ will be denoted by ∂^ℓ and the vector of all coordinate derivatives of order ℓ by D^ℓ. For a function f it then follows $D^\ell f = (\partial^\alpha f)_{|\alpha|=\ell}$, where α designates a multi-index. The notation for the Euclidean norm induced by a coordinate system of a coordinate dependent term $T^{B_1,\ldots,B_\ell}_{A_1,\ldots,A_k}$, not necessarily the components of a tensor, is given by

$$|(T^{B_1,\ldots,B_\ell}_{A_1,\ldots,A_k})|^2 = |T|^2_e = \textstyle\sum_{A_1,\ldots,A_k,B_1,\ldots,B_\ell} |T^{B_1,\ldots,B_\ell}_{A_1,\ldots,A_k}|^2.$$

To designate coordinate indices on timelike and spacelike submanifolds of N we use the following convention. Let Σ_0 be a spacelike submanifold of dimension m. Coordinates on Σ_0 will carry small Latin indices as $i, j, k, \ldots$ running run from 1 to m. An adapted coordinate system ∂_A on N will carry small Latin indices as $i, j, k, \ldots$ for directions tangent to Σ_0, Greek indices $\alpha, \beta, \ldots$ having the range $\{0, m+1, \ldots, n\}$ denoting normal directions to Σ_0. Small Latin letters as $a, b, c, \ldots$ will range from $m+1$ to n denoting spacelike normal directions to Σ_0.

Let Σ be a timelike $(m+1)$-dimensional submanifold of N. Local coordinates on Σ will carry Greek indices as $\mu, \nu, \lambda, \ldots$ and will run from 0 to m. The indices $a, b, c, \ldots$ described above will be used for directions normal to Σ being part of an adapted coordinate system.

In the following we first fix some notations concerning geometric quantities of timelike submanifolds. The next section will then discuss the notation for a spacelike submanifold. The metric and connection induced on Σ are given by

$$g := h\big|_\Sigma \quad \text{and} \quad \nabla_X Y := (\mathbf{D}_X Y)^\top \text{ for vector fields } X, Y \text{ tangent to } \Sigma.$$

The Christoffel symbols of ∇ will be denoted by Γ. The curvature will be denoted by Rm or R. To describe the structure of the submanifold we need a notion describing the interplay of the curvature of N and the submanifold. Such a notion is given by the exterior curvature of Σ in N. In the following definition we use the notation $\mathcal{T}(\Sigma)$ for smooth sections of the tangent bundle $T\Sigma$ of Σ and $\mathcal{N}(\Sigma)$ for smooth sections of the normal bundle $N\Sigma$ of Σ.

Definition 1.1 (2nd fundamental form)**.** Let X, Y be two vector fields on Σ extended to vector fields on N. Define a mapping

$$II : \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma) \to \mathcal{N}(\Sigma) \quad \text{by} \quad II(X,Y) := (\mathbf{D}_X Y)^{\perp}.$$

The tensor II is called *second fundamental form*. It is symmetric and independent of the extensions of the vector fields.

The relation between the curvature $\mathbf{Rm}$ of N and Rm of Σ is given by the *Gauß-equation*

$$g\big(R(X,Y)Z,W\big) = g\big(\mathbf{R}(X,Y)Z,W\big) + g\big(II(X,W), II(Y,Z)\big) - g\big(II(X,Z), II(Y,W)\big)$$

and the Codazzi-equation which needs a further definition. Set

$$(\nabla^{\perp}_X II)(Y,Z) := \nabla^{\perp}_X\big(II(Y,Z)\big) - II(\nabla_X Y, Z) - II(Y, \nabla_X Z) \qquad \text{with } \nabla^{\perp}_X W = (\mathbf{D}W)^{\perp}.$$

The *Codazzi-equation* is then given by

$$\big(\mathbf{R}(X,Y)Z\big)^{\perp} = (\nabla^{\perp}_X II)(Y,Z) - (\nabla^{\perp}_Y II)(X,Z)$$

We are now able to define the main notion of this thesis.

Definition 1.2 (Mean curvature)**.** The *mean curvature* $H \in \mathcal{N}(\Sigma)$ of a submanifold Σ is given by

$$H = \mathrm{tr}_g\, II \tag{1.1.1}$$

Remark 1.3. The usual definition of the mean curvature involves a factor dependent on the dimension. It was omitted here since we are only interested in the homogeneous equation.

The next section describes the geometric equations for an immersion of a spacelike submanifold which will play the role of an initial value for the membrane equation.

1.2 Geometry of immersions

We begin with a definition of the kind of manifolds we will use as initial data.

Definition 1.4. Let M^m be an m-dimensional manifold and $\varphi : M \to N$ be a mapping. The image of φ, denoted by $\Sigma_0 := \operatorname{im}\varphi$, is called a *regularly immersed submanifold of* N if the differential of φ is non-singular at every point of M .

Let Σ_0 be a m-dimensional spacelike regularly immersed submanifold of N. We denote geometric quantities referring to Σ_0 by symbols analog to those corresponding to Σ with a "$\circ$" to emphasize the role of initial value. Therefore the induced metric, connection, and the corresponding Christoffel symbols are denoted by $\mathring{g}$, $\mathring{\nabla}$, $\mathring{\Gamma}$, respectively.

Let $\varphi : M \to N$ denote the local diffeomorphism with image Σ_0 occurring in definition (1.4). Local coordinates on M will carry small Latin indices such as $i, j, k, \dots$ with range 1 to m. By abuse of language the metric, connection and Christoffel symbols induced on M by φ will also be denoted by $\mathring{g}$, $\mathring{\nabla}$ and $\mathring{\Gamma}$, respectively. The next considerations are derived from [ER93] and [CB99].

We first give a definition of a vector field along φ which will be used to compute the geometric quantities in this setting in dependence on the immersion φ and the ambient space N.

Definition 1.5. A mapping $\xi : M \to TN$ is said to be a *vector field along* φ if $\xi(p) \in T_{\varphi(p)}N$ for all $p \in M$. The bundle of vector fields along φ is denoted by φ^*TN. The connection $\mathbf{D}$ on N induces a covariant derivative on such vector fields.

Remark 1.6. For a smooth vector field $X \in \mathcal{T}(M)$ the mapping $p \mapsto d\varphi(X(p))$ is a vector field along φ.

The ambient space N induces a connection on φ^*TN which will be described in the next definition.

Definition 1.7. Let ξ be a vector field along φ and $v \in T_pM$. Choose coordinates y^A in a neighborhood of $\varphi(p) \in N$ giving the representation $\xi(q) = \lambda^A(q)\partial_A\big(\varphi(q)\big)$. Then the covariant derivative $\widehat{\nabla}_v\xi$ is given by

$$\widehat{\nabla}_v\xi = \Big(\partial_v\lambda^A + \lambda^B(p)d\varphi^C(v)\mathbf{\Gamma}^A_{BC}\big(\varphi(p)\big)\Big)\partial_A\big(\varphi(p)\big). \tag{1.2.1}$$

Remark 1.8. We summarize some properties of the covariant derivative on φ^*TN:

0. $\widehat{\nabla}$ is a connection on φ^*TN.
1. It is compatible with the metric on N, namely $\partial_v h(\xi,\eta) = h(\widehat{\nabla}_v\xi,\eta) + h(\xi,\widehat{\nabla}_v\eta)$ and
2. it is a restriction of the covariant derivative on N to φ^*TN. If ξ is of the form $\xi = \zeta\circ\varphi$ and ζ a vector field on N, then it follows that $\widehat{\nabla}_v\xi = \mathbf{D}_{d\varphi(v)}\zeta$.

To state local computations, we introduce coordinates x^j on M and use the notations $\bar{\xi}$ and Φ for the representations of the vector field ξ and φ w.r.t. the coordinates x and y, respectively, yielding that

$$\bar{\xi} = dy_{\varphi\circ x^{-1}}(\xi) \quad \text{and} \quad \Phi = y\circ\varphi\circ x^{-1}.$$

Using this notation we can state the derivative w.r.t. a coordinate basis vector field ∂_j on M and the derivative of the vector field $d\varphi(\partial_k)$. We have that

$$\widehat{\nabla}_j\xi = \big(\partial_j\bar{\xi}^A + \xi^B\partial_j\Phi^C\mathbf{\Gamma}^A_{BC}(\Phi)\big)\partial_A(\Phi) \tag{1.2.2a}$$

$$\text{and} \qquad \widehat{\nabla}_k\big(d\varphi(\partial_j)\big) = \big(\partial_k\partial_j\Phi^A + \partial_k\Phi^B\partial_j\Phi^C\mathbf{\Gamma}^A_{BC}(\Phi)\big)\partial_A(\Phi). \tag{1.2.2b}$$

The Christoffel symbols w.r.t. the connection $\widehat{\nabla}$ will be denoted by $\widehat{\Gamma}$. They are given by

$$\widehat{\Gamma}^A_{jB} = \partial_j \Phi^C \mathbf{\Gamma}^A_{BC}(\Phi). \tag{1.2.3}$$

For vector fields X and Y on M we define the *second fundamental form* of φ in this setting by

$$\mathring{I\!I}(X,Y) = \big(\widehat{\nabla}_X d\varphi(Y)\big)^{\perp}, \tag{1.2.4}$$

where the normal part is taken w.r.t. Σ_0, the image of φ. According to the Koszul formula, the following decomposition holds for $X, Y \in \mathcal{T}(M)$

$$\widehat{\nabla}_X d\varphi(Y) = \big(\widehat{\nabla}_X d\varphi(Y)\big)^{\top} + \big(\widehat{\nabla}_X d\varphi(Y)\big)^{\perp} = d\varphi(\mathring{\nabla}_X Y) + \mathring{I\!I}(X,Y).$$

For further reference we now state the representation of the second fundamental form and the induced Christoffel symbols $\mathring{\Gamma}^k_{ij}$ on M. They are given by

$$\mathring{I\!I}^A_{ij} = \partial_i\partial_j\Phi^A + \partial_i\Phi^B\partial_j\Phi^C\mathbf{\Gamma}^A_{BC}(\Phi) - \mathring{\Gamma}^k_{ij}\partial_k\Phi^A \tag{1.2.5}$$

and
$$\mathring{\Gamma}^k_{ij} = \mathring{g}^{k\ell}\partial_i\partial_j\Phi^A h_{AB}(\Phi)\partial_\ell\Phi^B + \mathring{g}^{k\ell}\partial_i\Phi^B\partial_j\Phi^C\mathbf{\Gamma}^A_{BC}(\Phi)h_{AD}(\Phi)\partial_\ell\Phi^D. \tag{1.2.6}$$

We introduce the notation of a norm to be applied to a covariant derivative of a vector field along φ as follows. Set

$$|\widehat{\nabla}^\ell\xi|^2_{\mathring{g},E} = \mathring{g}^{i_1j_1}\cdots\mathring{g}^{i_\ell j_\ell}E_{AB}\widehat{\nabla}_{i_1,\dots,i_\ell}\xi^A\widehat{\nabla}_{j_1,\dots,j_\ell}\xi^B \tag{1.2.7}$$

with the abbreviation $\widehat{\nabla}_{i_1,\dots,i_\ell} = \widehat{\nabla}_{i_1}\cdots\widehat{\nabla}_{i_\ell}$, where E denotes a Riemannian metric on the ambient manifold N. In this situation we introduce a further notation of coordinate dependent norm. Set $|\widehat{\nabla}^\ell\xi|^2_{e,e}$ equal to the RHS of the above term replacing $\mathring{g}$ and E by the Euclidean metric w.r.t. coordinates on M and N, respectively.

To simplify computations we introduce the term $S*T$ which denotes a linear combination of contractions of the tensors S and T, where we suppress factors of the metric h of the manifold N. We use this notation if the precise form is irrelevant for the arguments used. For example, with this notation, equation (1.2.2a) reads

$$\widehat{\nabla}\xi = \partial\xi + \widehat{\mathbf{\Gamma}} * \xi,$$

where $\widehat{\mathbf{\Gamma}}$ denote the Christoffel symbols defined in equation (1.2.3).

In the following section we will describe how we intend to access the geometric problem via PDE-techniques.

1.3 Reduction

Wishing to use standard PDE techniques it is convenient to introduce parametrizations. Let $F : U \subset \mathbb{R} \times M \to (N, h)$ denote an immersion with induced metric $g := F^*h$. The Christoffel symbols w.r.t. g will be denoted by Γ. The image of F corresponds to a solution

Σ of the geometric IVP (0.4.1). Again, by abuse of language we use the same notation for the metric on Σ and the induced metric on $U \subset \mathbb{R} \times M$. Let x^μ be coordinates on $\mathbb{R} \times M$ and y^A coordinates on N. Then the considerations from the previous section carry over to this situation. We denote the covariant derivative on the pullback bundle F^*TN by $\widehat{\nabla}^F$. It then follows from the formula (1.2.5) for the second fundamental form that the membrane equation (0.3.1) reads

$$g^{\mu\nu}\widehat{\nabla}^F_\mu \partial_\nu F - g^{\mu\nu}\big(\widehat{\nabla}^F_\mu \partial_\nu F\big)^\top = 0 \tag{1.3.1a}$$
$$\iff \qquad g^{\mu\nu}\partial_\mu\partial_\nu F^A + g^{\mu\nu}\partial_\mu F^B \partial_\nu F^C \mathbf{\Gamma}^A_{BC}(F) - \Gamma^\lambda \partial_\lambda F^A = 0. \tag{1.3.1b}$$

From the PDE viewpoint the latter equation seems to be quasilinear hyperbolic of second order. The hyperbolicity comes from the signature of the metric induced by F on $\mathbb{R} \times M$.

Based on equation (1.3.1b) we will consider the following existence and uniqueness problems.

Let Σ_0 be a regularly immersed m-dimensional submanifold with immersion $\varphi : M^m \to N$. Assume ν to be a unit timelike future-directed vector field normal to Σ_0. Let α be a function on M called *initial lapse* and let β be a vector field on M called *initial shift*.

Existence

Find a neighborhood U of $\{0\} \times M$ in $\mathbb{R} \times M$ and an immersion $F : U \subset \mathbb{R} \times M \to N$ solving the IVP

$$H(\operatorname{im} F) \equiv 0, \;\; F\big|_{t=0} = \varphi, \;\; \tfrac{d}{dt}F\big|_{t=0} = \alpha\, \nu \circ \varphi + d\varphi(\beta) \tag{1.3.2}$$

such that $\frac{d}{dt}F$ is timelike future-directed. The parameter t denotes the first component of $\mathbb{R} \times M$.

Uniqueness

Show that for an immersion $\bar{F}$ solving the IVP (1.3.2) for another choice $\bar{\varphi}$ of immersion of Σ_0 and another choice of initial lapse $\bar{\alpha}$ and initial shift $\bar{\beta}$ there exists a local diffeomorphism Ψ such that $F \circ \Psi^{-1} = \bar{F}$.

Before standard existence and uniqueness results for hyperbolic equations apply one issue needs to be solved. The invariance of the membrane equation under diffeomorphisms of a solution Σ leads to a degenerate equation. This can be seen by considering the contracted induced Christoffel symbols according to formula (1.2.6). It follows that

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = g^{\kappa\lambda}\big(g^{\mu\nu}\partial_\mu\partial_\nu F^A + g^{\mu\nu}\partial_\mu F^B\partial_\nu F^C \mathbf{\Gamma}^A_{BC}(F)\big)h_{AD}(F)\partial_\kappa F^D. \tag{1.3.3}$$

The first term cancels the tangential part of the leading term of the membrane equation (1.3.1b). Therefore a gauge is needed to remove the freedom of performing diffeomorphisms. Our choice is the harmonic map gauge taken from [FR00]. In this article this gauge is used in order to reduce the Einstein equations which also lead to a degenerate hyperbolic system of second order.

Definition 1.9. A solution $F : U \subset \mathbb{R} \times M \to N$ of the membrane equation (1.3.1b) is in *harmonic map gauge*, if the following condition is satisfied:

$$\mathrm{id} : \big(U, g = F^*h\big) \to \big(U, \hat{g}\big) \quad \text{is a harmonic map,} \tag{1.3.4}$$

where $\hat{g}$ is a fixed background metric on U. This condition will be called the *harmonic map gauge condition*.

Remark 1.10. In coordinates, this condition reads

$$g^{\mu\nu}(\Gamma^{\lambda}_{\mu\nu} - \hat{\Gamma}^{\lambda}_{\mu\nu}) = 0. \tag{1.3.5}$$

As a difference of two connections the term is a tensor and therefore the condition is invariant under change of coordinates.

Inserting the harmonic map gauge condition (1.3.5) into equation (1.3.1b) we derive the *reduced membrane equation*

$$g^{\mu\nu}\partial_\mu\partial_\nu F^A + g^{\mu\nu}\partial_\mu F^B\partial_\nu F^C \mathbf{\Gamma}^A_{BC}(F) - g^{\mu\nu}\hat{\Gamma}^{\lambda}_{\mu\nu}\partial_\lambda F^A = 0. \tag{1.3.6}$$

The reduced equation is hyperbolic since the dependency on the second-order derivatives of F in the Christoffel symbols was replaced by fixed functions. Let us now prove the equivalence of the equations.

Lemma 1.11. *The membrane equation* (1.3.1b) *together with condition* (1.3.5) *is equivalent to equation* (1.3.6)*.*

Proof. Straightforwardly, inserting condition (1.3.5) into the membrane equation (1.3.1b) gives us the reduced membrane equation (1.3.6).

Suppose the reduced membrane equation (1.3.6) holds. To show that the mean curvature vanishes we need to compute the contracted Christoffel symbols w.r.t. the induced metric. By inserting equation (1.3.6) into equation (1.3.3) we get that

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = g^{\kappa\lambda}g^{\mu\nu}\hat{\Gamma}^{\delta}_{\mu\nu}\partial_\delta F^A h_{AD}\partial_\kappa F^D = g^{\mu\nu}\hat{\Gamma}^{\lambda}_{\mu\nu}$$

Using this identity together with equation (1.3.6) gives us the desired result. □

Remark 1.12. This result can be derived in a somewhat more abstract fashion if we use equation (1.3.1a) instead. The reduced membrane equation (1.3.6) can then be read as $g^{\mu\nu}\widehat{\nabla}^F_\mu\partial_\nu F = g^{\mu\nu}\hat{\Gamma}^{\lambda}_{\mu\nu}\partial_\lambda F$. The RHS of the latter equation is tangential, therefore the normal part of the LHS has to vanish and the tangential part has to coincide with the RHS which yields us the desired identity (1.3.5).

Remark 1.13. An examination of other gauges for the Einstein equations (cf. [FR00]) shows that these are also possible here, since the contracted Christoffel symbols can be replaced by arbitrary fixed functions.

A similar coordinate dependent choice of gauge is the use of the so-called harmonic (or wave-) coordinates used by Y. Choquet-Bruhat (cf. [CB52]) to show well-posedness of the Einstein equations. Their gauge condition is the vanishing of the contracted Christoffel symbols.

Remark 1.14. This reduction process for the membrane equation is complete in the sense that no further equation is needed. This contrasts the situation for the Einstein equations where the propagation of the gauge condition is shown by using the Bianchi identity. In view of R. S. Hamilton's result for geometric parabolic flows (cf. [Ham82]) the membrane equation seems more closely analogous to mean curvature flow than to the Einstein equations. In the reduction process for both equations, the projector to the tangent space plays a fundamental role.

Background metric Throughout this text we use a *special background metric* $\hat{g}$. If the initial values of F are denoted by $F\big|_{t=0} = \varphi$ and $\frac{d}{dt}F\big|_{t=0} = \chi$, then we define

$$\hat{g} := -\lambda^2 dt^2 + \mathring{g}_{ij}(\beta^i dt + dx^i)(\beta^j dt + dx^j) \tag{1.3.7}$$
$$\text{with } \beta = \beta^i\partial_i,\ \beta^i = \mathring{g}^{ij} h\big(\chi, d\varphi(\partial_j)\big) \text{ and } -\lambda^2 = h(\chi,\chi) - \mathring{g}(\beta,\beta).$$

2 Existence and Uniqueness for Hyperbolic Equations of Second Order

Due to the signature of the metric, the reduced membrane equation (1.3.6) is a hyperbolic equation of second order. Nonlinear second-order equations whose coefficients and RHS depend on derivatives of the solution only up to first order are called quasilinear. In this section we develop an existence theory for such equations relying on Sobolev spaces. More precisely we search for solutions $u : [0,T'] \times \mathbb{R}^m \to \mathbb{R}^N$ of the initial value problem

$$g^{\mu\nu}(u, Du, \partial_t u)\partial_\mu\partial_\nu u = f(u, Du, \partial_t u),\ u\big|_{t=0} = u_0,\ \partial_t u\big|_{t=0} = u_1. \tag{2.0.1}$$

The equation is supposed to be hyperbolic in the sense that $g^{00} \leq -\lambda < 0$ and $g^{ij} \geq \mu\delta^{ij}$ for constants λ and μ. We will discuss the existence of a solution only within a small time interval. The solution will be sought within L^2-Sobolev spaces H^s for s large enough.

The strategy will mainly follow the work of T. Kato for symmetric hyperbolic systems (cf. [Kat75]). By using ideas from [FM72], [Tay96] and [SS98] we adapt it to second order equations obtaining a lower bound on the existence time. Following [HKM76] and [FM72] we consider asymptotic initial value problems. A solution to these problems satisfy the equation (2.0.1) and its difference to a linear function lies in a Sobolev space.

Throughout this section c will denote various constants depending on the dimensions involved.

2.1 Uniformly local Sobolev spaces

This section fixes some notations used for the existence theory. Let H^s denote the usual L^2-Sobolev space with norm $\|\,.\,\|_s$ given by

$$\|u\|_s^2 = \sum_{|\alpha|\leq s} \|\partial^\alpha u\|_{L^2}^2.$$

Further, let C^s designate the space of s-times continuously differentiable functions with norm $\|\,.\,\|_{C^s}$ defined by

$$\|u\|_{C^s}^2 = \sum_{|\alpha|\leq s} \|\partial^\alpha u\|_\infty^2 \tag{2.1.1}$$

and let C_b^s denote functions in C^s which have bounded derivatives up to order s. In the following definition we introduce spaces of functions which are only locally contained in a Sobolev space. These will be used to formulate conditions for the coefficients of the equation.

Definition 2.1 (Uniformly local L^p and Sobolev Spaces). 1. Suppose $1 \leq p < \infty$ and V is a finite dimensional vector space. Let $L^p_{\mathrm{ul}}(\mathbb{R}^m, V)$ denote the space of all functions u on $\mathbb{R}^m$ with values in V such that $\sup_{x\in\mathbb{R}^m} \|\varphi_x u\|_{L^p} < \infty$ where $\varphi \in C_c^\infty(\mathbb{R}^m)$ and $\varphi_x(y) = \varphi(y - x)$.

2. For an integer $s \geq 0$ we denote by $H^s_{\mathrm{ul}}(\mathbb{R}^m, V)$ the space of all functions $u \in L^2_{\mathrm{ul}}$ whose distributional derivatives $\partial^\alpha u$ of order $|\alpha| \leq s$ are in L^2_{ul}.
The space $H^s_{\mathrm{ul}}(\mathbb{R}^m, V)$ will be endowed with the norm $\|u\|_{s,\mathrm{ul}} = \sup_{|\alpha|\leq s} \|\partial^\alpha u\|_{L^2_{\mathrm{ul}}}$.

The next lemma lists some properties of the uniformly local Sobolev spaces concerning embeddings which are similar to properties of the usual Sobolev spaces.

Lemma 2.2. *1. The space $H^s_{\mathrm{ul}}(\mathbb{R}^m, V)$ is a Banach space and $H^s \subset H^s_{\mathrm{ul}}$. Let $\varphi \in C_c^\infty(\mathbb{R}^m)$. Then we have the equivalent norm $\sup_{x\in\mathbb{R}^m} \|\varphi_x u\|_s$.*

2. Some properties of the usual Sobolev spaces apply to the uniformly local case: If $s < \frac{m}{2}$, then $H^s_{\mathrm{ul}} \subset L^p_{\mathrm{ul}}$ for p with $1/2 - s/m \leq 1/p \leq 1/2$. If $s = \frac{m}{2}$ then $H^s_{\mathrm{ul}} \subset L^p_{\mathrm{ul}}$ for any p with $2 \leq p < \infty$ and if $s > \frac{m}{2}$ then $H^s_{\mathrm{ul}} \subset C^k_b$ for $k = s - \lfloor m/2 \rfloor - 1$, where $\lfloor m/2 \rfloor$ denotes the integer part of $m/2$.

*3. If $r = \min(s, t, s + t - \lfloor m/2 \rfloor - 1) > 0$, then $H^s_{\mathrm{ul}} H^t_{\mathrm{ul}} \subset H^r_{\mathrm{ul}}$ The LHS is either the product of two functions or a linear operator on V applied to a function.
With the same values of r, s, t it follows $H^s_{\mathrm{ul}} H^t \subset H^r$ and $H^s H^t_{\mathrm{ul}} \subset H^r$.*

As a consequence we state a result about the commutator of a differential operator and multiplication with a function analog to a result for the usual Sobolev space to be found in [Tay96].

Corollary 2.3. *Let $u \in H^s_{\mathrm{ul}}$ and $v \in H^{s-1}$ with $s > \frac{m}{2} + 1$. Then the following estimate holds*

$$\|[\partial^\alpha, u]v\|_{L^2} \leq c\|u\|_{s,\mathrm{ul}} \|v\|_{s-1} \qquad \text{for } |\alpha| \leq s. \tag{2.1.2}$$

Proof. From [Tay96] we take the expression

$$[\partial^\alpha, u]v = \partial^\alpha(uv) - u\partial^\alpha v = \textstyle\sum_{|\beta|+|\gamma|=s-1} c_{j\beta\gamma}(\partial^\beta \partial_j u)(\partial^\gamma v)$$

and therefore

$$|[\partial^\alpha, u]v|_{L^2} \leq c \textstyle\sum_{|\beta|+|\gamma|=s-1} |(\partial^\beta u_j)(\partial^\gamma v)|_{L^2}$$

with $\partial_j u = u_j$. The conditions $u \in H^s_{\mathrm{ul}}$ and $v \in H^{s-1}$ imply $\partial^\beta u_j \in H^{s-|\beta|-1}_{\mathrm{ul}}$ and $\partial^\gamma v \in H^{s-1-|\gamma|}$. To derive the desired result we want to use part 3 of lemma 2.2. Therefore, we need to check the orders of differentiation. We have

$$2s - 2 - |\beta| - |\gamma| = 2(s-1) - (s-1) > \tfrac{m}{2}$$

which ends the proof. □

We now turn to invertible elements within H^s_{ul}. For a vector space V we denote the space of linear bounded operators $L : V \to V$ by $B(V)$.

Lemma 2.4. *Let $s > \frac{m}{2}$ be an integer and $A \in H^s_{\mathrm{ul}}(\mathbb{R}^m, B(V))$. A necessary and sufficient condition for A to be invertible within H^s_{ul} is $\big(A(x)\big)^{-1} \in B(V)$ for all x and $|A^{-1}|_\infty \le \delta^{-1}$. Then it follows that*

$$\|A^{-1}\|_{s,\mathrm{ul}} \le c\delta^{-1}\big(1 + (\delta^{-1}\|A\|_{s,\mathrm{ul}})^s\big) \tag{2.1.3}$$

The next lemma states estimates for the composition of functions. The proof is an immediate consequence of the proof of theorem IV in [Kat75].

Lemma 2.5. *Assume $s > \frac{m}{2}$ to be an integer, $u \in H^s_{\mathrm{ul}}(\mathbb{R}^m, \mathbb{R}^N)$ and $F \in C^s_c(\mathbb{R}^N)$. Then it holds that*

$$\|F(u)\|_{s,\mathrm{ul}} \le c\|F\|_{C^s}(1 + \|u\|^s_{s,\mathrm{ul}})$$

with a constant c depending on N and m.

2.2 Linear Equation

The argument for solving the quasilinear equation will use estimates for the linearized equation. Therefore we establish existence for linear equations first using the result for linear symmetric hyperbolic systems of [Kat75]. To this end it is necessary to reduce the second-order equation to a first-order symmetric hyperbolic system. This will be done by a method taken from [FM72], where it is used to establish a reduction of the Einstein equations to a first-order symmetric hyperbolic system.

To keep this thesis self-contained we state the existence result for linear symmetric hyperbolic systems of first order from [Kat75] (cf. p. 189, Theorem I).

Theorem 2.6. *Let P be a Hilbert-space, $s > \frac{m}{2} + 1$ be an integer and let $1 \le s' \le s$. Assume the following conditions:*

$$A^\mu, B \in C\big([0,T], H^0_{\mathrm{ul}}\big(\mathbb{R}^m, B(P)\big)\big), 0 \le \mu \le m \tag{2.2.1a}$$

$$\|A^0(t)\|_{s,\mathrm{ul}} \le K_0,\ \|A^j(t)\|_{s,\mathrm{ul}},\ \|B(t)\|_{s,\mathrm{ul}} \le K \quad \text{for } 0 \le t \le T, 1 \le j \le m, \tag{2.2.1b}$$

$$\|A^0(t') - A^0(t)\|_{s-1,\mathrm{ul}} \le L|t' - t| \quad 0 \le t, t' \le T \tag{2.2.1c}$$

$$A^\mu(t,x) \text{ is symmetric for } 0 \le t \le T, x \in \mathbb{R}^m, 0 \le \mu \le m \tag{2.2.1d}$$

$$A^0(t,x) \ge \eta > 0 \text{ for all } t, x \text{ (uniformly positive definite)} \tag{2.2.1e}$$

$$F \in L^1\big([0,T], H^{s'}(\mathbb{R}^m, P)\big) \cap C\big([0,T], H^{s'-1}(\mathbb{R}^m, P)\big) \tag{2.2.1f}$$

$$v_0 \in H^{s'}(\mathbb{R}^m, P) \tag{2.2.1g}$$

Then the IVP

$$A^0\partial_t v + A^j\partial_j v + Bv = F,\ v\big|_{t=0} = v_0$$

has a solution

$$v \in C\big([0,T], H^{s'}(\mathbb{R}^m, P)\big) \cap C\big([0,T], H^{s'-1}(\mathbb{R}^m, P)\big).$$

The solution is unique in a larger class $C\big([0,T], H^1(\mathbb{R}^m, P)\big) \cap C\big([0,T], H^0(\mathbb{R}^m, P)\big)$.

The next step is to find suitable operators A^0, A^j and B connecting first-order systems with second-order equations. To this end we consider the matrices

$$C^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -g^{00} & 0 \\ 0 & 0 & (g^{ij})_{i,j} \end{pmatrix}, \quad C^j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2g^{j0} & (-g^{ij})_i \\ 0 & (-g^{ij})_i & 0 \end{pmatrix} \quad \text{for } 1 \le j \le m$$

and

$$D = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let V be a vector space. We define a member C of $B(V^{m+2})$ arising from a matrix $(c^k_\ell)_{1\le k,\ell\le m+2}$ by

$$(Cv)^k = \sum_{1\le \ell\le m+2} c^k_\ell v^\ell \quad \text{for } v \in V^{m+2}. \tag{2.2.2}$$

The following fact was established in [FM72].

Proposition 2.7. *Let $f \in C([0,T] \times \mathbb{R}^m, \mathbb{R}^N)$ and $(g^{\mu\nu})_{0\le\mu,\nu\le m}$ be a matrix-valued continuous function defined on $[0,T]\times\mathbb{R}^m$. Let A^0, A^j and B be the linear operators $\in B\big((\mathbb{R}^N)^{m+2}\big)$ arising from the matrices C^0, C^j and D by definition (2.2.2). Let $u_0 \in C^2([0,T]\times\mathbb{R}^m,\mathbb{R}^N)$ and $u_1 \in C^1([0,T]\times\mathbb{R}^m,\mathbb{R}^N)$.*

Then the initial value problem

$$A^0\partial_t v + A^j\partial_j v + Bv = F,\ v\big|_{t=0} = \big(u_0, u_1, (\partial_j u_0)_j\big) \tag{2.2.3}$$

with the RHS $F = (0, f, 0)$ is equivalent to the initial value problem

$$g^{\mu\nu}\partial_\mu\partial_\nu u = f,\ u\big|_{t=0} = u_0,\ \partial_t u\big|_{t=0} = u_1 \tag{2.2.4}$$

within C^1-solutions of (2.2.3) and C^2-solutions of (2.2.4).

Relying on this fact we state existence and uniqueness for linear second-order equations.

Theorem 2.8. *Let $s > \frac{m}{2} + 1$ be an integer. Assume that the following inequalities hold:*

$$g^{\mu\nu} \in C\big([0,T], H^0_{\mathrm{ul}}(\mathbb{R}^m)\big) \text{ for } 0 \le \mu,\nu \le m, \tag{2.2.5a}$$
$$\|(g^{\mu\nu})\|_{e,s,\mathrm{ul}} \le K \tag{2.2.5b}$$
$$\|g^{00}(t) - g^{00}(t')\|_{s-1,\mathrm{ul}} \le L|t-t'|,\ \big\|\big(g^{ij}(t)\big) - \big(g^{ij}(t')\big)\big\|_{e,s-1,\mathrm{ul}} \le L|t-t'| \tag{2.2.5c}$$
$$\big(g^{\mu\nu}\big) \text{ is symmetric and } -g^{00} \ge \eta > 0,\ \big(g^{ij}\big) \ge \eta I \tag{2.2.5d}$$
$$f \in L^1\big([0,T], H^s(\mathbb{R}^m,\mathbb{R}^N)\big) \cap C\big([0,T], H^{s-1}(\mathbb{R}^m,\mathbb{R}^N)\big) \tag{2.2.5e}$$

If $u_0 \in H^{s+1}(\mathbb{R}^m,\mathbb{R}^N)$ and $u_1 \in H^s(\mathbb{R}^m,\mathbb{R}^N)$, then the IVP (2.2.4) has a unique solution

$$u \in C\big([0,T], H^{s+1}(\mathbb{R}^m,\mathbb{R}^N)\big) \cap C^1\big([0,T], H^s(\mathbb{R}^m,\mathbb{R}^N)\big).$$

Proof. Consider the linear operators $A^0,\ A^j,\ B \in B\big((\mathbb{R}^N)^{m+2}\big)$ arising from the matrices C^0, C^j and D by definition (2.2.2). We have to check the conditions of theorem 2.6. Observe that we have for an operator $C \in B\big((\mathbb{R}^N)^{m+2}\big)$ arising from a matrix (c_i^j) by definition (2.2.2)

$$\|C\|_{s,\mathrm{ul}} \leq \|(c_i^j)\|_{e,s,\mathrm{ul}} \quad \text{and} \quad \langle v, Cv\rangle \geq \langle (\lambda^i), (c_i^j)(\lambda^j)\rangle,$$

where $(\mathbb{R}^N)^{m+2}$ is endowed with the Euclidean metric. Hence, it suffices to consider the matrices C^μ and D to obtain the conditions of theorem 2.6.

Constants are in L^2_{ul}, so condition (2.2.1a) holds for B. From assumption (2.2.5a) we get the rest of condition (2.2.1a). To obtain condition (2.2.1b) we use the Frobenius-norm of the matrices C^μ. With this norm we have

$$\|C^0\|^2_{e,s,\mathrm{ul}} \leq 1 + \|g^{00}\|^2_{s,\mathrm{ul}} + \|(g^{ij})\|^2_{e,s,\mathrm{ul}} \leq 1 + K^2.$$

The same consideration for C^j gives us $\|C^j\|^2_{e,s,\mathrm{ul}} \leq 4K^2$ and the matrix D is bounded by $\|D\|_{e,s,\mathrm{ul}} \leq 1$ and therefore condition (2.2.1b) follows.

The Lipschitz-condition (2.2.1c) can be obtained by assumption (2.2.5c) as follows:

$$\|C^0(t) - C^0(t')\|^2_{e,s,\mathrm{ul}} \leq \|g^{00}(t) - g^{00}(t')\|^2_{s,\mathrm{ul}} + \|(g^{ij}(t)) - (g^{ij}(t'))\|^2_{e,s,\mathrm{ul}} \leq 2L^2|t-t'|^2.$$

Positive definiteness of the matrix C^0 is given by $C^0 \geq \min(1,\eta)I$, where I is the identity matrix.

From Theorem 2.6 we obtain a solution $v = (w, w_0, w_j)$ of the IVP (2.2.3) and from the differentiability of v we derive $w \in C([0,T], H^{s+1}) \cap C^1([0,T], H^s)$. The Sobolev-embedding yields that Proposition 2.7 is applicable and therefore w solves the IVP (2.2.4). The equivalence of the first-order and second-order equations stated in Proposition 2.7 also shows that a solution u to the IVP (2.2.4) is unique. □

2.3 Quasilinear Equation

In this section we will develop an existence theory for the IVP (2.0.1). A key ingredient will be Banach's fixed point theorem which gives an outline of the proof. A complete metric space and a suitable mapping has to be constructed. The value of the mapping consists of the solution to the linearized equation and energy estimates will be established to show the desired properties.
For convenience we postpone the proofs of several more technical statements to section A of the appendix.

Let $s > \frac{m}{2} + 1$ be an integer and $W \subset H^{s+1}(\mathbb{R}^m, \mathbb{R}^N) \times H^s(\mathbb{R}^m, \mathbb{R}^N)$ be open. Suppose

$$\begin{aligned} (g^{\mu\nu}) : [0,T] \times W &\longrightarrow H^s_{\mathrm{ul}}\big(\mathbb{R}^m, \mathbb{R}^{(m+1)\times(m+1)}\big) \\ \text{and} \qquad f : [0,T] \times W &\longrightarrow H^s(\mathbb{R}^m, \mathbb{R}^N) \end{aligned} \tag{2.3.1}$$

to be nonlinear operators. For $v = (v_0, v_1) \in W$ we introduce the abbreviation $g^{\mu\nu}(t,v) = g^{\mu\nu}(t, v_0, Dv_0, v_1)$; $f(t,v)$ is defined analogously. The main result of this section is stated in the following theorem.

Theorem 2.9. *Suppose the following conditions hold:*

$$\left\|\left(g^{\mu\nu}(t,v)\right)\right\|_{e,s,\mathrm{ul}} \leq K \text{ and } \|f(t,v)\|_s \leq K_f \tag{2.3.2a}$$
$$\left\|\left(g^{\mu\nu}(t,v)\right) - \left(g^{\mu\nu}(t,w)\right)\right\|_{e,s-1,\mathrm{ul}} \leq \theta E_s(v-w) \tag{2.3.2b}$$
$$\left\|\left(g^{\mu\nu}(t,v)\right) - \left(g^{\mu\nu}(t,w)\right)\right\|_{e,0,\mathrm{ul}} \leq \theta' E_1(v-w) \tag{2.3.2c}$$
$$\left\|\left(g^{\mu\nu}(t,v)\right) - \left(g^{\mu\nu}(t',v)\right)\right\|_{e,s-1,\mathrm{ul}} \leq \nu|t-t'| \tag{2.3.2d}$$
$$\|f(t,v) - f(t,w)\|_{s-1} \leq \theta'_f E_s(v-w) \tag{2.3.2e}$$
$$\|f(t,v) - f(t,w)\|_{L^2} \leq \theta_f E_1(v-w) \tag{2.3.2f}$$
$$t \mapsto f(t,v) \text{ is continuous w.r.t. } H^{s-1} \text{ for all } v \in W \tag{2.3.2g}$$
$$\left(g^{\mu\nu}(t,v)\right) \text{ is symmetric and } g^{00}(t,v) \leq -\lambda,\ \left(g^{ij}(t,v)\right) \geq \mu\delta^{ij},\ \lambda,\mu > 0 \tag{2.3.2h}$$

for $v = (v_0, v_1), w = (w_0, w_1) \in W$.

If $(u_0, u_1) \in W$ *then there exists a constant* $0 < T' \leq T$ *and a unique solution*

$$u \in C\left([0,T'], H^{s+1}(\mathbb{R}^m, \mathbb{R}^N)\right) \cap C^1\left([0,T'], H^s(\mathbb{R}^m, \mathbb{R}^N)\right) \tag{2.3.3}$$

with $(u, \partial_t u) \in W$ *to the initial value problem* (2.0.1).

The strategy to proof this theorem will be to apply Banach's fixed point theorem to the map

$$\Phi : v \longmapsto \text{solution } u \text{ to the linearized equation } g^{\mu\nu}(t,v)\partial_\mu\partial_\nu u = f(t,v). \tag{2.3.4}$$

In the next lemma we derive ingredients for the definition of the metric space on which Φ will be defined.
Set $c_E = 2(\frac{2}{\mu} + \frac{1}{\lambda})$ and $\tau_s = \#\{\beta \in N_0^m : |\beta| \leq s\}$. For notational convenience we set $H^r = H^r(\mathbb{R}^m, \mathbb{R}^N)$ and

$$E_{r+1}(v-w) := \|v_0 - w_0\|_{r+1} + \|v_1 - w_1\|_r \text{ for } v = (v_0, v_1),\ w = (w_0, w_1) \in H^{r+1} \times H^r.$$

Lemma 2.10. *For arbitrary* $\mathring{u} = (u_0, u_1) \in W$ *there exist* $\delta > 0,\ 0 \leq \rho \leq \delta/3$ *and* $u_{00} = (y_0, y_1) \in W \cap (H^{s+2} \times H^{s+1})$ *such that*

$$E_{s+1}(v - u_{00}) \leq \delta \quad \Longrightarrow \quad (v_0, v_1) \in W$$
$$E_{s+1}(\mathring{u} - u_{00}) \leq \rho$$
$$\rho C^{1/2} \leq \delta/3$$

with

$$\hat{C} := \sup_{v \in W}(\mu + |g^{00}(0,v)|_\infty + \left|\left(g^{ij}(0,v)\right)\right|_{e,\infty}) \text{ and } C := 4c_E^{1/2}\tau_s^{1/2}\hat{C}. \tag{2.3.5}$$

We are now able to define the metric space.

Lemma 2.11. *Let L' be a constant chosen later and let*

$$\begin{aligned}Z_{\delta,L'} := \{u \in L^2([0,T'] \times \mathbb{R}^m, \mathbb{R}^N) : u\big|_{t=0} = u_0,\ \partial_t u\big|_{t=0} = u_1,\\ E_{s+1}(u(t) - u_{00}) = \|u(t) - y_0\|_{s+1} + \|\partial_t u(t) - y_1\|_s \leq \delta,\\ \|\partial_t u(t) - \partial_t u(t')\|_{s-1} \leq L'|t-t'|\}\end{aligned}$$

where $u_{00} = (y_0, y_1)$ from Lemma 2.10. $Z_{\delta,L'}$ equipped with the metric

$$d(u,v) = \sup_t \big(\|u(t) - v(t)\|_{H^1} + \|\partial_t u(t) - \partial_t v(t)\|_{L^2}\big)$$

is complete.

The next lemma establishes the fact that the mapping Φ is defined on $Z_{\delta,L'}$ for all choices of δ and L'.

Lemma 2.12. *The IVP for the linearized equation*

$$g^{\mu\nu}(t,v)\partial_\mu\partial_\nu u = f(t,v),\ u\big|_{t=0} = u_0,\ \partial_t u\big|_{t=0} = u_1 \tag{2.3.6}$$

for $v \in Z_{\delta,L'}$ has a solution.

Proof. We need to verify that the assumptions of the linear existence theorem 2.8 are met. It holds

$$\begin{aligned}\|\big(g^{\mu\nu}(t,v(t))\big) - \big(g^{\mu\nu}(t',v(t'))\big)\|_{e,s-1,\mathrm{ul}} \leq \nu|t-t'| + \theta E_s\big(v(t) - v(t')\big)\\ \text{with}\quad E_s\big(v(t) - v(t')\big) \leq \big(\delta + \|y_1\|_s + L'\big)|t-t'|\end{aligned} \tag{2.3.7}$$

derived from the conditions (2.3.2b), (2.3.2d) and the definition of the metric space $Z_{\delta,L'}$. Hence, the conditions (2.2.5a) and (2.2.5c) follow. Condition (2.2.5b) follows directly from condition (2.3.2a). The condition (2.2.5e) for f is satisfied since $f \in L^\infty([0,T], H^s)$. The continuity of f is shown in a manner analogous to that for the coefficients. □

The next proposition establishes second-order energy estimates for the linearized equation. To this end, first-order energy estimates taken from [Kat75] were adapted using ideas from [Tay96] and [SS98].

Proposition 2.13. *If u is a solution to the linearized equation (2.3.6), then we have the estimate*

$$E_{s+1}\big(u(t) - u_{00}\big) \leq e^{c_E\tau_s^2 C_2 t}\big(C^{1/2}E_{s+1}(\mathring{u} - u_{00}) + c_E\tau_s^2 C_1 t\big) \tag{2.3.8}$$

where

$$\begin{aligned}C_1 &= c\big((\mu+K)E_{s+2}(u_{00}) + K\big(1 + \big(\tfrac{1}{\lambda}K\big)^s\big)K_f + K^2\big(1 + \big(\tfrac{1}{\lambda}K\big)^s\big)E_{s+1}(u_{00})\big)\\ C_2 &= c\big(\mu + K + \nu + \theta(\delta + E_{s+1}(u_{00}) + L') + K^2\big(1 + \big(\tfrac{1}{\lambda}K\big)^s\big)\big)\end{aligned}$$

and the constant C is taken from Lemma 2.10.

In the next lemma it will be shown using the preceding energy estimate that the mapping Φ maps to the metric space $Z_{\delta,L'}$.

Lemma 2.14. *Φ maps $Z_{\delta,L'}$ to $Z_{\delta,L'}$ if L' and T' are chosen appropriately.*

Proof. Let $u = \Phi(v)$ for $v \in Z_{\delta,L'}$. To obtain the Lipschitz condition for $\partial_t u$ we estimate the second-order time derivative of u by using the modified equation (A.0.2). Using the bound for $(g^{00})^{-1}$ derived from Lemma 2.4 gives us a bound for $\partial_t^2 u$ dependent on the term $E_{s+1}(u(t) - u_{00})$ which is controlled by energy estimate (2.3.8).
In the limiting case $T' = 0$ the condition $u \in Z_{\delta,L'}$ reduces to

$$C^{1/2} E_{s+1}(\mathring{u} - u_{00}) < \delta \qquad (2.3.9a)$$

$$\text{and} \qquad c\left(1 + \left(\tfrac{1}{\lambda}K\right)^s\right)(2KC^{1/2}E_{s+1}(\mathring{u} - u_{00}) + 2KE_{s+1}(u_{00}) + K_f) < L'. \qquad (2.3.9b)$$

The first inequality (2.3.9a) is satisfied since $C^{1/2}E_{s+1}(\mathring{u} - u_{00}) \leq \rho \leq \delta/3$ by construction in Lemma 2.10. Condition (2.3.9b) can be satisfied choosing L' appropriately large. Due to the continuity of the right member of (2.3.8) the desired conditions hold for sufficiently small $T' > 0$. □

To show that Φ is a contraction we will show energy estimates for the difference equation of two solutions to the linearized equation.

Proposition 2.15. *Assume $u_1,\ u_2 \in Z_{\delta,L'}$ to be two solutions to the linear equation (2.3.6) for $v_1,\ v_2 \in Z_{\delta,L'}$ resp. Then the following estimate holds*

$$d(u_1, u_2) \leq c\, c_E T' e^{C_1 T'} \left(\theta_f + (E_{s+1}(u_{00}) + \delta + L')\theta'\right) d(v_1, v_2) \qquad (2.3.10)$$

with $C_1 = c\, c_E\big(\mu + K + \nu + \theta(\delta + E_{s+1}(u_{00}) + L')\big)$.

Proof. We consider the difference equation

$$g^{\mu\nu}(t, v_1)\partial_\mu\partial_\nu(u_1 - u_2) = \hat{f}$$
$$\text{with} \qquad \hat{f} = f(t, v_1) - f(t, v_2) + \big(g^{\mu\nu}(t, v_2) - g^{\mu\nu}(t, v_1)\big)\partial_\mu\partial_\nu u_2.$$

Following the proof of Proposition 2.13 we arrive at an energy estimate for the difference equation similar to (A.0.3). From the constants occurring we derive that, taking $u_{00} = 0$, only an estimate on $\|\hat{f}\|_{L^\infty L^2}$ is needed. We begin with the term including second-order derivatives of the solution u_2. They are bounded by the condition $u_2 \in Z_{\delta,L'}$ and the linearized equation solved by u_2 which gives us an estimate on the second time derivatives as in the proof of Lemma 2.14.

The assumptions (2.3.2f) and (2.3.2c) yield that the differences inheriting the RHS and the coefficients of the original equation can be estimated by a constant times the metric d. Hence, the desired result follows. □

We immediately obtain from the preceding energy estimate that the mapping Φ is a contraction.

Lemma 2.16. *Φ is a contraction w.r.t. the metric d, if T' is chosen small enough.*

We now turn to the proof of the existence theorem for quasilinear hyperbolic equations.

Proof of Theorem 2.9. From Lemma 2.11 we derive that the metric space $Z_{\delta,L'}$ is complete. The mapping Φ defined in (2.3.4) is defined on $Z_{\delta,L'}$ with values in $Z_{\delta,L'}$ for appropriate choices of the constants δ and L' showed in lemma 2.14. It is a continuous map since the solution to the linearized equation depends continuously on the initial data. We obtain from the preceding lemma that Φ is a contraction and therefore Banach's fixed point theorem yields the existence of a unique fixed point solving the quasilinear IVP (2.0.1). The proof of the differentiability property of such a solution is postponed to Lemma A.1. $\square$

Remark 2.17. To obtain a lower bound for the existence time T' of a solution observe that it has to satisfy the two inequalities

$$e^{c_1 T'}(c_2 + c_3 T') \leq \delta \qquad \text{and} \qquad c_4 T' e^{c_5 T'} \leq \zeta \quad \text{for a fixed } \zeta < 1 \tag{2.3.11}$$

with the constraint $c_2 \leq \delta/3$. Set

$$T' = \min\left(\tfrac{\delta}{3c_3}, \tfrac{1}{c_1}\ln(3/2), \tfrac{\zeta}{2c_4}, \tfrac{1}{c_5}\ln 2\right),$$

then previous inequalities (2.3.11) are satisfied. For convenience we record the constants occurring in this inequalities derived from the energy estimates (2.3.8) and (2.3.10)

$$\begin{aligned}
c_1 &= c\, c_E \tau_s^2\big(\mu + K + \nu + \theta(\delta + E_{s+1}(u_{00}) + L') + K^2\big(1 + \big(\tfrac{1}{\lambda}K\big)^s\big)\big)\\
c_3 &= c\, c_E \tau_s^2\big((\mu + K)E_{s+2}(u_{00}) + K\big(1 + \ \big(\tfrac{1}{\lambda}K\big)^s\big)K_f + K^2\big(1 + \big(\tfrac{1}{\lambda}K\big)^s\big)E_{s+1}(u_{00})\big)\\
c_4 &= c\, c_E\big(\theta_f + (E_{s+1}(u_{00}) + \delta + L')\theta'\big)\\
c_5 &= c\, c_E\big(\mu + K + \nu + \theta(\delta + E_{s+1}(u_{00}) + L')\big)
\end{aligned}$$

with $c_E = 2\big(\frac{2}{\mu} + \frac{1}{\lambda}\big)$.

Remark 2.18. If $\mathring{u} = (u_0, u_1) \in H^{s+2} \times H^{s+1}$ then we can choose $u_{00} = \mathring{u}$ in Lemma 2.10. A lower bound for the existence time T' is then given by

$$T' = \min\big(\tfrac{\delta}{2c_3}, \tfrac{1}{c_1}\ln(2), \tfrac{\zeta}{2c_4}, \tfrac{1}{c_5}\ln 2\big). \tag{2.3.12}$$

At the end of this section we will turn to higher regularity of solutions. We refer to [HKM76] and [FM72] for further reference.

Corollary 2.19. *Consider the Cauchy problem (2.0.1) with coefficients and a RHS having the domains described in (2.3.1). Let ℓ_0 and $r > \frac{m}{2} + 1 + \ell_0$ be integers. Suppose the coefficients and the RHS satisfy the assumptions of Theorem 2.9 with $s = r$. For $1 \leq \ell \leq \ell_0$ let the derivatives of the coefficients and the RHS of order ℓ satisfy the assumptions of Theorem 2.9 with $s = r - \ell$.*

Then the unique solution u to the IVP (2.0.1) has the property

$$u \in C^{2+\ell}\big([0,T'], H^{r-1-\ell}(\mathbb{R}^m, \mathbb{R}^N)\big) \quad \text{for } 0 \leq \ell \leq \ell_0. \tag{2.3.13}$$

Proof. The proof will be an induction on ℓ_0 and we will only show the first step, the general step follows in an analog manner.

Let $\ell_0 = 1$. Observe that from the existence Theorem 2.9 we obtain a solution u satisfying (2.3.3) with $s = r$ and therefore the second-order derivatives of u have the property

$$D^2 u,\ D\partial_t u \text{ and } \partial_t^2 u \in C\big([0,T'], H^{r-1}\big) \text{ with } r-1 > \tfrac{m}{2} + \ell_0 = \tfrac{m}{2} + 1.$$

By differentiating the equation solved by u we get that the resulting equation can be seen as a linear equation for $v = \partial_t u$. The second-order derivatives can be seen as RHS for the linear equation satisfying the conditions of the linear existence Theorem 2.8 with $s = r - 1$. From the uniqueness result of this theorem we obtain the desired differentiability property of $\partial_t u$. □

Remark 2.20. Due to the Sobolev embedding theorem it follows from property (2.3.13) that u is in fact a classical solution satisfying $u \in C^{2+\ell}([0,T'] \times \mathbb{R}^m)$ for $0 \le \ell \le \ell_0$.

2.3.1 Asymptotic equations

In this paragraph we discuss solutions to the IVP (2.0.1) which do not tend to 0 at infinity, but instead tend to a linear function. In the sequel, considerations follow roughly [HKM76] and [FM72] covering asymptotic solutions to the Einstein equations.

To make it precise let $w(t,x)$ be a linear function on $\mathbb{R}^m \times \mathbb{R}$ with $w(t,x) = w_0(x) + t w_1$. Denote the set of functions $v : \mathbb{R}^m \to \mathbb{R}^N$ satisfying $v - w_0 \in H^{s+1}$ by $H^{s+1}_{w_0}$ and let $H^s_{w_1}$ be defined analogously.

To simplify computations we use a special norm on uniformly Sobolev spaces. Let $\varphi \in C_c^\infty(\mathbb{R}^m)$ with $0 \le \varphi \le 1$ and $\int |\varphi|^2\, dx = 1$. Throughout this section we endow H^s_{ul} with the norm given by this test function as in lemma 2.2. The next lemma enlightens the role of this norm.

Lemma 2.21. *If $v \in H^s(\mathbb{R}^m, \mathbb{R}^N)$ and $v_0 \in \mathbb{R}^N$ is a constant vector, then $\|v + v_0\|_{s,\mathrm{ul}} \le \|v\|_s + |v_0|$.*

Proof. The claim follows from the observation

$$\int_{\mathbb{R}^m} \big|\varphi(y-x)\big(v(y) + v_0\big)\big|^2\, dx \le \|v\|_{L^2}^2 + |v_0|^2 + 2\|v\|_{L^2}\|\varphi_x v_0\|_{L^2}$$

where we used the Hölder inequality on the last term. □

We search for a solution u of the equation (2.0.1) where the coefficients and the RHS are assumed to be defined on $[0,T] \times \widetilde{W}$ with an open subset $\widetilde{W} \subset H^{s+1}_{w_0} \times H^s_{w_1}$.

Theorem 2.22. *Let*

$$g_{\mathrm{a}}^{\mu\nu}(t,\varphi_0,\varphi_1) := g^{\mu\nu}\big(t, w(t) + \varphi_0, Dw_0 + D\varphi_0, w_1 + \varphi_1\big) \tag{2.3.14}$$

and f_{a} be defined analogously. Assume $g_{\mathrm{a}}^{\mu\nu}$ and f_{a} to be defined on $[0,T] \times W \subset \mathbb{R} \times H^{s+1} \times H^s$ and to satisfy the assumptions of the existence Theorem 2.9.

Then there exists a constant $0 < T' \le T$ *and a unique solution* u *to the asymptotic IVP* (2.0.1) *with initial values* $(u_0, u_1) \in \widetilde{W} \subset H^{s+1}_{w_0} \times H^s_{w_1}$. *Further, it has the property*

$$u(t) - w(t) \in C([0,T'], H^{s+1}) \text{ and } \partial_t u - w_1 \in C([0,T'], H^s) \text{ for } 0 \le t \le T'. \tag{2.3.15}$$

Proof. The operators $g^{\mu\nu}_{\mathrm{a}}$ and f_{a} are supposed to be defined on $[0,T]\times W \subset \mathbb{R}\times H^{s+1}\times H^s$ and satisfy the assumptions of the existence theorem 2.9. It therefore follows the existence of a solution $\psi \in C([0,T], H^{s+1}) \cap C^1([0,T], H^s)$ to the IVP

$$g^{\mu\nu}_{\mathrm{a}}(t,\psi,\partial_t\psi)\partial_\mu\partial_\nu\psi = f_{\mathrm{a}}(t,\psi,\partial_t\psi),\ \psi\big|_{t=0} = u_0 - w_0,\ \partial_t\psi\big|_{t=0} = u_1 - w_1. \tag{2.3.16}$$

Let $u(t) = \psi(t) + w(t)$. Then we have $g^{\mu\nu}_{\mathrm{a}}(t,\psi,\partial_t\psi) = g^{\mu\nu}(t,u,Du,\partial_t u)$ and the same consideration holds for $f_{\mathrm{a}}(t,\psi,\partial_t\psi)$. Since $\partial_\mu\partial_\nu w(t) \equiv 0$ and $u\big|_{t=0} = \psi\big|_{t=0} + w_0$, $\partial_t u\big|_{t=0} = \partial_t\psi\big|_{t=0} + w_1$, the function u is a solution to the IVP (2.0.1). The differentiability claim of u follows from the property of ψ. □

Remark 2.23. Corollary 2.19 applies to the solution of the asymptotic equation (2.3.16) since the asymptotic is a linear function. the

2.4 Spatially local energy estimates

In this section we will develop uniqueness locally in space and time. To this end it is necessary to impose local conditions on the coefficients and the RHS of the quasilinear hyperbolic equation (2.0.1). The result will be obtained by means of local energy estimates in a similar way to the Sobolev-energy estimates of proposition 2.13.

Consider the coefficients $g^{\mu\nu}$ and the RHS f of equation (2.0.1) to be nonlinear operators defined as in (2.3.1) satisfying the assumptions of Theorem 2.9. We first fix some notations concerning the local energies to be used. Let $v = (v_0, v_1) \in W$ and define the energies $e_1(x,v)$ and $e_2(x,v)$ by

$$e_1(x,v) = |v_0(x)| + |v_1(x)| + |Dv_0(x)| \tag{2.4.1a}$$

$$\text{and} \quad e_2(x,v) = |v_0(x)|^2 + |v_1(x)|^2 + |Dv_0(x)|^2. \tag{2.4.1b}$$

Further we introduce a local energy adapted to the equation as follows

$$e(x,v) = |v_0|^2 + \langle -g^{00}v_1, v_1\rangle + \langle g^{ij}\partial_i v_0, \partial_j v_0\rangle.$$

The brackets $\langle\, .\, ,\, .\, \rangle$ denote the inner product of $\mathbb{R}^N$.

The assumptions on the coefficients give us equivalence of e to e_1 and e_2. We only state the parts of the equivalence which will be used in the sequel. It holds that

$$e_1^2 \le 4e_2 \le 4\big(1 + \max(\mu^{-1}, \lambda^{-1})\big)e.$$

Assume $u \in C([0,T'], H^{s+1}) \cap C^1([0,T'], H^s)$ to be a solution to the IVP (2.0.1) satisfying $(u, \partial_t) \in W$ for $0 \le t \le T'$. Suppose the RHS f satisfies the following additional local assumption

$$|f\big(t,x,v(x)\big)| \le K_f^{\mathrm{loc}} e_1(x,v) \tag{2.4.2}$$

for a function $v = (v_0, v_1) \in W$. Let $c_0 > 0$ be a constant to be chosen later and let $(t_0, x_0) \in [0,T] \times \mathbb{R}^m$ be arbitrary. Define the cone C and its section with the $\{t = \text{const}\}$-slices by

$$C = \{(x,t) \mid |x - x_0| < R(t)\} \quad \text{and} \quad C_t = \{x \mid |x - x_0| < R(t)\} \tag{2.4.3}$$

where $R(t) = -c_0(t - t_0)$. We want to show that the value of the solution u in the vertex (t_0, x_0) of the cone only depends on the values of u in the cone C.

To this end we will consider the quantity

$$E(t) = E\big(x, u(t,x), \partial_t u(t,x)\big) := \int_{C_t} e\big(x, u(x), \partial_t u(x)\big)\, dx$$

the integral of the adapted energy e over a spatial part of the previously introduced cone. To estimate the deviation of $E(t)$ in time we consider the $\limsup_t$. Following [Eva98] we make use of the Co-area formula adapted to our case. If v is a continuous real-valued function defined on $\mathbb{R}^m$, then

$$\frac{d}{dt}\bigg(\int_{C_t} v\, dx\bigg) = -c_0 \int_{\partial C_t} v\, dS.$$

Thus

$$\begin{aligned}
&\limsup_\tau \tfrac{1}{\tau}\big(E(t+\tau) - E(t)\big) = \\
&\qquad \int_{C_t}\bigg(2\langle u, \partial_t u\rangle - \langle \limsup_\tau \tfrac{1}{\tau}\big(g^{00}(u(t+\tau)) - g^{00}(u(t))\big)\partial_t u, \partial_t u\rangle \\
&\qquad\qquad + \langle \limsup_\tau \tfrac{1}{\tau}\big(g^{ij}(u(t+\tau)) - g^{ij}(u(t))\big)\partial_i u, \partial_j u\rangle \\
&\qquad\qquad - 2\langle g^{00}\partial_t^2 u, \partial_t u\rangle + 2\langle g^{ij}\partial_i\partial_t u, \partial_j u\rangle \bigg)\, dx - c_0 \int_{\partial C_t} e(t)\, dS
\end{aligned} \tag{2.4.4}$$

The $\limsup_t$ applied to the coefficients can be estimated via

$$|u(t) - u(t')|_s \leq |\partial_t u|_{L^\infty H^s} |t - t'|$$

and an analogous result holds for $\partial_t u$. Therefore

$$\limsup_\tau \tfrac{1}{\tau}\big|(g^{\mu\nu})(t+\tau, u(t+\tau)) - (g^{\mu\nu})(t, u(t))\big| \leq c\nu|\tau| + c\theta\big(|\partial_t u|_{L^\infty H^s} + |\partial_t^2 u|_{L^\infty H^{s-1}}\big).$$

We calculate via integration by parts taking care of the boundary terms

$$\begin{aligned}
\int_{C_t} \langle g^{ij}\partial_i\partial_j u, \partial_t u\rangle + \langle g^{ij}\partial_i\partial_t u, \partial_j u\rangle = \int_{C_t} \big(&\langle g^{ij}\partial_i\partial_j u, \partial_t u\rangle - \langle g^{ij}\partial_i\partial_j u, \partial_t u\rangle \\
&- \langle \partial_i g^{ij}\partial_t u, \partial_j u\rangle\big)\, dx + \int_{\partial C_t} \langle g^{ij}\partial_j u, \partial_t u\rangle \nu_i\, dS
\end{aligned}$$

where ν denotes the unit outer normal to ∂C_t. Analogously we obtain

$$\int_{C_t} 2\langle g^{0j}\partial_j\partial_t u, \partial_t u\rangle \, dx = -\int_{C_t} \langle \partial_j g^{0j}\partial_t u, \partial_t u\rangle \, dx + \int_{\partial C_t} \langle g^{0j}\partial_t u, \partial_t u\rangle \nu_j \, dS.$$

These identities yield

$$\begin{aligned}\limsup_{\tau} \tfrac{1}{\tau}\big(E(t+\tau)-E(t)\big) \le\; & -\int_{C_t} \Big(\langle \partial_j g^{0j}\partial_t u, \partial_t u\rangle + \langle \partial_i g^{ij}\partial_t u, \partial_j u\rangle\Big)\, dx \\ & + \int_{C_t} \Big(c\theta\big(|\partial_t u|_{L^\infty H^s} + |\partial_t^2 u|_{L^\infty H^{s-1}}\big)\big(|\partial_t u|^2 + |Du|^2\big) + \langle 2u - f, \partial_t u\rangle\Big)\, dx \\ & + \int_{\partial C_t} \Big(-c_0\, e(t) + 2\langle g^{ij}\partial_j u, \partial_t u\rangle \nu_i + 2\langle g^{0j}\partial_t u, \partial_t u\rangle \nu_j\Big)\, dS\end{aligned}$$

It now follows from the local condition (2.4.2) for the RHS and the equivalence of the energies $e_2(t)$ and $e(t)$ that

$$\begin{aligned}\limsup_{\tau} \tfrac{1}{\tau}\big(E(t+\tau)-E(t)\big) \le\; & -\int_{C_t} \big(\langle \partial_j g^{0j}\partial_t u, \partial_t u\rangle + \langle \partial_i g^{ij}\partial_t u, \partial_j u\rangle\big)\, dx \\ + \int_{C_t} \Big(\big[c\theta\big(|\partial_t u|_{L^\infty H^s} + |\partial_t^2 u|_{L^\infty H^{s-1}}\big) & + 8(1+K_f^{\mathrm{loc}})\big]\big(1+\max(\mu^{-1},\lambda^{-1})\big)e(t)\Big)\, dx \\ + \int_{\partial C_t} \Big(-c_0\, e(t) + 2\langle g^{ij}\partial_j u, \partial_t u\rangle \nu_i & + 2\langle g^{0j}\partial_t u, \partial_t u\rangle \nu_j\Big)\, dS.\end{aligned}$$

The first integral can be estimated by condition (2.3.2a) using the Sobolev embedding arriving at

$$\langle \partial_j g^{0j}\partial_t u, \partial_t u\rangle + \langle \partial_i g^{ij}\partial_t u, \partial_j u\rangle \le cKe_1^2.$$

We infer from this inequality an estimate for the deviation of the energy $E(t)$ by term involving the energy itself and boundary terms as follows

$$\begin{aligned}\limsup_{\tau} \tfrac{1}{\tau}\big(E(t+\tau)-E(t)\big) \le & \int_{\partial C_t} \Big(-c_0\, e(t) + 2\langle g^{ij}\partial_j u, \partial_t u\rangle \nu_i + 2\langle g^{0j}\partial_t u, \partial_t u\rangle \nu_j\Big)\, dS \\ + \big(4cK + c\theta\big(|\partial_t u|_{L^\infty H^s} & + |\partial_t^2 u|_{L^\infty H^{s-1}}\big) + 8(1+K_f^{\mathrm{loc}})\big)\big(1+\max(\mu^{-1},\lambda^{-1})\big)E(t).\end{aligned}$$

The idea is now to adjust the constant c_0 such that the boundary term becomes non-positive. Denoting components of $\mathbb{R}^N$ by the index A it follows for the second boundary term by using the generalized Hölder inequality

$$|\nu_i g^{ij}\partial_j u^A| \le \big(g^{ij}\partial_i u^A \partial_j u^A\big)^{1/2}\big(g^{ij}\nu_i\nu_j\big)^{1/2}.$$

The last term can be estimated by $|(g^{ij})|_e^{1/2}$ since ν has unit Euclidean length. Summation over A gives us

$$\Big(\sum_A |\nu_i g^{ij}\partial_j u^A|^2\Big)^{1/2} \le |(g^{ij})|_e^{1/2}\Big(\sum_A g^{ij}\partial_i u^A \partial_j u^A\Big)^{1/2} = |(g^{ij})|_e^{1/2}\langle g^{ij}\partial_i u, \partial_j u\rangle^{1/2}.$$

Hence
$$2\langle g^{ij}\partial_j u, \partial_t u\rangle \nu_i \le 2|(g^{ij})|_e^{1/2}|\partial_t u|\langle g^{ij}\partial_i u, \partial_j u\rangle^{1/2}.$$
By a similar device we conclude $2\langle g^{0j}\partial_t u, \partial_t u\rangle\nu_j \le 2|(g^{0j})|_e|\partial_t u|^2$ and this yields for the integrand of the boundary term

$$\begin{aligned}
&- c_0\, e(t) + 2\langle g^{ij}\partial_j u, \partial_t u\rangle\nu_i + 2\langle g^{0j}\partial_t u, \partial_t u\rangle\nu_j \\
&\le\ - c_0\, e(t) + 2|(g^{0j})|_e|\partial_t u|^2 + 2|(g^{ij})|_e^{1/2}|\partial_t u|\langle g^{ij}\partial_i u, \partial_j u\rangle^{1/2} \\
&\le -c_0\, e(t) + 2|(g^{0j})|_e\tfrac{1}{\lambda}\langle -g^{00}\partial_t u, \partial_t u\rangle + 2|(g^{ij})|_e^{1/2}\big(\tfrac{1}{\lambda}\langle -g^{00}\partial_t u, \partial_t u\rangle\big)^{1/2}\langle g^{ij}\partial_i u, \partial_j u\rangle^{1/2} \\
&\le -c_0\, e(t) + \big(2|(g^{0j})|_e + |(g^{ij})|_e\big)\tfrac{1}{\lambda}\langle -g^{00}\partial_t u, \partial_t u\rangle + \langle g^{ij}\partial_i u, \partial_j u\rangle, \qquad (2.4.5)
\end{aligned}$$

where we used the condition $g^{00} \le -\lambda$. The Sobolev embedding provides us with a local estimate $|(g^{\mu\nu})|_e \le c|(g^{\mu\nu})|_{e,s,\mathrm{ul}} \le cK$. By choosing

$$c_0 = 1 + 3cK\tfrac{1}{\lambda}, \qquad (2.4.6)$$

the desired non-positivity of (2.4.5) follows. Here, the constant c only depends on the embedding $H^{s-1}_{\mathrm{ul}} \hookrightarrow C^0_b$. Applying Gronwall's lemma yields the following result.

Lemma 2.24. *Let c_0 be defined by (2.4.6) and $0 \le t_0 < t_1$. Then the following estimate holds*

$$\begin{aligned}
&E(t) \le e^{C_1(t_1-t_0)}E(t_0) \quad \text{for } 0 \le t_0 \le t \le t_1 \qquad (2.4.7)\\
&\text{with}\quad C_1 = \big(4cK + c\theta\big(|\partial_t u|_{L^\infty H^s} + |\partial_t^2 u|_{L^\infty H^{s-1}}\big) + 8(1 + K_f^{\mathrm{loc}})\big)\big(1 + \max(\mu^{-1}, \lambda^{-1})\big).
\end{aligned}$$

A consequence of this estimate is that a solution depends only on its values within the cone C introduced in (2.4.3).

Theorem 2.25 (Domain of dependence)**.** *Assume the nonlinear operators $g^{\mu\nu}$ and f defined in (2.3.1) satisfy the assumptions of Theorem 2.9. Suppose the local condition (2.4.2) for the RHS holds and set c_0 as in (2.4.6). Let the initial data satisfy $u_0 \equiv 0$ and $u_1 \equiv 0$ on $B_{c_0 t_0}(x_0)$.*

Then $u \equiv 0$ on the closure of the cone with base $B_{c_0 t_0}(x_0)$ and vertex (t_0, x_0).

Proof. From the local energy estimate (2.4.7) we derive $E(t) = 0$ for $0 \le t \le t_0$ since $E(0) = 0$. It follows from the assumptions on the coefficients that $\partial_t u \equiv 0$ and $Du \equiv 0$ on the spatial part C_t for all t. The vanishing of the initial values gives the desired result. □

Theorem 2.25 immediately yields the following local uniqueness result.

Theorem 2.26 (Local Uniqueness)**.** *Let the coefficients $g^{\mu\nu}$ and the RHS f of the hyperbolic equation (2.0.1) satisfy the assumptions of the existence theorem 2.9. Suppose $g^{\mu\nu}$ and f admit constants θ_f^{loc} and θ^{loc} satisfying*

$$\begin{aligned}
\big|f\big(t,x,v(x)\big) - f\big(t,x,w(x)\big)\big| &\le \theta_f^{\mathrm{loc}} e_1(x, v-w) \\
\big|\big(g^{\mu\nu}(t,x,v(x))\big) - \big(g^{\mu\nu}(t,x,w(x))\big)\big| &\le \theta^{\mathrm{loc}} e_1(x, v-w)
\end{aligned} \qquad (2.4.8)$$

for $v, w \in W$ and e_1 defined in (2.4.1a). Let c_0 be defined as in (2.4.6). Assume u_1 and u_2 to be two solutions of the hyperbolic equation (2.0.1) satisfying $u_1\big|_{t=0} = u_2\big|_{t=0}$ and $\partial_t u_1\big|_{t=0} = \partial_t u_2\big|_{t=0}$ on $B_R(x_0)$.

Then $u_1 = u_2$ on the closure of the cone with base $B_R(x_0)$ and vertex $(R/c_0, x_0)$.

Proof. Consider the difference equation

$$g^{\mu\nu}(u_1)\partial_\mu\partial_\nu(u_1 - u_2) = f(u_1) - f(u_2) + \big(g^{\mu\nu}(u_2) - g^{\mu\nu}(u_1)\big)\partial_\mu\partial_\nu u_2. \tag{2.4.9}$$

The result follows from applying Theorem 2.25 to this equation. □

Remark 2.27. From the proof of the energy estimate (2.4.7) we derive that the slope c_0 of the cone on which uniqueness holds depends on the coefficients of the equation stated in (2.0.1). Therefore, as can be seen from the difference equation (2.4.9), c_0 only depends on the solution u_1. By interchanging the role of the two solutions it is possible to achieve dependency only on u_2.

3 Minkowski space

In this section we will construct a solution to the Cauchy problem (1.3.2) for an immersion of the initial submanifold in the case where the ambient manifold equals the $(n+1)$-dimensional Minkowski space $\mathbb{R}^{n,1}$. Only fixed direction, lapse and shift will be treated; they will be combined in a timelike vector field serving as initial velocity.

The layout of this section is as follows. In section 3.1 a graph representation for spacelike submanifolds in Minkowski space will be developed independently of the codimension. A first solution to the Cauchy problem will be given in section 3.2 for given coordinates. It will provide a connection between assumptions on the initial values in given coordinates and the existence theory for quasilinear second-order hyperbolic equations. In section 3.3 we will derive a solution to the Cauchy problem (1.3.2).

The standard basis of the Minkowski space will be denoted by $\tau_0, \dots, \tau_n$, where τ_0 denotes the timelike direction. The metric on $\mathbb{R}^{n,1}$ will be denoted by $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$ or η with components $\eta_{AB} = \operatorname{diag}(-1, 1, \dots, 1)$ and we set $\varepsilon_A = \eta_{AA}$. Recall that we use the notation $\langle v \rangle = \big(-\langle\!\langle v, v\rangle\!\rangle\big)^{1/2}$ for a timelike vector v.

3.1 Special graph representation

In this section we develop coordinates on the initial submanifold Σ_0. This will be done by considering a graph representation similar to representations for hypersurfaces of Euclidean space derived by A. Stahl in [Sta96].

Let Σ_0 be a regularly immersed submanifold of dimension m. Assume M to be an m-dimensional manifold and $\varphi : M \to \mathbb{R}^{n,1}$ to be an immersion with $\operatorname{im}\varphi = \Sigma_0$. Let $p_0 \in M$ and $z : U \subset M \to \Omega \subset \mathbb{R}^m$ be a chart centered at p_0. Since φ is a spacelike immersion we have that the mapping $\varphi \circ z^{-1}$ has rank m and we only need a renumbering of the spacelike directions to get

$$\det\big(d(\varphi^1 \circ z^{-1}, \dots, \varphi^m \circ z^{-1})\big)(0) \neq 0.$$

The inverse function theorem now yields the existence of a diffeomorphism

$$\Psi = (\varphi^1 \circ z^{-1}, \dots, \varphi^m \circ z^{-1})^{-1} : V \subset \mathbb{R}^m \to W \subset \Omega.$$

By setting $x = \Psi^{-1} \circ z$ we get a graph representation $\varphi \circ x^{-1}(w) = w^j \tau_j + u^\alpha(w)\tau_\alpha$ in a neighborhood of p_0. Applying an orthogonal transformation to the spacelike directions of the Minkowski space gives us the property $Du^a(0) = 0$ for all a, where a denotes the spacelike directions of the graph. Together with this orthogonal transformation we derive the representation

$$\Phi(w) := v \circ \varphi \circ x^{-1}(w) = w^j \tau_j + u^\alpha(w)\tau_\alpha \tag{3.1.1}$$

where v is a choice of coordinates for the Minkowski space. A representation of the geometry is as follows:

tangent vectors	$\partial_j \Phi^A \tau_A = \tau_j + \partial_j u^\alpha \tau_\alpha$	(3.1.2a)
normal vectors	$N_\alpha = \tau_\alpha - \varepsilon_\alpha \partial_k u_\alpha \delta^{k\ell} \tau_\ell$	(3.1.2b)
induced metric	$\mathring{g}_{ij} = \delta_{ij} + \partial_i u^\alpha \eta_{\alpha\beta} \partial_j u^\beta$	(3.1.2c)
induced Christoffel symbols	$\mathring{\Gamma}^k_{ij} = g^{\ell k} \partial_i \partial_j u^\alpha \eta_{\alpha\beta} \partial_\ell u^\beta$	(3.1.2d)

and the second fundamental form:

$$\mathring{I\!I}_{ij} = \partial_i \partial_j u^\alpha \tau_\alpha - \mathring{g}^{\ell k} \partial_i \partial_j u^\alpha \eta_{\alpha\beta} \partial_\ell u^\beta (\tau_k + \partial_k u^\delta \tau_\delta). \tag{3.1.3}$$

The choice of the normal vectors N_α is based on the construction of Gram-Schmidt. Instead of using the tangential vectors $\tau_j + \partial_j u^\alpha \tau_\alpha$ we only use the vectors τ_j to develop the tangential part of τ_α. We take all vectors τ_α which are normal to the domain of the graph and subtract the tangential part as follows

$$\langle\!\langle \tau_\alpha, \tau_\ell + \partial_\ell u^\beta \tau_\beta \rangle\!\rangle = \varepsilon_\alpha \partial_\ell u_\alpha \overset{!}{=} \langle\!\langle \lambda^k \tau_k, \tau_\ell + \partial_\ell u^\beta \tau_\beta \rangle\!\rangle = \lambda^k \delta_{k\ell}.$$

This consideration gives us the normal vectors N_α as defined above.

The following lemma establishes control over the eigenvalues of the induced metric $\mathring{g}_{ij}$ involving the graph functions.

Lemma 3.1. *The components $\mathring{g}_{ij}$ defined in (3.1.2c) of the induced metric satisfies*

$$(1 - |Du^0|^2_e)\delta_{ij} \leq \mathring{g}_{ij} \leq (1 + |Du|^2_e)\delta_{ij}. \tag{3.1.4}$$

Proof. The second estimate comes from a straight-forward calculation using the definition (3.1.2c).

To show the positive definiteness we examine the last term of the definition. It reads

$$-\partial_i u^0 \partial_j u^0 + \textstyle\sum_a (Du^a \otimes Du^a)_{ij},$$

where the index a denotes the spacelike directions normal to the submanifold. The second term is ≥ 0, it remains therefore to consider the first term. An orthogonal transformation $O = (v_1, \ldots, v_m)$ with $v_1 = \frac{Du^0}{|Du^0|_e}$ gives us

$$\big(O^*(\delta_{ij} - \partial_i u^0 \partial_j u^0)O\big)_{k\ell} = \delta_{k\ell} - Du^0[v_k]Du^0[v_\ell] = \delta_{k\ell} - \delta_{k\ell 1}|Du^0|^2$$

and the desired estimate follows. □

Remark 3.2. At the origin we have the condition $Du^a(0) = 0$ for all a, the proof therefore yields that $1 - |Du^0(0)|^2$ is the smallest eigenvalue of $\mathring{g}_{ij}(0)$. Since the submanifold is assumed to be spacelike and the graph representation is valid at least in a neighborhood of the origin, we derive $|Du^0(0)| < 1$.

The next lemma will establish a bound for second derivatives of the graph functions in terms of first derivatives and the norm of the second fundamental form of the immersion.

Lemma 3.3. *The following estimate holds*

$$|D^2 u|_e \leq m^{1/2}(n+1-m)^{1/2}\big(1+|Du|^2\big)^2 |\mathring{I\!I}|_{\mathring{g},e}. \tag{3.1.5}$$

Proof. Let $n_{\alpha\beta} = \langle\!\langle N_\alpha, N_\beta \rangle\!\rangle$ and let h^α_{ij} be the coefficients of the second fundamental form w.r.t. the chosen normal vectors N_α. From representation (3.1.3) for the second fundamental form we get $h^\alpha_{ij} = n^{\alpha\beta}\epsilon_\beta \partial_i \partial_j u_\beta$. We further set $e_{\alpha\beta} = e(N_\alpha, N_\beta)$ with $e_{\alpha\beta} \geq C_e \delta_{\alpha\beta}$, where e denotes the Euclidean metric on $\mathbb{R}^{n+1}$. The value of C_e and the invertability of the matrix $n_{\alpha\beta}$ will be shown later.

We derive from the expression $|\mathring{I\!I}|^2_{\mathring{g},e} = \mathring{g}^{ij}\mathring{g}^{\ell k} h^\alpha_{i\ell} h^\beta_{jk} e_{\alpha\beta}$ that

$$|D^2 u|^2_e \leq |(\mathring{g}_{ij})|^2_e |(n_{\alpha\beta})|^2_e C_e^{-1} |\mathring{I\!I}|^2_{\mathring{g},e} \tag{3.1.6}$$

From a consideration similar to the proof of Lemma 3.1 we get $n_{\alpha\beta} \leq (1+|Du|^2_e)\delta_{\alpha\beta}$. This yields for the norms of $\mathring{g}_{ij}$ and $n_{\alpha\beta}$

$$|(\mathring{g}_{ij})|_e \leq m^{1/2}(1+|Du|^2_e) \quad \text{and} \quad |(n_{\alpha\beta})|_e \leq (n+1-m)^{1/2}(1+|Du|^2_e).$$

It remains to estimate the matrices $e_{\alpha\beta}$ and $n_{\alpha\beta}$. We compute $n_{00} = -1 + |Du^0|^2$ and $n_{ab} = \delta_{ab} + \partial_k u_a \delta^{k\ell} \partial_\ell u_b$. From Remark 3.2 and Lemma B.1 we get that $n_{\alpha\beta}$ is invertible in a neighborhood of 0. To establish the positive definiteness of $e_{\alpha\beta}$ let $(v^\alpha) \in \mathbb{R}^{n+1-m}$, then it follows that

$$v^\alpha e_{\alpha\beta} v^\beta \geq |v|^2 + \big(|v^0||Du^0| - |w|\big)^2$$

where we used the abbreviation $w = (w_j)$ and $w_j = v^a \partial_j u_a$. Therefore, the desired result follows with $C_e = 1$. □

At this stage it is convenient to introduce the following definition.

Definition 3.4. A submanifold Σ_0 of the Minkowski space $\mathbb{R}^{n,1}$ is called *uniformly spacelike with bounded curvature*, if there exist constants ω_1, C_0 such that

$$\inf\{-\eta(\gamma,\tau_0) : \gamma \text{ timelike future-directed unit normal to } \Sigma_0\} \le \omega_1$$
$$\text{and} \quad |\mathring{I\!I}|_{\mathring{g},e} \le C_0.$$

Remark 3.5. By abuse of notation we use $\mathring{g}$ to denote the induced metric on Σ_0 and on M as well as we use $\mathring{I\!I}$ to denote the second fundamental form of Σ_0 and of the immersion φ as a tensor along φ.

From now on we suppose the submanifold Σ_0 to be uniformly spacelike with bounded curvature. The following lemma provides a definition of a unit timelike normal for which the assumption on unit timelike normals to the submanifold Σ_0 of Definition 3.4 can be used.

Lemma 3.6. *Set $\nu_0 = \langle N_0\rangle^{-1} N_0$ with the normal vector N_0 defined in (3.1.2b). Then it follows that $-\eta(\nu_0(0), \tau_0) \le \omega_1$.*

Proof. We consider a perturbation $\gamma = \langle \nu_0 + \lambda^a N_a\rangle^{-1}(\nu_0 + \lambda^a N_a)$ of ν_0 with scalars λ^a. At the origin we have $Du^a(0) = 0$ for all a and therefore $N_a(0) = \tau_a$ for all a. This yields $\langle \nu_0 + \lambda^a N_a\rangle^2 = 1 - \lambda^a \delta_{ab} \lambda^b$ since $\nu_0 \perp \tau_a$. Hence

$$-\eta(\tau_0, \gamma) = (1 - \lambda^a \delta_{ab} \lambda^b)^{-1/2}\big(-\eta(\tau_0, \nu_0)\big) > -\eta(\tau_0, \nu_0)$$

if $(\lambda^a) \neq 0$. ☐

The Definition 3.4 yields the estimate

$$-\eta\big(\tau_0, \nu_0(0)\big) = \langle N_0(0)\rangle^{-1} = (1 - |Du^0(0)|^2)^{-1/2} \le \omega_1. \tag{3.1.7}$$

The following lemma is the key ingredient for developing estimates on the graph representation of uniformly spacelike submanifolds with bounded curvature. it follows that the strategy of [Sta96] establishing similar estimates for Euclidean hypersurfaces.

Lemma 3.7. *Let $\lambda \ge 800$ and $\rho_1 > 0$ be an arbitrary constant. Suppose Σ_0 has bounded curvature with a bound C_0 of the form $C_0 = (2\lambda m^{1/2}(n+1-m)^{1/2}\rho_1)^{-1}$. Assume x to be the coordinates on M which are part of the graph representation (3.1.1). Furthermore, suppose $y \in \mathbb{R}^m$ such that $|y| = r < \rho_1$ and the set $\{\tau y : 0 \le \tau \le 1\}$ to be contained in the image of x. Define a function v by $v(z) = \frac{1}{\sqrt{1-|Du^0|^2}}$ then the following inequalities hold*

$$\big(1 + |Du(y)|_e^2\big)^{1/2} < (1 + 2|Du^0(0)|^2)^{1/2} + \tfrac{1}{2}\big(\tfrac{r}{\lambda\rho_1}\big)^2 \le B_1 \tag{3.1.8a}$$
$$|D^2 u|_e \le \tfrac{1}{2\lambda\rho_1} B_1^4 \tag{3.1.8b}$$
$$v(y) < \omega_1 + \big(1 + B_1^4 \tfrac{r}{2\lambda\rho_1}\big) B_1^4 \tfrac{r}{\lambda\rho_1} \tag{3.1.8c}$$

where $B_1 = 3^{1/2} + \frac{1}{2\lambda^2}$.

Proof. To show the first inequality we consider the bound (3.1.5) for the second derivatives of the graph functions u^α. The expression of the bound C_0 for the curvature yields

$$|D^2u|_e \le \tfrac{1}{2\lambda\rho_1}(1+|Du|^2)^2. \tag{3.1.9}$$

From the mean value theorem along the direction $e_y = \frac{y}{|y|_e}$ and the Hölder inequality we get

$$|\partial_j u^\alpha(re_y)| \le |\partial_j u^0(0)| + \left(\textstyle\int_0^r \sum_i |\partial_i\partial_j u^\alpha|^2 \, d\tau \, r\right)^{1/2}$$

since $Du^a(0) = 0$. Summation over j and α gives us

$$|Du(re_y)| \le |Du^0(0)| + \left(\int_0^r |D^2u|_e^2 \, d\tau \, r\right)^{1/2}. \tag{3.1.10}$$

This yields the following integral inequality

$$1+|Du(y)|^2 \le 1 + 2|Du^0(0)|^2 + \tfrac{r}{2\lambda^2\rho_1^2}\int_0^r (1+|Du|^2)^4 \, d\tau.$$

We solve the corresponding ODE giving us the solution $f(t) = \left(t_0 - 3\frac{r}{2\lambda^2\rho_1^2}\, t\right)^{-1/3}$. We want to use a Taylor expansion for $f^{1/2}$; it is therefore necessary to estimate the derivative. This can be done via the choice of the parameter λ, so that it follows that

$$f'(\xi) \le \tfrac{r}{2\lambda^2\rho_1^2}\left(t_0 - 3\tfrac{r}{2\lambda^2\rho_1^2}\, t\right)^{-1/3} \le 2\,\tfrac{r}{2\lambda^2\rho_1^2} \text{ for } 0 \le \xi < r$$

The Taylor expansion for $f^{1/2}$ now yields the desired estimates (3.1.8a) and (3.1.8b).

We will now consider the function v. Observe that this function is bounded at the origin by virtue of Lemma 3.6, namely by estimate (3.1.7). Again we will use an ODE comparison argument, therefore we need to estimate the derivative of v. it follows that

$$|Dv| \le v^3\left(\textstyle\sum_j |\sum_i \partial_i u^0 \partial_i\partial_j u^0|^2\right)^{1/2} \le v^3|Du^0||D^2u^0|.$$

From the inequality (3.1.10) we derive together with the bound (3.1.8b) for the second derivatives

$$|Dv| \le v^3\Lambda \quad \text{with} \quad \Lambda = \left(|Du^0(0)| + B_1^4\tfrac{r}{2\lambda\rho_1}\right)B_1^4\tfrac{1}{2\lambda\rho_1}$$

This gives us an integral inequality analog of the above case. The corresponding ODE is solved by $\tilde{f}(t) = (w_1 - 2\Lambda t)^{-1/2}$. Analogously to the argument for the function f we estimate $(\omega_1 - 2\Lambda t)^{-3/2}$ by 2, which is possible due to the value of λ. This yields the last estimate. $\square$

Remark 3.8. Assume we have an uniformly spacelike submanifold with an extrinsic curvature bounded by an arbitrary constant C_0. If we set

$$\rho_1 = (2\lambda m^{1/2}(n+1-m)^{1/2}C_0)^{-1},$$

then Lemma 3.7 applies to this general situation.

Remark 3.9. Lemma 3.7 also shows that for an uniformly spacelike submanifold with bounded curvature the graph representation as described above exists at least in the ball $B_{\rho_1}(0)$. There are two reasons for the graph representation to fail. The first one is the unboundedness of $|Du^a|$ for a spacelike direction and the second one is the convergence $|Du^0| \to 1$ so that the matrix $\mathring{g}_{ij}$ becomes degenerate.

The previous lemma shows that these two possibilities can not occur as long as we stay in the Euclidean ball with radius ρ_1 about 0. In fact, it gives us control over the eigenvalues of the induced metric within this ball

$$\tilde{G}_1 \delta_{ij} \le \mathring{g}_{ij} \le \tilde{G}_2 \delta_{ij} \tag{3.1.11a}$$

$$\text{with} \quad \tilde{G}_1 = \left[\omega_1 + \left(1 + B_1^4 \tfrac{1}{2\lambda}\right) B_1^4 \tfrac{1}{\lambda}\right]^{-2} \text{ and } \tilde{G}_2 = B_1^2. \tag{3.1.11b}$$

Observe that these constants are independent of the origin p_0. In particular estimate (3.1.11a) leads to a comparison of Euclidean balls and balls w.r.t. $\mathring{g}$. It holds that

$$B^e_{\tilde{G}_2^{-1/2} r}\big(x(p)\big) \subset x\big(B_r^{\mathring{g}}(p)\big) \subset B^e_{\tilde{G}_1^{-1/2} r}\big(x(p)\big) \tag{3.1.12}$$

for a radius $r > 0$ such that the middle term makes sense.

We will need estimates for higher derivatives of the graph functions depending only on curvature bounds. We begin with a representation of the partial derivatives of the second fundamental form in terms of the covariant derivative, the Christoffel symbols of the ambient space and the induced one

Lemma 3.10. *Set $A = \{\mathring{I\!I}{}^B_{ij}\}$. Then the following identity holds*

$$\partial^k A = \widehat{\nabla}^k A + \sum \partial^{\alpha_1} \mathring{\Gamma} * \cdots * \partial^{\alpha_p} \mathring{\Gamma} * \partial^{\beta_1} \widehat{\Gamma} * \cdots * \partial^{\beta_q} \widehat{\Gamma} * \widehat{\nabla}^\ell A \tag{3.1.13}$$

where the sum ranges over a certain subset of all tuples

$$(p, \alpha_i, q, \beta_j, \ell) \quad \textit{satisfying} \quad p + \sum \alpha_i + q + \sum \beta_j + \ell = k + 1.$$

Proof. The proof will be an induction, so let $k = 1$. Then the definition (1.2.1) and the expression (1.2.5) yield

$$\widehat{\nabla} A = \partial A + A * \widehat{\Gamma} + A * \mathring{\Gamma}.$$

We have $p = q = 1$ and therefore $p + q = k + 1$.

$k \to k+1$: We compute $\partial^{k+1} A = \partial \partial^k A$ and insert the assumption on k which leaves us with considering the derivative of the sum. We begin with the last term involving derivatives of A:

$$\partial \widehat{\nabla}^\ell A = \widehat{\nabla}^\ell A + \widehat{\nabla}^\ell A * \widehat{\Gamma} + \widehat{\nabla}^\ell A * \mathring{\Gamma}$$

The terms on the RHS can be described by $\ell \to \ell + 1$, $p \to p + 1$ or $q \to q + 1$, therefore they fit into the pattern for the sum replacing k by $k + 1$. The rest of the sum can be treated with the product rule increasing one of the indices α_i or β_j. □

The following lemma establishes estimates for higher derivatives of the graph functions u^α by inequalities for the terms involved in the preceding lemma. It will be done only for the Minkowski space.

Lemma 3.11. *Let k be an integer and assume the immersion φ to satisfy the following.*

There exist constants $C^\varphi_0, \dots, C^\varphi_k$ such that $|\widehat{\nabla}^\ell \mathring{I\!I}|_{\mathring{g},e} \le C^\varphi_\ell$ for $0 \le \ell \le k$.

Then there exists a constant C^u_{k+2} such that $|D^{k+2}u|_e \le C^u_{k+2}$.

Proof. According to $\mathring{I\!I}{}^B_{ij} = h^\alpha_{ij} N^B_\alpha$ we have the expression

$$\varepsilon_\beta \partial^k \partial_i \partial_j u_\beta = \partial^k \mathring{I\!I}{}^B_{ij}\, \eta_{BC}\, N^C_\beta + \sum_{\substack{k_1+k_2=k,\\ k_1<k}} \partial^{k_1} \mathring{I\!I}{}^B_{ij}\, \eta_{BC}\, \partial^{k_2} N^C_\beta . \tag{3.1.14}$$

Since second derivatives are involved in the representation of the second fundamental form and the definition of the normal N_α only contains first derivatives of the graph function we have to estimate the first term on the RHS to apply an induction. The case $k = 0$ was done in Lemma 3.7.

We start with the expression (3.1.13) for derivatives of the second fundamental form. Since we work in Minkowski space only the induced Christoffel symbols needs to be treated. From expression (3.1.2d) we get that derivatives of order ℓ can be estimated by derivatives of the graph function up to order $\ell + 2$ and in (3.1.13) there occur only derivatives up to order $k - 1$. The matrix $\mathring{g}^{ij}$ occurring in the Christoffel symbols can be estimated by Corollary B.4 using inequality (3.1.11a). It is a lower order term involving first derivatives of the graph functions.

The Euclidean norm of the derivatives of the second fundamental form can be estimated using the comparison (3.1.11a) for the induced metric. Therefore, the first term on the RHS is bounded if derivatives of the second fundamental form up to order k are bounded.

Derivatives of order ℓ of the components of the normal N_β are bounded by derivatives of order $\ell + 1$ of the graph functions, thus the result follows by induction. □

3.2 Solutions for fixed coordinates

In this section we will consider the Cauchy problem (1.3.2) for the membrane equation in a given family of coordinates on the initial submanifold and on the Minkowski space. The arguments used will rely on the theory for existence of hyperbolic equations. We will adopt the notation used in section 2 and especially in section 2.3.

In the following definition we introduce a such a family of coordinates in which we search for a solution to the membrane equation.

Definition 3.12. Let N be a manifold and Σ_0 be a submanifold. Suppose M to be a manifold and $\varphi : M \to N$ to be an immersion of Σ_0. A set $(U_\lambda, x_\lambda, V_\lambda, y_\lambda)_{\lambda\in\Lambda}$ is called a *decomposition of φ* if $(U_\lambda, x_\lambda)_{\lambda\in\Lambda}$ is an atlas for M, and (V_λ, y_λ) are charts on N satisfying $\varphi(U_\lambda) \subset V_\lambda$ for all $\lambda \in \Lambda$.

Assume M to be an m-dimensional manifold and $\varphi : M \to \mathbb{R}^{n,1}$ to be an immersion (called *initial immersion*) of the initial submanifold Σ_0. Since we treat the Cauchy problem (1.3.2) for fixed direction, lapse and shift we consider an initial velocity combined in a timelike vector field $\chi : M \to T\mathbb{R}^{n,1}$ along φ. We will use an integer $s > \frac{m}{2} + 1$ to state differentiability properties.

Let $(U_\lambda, x_\lambda, V_\lambda, y_\lambda)_{\lambda\in\Lambda}$ be a decomposition of φ and let

$$\Phi_\lambda(z) = y_\lambda \circ \varphi \circ x_\lambda^{-1}(z) \quad \text{and} \quad \chi_\lambda(z) = (dy_\lambda)_{\varphi\circ x_\lambda^{-1}(z)}(\chi) \tag{3.2.1}$$

be the representations of φ and χ in these coordinates for $\lambda \in \Lambda$. The functions Φ_λ and χ_λ will be the initial data for our first main existence result.

We make the following uniformity assumptions.

Assumptions 3.13. *1. The initial immersion and the decomposition admit constants ω_1 and ρ_1 such that for each $\lambda \in \Lambda$ the image of the coordinates $x_\lambda(U_\lambda)$ contain a Euclidean ball with radius ρ_1 about 0. Further, the representation of the induced metric $\mathring{g}_{ij}$ on M w.r.t. the coordinates x_λ satisfies in the center $\mathring{g}_{ij}(0) \geq \omega_1^{-2}\delta_{ij}$.*

2. There exist constants C_{w_0} and $\tilde{C}_\ell^\varphi$ for $2 \leq \ell \leq s+2$ such that for each $\lambda \in \Lambda$

$$|D\Phi_\lambda(0)|_e \leq C_{w_0}, \ |D^2\Phi_\lambda|_e \leq \tilde{C}_2^\varphi \tfrac{1}{\rho_1}, \ |D^\ell\Phi_\lambda|_e \leq \tilde{C}_\ell^\varphi \ \textit{for } 3 \leq \ell \leq s+2. \tag{3.2.2}$$

3. Further the initial velocity χ_λ satisfies for each $\lambda \in \Lambda$:

$$\eta(\chi_\lambda, \chi_\lambda) \leq -L_2 \quad \textit{and} \quad |D^\ell\chi_\lambda|_{e,e} \leq \tilde{C}_\ell^\chi \ \textit{for } 0 \leq \ell \leq s+1 \tag{3.2.3}$$

with constants $\tilde{C}_\ell^\chi, L_2$.

The following theorem is the first main result for the initial value problem (1.3.2) for the membrane equation in a given family of coordinates.

Theorem 3.14. *Let the decomposition $(U_\lambda, x_\lambda, V_\lambda, y_\lambda)_{\lambda\in\Lambda}$ and the initial data φ_λ and χ_λ for the membrane equation satisfy the assumptions 3.13.*

Then there exist constants $\bar{T} > 0$, $0 < \theta < 1$ and a family (F_λ) of bounded C^2-immersions $F_\lambda : [-\bar{T}, \bar{T}] \times B^e_{\theta\rho_1/2}(0) \subset \mathbb{R} \times x_\lambda(U_\lambda) \to \mathbb{R}^{n+1}$ solving the reduced membrane equation

$$g^{\mu\nu}\partial_\mu\partial_\nu F^A - g^{\mu\nu}\hat{\Gamma}^\lambda_{\mu\nu}\partial_\lambda F^A = 0 \tag{3.2.4}$$

w.r.t. the background metric $\hat{g}$ defined in (1.3.7) and attaining the initial values

$$F_\lambda\big|_{t=0} = \Phi_\lambda \quad \textit{and} \quad \partial_t F_\lambda\big|_{t=0} = \chi_\lambda. \tag{3.2.5}$$

Let F_λ and $\bar{F}_\lambda$ be two such solutions defined on the image of the coordinates x_λ. Assume $z \in \mathbb{R}^m$ to be a point contained in the image of x_λ. If F_λ and $\bar{F}_\lambda$ attain the initial values Φ_λ and χ_λ on a ball $B^e_r(z)$, then they coincide on the double-cone with base $B^e_r(z)$ and slope c_0.

Remark 3.15.
1. A lower bound for the existence time $\bar{T} > 0$ will be given in Remark 3.31.
2. A precise value for the slope c_0 of the uniqueness cone will be defined in (3.2.33).

Remark 3.16. Let ℓ_0 be an integer. Assume that the initial values and the decomposition satisfy the assumptions 3.13 with an integer $r = s+\ell_0 > \frac{m}{2}+1+\ell_0$ instead of an $s > \frac{m}{2}+1$. Then the family (F_λ) of solutions to the reduced membrane equation are immersions of class $C^{2+\ell_0}$.

We will begin with the construction of a solution to equation (3.2.4) in fixed charts x_λ and y_λ. The strategy will be to obtain a formulation of the equation to which Theorem 2.22 applies. The construction will provide us with an estimate on the existence time and the part of the image of the coordinates x_λ on which the solution attains the initial data in dependency on the constants occurring in the assumptions 3.13.

Let $\lambda \in \Lambda$ be fixed. Since the existence result for hyperbolic equations obtained in section 2 is only suitable for functions defined on all of $\mathbb{R}^m$, we have to extend the functions Φ_λ and χ_λ. To obtain such an extension we introduce a cut-off function $\zeta \in C_c^\infty(\mathbb{R}^m)$ with the property that for a constant $0 < \theta < 1$ we have $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ on $B_{\theta\rho_1/2}(0)$ and $\zeta \equiv 0$ outside $B_{\theta\rho_1}(0)$. The derivatives of ζ are bounded by

$$|D^\ell \zeta|_e \leq \tilde{C}_\ell (\theta\rho_1)^{-\ell}, \tag{3.2.6}$$

where $\tilde{C}_\ell$ denote constants independent of θ and ρ_1.

Define a linear function $w(t,x)$ on $\mathbb{R}^{m+1}$ by

$$w(t,x) = w_0(x) + t\, w_1 \tag{3.2.7}$$
$$\text{with} \quad w_0(x) = x^\ell \partial_\ell \Phi_\lambda(0) \text{ for } x \in \mathbb{R}^m \quad \text{and} \quad w_1 = \chi_0 = \chi_\lambda(0).$$

This function is defined as the tangent plane of the function Φ_λ at the origin moving with the constant velocity χ_0. The linear function $w(t,x)$ satisfies the membrane equation (1.3.1b) w.r.t. the coordinates x_λ and y_λ. The idea is now to apply the cut-off function ζ to terms of the reduced equation (3.2.4) which are independent of the solution. Then we are able to search for solutions within Sobolev-space perturbations of $w(t,x)$. The only independent term in equation (3.2.4) are the Christoffel symbols of the background metric $\hat{g}$ defined by (1.3.7). Since the definition only contains the initial values, we begin with a cut-off process for them. Set

$$\mathring{\Phi}(x) = \zeta(x)\big(\Phi_\lambda(x) - w_0(x)\big) \qquad \text{for } x \in \mathbb{R}^m \tag{3.2.8a}$$
$$\text{and} \qquad \mathring{\chi}(x) = \zeta(x)\big(\chi_\lambda(x) - \chi_0\big) \qquad \text{for } x \in \mathbb{R}^m. \tag{3.2.8b}$$

These functions constitute the interpolation of the function Φ_λ and its tangent plane at the origin and the velocity χ_λ with the velocity at the origin. From these functions we derive a cut-off of the background metric $\hat{g}$. Let $\hat{a}_{\mu\nu}$ be the matrix with the components

$$\hat{a}_{00} = (\chi_0+\mathring{\chi})^A \eta_{AB} (\chi_0+\mathring{\chi})^B,\ \hat{a}_{0j} = (\chi_0+\mathring{\chi})^A \eta_{AB} \partial_j \mathring{\Phi}^B,\ \hat{a}_{ij} = \partial_i \mathring{\Phi}^A \eta_{AB} \partial_j \mathring{\Phi}^B. \tag{3.2.9}$$

This metric coincides with the background metric on the ball $B_{\theta\rho_1/2}(0)$. The Christoffel symbols of this metric will be denoted by $\hat{\gamma}^{\lambda}_{\mu\nu}$.

We search for a function $F : \mathcal{V} \subset \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ defined on a neighborhood of $\{t = 0\}$ solving the IVP

$$g^{\mu\nu}(DF, \partial_t F)\partial_\mu\partial_\nu F^A = f^A(t, DF, \partial_t F) = g^{\mu\nu}(DF, \partial_t F)\hat{\gamma}^{\lambda}_{\mu\nu}(t)\partial_\lambda F^A,$$
$$F\big|_{t=0} = w_0 + \mathring{\Phi},\ \partial_t F\big|_{t=0} = w_1 + \mathring{\chi}. \quad (3.2.10)$$

The coefficient matrix $g^{\mu\nu}(DF, \partial_t F)$ is defined as the inverse of the matrix

$$g_{\mu\nu}(DF, \partial_t F) = \partial_\mu F^A \eta_{AB} \partial_\nu F^B \quad (3.2.11)$$

corresponding to the pullback metric $F^*\eta$. For notational convenience we set $g_{\mu\nu}(F) = g_{\mu\nu}(DF, \partial_t F)$ and $f(F) = f(t, DF, \partial_t F)$, if the exact dependency is not important for the argument. We will show the following proposition.

Proposition 3.17. *There exist a constant $T' > 0$ and a unique C^2-solution $F : [-T', T'] \times \mathbb{R}^m \to \mathbb{R}^{n+1}$ of the IVP (3.2.10) satisfying*

$$F(t) - w(t) \in C([-T', T'], H^{s+1}),\ \partial_t F(t) - w_1 \in C([-T', T'], H^s). \quad (3.2.12)$$

From this proposition it follows immediately the existence claim of Theorem 3.14 by setting $F_\lambda := F$. The cut-off process yields that a solution of equation (3.2.10) solves the reduced membrane equation (3.2.4) and attains the initial values (3.2.5) within the ball $B_{\theta\rho_1/2}(0)$.

The differentiability properties (3.2.12) suggest that we make use of Theorem 2.22 discussing asymptotic equations. To obtain the conditions of Theorem 2.9 for the asymptotic coefficients and RHS (cf. (2.3.14)) we will follow the treatment of the Cauchy problem for the Einstein equations in [HKM76].

Let $\Omega \subset \mathbb{R}^{m(n+1)} \times \mathbb{R}^{n+1}$ be a set chosen later and define for $(Y, X) \in \Omega$ with $Y = (Y_k)$ the matrix $g^{\mathrm{a}}_{\mu\nu}$ by

$$g^{\mathrm{a}}_{0\ell}(Y, X) := (\partial_t w + X)^A \eta_{AB} (\partial_\ell w + Y_\ell)^B, \quad (3.2.13)$$

where the other parts g^{a}_{00} and $g^{\mathrm{a}}_{k\ell}$ are defined analogously (see (3.2.9)). The inverse of $g^{\mathrm{a}}_{\mu\nu}$ will be denoted by $g^{\mu\nu}_{\mathrm{a}}$. In an analogous way we define a function f_{a} by

$$f^A_{\mathrm{a}}(t, Y, X) := g^{\mu\nu}_{\mathrm{a}}(Y, X)\big(\hat{\gamma}^0_{\mu\nu}(t)(\partial_t w + X)^A + \hat{\gamma}^{\ell}_{\mu\nu}(t)(\partial_\ell w + Y_\ell)^A\big) \quad (3.2.14)$$

As in [HKM76] the set Ω will be used to ensure that the matrix $g^{\mu\nu}_{\mathrm{a}}$ has the desired signature $(-+\cdots+)$.

These definitions give rise to the following operators

$$g^{\mathrm{a}}_{\mu\nu}(\varphi_0, \varphi_1) = g^{\mathrm{a}}_{\mu\nu}(D\varphi_0, \varphi_1),\ g^{\mu\nu}_{\mathrm{a}}(\varphi_0, \varphi_1) = g^{\mu\nu}_{\mathrm{a}}(D\varphi_0, \varphi_1)$$
$$\text{and}\quad f_{\mathrm{a}}(t, \varphi_0, \varphi_1) = f_{\mathrm{a}}(t, D\varphi_0, \varphi_1)$$

with domain $W \subset H^{s+1} \times H^s$ chosen later in dependency on Ω. Observe that the choice of the background metric $\hat{g}$ (cf. (1.3.7)) gives us that the RHS f_{a} in fact does not depend explicitly on the time parameter t. These operators correspond to the definition (2.3.14) of asymptotic coefficients for the IVP (3.2.10). Therefore, if we find a solution ψ of the following *asymptotic equation*

$$g_{\mathrm{a}}^{\mu\nu}(D\psi, \partial_t\psi)\partial_\mu\partial_\nu\psi^A = f_{\mathrm{a}}(t, D\psi, \partial_t\psi) \tag{3.2.15a}$$

$$\text{with initial values} \quad \psi^A\big|_{t=0} = \mathring{\Phi}^A \text{ and } \partial_t\psi^A\big|_{t=0} = \mathring{\chi}^A \tag{3.2.15b}$$

then $\psi(t) + w(t)$ is a solution to IVP (3.2.10).

To obtain suitable definitions of Ω and W, the strategy will be first to choose the set Ω such that each $(Y, X) \in \Omega$ satisfies $g^a_{00}(Y, X) \leq -L_2(1 - r_0)$ and $g^a_{k\ell}(Y, X) \geq \omega_1^{-2}(1 - R_0)\delta_{k\ell}$ with fixed constants $0 < r_0, R_0 < 1$. Then we define $W \subset H^{s+1} \times H^s$ in a way such that $(\varphi_0, \varphi_1) \in W$ yields $(D\varphi_0, \varphi_1) \in \Omega$ pointwise. The exact definition of W will be given in the next section.

According to condition (2.3.2h) we need to show that g^a_{00} can be bounded away from 0 and the submatrix g^a_{ij} is positive definite. Lemma B.2 then yields the desired estimates.

Lemma 3.18. *For the matrix $g^{\mathrm{a}}_{\mu\nu}(Y, X)$ the following inequalities hold*

$$g^a_{k\ell} \geq \big(\omega_1^{-2} - 2|Y|_e(|Y|_e + |Dw_0|_e)\big)\delta_{k\ell} \quad \textit{and} \quad g^a_{00} \leq \langle\!\langle \chi_0, \chi_0 \rangle\!\rangle + 2|X|_e(|\chi_0|_e + |X|_e),$$

where we set $|Y|_e := (\sum_k |Y_k|^2)^{1/2}$.

Proof. The estimates follow from the definition of the matrix in (3.2.13) and the condition $\mathring{g}_{ij} = g_{ij}(0) \geq \omega_1^{-2}\delta_{ij}$ for the induced metric at the origin. □

Our starting point will be the following definition

$$\Omega := B^e_{\delta_1}(0) \times B^e_{\delta_2}(0) \subset \mathbb{R}^{m(n+1)} \times \mathbb{R}^{n+1} \tag{3.2.16}$$

with constants $\delta_1, \delta_2 > 0$ to be chosen with the help of the estimates established in Lemma 3.18. Let $(Y, X) \in \Omega$. Then Lemma 3.18 yields

$$\begin{aligned} g^a_{k\ell}(Y, X) &\geq \omega_1^{-2}\big(1 - 2\omega_1^2\delta_1(C_{w_0} + \delta_1)\big)\delta_{k\ell} \\ \text{and} \qquad g^a_{00}(Y, X) &\leq -L_2\big(1 - 2L_2^{-1}\delta_2(\tilde{C}^\chi_0 + \delta_2)\big). \end{aligned} \tag{3.2.17}$$

Here, we used the assumptions for χ_λ and Φ_λ. These estimates provide us with the following conditions for δ_1 and δ_2

$$2\delta_1(C_{w_0} + \delta_1) \leq \omega_1^{-2}R_0 \quad \text{and} \quad 2\delta_2(\tilde{C}^\chi_0 + \delta_2) \leq L_2 r_0. \tag{3.2.18}$$

Let $\delta_1, \delta_2 > 0$ be two constants satisfying the preceding inequalities. This choice provides us with a definition of the set Ω such that for $(Y, X) \in \Omega$, the following holds:

$$g^a_{00}(Y, X) \leq -\tilde{\lambda} := -L_2(1 - r_0) \text{ and } g^a_{k\ell}(Y, X) \geq \tilde{\mu}\delta_{k\ell} := \omega_1^{-2}(1 - R_0)\delta_{k\ell}. \tag{3.2.19}$$

From the definition of Ω we infer bounds for the norms of the matrices $(g^{\mathrm{a}}_{\mu\nu})$ and $(g^{\mu\nu}_{\mathrm{a}})$, and estimates for g^{00}_a and g^{ij}_a needed by condition (2.3.2h) of the existence theorem. For notational convenience we define

$$K_0 := (C^2_{w_0} + \delta^2_1)^{1/2} \qquad \Longrightarrow \qquad |Dw_0 + Y|_e \le K_0 \tag{3.2.20a}$$
$$K_1 := \big((\tilde{C}^{\chi}_0)^2 + \delta^2_2\big)^{1/2} \qquad \Longrightarrow \qquad |\chi_0 + X|_e \le K_1 \tag{3.2.20b}$$

for $(Y,X) \in \Omega$. With the help of Lemma B.2 we obtain the desired estimates for g^{00}_a, (g^{ij}_a) and the norm of the inverse matrix. Estimates for several submatrices of the matrix $g^{\mathrm{a}}_{\mu\nu}$ will be needed. We obtain such estimates in the following lemma.

Lemma 3.19. *Assume $(Y,X) \in \Omega$. Then the following inequalities hold*

$$|g^a_{00}(Y,X)| \le K_1^2, \qquad \big|\big(g^{\mathrm{a}}_{0\ell}(Y,X)\big)\big|_e \le K_0 K_1, \tag{3.2.21a}$$
$$\big|\big(g^a_{ij}(Y,X)\big)\big|_e \le K_0^2 \qquad \textit{and} \qquad \big|\big(g^{\mathrm{a}}_{\mu\nu}(Y,X)\big)\big|_e \le K_1^2 + K_0^2. \tag{3.2.21b}$$

Proof. We begin with the second estimate since all terms occurring in the matrix $g^{\mathrm{a}}_{\mu\nu}$ are involved (see definition (3.2.13)). We have

$$\begin{aligned} \textstyle\sum_\ell |g^{\mathrm{a}}_{0\ell}|^2 &\le \big(|X| + |\chi_0|\big)^2 |\eta|^2 \textstyle\sum_\ell \big(|Y_\ell| + |\partial_\ell w_0|\big)^2 \\ &\le \big(|X| + |\chi_0|\big)^2 \big(|Y| + |Dw_0|\big)^2. \end{aligned}$$

Hence, the inequality for $g^{\mathrm{a}}_{0\ell}$ follows from the definition of the constants K_0 and K_1. A similar argument yields the other estimates. □

The preceding lemma and Lemma B.2 immediately yield the following result.

Lemma 3.20. *Assume $(Y,X) \in \Omega$. Then the following inequalities hold for the coefficients $g^{\mu\nu}_{\mathrm{a}}(Y,X)$*

$$\big|\big(g^{\mu\nu}_{\mathrm{a}}(Y,X)\big)\big|^2_e \le \tilde{\lambda}^{-2} + 2\tilde{\lambda}^{-2}\tfrac{m}{\tilde{\mu}^2} K_0^2 K_1^2 + \tfrac{m}{\tilde{\mu}^2} \qquad =: \Delta^{-2}, \tag{3.2.22a}$$
$$g^{00}_{\mathrm{a}}(Y,X) \le -K_1^{-2}\big(1 + \tfrac{m^{1/2}}{\tilde{\mu}} K_0^2\big)^{-1} \qquad =: -\lambda \tag{3.2.22b}$$
$$\textit{and} \qquad g^{ij}_{\mathrm{a}}(Y,X) \ge K_0^{-2}\big(1 + \tilde{\lambda}^{-1} K_1^2\big)^{-1} \delta^{ij} \qquad =: \mu\delta^{ij} \tag{3.2.22c}$$

where we used the constants $K_0, K_1, \tilde{\lambda}$ and $\tilde{\mu}$ defined in (3.2.20a), (3.2.20b) and (3.2.19).

3.2.1 Sobolev estimates

Having in mind the estimates following from the definition of Ω we are now able to define the domain $W \subset H^{s+1} \times H^s$ of the coefficients and the RHS of the asymptotic equation (3.2.10). The construction should be done in such a way that if $(\varphi_0, \varphi_1) \in W$, then the estimates stemming from the definition of Ω are applicable. We start with balls around the initial values (3.2.15b) of the asymptotic equation. Let $\rho > 0$ be a constant chosen later and define

$$W := B_\rho(\mathring{\Phi}) \times B_\rho(\mathring{\chi}). \tag{3.2.23}$$

The goal is to ensure the following property

$$(\varphi_0, \varphi_1) \in W \quad \Longrightarrow \quad (D\varphi_0, \varphi_1) \in \Omega \text{ pointwise.} \tag{3.2.24}$$

Two steps are necessary to make the definition of W reasonable. Firstly, the initial values have to satisfy condition (3.2.24), and secondly there needs to be enough space left for members of W. The first step will be obtained by the following lemma.

Lemma 3.21. *The following inequalities hold for the initial values $\mathring{\Phi}$ and $\mathring{\chi}$*

$$|D\mathring{\Phi}|_e \leq \tilde{C}_2^{\varphi}\theta\big(1 + \tilde{C}_1\big) \qquad \textit{and} \qquad |\mathring{\chi}|_e \leq \tilde{C}_1^{\chi}\rho_1\theta. \tag{3.2.25}$$

The constants are taken from the assumptions (3.2.2) and (3.2.3), and from the bounds (3.2.6) for the cut-off function.

Proof. The definition of the interpolated function $\mathring{\Phi}$ yields

$$|D\mathring{\Phi}|_e \quad \leq \quad \zeta|D\Phi_\lambda - D\Phi_\lambda(0)|_e + |D\zeta|_e|\Phi_\lambda - x^\ell\partial_\ell\Phi_\lambda(0)|_e.$$

To obtain an estimate for the two terms we use the bound for the second derivative of Φ_λ. The mean value theorem and the Hölder inequality yield for $|x| =: r < \rho_1$

$$|D\Phi_\lambda - D\Phi_\lambda(0)|_e \leq \tilde{C}_2^{\varphi}\tfrac{r}{\rho_1} \quad \text{and} \quad |\Phi_\lambda - x^\ell\partial_\ell\Phi_\lambda(0)|_e \leq \tilde{C}_2^{\varphi}\tfrac{r^2}{\rho_1}.$$

The fist estimate follow by considering the bounds for cut-off function ζ and $r \leq \theta\rho_1$.

The estimate for the initial velocity can be derived via a similar device using the bound for $D\chi_\alpha$. □

Now we choose θ small enough that the RHS of the first inequality in (3.2.25) is less than $\delta_1/2$ and the RHS of the second inequality is less than $\delta_2/2$. Furthermore, due to the Sobolev embedding theorem it is possible to adapt ρ so that condition (3.2.24) is fulfilled.

To obtain estimates meeting the conditions (2.3.2a) to (2.3.2f) we need bounds on $\|\mathring{\Phi}\|_{s+1}$ and $\|\mathring{\chi}\|_s$. These can be derived by a similar device as in the proof of Lemma 3.21 via the assumptions 3.13 on the representations of φ and χ w.r.t. the given decomposition and the bounds for derivatives of the cut-off function ζ. Therefore, constants $D_0, \tilde{D}_0, D_1$ and $\tilde{D}_1$ exist such that

$$\|\mathring{\Phi}\|_{s+1} \leq D_0,\ \|D^2\mathring{\Phi}\|_s \leq \tilde{D}_0 \qquad \text{and} \qquad \|\mathring{\chi}\|_s \leq D_1,\ \|D\mathring{\chi}\|_s \leq \tilde{D}_1. \tag{3.2.26}$$

The bounds $\tilde{D}_0$ and $\tilde{D}_1$ are needed to estimate the Christoffel symbols of the cut-off background metric $\hat{a}_{\mu\nu}$.

We define constants similar to K_0 and K_1 declared in (3.2.20a) and (3.2.20b) we define for $(\varphi_0, \varphi_1) \in W$

$$K_{0,s} := (C_{w_0}^2 + \rho^2 + D_0^2)^{1/2} \qquad \Longrightarrow \qquad \|Dw_0 + D\varphi_0\|_{s,\mathrm{ul}} \leq K_{0,s} \tag{3.2.27a}$$

$$K_{1,s} := \big((\tilde{C}_0^{\chi})^2 + \rho^2 + D_1^2\big)^{1/2} \qquad \Longrightarrow \qquad \|\chi_0 + \varphi_1\|_{s,\mathrm{ul}} \leq K_{1,s} \tag{3.2.27b}$$

where we used the property of the special norm for the uniformly local Sobolev spaces H^s_{ul} derived in Lemma 2.21.

The next lemma states generic estimates for the coefficients and the RHS of the equation in (3.2.10) depending on the unknown function F.

Lemma 3.22. *The following estimates hold for two functions F and $\bar{F}$*

$$|f(F)| \leq \big|\big(g^{\mu\nu}(F)\big)\big|\,|(\hat{\gamma}^{\lambda}_{\mu\nu})|\big(|\partial_t F|^2 + |DF|^2\big)^{1/2} \tag{3.2.28a}$$

$$\begin{aligned} |f(F) - f(\bar{F})| \leq \big|\big(g^{\mu\nu}(F)\big) - \big(g^{\mu\nu}(\bar{F})\big)\big||(\hat{\gamma}^{\lambda}_{\mu\nu})|(|\partial_t \bar{F}|^2 + |D\bar{F}|^2)^{1/2} \\ + \big|\big(g^{\mu\nu}(F)\big)\big|\,|(\hat{\gamma}^{\lambda}_{\mu\nu})|(|\partial_t V| + |DV|) \end{aligned} \tag{3.2.28b}$$

$$\begin{aligned} \big|\big(g_{\mu\nu}(F)\big) - \big(g_{\mu\nu}(\bar{F})\big)\big|_e \leq (|\partial_t V| + |DV|)\big((|\partial_t \bar{F}|^2 + |D\bar{F}|^2)^{1/2} \\ + (|\partial_t F|^2 + |DF|^2)^{1/2}\big), \end{aligned} \tag{3.2.28c}$$

where we set $V := F - \bar{F}$.

Proof. The inequalities follow from a straight-forward calculation using the fact

$$u_1 \cdots u_n - v_1 \cdots v_n = \textstyle\sum_j u_1 \cdots u_{j-1}(u_j - v_j)v_{j+1} \cdots v_n. \qquad \square$$

These generic estimates will be used to derive a bound for the RHS f_{a} and Lipschitz estimates for the coefficients and the RHS of the asymptotic equation. Recall the following notation introduced in section 2.3

$$E_r(u - v) = \|u_0 - v_0\|_{r+1} + \|u_1 - v_1\|_r \text{ for } (u_0, u_1), (v_0, v_1) \in H^{r+1} \times H^r.$$

Sobolev norm estimates for the coefficients will be obtained in a way similar to that for pointwise estimates. We begin with estimates for the matrix $g^{\mathrm{a}}_{\mu\nu}$ and then turn to the inverse $g^{\mu\nu}_{\mathrm{a}}$.

Lemma 3.23. *Let $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in W$. Then the following estimates hold*

$$\big\|\big(g^{\mathrm{a}}_{\mu\nu}(\varphi_0, \varphi_1)\big)\big\|_{e,s,\mathrm{ul}} \leq K^2_{0,s} + K^2_{1,s} =: \tilde{K} \tag{3.2.29a}$$

$$\big\|\big(g^{\mathrm{a}}_{\mu\nu}(\varphi_0, \varphi_1)\big) - \big(g^{\mathrm{a}}_{\mu\nu}(\psi_0, \psi_1)\big)\big\|_{e,s-1,\mathrm{ul}} \leq \tilde{\theta} E_s\big((\varphi_0, \varphi_1) - (\psi_0, \psi_1)\big) \tag{3.2.29b}$$

$$\big\|\big(g^{\mathrm{a}}_{\mu\nu}(\varphi_0, \varphi_1)\big) - \big(g^{\mathrm{a}}_{\mu\nu}(\psi_0, \psi_1)\big)\big\|_{e,0,\mathrm{ul}} \leq \tilde{\theta}' E_1\big((\varphi_0, \varphi_1) - (\psi_0, \psi_1)\big) \tag{3.2.29c}$$

where $\tilde{\theta} = 2^{3/2}\,(K_{0,s} + K_{1,s})$ and $\tilde{\theta}' = 2^{3/2}\,(K_0 + K_1)$.

Proof. Replacing the pointwise norms for Y and X in the proof of Lemma 3.19 by the norm $\|\,.\,\|_{s,\mathrm{ul}}$ for $D\varphi_0$ and φ_1 yields the first bound.

To obtain the Lipschitz estimates we consider the generic estimate (3.2.28c) replacing

$$\partial_t F \mapsto \varphi_1,\ DF \mapsto D\varphi_0,\ \partial_t \bar{F} \mapsto \psi_1 \quad \text{and} \quad D\bar{F} \mapsto D\psi_0. \tag{3.2.30}$$

The last estimate follows then by using the L^∞-norm and the second estimate follows by using the norm $\|\,.\,\|_{s,\mathrm{ul}}$. $\square$

The preceding lemma provides us with estimates for the coefficients of the asymptotic equation (3.2.15a) meeting the conditions of the existence Theorem 2.9. These will be stated in the next lemma.

Lemma 3.24. *Suppose* $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in W$. *Then we have*

$$\big\|\big(g_{\mathrm{a}}^{\mu\nu}(\varphi_0, \varphi_1)\big)\big\|_{e,s,\mathrm{ul}} \le c\Delta^{-1}\big(1 + (\Delta^{-1}\tilde{K})^s\big) \qquad =: K \tag{3.2.31a}$$

$$\big\|\big(g_{\mathrm{a}}^{\mu\nu}(\varphi_0, \varphi_1)\big) - \big(g_{\mathrm{a}}^{\mu\nu}(\psi_0, \psi_1)\big)\big\|_{e,s-1,\mathrm{ul}} \le \theta E_s\big((\varphi_0, \varphi_1) - (\psi_0, \psi_1)\big) \tag{3.2.31b}$$

$$\big\|\big(g_{\mathrm{a}}^{\mu\nu}(\varphi_0, \varphi_1)\big) - \big(g_{\mathrm{a}}^{\mu\nu}(\psi_0, \psi_1)\big)\big\|_{e,0,\mathrm{ul}} \le \theta' E_1\big((\varphi_0, \varphi_1) - (\psi_0, \psi_1)\big) \tag{3.2.31c}$$

where the Lipschitz constants are given by $\theta := K^2\tilde{\theta}$ *and* $\theta' := \Delta^{-2}\tilde{\theta}'$.

Proof. The first inequality follows directly from Lemma 2.4 and the estimates (3.2.22a) and (3.2.29a). The other inequalities follow from the observation

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$$

for matrices A and B and the Lipschitz estimates (3.2.29b) and (3.2.29c) for $g^{\mathrm{a}}_{\mu\nu}$. □

We now head to the RHS f_{a} of the asymptotic equation (3.2.15a). To obtain the desired estimates we have to control the Christoffel symbols of the cut-off background metric $\hat{a}$. To control derivatives of the Christoffel symbols up to order s we need bounds for derivatives of the initial values $\mathring{\Phi}$ and $\mathring{\chi}$ up to order $s+2$ and $s+1$ resp. These were stated in (3.2.26).

A Lipschitz estimate for the RHS f_1 in L^2 will use local bounds for the Christoffel symbols $\hat{\gamma}^{\lambda}_{\mu\nu}$. To this end we need local estimates for $D^2\mathring{\Phi}$ and $D\mathring{\chi}$ which will be derived in the next lemma.

Lemma 3.25. *The following inequalities hold*

$$|D^2\mathring{\Phi}|_e \le \tilde{C}_2^{\varphi}\tfrac{1}{\rho_1}\big(1 + \tilde{C}_1 + \tilde{C}_2\big) \quad \textit{and} \quad |D\mathring{\chi}|_e \le \tilde{C}_1^{\chi}(1 + \tilde{C}_1). \tag{3.2.32}$$

Proof. The result follows from

$$|D^2\mathring{\Phi}|_e \le \zeta|D^2\Phi_\alpha|_e + |D\zeta|_e|D\Phi_\alpha - D\Phi_\alpha(0)|_e + |D^2\zeta|_e|\Phi_\alpha - x^\ell\partial_\ell\Phi_\alpha(0)|_e.$$

using the bounds for the derivatives of the cut-off function and the bounds for $\mathring{\Phi}$ established in the proof of Lemma 3.21. A similar device gives us the second inequality. □

Using this preparation we are able to estimate the Christoffel symbols of $\hat{a}$.

Proposition 3.26. *The following inequalities hold*

$$\|(\hat{\gamma}^{\lambda}_{\mu\nu})\|_{e,s} \le C^{\hat{\gamma}} \quad \textit{and} \quad |(\hat{\gamma}^{\lambda}_{\mu\nu})|_e \le C_0^{\hat{\gamma}},$$

where

$$\begin{aligned} C^{\hat{\gamma}} = 2K\big(\tilde{D}_1\big(2((\tilde{C}_0^{\chi})^2 + D_1^2)^{1/2} + 4(C_{w_0}^2 + D_0^2)^{1/2}\big) \\ + \tilde{D}_0\big(4((\tilde{C}_0^{\chi})^2 + D_1^2)^{1/2} + 2(C_{w_0}^2 + D_0^2)^{1/2}\big)\big). \end{aligned}$$

and $C_0^{\hat{\gamma}}$ *arises from* $C^{\hat{\gamma}}$ *by applying the replacements*

$$K \mapsto \Delta^{-1}, \;\; D_0 \mapsto \delta_1/2, \;\; D_1 \mapsto \delta_2/2$$

and $\tilde{D}_0$, $\tilde{D}_1$ *are respectively replaced by the bounds for* $|D\mathring{\chi}|_e$ *and* $|D^2\mathring{\Phi}|_e$ *stated in* (3.2.32).

Proof. Taking a generic norm for $\hat{\gamma}^{\lambda}_{\mu\nu}$ we see that it is necessary to estimate $\hat{a}^{\mu\nu}$ and $D\hat{a}_{\mu\nu}$. Observe that Lemma 4.20 also applies to the matrix $\hat{a}^{\mu\nu}$ so that the H^s-norm of $\hat{a}^{\mu\nu}$ is bounded by K. To obtain a bound for derivatives of $\hat{a}_{\mu\nu}$ we compute the following representative part of the derivative of $\hat{a}$

$$\partial_k \hat{a}_{0\ell} = \partial_k \mathring{\chi}^A \eta_{AB} (\partial_\ell w_0 + \partial_\ell \mathring{\Phi})^B + (\chi_0 + \mathring{\chi})^A \eta_{AB} \partial_k \partial_\ell \mathring{\Phi}^B.$$

We see that the highest-order terms can be estimated by the bounds $\tilde{D}_0$ and $\tilde{D}_1$ for the H^s-norm of $D^2\mathring{\Phi}$ and $D\mathring{\chi}$, respectively.

Observe that the choice of θ ensures that $|D\mathring{\Phi}|_e < \delta_1/2$ and $|\mathring{\chi}|_e < \delta_2/2$. By the same argument as for the first inequality using a pointwise norm we infer the second inequality from the local bounds for $g^{\mu\nu}$ stated in (3.2.22a) and the bounds for the initial values stated in (3.2.25) and (3.2.32). □

The preceding considerations yield estimates for the RHS of the asymptotic equation (3.2.15a) meeting the conditions (2.3.2a), (2.3.2e) and (2.3.2f) of the existence Theorem 2.9.

Lemma 3.27. *Let $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in W$. Then the following estimates hold*

$$\begin{aligned} \|f_{\mathrm{a}}(t, \varphi_0, \varphi_1)\|_s &\le K_f, \\ \|f_{\mathrm{a}}(t, \varphi_0, \varphi_1) - f_{\mathrm{a}}(t, \psi_0, \psi_1)\|_{s-1} &\le \theta'_f E_s\big((\varphi_0, \varphi_1) - (\psi_0, \psi_1)\big) \\ \|f_{\mathrm{a}}(t, \varphi_0, \varphi_1) - f_{\mathrm{a}}(t, \psi_0, \psi_1)\|_{L^2} &\le \theta_f E_1\big((\varphi_0, \varphi_1) - (\psi_0, \psi_1)\big) \end{aligned}$$

with $K_f = K C^{\hat{\gamma}} (K_{0,s}^2 + K_{1,s}^2)^{1/2}$, $\theta'_f = \theta C^{\hat{\gamma}} (K_{0,s}^2 + K_{1,s}^2)^{1/2} + K C^{\hat{\gamma}}$ and θ_f arises from θ'_f by applying the replacements

$$\theta \mapsto \theta', \;\; K \mapsto \Delta^{-1}, \;\; C^{\hat{\gamma}} \mapsto C^{\hat{\gamma}}_0, \;\; (K_{0,s}, K_{1,s}) \mapsto (K_0, K_1).$$

Proof. The first statement can be derived using the generic estimate (3.2.28a) and proposition 3.26.

The Lipschitz estimates can be derived in a similar way to those for the coefficients stated in Lemma 3.23 regarding the generic estimate (3.2.28b). □

To obtain uniqueness of solutions to the IVP (3.2.10) we need to show the conditions (2.4.8) for coefficients and RHS of the asymptotic equation (3.2.15a) are such that the uniqueness Theorem 2.26 is applicable.

Lemma 3.28. *The coefficients $g^{\mu\nu}_{\mathrm{a}}$ and the RHS f_{a} of equation (3.2.15a) satisfy the conditions (2.4.8).*

Proof. The claim follows from the observation that the Lipschitz estimates for the L^2-norm of the RHS and the coefficients do in fact involve local constants. Therefore, the desired constants are given by θ'_f from Lemma 3.27 and θ' from Lemma 3.24. □

Remark 3.29. To make this argument possible we derived a local estimate for the Christoffel symbols $\hat{\gamma}^{\lambda}_{\mu\nu}$ in proposition 3.26.

We will now give a proof of the main result of this section. The steps existence and uniqueness will be done separately, where the proof of existence reduces to a proof of Proposition 3.17.

Proof of Proposition 3.17. The Lemmata 3.20, 3.24 and 3.27 show that the assumptions of the asymptotic existence Theorem 2.22 are satisfied. Therefore, we infer the existence of a solution F to the IVP (3.2.10) with the desired properties provided by (2.3.15).

Lemma 3.28 provides us with the conditions needed to apply the local uniqueness Theorem 2.26 to the asymptotic equation (3.2.15a). Uniqueness of solutions therefore follows by considering balls of arbitrary radius about each point $\in \mathbb{R}^m$. □

Remark 3.30. The solution is given by $F(t) = w(t) + \psi(t)$, where $\psi(t)$ solves equation (3.2.15a) with initial values given by (3.2.15b). From the existence Theorem 2.9 applied to the asymptotic equation we derive that

$$(DF(t) - Dw_0, \partial_t F(t) - w_1) \in W \subset H^{s+1} \times H^s.$$

Therefore, all estimates on the coefficients remain valid for $t > 0$.

Remark 3.31. A lower bound for the existence time T' is given in Remark 2.18 with the constants defined in remark 2.17. The constant c_E plays an important role controlling the inverse of the existence time. For convenience we state it in terms of quantities occurring in our estimates. It is given by

$$\begin{aligned} c_E = 2\big(2(C^2_{w_0} + \delta_1^2)\big(1 + L_2^{-1}(1-r_0)^{-1}\big((\tilde{C}_0^\chi)^2 + \delta_2^2\big)\big) \\ + \big((\tilde{C}_0^\chi)^2 + \delta_2^2\big)\big(1 + m^{1/2}\omega_1^2(1-R_0)^{-1}(C^2_{w_0} + \delta_1^2)\big)\big). \end{aligned}$$

Remark 3.32. From Theorem 2.9 we only obtain a solution for $t > 0$, but hyperbolic equations are invariant under time reversal, so that for $t < 0$ the function $F(-t, z)$ is a solution to the equation in (3.2.10) if the ingredients are defined for negative values of t. The definition (1.3.7) of the background metric $\hat{g}$ ensures that the RHS is defined for all values of t. This also shows the uniqueness claim by considering double-cones, i.e. forward and backward w.r.t. the time parameter.

Remark 3.33. From the assumptions on the initial values we get that the cut-off background metric $\hat{a}_{\mu\nu}$ defined by (3.2.9) is C_c^{r+1}. Therefore the Christoffel symbols of $\hat{a}$ occurring in the RHS of the asymptotic equation defined by (3.2.14) are C_c^r with $r = s + \ell_0$. This shows that derivatives of the RHS up to order ℓ_0 are in C_c^s with $s > \frac{m}{2} + 1$ as it is necessary to obtain the estimates for the RHS in Lemma 3.27. Since the coefficients and the RHS in fact do not depend on the time parameter, remark 3.16 follows from corollary 2.19.

Proof of the uniqueness claim in Theorem 3.14. The strategy will be to compare an arbitrary C^2-solution $\bar{F}_\lambda$, defined in a chart x_λ, with the solution constructed in proposition 3.17 denoted by F_λ. Since the coefficients of the asymptotic equation (3.2.15a) equal the coefficients of the equation solved by F_λ defined on a different domain, Lemma 3.28 yields the local constants desired by the uniqueness Theorem 2.26. The proof of the theorem provides us with an expression for the slope c_0 of the cone on which uniqueness holds by choosing $u_1 = F_\lambda$ in the difference equation (2.4.9). It is given by the following consideration

$$\begin{aligned} 1 + \big(2\big|\big(g^{0j}(F_\lambda)\big)\big|_e + \big|\big(g^{ij}(F_\lambda)\big)\big|_e\big)\tfrac{1}{\lambda} \\ \leq 1 + m^{1/2}\tilde{\mu}^{-1}K_1^2\big(2\tilde{\lambda}^{-1}K_0K_1 + 1\big)\big(1 + m^{1/2}\tilde{\mu}^{-1}K_0^2\big) =: c_0, \end{aligned} \tag{3.2.33}$$

where we used estimate (2.4.5) and Lemma 3.20. □

3.2.2 Properties of a solution

In this section we will develop some estimates for the family of solutions to the membrane equation obtained in Theorem 3.14. The goal is to show that for a small enough time parameter and radius the solutions are in fact embeddings and therefore represent a submanifold.

Let $\lambda \in \Lambda$ be fixed. We begin with an estimate for the second fundamental form of a solution $F := F_\lambda$ obtained by Theorem 3.14. To measure the second fundamental form a Riemannian metric on the pre-image of F is necessary. A natural choice is the pullback of the Riemannian metric on the ambient manifold, so let $\hat{e} := F^*E$. To gain advantage from the estimates following from the construction of the solution a comparison between $\hat{e}$ and the Euclidean metric w.r.t. coordinates will be established in the next lemma.

Lemma 3.34. *The matrix $\hat{e}_{\mu\nu}$ satisfies*

$$\begin{aligned} \mu_e\delta_{\mu\nu} \leq \hat{e}_{\mu\nu} \leq M_e\delta_{\mu\nu} \\ \textit{with} \quad \mu_e = (1 - \delta_0/2)\min\big(L_2(1 - r_0), (1 + \delta_0/2)^{-1}\omega_1^{-2}(1 - R_0)\big) \\ \textit{and} \quad M_e = (K_0^2 + K_1^2)C_0^h. \end{aligned}$$

Proof. Positive definiteness follows from the comparison of E_{AB} with the Euclidean metric stated in (4.1.10) and the property of $\partial_t F$ and DF to be timelike and spacelike, respectively, stated in (3.2.19).

The bound for $\hat{e}_{\mu\nu}$ follows from the bounds for $\partial_t F$ and DF stated in (3.2.20b) and (3.2.20a), respectively, and the bound for E_{AB}, which coincides with the one for h_{AB} (cf. (4.1.8a)). □

This yields a bound for the second fundamental form of the mapping F by using second-order estimates.

Proposition 3.35. *The second fundamental form of a solution F of the IVP (3.2.10) obtained by Theorem 3.14 satisfies the following uniform inequality*

$$|I\!I|_{\hat{e},E} \leq C_{I\!I}.$$

Proof. We first consider $|\mathring{I\!I}|_{e,e}$ with the representation (1.2.5) for the second fundamental form of F. Estimates for second-order derivatives of the solution F to the reduced membrane equation can be obtained via the definition of W (cf. (3.2.23)) and the Sobolev embedding theorem. A bound for the second-order time derivative follows by considering the equation and applying the bounds for the coefficients and the RHS. Bounds for first-order derivatives of F follow from the definition of Ω in (3.2.16). Applying bounds for the induced metric $g_{\mu\nu}$, bounds for the metric h_{AB} and its Christoffel symbols $\mathbf{\Gamma}^A_{BC}$ established in (3.2.22a), (4.1.8a) and (4.1.17), respectively, gives an estimate for $|\mathring{I\!I}|_{e,e}$. The desired bound follows from the comparison of $\hat{e}$ and E with the Euclidean metric stated in Lemma 3.34 and (4.1.10), respectively. □

Proposition 3.36. *Suppose F is a solution of the IVP (3.2.10) for the membrane equation obtained by Theorem 3.14.*
Then there exist constants $0 < \tilde{T} \leq T'$ and θ' depending on the bounds for second-order derivatives and the estimates ensuring the timlikeness of $\partial_t F$ and the spacelikeness of DF stated in (3.2.19) such that $F : [-\tilde{T}, \tilde{T}] \times B^e_{\theta'\rho_1/2}(0)$ is an embedding.

Proof. The claim follows from the inverse function theorem. An inspection of the proof to be found in [Spi65] shows that the region, in which the function is invertible depends on a bound for the inverse of the differential at the origin and a bound for the second-order derivatives. □

3.3 Gluing local solutions

The goal of this section is to obtain existence and uniqueness of solutions to the Cauchy problem (1.3.2) for fixed initial immersion, direction, lapse and shift. From the uniformity assumptions imposed on the initial data it will follow that the solution obtained as an immersion has the domain $[-T, T] \times M$ with a fixed constant $T > 0$.

Recall the notations of the main problem (0.4.1). Assume M to be an m-dimensional manifold and $\varphi : M \to \mathbb{R}^{n,1}$ to be an immersion of Σ_0. As in section 3.2 we use a timelike vector field $\chi : M \to T\mathbb{R}^{n,1}$ along φ as initial velocity for the Cauchy problem (1.3.2). Recall that τ_0 denotes the timelike direction of the Minkowski space and $\widehat{\nabla}$ denotes the connection on the pullback bundle $\varphi^* T\mathbb{R}^{n,1}$.

Let $s > \frac{m}{2} + 1$ be an integer. We make the following uniform assumptions on the initial data φ and χ.

Assumptions 3.37.

There exist constants $\omega_1, C^\varphi_0, \ldots, C^\varphi_s$ such that

$$\inf\{-\eta(\gamma, \tau_0) : \gamma \text{ timelike future-directed unit normal to } \Sigma_0\} \leq \omega_1 \tag{3.3.1a}$$

$$\text{and} \quad |\widehat{\nabla}^\ell \mathring{I\!I}|_{\mathring{g},e} \leq C^\varphi_\ell \text{ for } 0 \leq \ell \leq s.$$

There exist constants $L_1, L_2, L_3, C^\chi_1, \ldots, C^\chi_{s+1}$ such that

$$-L_1 \leq \eta(\chi, \chi) \leq -L_2, \quad -\eta\big(\tfrac{\chi}{\langle\chi\rangle}, \tau_0\big) \leq L_3, \text{ where } \langle\chi\rangle^2 = -\eta(\chi, \chi) \tag{3.3.1b}$$

$$\text{and} \quad |\widehat{\nabla}^\ell \chi|_{\mathring{g},e} \leq C^\chi_\ell \text{ for } 1 \leq \ell \leq s+1.$$

Remark 3.38. Suppose the initial submanifold Σ_0 to be compact. Then the assumptions are satisfied since we can pick a finite covering of the submanifold and choose the largest constants occurring in the finitely many subsets.

The next theorem states the main result of this section providing a solution to the IVP (1.3.2) in Minkowski space.

Theorem 3.39. *Suppose the initial data $\varphi \in C^{s+2}$ and $\chi \in C^{s+1}$ satisfy the assumptions 3.37.*

Then there exist a constant $T > 0$ and a C^2-solution $F : [-T, T] \times M \to \mathbb{R}^{n,1}$ of the membrane equation

$$g^{\mu\nu}\partial_\mu\partial_\nu F^A - \Gamma^\lambda \partial_\lambda F^A = 0 \tag{3.3.2}$$

in harmonic map gauge w.r.t. the background metric defined by (1.3.7) attaining the initial values

$$F\big|_{t=0} = \varphi \qquad \text{and} \qquad \tfrac{d}{dt}F\big|_{t=0} = \chi. \tag{3.3.3}$$

Let F and $\bar{F}$ be two such solutions. Then there exists a constant T_0 such that F and $\bar{F}$ coincide for $-\min(T, T_0) \le t \le \min(T, T_0)$.

Remark 3.40.
1. The precise value of the constant T_0 will be given in (3.3.9).
2. The choice of the background metric to be defined by the initial values (3.3.3) is not optimal in the sense that we need bounds for derivatives of one order higher for the initial values to use the existence Theorem 2.9 for hyperbolic equations.

Remark 3.41. From Remark 3.16 we obtain that, providing $s > \frac{m}{2} + 1 + \ell_0$ with an integer ℓ_0, the solution $F : [-T, T] \times M \to \mathbb{R}^{n,1}$ is in fact of class $C^{2+\ell_0}$.

Remark 3.42. The theorem applies to the situation, where the assumptions 3.37 are only valid in a neighborhood of the initial submanifold Σ_0.

To obtain the existence claim we will show that the special graph representation derived in section 3.1 satisfies the assumptions 3.13 and therefore Theorem 3.14 gives a family of solutions in a decomposition consisting of special graph representations. The desired estimates will be derived in the following proposition.

Proposition 3.43. *Let the initial data $\varphi \in C^{s+2}$ and $\chi \in C^{s+1}$ satisfy the assumptions 3.37. Let $p \in M$ and suppose x, y are coordinates for M and $\mathbb{R}^{n,1}$, respectively, such that $y \circ \varphi \circ x^{-1}$ is the special graph representation (3.1.1) obtained in section 3.1.*

Then the coordinates x, y and the representations Φ and χ_{xy} of φ and χ w.r.t. these charts satisfy the conditions 3.13.

Proof. The proof is divided in three steps. In the first step we will show the parts 1 and 2 of the assumptions concerning the chart x and the representation Φ. In the last two steps we will derive the assumptions concerning the representation of the initial velocity.

1. The first condition on the chart x follows from Lemma 3.7 yielding that a Euclidean ball with radius ρ_1 about 0 is contained in the image of x. The positive definiteness of the induced metric at the origin can be obtained from Lemma 3.6 by using that φ is an immersion of a uniformly spacelike submanifold with bounded curvature defined in 3.4.

   The first condition of (3.2.2) for derivatives of the representation Φ can be derived from $|Du^0|_e < 1$ and $Du^a(0) = 0$ for all a, which gives us $|D\Phi(0)|_e^2 < 1 + m =: C_{w0}^2$. The second condition was established in Lemma 3.7 and bounds for higher derivatives of Φ were derived in Lemma 3.11.

2. The desired inequality (3.2.3) for $\eta(\chi_{xy}, \chi_{xy})$ is satisfied by assumption. To estimate $|\chi_{xy}|_e$ we compute

$$-(\chi_{xy}^0)^2 + \textstyle\sum_{\underline{a}} (\chi_{xy}^{\underline{a}})^2 \leq -L_2 \quad \Longrightarrow \quad \textstyle\sum_{\underline{a}} (\chi_{xy}^{\underline{a}})^2 \leq -L_2 + (\chi_{xy}^0)^2.$$

   The assumption on the angle yields

$$-\langle\!\langle \chi/\langle\chi\rangle, \tau_0 \rangle\!\rangle \leq L_3 \quad \Longrightarrow \quad \chi_{xy}^0 \leq \langle\chi\rangle L_3 \leq L_1^{1/2} L_3.$$

   Since χ is supposed to be future directed, we have $\langle\!\langle \chi_{xy}, \tau_0 \rangle\!\rangle = -\chi_{xy}^0 < 0$. Hence

$$|\chi_{xy}|_e^2 = (\chi_{xy}^0)^2 + \textstyle\sum_{\underline{a}} (\chi_{xy}^{\underline{a}})^2 \leq -L_2 + 2(\chi_{xy}^0)^2 \leq -L_2 + 2L_1 L_3^2 := (\tilde{C}_0^\chi)^2. \quad (3.3.4)$$

3. Bounds for higher derivatives of the representation χ_{xy} can be obtained by a similar device to the proof for higher derivatives of the graph functions stated in Lemma 3.11. We start with the identity

$$\partial^k \chi_{xy} = \widehat{\nabla}^k \chi_{xy} + \textstyle\sum \partial^{\alpha_1} \mathring{\Gamma} * \cdots * \partial^{\alpha_p} \mathring{\Gamma} * \widehat{\nabla}^\ell \chi_{xy}$$

   for the components of the initial velocity χ_{xy}^B which follows analogously to equation (3.1.13) for the second fundamental form. The sum ranges over a certain subset of all tuples (p, α_i, ℓ) such that $p + \sum \alpha_i + \ell = k + 1$. We infer from this expression that coordinate derivatives of the initial velocity are bounded, if the covariant derivatives are bounded. The Christoffel symbols can be estimated as in the proof of Lemma 3.11. Observe that also the full range of bounded derivatives of the second fundamental form is needed. □

Remark 3.44. The preceding proposition shows that any decomposition consisting of special graph representations of φ satisfies the conditions of Theorem 3.14.

To obtain a solution of the form $F : [-T, T] \times M \to \mathbb{R}^{n,1}$, we need to glue solutions obtained in different charts of a decomposition. This can be done if the family of solutions obtained by Theorem 3.14 coincide on common domains of coordinates on M, which will be shown in the sequel.

Let (U, x, V, y) and $(\hat{U}, \hat{x}, \hat{V}, \hat{y})$ be two decompositions of φ such that the representations of the initial data and the decompositions satisfy the assumptions 3.13 possibly with different constants. Let Φ, χ_{xy} and $\hat{\Phi}$, $\chi_{\hat{x},\hat{y}}$ denote the representations of the initial data φ and χ w.r.t. the decompositions (U,x,V,y) and $(\hat{U},\hat{x},\hat{V},\hat{y})$ resp. Assume the images of the coordinates x and $\hat{x}$ to contain a Euclidean ball of radius ρ_1 and $\hat{\rho}_1$ resp. Let F_0 and $\hat{F}$ be the solutions to the IVP (3.2.10) obtained by Proposition 3.17. Assume the function F_0 to be defined on a ball $B^e_{\theta\rho_1}(0)$ and analogously $\hat{F}$ to be defined on $B^e_{\hat{\theta}\hat{\rho}_1}(0)$. We want to compare the two solutions, so suppose $U \cap \hat{U} \neq \varnothing$.

To derive a quantitative description of an uniqueness result a further condition on the coordinates x and $\hat{x}$ are needed. Let the following assumption be satisfied

$$\text{There exist constants } G_1, G_2 \text{ such that the components of the induced metric w.r.t. } x \text{ satisfy } G_1\delta_{ij} \leq \mathring{g}_{ij} \leq G_2\delta_{ij}. \tag{3.3.5}$$

Suppose the components of $\mathring{g}$ w.r.t. the coordinates $\hat{x}$ satisfy the same assumption with the constants G_1, G_2 replaced by $\hat{G}_1, \hat{G}_2$, respectively. Denote the change of coordinates by $\hat{u}$ and u such that $\hat{x} = u \circ x$ and $\hat{y} = \hat{u} \circ y$. Set

$$G(t,z) = \hat{u}^{-1} \circ \hat{F}\big(t, u(z)\big). \tag{3.3.6}$$

Then G is defined on the image of the chart x and can by compared with F_0. We have to show that G is a solution to the reduced membrane equation (1.3.6) and that the initial values of G and F_0 coincide wherever G is defined. This will be done in the next lemma.

Lemma 3.45. *The function G defined in (3.3.6) satisfies the reduced membrane equation (1.3.6) with the background metric defined by (1.3.7) and the initial values of F_0 and G coincide.*

Proof. The membrane equation (1.3.1b) is coordinate independent as well as is the gauge condition (1.3.5). Therefore both equations remain valid after a change of coordinates.

We compute the initial values for G arriving at

$$\begin{aligned} G(0,z) = \hat{u}^{-1} \circ \hat{F}\big(0,u(z)\big) &= \hat{u}^{-1} \circ \hat{y} \circ \varphi\big(\hat{x}^{-1} \circ u(z)\big) = y \circ \varphi\big(x^{-1}(z)\big) = \Phi(z) \\ \text{and} \quad d\hat{u}^{-1}\big(\partial_t \hat{F}\big(0,u(z)\big)\big) &= d(\hat{u}^{-1} \circ \hat{y})\big(\chi(\hat{x}^{-1} \circ u(z))\big) = \chi_{xy}(z). \end{aligned}$$ ☐

The next proposition will show equality of F_0 and G providing a quantitative description of the cone on which equality holds.

Proposition 3.46. *Let $q \in M$ such that*

$$B^{\mathring{g}}_r(q) \subset B^{\mathring{g}}_{G_1^{1/2}\theta\rho_1/2}\big(x^{-1}(0)\big) \cap B^{\mathring{g}}_{\hat{G}_1^{1/2}\hat{\theta}\hat{\rho}_1/2}\big(\hat{x}^{-1}(0)\big).$$

Then $F_0 = G$ within the cone

$$C = \big\{(t,z) : |z - x(q)|_e \leq -c_0 t + G_2^{-1/2} r\big\} \subset \mathbb{R}^{1+m}.$$

Proof. The additional condition (3.3.5) for both decompositions provides us with a comparison between Euclidean balls and balls w.r.t. the metric $\mathring{g}$. it follows that

$$B^e_{G_2^{-1/2}r}\big(\hat{x}(q)\big) \subset x\big(B^{\mathring{g}}_r(q)\big) \subset B^e_{\theta\rho_1/2}\big(x^{-1}(0)\big),$$

where the Euclidean balls are taken w.r.t. the chart x and an analog result holds for the chart $\hat{x}$. We conclude that the transferred solution G is at least defined on $B^e_{G_2^{-1/2}r}\big(x(q)\big)$ attaining the initial values Φ and χ_{xy} in that region. From the uniqueness claim of Theorem 3.14 it follows that $F_0 = G$ on the desired cone with the slope c_0 defined by (3.2.33). □

Remark 3.47. We only considered positive values of t, but the same holds for negative values by applying a time reversal (cf. Remark 3.32).

It it now possible to construct a solution to the IVP (1.3.2) with a domain of the form $[-T, T] \times M$.

Proof of the existence claim in 3.39. The first step consists of constructing a suitable covering of the manifold M. For $p \in M$ consider charts x_p on M and y_p on $\mathbb{R}^{n,1}$ such that $y_p \circ \varphi \circ x_p^{-1}$ is the special graph representation with center p constructed in section 3.1. From Proposition 3.43 we derive that the representations of the initial data w.r.t. x_p and y_p satisfy the conditions 3.13 with constants independent of p. Hence, Proposition 3.17 yields the existence of a solution F_p defined on the image of x_p and attaining the initial data on the ball $B^e_{\theta\rho_1/2}(0)$ with a constant θ independent of the point p.

Observe that the special graph representation about each point satisfies uniformly the additional assumption (3.3.5) which is stated in Remark 3.9. Let $\tilde{G}_1$ and $\tilde{G}_2$ denote the constants defined in (3.1.11b) providing a comparison of Euclidean balls and balls w.r.t. $\mathring{g}$.

Let $\mathfrak{U}$ be the family of balls $B^{\mathring{g}}_{\tilde{G}_1^{1/2}\theta\rho_1/4}(p)$ for $p \in M$. Choose a locally finite covering of M subordinate to $\mathfrak{U}$ and denote it by

$$(W_\lambda)_{\lambda\in\Lambda} \text{ with } W_\lambda = B^{\mathring{g}}_{\tilde{G}_1^{1/2}\theta\rho_1/4}(p_\lambda). \tag{3.3.7}$$

Consider the decomposition $(U_\lambda, x_\lambda, V_\lambda, y_\lambda)_{\lambda\in\Lambda}$ of φ where $y_\lambda \circ \varphi \circ x_\lambda^{-1}$ is the special graph representation (3.1.1) with center p_λ. From Theorem 3.14 we get a family (F_λ) of solutions defined on $[-T', T'] \times B^e_{\theta\rho_1/2}(0)$, where the constant T' is independent of λ and the ball $B^e_{\theta\rho_1/2}(0)$ is a subset of the image of x_λ.

Define a mapping $F : U \subset \mathbb{R} \times M \to \mathbb{R}^{n,1}$ by

$$F(t, p) = y_\lambda^{-1}\big(F_\lambda(t, x_\lambda(p))\big) \quad \text{if } p \in W_\lambda. \tag{3.3.8}$$

Here, the set U is a neighborhood of $\{0\} \times M$. A priori it is not clear whether the set U has the form $[-T, T] \times M$ for a fixed constant $T > 0$.

We need to show that $F(t, p)$ is well-defined. Let $q \in W_{\lambda_1} \cap W_{\lambda_2}$ for $\lambda_1, \lambda_2 \in \Lambda$. By enlarging the radii of the balls W_{λ_1} and W_{λ_2} it follows that

$$B^{\mathring{g}}_{\tilde{G}_1^{1/2}\theta\rho_1/4}(q) \subset B^{\mathring{g}}_{\tilde{G}_1^{1/2}\theta\rho_1/2}(p_{\lambda_1}) \cap B^{\mathring{g}}_{\tilde{G}_1^{1/2}\theta\rho_1/2}(p_{\lambda_2}).$$

Denote changes of coordinates by $\hat{u} \circ y_{\lambda_1} = y_{\lambda_2}$ and $u \circ x_{\lambda_1} = x_{\lambda_2}$. Proposition 3.46 now yields that

$$y_\lambda^{-1}\big(F_\lambda(t, x_\lambda(q))\big) = y_\lambda^{-1} \circ \hat{u}^{-1} \circ F_{\lambda_2}\big(t, u \circ x_\lambda(q)\big) = y_{\lambda_2}^{-1}\big(F_{\lambda_2}(t, x_{\lambda_2}(q))\big)$$

provided t satisfies

$$0 \leq t \leq T_0 := \tfrac{1}{4c_0}\tilde{G}_2^{-1/2}\tilde{G}_1^{1/2}\theta\rho_1. \tag{3.3.9}$$

Applying Remark 3.47 gives us that the values of $F(t,p)$ coincide also for negative t.

The preceding considerations yield that there exist a constant $T > 0$ such that the mapping F given by (3.3.8) is defined on $[-T, T] \times M$. $\square$

Remark 3.48. The proof provides an estimate for the existence time T of the constructed solution F. We get $T \geq \min(T', T_0)$ with the lower bound T' from Remark 3.31 and T_0 introduced in (3.3.9). Further, we can give an expression for the constant c_E based on the assumptions 3.37. From estimate (3.3.4) we derive the bound $\tilde{C}_0^\chi$ and the value $C_{w_0}^2 = 1{+}m$ follows from the special graph representation. This yields the following expression for c_E controlling the existence time T'

$$\begin{aligned} c_E = 4(1+m+\delta_1^2)\big(1 + L_2^{-1}(1-r_0)^{-1}\big(-L_2 + 2L_1L_3^2 + \delta_2^2\big)\big) \\ + 2\big(-L_2 + 2L_1L_3^2 + \delta_2^2\big)\big(1 + m^{1/2}\omega_1^2(1-R_0)^{-1}(1+m+\delta_1^2)\big). \end{aligned} \tag{3.3.10}$$

The constants controlling the angle for the initial submanifold and the initial velocity enter the constant c_E having the effect that the existence time shrinks if the initial submanifold or the initial velocity are closer to the light cone. The bound for the second fundamental form enters the definition of T_0 through the radius ρ_1.

Remark 3.49. By Proposition 3.36 it follows that local solutions obtained by Theorem 3.14 can be restricted such that they are embeddings, provided that the initial submanifold is locally embedded. By using this shrinked domain we see that a solution defined by (3.3.8) is also locally an embedding.

Remark 3.50. The construction provides us with a lower bound for the supremum over length of timelike curves in $\operatorname{im} F$. Consider the curve $\gamma(t) := F(t,p)$ for $p \in M$. To derive a lower bound for the length of γ we introduce the function $w(r)$ by

$$w(r) = \textstyle\int_0^r \big(-h(\dot\gamma,\dot\gamma)\big)^{1/2}\, d\sigma = \int_0^r \big(-g_{00}(\sigma,p)\big)^{1/2}\, d\sigma.$$

The construction of F based on Proposition 4.15 yields that $\dot{w}(r) \geq \tilde{\lambda}^{1/2}$ as can be seen from (3.2.19) where the constant $\tilde\lambda > 0$ is defined. The derivative of w is non-zero, therefore an inverse function $\rho(s)$ to $w(r)$ exists. The curve $\tilde\gamma(s) = \gamma\big(\rho(s)\big)$ is parametrized by proper time. Since the inverse function is defined at least on the interval $[0, w(T)]$, the length of γ is at least $T\tilde\lambda^{1/2}$.

The preceding consideration is valid for each $p \in \Sigma_0$ which follows from the uniformity of the assumptions on Σ_0.

Proof of the uniqueness claim in 3.39. Suppose $\bar{F} : [-T, T] \times M \to \mathbb{R}^{n,1}$ is a C^2-solution to the membrane equation in harmonic map gauge attaining the initial values (3.3.3). To show uniqueness we compare this solution with the solution $F : [-T, T] \times M \to \mathbb{R}^{n,1}$ defined by (3.3.8).

Consider the decomposition $(U_\lambda, x_\lambda, V_\lambda, y_\lambda)_{\lambda\in\Lambda}$ of φ used in the construction of F. Proposition 3.43 yields that the assumptions 3.13 are satisfied uniformly in λ. If $q \in M$, then we find a $\lambda \in \Lambda$ such that the image of the chart x_λ contains a ball of radius $\tilde{G}_2^{-1/2}\tilde{G}_1^{1/2}\theta\rho_1/4$ about $z = x_\lambda(q)$ (see Remark 3.9). From the uniqueness result of Theorem 3.14 we conclude that the representations of $\bar{F}$ and F w.r.t. x_λ and y_λ coincide on a double-cone with height $2T_0$ as defined in (3.3.9). This yields $\bar{F}(t, q) = F(t, q)$ for $-\min(T, T_0) \le t \le \min(T, T_0)$. $\square$

4 Generalization to a Lorentzian manifold

In this section we will consider the Cauchy problem (1.3.2) for immersions in the case of a Lorentzian ambient manifold. Restrictions on the causal structure of the ambient manifold will be made by assuming that a time function exists, i.e. a function whose gradient is timelike everywhere. As a consequence of this assumption the ambient manifold is globally hyperbolic (see e.g. [O'N83] or [Wal84] for further reference). Such manifolds carry a causal structure similar to the Minkowski space providing the possibility to take advantage of the results obtained in section 3.

The layout of this section is based on section 3. To obtain a graph representation (section 4.2) with similar properties than derived in section 3.1 it is necessary to construct coordinates on the ambient manifold which will be done in section 4.1.

Some notations taken from [CK93] and [Bar84] will be used. Let N be an $(n+1)$-dimensional smooth manifold endowed with a Lorentzian metric h. From the notations 1.1 we adopt the designations for the Christoffel symbols and the curvature of h. An assumption on the causal structure will be made by the following term.

Definition 4.1. A real-valued function τ on N is called *time function* if its gradient $\mathbf{D}\tau$ is timelike everywhere. Such a time function induces a *time foliation* of N by its levelsets $\mathscr{S}_\tau$, which are spacelike hypersurfaces. The induced connection on a slice $\mathscr{S}_\tau$ will be denoted by $\mathcal{D}$, the corresponding Christoffel symbols by γ and the curvature by $\mathcal{R}$.

We further introduce the *lapse* ψ of the foliation by

$$\psi^{-2} = -h(\mathbf{D}\tau, \mathbf{D}\tau) \tag{4.0.1}$$

and derive the unit future directed normal to the time foliation by

$$\widehat{T} = -\psi\mathbf{D}\tau. \tag{4.0.2}$$

The dual vector field ∂_τ to the differential $d\tau$ of the time function is given by

$$\partial_\tau = -\psi^2\mathbf{D}\tau \quad \Rightarrow \quad \partial_\tau = \psi\widehat{T}.$$

The existence of such a function provides us with a possibility of introducing a Riemannian metric.

Definition 4.2. Suppose N admits a time function τ. We consider the decomposition into tangential and normal part w.r.t. the slices, namely

$$v \in T_qN \Rightarrow v = v^\top + v^\perp = v^\top + \lambda \widehat{T}.$$

We introduce a Riemannian metric E *associated to the time foliation* by a flip of the sign of the unit normal $\widehat{T}$, more precisely

$$E(v,w) := h\big(v^\top, w^\top\big) + \lambda\mu, \quad \text{where } v^\perp = \lambda\widehat{T} \text{ and } w^\perp = \mu\widehat{T}. \tag{4.0.3}$$

Since the slices are spacelike this metric is Riemannian.

In the sequel we assume N to admit a time function.

4.1 Special coordinates

The coordinates constructed in this section will make contact to the results for the Minkowski space in the sense that the representation of the metric h coincides in the center of the coordinates with the Minkowski metric. These coordinates will be used to establish a special graph representation similar to section 3.1.

Pick a point $q_0 \in N$ and set $\tau_0 = \tau(q_0),\ \psi_0 = \psi(q_0)$. To adjust the lapse we introduce a new time function $\tilde\tau = \psi_0\tau$ with the corresponding lapse and dual vector field to $d\tilde\tau$ defined as follows

$$\begin{aligned} \tilde\psi^{-2} &= -h(\mathbf{D}\tilde\tau, \mathbf{D}\tilde\tau) = \psi_0^2\psi^{-2} \\ \partial_0 &= \tilde\psi\widetilde{T} = \psi_0^{-1}\psi\widehat{T} = \psi_0^{-1}\partial_\tau. \end{aligned}$$

The following calculation shows that the unit normals w.r.t. both time functions coincide

$$-\tilde\psi\mathbf{D}\tilde\tau = -\psi_0^{-1}\psi\psi_0\mathbf{D}\tau = \widehat{T}.$$

Now we have a lapse equal to 1 in the point q_0. The slices remain the same hypersurfaces, only the indexing has changed by means of $\tilde\tau(q) = \sigma \Leftrightarrow \tau(q) = \psi_0^{-1}\sigma$.

Let Φ_s be the flow of the vector field ∂_0 emanating from a neighborhood U of $q_0 \in \mathscr{S}_{\tau_0}$. If $\gamma(\sigma)$ is a part of the flow with $\gamma(0) = q \in U \subset \mathscr{S}_{\tau_0}$ and $\dot\gamma(\sigma) = \partial_0\big(\gamma(\sigma)\big)$, then

$$\tilde\tau\big(\gamma(\sigma)\big) - \tilde\tau(q_0) = \textstyle\int_\gamma d\tilde\tau = \int_0^\sigma d\tilde\tau(\dot\gamma)\, d\tau = \sigma.$$

Therefore, the flow Φ_s induces a diffeomorphism from $U \subset \mathscr{S}_{\tau_0}$ onto its image contained in $\mathscr{S}_{t_0+\psi_0^{-1}s}$.

We now construct coordinates on N as follows. Pick normal coordinates z w.r.t. the slice $\mathscr{S}_{\tau_0}$ with center q_0 and define

$$y(q) = \big(\tilde\tau(q) - \tilde\tau_0, z \circ \Phi_{-(\tilde\tau(q)-\tilde\tau_0)}(q)\big) \tag{4.1.1}$$

where $\tilde{\tau}_0 = \tilde{\tau}(q_0)$. The basis vector fields of the tangent space of N are given by

$$dy^{-1}_{(s,x)}(e_A) = \begin{cases} \partial_0\big(y^{-1}(s,x)\big), & A = 0 \\ \partial_{\underline{b}}(s,x) = d\Phi_s \circ dz_x^{-1}(e_{\underline{b}}), & A = \underline{b} \text{ with } 1 \leq \underline{b} \leq n. \end{cases}$$

The definition of the spatial vector fields $\partial_{\underline{b}}$ yields that $\mathcal{L}_{\partial_0}\partial_{\underline{b}}(s,x) = 0$. The metric h has therefore the following representation in these coordinates

$$h_{AB}(s,x) = \begin{pmatrix} -\psi_0^{-2}\psi^2(s,x) & 0 \\ 0 & h_{\underline{a}\underline{b}}(s,x) \end{pmatrix} \quad \text{with} \quad h_{AB}(0,0) = \eta_{AB} \tag{4.1.2}$$

where $h_{\underline{a}\underline{b}}(0,x)$ is the representation of the induced metric on the slices w.r.t. the normal coordinates z. The zeros come from the fact that the gradient of the time function is normal to the slices.

We seek quantitative control over the representation of the metric and higher derivatives of it. This will be done by first considering the metric components and then the Christoffel symbols $\mathbf{\Gamma}^A_{BC}$.

A quantitative description makes it necessary to impose the following conditions on (N,h).

$$\begin{aligned} &\text{There exist constants } C_0^N, C_1, C_2 \text{ and } C_1^\tau \text{ such that} \\ &|\mathbf{Rm}|_E \leq C_0^N, \ C_1 \leq \psi \leq C_2 \quad \text{and} \quad |\mathbf{D}(\mathbf{D}\tau)|_E \leq C_1^\tau, \end{aligned} \tag{4.1.3}$$

where $\mathbf{D}(\mathbf{D}\tau)$ denotes the $(1,1)$-tensor obtained by applying the covariant derivative to the gradient of τ. Then we apply the norm w.r.t. the metric E, introduced in definition 4.2, to this tensor. These conditions yield a bound for the curvature of the slices due to the Gauß equations $|\mathcal{R}| \leq c(|\mathbf{Rm}|_E + |k|^2)$, where k denotes the $(0,2)$-version of the second fundamental form of the slices w.r.t. $\widehat{T}$. The norm of k is taken w.r.t. the Riemannian metric on the slices induced by h. A bound for the extrinsic curvature of the slices can be derived via the following computation for $v, w \in T\mathscr{S}_\tau$

$$\begin{aligned} k(v,w) &= -h(\widehat{T}, \mathbf{D}_v w) = h(\mathbf{D}_v\widehat{T}, w) \\ \text{and} \qquad \mathbf{D}_v\widehat{T} &= -v(\psi)\widehat{T} - \psi\mathbf{D}_v(\mathbf{D}\tau). \end{aligned}$$

If inserted the term involving a derivative of the lapse function vanishes since $w \perp T$ and the last term of the second line is bounded by assumption (4.1.3).

The second fundamental form k can also be seen to be bounded via the representation of $\mathbf{D}(\mathbf{D}\tau)$ in this coordinates, namely

$$\begin{aligned} S_{\underline{a}}^{\underline{b}} &= h^{\underline{b}\underline{c}} h(\mathbf{D}_{\underline{a}}(\mathbf{D}\tau), \partial_{\underline{c}}) = -\tilde{\psi}^{-1} k_{\underline{a}}{}^{\underline{b}} \\ S_A^0 &= h^{00} h(\mathbf{D}_A(\mathbf{D}\tau), \partial_0) = \tfrac{1}{2}\partial_A \tilde{\psi}^2, \\ \text{and} \qquad S_0^{\underline{b}} &= g^{\underline{a}\underline{b}} h(\mathbf{D}_0(\mathbf{D}\tau), \partial_{\underline{a}}) = -\tilde{\psi}^{-1}\mathcal{D}^{\underline{b}}\tilde{\psi}, \end{aligned}$$

where we set $S_A{}^B = \mathbf{D}_A\mathbf{D}^B\tau$.

We begin with estimates for the metric components and the Christoffel symbols w.r.t. the normal coordinates z on the slice $\mathscr{S}_{\tau_0}$. It is a well known fact that the components of the metric admits a Taylor expansion where the coefficients are derived from linear combinations of the curvature components. The error term can also be estimated by such combinations. The Taylor expansion can be found in [Wil93]. It follows for the components of the metric that

$$h_{\underline{a}\underline{b}}(0,x) - \delta_{\underline{a}\underline{b}} = C_{\underline{a}\underline{c}\underline{d}\underline{b}} x^{\underline{c}} x^{\underline{d}}.$$

If we assume $\frac{1}{3}\delta_{\underline{a}\underline{b}} \le h_{\underline{a}\underline{b}}(0,x) \le 3\delta_{\underline{a}\underline{b}}$, then there exist a constant c_1 such that

$$\left|\left(h_{\underline{a}\underline{b}}(0,x)\right) - \left(\delta_{\underline{a}\underline{b}}\right)\right|_e \le c_1 \sup_{w\in B_{|x|}(0)} |\mathcal{R}(w)|\,|x|^2.$$

From the Gauß-equations we derive that the curvature of the slices is bounded by the curvature of N and the derivative of the gradient of the time function τ. By choosing $\rho_0 = c_2\big(C_0^N + (C_1^\tau)^2\big)^{-1/2}$ with a constant c_2 and the constants occurring in the assumptions (4.1.3) it follows

$$\left(1 - \tfrac{1}{6}\tfrac{|x|^2}{\rho_0^2}\right)\delta_{\underline{a}\underline{b}} \le h_{\underline{a}\underline{b}}(0,x) \le \left(1 + \tfrac{1}{6}\tfrac{|x|^2}{\rho_0^2}\right)\delta_{\underline{a}\underline{b}}. \tag{4.1.4}$$

with $|x| < \rho_0$. We further need the evolution of the metric components to get a bound on the whole domain of the coordinates. From the vanishing Lie derivative we derive that the covariant derivative of the vector fields ∂_0 and $\partial_{\underline{a}}$ commute. Hence a similar device as stated in [CK93], [FR00] p. 138 (or [HP96] for the Riemannian case) gives us that

$$\tfrac{d}{ds} h_{\underline{a}\underline{b}} = 2\tilde{\psi} k_{\underline{a}\underline{b}}. \tag{4.1.5}$$

Assuming that $\frac{1}{3}\delta_{\underline{a}\underline{b}} \le h_{\underline{a}\underline{b}}(s,x) \le 3\delta_{\underline{a}\underline{b}}$ gives us a bound for the RHS of equation (4.1.5) in terms of the bound for the lapse and the derivative of the gradient of the time function.

By shrinking the constant c_2 in the definition of ρ_0 and assuming $|s| < \rho_0$ we arrive at

$$\left|\left(h_{\underline{a}\underline{b}}(s,x)\right) - \left(h_{\underline{a}\underline{b}}(0,x)\right)\right| < \tfrac{1}{6}\tfrac{|s|}{\rho_0}.$$

For abbreviation we set $\lambda(s,x) = \frac{1}{6}\frac{|s|}{\rho_0} + \frac{1}{6}\frac{|x|^2}{\rho_0^2} < 2^{-1}\frac{2}{3} =: \frac{1}{2}\delta_0$ then the inequalities (4.1.4) yield estimates for $h_{\underline{a}\underline{b}}(s,x)$ as follows

$$\begin{aligned}(1-\delta_0/2)\delta_{\underline{a}\underline{b}} \le \big(1-\lambda(s,x)\big)\delta_{\underline{a}\underline{b}} \le h_{\underline{a}\underline{b}}(s,x)&\\ \le \big(1+\lambda(s,x)\big)\delta_{\underline{a}\underline{b}} \le (1+\delta_0/2)\delta_{\underline{a}\underline{b}}.&\end{aligned} \tag{4.1.6}$$

With $\mathbf{D}(\mathbf{D}\tau)$ also the full gradient of the lapse function $\tilde{\psi}$ is bounded. Hence

$$|h_{00}(s,x) - h_{00}(0,0)| \le 2C_1^\tau C_1^{-1}\big(|s| + 2C_1^{-1}|x|\big).$$

By further shrinking the radius ρ_0 and setting $\delta(s,x) = \frac{1}{6}\frac{1}{\rho_0}\big(|s|+|x|\big) < \delta_0/2$ we obtain

$$-1-\delta_0/2 \le -1-\delta(s,x) \le h_{00} \le -1+\delta(s,x) \le -1+\delta_0/2, \tag{4.1.7a}$$

$$(1-\delta_0/2)^{1/2} \le \big(1-\delta(s,x)\big)^{1/2} \le \tilde{\psi} \le \big(1+\delta(s,x)\big)^{1/2} \le (1+\delta_0/2)^{1/2}. \tag{4.1.7b}$$

Therefore, the coordinates y exist in a cylinder $(s,x)\in[-\rho_0,\rho_0]\times B_{\rho_0}(0)\subset\mathbb{R}\times\mathbb{R}^n$. For the norms of h_{AB} and its inverse we derive from the preceding estimates

$$|(h_{AB})|_e \le 2^{1/2}(1+\delta_0) \qquad =: C_0^h \tag{4.1.8a}$$
$$\text{and}\quad |(h^{AB})|_e \le 2^{1/2}(1-\delta_0)^{-1} \qquad =: \Delta_h^{-1}. \tag{4.1.8b}$$

For later reference we state an estimate for the difference $h_{AB}-h_{AB}(0,0)$ within the domain of the coordinates. it holds that

$$|(h_{AB})-(\eta_{AB})|_e \le (\delta(s,x)^2+\lambda(s,x)^2)^{1/2}\le \delta_0\tfrac{1}{2\rho_0}(|s|+|x|). \tag{4.1.9}$$

From this consideration it follows a comparison estimate for the metric E (see definition 4.2) and the Euclidean metric in these coordinates. We have

$$\begin{aligned}(1-\delta_0/2)\delta_{AB}\le \big(1-\delta(s,x)\big)\delta_{AB}\le E_{AB}&\\ \le \big(1+\delta(s,x)\big)\delta_{AB}\le(1+\delta_0/2)\delta_{AB}.&\end{aligned} \tag{4.1.10}$$

Observe that $|(E_{AB})|$ and $|(h_{AB})|$ coincide and therefore, the estimates (4.1.8a) and (4.1.8b) also hold for E_{AB}.

The next step is an estimate for the Christoffel symbols $\gamma^{\underline{a}}_{\underline{b}\underline{c}}$ of the slice $\mathscr{S}_{\tilde\tau_0}$. A similar expression as for the metric components exists for the Christoffel symbols in Riemannian normal coordinates. It holds that (see e.g. [Nes99])

$$\gamma^{\underline{c}}_{\underline{a}\underline{b}}(0,x)-\gamma^{\underline{c}}_{\underline{a}\underline{b}}(0,0)=-\tfrac{2}{3}\mathcal{R}^{\underline{c}}{}_{(\underline{a}\underline{b})\underline{d}}(0)x^{\underline{d}}+\tilde{C}^{\underline{c}}{}_{\underline{a}\underline{b}\underline{d}\underline{e}}x^{\underline{d}}x^{\underline{e}},$$

where we used the anti-commutator $T_{(ij)}=\frac12(T_{ij}+T_{ji})$. To control the last term of the preceding equation we have to assume the following.

$$\begin{gathered}\text{There exist constants } C_2^\tau \text{ and } C_1^N \text{ such that}\\ |\mathbf{D}\,\mathbf{Rm}|_E\le C_1^N \quad\text{and}\quad |\mathbf{D}^2(\mathbf{D}\tau)|_E\le C_2^\tau.\end{gathered} \tag{4.1.11}$$

By shrinking the value of the radius ρ_0 in the following manner

$$\rho_0=c_3\left((C_0^N+(C_1^\tau)^2)^{-1/2}+(C_1^N+C_1^\tau C_2^\tau)^{-1/3}\right)$$

with a constant c_3 it follows from the Gauß-equations that there exist a constant c_4 such that

$$\left|\left(\gamma^{\underline{c}}_{\underline{a}\underline{b}}(0,x)\right)\right|_e\le c_4\tfrac{|x|}{\rho_0^2} \tag{4.1.12}$$

assuming $|x|<\rho_0$.

To control the time evolution of the Christoffel symbols we compute the evolution of $h(\mathbf{D}_{\underline{a}}\partial_{\underline{b}},\partial_{\underline{c}})$ along the flow of the vector field ∂_0. it holds that

$$\begin{aligned}\tfrac{d}{ds}h(\mathbf{D}_{\underline{a}}\partial_{\underline{b}},\partial_{\underline{c}}) &= h(\mathbf{D}_0\mathbf{D}_{\underline{a}}\partial_{\underline{b}},\partial_{\underline{c}})+h(\mathbf{D}_{\underline{a}}\partial_{\underline{b}},\mathbf{D}_0\partial_{\underline{c}}).\\ &= \mathbf{R}_{0\underline{a}\underline{b}\underline{c}}+h(\mathbf{D}_{\underline{a}}\mathbf{D}_0\partial_{\underline{b}},\partial_{\underline{c}})+h(\mathbf{D}_{\underline{a}}\partial_{\underline{b}},\mathbf{D}_0\partial_{\underline{c}})\end{aligned}$$

The vanishing of the Lie derivative $\mathcal{L}_{\partial_0}\partial_{\underline{b}}$ yields the commutativity of the covariant derivative $\mathbf{D}_0\partial_{\underline{b}} = \mathbf{D}_{\underline{b}}\partial_0$. By inserting $\partial_0 = \tilde{\psi}\widehat{T}$ we arrive at

$$h(\mathbf{D}_{\underline{a}}\mathbf{D}_0\partial_{\underline{b}}, \partial_{\underline{c}}) \;=\; \partial_{\underline{b}}\tilde{\psi}k_{\underline{ac}} + \partial_{\underline{a}}\tilde{\psi}k_{\underline{bc}} + \tilde{\psi}h(\mathbf{D}_{\underline{a}}\mathbf{D}_{\underline{b}}\widehat{T}, \partial_{\underline{c}}). \tag{4.1.13}$$

These terms are controlled by the bounds for the lapse, the curvature of N and the bounds for the first and the second derivative of the gradient of the time function. Since $\mathbf{D}_{\underline{a}}\widehat{T}$ is tangential to the slices, we obtain for the other term

$$h\big(\mathbf{D}_{\underline{a}}\partial_{\underline{b}}, \mathbf{D}_{\underline{c}}(\tilde{\psi}T)\big) = -\mathbf{D}_{\underline{c}}\tilde{\psi}\, k_{\underline{ab}} + \tilde{\psi}\, \gamma^{d}_{\underline{ab}}\, k_{\underline{dc}}. \tag{4.1.14}$$

Integrating and using estimate (4.1.12) yields the existence of constants C_3 and C_4 such that it holds that

$$\big|\big(\gamma^{c}_{\underline{ab}}(s,x)\big)\big|_e \leq c_4\tfrac{|x|}{\rho_0^2} + C_3\tfrac{s}{\rho_0^2} + C_4\tfrac{1}{\rho_0}\int_0^s \big|\big(\gamma^{c}_{\underline{ab}}(\tau,x)\big)\big|_e\, d\tau. \tag{4.1.15}$$

From Gronwall's lemma we derive a constant C_0' such that

$$\big|\big(\gamma^{c}_{\underline{ab}}(s,x)\big)\big|_e \leq C_0'\tfrac{1}{\rho_0^2}(|s| + |x|). \tag{4.1.16}$$

Since $\partial_0 \perp \partial_{\underline{a}}$ the Christoffel symbols of N split as follows $\mathbf{\Gamma}^{a}_{\underline{bc}} = \gamma^{a}_{\underline{bc}}$. The other parts of the Christoffel symbols $\mathbf{\Gamma}^{A}_{BC}$ only involve the second fundamental form of the slices and derivatives of the lapse. The evolution along a straight line in the normal coordinates on the slices and the evolution along the flow of the vector field ∂_0 can already be estimated by assumptions (4.1.3) and (4.1.11). These considerations yield an estimate similar to (4.1.16) for the Christoffel symbols on the slices as follows

$$\big|\big(\mathbf{\Gamma}^{A}_{BC}(s,x)\big)\big|_e \leq C_0^{\mathbf{\Gamma}}\tfrac{1}{2\rho_0^2}(|s| + |x|) \qquad \text{for } |s|, |x| < \rho_0 \tag{4.1.17}$$

with a constant $C_0^{\mathbf{\Gamma}}$.

Higher derivatives of the metric h_{AB} can be estimated via this considerations, if bounds for higher derivatives of the curvature of N and for derivatives of the gradient of the lapse function are imposed. From the Taylor expansion for the components of the metric and the Christoffel symbols we derive that the ℓ-th derivative of the metric coefficients h_{AB} is bounded provided that bounds for derivatives of the curvature of N up to order ℓ and bounds for derivatives of the gradient of the time function up to one order higher exist. We summarize the result for later reference. Let k_0 be an integer and suppose the following assumptions hold

$$\begin{aligned} &\text{There exist constants } C_1, C_2, C_1^{\tau}, \ldots, C_{k_0+1}^{\tau}, C_0^{N}, \ldots, C_{k_0}^{N} \text{ such that} \\ &\qquad |\mathbf{D}^{\ell}\,\mathbf{Rm}|_E \leq C_\ell^N \text{ for } 0 \leq \ell \leq k_0, \quad C_1 \leq \psi \leq C_2 \\ &\text{and} \quad |\mathbf{D}^{\ell}(\mathbf{D}\tau)|_E \leq C_\ell^{\tau} \text{ for } 1 \leq \ell \leq k_0 + 1. \end{aligned} \tag{4.1.18}$$

Then there exist constants $C_1^h, \ldots, C_{k_0}^h$ such that

$$|(D^{\ell}h_{AB})|_e \leq C_\ell^h = C_\ell^h(C_1, C_2, C_1^{\tau}, \ldots, C_{\ell+1}^{\tau}, C_0^N, \ldots, C_\ell^N) \quad \text{for } 1 \leq \ell \leq k_0. \tag{4.1.19}$$

Remark 4.3. Suppose N admits a time function τ only in a neighborhood V of a point q. Let $\rho > 0$ be a constant satisfying $B^E_\rho(q) \subset V$, where E denotes the flipped metric. According to the comparison of balls w.r.t. E and Euclidean balls in $\mathbb{R}^{n+1}$ in the constructed coordinates we can shrink ρ_0 depending on ρ such that the coordinates remain to contain the cylinder $[-\rho_0, \rho_0] \times B_{\rho_0}(0) \subset \mathbb{R} \times \mathbb{R}^n$.

4.2 Special graph representation

In this section we develop a graph representation of a spacelike submanifold in analog manner to the special graph representation in Minkowski space in section 3.1.

Let Σ_0 be a spacelike submanifold in N. Suppose M to be an m-dimensional manifold and $\varphi : M \to N$ to be an immersion of Σ_0. The induced metric on M will be denoted by $\mathring{g}$, the corresponding Levi-civita connection and Christoffel symbols by $\mathring{\nabla}$ and $\mathring{\Gamma}$ respectively.

Let $p_0 \in M$ and choose coordinates on M with center p_0. Let y be the special coordinates constructed in the preceding section with center $q_0 = \varphi(p_0)$. After possibly changing the order of the spatial directions of the coordinates y and applying an orthogonal transformation to the spatial directions we get a graph representation analog to the case of the Minkowski space of the form

$$\Phi(w)^A e_A = y \circ \varphi \circ x^{-1}(w) = w^j e_j + u^\alpha(w) e_\alpha \tag{4.2.1}$$

with the property $Du^a(0) = 0$ for all a. The index of the graph functions u^α will be lowered with the Euclidean metric.

We state a representation of the geometry in this setting as follows

$$\begin{aligned}
&\text{tangent vectors} & \partial_j \Phi^A e_A &= e_j + \partial_j u^\alpha e_\alpha && \text{(4.2.2a)}\\
&\text{normal vectors} & N_\alpha &= (h^{Ak}\partial_k u^\alpha - h^{A\alpha}) e_A && \text{(4.2.2b)}\\
&\text{induced metric} & \mathring{g}_{ij} &= h_{ij} + h_{ib}\partial_j u^b + h_{aj}\partial_i u^a && \\
& & &\quad + \partial_i u^a h_{ab} \partial_j u^b + \partial_i u^0 h_{00} \partial_j u^0. && \text{(4.2.2c)}
\end{aligned}$$

The Christoffel symbols $\mathring{\Gamma}^k_{ij}$ and the second fundamental form $\mathring{I\!I}^A_{ij}$ are defined according to the expressions (1.2.6) and (1.2.5).

The chosen timelike normal can be rewritten to enlighten the role of the unit normal $\widehat{T}$ as follows. Let

$$\widehat{N}_0 := \tilde{\psi}(h^{Ak}\partial_k u^0 - h^{A0}) e_A = \widehat{T} + \tilde{\psi} h^{Ak} \partial_k u_0 e_A. \tag{4.2.3}$$

Then the last part cancels the tangential part of $\widehat{T}$

$$h(\widehat{T}, \partial_j \Phi) = \tilde{\psi}^{-1}(h_{0j} + \partial_j u^\alpha h_{0\alpha}) = -\tilde{\psi} \partial_j u_0.$$

Observe that the Minkowski space admits a time function defined as the time component and the unit normal to the corresponding time foliation is the timelike direction τ_0. Thus, the preceding consideration shows that $\widehat{N}_0$ is the correspondent to the timelike normal N_0 of the Minkowski case defined in (3.1.2b).

In the following lemma we derive comparison estimates for the metric components to the Euclidean metric.

Lemma 4.4. *Let* (N,h) *satisfy the assumption* (4.1.3)*. Assume* $|\Phi^0|, |(\Phi^{\underline{a}})|_e < \rho_0$*. Then the induced metric satisfies the estimates* (3.1.4) *at the center* p_0 *of the chart and*

$$\begin{aligned}\big(1 - |Du^0|^2 - (1+|Du|^2)\delta_0 \tfrac{2}{\rho_0}(|\Phi^0| + |(\Phi^{\underline{a}})|_e)\big)\delta_{ij}\\ \le \mathring{g}_{ij} \le (1+|Du|^2)(1+\delta_0)\delta_{ij}.\end{aligned} \tag{4.2.4}$$

Proof. The first claim follows from the fact that the metric h_{AB} coincides with the Minkowski metric at the center and $Du^a(0) = 0$ for all a. To obtain the positive definiteness we write the induced metric in the following form

$$\mathring{g}_{ij} = \partial_i \Phi^A \eta_{AB} \partial_i \Phi^B + \partial_i \Phi^A (h_{AB}(\Phi) - \eta_{AB}) \partial_j \Phi^B.$$

To the first part of this equation estimate (3.1.4) applies and the second part can be treated with estimate (4.1.9). From the same expression for $\mathring{g}_{ij}$ and again estimate (4.1.9) the second inequality of the claim follows. □

The next step will be to derive an estimate for the second derivatives of the graph functions u^α.

Lemma 4.5. *Let* (N,h) *satisfy the assumptions* (4.1.3) *and* (4.1.11)*. Let* $|\Phi^0|, |(\Phi^{\underline{a}})|_e < \rho_0$*. Then the following inequality holds for the second derivatives of the graph functions*

$$|D^2 u|_e \le C_3 |\mathring{I\!I}|_{\mathring{g},E}(1+|Du|^2)^2 + C_0^{\mathbf{\Gamma}} \tfrac{1}{\rho_0}(1+|Du|^2)^{3/2}, \tag{4.2.5}$$

where $C_3 = 2(1+\delta_0/2)^{1/2}\Delta_h^{-1} m^{1/2}(1+\delta_0)$*.*

Proof. Let $n_{\alpha\beta} = h(N_\alpha, N_\beta)$ with the normal vector fields N_α defined in (4.2.2b) analog to the Minkowski case. Let further h_{ij}^α be the coefficients satisfying $\mathring{I\!I}_{ij} = h_{ij}^\alpha N_\alpha$. Then we get the following expression for the second derivatives of the graph functions

$$-\partial_i \partial_j u_\gamma = n_{\gamma\alpha} h_{ij}^\alpha - \partial_i \Phi^B \partial_j \Phi^C (\mathbf{\Gamma}_{BC}^k \partial_k u_\gamma - \mathbf{\Gamma}_{BC}^\gamma). \tag{4.2.6}$$

It differs from the expression in the Minkowski case by the last term on the RHS involving the Christoffel symbols $\mathbf{\Gamma}_{BC}^A$ of the ambient manifold. By introducing the matrix $e_{\alpha\beta} = E(N_\alpha, N_\beta)$ with $e_{\alpha\beta} \ge C_e \delta_{\alpha\beta}$ the first term can be estimated by the RHS of inequality (3.1.6) for the Minkowski case. From estimate (4.1.8b) we get

$$|(n_{\alpha\beta})|_e \le 2\Delta_h^{-1}(1+|Du|^2).$$

To obtain an expression for the constant C_e, we derive with the abbreviations $w_k = \partial_k u_a v^a$ and $w_k^0 = v^0 \partial_k u^0$ for a vector $(v^\alpha) \in \mathbb{R}^{n+1-m}$ that

$$v^\alpha e_{\alpha\beta} v^\beta = \tilde{\psi}^{-2}(v^0)^2 + \Big\langle \Big(\begin{smallmatrix} -v_a \\ w_k + w_k^0 \end{smallmatrix}\Big), (h^{\underline{ab}}) \Big(\begin{smallmatrix} -v_b \\ w_\ell + w_\ell^0 \end{smallmatrix}\Big) \Big\rangle$$

Estimate (4.1.6) for the components $(h_{\underline{ab}})$ now yields that

$$\Big\langle \big(\tfrac{-v_a}{w_k+w_k^0}\big), (h^{\underline{ab}})\big(\tfrac{-v_b}{w_\ell+w_\ell^0}\big)\Big\rangle \geq (1+\delta_0/2)^{-1}|(v^a)|^2. \tag{4.2.7}$$

Using estimate (4.1.7b) for the lapse gives us $C_e = (1+\delta_0/2)^{-1}$. The term on the RHS of equation (4.2.6) involving the Christoffel symbols can be estimated by means of the bound stated in (4.1.17). □

Replacing the Euclidean metric e by the Riemannian metric E defined in 4.2 gives us an analog definition for a uniformly spacelike submanifold with bounded curvature.

Definition 4.6. A spacelike submanifold Σ_0 of a Lorentzian manifold (N,h) is called *uniformly spacelike submanifold with bounded curvature* if there exist constants ω_1, C_0 such that

$$\inf\{-h(\gamma,\widehat{T}) : \gamma \text{ future-directed unit timelike normal to } \Sigma_0\} \leq \omega_1 \tag{4.2.8a}$$

$$\text{and} \quad |\mathring{I\!I}|_{\mathring{g},E} \leq C_0^{\varphi}. \tag{4.2.8b}$$

From now on we suppose the submanifold Σ_0 to be uniformly spacelike with bounded curvature. Analog to Lemma 3.6 in the Minkowski case a choice of a unit timelike normal to the graph is needed to gain an estimate from the condition on the angle.

Lemma 4.7. *Recall the definition (4.2.3) for the timelike vector field $\widehat{N}_0$. Set*

$$\nu_0 := \langle \widehat{N}_0\rangle^{-1}\widehat{N}_0 \quad \textit{with} \quad \langle \widehat{N}_0\rangle^{-1} = (1-\tilde{\psi}^2\partial_k u_0 h^{k\ell}\partial_\ell u_0)^{-1/2}. \tag{4.2.9}$$

Then it follows from condition (4.2.8a) that $-h\big(\nu_0(0),\widehat{T}\big) \leq \omega_1$.

Proof. Let $\gamma = \langle \nu_0 + \lambda^a N_a\rangle^{-1}(\nu_0+\lambda^a N_a)$ for $(\lambda^a)\neq 0$ a perturbation of ν_0 using the other normal vectors N_a defined in (4.2.2b). At the origin we have $N_a(0) = -h^{Aa}e_A$ and $h(N_a,\widehat{T}) = 0$. This yields $\langle \nu_0(0)+\lambda^a N_a(0)\rangle^2 = 1-\lambda_a h^{ab}\lambda_b < 1$, where we lowered with the Euclidean metric. It follows that $-h\big(\gamma(0),\widehat{T}\big) > -h\big(\nu_0(0),\widehat{T}\big)$. □

An analog to Lemma 3.7 contains an additional estimate to ensure that the graph stays within the chosen coordinates on the ambient manifold N. Let

$$v(x) := \big(1-|Du^0|^2-(1+|Du|^2)\delta_0\tfrac{1}{2\rho_0}(|\Phi^0|+|(\Phi^{\underline{a}})|_e)\big)^{-1/2} \tag{4.2.10}$$

$$\text{and} \qquad w(x) := \big(1-\tfrac{|\Phi|_e^2}{\rho_0^2}\big)^{-1/2}. \tag{4.2.11}$$

In the definition of the function w we use the Euclidean norm on $\mathbb{R}^{n+1}$ for the representation (Φ^A). This ensures that if $|(s,x)|_e < \rho_0$ then the pair (s,x) is contained in the cylinder $[-\rho_0,\rho_0]\times B_{\rho_0}(0)\subset \mathbb{R}\times\mathbb{R}^n$ where the special coordinates constructed in section 4.1 are defined.

Lemma 4.8. *Let (N,h) satisfy the assumptions (4.1.3) and (4.1.11). Let $\lambda \geq 250000$ be a fixed constant. Then there exist a constant $\rho_1 > 0$ such that for $z \in \mathbb{R}^m$, $|z| =: r < \rho_1$ and $\{\tau z : 0 \leq \tau \leq 1\}$ contained in the image of the coordinates x on M, which are part of the graph representation (4.2.1), the inequalities (3.1.8a) and (3.1.8b) (taking the new values of λ, ρ_1 and B_1 into account) and the following estimates hold*

$$w(z) \leq 1 + \left(\tfrac{r}{2\lambda\rho_1}\right)^2 \qquad \textit{and} \qquad v(z) \leq \omega_1 + \tfrac{r}{2\lambda\rho_1}. \tag{4.2.12}$$

Proof. We begin with

$$\rho_1 := (2\lambda)^{-1} \max(C_3 C_0^{\varphi}, \rho_0^{-1} C_0^{\mathbf{\Gamma}})^{-1}.$$

This definition yields the same bound for the second derivatives of the graph functions as stated in (3.1.9). Therefore, from the proof of Lemma 3.7 we derive the estimates (3.1.8a) and (3.1.8b).

We head to the function $w(z)$ defined in (4.2.11). We will apply an ODE comparison argument similar to the argument used in the proof of Lemma 3.7 for the function v. The differential of w can be estimated by

$$|Dw| \leq w^3 \tfrac{1}{\rho_0^2} |\Phi||D\Phi|.$$

With the Taylor expansion for Φ we derive

$$|Dw| \leq w^3 m \tfrac{r}{\rho_0^2}(1 + |Du|^2).$$

By using estimate (3.1.8a) it follows that the last term is bounded by B_1^2. Shrinking the radius ρ_1 such that

$$\rho_1 \leq \rho_0 (4\lambda^2 m B_1^2)^{-1/2} \tag{4.2.13}$$

and using an ODE comparison argument provides us with the desired estimate.

We now look at the function $v(z)$ defined in (4.2.11). To use an ODE comparison argument we have to estimate the deviation of v which will be done by considering the $\limsup$ since a norm is involved. In the proof of Lemma 3.7 it was shown that

$$\left|D(|Du^\alpha|^2)\right| \leq 2|Du^\alpha||D^2 u^\alpha|$$

and the inequalities (3.1.8a) and (3.1.8b) yield an estimate for the RHS. It remains to regard the $\limsup$ of $|\Phi^0| + |(\Phi^{\underline{a}})|$ which can be controlled by

$$|D\Phi|_e \leq m^{1/2}(1 + |Du|^2)^{1/2} \tag{4.2.14}$$

Using the inequalities (3.1.10), (3.1.8a), (3.1.8b) established in the preceding step and $|\Phi^0|, |(\Phi^{\underline{a}})|_e < \rho_0$ yields

$$|Dv| \leq v^3\big((1+\delta_0)(1 + B_1^4 \tfrac{r}{2\lambda\rho_1}) B_1^4 \tfrac{1}{2\lambda\rho_1} + \delta_0 \tfrac{1}{2\rho_0} m^{1/2} B_1^3\big).$$

By shrinking the radius ρ_1 we achieve that $|Dv| \leq v^3 \frac{1}{2\lambda\rho_1}$. From the definition of ν_0 in (4.2.9) and lemma 4.7, we derive that $v(0) \leq \omega_1$. This gives us the desired result by an ODE comparison argument. □

Remark 4.9. Analog to Remark 3.9 the preceding lemma yields that the graph representation exists in the ball $B_{\rho_1}(0) \subset \mathbb{R}^m$. The additional possibility for the graph representation to fail, leaving the image of the special coordinates on N, was excluded by estimating the function $w(z)$.

Set

$$G_1 = \left(\omega_1 + \tfrac{2}{\lambda^2}\right)^{-2} \quad \text{and} \quad G_2 = B_1^2 \tag{4.2.15}$$

then it holds that

$$G_1 \delta_{ij} \leq \mathring{g}_{ij} \leq G_2 \delta_{ij} \quad \text{within the ball } B_{\rho_1}(0) \subset \mathbb{R}^m. \tag{4.2.16}$$

To obtain bounds for higher derivatives, derived in the next lemma, of the graph functions we make use of Lemma 3.10.

Lemma 4.10. *Let k be an integer and assume (N, h) to satisfy the inequalities (4.1.18) for $k_0 = k$. Let the immersion φ satisfy the the following assumption.*

There exist constants $C_0^\varphi, \ldots, C_k^\varphi$ such that $|\widehat{\nabla}^\ell \mathring{I\!I}|_{\mathring{g},E} \leq C_\ell^\varphi$ for $0 \leq \ell \leq k$.

Then there exists a constant C_{k+2}^u such that $|D^{k+2}u|_e \leq C_{k+2}^u$.

Proof. As in the proof of the analog statement in Lemma 3.11 we make use of expression (3.1.13) derived in Lemma 3.10 and adapt expression (3.1.14) derived in Lemma 3.11 to the case of a Lorentzian ambient manifold. The proof will be done via an induction on the order of differentiation. The case $k = 0$ was derived in Lemma 4.8.

We begin with an expression similar to (3.1.14). it holds that

$$-\partial^k \partial_i \partial_j u_\beta = \partial^k \mathring{I\!I}_{ij}^B \, h_{BC}(\Phi) \, N_\beta^C + \sum_{\substack{k_1+k_2+k_3=k, \\ k_1<k}} \partial^{k_1} \mathring{I\!I}_{ij}^B \, \partial^{k_2} \big(h_{BC}(\Phi)\big) \, \partial^{k_3} N_\beta^C. \tag{4.2.17}$$

Only the first term on the RHS contains derivatives of the graph functions up to order $k+2$, the others are lower order terms since the normal N_β involves first derivatives and the representation h_{AB} involves the graph functions itself. Derivatives of the components h_{AB} are bounded up to order k which follows from inequality (4.1.19).

According to expression (3.1.13) we need to estimate the induced Christoffel symbols $\mathring{\Gamma}$ computed in (1.2.6) and the Christoffel symbols $\widehat{\Gamma}$ for the connection on the pullback bundle $\varphi^* TN$ defined in (1.2.3).

Derivatives of the induced Christoffel symbols up to order $k-1$ need to be estimated. The first term of expression (1.2.6) is controlled by derivatives of the induced metric up to order $k-1$, by derivatives of the graph functions up to order $k+1$ and by bounds for the metric h_{AB}. Thus, we get bounds using Corollary B.4 and estimates (4.2.16) and (4.1.19). The second term involves Christoffel symbols of N, therefore we need one order of differentiation more of the metric h_{AB} to control it.

Definition (1.2.3) for the Christoffel symbols $\widehat{\Gamma}$ yields that their derivatives of order $k-1$ can be estimated by derivatives of the graph functions and of the metric h_{AB} up to order k which is covered by the preceding consideration.

It remains to treat the last term of expression (3.1.13) involving covariant derivatives of the second fundamental form. Opposing to the Minkowski space it is necessary to estimate $|\widehat{\nabla}^\ell \mathring{I\!I}|_{\mathring{g},e}$ by the term $|\widehat{\nabla}^\ell \mathring{I\!I}|_{\mathring{g},E}$ which can be done via the estimates (4.1.10) for the components of the metric E. □

4.3 Solutions for fixed coordinates

In this section we will consider the Cauchy problem (1.3.2) for submanifolds Σ_0 of a globally hyperbolic ambient manifold (N, h).

We use the notations of the main problem regarding the initial submanifold and the initial direction ν. Assume M to be an m-dimensional manifold and $\varphi : M \to N$ to be an immersion of Σ_0. As initial velocity for the Cauchy problem we combine initial direction, lapse and shift in a timelike vector field $\chi : M \to TN$ along φ. Let $s > \frac{m}{2} + 1$ be an integer fixing differentiability properties in the sequel.

Assume $(U_\lambda, x_\lambda, V_\lambda, y_\lambda)_{\lambda\in\Lambda}$ to be a decomposition of φ according to definition 3.12. For $\lambda \in \Lambda$ let Φ_λ and χ_λ denote the representations of φ and χ in these coordinates analog to (3.2.1).

We make the following uniformity assumptions on the coordinates $(V_\lambda, y_\lambda)_{\lambda\in\Lambda}$ for N and the representation of h in these coordinates.

Assumptions 4.11. *There exist constants $\delta_0 < 1,\ \rho_0$ such that for each $\lambda \in \Lambda$ the image of y_λ contains a cylinder $\big(y^0_\lambda, (y^a_\lambda)_{\underline{a}}\big) = (s, x) \in [-\rho_0, \rho_0] \times B_{\rho_0}(0) \subset \mathbb{R} \times \mathbb{R}^n$ and the representation of the metric h satisfies in these coordinates*

$$\big|\big(h_{AB}(s,x)\big) - (\eta_{AB})\big|_e \le \delta_0 \tfrac{1}{2\rho_0}(|s| + |x|) \tag{4.3.1a}$$

and

$$\big|(D^\ell h_{AB})\big|_e \le C^h_\ell \quad \text{for } 1 \le \ell \le s+1 \tag{4.3.1b}$$

with constants C^h_ℓ independent of $\lambda \in \Lambda$.

The following theorem states the first main result of this section, a generalization of Theorem 3.14.

Theorem 4.12. *Let the coordinates $(V_\lambda, y_\lambda)_{\lambda\in\Lambda}$ on N, which are a part of the decomposition*
$(U_\lambda, x_\lambda, V_\lambda, y_\lambda)_{\lambda\in\Lambda}$, and the representation of the metric h in these coordinates satisfy the assumptions 4.11. Let the decomposition $(U_\lambda, x_\lambda, V_\lambda, y_\lambda)_{\lambda\in\Lambda}$ and the initial data φ_λ and χ_λ for the reduced membrane equation satisfy the assumptions 3.13 with η_{AB} replaced by h_{AB}.

Then there exist constants $0 < T \le T_1$, $0 < \theta < 1$ and a family (F_λ) of bounded C^2-immersions $F_\lambda : [-T, T] \times B^e_{\theta\rho_1/2}(0) \subset \mathbb{R} \times x_\lambda(U_\lambda) \to \mathbb{R}^{n+1}$ solving the reduced membrane equation

$$g^{\mu\nu}\partial_\mu\partial_\nu F^A - g^{\mu\nu}\hat{\Gamma}^\lambda_{\mu\nu}\partial_\lambda F^A + g^{\mu\nu}\partial_\mu F^B \partial_\nu F^C \mathbf{\Gamma}^A_{BC}(F) = 0$$

w.r.t. the background metric $\hat{g}$ defined in (1.3.7) and attaining the initial values

$$F_\lambda\big|_{t=0} = \Phi_\lambda \quad \text{and} \quad \partial_t F_\lambda\big|_{t=0} = \chi_\lambda.$$

Let F_λ and $\bar{F}_\lambda$ be two such solutions defined on the image of the same coordinates. Assume $z \in \mathbb{R}^m$ be a point contained in the image of x_λ. If F_λ and $\bar{F}_\lambda$ attain the initial values Φ_λ and χ_λ on a ball $B^e_r(z)$, then they coincide on the truncated double-cone with base $B^e_r(z)$, slope c_0 and maximal height $2T_1$.

Remark 4.13. In contrast to the Minkowski case uniqueness only holds on a truncated double-cone with the maximal height given by a constant T_1 which will be defined in (4.3.19).

Remark 4.14. Let ℓ_0 be an integer. Assume that the initial values and the decomposition satisfy the assumptions of Theorem 4.12 with an integer $r = s + \ell_0 > \frac{m}{2} + 1 + \ell_0$ instead of an $s > \frac{m}{2} + 1$. Then the family (F_λ) of solutions to the reduced membrane equation are immersions of class $C^{2+\ell_0}$.

Let $\lambda \in \Lambda$ be fixed. From section 3.2 we adopt the definition of the cut-off function ζ with the bounds for derivatives of ζ in (3.2.6), the asymptotics $w(t)$ defined in (3.2.7) and the functions $\mathring{\Phi}$ and $\mathring{\chi}$ defined in (3.2.8a) and (3.2.8b) respectively.

To handle the metric of the ambient manifold we define another cut-off function $\tilde{\zeta} \in C^\infty_c(\mathbb{R}^{n+1})$ satisfying $\tilde{\zeta} \equiv 1$ on $B_{\tilde{\theta}\rho_0/2}(0) \subset \mathbb{R}^{n+1}$ and $\tilde{\zeta} \equiv 0$ outside $B_{\tilde{\theta}\rho_0}(0) \subset \mathbb{R}^{n+1}$. The bounds for derivatives of $\tilde{\zeta}$ are given by

$$|D^\ell \tilde{\zeta}|_e \leq \tilde{C}'_\ell (\tilde{\theta}\rho_0)^{-\ell} \tag{4.3.2}$$

with constants $\tilde{C}'_\ell$ independent of $\tilde{\theta}$ and ρ_0. We use this function to define an interpolated metric as follows

$$\hat{h}_{AB}(z) = \eta_{AB} + \mathring{h}_{AB}(z) \quad \text{with} \quad \mathring{h}_{AB}(z) := \tilde{\zeta}(z)\big(h_{AB}(z) - \eta_{AB}\big) \text{ for } z \in \mathbb{R}^{n+1}. \tag{4.3.3}$$

The Christoffel symbols w.r.t. $\hat{h}_{AB}$ will be denoted by $\widehat{\boldsymbol{\Gamma}}^A_{BC}$. We define the matrix $\hat{a}_{\mu\nu}$ as in (3.2.9) taking the metric $\hat{h}_{AB}$ instead of the Minkowski metric η_{AB}. The Christoffel symbols of $\hat{a}_{\mu\nu}$ will be denoted by $\hat{\gamma}^\lambda_{\mu\nu}$.

We search for a solution to the IVP

$$\begin{gathered} g^{\mu\nu}(F, DF, \partial_t F)\partial_\mu\partial_\nu F^A = f^A(t, F, DF, \partial_t F) + \mathbf{f}^A(t, F, DF, \partial_t F) \\ F\big|_{t=0} = w_0 + \mathring{\Phi}, \;\; \partial_t F\big|_{t=0} = w_1 + \mathring{\chi}, \end{gathered} \tag{4.3.4a}$$

$$\text{with } f^A(t, F, DF, \partial_t F) = g^{\mu\nu}(F, DF, \partial_t F)\hat{\gamma}^\lambda_{\mu\nu}(t)\partial_\lambda F^A \tag{4.3.4b}$$

$$\text{and } \mathbf{f}^A(F, DF, \partial_t F) = -\, g^{\mu\nu}(F, DF, \partial_t F)\partial_\mu F^B \partial_\nu F^C \widehat{\boldsymbol{\Gamma}}^A_{BC}(F). \tag{4.3.4c}$$

The coefficients are defined as the inverse of the matrix

$$g_{\mu\nu}(F, DF, \partial_t F) = \partial_\mu F^A \hat{h}_{AB}(F)\partial_\nu F^B. \tag{4.3.5}$$

Here, the coefficients depend on F itself, not only on the derivatives opposing to the Minkowski case. For notational convenience we set

$$g_{\mu\nu}(F) = g_{\mu\nu}(F, DF, \partial_t F), \;\; f(F) = f(t, F, DF, \partial_t F) \text{ and } \mathbf{f}(F) = \mathbf{f}(F, DF, \partial_t F).$$

The following proposition states an analog result to proposition 3.17, from which we immediately derive the existence claim of Theorem 4.12.

Proposition 4.15. *There exist a constant $T' > 0$ and a unique C^2-solution $F : [-T', T'] \times \mathbb{R}^m \to \mathbb{R}^{n+1}$ to the IVP (4.3.4a) satisfying*

$$F(t) - w(t) \in C([-T', T'], H^{s+1}),\ \partial_t F(t) - w_1 \in C([-T', T'], H^s). \tag{4.3.6}$$

The strategy to obtain a solution to the IVP (4.3.4a) will be taken from section 3.2. We begin with a definition of coefficients and RHS of an asymptotic equation to which Theorem 2.22 will be applied. Since F itself enters the coefficients, we have to add another slot to the definition of Ω in (3.2.16). Let $\Omega \subset \mathbb{R}^{n+1} \times \mathbb{R}^{m(n+1)} \times \mathbb{R}^{n+1}$ be a set chosen later and define for $(t, v, Y, X) \in \mathbb{R} \times \Omega$ with $Y = (Y_k)$

$$g^{\mathrm{a}}_{0\ell}(t, v, Y, X) := (\partial_t w + X)^A \hat{h}_{AB}\big(w(t) + v\big)(\partial_\ell w + Y_\ell)^B. \tag{4.3.7}$$

The other parts g^a_{00} and $g^a_{k\ell}$ are defined analogously. Observe that in this case the coefficients depend explicitly on the time parameter t. Let the functions f_{a} and $\mathbf{f}_{\mathrm{a}}$ be defined in an analog manner to the RHS of the asymptotic equation for the Minkowski space in (3.2.14) regarding the definitions (4.3.4b) and (4.3.4c). As in section 3.2 the set Ω will be used to ensure that the matrix $g^{\mu\nu}_{\mathrm{a}}$ has the desired signature $(-+\cdots+)$.

These definitions give rise to the following operators

$$g^{\mathrm{a}}_{\mu\nu}(t, \varphi_0, \varphi_1) = g^{\mathrm{a}}_{\mu\nu}(t, \varphi_0, D\varphi_0, \varphi_1),\ g^{\mu\nu}_{\mathrm{a}}(t, \varphi_0, \varphi_1),$$
$$f_{\mathrm{a}}(t, \varphi_0, \varphi_1) \quad \text{and} \quad \mathbf{f}_{\mathrm{a}}(t, \varphi_0, \varphi_1), \tag{4.3.8}$$

where the last terms are defined analogously. The operators have a domain $[0, T_1] \times W \subset \mathbb{R} \times H^{s+1} \times H^s$, where T_1 is a constant and W will be chosen later in dependency on Ω. For later reference we state the asymptotic equation to be solved. It reads

$$g^{\mu\nu}_{\mathrm{a}}(t, \psi, \partial_t\psi)\partial_\mu\partial_\nu\psi^A = f^A_a(t, \psi, \partial_t\psi) + \mathbf{f}^A_a(t, \psi, \partial_t\psi) \tag{4.3.9a}$$
$$\text{with } f^A_{\mathrm{a}}(t, \psi, \partial_t\psi) = g^{\mu\nu}_{\mathrm{a}}\hat{\gamma}^\lambda_{\mu\nu}(\partial_\lambda w + \partial_\lambda\psi)^A \tag{4.3.9b}$$
$$\text{and } \mathbf{f}^A_{\mathrm{a}}(t, \psi, \partial_t\psi) = -\, g^{\mu\nu}_{\mathrm{a}}(\partial_\mu w + \partial_\mu\psi)^B(\partial_\nu w + \partial_\nu\psi)^C\widehat{\mathbf{\Gamma}}^A_{BC}\big(w(t) + \psi\big), \tag{4.3.9c}$$

The initial values of ψ are given by (3.2.15b) regarding the new definition of these functions.

Following the strategy of section 3.2 for the Minkowski case we begin with an analog to Lemma 3.18 establishing estimates for the spacelike and timelike part of the inverse of the coefficients.

Lemma 4.16. *The following estimates hold for the matrix $g^{\mathrm{a}}_{\mu\nu}(t, v, Y, X)$*

$$g^a_{k\ell} \geq \big(\omega_1^{-2} - 2|Y|_e(|Y|_e + |Dw_0|_e) - \delta_0\tilde{\theta}(|Dw_0|^2_e + |Y|^2_e)\big)\delta_{k\ell}$$
$$\text{and} \qquad g^a_{00} \leq h(\chi_0, \chi_0) + 2(1 + \delta_0\tilde{\theta})|X|_e(|\chi_0|_e + |X|_e).$$

Proof. Both members of the matrix $g^{\mathrm{a}}_{\mu\nu}$ can be divided into a part which is covered by Lemma 3.18 and a part depending on $\mathring{h}_{AB}$. The latter part can be estimated by condition (4.3.1a) on the representation of the metric h, where we use $|s| < \tilde{\theta}\rho_0$ and $|x| < \tilde{\theta}\rho_0$ due to the presence of the cut-off function $\tilde{\zeta}$. □

Let Ω be defined as follows

$$\Omega := \mathbb{R}^{n+1} \times B^e_{\delta_1}(0) \times B^e_{\delta_2}(0) \subset \mathbb{R}^{n+1} \times \mathbb{R}^{m(n+1)} \times \mathbb{R}^{n+1}, \tag{4.3.10}$$

where δ_1 and δ_2 will be chosen in the sequel. There is no condition for the first slot, since the cut-off process ensures that the matrix $\hat{h}_{AB}$ entering the definition of the coefficients is defined on all of $\mathbb{R}^{n+1}$. We derive from lemma 4.16 and the assumptions 3.13 the following estimates

$$\begin{aligned} & g^a_{k\ell}(v,Y,X) \geq \omega_1^{-2}\big(1 - 2\omega_1^2\delta_1(C_{w_0}+\delta_1) - \omega_1^2\delta_0\tilde{\theta}(C^2_{w_0}+\delta_1^2)\big)\delta_{k\ell}, \\ \text{and} \quad & g^a_{00}(v,Y,X) \leq -L_2\big(1 - 2L_2^{-1}(1+\delta_0\tilde{\theta})\delta_2(\tilde{C}^X_0+\delta_2)\big) \text{ for } (v,Y,X)\in\Omega. \end{aligned}$$

Compared to the estimates (3.2.17) two additional parts occur which can be controlled via the parameter $\tilde{\theta}$.

Let $0 < R_0, r_0 < 1$ be fixed constants. The goal is to derive radii δ_1 and δ_2 such that the matrix $g^{\mathrm{a}}_{\mu\nu}$ satisfies similar estimates to (3.2.19). First we choose δ_1 and δ_2 according to the inequalities (3.2.18) replacing the RHS by $\omega_1^{-2}R_0/2$ and $L_2 r_0/2$ respectively. Now we choose $\tilde{\theta}$ so small that the terms in $g^a_{k\ell}$ and g^a_{00} involving this constant are also $\leq \omega_1^{-2}R_0/2$ and $\leq L_2 r_0/2$ respectively. This ensures that the components g^a_{00} and $g^a_{k\ell}$ satisfy the inequalities (3.2.19).

In the sequel we adopt the definition of the bounds K_0 and K_1 in (3.2.20a) and (3.2.20b) respectively. Analog to Lemma 3.20 the estimates (3.2.19) and condition (4.3.1a) on the matrix h_{AB} yield the following result.

Lemma 4.17. *Assume $(v,Y,X)\in\Omega$. Then it follows for the coefficients $g^{\mu\nu}_{\mathrm{a}}(v,Y,X)$*

$$|(g^{\mu\nu}_{\mathrm{a}})|^2_e \leq \tilde{\lambda}^{-2} + 2\tilde{\lambda}^{-2}\tfrac{m}{\tilde{\mu}^2}K_0^2K_1^2(1+\delta_0\tilde{\theta})^2 + \tfrac{m}{\tilde{\mu}^2} \qquad =:\Delta^{-2}, \tag{4.3.11a}$$

$$g^{00}_a \leq -K_1^{-2}\big(1+\tfrac{m^{1/2}}{\tilde{\mu}}K_0^2(1+\delta_0\tilde{\theta})\big)^{-1}(1+\delta_0\tilde{\theta})^{-1} \qquad =: -\lambda \tag{4.3.11b}$$

$$\textit{and} \quad g^{ij}_a \geq K_0^{-2}\big(1+\tilde{\lambda}^{-1}K_1^2(1+\delta_0\tilde{\theta})\big)^{-1}(1+\delta_0\tilde{\theta})^{-1}\delta^{ij} \qquad =:\mu\delta^{ij}. \tag{4.3.11c}$$

4.3.1 Sobolev estimates

Since the definition of Ω in (4.3.10) contains no constraint for the first slot, we can follow the arguments in section 3.2.1 for the Minkowski case, beginning with an analog definition of the domain W done in (3.2.23) involving balls around the initial values of the asymptotic equation.

Based on the observation that Lemma 3.21 remains valid, since we only used the assumptions (3.2.2) and (3.2.3), our first choice of the parameter θ follows the same way as in section 3.2.1.

In contrast to the Minkowski case we can not be sure that the unmodified reduced equation (1.3.6) will be solved within the ball $B^e_{\theta\rho_1/2}(0)\subset\mathbb{R}^m$ due to the bounded image of the coordinates on the ambient manifold. To handle this problem we consider the initial value $\mathring{\Phi}$. We adjust the parameter θ to ensure that the function $\mathring{\Phi}$ lies in the cylinder

$[-\tilde{\theta}\rho_0/4, \tilde{\theta}\rho_0/4] \times B_{\tilde{\theta}\rho_0/4}(0) \subset \mathbb{R} \times \mathbb{R}^n$. This can be done by the following consideration. From assumptions (3.2.2) for Φ we obtain via a Taylor expansion

$$|\Phi(z)|_e \leq r(C_{w_0} + \tilde{C}_2^{\varphi} \tfrac{r}{\rho_1}) \quad \text{if } z \in \mathbb{R}^m \text{ with } |z| =: r < \rho_1. \tag{4.3.12}$$

We shrink the parameter θ such that $|\Phi(z)|_e < \tilde{\theta}\rho_0/4$ within the ball $B^e_{\theta\rho_1/2} \subset \mathbb{R}^m$. This choice of θ ensures that there is enough space for time evolution.

Now we choose the constant ρ small enough such that $(\varphi_0, \varphi_1) \in W$, then it follows $(\varphi_0, D\varphi_0, \varphi_1) \in \Omega$ pointwise. This gives us a definition of W analog to (3.2.23).

In the sequel we adopt the notion for the bounds $D_0, D_1, \tilde{D}_0$ and $\tilde{D}_1$ in (3.2.26) and the definitions of $K_{0,s}$ and $K_{1,s}$ in (3.2.27a) and (3.2.27b). To obtain estimates on the argument $w(t) + \varphi_0$ of the matrix $\hat{h}_{AB}$ and its Christoffel symbols we needed to introduce a bound T_1 on the time parameter t. For notational convenience we introduce for $t \leq T_1$ and $\varphi_0 \in B_\rho(\mathring{\Phi})$

$$\begin{aligned}\|w(t) + \varphi_0\|_{s,\mathrm{ul}} &\leq c\big(|Dw_0|^2 + T_1^2|w_1|^2 + \|\varphi_0\|_s^2\big)^{1/2} \\ &\leq c\big(C_{w_0}^2 + T_1^2(\tilde{C}_0^{\chi})^2 + \rho^2 + D_0^2\big)^{1/2} =: C_{\mathrm{arg}}.\end{aligned} \tag{4.3.13}$$

The constant c depends on the test function used to define the spaces H^s_{ul} (see beginning of section 2.3.1).

To derive Sobolev estimates for the coefficients and the RHS, bounds for the Sobolev norm of the interpolated metric $\hat{h}_{AB}$ and its Christoffel symbols $\widehat{\boldsymbol{\Gamma}}^A_{BC}$ are needed. These will be obtained by local bounds for derivatives of $\hat{h}_{AB}(z)$ for $z \in \mathbb{R}^{n+1}$ to be shown in the next lemma.

Lemma 4.18. *The following estimates hold for the matrix $\hat{h}_{AB}$ within $\mathbb{R}^{n+1}$*

$$|(\hat{h}_{AB})| \leq n + 1 + \delta_0\tilde{\theta} =: C_0^{\hat{h}}, \qquad |(\hat{h}^{AB})| \leq \Delta_{\hat{h}}^{-1}, \qquad |(D\hat{h}_{AB})| \leq C_{d,0}^{\hat{h}}, \tag{4.3.14a}$$

$$|(D\hat{h}_{AB})| \leq C_{d,0}^{\hat{h}}, \qquad |(\hat{h}_{AB})|_{C^s} \leq C_s^{\hat{h}}, \qquad |(D\hat{h}_{AB})|_{C^s} \leq C_{d,s}^{\hat{h}}. \tag{4.3.14b}$$

Proof. The first bound in (4.3.14a) can be derived from the condition for the difference $h_{AB} - \eta_{AB}$ in (4.3.1a). The same condition yields that lemma B.2 is applicable providing the second bound.

Estimates for higher derivatives of $\hat{h}_{AB}$ follow from the bounds for the cut-off function $\tilde{\zeta}$ in (4.3.2) and from condition (4.3.1b) for derivatives of the metric h_{AB} within the ball $B^e_{\rho_0}(0) \subset \mathbb{R}^{n+1}$. □

These bounds can be used to establish Sobolev norm estimates for the matrix $\hat{h}_{AB}$ occurring and its Christoffel symbols.

Lemma 4.19. *Let $t \leq T_1$ and $\varphi_0, \psi_0 \in B_\rho(\mathring{\Phi}) \subset H^{s+1}$. Then the matrix $\hat{h}_{AB}$ and its Christoffel symbols $\widehat{\boldsymbol{\Gamma}}^A_{BC}$ satisfy*

$$\big\|\big(\hat{h}_{AB}\big(w(t)+\varphi_0\big)\big)\big\|_{e,s,\mathrm{ul}} \leq c\, C^{\hat{h}}_s(1+C^s_{\arg}) \qquad =: \tilde{K}_h, \tag{4.3.15a}$$

$$\big\|\big(\hat{h}^{AB}\big(w(t)+\varphi_0\big)\big)\big\|_{e,s,\mathrm{ul}} \leq c\, \Delta^{-1}_{\hat{h}}\big(1+(\Delta^{-1}_{\hat{h}}(C^{\hat{h}}_s(1+C^s_{\arg}))^s\big) \qquad =: K_h, \tag{4.3.15b}$$

$$\big\|\big(\widehat{\boldsymbol{\Gamma}}^A_{BC}\big(w(t)+\varphi_0\big)\big)\big\|_{e,s} \leq c\, K_h C^{\hat{h}}_{d,s}(1+C^s_{\arg}) \qquad =: C_{\widehat{\boldsymbol{\Gamma}},s}, \tag{4.3.15c}$$

$$\big\|\big(\widehat{\boldsymbol{\Gamma}}^A_{BC}\big(w(t)+\varphi_0\big)\big)\big\|_{e} \leq 2\Delta^{-1}_{\hat{h}} C^{\hat{h}}_{d,0} \qquad =: C_{\widehat{\boldsymbol{\Gamma}},0} \tag{4.3.15d}$$

with $C_{\arg}$ defined in (4.3.13). Further, the following Lipschitz estimates hold

$$\big\|\big(\hat{h}_{AB}\big(w(t)+\varphi_0\big)\big) - \big(\hat{h}_{AB}\big(w(t)+\psi_0\big)\big)\big\|_{e,s-1} \leq \mathrm{Lip}_{\hat{h},s-1}\|\varphi_0-\psi_0\|_{s-1} \tag{4.3.16a}$$

$$\big|\big(\hat{h}_{AB}\big(w(t)+\varphi_0\big)\big) - \big(\hat{h}_{AB}\big(w(t)+\psi_0\big)\big)\big|_{e} \leq C^{\hat{h}}_{d,0}|\varphi_0-\psi_0| \tag{4.3.16b}$$

$$\big\|\big(\hat{h}_{AB}\big(w(t)+\varphi_0\big)\big) - \big(\hat{h}_{AB}\big(w(t')+\varphi_0\big)\big)\big\|_{e,s-1} \leq \mathrm{Lip}_{\hat{h},t}|t-t'| \tag{4.3.16c}$$

$$\big\|\big(\widehat{\boldsymbol{\Gamma}}^A_{BC}\big(w(t)+\varphi_0\big)\big) - \big(\widehat{\boldsymbol{\Gamma}}^A_{BC}\big(w(t)+\psi_0\big)\big)\big\|_{e,s-1} \leq \mathrm{Lip}_{\widehat{\boldsymbol{\Gamma}},s-1}\|\varphi_0-\psi_0\|_{s-1} \tag{4.3.16d}$$

$$\big\|\big(\widehat{\boldsymbol{\Gamma}}^A_{BC}\big(w(t)+\varphi_0\big)\big) - \big(\widehat{\boldsymbol{\Gamma}}^A_{BC}\big(w(t')+\varphi_0\big)\big)\big\|_{e,s-1} \leq \mathrm{Lip}_{\widehat{\boldsymbol{\Gamma}},t}|t-t'| \tag{4.3.16e}$$

$$\big|\big(\widehat{\boldsymbol{\Gamma}}^A_{BC}\big(w(t)+\varphi_0\big)\big) - \big(\widehat{\boldsymbol{\Gamma}}^A_{BC}\big(w(t)+\psi_0\big)\big)\big|_{e} \leq \mathrm{Lip}_{\widehat{\boldsymbol{\Gamma}},0}|\varphi_0-\psi_0| \tag{4.3.16f}$$

where the occurring constants are defined by

$$\begin{aligned}
\mathrm{Lip}_{\hat{h},s-1} &= c\, C^{\hat{h}}_{d,s}\big(1+(C^2_{\arg}+3(\rho^2+D^2_0))^{s/2}\big)\\
\mathrm{Lip}_{\hat{h},t} &= c\, C^{\hat{h}}_{d,s}(\tilde{C}^{\mathcal{X}}_0)(1+(C^2_{\arg}+3T^2_1(\tilde{C}^{\mathcal{X}}_0)^2)^{s/2})\\
\mathrm{Lip}_{\widehat{\boldsymbol{\Gamma}},s-1} &= c\, K^2_h\tilde{\theta}_h C^{\hat{h}}_{d,s}(1+C^s_{\arg}) + c\, K_h C^{\hat{h}}_{d^2,s}(1+(C^2_{\arg}+3(\rho^2+D^2_0))^{s/2})\\
\mathrm{Lip}_{\widehat{\boldsymbol{\Gamma}},t} &= c\, K^2_h \mathrm{Lip}_{\hat{h},t} C^{\hat{h}}_{d,s}(1+C^s_{\arg}) + c\, K_h C^{\hat{h}}_{d^2,s}(\tilde{C}^{\mathcal{X}}_0)(1+(C^2_{\arg}+3T^2_1(\tilde{C}^{\mathcal{X}}_0)^2)^{s/2})\\
\mathrm{Lip}_{\widehat{\boldsymbol{\Gamma}},0} &= 2\big(\Delta^{-2}_h(C^{\hat{h}}_{d,0})^2+\Delta^{-1}_{\hat{h}}C^{\hat{h}}_{d^2,0}\big).
\end{aligned}$$

Proof. The bound (4.3.15a) follows from Lemma 2.5 using the C^s-bounds stated in the previous lemma. The bound for the inverse follows from Lemma 2.4. Again, Lemma 2.5 yields the third estimate making use of the C^s-bounds for the derivative of the metric $\hat{h}_{AB}$.

To obtain the Lipschitz estimates we use a similar device as in [Kat75] section 5.2. Estimate (4.3.16b) follows directly from the bound for $D\hat{h}_{AB}$ stated in (4.3.14a). Inequality (4.3.16a) can be derived from a device similar to the proof of Lemma 2.5 regarding the C^{s-1}-bound for the derivative of the metric $D\hat{h}_{AB}$ stated in (4.3.14b) and the estimate for the argument $w(t)+\varphi_0$ stated in (4.3.13). To treat the argument of $D\hat{h}_{AB}$, namely $\lambda\varphi_0+(1-\lambda)\psi_0$ for $0\leq\lambda\leq 1$ we have to insert $2(\rho^2+D^2_0)^{1/2}$ instead of $(\rho^2+D^2_0)^{1/2}$. Therefore, an additional term appears in the estimate. The Lipschitz estimate (4.3.16c) w.r.t. the time parameter t follows from the same considerations. Here we have to use twice the radius to estimate the argument $\lambda w(t)+(1-\lambda)w(t')$ which gives the additional term $3T^2_1(\tilde{C}^{\mathcal{X}}_0)^2$.

The Lipschitz estimates for the Christoffel symbols follow from considering a generic norm

$$\begin{aligned}
&\big|\big(\widehat{\mathbf{\Gamma}}^A_{BC}(w(t)+\varphi_0)\big)-\big(\widehat{\mathbf{\Gamma}}^A_{BC}(w(t)+\psi_0)\big)\big|_e \\
&\qquad\le 2\big|\big(\hat{h}^{AB}(w(t)+\varphi_0)\big)-\big(\hat{h}^{AB}(w(t)+\psi_0)\big)\big|_e\big|\big(D\hat{h}_{AB}(w(t)+\psi_0)\big)\big|_e \\
&\qquad\qquad+2\big|\big(\hat{h}^{AB}(w(t)+\varphi_0)\big)\big|_e\big|\big(D\hat{h}_{AB}(w(t)+\varphi_0)\big)-\big(D\hat{h}_{AB}(w(t)+\psi_0)\big)\big|_e
\end{aligned}$$

Inserting the bounds and Lipschitz estimates for the metric $\hat{h}_{AB}$, namely the inequalities (4.3.15b), (4.3.16a) and (4.3.16b), we derive the estimates (4.3.16d) and (4.3.16f). A similar generic estimate and Lipschitz estimate (4.3.16c) yield the remaining estimate (4.3.16e). The second term includes a Lipschitz estimate for $D\hat{h}_{AB}$ which can be established as the Lipschitz estimate for $\hat{h}_{AB}$ using the constant $C^{\hat{h}}_{d^2,s}$ instead of $C^{\hat{h}}_{d,s}$. □

The preceding lemma provides us with Sobolev norm bounds for the matrix $g^{\mathrm{a}}_{\mu\nu}$ and the coefficients.

Lemma 4.20. *The Lemmata 3.23 and 3.24 remain valid if the constants are changed in the following way*

$$\begin{aligned}
\tilde{K}&=\tilde{K}_h(K^2_{0,s}+K^2_{1,s}),\\
\tilde{\theta}&=2^3\,\tilde{K}_h(K_{0,s}+K_{1,s})+2^3\,\mathrm{Lip}_{\hat{h},s-1}(K^2_{0,s}+K^2_{1,s})\\
\tilde{\theta}'&=2^3\,C^{\hat{h}}_0(K_0+K_1)+2^3\,C^{\hat{h}}_{d,0}(K^2_0+K^2_1),\\
K&=c\Delta^{-1}\big(1+(\Delta^{-1}\tilde{K})^s\big),\quad \theta=K^2\tilde{\theta}\quad \textit{and}\quad \theta'=\Delta^{-2}\tilde{\theta}'.
\end{aligned}\tag{4.3.17}$$

Proof. The first claim follows from the proof of Lemma 3.23 taking care of the additional terms involving the interpolated metric $\hat{h}_{AB}$ and its Christoffel symbols which can be estimated by the Sobolev estimates obtained in lemma 4.19.

In the same way as in the proof of Lemma 3.24, estimates for the coefficients follow from estimates for the matrix $g^{\mathrm{a}}_{\mu\nu}$. □

The Lemmata 4.17 and 4.20 yield that the coefficients $g^{\mu\nu}_{\mathrm{a}}$ satisfy the conditions (2.3.2a) to (2.3.2c) and (2.3.2h) of the existence Theorem 2.9.

In Minkowski space there were no explicit dependency of the coefficients on the time parameter t. Here, the coefficients need to satisfy condition (2.3.2d).

Lemma 4.21. *Let $t\le T_1$ and $(\varphi_0,\varphi_1)\in W$.*

Then the matrix $g^{\mathrm{a}}_{\mu\nu}$ satisfies

$$\|(g^{\mathrm{a}}_{\mu\nu}(t,\varphi_0,\varphi_1))-(g^{\mathrm{a}}_{\mu\nu}(t',\varphi_0,\varphi_1))\|_{e,s-1,\mathrm{ul}}\le\tilde{\nu}|t-t'|$$

with $\tilde{\nu}=\mathrm{Lip}_{\hat{h},t}(K^2_{0,s}+K^2_{1,s})$.

Proof. Since $\chi_0+\varphi_1$ and $Dw_0+D\varphi_0$ do not depend on the time parameter it suffices to obtain an estimate on the difference of the metric $\hat{h}_{AB}$ which is provided by inequality (4.3.16c). □

Remark 4.22. The Lipschitz constant ν for the coefficients $g_{\mathrm{a}}^{\mu\nu}$ of the asymptotic equation (3.2.15a) meeting the condition (2.3.2d) is given by $K^2\tilde{\nu}$.

We will now give estimates for the RHS of equation (4.3.9a). The first part f_{a}, defined analog to the RHS of the asymptotic equation (3.2.15a) in Minkowski space, can be estimated as in lemma 3.27 using the estimates for the coefficients provided that there exist bounds for the Christoffel symbols $\hat{\gamma}_{\mu\nu}^{\lambda}$ of the metric $\hat{a}_{\mu\nu}$. These bounds will be derived in the next lemma.

Lemma 4.23. *The Christoffel symbols of the matrix $\hat{a}_{\mu\nu}$ satisfy the following inequalities*

$$\|(\hat{\gamma}_{\mu\nu}^{\lambda})\|_{e,s} \le C^{\hat{\gamma}} \quad \textit{and} \quad |(\hat{\gamma}_{\mu\nu}^{\lambda})|_e \le C_0^{\hat{\gamma}},$$

where

$$\begin{aligned}(2K)^{-1}C^{\hat{\gamma}} &= \tilde{D}_1\tilde{K}_h\big(2((\tilde{C}_0^{\chi})^2 + D_1^2)^{1/2} + 4(C_{w_0}^2 + D_0^2)^{1/2}\big) \\ &\quad + \tilde{D}_0\tilde{K}_h\big(4((\tilde{C}_0^{\chi})^2 + D_1^2)^{1/2} + 2(C_{w_0}^2 + D_0^2)^{1/2}\big) + c\,C_{d,s}^{\hat{h}}(1 + C_{\arg}^s)\big(((\tilde{C}_0^{\chi})^2 + D_1^2) \\ &\qquad + (C_{w_0}^2 + D_0^2) + 4((\tilde{C}_0^{\chi})^2 + D_1^2)^{1/2}(C_{w_0}^2 + D_0^2)^{1/2}\big)\end{aligned}$$

and $C_0^{\hat{\gamma}}$ arises from $C^{\hat{\gamma}}$ by applying the replacements

$$K \mapsto \Delta^{-1},\ \tilde{K}_h \mapsto \Delta_{\hat{h}}^{-1},\ (D_0, D_1) \mapsto (\delta_1, \delta_2),\ c\,C_{d,s}^{\hat{h}}(1 + C_{\arg}^s) \mapsto C_{d,0}^{\hat{h}}$$

and $\tilde{D}_0$, $\tilde{D}_1$ are replaced by the bounds for $|D\mathring{\chi}|_e$ and $|D^2\mathring{\Phi}|_e$ stated in (3.2.32).

Proof. The proof follows by considering the additional terms $\|(\hat{h}_{AB})\|_{e,s,\mathrm{ul}}$ and $\|(D\hat{h}_{AB})\|_{e,s}$ occurring in $\|D\hat{a}\|_{e,s,\mathrm{ul}}$. The first term was estimated in (4.3.15a). Lemma 2.5 and the C^s-bound for $D\hat{h}_{AB}$ stated in (4.3.14b) yield the following estimate

$$\|(D\hat{h}_{AB})\|_{e,s} \le c\,C_{d,s}^{\hat{h}}(1 + C_{\arg}^s).$$

Further, we make use of the H^s-bounds $\tilde{D}_0$ and $\tilde{D}_1$ for $D\mathring{\chi}$ and $D^2\mathring{\Phi}$ as introduced in (3.2.26) and the bounds

$$\|Dw_0 + Dv_0\|_{s,\mathrm{ul}} \le (C_{w_0}^2 + D_0^2)^{1/2} \quad \text{and} \quad \|\chi_0 + \mathring{\chi}\|_{s,\mathrm{ul}} \le ((\tilde{C}_0^{\chi})^2 + D_1^2)^{1/2}.$$

The replacement is possible if we apply the local bounds for $\hat{h}_{AB}$ stated in (4.3.14a). □

The Lemmata 3.27 and 4.23 yield that f_{a} satisfies the assumptions (2.3.2a) and (2.3.2e) to (2.3.2g) of the existence Theorem 2.9.
To obtain estimates for the additional term $\mathbf{f}_{\mathrm{a}}$, we first state generic estimates to be proved similar to the ones in Lemma 3.22.

Lemma 4.24. *For two functions F and $\bar{F}$ the following inequalities hold*

$$|\mathbf{f}(F)| \leq \big|\big(g^{\mu\nu}(F)\big)\big|(|\partial_t F|^2 + |DF|^2)\big|\big(\widehat{\mathbf{\Gamma}}^A_{BC}(F)\big)\big| \tag{4.3.18a}$$

$$\begin{aligned}|\mathbf{f}(F) - \mathbf{f}(\bar{F})| \leq{}& \big|\big(g^{\mu\nu}(F)\big) - \big(g^{\mu\nu}(\bar{F})\big)\big|\,(|\partial_t \bar{F}|^2 + |D\bar{F}|^2)\big|\big(\widehat{\mathbf{\Gamma}}^A_{BC}(\bar{F})\big)\big| \\ &+ \big|\big(g^{\mu\nu}(F)\big)\big|(|\partial_t V| + |DV|)(|\partial_t \bar{F}|^2 + |D\bar{F}|^2)^{1/2}\big|\big(\widehat{\mathbf{\Gamma}}^A_{BC}(\bar{F})\big)\big| \\ &+ \big|\big(g^{\mu\nu}(F)\big)\big|(|\partial_t F|^2 + |DF|^2)^{1/2}(|\partial_t V| + |DV|)\big|\big(\widehat{\mathbf{\Gamma}}^A_{BC}(\bar{F})\big)\big| \\ &+ \big|\big(g^{\mu\nu}(F)\big)\big|(|\partial_t F|^2 + |DF|^2)\big|\big(\widehat{\mathbf{\Gamma}}^A_{BC}(F)\big) - \big(\widehat{\mathbf{\Gamma}}^A_{BC}(\bar{F})\big)\big|,\end{aligned} \tag{4.3.18b}$$

where we set $V = F - \bar{F}$.

Remark 4.25. As for the asymptotic equation in Minkowski space this estimates will be used with the replacements given in (3.2.30).

The next lemma establishes statements analog to Lemma 3.27 for the additional term $\mathbf{f}_{\mathrm{a}}$ from which the conditions of Theorem 2.9 can be obtained.

Lemma 4.26. *Let $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in W$.*
Then the term $\mathbf{f}_{\mathrm{a}}$ satisfies the following estimates

$$\begin{aligned}\|\mathbf{f}_{\mathrm{a}}(t, \varphi_0, \varphi_1)\|_s &\leq K_{\mathbf{f}}, \\ \|\mathbf{f}_{\mathrm{a}}(t, \varphi_0, \varphi_1) - \mathbf{f}_{\mathrm{a}}(t, \psi_0, \psi_1)\|_{s-1} &\leq \theta'_{\mathbf{f}} E_s\big((\varphi_0, \varphi_1) - (\psi_0, \psi_1)\big) \\ \|\mathbf{f}_{\mathrm{a}}(t, \varphi_0, \varphi_1) - \mathbf{f}_{\mathrm{a}}(t, \psi_0, \psi_1)\|_{L^2} &\leq \theta_{\mathbf{f}} E_1\big((\varphi_0, \varphi_1) - (\psi_0, \psi_1)\big) \\ \|\mathbf{f}_{\mathrm{a}}(t, \varphi_0, \varphi_1) - \mathbf{f}_{\mathrm{a}}(t', \varphi_0, \varphi_1)\|_{s-1} &\leq \nu_{\mathbf{f}}|t - t'|\end{aligned}$$

with

$$\begin{aligned}K_{\mathbf{f}} &= C_{\widehat{\mathbf{\Gamma}},s} K (K_{1,s}^2 + K_{0,s}^2), \\ \nu_{\mathbf{f}} &= K^2 \tilde{\nu} C_{\widehat{\mathbf{\Gamma}},s} (K_{1,s}^2 + K_{0,s}^2) + K(K_{1,s}^2 + K_{0,s}^2)\mathrm{Lip}_{\widehat{\mathbf{\Gamma}},t}, \\ \theta'_{\mathbf{f}} &= \theta C_{\widehat{\mathbf{\Gamma}},s}(K_{1,s}^2 + K_{0,s}^2) + 2K C_{\widehat{\mathbf{\Gamma}},s}(K_{1,s}^2 + K_{0,s}^2)^{1/2} + K(K_{1,s}^2 + K_{0,s}^2)\mathrm{Lip}_{\widehat{\mathbf{\Gamma}},s-1}\end{aligned}$$

and $\theta_{\mathbf{f}}$ arises from $\theta'_{\mathbf{f}}$ by applying the replacements

$$\theta \mapsto \theta',\ (K_{0,s}, K_{1,s}) \mapsto (K_0, K_1),\ C_{\widehat{\mathbf{\Gamma}},s} \mapsto C_{\widehat{\mathbf{\Gamma}},0},\ \mathrm{Lip}_{\widehat{\mathbf{\Gamma}},s-1} \mapsto \mathrm{Lip}_{\widehat{\mathbf{\Gamma}},0} \text{ and } K \mapsto \Delta^{-1}.$$

Proof. The inequalities follow from the generic estimates (4.3.18a) and (4.3.18b) using the estimates for the Christoffel symbols $\widehat{\mathbf{\Gamma}}^A_{BC}$ as in (4.3.15c), (4.3.16d) and the corresponding local estimates (4.3.15d) and (4.3.16f). The Lipschitz constant $\nu_{\mathbf{f}}$ is obtained by inserting the Lipschitz constants for $\widehat{\mathbf{\Gamma}}^A_{BC}$ and $g^{\mathrm{a}}_{\mu\nu}$ stated in (4.3.16e) and Lemma 4.21. □

Remark 4.27. An examination of the proofs of the Lemmata 4.19, 4.20 and the preceding lemma reveals that the statement of Lemma 3.28 remains valid. Therefore, the local conditions (2.4.8) needed to apply the local uniqueness Theorem 2.26 to the asymptotic equation (4.3.9a) are satisfied.

We will now give a proof for the main result of this section, an analog to Theorem 3.14. In the treatment of the Minkowski case the main result followed from an analog to Proposition 4.15. Here, we need to apply one step more to be sure that solutions obtained by this proposition indeed satisfy the membrane equation.

Proof of Proposition 4.15. The Lemmata 4.17, 4.20, 4.23 and 4.26 show that the conditions of the existence Theorem 2.9 are satisfied for the asymptotic equation (4.3.9a). From Theorem 2.22 we derive a solution F to the IVP (4.3.4a) with the properties stated in (4.3.6). Applying Remark 3.32 provides us with a solution for negative values of t.

Regarding Remark 4.27 uniqueness of solutions can be shown as in the proof of Proposition 3.17 by applying the local uniqueness result 2.26. $\square$

Remark 4.28.
1. Remark 3.30 remains valid. All estimates for the coefficients and the RHS are valid for $t > 0$ and $t < 0$.

2. As in Remark 3.31 we get a lower bound for the existence time of the solution. The constant c_E has to be changed according to the new definition of the constants λ and μ in (4.3.11b) and (4.3.11c). Therefore, some factors of $(1 + \delta_0\tilde{\theta})$ occur in the definition of c_E.

Remark 4.29. In contrast to the Minkowski case the coefficients and RHS of the asymptotic equation depend on the time parameter through the components of the metric on the ambient manifold (cf. (4.3.7)).

It suffices to consider the components of the metric h_{AB} since the coefficients and the RHS differ from the coefficients and the RHS in the Minkowski case with terms depending on this metric. From the assumption (4.11) we get that derivatives of h_{AB} are bounded up to order $s + 3 + \ell_0$. This yields that derivatives of order $0 \leq \ell \leq \ell_0$ admit the bounds stated in Lemma 4.18. From the fact that the Sobolev-norm estimates derived in Lemma 4.19 are shown by only using the bounds of Lemma 4.18 we obtain the desired result from corollary 2.19.

In the sequel we will proof Theorem 4.12. In contrast to the Minkowski case existence does not follow immediately from Proposition 4.15, since the solution can leave the image of the coordinates on the ambient manifold.

Proof of the existence claim in Theorem 4.12. In the proof of Proposition 4.15 the parameter T_1 was left free (cf. definition (4.3.8)). Set

$$T_1 := K_1^{-1}\tilde{\theta}\rho_0/4. \tag{4.3.19}$$

Then it follows that a solution $F(t, z)$ obtained from Proposition 4.15 lies in $B^e_{\tilde{\theta}\rho_0/2}(0) \subset \mathbb{R}^{n+1}$ for $(t, z) \in [0, T_1] \times B^e_{\theta\rho_1/2}(0) \subset \mathbb{R} \times \mathbb{R}^m$. Therefore it satisfies the unmodified reduced membrane equation in that region. The same holds for negative parameter t provided $-t \leq T_1$. $\square$

Proof of the uniqueness claim in Theorem 4.12. The claim follows from similar considerations as the proof of the uniqueness part of Theorem 3.14. From Remark 4.27 we conclude that the local uniqueness result 2.26 is applicable. The slope c_0 of the cone on which uniqueness holds has to be adapted according to estimate (2.4.5) and lemma 4.17. □

Remark 4.30. For later reference we state the exact value of the slope. It is given by

$$c_0 = 1 + m^{1/2}\tilde{\mu}^{-1}K_1^2\big(2\tilde{\lambda}^{-1}K_0K_1(1+\delta_0\tilde{\theta})+1\big) \cdot \big(1+m^{1/2}\tilde{\mu}^{-1}K_0^2(1+\delta_0\tilde{\theta})\big)(1+\delta_0\tilde{\theta}). \quad (4.3.20)$$

4.4 Gluing local solutions

In this section we will develop solutions to the Cauchy problem (1.3.2) analog to section 3.3. Based on the observation that the membrane equation (0.3.1) is invariant under scaling, i.e. conformal transformations of the ambient manifold with a constant, the result will be stated in a scale invariant manner.

Let $R > 0$ be the scale. Recall the notations of the main problem (0.4.1). Assume M to be an m-dimensional manifold and $\varphi : M \to N$ to be an immersion of Σ_0. The timelike vector field along φ denoted by $\chi : M \to TN$ will serve as initial velocity combining initial direction, lapse and shift.

Let $s > \frac{m}{2}+1$ be an integer. We make the following uniform assumptions on the ambient manifold (N,h) and the initial data φ and χ.

Assumptions 4.31. *The ambient manifold (N,h) endowed with time function τ satisfies the following condition*

$$\begin{aligned} &\textit{There exist constants } C_1, C_2, C_1^\tau, \dots, C_{s+2}^\tau, C_0^N, \dots, C_{s+1}^N \\ &\textit{independent of } R \textit{ such that} \\ &C_1 \le R^{-1}\psi \le C_2, \quad R^{2+\ell}|\mathbf{D}^\ell \mathbf{Rm}|_E \le C_\ell^N \quad \textit{for} \quad 0 \le \ell \le s+1 \\ \textit{and} \quad &R^{1+\ell}|\mathbf{D}^\ell(\mathbf{D}\tau)|_E \le C_\ell^\tau \quad \textit{for} \quad 1 \le \ell \le s+2, \end{aligned} \quad (4.4.1a)$$

where $\mathbf{D}(\mathbf{D}\tau)$ denotes the $(1,1)$-tensor obtained by applying the covariant derivative to the gradient of τ.

$$\begin{aligned} &\textit{There exist constants } \omega_1, C_0^\varphi, \dots, C_s^\varphi \textit{ independent of } R \textit{ such that} \\ &\inf\{-h(\gamma,\widehat{T}) : \gamma \textit{ timelike future-directed unit normal to } \Sigma_0\} \le \omega_1 \\ \textit{and} \quad &R^{\ell+1}|\widehat{\nabla}^\ell \mathring{I\!I}|_{\mathring{g},E} \le C_\ell^\varphi \textit{ for } 0 \le \ell \le s. \end{aligned} \quad (4.4.1b)$$

$$\begin{aligned} &\textit{There exist constants } L_1, L_2, L_3, C_1^\chi, \dots, C_{s+1}^\chi \textit{ independent of } R \textit{ such that} \\ &-L_1 \le R^{-2}h(\chi,\chi) \le -L_2, \quad -h\big(\tfrac{\chi}{\langle\chi\rangle},\widehat{T}\big) \le L_3, \textit{ where } \langle\chi\rangle^2 = -h(\chi,\chi) \\ \textit{and} \quad &R^\ell|\widehat{\nabla}^\ell\chi|_{\mathring{g},E} \le C_\ell^\chi \textit{ for } 1 \le \ell \le s+1. \end{aligned} \quad (4.4.1c)$$

The following result states existence and uniqueness for solutions to the Cauchy problem (1.3.2) in analogy to Theorem 3.39 in Minkowski space.

Theorem 4.32. *Let the metric $h \in C^{s+3}$ and the time function $\tau \in C^{s+3}$ of the ambient manifold N satisfy the assumptions (4.4.1a). Suppose the initial data $\varphi \in C^{s+2}$ and $\chi \in C^{s+1}$ satisfy the assumptions (4.4.1b) and (4.4.1c) respectively.*

Then there exist a constant C_0 independent of R and a C^2-solution $F : [-RC_0, RC_0] \times M \to N$ to the membrane equation

$$g^{\mu\nu}\partial_\mu\partial_\nu F^A - \Gamma^\lambda \partial_\lambda F^A + g^{\mu\nu}\partial_\mu F^B \partial_\nu F^C \mathbf{\Gamma}^A_{BC}(F) = 0$$

in harmonic map gauge w.r.t. the background metric defined in (1.3.7) attaining the initial values

$$F\big|_{t=0} = \varphi \qquad \text{and} \qquad \tfrac{d}{dt}F\big|_{t=0} = \chi.$$

If F and $\bar{F}$ are two such solutions, then they coincide for $-R\min(\bar{C}_1, \bar{C}_2) \le t \le R\min(\bar{C}_1, \bar{C}_2)$. The constants $\bar{C}_1$ and $\bar{C}_2$ are given by the constants T_0 and T_1 defined in respectively (3.3.9) and (4.3.19) at scale $R = 1$.

Remark 4.33. Let $s > \frac{m}{2} + 1 + \ell_0$. Then the solution $F : [-RC_0, RC_0] \times M \to N$ is of class $C^{2+\ell_0}$.

Remark 4.34. The theorem applies to the situation, where the assumptions 4.31 are only valid in a neighborhood of the initial submanifold Σ_0.

The strategy will be to show the claim of the theorem for $R = 1$ first, and then examine the behaviour of a solution to the membrane equation under scaling. However, we will start with the second step.

Lemma 4.35. *The membrane equation (0.3.1) is invariant under scaling.*

Proof. Let (N, h) be a Lorentzian manifold and Σ be a timelike submanifold of N satisfying the membrane equation. Let $R > 0$ be a constant and define $\bar{h} := R^2 h$. Since the membrane equation is pointwise we pick a point $p \in \Sigma$ and let $F : V \subset \mathbb{R}^{m+1} \to N$ be an embedding of Σ in a neighborhood of p. As an embedding of a submanifold satisfying the membrane equation, $F^A := y^A \circ F$ satisfies equation (1.3.1b) for given coordinates y on N.

We introduce the following coordinates on $\mathbb{R}^{m+1}$ and N. Set $\bar{x}^\mu = Rx^\mu$, where $x = (x^\mu)$ denote the standard coordinates on $\mathbb{R}^{m+1}$, and $\bar{y}^A = Ry^A$. Let

$$\bar{F}^A(z) = \bar{y}^A \circ F \circ \bar{x}^{-1}(z) = Ry^A \circ F \circ x(R^{-1}z) = RF^A(R^{-1}z).$$

We will show that $\bar{F}^A$ satisfies the membrane equation in the coordinates $\bar{x}$ and $\bar{y}$ on $(N, \bar{h})$. Then it follows that Σ satisfies the membrane equation in $(N, \bar{h})$, since $\bar{F}^A$ is also an embedding of Σ.

To this end we compute the ingredients of equation (1.3.1b). The components of the metric $\bar{h}$ w.r.t. the coordinates $\bar{y}$ are

$$\bar{h}_{AB}(w) = \bar{h}\big(\bar{\partial}_A(w), \bar{\partial}_B(w)\big) = R^2 h\big(R^{-1}dy^{-1}_{R^{-1}w}(e_A), R^{-1}dy^{-1}_{R^{-1}w}(e_B)\big) = h_{AB}(R^{-1}w)$$

where h_{AB} are the components of the metric h w.r.t. the chart y on N. With a similar consideration it follows that $\bar{\mathbf{\Gamma}}^A_{BC}(w) = R^{-1}\mathbf{\Gamma}^A_{BC}(R^{-1}w)$.

These expressions yield for the induced metric $g = F^*h$ and $\bar{g} = \bar{F}^*(\bar{h}_{AB})$ on $\mathbb{R}^{m+1}$

$$\begin{aligned}\bar{g}_{\mu\nu}(\bar{F})(z) &= \partial_\mu \bar{F}^A(z)\bar{h}_{AB}\big(\bar{F}(z)\big)\partial_\nu \bar{F}^B(z)\\ &= \partial_\mu F^A(R^{-1}z)h_{AB}\big(F^A(R^{-1}z)\big)\partial_\nu F^B(R^{-1}z) = g_{\mu\nu}(F)(R^{-1}z).\end{aligned}$$

Similar to the argument giving the representation for $\bar{\mathbf{\Gamma}}^A_{BC}$ it follows for the contracted Christoffel symbols of $\bar{g}$ that

$$\bar{\Gamma}^\lambda(\bar{F})(z) = R^{-1}\Gamma^\lambda(F)(R^{-1}z).$$

We derive that the mean curvature operator of $\bar{F}$, seen as a differential operator depending on the coordinates, computed in the coordinates $\bar{x}$ and $\bar{y}$ has the form

$$H\big(\bar{F}(z)\big) = R^{-1}H\big(F(R^{-1}z)\big).$$

Hence, the desired result follows. □

After showing the scale invariance of the equation we will investigate the behaviour of the assumptions of Theorem 4.32 under scaling.

Lemma 4.36. *Let N be a Lorentzian manifold satisfying the assumptions (4.4.1a) at scale $R = 1$. Set $\bar{h} = R^2 h$.*
Then $(N, \bar{h})$ satisfies the assumptions (4.4.1a) at scale $R > 0$.

Proof. To derive the scaling behaviour of the occurring terms we consider coordinates y^A and $\bar{y}^A = Ry^A$ on N. Coordinate derivatives give a factor of R^{-1}, therefore it follows from the definition of the lapse that it scales with R^1. The factors of R cancel such that the constants C_1 and C_2 are independent of R. The same argument shows that the curvature scales with R^{-2} and every covariant derivative gives a factor of R^{-1}. We conclude that the terms involving the curvature and covariant derivatives of the time function satisfy the desired inequalities with constants independent of R. □

Based on the Lemmata 4.35 and 4.36 the next proposition will establish the claim of Theorem 4.32 concerning the scaling property of solutions to the Cauchy problem (1.3.2).

Proposition 4.37. *Suppose Theorem 4.32 holds at scale $R = 1$. Then it also holds for an arbitrary scaling constant $R > 0$.*

Proof. Let h, τ, φ and χ satisfy the assumptions of Theorem 4.32 at scale R. Let $\bar{h} = R^{-2}h$. From lemma 4.36 we derive that $(N, \bar{h})$ satisfies the assumptions (4.4.1a) at scale $R = 1$ and from the proof we infer that φ and χ also satisfy the assumptions (4.4.1b) and (4.4.1c) at scale $R = 1$.

Theorem 4.32 at scale $R = 1$ yields that there exist a constant C_0 and a solution $\bar{F} : [-C_0, C_0] \times M \to N$ of the membrane equation (1.3.1b) in $(N, \bar{h})$ attaining the initial values $\bar{F}\big|_{\bar{t}=0} = \varphi$ and $\frac{d}{d\bar{t}}\bar{F}\big|_{\bar{t}=0} = \chi$.

Set $F(R^{-1}t,p) = \bar{F}(\bar{t},p)$ defined on $[-RC_0, RC_0] \times M$. Since the image remains the same we conclude from Lemma 4.35 that F also satisfies the membrane equation. Since $\bar{F}\big|_{\bar{t}=0} = \varphi$, it follows that $F\big|_{t=0} = \varphi$. The initial velocity of F can be computed via

$$\tfrac{d}{dt}F(t,p) = \tfrac{d}{dt}\bar{F}(R\bar{t},p) = R^{-1}\tfrac{d}{d\bar{t}}\bar{F}(R\bar{t},p) = \chi(p) \quad \text{for } p \in M.$$

The considerations for the existence claim show that only a scaling of the time parameter is necessary which provides the uniqueness claim. □

The preceding result yields that it remains to investigate Theorem 4.32 at scale $R = 1$. Following the method of section 3.3 we will show in the next proposition that the special coordinates of N introduced in section 4.1 satisfy the conditions 4.11 and the graph representation derived in section 4.2 with help of the special coordinates satisfy the conditions 3.13.

Proposition 4.38. *Suppose $h, \tau \in C^{s+3}$, $\varphi \in C^{s+2}$ and $\chi \in C^{s+1}$ satisfy the assumptions 4.31. Let $p \in M$. Suppose x, y are coordinates for M and special coordinates on N introduced in section 4.1 such that $y \circ \varphi \circ x^{-1}$ is the special graph representation obtained in section 4.2 with center p.*

Then y and the representation of h w.r.t. y satisfy the conditions 4.11. Further the coordinates x, y and the representations Φ and χ_{xy} w.r.t. x and y satisfy the conditions 3.13 with η_{AB} replaced by the representation of h.

Proof. We will first show that the special coordinates y and the representation of h in y have the desired property. The image of the coordinates y contain a cylinder $[-\rho_0, \rho_0] \times B^e_{\rho_0}(0) \subset \mathbb{R} \times \mathbb{R}^n$ which follows from the estimates (4.1.6) and (4.1.7a).

The choice of the parameters within the construction estimate (4.1.9) yields for $\delta_0 = \frac{2}{3}$. Hence, condition (4.3.1a) is satisfied. Bounds for derivatives of the representation of the metric h w.r.t. y can be obtained from (4.1.19). We see that the assumptions (4.4.1a) meet the conditions (4.1.18) giving us the boundedness of the desired order of derivatives of h.

We now head to the conditions 3.13. The parts 1 and 2 can be obtained by the same arguments as in the first step of the proof of Proposition 3.43 taking the Lemmata 4.8, 4.7 and 4.10 into account.

To derive the conditions for the initial velocity we first consider the second step of the proof of Proposition 3.43 establishing the boundedness of $|\chi_{xy}|_e$. By using the representation of h in the special coordinates (cf. (4.1.2)) we get from the assumptions (4.4.1c) in a similar way the boundedness of $|\chi_{xy}|_E$. Estimate (4.1.10) for E_{AB} now yields the desired boundedness of the Euclidean norm.

Following part 3 of the proof of proposition 3.43 we use Lemma 3.10 to obtain the following identity for derivatives of the initial velocity

$$\partial^k \chi_{xy} = \widehat{\nabla}^k \chi_{xy} + \sum \partial^{\alpha_1} \mathring{\Gamma} * \cdots * \partial^{\alpha_p} \mathring{\Gamma} * \partial^{\beta_1} \widehat{\Gamma} * \cdots * \partial^{\beta_q} \widehat{\Gamma} * \widehat{\nabla}^{\ell} \chi_{xy}. \tag{4.4.2}$$

This expression can be estimated analog to a similar expression for the second fundamental form in Lemma 4.10. Thus, the desired bounds for derivatives of χ_{xy} up to order $s+1$ follow. □

The next proposition shows that solutions obtained by Theorem 4.12 are independent of the specific decomposition.

Proposition 4.39. *Proposition 3.46 remains valid, if the slope c_0 given by (4.3.20) is used to define the cone C and the cone is truncated to points (t, x) satisfying $t \leq T_1$ with the constant T_1 defined in (4.3.19).*

Proof. The choice of T_1 ensures that a solution $F(t, z)$ gained from Proposition 4.15 lies in $B_{\tilde{\theta}\rho_0/2}(0) \subset \mathbb{R}^{n+1}$ for $(t, z) \in [0, T_1] \times B^e_{\theta\rho_1/2}(0) \subset \mathbb{R} \times \mathbb{R}^m$. Therefore it satisfies the unmodified reduced membrane equation in that region. Lemma 3.45 yields that the proof of Proposition 3.46 goes through using the uniqueness result from Theorem 4.12 and regarding Remark 3.32. □

We now head to the proof of the main result of this section providing existence to the Cauchy problem (1.3.2).

Proof of the existence claim in Theorem 4.32. We follow the strategy to show the existence claim in Theorem 3.39. The special graph representation satisfies the additional condition (3.3.5) which is stated in estimate (4.2.16). The rest of the proof is analog to the proof of Theorem 3.39 using the corresponding results from Proposition 4.38, Theorem 4.12 and the preceding proposition. □

Remark 4.40. The new constraint T_1 defined in (4.3.19) for the time parameter has to be added to the statements of Remark 3.48. This yields the following lower bound for the existence time T. It holds that $T \geq \min(T', T_0, T_1)$ with T' derived from Proposition 4.15 and T_0 from (3.3.9).

The value of the constant c_E (cf. Remark 2.18 and (3.3.10)), controlling the inverse of the constant T', has to be changed according to the estimates of Lemma 4.17. Not only the angles and the bound for the second fundamental form enter the existence time. Through ρ_0, also the bounds for the curvature and the gradient of the time function are involved.

Remark 4.41. As in the context of the Minkowski space the construction of a solution to the IVP (1.3.2) provides us with a lower bound for the proper time of timelike curves (cf. Remark 3.50).

Proof of the uniqueness claim in Theorem 4.32. In analogy to the proof of the uniqueness claim in Theorem 3.39 the result follows by using the uniqueness result of Theorem 4.12. □

5 Prescribed initial lapse and shift

5.1 Existence and uniqueness

The goal of this section is to show that solutions of the membrane equation represented as immersions are independent of the choice of immersion of the initial submanifold Σ_0, as well as of the initial lapse α and shift β. To this end we will consider the Cauchy problem

(1.3.2) and, after establishing existence for an arbitrary choice of initial lapse and shift, compare solutions with different initial values.

We will use the following norm on tensors. Let (M, g) and $(\widetilde{M}, \tilde{g})$ be two Riemannian manifolds. Suppose $\psi : M \to \widetilde{M}$ is a C^1-mapping. Then we denote by $|d\psi|_{g,\tilde{g}}$ the norm induced on $T^*M \otimes T\widetilde{M}$ given by

$$|d\psi|^2_{g,\tilde{g}} = g^{ij}\tilde{g}_{k\ell}(d\psi)^k_i(d\psi)^\ell_j \quad \text{with} \quad (d\psi)^k_i = \tilde{x}^k\big(d\psi(\partial_i)\big). \tag{5.1.1}$$

$\tilde{x}^k$ are assumed to be coordinates on $\widetilde{M}$ and ∂_i are the basis vector fields arising from coordinates on M. In a similar way we introduce norms for higher derivatives.

Recall the notations of the main problem concerning the ambient space (N, h), Σ_0 and initial direction ν (cf. chapter 0.3).

Assume Σ_0 to be a regularly immersed submanifold of dimension m and suppose $\varphi : M^m \to N$ is an immersion defined on an m-dimensional manifold M satisfying $\operatorname{im}\varphi = \Sigma_0$. Let ν be a timelike future-directed unit vector field along φ normal to Σ_0. Let $\alpha > 0$ be a function on M and let β be a vector field on M. We make the following uniformity assumptions on initial direction, lapse and shift.

Assumptions 5.1. • *There exist constants C^ν_ℓ and L_3 such that*

$$-h(\nu, \widehat{T}) \le L_3 \quad \textit{and} \quad |\widehat{\nabla}^\ell \nu|_{\mathring{g},E} \le C^\nu_\ell \quad \textit{for } 1 \le \ell \le s+1, \tag{5.1.2}$$

where $\widehat{T}$ denotes the unit normal to the slices of the time foliation defined by (4.0.2).

- *There exist constants L_2, C^α_ℓ and C^β_ℓ such that*

$$-\alpha^2 + |\beta|^2_{\mathring{g}} \le -L_2 \tag{5.1.3a}$$

$$\begin{aligned} |\mathring{\nabla}^\ell \beta|_{\mathring{g}} &\le C^\beta_\ell \quad \textit{for } 1 \le \ell \le s+1 \\ \textit{and} \quad |\mathring{\nabla}^\ell \alpha|_{\mathring{g}} &\le C^\alpha_\ell \quad \textit{for } 0 \le \ell \le s+1. \end{aligned} \tag{5.1.3b}$$

The following result provides existence for given initial direction, lapse, and shift and uniqueness for solutions of the membrane equation with different prescribed immersions of the initial submanifold, initial lapse, and shift.

Theorem 5.2. *Let the data $h, \tau \in C^{s+3}$ satisfy the assumptions (4.4.1a), let $\varphi \in C^{s+2}$ satisfy the assumptions (4.4.1b) and let $\alpha, \beta, \nu \in C^{s+1}$ satisfy the assumptions 5.1.*

Then there exist a constant $T > 0$ and a C^2-solution $F : [-T, T] \times M \to N$ of the membrane equation

$$g^{\mu\nu}\partial_\mu\partial_\nu F^A - \Gamma^\lambda\partial_\lambda F^A + g^{\mu\nu}\partial_\mu F^B \partial_\nu F^C \mathbf{\Gamma}^A_{BC}(F) = 0 \tag{5.1.4a}$$

in harmonic map gauge w.r.t. the background metric $\hat{g}$ defined by

$$\hat{g} = -\alpha^2 dt^2 + \mathring{g}_{ij}(\beta^i dt + dx^i)(\beta^j dt + dx^j) \tag{5.1.4b}$$

attaining the initial values

$$F\big|_{t=0} = \varphi \quad \textit{and} \quad \tfrac{d}{dt}F\big|_{t=0} = \alpha\,\nu + d\varphi(\beta). \tag{5.1.4c}$$

Suppose $\bar\varphi : M \to N$ *is another immersion of the initial submanifold* Σ_0 *satisfying the assumptions (4.4.1b) and there exists a local diffeomorphism* $\psi_0 : M \to M$ *with* $\varphi \circ \psi_0^{-1} = \bar\varphi$ *and admitting constants* C_1^ψ *and* C_2^ψ *such that*

$$|d\psi_0|_{\mathring{g},\mathring{\bar g}} \le C_1^\psi \quad \textit{and} \quad |d^2\psi_0|_{\mathring{g},\mathring{\bar g}} \le C_2^\psi, \tag{5.1.5}$$

where we set $\mathring{\bar g} = \bar\varphi^* h$ *and use the norm introduced in (5.1.1). Let the initial direction* ν *be defined on* Σ_0*. Further let* $\bar\alpha > 0$ *and* $\bar\beta$ *denote another choice of initial lapse and shift satisfying the assumptions (5.1.3a) and (5.1.3b). Assume* $\bar F : [-T,T] \times M \to N$ *to be the solution of the membrane equation (5.1.4a) in harmonic map gauge w.r.t. the background metric* $\hat{\bar g}$ *defined by replacing* $\mathring{g}, \alpha, \beta$ *with* $\mathring{\bar g}, \bar\alpha$ *and* $\bar\beta$ *in definition (5.1.4b) originating from the existence claim and attaining the initial values*

$$\bar F\big|_{\bar t=0} = \bar\varphi \quad \textit{and} \quad \tfrac{d}{dt}\bar F\big|_{\bar t=0} = \bar\alpha\,\nu\circ\bar\varphi + d\bar\varphi(\bar\beta). \tag{5.1.6}$$

Then, for all $p \in M$*, there exists a local diffeomorphism* Ψ *about* $(0,p) \in [-T,T] \times M$ *such that* $F \circ \Psi^{-1}$ *and* $\bar F$ *coincide.*

Remark 5.3. The solution F mentioned in the uniqueness statement is assumed to have the initial values

$$F\big|_{t=0} = \varphi \quad \text{and} \quad \tfrac{d}{dt}F\big|_{t=0} = \alpha\,\nu\circ\varphi + d\varphi(\beta). \tag{5.1.7}$$

Since the initial submanifold is locally embedded it is possible to replace the vector field along φ with the composition of the initial direction defined on the initial submanifold and φ.

The proof has the following structure. Firstly, the existence claim is proven similar to that of Theorem 4.32 Due to the combination of initial direction, initial lapse, and shift the result of Theorem 4.32 is not directly applicable. However, we make contact with the existence result of Theorem 4.12. Afterwards we will consider the uniqueness claim.

Let x be a chart on M with center $p \in M$ and let y be a chart on N with center $\varphi(p) \in N$. Let Φ be the expression $y \circ \varphi \circ x^{-1}$. Suppose α_x, β_x are the representations of α and β w.r.t. the chart x, and ν_{xy} is the representation of νw.r.t. the charts x and y. We make the following uniformity assumptions on the representations of direction, lapse and shift.

Assumptions 5.4.

- *For a positive constant* L_2 *the following inequality holds*

$$-\alpha_x^2 + \beta_x^i \mathring{g}_{ij} \beta_x^j \le -L_2, \tag{5.1.8}$$

where $\mathring{g}_{ij}$ *denotes the representation of the induced metric* $\mathring{g}$ *w.r.t. the coordinates* x*.*

- *There exist constants* $\tilde{C}_\ell^\alpha, \tilde{C}_\ell^\beta, \tilde{C}_\ell^\nu$ *such that*

$$\begin{aligned} & |D^\ell \alpha_x|_e \le \tilde{C}_\ell^\alpha, \quad |D^\ell \beta_x|_e \le \tilde{C}_\ell^\beta \\ \text{and} \quad & |D^\ell \nu_{xy}|_{e,e} \le \tilde{C}_\ell^\nu \quad \text{for } 0 \le \ell \le s+1. \end{aligned} \tag{5.1.9}$$

The next lemma will show that these assumptions are sufficient to get the conditions (3.2.3) for the initial velocity defined by

$$\chi_{xy} := \alpha_x \nu_{xy} + \beta_x^j \partial_j \Phi, \tag{5.1.10}$$

the representation of the initial velocity given in (5.1.4c) w.r.t. the charts x and y.

Lemma 5.5. *Suppose y and the representation of the metric h in these coordinates satisfy the assumptions 4.11. Let the chart x and the representation Φ satisfy part 1 and 2 of the assumptions 3.13. Assume the representations α_x, β_x and ν_{xy} to satisfy the conditions 5.4. Then χ_{xy} defined in* (5.1.10) *satisfies part 3 of the assumptions 3.13.*

Remark 5.6. If ν is replaced by $\nu \circ \varphi$ then the lemma remains valid.

Proof. The condition $h(\chi_{xy}, \chi_{xy}) \le -L_2$ follows immediately from inequality (5.1.8). The bounds for χ_{xy} desired by assumption (3.2.3) follow from the assumptions for derivatives of direction, lapse, and shift using the bounds for derivatives of Φ stated in (3.2.2). A bound for $D\Phi$ follows from a Taylor expansion by considering the bound for the second derivative. □

The preceding lemma yields that Theorem 4.12 is applicable. In the next step we will show that the assumptions 5.1 as they are independent of coordinates lead to the conditions 5.4. This step is analogous to the propositions 3.43 and 4.38.

Lemma 5.7. *Let the metric $h \in C^{s+3}$ and the time function $\tau \in C^{s+3}$ satisfy the assumptions* (4.4.1a)*. Assume the initial immersion $\varphi \in C^{s+2}$ to satisfy the assumptions* (4.4.1b) *and $\nu \circ \varphi, \alpha, \beta \in C^{s+1}$ to satisfy the assumptions 5.1.*

Let $p \in M$ and let x be a chart on M with center p. Let y be the special coordinates on N introduced in section 4.1 and assume $y \circ \varphi \circ x^{-1}$ to be the special graph representation obtained in section 4.2.

Then the representations Φ, α_x, β_x, and ν_{xy} of φ, α, β, and ν w.r.t. these charts satisfy the assumptions 5.4.

Remark 5.8. Here, the Remark 5.6 for the preceding lemma also applies. The bounds follow from the bounds for the representation of the immersion φ.

Proof. The first condition (5.1.8) follows immediately from (5.1.3a). A bound for the lapse α is given in condition (5.1.3b). Hence, we can derive the inequality $|\beta|^2_{\mathring{g}} \le -L_2 + (C_0^\alpha)^2$ from condition (5.1.3a). To obtain an estimate for the Euclidean norm in coordinates we use the comparison of the Euclidean metric in coordinates and the metric $\mathring{g}$ on M stated in (4.2.16). Similar to the argument used in the proof of proposition 4.38 (cf. Proposition 3.43) it follows from condition (5.1.2) that $|\nu|_E$ is bounded. The comparison of the metric

E with the Euclidean metric in the special coordinates stated in (4.1.10) provides a bound for $|\nu_{xy}|_e$.

The conditions for derivatives of the representation of the initial direction ν follow from a similar consideration as in the proof of Proposition 4.38. An identity similar to (4.4.2) also holds for the representations of lapse and shift involving the induced covariant derivative on M and its Christoffel symbols. Thus, the desired bounds follow from a device similar to the proof of Lemma 4.10. □

The preceding lemmata yield the existence claim of Theorem 5.2 as we will now see.

Proof of the existence claim of Theorem 5.2. We will appeal to the proof of Theorem 4.32. We need to verify whether the steps taken there can be paralleled here. From Lemma 5.5 we derive that, providing the assumptions 5.4 are satisfied, Theorem 4.12 applies. If follows from Lemma 5.7 that, providing the assumptions 5.1 are satisfied, the conditions 5.4 are met by decompositions built by special graph representations. Therefore, the result follows. □

We now head to the uniqueness claim of Theorem 5.2.

Let F and $\bar{F}$ be two solutions of the membrane equation (5.1.4a) attaining the initial values (5.1.7) and (5.1.6), respectively. The solutions F and $\bar{F}$ are assumed to be in harmonic map gauge w.r.t. the background metrics $\hat{g}$ and $\hat{\bar{g}}$ defined by the initial values as in (5.1.4b), respectively. Let $g := F^*h$ and $\bar{g} := \bar{F}^*h$.

Our strategy will be to show that there exists a local diffeomorphism Ψ such that $F \circ \Psi^{-1}$ and $\bar{F}$ satisfy the reduced membrane equation (1.3.6) in harmonic map gauge w.r.t. the same background metric and attaining the same initial values.

In the following proposition we will derive conditions for such a diffeomorphism Ψ.

Proposition 5.9. *Let $p \in M$, $U \subset M$ a neighborhood of p and $V \subset M$ a neighborhood of $\psi_0(p) \in M$. Suppose there exist constants $0 < T', \tilde{T}' \leq T$ and a diffeomorphism $\Psi : (-T', T') \times U \to (-\tilde{T}', \tilde{T}') \times V$ such that*

$$\Psi : \big((-T', T') \times U, g\big) \to \big((-\tilde{T}', \tilde{T}') \times V, \hat{\bar{g}}\big) \text{ is a harmonic map.} \tag{5.1.11}$$

Assume further that the inverse satisfies the initial conditions

$$\begin{aligned} \Psi^{-1}\big|_{\tilde{t}=0} &= (0, \psi_0^{-1}) \quad \text{and} \quad \partial_{\tilde{t}}\Psi^{-1}\big|_{\tilde{t}=0} = \hat{\lambda}\partial_t + \hat{\chi} \\ \text{with } \hat{\lambda}(p) &= \tfrac{\bar{\alpha}(p)}{\alpha(\psi_0^{-1}(p))} \quad \text{and} \quad \hat{\chi}(p) = d(\psi_0^{-1})_p(\bar{\beta}) - \hat{\lambda}(p)\beta(\psi_0^{-1}(p)). \end{aligned} \tag{5.1.12}$$

Then $F \circ \Psi^{-1}$ satisfies the reduced membrane equation (1.3.6) w.r.t. the background metric $\hat{\bar{g}}$ attaining the initial values of $\bar{F}$.

Proof. From the harmonic map equation satisfied by Ψ we derive the condition for the solution $F \circ \Psi^{-1}$ of the membrane equation to be in harmonic map gauge w.r.t. the background metric $\hat{\bar{g}}$ (cf. (1.3.4)). This can be seen by computing the contracted Christoffel symbols of the metric $(F \circ \Psi^{-1})^*h$ to satisfy condition (1.3.5) w.r.t. the background metric $\hat{\bar{g}}$.

The next step is to show that the initial values of Ψ^{-1} give us the appropriate initial values for $F \circ \Psi^{-1}$. We compute

$$\begin{aligned}\partial_{\bar{t}}(F \circ \Psi^{-1})\big|_{\bar{t}=0}(p) &= \hat{\lambda}(p)\partial_t F\big|_{t=0}(\psi_0^{-1}(p)) + d\varphi_{\psi_0^{-1}(p)}(\hat{\chi}) \\ &= \hat{\lambda}(p)\big(\alpha(\psi_0^{-1}(p))\nu \circ \varphi(\psi_0^{-1}(p)) + d\varphi_{\psi_0^{-1}(p)}(\beta)\big) + d\varphi_{\psi_0^{-1}(p)}(\hat{\chi}) \\ &= \hat{\lambda}(p)\alpha(\psi_0^{-1}(p))\nu \circ \tilde{\varphi}(p) + \hat{\lambda}(p) d\varphi_{\psi_0^{-1}(p)}(\beta) + d(\varphi \circ \psi_0^{-1})_p(\bar{\beta}) - \hat{\lambda}(p) d\varphi_{\psi_0^{-1}(p)}(\beta)\end{aligned}$$

Since the second and the last term cancel the result follows. □

As an immediate consequence we can show uniqueness.

Proof of the uniqueness claim of Theorem 5.2. As for the existence claim Theorem 4.32 is not directly applicable. The Lemmata 5.5 and 5.7 yield that the uniqueness result of Theorem 4.12 is applicable to a decomposition consisting of special graph representations.

Assuming the existence of a diffeomorphism Ψ satisfying the assumptions of Proposition 5.9 provides us with two solutions satisfying the reduced membrane equation (1.3.6) w.r.t. the same background metric and attain the same initial values. Following the proof of the uniqueness claim of Theorem 4.32 gives the desired result. □

The next section is devoted to construct a local diffeomorphism satisfying the condition (5.1.11) in Proposition 5.9.

5.2 Construction of a reparametrization

From Proposition 5.9 we derive a harmonic map equation to be satisfied. Henceforth, we obtain the diffeomorphism by constructing an immersion solving this equation.

The following designation concerning coordinates will be used throughout this section. Coordinates on $(\mathbb{R} \times M, g)$ will be denoted by x^μ with indices $\mu, \nu, \lambda, \dots$ and coordinates on $(\mathbb{R} \times M, \hat{\bar{g}})$ will be denoted by $\bar{x}^\delta$ with indices $\delta, \varepsilon, \kappa, \dots$.

Condition (5.1.11) for the desired diffeomorphism leads to the following harmonic map equation in coordinates

$$g^{\mu\nu}\partial_\mu\partial_\nu\Psi^\delta = \Gamma^\lambda\partial_\lambda\Psi^\delta - g^{\mu\nu}\partial_\mu\Psi^\varepsilon\partial_\nu\Psi^\kappa\widehat{\bar{\Gamma}}^\delta_{\varepsilon\kappa}(\Psi), \tag{5.2.1}$$

where Γ^λ denote the contracted Christoffel symbols of g and $\widehat{\bar{\Gamma}}^\delta_{\varepsilon\kappa}$ denote the Christoffel symbols of the metric $\hat{\bar{g}}$. Observe that this is a semilinear equation, since the coefficients are fixed. The solution F of the membrane equation is assumed to be in harmonic map gauge w.r.t. $\hat{g}$. Therefore, the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ on the RHS of the equation can be replaced by the Christoffel symbols $\widehat{\Gamma}$ of $\hat{g}$ (cf. condition (1.3.5)). The initial values are given by the initial values for the inverse stated in (5.1.12). Hence, the initial values for Ψ are as follows

$$\Psi\big|_{t=0} = (0, \psi_0), \ \tfrac{d}{dt}\Psi\big|_{t=0} = \hat{\alpha}\tfrac{d}{dt} + \hat{\beta} \tag{5.2.2a}$$

$$\text{with} \ \ \hat{\alpha}(p) = \tfrac{\alpha(p)}{\bar{\alpha}(\psi_0(p))} \ \ \text{and} \ \ \hat{\beta}(p) = -\hat{\alpha}(p)\bar{\beta}(\psi_0(p)) + d(\psi_0)_p(\beta). \tag{5.2.2b}$$

The following proposition states the main result of this section providing a solution of the initial value problem for Ψ.

Proposition 5.10. *Let the assumptions of the uniqueness claim of Theorem 5.2 be satisfied.*

Then there exist a constant $\bar{T} > 0$ and a C^2-immersion $\Psi : [-\bar{T}, \bar{T}] \times M \to \mathbb{R} \times M$ satisfying the equation (5.2.1) and attaining the initial values (5.2.2a).

If Ψ and $\bar{\Psi}$ are two such solutions, then they coincide for $-\min(T_0, \tilde{T}_1) \leq t \leq \min(T_0, \tilde{T}_1)$, where $T_0, \tilde{T}_1 > 0$ are constants.

Remark 5.11. The inverse function theorem yields that for $|t|$ small enough $\Psi(t,p)$ is a diffeomorphism around each point $p \in M$. It therefore has the properties described in proposition 5.9.

The constant T_0 is defined in (3.3.9) with the constant c_0 being defined in (4.3.20) and $\tilde{T}_1$ being defined similarly to (4.3.19).

The method to solve the IVP will follow the same strategy as we used in the sections 4.3 and 4.4 in order to solve the membrane equation. We begin with a local result similar to Theorem 4.12.

Let $(U_\lambda, x_\lambda, V_\lambda, y_\lambda)_{\lambda \in \Lambda}$ be a decomposition of φ (cf. Definition 3.12) and $(\bar{U}_\sigma, \bar{x}_\sigma, \bar{V}_\sigma, \bar{y}_\sigma)_{\sigma \in \Lambda'}$ be a decomposition of $\bar{\varphi}$. Let $\Phi_\lambda, \bar{\Phi}_\sigma$ be the representations of φ and $\bar{\varphi}$ w.r.t. the charts $x_\lambda,\ y_\lambda$ and $\bar{x}_\sigma,\ \bar{y}_\sigma$ as defined in (3.2.1). Let $(\bar{\psi}_0)_{\lambda\sigma}$ denote the expression $\bar{x}_\sigma \circ \psi_0 \circ x_\lambda^{-1}$. Assume $\alpha_\lambda, \beta_\lambda$ to be the representations of α and β w.r.t. the charts x_λ and define $\bar{\alpha}_\sigma, \bar{\beta}_\sigma$ in an analogous manner. Further let ν_λ be the representation of $\nu \circ \varphi$ w.r.t. the charts x_λ and y_λ and let $\bar{\nu}_\lambda$ be the representation of $\nu \circ \bar{\varphi}$. Further let $\hat{\alpha}_{\lambda\sigma}$ and $\hat{\beta}_{\lambda\sigma}$ be the expressions

$$\hat{\alpha}_{\lambda\sigma}(z) = \frac{\alpha_\lambda(z)}{\bar{\alpha}_\sigma((\bar{\psi}_0)_{\lambda\sigma}(z))} \tag{5.2.3a}$$

$$\text{and} \quad \hat{\beta}_{\lambda\sigma}(z) = -\hat{\alpha}_\lambda(z)\bar{\beta}_\sigma \circ (\bar{\psi}_0)_{\lambda\sigma}(z) + d(\bar{\psi}_0)_{\lambda\sigma}\big(\beta_\lambda(z)\big). \tag{5.2.3b}$$

The assumption on the representation of the local diffeomorphism ψ_0 is as follows.

$$\begin{gathered} \text{There are constants } \tilde{C}_1^\psi \text{ and } \tilde{C}_2^\psi \text{ such that} \\ |D(\bar{\psi}_0)_{\lambda\sigma}|_e \leq \tilde{C}_1^\psi \quad \text{and} \quad |D^2(\bar{\psi}_0)_{\lambda\sigma}|_e \leq \tilde{C}_2^\psi. \end{gathered} \tag{5.2.4}$$

The following proposition includes a local existence and uniqueness result for the harmonic map equation (5.2.1) with the initial values (5.2.2a).

Proposition 5.12. *We make the following assumptions uniformly in $\lambda \in \Lambda$. Suppose y_λ and the representation of the metric h in these coordinates satisfy the assumptions 4.11. Let the chart x_λ and the representation Φ_λ satisfy part 1 and 2 of the assumptions 3.13. Assume the representations $\alpha_\lambda, \beta_\lambda$, and ν_λ to satisfy the conditions 5.4.*
Assume $\bar{y}_\sigma, \bar{x}_\sigma, \bar{\Phi}_\sigma, \bar{\alpha}_\sigma, \bar{\beta}_\sigma, \bar{\nu}_\sigma$ to satisfy the same assumptions uniformly in $\sigma \in \Lambda'$ possibly with different constants.
Let the representation $(\bar{\psi}_0)_{\lambda\sigma}$ of the local diffeomorphism ψ_0 satisfy the assumptions (5.2.4).

Then there exist constants $\bar{T}' > 0$, $0 < \hat{\theta} < 1$ and a family (Ψ_λ) of bounded C^2-immersions $\Psi_\lambda : [-\bar{T}', \bar{T}'] \times B^e_{\hat{\theta}\rho_1/2}(0) \subset \mathbb{R} \times x_\lambda(U_\lambda) \to \mathbb{R}^{m+1}$ solving equation (5.2.1) and attaining the initial values

$$\Psi_\lambda(0, z) = (\bar{\psi}_0)_{\lambda\sigma} \text{ for } z \in B^e_{\hat{\theta}\rho_1/2}(0) \quad \text{and} \quad \partial_t \Psi_\lambda\big|_{t=0} = \begin{pmatrix} \hat{\alpha}_{\lambda\sigma} \\ \hat{\beta}_{\lambda\sigma} \end{pmatrix}. \tag{5.2.5}$$

If Ψ_λ and $\bar{\Psi}_\lambda$ are two such solutions defined on the image of the same chart with the same initial values in $B^e_r(z)$, then they coincide on the double-cone with basis $B^e_r(z)$ and slope c_0 defined by (4.3.20).

The proof of the above proposition will occupy the rest of this section. We will follow the strategy of section 4.3, especially the proof of the local result presented in Theorem 4.12. Since solutions constructed in Theorem 5.2 are C^2, a cut-off process does not lead to the conditions of the existence Theorem 2.9. We circumvent this issue by using the result of proposition 3.17. Assume $\lambda \in \Lambda$ and $\sigma \in \Lambda'$ to be fixed such that the definition of $(\bar{\psi}_0)_{\lambda\sigma}$ makes sense. Let F_λ denotes the representation of the solution F w.r.t. the charts x_λ and y_λ. Then we derive from Theorem 4.12 that F_λ coincides with the solution F_0 constructed in Proposition 3.17 on the cone with basis $B^e_{\hat{\theta}\rho_1/2}(0)$ and slope c_0 as described in the uniqueness statement of Proposition 5.12. From the proof of Proposition 3.17 it follows that the induced metric w.r.t. F_0 satisfies the assumptions of Theorem 2.9. Let $g^0_{\mu\nu}$ be the coefficients defined in (4.3.5) w.r.t. the solution F_0.

The cut-off function that comes along with the construction of F_0 will be used to define cut-off functions corresponding to the initial values $\Phi_\lambda, \alpha_\lambda, \nu_\lambda$ and β_λ and the initial values with a bar. These definitions give rise to a cut-off process for the background metrics $\hat{g}$ and $\hat{\bar{g}}$ providing us with matrices $\hat{a}$ and $\hat{\bar{a}}$ analogous to (3.2.9). The Christoffel symbols of $\hat{a}$ and $\hat{\bar{a}}$ will be denoted by $\hat{\gamma}^\lambda_{\mu\nu}$ and $\hat{\bar{\gamma}}^\varepsilon_{\delta\kappa}$, respectively.

In the sequel we will develop the asymptotic equation to be solved. For notational convenience we set $\bar{\psi}_0 := (\bar{\psi}_0)_{\lambda\sigma}$. Define a linear function $w(t,x)$ by

$$w(t,x) = w_0(x) + t w_1 \tag{5.2.6}$$

$$\text{with} \quad w_0(z) = x^\ell \partial_\ell \bar{\psi}_0(0) \text{ for } x \in \mathbb{R}^m \quad \text{and} \quad w_1 = \begin{pmatrix} \hat{\alpha}_{\lambda\sigma}(0) \\ \hat{\beta}_{\lambda\sigma}(0) \end{pmatrix}.$$

Set

$$\mathring{\psi}_0 := \hat{\zeta}(\bar{\psi}_0 - w_0) \quad \text{and} \quad \mathring{X} := \hat{\zeta}\left(\left(\begin{smallmatrix} \hat{\alpha}_{\lambda\sigma} \\ \hat{\beta}_{\lambda\sigma} \end{smallmatrix}\right) - w_1\right)$$

where $\hat{\zeta}$ is a cut-off function with the same properties as the cut-off function ζ used in the proof of Proposition 3.17 with the parameter θ replaced by $\hat{\theta}$.

We will now define the operator corresponding to the RHS of equation (5.2.1). Let $\Omega \subset \mathbb{R}^{m+1} \times \mathbb{R}^{m(m+1)} \times \mathbb{R}^{m+1}$ be a set chosen later and define

$$f^\delta_a(t, v, Y, X) := g_0^{\mu\nu}(t)\big(\hat{\gamma}^0_{\mu\nu}(t)(\partial_t w + X)^\delta + \hat{\gamma}^\ell_{\mu\nu}(t)(\partial_\ell w + Y_\ell)^\delta\big)$$

for $(t, v, Y, X) \in \mathbb{R} \times \Omega$. Define $\mathbf{f}_a^\delta(t, v, Y, X)$ as in (4.3.9c) replacing the metric $g_a^{\mu\nu}$ by the fixed metric $g_0^{\mu\nu}$ and replacing the Christoffel symbols $\widehat{\boldsymbol{\Gamma}}_{BC}^A$ by the Christoffel symbols $\hat{\bar{\gamma}}_{\varepsilon\kappa}^\delta\big(w(t) + v\big)$. These definitions immediately reveal the similarity to equation (4.3.9a). Analogously to definition (4.3.8) we obtain from the preceding definitions of f_a and $\mathbf{f}_a$ operators defined on a set $[0, \bar{T}_1] \times W \subset \mathbb{R} \times H^{s+1} \times H^s$. The operators will carry the same names and it will be clear from the context which notation is used. The constant $\bar{T}_1$ and the form of W will be chosen later depending on Ω. For later reference we state the asymptotic IVP to be solved in the sequel

$$\begin{aligned} g_0^{\mu\nu}(t)\partial_\mu\partial_\nu\psi^\delta =\; & f_a^\delta(t, \psi, D\psi, \partial_t\psi) + \mathbf{f}_a^\delta(t, \psi, D\psi, \partial_t\psi) \\ & \psi\big|_{t=0} = \mathring{\psi}_0 \quad \text{and} \quad \partial_t\psi\big|_{t=0} = \mathring{X} \end{aligned} \tag{5.2.7a}$$

$$\text{with } f_a^\delta(t, \psi, D\psi, \partial_t\psi) = g_0^{\mu\nu}(t)\hat{\gamma}_{\mu\nu}^\lambda(t)(\partial_\lambda w + \partial_\lambda\psi)^\delta \tag{5.2.7b}$$

$$\text{and } \mathbf{f}_a^\delta(t, \psi, D\psi, \partial_t\psi) = -g_0^{\mu\nu}(t)(\partial_\mu w + \partial_\mu\psi)^\varepsilon(\partial_\nu w + \partial_\nu\psi)^\kappa\hat{\bar{\gamma}}_{\varepsilon\kappa}^\delta\big(w(t) + \psi\big). \tag{5.2.7c}$$

In contrast to section 4.3 the set Ω will only be used to control the differential of the mapping $w(t) + \psi(t)$, since the coefficients of equation (5.2.7a) are fixed. We consider the matrix

$$b_{0\ell}(v, Y, X) = (w_1 + X)^\delta\hat{\bar{a}}_{\delta\varepsilon}(\partial_\ell w_0 + Y_\ell)^\varepsilon \quad \text{for } (v, Y, X) \in \Omega$$

where the other parts b_{00} and $b_{k\ell}$ are defined analogously. Recall that the matrix $\hat{\bar{a}}_{\delta\lambda}$ is independent of the time parameter. For notational convenience we set

$$\mathring{b}_{0\ell} = w_1^\delta\hat{\bar{a}}_{\delta\varepsilon}(0)\partial_\ell w_0^\varepsilon \qquad \mathring{b}_{00} \text{ and } \mathring{b}_{k\ell} \text{ being defined analogously.}$$

The argument of $\hat{\bar{a}}$ corresponds to the center of the chart $\bar{x}_\sigma$.

The following lemma establishes estimates for components of the matrix $b_{\mu\nu}$ which will be used to derive a definition of the set Ω.

Lemma 5.13. *The following inequalities hold*

$$\begin{aligned} \mathring{b}_{00} &\le -L_2, & b_{00} &\le \mathring{b}_{00} + 2|X|\big(C_{w_0}^2 + (\tilde{C}_0^\alpha\tilde{C}_0^\nu + \tilde{C}_0^\beta)^2\big)(1 + \tilde{\theta}\delta_0)(|w_1| + |X|), \\ \mathring{b}_{ij} &\ge \omega_1^{-2}\delta_{ij}, & b_{ij} &\ge \mathring{b}_{ij} - 2|Y|\big(C_{w_0}^2 + (\tilde{C}_0^\alpha\tilde{C}_0^\nu + \tilde{C}_0^\beta)^2\big)(1 + \tilde{\theta}\delta_0)(|Dw_0| + |Y|)\delta_{ij}. \end{aligned}$$

Proof. From the definitions of the initial values $\hat{\alpha}$ and $\hat{\beta}$ in (5.2.2b) it follows that $\mathring{b}_{00} = -\alpha_\lambda^2(0) + \beta_\lambda^k\mathring{g}_{k\ell}\beta_\lambda^\ell$ and $\mathring{b}_{ij} = \mathring{g}_{ij}(0)$. The inequalities on the LHS therefore follow from assumption (5.1.8) and part 1 of the assumptions 3.13.

The other estimates can be obtained from the proof of Lemma 4.16. $\square$

Define

$$\Omega = \mathbb{R}^{m+1} \times B_{\delta_1}^e(0) \times B_{\delta_2}^e(0) \subset \mathbb{R}^{m+1} \times \mathbb{R}^{m(m+1)} \times \mathbb{R}^{m+1} \tag{5.2.8}$$

with constants δ_1 and δ_2 such that if $(v, Y, X) \in \Omega$ then we have

$$b_{00}(v, Y, X) \le -L_2(1 - r_0) \quad \text{and} \quad b_{ij}(v, Y, X) \ge \omega_1^{-2}(1 - R_0)$$

for fixed constants $0 < r_0, R_0 < 1$. This can be done in a similar way as for the constants δ_1 and δ_2 in definition (4.3.10).

To ensure that the initial values $\mathring{\psi}_0$ and $\mathring{X}$ can be controlled by the parameter $\hat{\theta}$, a statement similar to Lemma 3.21 is needed. Let the representations $\bar{\alpha}_\sigma, \bar{\beta}_\sigma$ and $\bar{\nu}_\sigma$ satisfy the assumptions (5.1.9) with constants denoted by $\tilde{C}^{\bar{\alpha}}_\ell, \tilde{C}^{\bar{\beta}}_\ell$, and $\tilde{C}^{\bar{\nu}}_\ell$, respectively.

Lemma 5.14. *The following inequalities hold*

$$|D\mathring{\psi}_0| \leq \tilde{C}^\psi_2 \hat{\theta}(1+\tilde{C}_1) \qquad \textit{and} \qquad |\mathring{X}| \leq C^X_1 \rho_1 \hat{\theta}$$
$$\textit{with} \quad C^X_1 = \tilde{C}^\psi_2 \tilde{C}^\beta_0 + \tilde{C}^\psi_1 \tilde{C}^\beta_1 + L_2^{-2}\big(\tilde{C}^\alpha_1 \tilde{C}^{\bar{\alpha}}_0 + \tilde{C}^\alpha_0 \tilde{C}^{\bar{\alpha}}_1\big)(1+\tilde{C}^{\bar{\beta}}_0) + L_2^{-1/2} \tilde{C}^\alpha_0 \tilde{C}^{\bar{\beta}}_1 \tilde{C}^\psi_1.$$

Proof. The proof follows from a device similar to that used in the proof of Lemma 3.21 taking into account the conditions (5.1.9) on $\alpha_\lambda, \bar{\alpha}_\sigma, \beta_\lambda, \bar{\beta}_\sigma$, and $(\bar{\psi}_0)_{\lambda\sigma}$. □

We choose the parameter $\hat{\theta}$ small enough such that $|\mathring{X}| < \delta_2/2$, $|D\mathring{\psi}_0| < \delta_1/2$ and $|\bar{\psi}_0(z)|_e \leq \theta\rho_1/4$ for $|z| < \hat{\theta}\rho_1/2$ which can be done via Taylor expansion. This yields that the asymptotic IVP (5.2.7a) coincides with the IVP consisting of equation (5.2.1) and the initial values defined by (5.2.5), in the region $B^e_{\hat{\theta}\rho_1/2}(0) \subset \mathbb{R}^m$.

Let ρ be chosen in a way such that for $(\varphi_0, \varphi_1) \in W := B_\rho(\mathring{\psi}_0) \times B_\rho(\mathring{X})$ it follows that $|D\varphi_0| < \delta_1$ and $|\varphi_1| < \delta_2$.

From part 1 of remark 4.28 we get that $g^{\mu\nu}_0$ satisfies the conditions (2.2.5a) to (2.2.5d) of the linear existence Theorem 2.8. The construction of the solution F_0 in Proposition 4.15 further provides bounds for the Christoffel symbols of the matrix $\hat{a}_{\mu\nu}$ specified in Lemma 4.23. The Lemmata 4.17, 4.21 and estimate (4.3.17) yield that the conditions (2.3.2a) and (2.3.2e) through (2.3.2g) for the first part f_a of equation (5.2.7a) given by (5.2.7b) can by deduced by the same device as used in section 4.3.1.

It therefore remains to derive the preceding conditions for the second part $\mathbf{f}_a$ of the RHS defined by (5.2.7c). A careful examination of the proof of Lemma 4.26 shows that the estimates are derived from the Sobolev estimates for the metric $\hat{h}_{AB}$ and its Christoffel symbols. From the proof of Lemma 4.19 we deduce that only the local bounds for h_{AB} stated in Lemma 4.18 were used. Hence, taking the structure of $\mathbf{f}_a$ into account, similar local estimates for the metric $\hat{\bar{a}}$ lead to the conditions for $\mathbf{f}_a$ needed by Theorem 2.9. The local bound will be derived in the next lemma.

Lemma 5.15. *The metric $\hat{\bar{a}}$ satisfies the inequalities*

$$|(\hat{\bar{a}}_{\mu\nu})|_e \leq C_{\hat{a}_2,0}, \qquad |(D\hat{\bar{a}}_{\mu\nu})|_e \leq C_{D\hat{a}_2,0}, \qquad |(D^2\hat{\bar{a}}_{\mu\nu})|_e \leq C_{D^2\hat{a}_2,0},$$
$$|(\hat{\bar{a}}_{\mu\nu})|_{e,C^s} \leq C_{\hat{a}_2,s}, \qquad \textit{and} \qquad |(D\hat{\bar{a}}_{\mu\nu})|_{e,C^s} \leq C_{D\hat{a}_2,s}.$$

Proof. Let $\bar{\chi}$ denote the expression arising from definition (5.1.10) by replacing Φ, α_x, β_x, and ν_{xy} with $\bar{\Phi}_\sigma, \bar{\alpha}_\sigma, \bar{\beta}_\sigma$, and $\bar{\nu}_\sigma$, respectively. Suppose $\mathring{\bar{\Phi}}$ and $\mathring{\bar{\chi}}$ are defined as in (3.2.8a) and (3.2.8b) by replacing Φ and χ with $\bar{\Phi}$ and $\bar{\chi}$. The matrix $\hat{\bar{a}}_{\mu\nu}$ is defined as in (3.2.9) by replacing $\mathring{\Phi}$ and $\mathring{\chi}$ with $\mathring{\bar{\Phi}}$ and $\mathring{\bar{\chi}}$. The proof of Proposition 3.26 implies that we need bounds for the representations

$$|D\mathring{\bar{\chi}}|_\infty, \; |D^2\mathring{\bar{\chi}}|_\infty, \; |D\mathring{\bar{\chi}}|_{C^s}, \; |D^2\mathring{\bar{\Phi}}|_\infty, \; |D^3\mathring{\bar{\Phi}}|_\infty \quad \text{and} \quad |D^2\mathring{\bar{\Phi}}|_{C^s}.$$

These can be derived from the bounds for the cut-off function ζ in (4.3.2) and from the assumptions on $\bar{\Phi}_\sigma$ and $\bar{\alpha}_\sigma, \bar{\beta}_\sigma, \bar{\nu}_\sigma$ stated in (3.2.2) and (5.1.9). □

Proof of Proposition 5.12. The preceding considerations show that the IVP (5.2.7a) satisfies the assumptions of Theorem 2.9. From the asymptotic existence Theorem 2.22 we derive a solution $\Psi(t) = w(t) + \psi(t)$ of equation (5.2.1) attaining the initial values (5.2.2a) within the ball $B^e_{\hat{\theta}\rho_1/2}(0)$. From the choice of the domain W of the RHS based on the set Ω defined in (5.2.8) we obtain that the differential of $\Psi(t)$ is invertible for all (t, x) in the domain of Ψ.

In analogy to the proof of Theorem 4.12 we intend to impose a bound on the time parameter by the constant $\tilde{T}_1$ defined by $\tilde{T}_1 = K_1^{-1}\theta\rho_1/4$. This choice guarantees that $\Psi(t, z)$ solves the original equation (5.2.1) for $|z| < \hat{\theta}\rho_1/2$ and $0 \le t \le \tilde{T}_1$ if it exists up to this time. Remark 3.32 applies so that the solution can be extended to negative time parameters.

The proof of the uniqueness claim is analogous to the proof of the uniqueness result in Theorem 4.12. Since the coefficients are taken from the solution obtained by Proposition 4.15, the value of the slope c_0 of the uniqueness cone is described in the proof of the uniqueness result of Theorem 4.12. □

In Lemma 5.7 we obtained the transition from the assumptions 5.1 to the local assumptions 5.4 via the special graph representation introduced in section 4.2. The next step is to show that the assumptions (5.1.5) on the diffeomorphism ψ_0 localize in a way that the assumptions (5.2.4) are satisfied.

Lemma 5.16. *Suppose $\psi_0 \in C^{s+2}$ satisfies the assumptions (5.1.5). Let $p \in M$. Suppose x are coordinates on M with center p and $\bar{x}$ are coordinates with center $\psi_0(p)$. Let y and $\bar{y}$ be the special coordinates on N introduced in section 4.1 such that $y \circ \varphi \circ x^{-1}$ and $\bar{y} \circ \bar{\varphi} \circ \bar{x}^{-1}$ are the special graph representations of φ and $\bar{\varphi}$ obtained in section 4.2. Set $\bar{\psi}_0 = \bar{x} \circ \psi_0 \circ x^{-1}$. Then $\bar{\psi}_0$ satisfies the local assumptions (5.2.4).*

Proof. The claim follows by considering the comparison of the eigenvalues of the representations of the metrics $\mathring{g}$ and $\mathring{\bar{g}}$ in the special graph representation with the Euclidean metric stated in (4.2.16). □

Using considerations elaborated for the membrane equation in section 4.4 we are now able to give a proof of the main result of this section.

Proof of Proposition 5.10.

- <u>Existence</u>

  We can show analogously to the proof of Proposition 4.39 that solutions Ψ_λ derived in proposition 5.12 are independent of the specific decomposition. A solution can be transferred to another coordinate system since equation (5.2.1) is invariant under change of coordinates. The parameter $\tilde{T}_1$ controlling the region where the constructed solution solves the unmodified equation was set to $K_1^{-1}\theta/4$ in Proposition 5.12. The constant T_0 controlling the height of the uniqueness cone depends on the slope c_0. Since we used the coefficients of a solution derived from proposition 4.15, the definition of c_0 made in (4.3.20) can be used. Therefore, it is possible to construct a solution Ψ defined on an interval $\times M$ just as we have proceeded in the proof of Theorem 4.32 in the case of the membrane equation.

- Uniqueness
  Uniqueness follows as in the proof of the uniqueness result of Theorem 4.32, i. e. using the local uniqueness result of Proposition 5.12. □

6 Main results

In this section we will consider existence and uniqueness for the membrane equation (0.3.1) in purely geometric terms. An existence result will be obtained including a geometric notion of time of existence of a solution. The uniqueness result will show that two submanifolds solving the IVP coincide on a neighborhood of the initial submanifold.

6.1 Existence

In this section we present the main existence theorem for the Cauchy problem (0.4.1) in Theorem 6.5. The statement is given in a scale invariant way providing the scaling behaviour of the time of existence of a solution. Furthermore, it is shown that a smooth solution exists for smooth ambient manifolds and smooth initial data.

Throughout this section we use the following assumptions. Let N^{n+1} be an $(n+1)$-dimensional manifold endowed with a Lorentzian metric h. Suppose Σ_0 is an m-dimensional spacelike regularly immersed submanifold of N with $m \leq n-1$. Suppose $\varphi : M^m \to N$ is an immersion with $\operatorname{im}\varphi = \Sigma_0$. Let ν be a timelike future-directed unit vector field along φ normal to Σ_0. Recall that we use $\mathbf{D}$ to denote the Levi-Civita connection on N corresponding to h.

Let $s_0 \geq 2$ be an integer and let $R > 0$ be the scale.

Assumptions 6.1 (on the ambient space). *Let N admit a time function τ. Let ψ denote the lapse of the time foliation induced by τ as defined by (4.0.1) and let E denote the flipped Riemannian metric defined in 4.2.*

Suppose there are constants C_1, C_2, C_ℓ^N and C_ℓ^τ independent of R such that the following inequalities are satisfied

$$C_1 \leq R^{-1}\psi \leq C_2,\ \ R^{2+\ell}|\mathbf{D}^\ell\,\mathbf{Rm}|_E \leq C_\ell^N \quad \text{for } 0 \leq \ell \leq s_0+1$$
$$\text{and} \quad R^{1+\ell}|\mathbf{D}^\ell(\mathbf{D}\tau)|_E \leq C_\ell^\tau \quad \text{for } 1 \leq \ell \leq s_0+2$$

where $\mathbf{D}(\mathbf{D}\tau)$ denotes the $(1,1)$-tensor obtained by applying the covariant derivative to the gradient of the time function τ.

Assumptions 6.2 (on the initial submanifold). *Suppose there exist constants ω_1, C_ℓ^φ independent of R such that*

$$\inf\{-h(\gamma,\widehat{T}) : \gamma \text{ timelike future-directed unit normal to } \Sigma_0\} \leq \omega_1,$$
$$R^{\ell+1}|\widehat{\nabla}^\ell \mathring{I\!I}|_{\mathring{g},E} \leq C_\ell^\varphi \quad \text{for } 0 \leq \ell \leq s_0$$

where $\widehat{T}$ denotes the future-directed timelike unit normal to the time foliation on N and $\mathring{II}$ denotes the second fundamental form of Σ_0.

Assumptions 6.3 (on the initial direction)**.** *Suppose there exist constants L_3, C^ν_ℓ independent of R such that*

$$-h(\nu, \widehat{T}) \le L_3 \quad \text{and} \quad R^\ell \big|\widehat{\nabla}^\ell \nu\big|_{\mathring{g},E} \le C^\nu_\ell \qquad \text{for } 1 \le \ell \le s_0 + 1$$

The next definition gives a notion of "time of existence" which respects the geometric behaviour of a solution in contrast to the time parameter obtained in the existence theorems 3.39, 4.32 and 5.2.

Definition 6.4 (Time of existence)**.** Let Σ be a solution of the IVP (0.4.1). The *time of existence* τ_Σ of Σ is given by

$$\tau_\Sigma := \inf_{p \in \Sigma_0} \sup\{\text{length of all timelike future-directed} \\ \text{curves in } \Sigma \text{ emanating from } p\}.$$

Our aim is an existence result for the IVP (0.4.1) which includes a lower bound on the time of existence according to the preceding definition. The following assumptions on ambient manifold, initial submanifold, and initial direction will provide us with such a lower bound.

Theorem 6.5. *Let $s > \frac{m}{2} + 1$ be an integer and let $\rho > 0$ be a constant. Assume that for each point $q \in \Sigma_0$ there exists a neighborhood $V \subset N$ of p such that N admits a time function τ in V. Let E denote the flipped metric on V defined in 4.2. Let $B^E_{R\rho}(q) \subset V$ and suppose $h, \tau \in C^{s+3}$ satisfy the assumptions 6.1 in V with $s_0 = s$ and constants independent of q. Let Σ_0 be of class C^{s+2} and let the immersion φ with image Σ_0 satisfy the assumptions 6.2 with $s_0 = s$ and constants independent of q. Let the initial direction $\nu \in C^{s+1}$ satisfy the assumptions 6.3 with $s_0 = s$ and constants independent of q.*

Then there exists an open $(m+1)$-dimensional regularly immersed Lorentzian submanifold Σ of class C^2 satisfying the IVP

$$H(\Sigma) \equiv 0, \;\; \Sigma_0 \subset \Sigma, \;\; \text{and } \nu \text{ is tangential to } \Sigma.$$

Furthermore, there exists a constant $\delta > 0$ such that

$$\tau_\Sigma \ge R\,\delta$$

Proof. Then φ satisfies the assumptions (4.4.1b) and $\nu \circ \varphi$ satisfies the assumptions (5.1.2). From the assumptions on the time function of the ambient manifold we get that the special coordinates introduced in section 4.1 are defined in a ball depending the radius ρ.

Let F denote the solution of the IVP (1.3.2) obtained from Theorem 5.2 with initial lapse equal to 1 and initial shift equal to 0. A lower bound for the time of existence of the solution $\Sigma := \operatorname{im} F$ is obtained by the uniformity of the assumptions by taking remark 4.41 into account. □

Remark 6.6. The proof shows that the theorem applies to the situation where the assumptions are valid only locally with constants depending on the specific point.

In the sequel we will specialize the result of Theorem 6.5 to various types of initial submanifolds which will be subject to the next definitions.

Definition 6.7. Let Σ_0 be a regularly immersed submanifold of dimension m being the image of an immersion $\varphi : M^m \to N$. Σ_0 is called a *locally embedded* submanifold, if for every point $q \in \Sigma_0$ there exist open sets $q \in V \subset N$ and $U \subset M$ such that

$$\varphi : U \to \varphi(U) \text{ is a diffeomorphism and } \quad \varphi^{-1}(V \cap \Sigma_0) = U. \tag{6.1.1}$$

Definition 6.8. Let Σ_0 be a regularly immersed submanifold of dimension m being the image of an immersion $\varphi : M^m \to N$. Σ_0 is called *regularly immersed with locally finite intersections*, if for every point $q \in \Sigma_0$ there exist a neighborhood $V \subset N$ of q and finitely many open pairwise disjoint sets $U_\ell \subset M$ such that

$$\varphi : U_\ell \to \varphi(U_\ell) \text{ is a diffeomorphism for every } \ell \text{ and } \quad \varphi^{-1}(V \cap \Sigma_0) = \bigcup U_\ell. \tag{6.1.2}$$

The following corollaries will show that a solution to the IVP (0.4.1) for the membrane equation can be constructed in such a way that these properties of initial submanifolds are preserved.

Corollary 6.9. *Let $s > \frac{m}{2} + 1$ be an integer. Let Σ_0 be locally embedded and for a point $q \in \Sigma_0$ let $U \subset M$ and $V \subset N$ denote sets satisfying the conditions (6.1.1) of definition 6.7. Suppose that for every point $q \in \Sigma_0$ the immersion $\varphi \in C^{s+2}$ satisfies the assumptions 6.2 in U, and N with metric $h \in C^{s+3}$ and time function $\tau \in C^{s+3}$ defined in V satisfies the assumptions 6.1 in V with $s_0 = s$. Assume further the initial direction $\nu \in C^{s+1}$ to satisfy the assumptions 6.3 in V with $s_0 = s$.*

Then there exists a locally embedded timelike $(m+1)$-dimensional submanifold Σ of class C^2 solving the IVP

$$H(\Sigma) \equiv 0, \ \Sigma_0 \subset \Sigma, \ \textit{and } \nu \textit{ is tangential to } \Sigma.$$

Proof. We will appeal to the proof of Theorem 4.32. From the assumptions on N we obtain existence of the special coordinates constructed in section 4.1, at least locally in V. Therefore, the special graph representation of φ introduced in section 4.2 can be constructed in a neighborhood of $\varphi^{-1}(q) = p \in U$. This is possible by the assumptions 6.2 on Σ_0.

Proposition 4.38 yields that the local existence Theorem 4.12 for the reduced membrane equation can be applied. From Proposition 4.39 we get that the family of solutions is independent of the chosen coordinates. It follows that for every point $q \in \Sigma_0$ there exist neighborhoods $q \in V \subset N$ and $\varphi^{-1}(q) \in U \subset M$ and a solution F_q of the membrane equation defined on a neighborhood W of $\{0\} \times U$ in $\mathbb{R} \times M$ with values in V. By considering proposition 3.36 we may shrink the domain of the solution F_q such that F_q is a diffeomorphism onto its image.

Consider the family $\mathfrak{U} = \left(\tilde{U}(\varphi^{-1}(q))\right)_{q \in \Sigma_0}$, where $\tilde{U}(\varphi^{-1}(q))$ is the neighborhood of $\varphi^{-1}(q)$ on which the local solution associated to $\varphi^{-1}(q)$ is defined. Choose a locally finite covering subordinate to $\mathfrak{U}$. Use this covering to define a mapping $F : W \subset \mathbb{R} \times M \to N$ by (3.3.8). This construction is well-defined by virtue of proposition 4.39.

By construction it follows that $\Sigma := \operatorname{im} F$ is a locally embedded submanifold of class C^2. □

To obtain a solution for regularly immersed initial submanifolds with finite intersection it is necessary to change the way in which the local solutions are pieced together.

Corollary 6.10. *Let $s > \frac{m}{2} + 1$ be an integer. Let Σ_0 be regularly immersed with locally finite intersections and for a point $q \in \Sigma_0$ let $U_\ell \subset M$ and $V \subset N$ denote sets satisfying the conditions (6.1.2) of definition 6.8. Suppose that for every point $q \in \Sigma_0$ the immersion $\varphi \in C^{s+2}$ satisfies the assumptions 6.2 in U_ℓ for every ℓ, and N with metric $h \in C^{s+3}$ and time function $\tau \in C^{s+3}$ defined in V satisfies the assumptions 6.1 in V with $s_0 = s$. Assume the initial direction $\nu \in C^{s+1}$ to satisfy the assumptions 6.3 in each U_ℓ with $s_0 = s$.*

Then there exists a timelike $(m+1)$-dimensional regularly immersed submanifold Σ with locally finite intersections of class C^2 solving the IVP

$$H(\Sigma) \equiv 0, \;\; \Sigma_0 \subset \Sigma, \;\; \textit{and } \nu \textit{ is tangential to } \Sigma.$$

Proof. We mimic the proof of Corollary 6.9. For each $q \in \Sigma_0$ we pick the finitely many $U_{q,\ell}$ and solve the membrane equation in $U_{q,\ell}$ with values in V. We shrink the domain of the solutions $F_{q,\ell}$ to obtain embeddings. By shrinking the set $V \subset N$ to a subset $\tilde{V}_q \subset N$ we achieve that

$$\left(F_{q,\ell}\right)^{-1}\left(\tilde{V}_q \cap \operatorname{im} F_{q,\ell}\right) = W_{q,\ell} \subset \operatorname{domain}(F_{q,\ell}) \subset \mathbb{R} \times U_{q,\ell} \quad \text{for all } \ell.$$

Consider the family $\mathfrak{U} = \left(\tilde{V}_q \cap \Sigma_0\right)_{q \in \Sigma_0}$. Choose a locally finite covering $\left(\tilde{V}_{q_\lambda} \cap \Sigma_0\right)_{\lambda \in \Lambda}$ of Σ_0 subordinate to $\mathfrak{U}$. Let $\tilde{U}_{q,\ell}$ denote the part of $W_{q,\ell}$ which belongs to $\{0\} \times M$. Consider the family $\left(\tilde{U}_{q,\ell}\right)_{q \in \Sigma_0}$. Then the family $(\tilde{U}_{q_\lambda,\ell})_\lambda$ is a locally finite covering of M subordinate to $\left(U_{q,\ell}\right)_{q \in \Sigma_0}$ due to the finiteness of the sets $U_{q,\ell}$ for fixed q.

We use this covering to define a mapping $F : W \subset \mathbb{R} \times M \to N$ by (3.3.8); proposition 4.39 shows that it is well-defined.

By construction it follows that $\Sigma := \operatorname{im} F$ is regularly immersed with locally finite intersections and of class C^2. □

We show that smooth data lead to a smooth solution of the IVP (0.4.1) for the membrane equation again respecting the type of the initial submanifold.

Corollary 6.11. *Assume (N, h) to be smooth and suppose Σ_0 is smooth*

1. *regularly immersed*

2. *locally embedded*

3. *regularly immersed with locally finite intersections.*

Suppose N admits a smooth time function τ in a neighborhood of the initial submanifold Σ_0. Assume the initial direction ν to be smooth.

Then there exists an open smooth $(m+1)$-dimensional timelike

1. *regularly immersed submanifold Σ*
2. *locally embedded submanifold Σ*
3. *regularly immersed submanifold Σ with locally finite intersections, respectively,*

satisfying the IVP

$$H(\Sigma) \equiv 0, \ \Sigma_0 \subset \Sigma, \ \textit{and } \nu \textit{ is tangential to } \Sigma.$$

Proof. For each integer ℓ_0 it follows from the smoothness of h, τ, the immersion φ, and the initial direction that the assumptions 6.1 to 6.3 are satisfied for $s_0 = s > \frac{m}{2} + 1 + \ell_0$ in a neighborhood about each point $q \in \Sigma_0$. Depending on Σ_0 to be regularly immersed, locally embedded, or regularly immersed with locally finite intersections we apply theorem 6.5 and Remark 6.6 or the corollaries 6.9 and 6.10, respectively, and obtain a solution Σ of class $C^{2+\ell_0}$ by taking remark 4.33 into account. By considering the constructions made in the proof of corollary 6.10 and Theorem 6.5 the other cases follow. □

6.2 Uniqueness

In this section we will consider the uniqueness claim (0.4.2) of the main problem. In the preceding section, we showed that the construction of a solution accomplished in Theorem 5.2 are independent of the choice of an immersion of the initial submanifold, as well as of the initial lapse and shift. It therefore remains to construct an immersion of a solution to the membrane equation which is in harmonic map gauge w.r.t. the background metric defined by its initial values as in (1.3.7).

Throughout this section we use the following assumptions. Let N^{n+1} be an $(n+1)$-dimensional manifold endowed with a Lorentzian metric h. Suppose Σ_0 is a m-dimensional spacelike locally embedded submanifold of N. Suppose $\varphi : M^m \to N$ is an immersion with $\operatorname{im}\varphi = \Sigma_0$ satisfying the conditions (6.1.1) of Definition 6.7. Let ν be a timelike future-directed unit vector field on Σ_0 normal to Σ_0.

Theorem 6.12. *Assume (N, h) to be smooth and suppose Σ_0 is smooth. Let N admit a smooth time function τ in a neighborhood of the initial submanifold Σ_0. Suppose the initial direction ν is smooth.*

Let Σ_1 and Σ_2 be two open smooth $(m+1)$-dimensional locally embedded Lorentzian submanifolds of N solving the IVP

$$H(\Sigma) \equiv 0, \ \Sigma_0 \subset \Sigma, \ \textit{and } \nu \textit{ is tangential to } \Sigma.$$

Then there exists a neighborhood $\Sigma_0 \subset V \subset N$ of Σ_0 such that

$$V \cap \Sigma_1 = V \cap \Sigma_2.$$

Our strategy to prove this theorem will be to compare an arbitrary solution with the solution constructed in the previous section. To apply the uniqueness result of section 4.4 we need to construct an immersion satisfying the IVP (1.3.2).

Proposition 6.13. *Let (N, h), Σ_0 and ν satisfy the assumptions of Theorem 6.12. Let Σ be a smooth locally embedded solution to the IVP (0.4.1).*

Then there exists an immersion $F : W \subset \mathbb{R} \times M \to N$ with $\operatorname{im} F \subset \Sigma$ is a locally embedded submanifold. Furthermore, F has the properties that $\frac{d}{dt}F$ is timelike and that $F(t) : M \to N$ has a spacelike image. The initial values of F are given by $F\big|_{t=0} = \varphi$ and $\frac{d}{dt}F\big|_{t=0} = \nu \circ \varphi$.

Proof. Let $p \in M$ and let $\gamma_{\varphi(p)}(t)$ be a geodesic in Σ attaining the initial values $\gamma_{\varphi(p)}(0) = \varphi(p)$ and $\dot\gamma_{\varphi(p)}(0) = \nu \circ \varphi(p)$. Set $F(t, p) = \gamma_{\varphi(p)}(t)$, then F is an immersion, since φ is assumed to be an immersion and ν is assumed to be unit timelike. The claim about the initial values of F follows from the properties of the geodesic.

This construction is similar to that of Gaussian coordinates (see e.g. [Wal84]). In an analogous way as it is shown that those are coordinates it follows that $\frac{d}{dt}F$ is timelike and that $F(t) : M \to N$ has a spacelike image. From the same argument we derive that the geodesics do not cross in a neighborhood of any point in Σ_0 as long as φ is an embedding and Σ is an embedded submanifold around that point. □

The following proposition makes contact with the reduction in section 1.3. If $\hat g$ denotes the special background metric on $\mathbb{R} \times M$ defined in (1.3.7), we will construct a reparametrization of F such that it satisfies the membrane equation in harmonic map gauge w.r.t. $\hat g$.

Proposition 6.14. *Let (N, h), Σ_0 and ν satisfy the assumptions of Theorem 6.12. Let Σ be a smooth locally embedded solution to the IVP (0.4.1). Suppose $F : W \subset \mathbb{R} \times M \to N$ of Σ is the immersion with locally embedded image constructed in Proposition 6.13.*

Then, for all $p \in M$, there exists a local diffeomorphism Ψ defined on a neighborhood of $(0, p) \in \mathbb{R} \times M$ such that $F \circ \Psi^{-1}$ is a solution of the membrane equation in harmonic map gauge w.r.t. the background metric $\hat g$ defined by (1.3.7) using the initial values of F. Further, the initial values of $F \circ \Psi^{-1}$ coincide with the initial values of F.

Proof. The condition (1.3.5) for a solution F to be in harmonic map gauge computed for $F \circ \Psi^{-1}$ leads to a harmonic map equation analogous to (5.2.1) with $\hat{\hat g}$ replaced by $\hat g$. We use the coordinates x on M and y on N belonging to the special graph representation of section 4.2. We have to show that the metric $g = F^*h$ has the property that the existence Theorem 2.9 is applicable. To this end we will show that it satisfies estimates analog to (4.3.1a) and (4.3.1b) in the coordinates x.

Since the geodesic used to define the immersion F in the proof of the previous proposition remains orthogonal to the slices of constant parameter t the metric has the following form

$$g_{\mu\nu}(t, z) = \begin{pmatrix} -1 & 0 \\ 0 & g_{ij}(t, z) \end{pmatrix}.$$

The components $g_{ij}(0, z)$ are the expression of the metric induced on M by φ w.r.t. the coordinates x which belong to the special graph representation. Therefore, from estimate

(4.2.4) and lemma 4.8 we infer control over the metric at $t = 0$. The derivatives of the matrix $\big(g_{\mu\nu}(t,z)\big)$ can be bounded up to order $s+1$ if the domain of the coordinates x and the parameter t are bounded appropriately. From this fact the remaining estimates follow.

A cut-off process analogous to (4.3.3) shows that the metric components of g and the corresponding Christoffel symbols can be estimated to meet the conditions of Theorem 2.9. Existence of a solution Ψ to the harmonic map equation (5.2.1) follows from the consideration made in section 5.2, where we used the initial values

$$\Psi\big|_{t=0}(z) = (0,z) \quad \text{and} \quad \partial_t\Psi\big|_{t=0}(z) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right). \qquad \square$$

We are now in the position to give a proof of the main uniqueness result.

Proof of Theorem 6.12. Let F_0 be the solution to the IVP (1.3.2) with initial values $F_0\big|_{t=0} = \varphi$ and $\frac{d}{dt}F_0\big|_{t=0} = \nu\circ\varphi$ constructed in Corollary 6.9. Let $\widehat{\Sigma} := \operatorname{im} F_0$ denote the locally embedded image of the solution F_0. Our strategy will be to compare a smooth solution Σ to the IVP (0.4.1) with the solution $\widehat{\Sigma}$.

From Proposition 6.13 we obtain that Σ admits an immersion $F : W \subset \mathbb{R}\times M \to N$ with locally embedded image and initial values $F\big|_{t=0} = \varphi$ and $\frac{d}{dt}F\big|_{t=0} = \nu\circ\varphi$ in a neighborhood of the initial submanifold Σ_0.

From Proposition 6.14 we obtain a local diffeomorphism Ψ of $\mathbb{R}\times M$ defined in a neighborhood of $\{0\}\times M$ such that $F\circ\Psi^{-1}$ is in harmonic map gauge w.r.t. the background metric defined by the initial values of F. We shrink the domain of this mapping to ensure that F is a diffeomorphism, defined on the image of Ψ, onto its image.

By applying the uniqueness result of Theorem 4.32 and taking Remark 4.34 into account we obtain that locally $F\circ\Psi^{-1}$ and F_0 coincide. Since F is an embedding of a portion of Σ, also $F\circ\Psi^{-1}$ is a local embedding of Σ. Hence, the desired result follows. $\square$

Remark 6.15. The existence and uniqueness results for the membrane equation 0.3.1 are kept in purely geometric terms. This indicates that a notion of maximal developments as introduced by Y. Choquet-Bruhat (cf. [CBG69]) in the context of the Einstein equations should also be possible for the membrane equation.

Appendix A Proofs of the statements in section 2.3

In this section we state the proofs of the lemmas and propositions used to the existence Theorem 2.9 for the hyperbolic IVP (2.0.1).

Proof of Lemma 2.10. We know from the assumptions on the coefficients that

$$C \geq 4c_E^{1/2}\tau_s^{1/2}(\lambda + (m+1)\mu).$$

We have to distinguish between two cases:

Case 1. $4c_E^{1/2}\tau_s^{1/2}(\lambda + (m+1)\mu) \geq 1$: Choose $r > 0$ as the largest radius satisfying $(v_0, v_1) \in B_r(u_0) \times B_r(u_1)$, then $(v_0, v_1) \in W$. Set $\delta = r/2$. C is bounded by the assumption $\|(g^{\mu\nu})\|_{e,s,\mathrm{ul}} \leq K$ on W. Therefore we can find a $\rho > 0$ such that $\rho C^{1/2} \leq \delta/3$. In this case we get directly that $\rho \leq \delta/3$. Since $H^{s+2} \times H^{s+1}$ is dense in $H^{s+1} \times H^s$ we find u_{00} such that $E_{s+1}(\mathring{u} - u_{00}) \leq \rho$. The first condition comes from

$$E_{s+1}(v - \mathring{u}) \leq E_{s+1}(v - u_{00}) + E_{s+1}(\mathring{u} - u_{00}) \leq \delta + \rho \leq 4\delta/3 < r$$

Case 2. $4c_E^{1/2}\tau_s^{1/2}(\lambda + (m+1)\mu) \leq 1$: Choose δ as above. We can find a ρ such that $\rho C^{1/2} \leq 4c_E^{1/2}\tau_s^{1/2}(\lambda + (m+1)\mu)\delta/3 \leq \delta/3$. $C \geq 4c_E^{1/2}\tau_s^{1/2}(\lambda + (m+1)\mu)$ gives us now that $\rho \leq \delta/3$. The rest is done as in the above case.

□

Proof of Lemma 2.11. Let (u_ℓ) be a Cauchy-sequence in $(Z_{\delta,L'}, d)$. Then (u_ℓ) , $(\partial_t u_\ell)$ are Cauchy-sequences in $C([0,T], H^1)$ and $C([0,T], L^2)$ resp.

We conclude $u_\ell \to u$ in $C([0,T], H^1)$ and $\partial_t u_\ell \to w$ in $C([0,T], L^2)$. By the fundamental theorem of calculus we get

$$u_\ell(t+\tau) = u_\ell(\tau) + \int_\tau^{t+\tau} \partial_t u_\ell(s)\, ds \qquad \text{in } L^2$$

Passing to the limit yields

$$u(t+\tau) = u(\tau) + \int_\tau^{t+\tau} w(s)\, ds \qquad \text{in } L^2$$

Therefore $\partial_t u = w$.

It follows from $E_{s+1}(u_\ell - u_{00}) \leq \delta$ that (u_ℓ) is a bounded sequence in the Hilbert-space H^{s+1} and $(\partial_t u_\ell)$ analogously in H^s. We have therefore weak convergence of subsequences. We consider the sequence (u_ℓ), the other is treated analogously. It holds that $u_{\ell_n} \rightharpoonup v$ in H^{s+1} pointwise and $\partial_t u_{\ell_n} \rightharpoonup h$ in H^s. Again, the fundamental theorem of calculus gives us after passing to the weak limit in H^s that $\partial_t v = h$. Since we have a limit in L^2-norm of u_{ℓ_n} and $\partial_t u_{\ell_n}$ we conclude that $u = v$.

Consider the normed space $H^{s+1} \times H^s$ with norm $\|\,.\,\| = (\|\,.\,\|_{s+1}^2 + \|\,.\,\|_s^2)^{1/2}$. This is a Hilbert-space with equivalent norm $\|\,.\,\|_{s+1} + \|\,.\,\|_s$. We get from Riesz' theorem that weak convergence is equivalent to weak convergence by components, so that $(u_{\ell_n}, \partial_t u_{\ell_n}) \rightharpoonup (u, \partial_t u)$ here. It follows

$$\begin{aligned} \delta \geq \liminf E_{s+1}(u_{\ell_n} - u_{00}) &\geq \tfrac{1}{\sqrt{2}} \liminf \|(u_{\ell_n} - y_0, \partial_t u_{\ell_n}) - y_1\| \\ &\geq \tfrac{1}{\sqrt{2}} \|(u - y_0, \partial_t u - y_1)\| \geq E_{s+1}(u - u_{00}) \text{ pointwise in } t. \end{aligned}$$

It was shown that $\partial_t u_\ell \rightharpoonup \partial_t u$ in H^s. By virtue of Rellich's theorem we get $\partial_t u_\ell \to \partial_t u$ in H^{s-1} and therefore

$$\begin{aligned}\|\partial_t u(t) - \partial_t u(t')\|_{s-1} \le \|\partial_t u(t) - \partial_t u_\ell(t)\|_{s-1} + \|\partial_t u_\ell(t) - \partial_t u_\ell(t')\|_{s-1} \\ + \|\partial_t u_\ell(t') - \partial_t u(t')\|_{s-1} \le 2\varepsilon + L'|t-t'|\end{aligned}$$

if ℓ is chosen large enough. □

Proof of Proposition 2.13. To derive estimate (2.3.8) we introduce the energy

$$\begin{aligned}E(u-u_{00}) = \mu\|u-y_0\|_{L^2}^2 - \langle g^{00}(v,Dv)(\partial_t u - y_1), \partial_t u - y_1\rangle \\ + \langle g^{ij}(v,Dv)\partial_i(u-y_0), \partial_j(u-y_0)\rangle \quad \text{(A.0.1)}\end{aligned}$$

Here $\langle\, ,\rangle$ denotes the L^2 scalar product. This energy is equivalent to $E_1(u-u_{00})$, namely

$$E_1^2(u-u_{00}) \le c_E E(u-u_{00}) \text{ and}$$
$$E(u-u_{00}) \le (\mu + \|g^{00}\|_\infty + \|(g^{ij})\|_{e,\infty})E_1^2(u-u_{00}) \le \hat{C}, E_1^2(u-u_{00})$$

where c_E is defined prior to Lemma 2.10 and $\hat{C}$ is defined in Lemma 2.10.

We want to estimate the change of E in time. E is not differentiable because of the coefficients, so we have to use the $\limsup$. By considering the modified equation

$$g^{00}\partial_t^2 u + 2g^{0j}\partial_j(\partial_t u - y_1) + g^{ij}\partial_i\partial_j(u-y_0) = f - 2g^{0j}\partial_j y_1 - g^{ij}\partial_i\partial_j y_0 \qquad \text{(A.0.2)}$$

and omitting the argument $u-u_{00}$ of E we get

$$\begin{aligned}&\limsup_\tau \tfrac{1}{\tau}\big(E(t+\tau) - E(t)\big) = \\ &\qquad 2\langle -f + 2g^{0j}\partial_j(\partial_t u - y_1) + g^{ij}\partial_i\partial_j(u-y_0) + 2g^{0j}\partial_j y_1 + g^{ij}\partial_i\partial_j y_0, \partial_t u - y_1\rangle \\ &\quad + 2\langle \partial_i(\partial_t u - y_1), g^{ij}\partial_j(u-y_0)\rangle + 2\langle \partial_i y_1, g^{ij}\partial_j(u-y_0)\rangle \\ &\quad + 2\mu\langle u-y_0, \partial_t u - y_1\rangle + 2\mu\langle u-y_0, y_1\rangle \\ &\quad - \langle \partial_t u - y_1, \limsup \tfrac{1}{\tau}\big(g^{00}(t+\tau) - g^{00}(t)\big)\partial_t u - y_1\rangle \\ &\quad + \langle \partial_i(u-y_0), \limsup \tfrac{1}{\tau}\big(g^{ij}(t+\tau) - g^{ij}(t)\big)\partial_j(u-y_0)\rangle\end{aligned}$$

Terms with second-order derivatives need a closer examination. It holds that

$$\langle \partial_i(\partial_t u - y_1), g^{ij}\partial_j(u-y_0)\rangle = -\langle \partial_t u - y_1, g^{ij}\partial_i\partial_j(u-y_0)\rangle - \langle \partial_t u - y_1, \partial_i g^{ij}\partial_j(u-y_0)\rangle$$

and

$$\langle g^{0j}\partial_j(\partial_t u - y_1), \partial_t u - y_1\rangle = -\langle \partial_t u - y_1, g^{0j}\partial_j(\partial_t u - y_1)\rangle - \langle \partial_j g^{0j}(\partial_t u - y_1), \partial_t u - y_1\rangle.$$

Therefore, the terms consisting $g^{0j}\partial_j$ and $g^{ij}\partial_i\partial_j$ cancel. The coefficients are supposed to be Lipschitz-continuous w.r.t. $H^{s-1}_{\mathrm{ul}} \hookrightarrow C_b$. Hence

$$\big\|\big(g^{\mu\nu}(t,v(t))\big) - \big(g^{\mu\nu}(t',v(t'))\big)\big\|_{e,s-1,\mathrm{ul}} \le \nu|t-t'| + \theta E_s(v(t) - v(t'))$$

which can be estimated by means of inequality (2.3.7). This estimate and the Sobolev embedding theorem applied to $\|(g^{\mu\nu})\|_\infty$ and $\|(Dg^{\mu\nu})\|_\infty$ yield

$$\begin{aligned}\limsup_{\tau} \tfrac{1}{\tau}\big(E(t+\tau)-E(t)\big) \quad &\le \quad c\big(\|f\|_{L^2}+(\mu+K)E_1(u_{00})\big)E_1(u-u_{00})\\ &\qquad + c\big(\mu+K+\nu+\theta(\delta+E_{s+1}(u_{00})+L')\big)E_1^2(u-u_{00}).\end{aligned}$$

By using the equivalence of E and E_1 it follows that

$$E^{1/2}(t) \le e^{\hat{C}_2 t}\big(E^{1/2}(0)+\hat{C}_1 t\big)$$

with $\qquad \hat{C}_1 = c\, c_E^{1/2}\big(\|f\|_{L^\infty L^2}+(\mu+K)E_1(u_{00})\big)$

and $\qquad \hat{C}_2 = c\, c_E\big(\mu+K+\nu+\theta(\delta+E_{s+1}(u_{00})+L')\big).$

For the energy E_1 we have the following estimate.

$$E_1\big(u(t)-u_{00}\big) \le c_E^{1/2}E^{1/2}(t) \le c_E^{1/2}e^{\hat{C}_2 t}\big(\tilde{C}E_1(\mathring{u}-u_{00})+\hat{C}_1 t\big). \tag{A.0.3}$$

To obtain estimates for higher derivatives of the solution u we have to take the differentiability of u into account. Define a new energy $\tilde{E}$ by

$$\tilde{E} = \sum_{|\beta|\le s} E^\beta = \sum_{|\beta|\le s} E(\partial^\beta u). \tag{A.0.4}$$

From the equivalence of the energies E and E_1 it follows that

$$E_{s+1} \le \sum_{|\beta|\le s} E_1(\partial^\beta u) \le 2E_{s+1}$$

and therefore

$$E_{s+1}^2 \le \tau_s \sum_{|\beta|\le s} E_1^2(\partial^\beta u) \le c_E\tau_s\tilde{E} \quad \text{and} \quad \tilde{E} \le \sum_{|\beta|\le s} \tilde{C}E_1^2(\partial^\beta u) \le 4\tilde{C}E_{s+1}^2, \tag{A.0.5}$$

where τ_s is defined prior to Lemma 2.10.

We want to estimate $\tilde{E}$ but we cannot handle the occurring term $\partial^\beta\partial_t^2 u$, so we have to use a mollified version of the energy. Suppose J_ε is a standard Friedrich's mollifier for $\mathbb{R}^m$. Define

$$\tilde{E}_\varepsilon = \sum_{|\beta|\le s} E_\varepsilon^\beta = \sum_{|\beta|\le s} E(\partial^\beta J_\varepsilon u).$$

This mollified energy satisfies $\tilde{E}_\beta \to \tilde{E}$ since $J_\varepsilon h \to h$ in H^ℓ if $h \in H^\ell$. It is not admitted to differentiate the equation and then apply the mollifier because the equation holds only in H^{s-1}. We have to find an equation for the term $g^{00}\partial^\beta J_\varepsilon \partial_t^2 u$ occurring in the $\limsup$ of the energy $\tilde{E}_\varepsilon$ which holds within L^2. By virtue of Lemma 2.1.3 it follows that $\frac{1}{g^{00}} \in H^s$, so we can use the equation

$$g^{00}\partial^\beta J_\varepsilon\partial_t^2 u = g^{00}\partial^\beta J_\varepsilon\big(-\tfrac{2}{g^{00}}g^{0j}\partial_j\partial_t u - \tfrac{g^{ij}}{g^{00}}\partial_i\partial_j u + \tfrac{1}{g^{00}}f\big) \tag{A.0.6}$$

It gives us for example

$$g^{00}\partial^\beta J_\varepsilon \tfrac{g^{ij}}{g^{00}}\partial_i\partial_j u = g^{ij}\partial^\beta J_\varepsilon \partial_i\partial_j u + g^{00}\big[\partial^\beta, \tfrac{g^{ij}}{g^{00}}\big] J_\varepsilon \partial_i\partial_j u + g^{00}\partial^\beta\big[J_\varepsilon, \tfrac{g^{ij}}{g^{00}}\big]\partial_i\partial_j u. \quad \text{(A.0.7)}$$

The commutators can be estimated by inequality (2.1.2) and we arrive at

$$\begin{aligned}\big\|g^{00}\big[\partial^\beta, \tfrac{g^{ij}}{g^{00}}\big] J_\varepsilon \partial_i\partial_j u\big\|_{L^2} &\le \|g^{00}\|_\infty \big\|[\partial^\beta, \tfrac{g^{ij}}{g^{00}}] J_\epsilon \partial_i\partial_j u\big\|_{L^2} \\ &\le c\|g^{00}\|_\infty \|(g^{ij})\|_{e,s,\mathrm{ul}} \big\|\tfrac{1}{g^{00}}\big\|_{s,\mathrm{ul}} \|u\|_{s+1}\end{aligned}$$

and by the properties of the mollifier it follows that

$$\big\|g^{00}\partial^\beta\big[J_\varepsilon, \tfrac{g^{ij}}{g^{00}}\big]\partial_i\partial_j u\big\|_{L^2} \le \|g^{00}\|_\infty \big\|\big[J_\varepsilon, \tfrac{g^{ij}}{g^{00}}\big]\partial_i\partial_j u\big\|_s \le c\|g^{00}\|_\infty \big\|\tfrac{(g^{ij})}{g^{00}}\big\|_{e,C^1}\|u\|_{s+1}. \quad \text{(A.0.8)}$$

Our goal is to estimate $E_{s+1}(u-u_{00})$, so we have to modify the RHS of equation (A.0.7) in the same way as equation (2.0.1) was modified to obtain equation (A.0.6). No differentiability issues occur since we required y_0 to be H^{s+2} and y_1 to be H^{s+1}. The commutators, where u is replaced by $u-u_{00}$ and u_{00} can be estimated by inequality (A.0.8) providing us with the bounds

$$cK^2\big\|\tfrac{1}{g^{00}}\big\|_{s,\mathrm{ul}} E_{s+1}(u-u_{00}) \quad \text{and} \quad cK^2\big\|\tfrac{1}{g^{00}}\big\|_{s,\mathrm{ul}} E_{s+1}(u_{00})$$

The term in equation (A.0.6) containing the RHS f can be estimated by

$$\big\|g^{00}\partial^\beta \tfrac{1}{g^{00}} f\big\|_{L^2} \le c\|g^{00}\|_\infty \big\|\tfrac{1}{g^{00}}\big\|_{s,\mathrm{ul}}\|f\|_s \le cK\big\|\tfrac{1}{g^{00}}\big\|_{s,\mathrm{ul}}\|f\|_{\infty,s}$$

The terms without a commutator can be treated as in the above case. We conclude the following inequality for the mollified energy

$$\limsup_\tau \tfrac{1}{\tau}\big(E_\varepsilon^\beta(t+\tau) - E_\varepsilon^\beta(t)\big) \le C_1 E_{s+1}(u-u_{00}) + C_2 E_{s+1}^2(u-u_{00}).$$

In the definition of C_1 and C_2, c denotes a constant depending on m, k and on the mollifier. Using the equivalence stated in (A.0.5) we derive

$$\begin{aligned}\limsup_\tau \tfrac{1}{\tau}\big(\tilde{E}_\varepsilon(t+\tau) - \tilde{E}_\varepsilon(t)\big) &\le \sum_{|\beta|\le k-1} \limsup_\tau \tfrac{1}{\tau}\big(E_\varepsilon^\beta(t+\tau) - E_\varepsilon^\beta(t)\big) \\ &\le c_E^{1/2}\tau_{k-1}^{3/2} C_1 \tilde{E}^{1/2} + c_E \tau_{k-1}^2 C_2 \tilde{E}. \quad \text{(A.0.9)}\end{aligned}$$

To perform the limit $\varepsilon \to 0$ we integrate this inequality

$$\tilde{E}_\varepsilon(t+\tau) \le \tilde{E}_\varepsilon(t) + \int_t^{t+\tau} \big(c_E^{1/2}\tau_{k-1}^{3/2} C_1 \tilde{E}^{1/2} + c_E\tau_{k-1}^2 C_2 \tilde{E}\big)\, ds$$

Taking the limit and again considering the $\limsup$ we obtain

$$\limsup_\tau \tfrac{1}{\tau}\big(\tilde{E}(t+\tau) - \tilde{E}(t)\big) \le c_E^{1/2}\tau_{k-1}^{3/2} C_1 \tilde{E}^{1/2}(t) + c_E\tau_{k-1}^2 C_2 \tilde{E}(t).$$

With the help of considerations to be found in [Sog95], we get

$$\limsup_{\tau} \tfrac{1}{\tau}\big(\tilde{E}^{1/2}(t+\tau) - \tilde{E}^{1/2}(t)\big) \le c_E^{1/2}\tau_{k-1}^{3/2}C_1 + c_E\tau_{k-1}^2 C_2\tilde{E}^{1/2}$$

and further

$$\tilde{E}^{1/2}(t) \le e^{c_E\tau_{k-1}^2 C_2 t}(\tilde{E}^{1/2}(0) + c_E^{1/2}\tau_{k-1}^{3/2}C_1 t).$$

Using again the equivalence stated in (A.0.5) it follows

$$E_{s+1}(u-u_{00}) \le c_E^{1/2}\tau_s^{1/2}\tilde{E}^{1/2}(t) \le e^{c_E\tau_s^2 C_2 t}\big(C^{1/2}E_{s+1}(\mathring{u}-u_{00}) + c_E\tau_s^2 C_1 t\big),$$

where the constant C is taken from Lemma 2.10. □

Lemma A.1. *A solution u to the quasilinear second-order equation (2.0.1) obtained by Theorem 2.9 satisfies $u \in C([0,T'],H^{s+1}) \cap C^1([0,T'],H^s)$.*

Proof. Firstly we will show that we have weak continuity. Only u itself will be treated, the case for $\partial_t u$ is analog. If $t_n \to t$ is a sequence, then we know $\big(u(t_n)\big)$ is bounded in H^{s+1}. There is a weakly convergent subsequence $u(t_{n_k}) \rightharpoonup v \in H^{s+1}$. By Rellich's theorem it follows that $u(t_{n_k}) \to v$ in H^s. Since $u \in C([0,T],H^s)$ it holds $v = u(t)$. We conclude that every convergent subsequence of $\big(u(t_n)\big)$ has the limit $u(t)$. If we can find a subsequence not converging to $u(t)$, then we can apply the argument since this subsequence is also bounded in H^{s+1}.

The next step is to show that $\|u(t)\|_{s+1}$ is continuous. This will be done with the help of the continuity of the energy $\tilde{E}$ (cf. A.0.4). Integrating estimate (A.0.9) for the $\limsup$ of $\tilde{E}_\varepsilon$ gives us for $\tau > 0$

$$\tilde{E}_\varepsilon(t+\tau) \le \tilde{E}_\varepsilon(t) + \int_t^{t+\tau} \big(c_E^{1/2}\tau_{k-1}^{3/2}C_1\tilde{E}_\varepsilon^{1/2} + c_E\tau_{k-1}^2 C_2\tilde{E}_\varepsilon\big)\,ds.$$

Passing to the limit $\varepsilon \to 0$ we derive $\lim_{\tau\to 0}\tilde{E}(t+\tau) \le \tilde{E}(t)$. The same argument gives us $\lim_{\tau\to 0}\tilde{E}(t-\tau) \ge \tilde{E}(t)$. The other parts of the semi-continuity follow from the invariance of the equation under time reversal. It follows

$$\lim_{\tau\to 0}\tilde{E}(T-(r+\tau)) \le \tilde{E}(T-r) \text{ and } \lim_{\tau\to 0}\tilde{E}(T-(r-\tau)) \ge \tilde{E}(T-r).$$

By setting $T - r = t$, the desired continuity of the energy $\tilde{E}$ follows.

We have now that $\tilde{E}$ is continuous and $(u,\partial_t u) \in C^w([0,T],H^s) \times C^w([0,T],H^{s-1})$. Only the highest derivatives are interesting, so we assume $|\beta| = s$. We will show that

$$\begin{aligned}\mu\|\partial^\beta(u(t_n)-u(t))\|_{L^2}^2 - \langle g^{00}\partial^\beta(\partial_t u(t_n) - \partial_t u(t)), \partial^\beta(\partial_t u(t_n) - \partial_t u(t))\rangle \\ + \langle g^{ij}\partial_i\partial^\beta(u(t_n)-u(t)), \partial_j\partial^\beta(u(t_n)-u(t))\rangle\end{aligned}$$

converges to 0, if $t_n \to t$. Expanding this term gives us

$$\begin{aligned}
&\mu\|\partial^\beta(u(t_n)-y_0)\|_{L^2}^2+\mu\|\partial^\beta(u(t)-y_0)\|_{L^2}^2-2\mu\langle\partial^\beta(u(t_n)-y_0),\partial^\beta(u(t)-y_0)\rangle\\
&-\langle g^{00}(t)\partial^\beta(\partial_t u(t_n)-y_1),\partial^\beta(\partial_t u(t_n)-y_1)\rangle-\langle g^{00}(t)\partial^\beta(\partial_t u(t)-y_1),\partial^\beta(\partial_t u(t)-y_1)\rangle\\
&+2\langle g^{00}(t)\partial^\beta(\partial_t u(t_n)-y_1),\partial^\beta(\partial_t u(t)-y_1)\rangle+\langle g^{ij}(t)\partial_i\partial^\beta(u(t_n)-y_0),\partial_j\partial^\beta(u(t_n)-y_0)\rangle\\
&+\langle g^{ij}(t)\partial_i\partial^\beta(u(t)-y_0),\partial_j\partial^\beta(u(t)-y_0)\rangle-2\langle g^{ij}(t)\partial_i\partial^\beta(u(t_n)-y_0),\partial_j\partial^\beta(u(t)-y_0)\rangle
\end{aligned}$$

The mixed terms converge to $-2E(u(t))$ due to the weak continuity. Exemplarily we consider the following term to examine the other terms without t and t_n mixed. It holds that

$$\begin{aligned}
&-\langle g^{00}(t)\partial^\beta(\partial_t u(t_n)-y_1),\partial^\beta(\partial_t u(t_n)-y_1)\rangle=\\
&\qquad-\langle g^{00}(t_n)\partial^\beta(\partial_t u(t_n)-y_1),\partial^\beta(\partial_t u(t_n)-y_1)\rangle\\
&\qquad\quad-\langle(g^{00}(t)-g^{00}(t_n))\partial^\beta(\partial_t u(t_n)-y_1),\partial^\beta(\partial_t u(t_n)-y_1)\rangle.
\end{aligned}$$

The last term has to vanish in the limit which follows from

$$\begin{aligned}
&|\langle(g^{00}(t)-g^{00}(t_n))\partial^\beta(\partial_t u(t_n)-y_1),\partial^\beta(\partial_t u(t_n)-y_1)\rangle|\\
&\qquad\le\|g^{00}(t)-g^{00}(t_n)\|_\infty(\|\partial_t u(t_n)\|_s+\|y_1\|_s)^2.
\end{aligned}$$

The assumptions for the linear equation yield that the coefficients are Lipschitz w.r.t. $H^{s-1}_{\mathrm{ul}} \hookrightarrow C_b$. The structure of the expanded term is

$$E(u(t_n))+E(u(t))+\text{mixed terms}+\text{terms involving } g^{\mu\nu}(t)-g^{\mu\nu}(t_n).$$

This converges to 0. □

Appendix B Matrix computations

First, we state some computations for the inverse of the metric.

Lemma B.1. *Suppose $(a_{\mu\nu})$ is a symmetric $(m+1)\times(m+1)$-matrix satisfying $a_{00}<0$ and $(a_{ij})>0$ where (a_{ij}) is the submatrix of $(a_{\mu\nu})$. Let $(\bar a^{ij})$ denote the inverse of the submatrix (a_{ij}). Then we have for the components of the inverse $(a^{\mu\nu})$*

$$a^{00}=\tfrac{1}{a_{00}}\big(1+\tfrac{1}{a_{00}}a_{0i}a_{0j}a^{ij}\big),\qquad a^{0j}=-\tfrac{1}{a_{00}}a^{jk}a_{0k},$$
$$a^{ik}\big(a_{kj}-\tfrac{1}{a_{00}}a_{0k}a_{0j}\big)=\delta^i_j$$

or

$$a^{00}=(a_{00}-a_{0i}a_{0j}\bar a^{ij})^{-1},\qquad a^{0j}=-a^{00}a_{0k}\bar a^{jk},$$
$$a^{ij}=a^{00}\bar a^{i\ell}a_{0\ell}a_{0k}\bar a^{kj}+\bar a^{ij}$$

Proof. The results follow from the definition of the inverse after tedious computations. □

Lemma B.2. *Assume $A = (a_{\mu\nu})$ is an $m+1 \times m+1$ matrix with $a_{00} \le -C_1 < 0$ and $a_{ij} \ge C_2\delta_{ij}$ where C_1 and C_2 are positive constants. Let $\bar{a}^{ij}$ denote the inverse to the submatrix a_{ij}. Then it follows*

$$a^{00} \le -\big(|a_{00}| + m^{1/2}C_2^{-1}|(a_{0\ell})|_e^2\big)^{-1}, \qquad a^{ij} \ge \big(|(a_{ij})|_e + C_1^{-1}|(a_{0\ell})|_e^2\big)^{-1}\delta^{ij}$$

and for the norm of the inverse it holds

$$|A^{-1}|_e^2 \le C_1^{-2} + 2mC_1^{-2}C_2^{-2}|(a_{0\ell})|_e^2 + mC_2^{-2}.$$

Proof. Consider first the inverse $(\bar{a}^{ij})$ of the submatrix. The following estimates hold

$$|(a_{ij})|_e^{-1}\delta^{ij} \le \bar{a}^{ij} \le C_2^{-1}\delta^{ij}.$$

Therefore we have an estimate for the norm $|(\bar{a}^{ij})|_e \le m^{1/2}C_2^{-1}$. We are now heading to the first claim. From Lemma B.1 we get a description of a^{00} so we have to estimate $a_{00} - a_{0i}a_{0j}\bar{a}^{ij}$. Estimating the absolute value gives us

$$|a_{00} - a_{0i}a_{0j}\bar{a}^{ij}| = -a_{00} + a_{0i}a_{0j}\bar{a}^{ij} \le |a_{00}| + |(a_{0\ell})|_e^2|(\bar{a}^{ij})|_e$$

and $-a_{00} + a_{0i}a_{0j}\bar{a}^{ij} \ge C_1$. Therefore the estimates on a^{00} follow.

For the positive definiteness of a^{ij} consider the norm of the inverse $a_{ij} - \frac{1}{a_{00}}a_{0i}a_{0j}$. The inequality $c^{ij} \ge |(c_{ij})|_e^{-1}\delta^{ij}$ for a matrix (c_{ij}) gives the desired estimate. For the norm of (a^{ij}) consider the positive definiteness of the inverse. We have $(a_{0i}a_{0j}) \ge 0$ and therefore $a_{ij} - \frac{1}{a_{00}}a_{0i}a_{0j} \ge C_2\delta_{ij}$. The norm of $a^{0\ell}$ can be estimated via $|(a^{0\ell})|_e \le C_1^{-1}|(a_{0\ell})|_e|(\bar{a}^{ij})|_e$. To derive an estimate for the norm of the full matrix $(a^{\mu\nu})$ we assemble the results. It follows

$$\textstyle\sum_{\mu\nu}|a^{\mu\nu}|^2 \;=\; |a^{00}|^2 \;+\; 2|(a^{0\ell})|_e^2 \;+\; |(a^{ij})|_e^2 \;\le\; C_1^{-2} \;+\; 2C_1^{-2}|(a_{0\ell})|_e^2|(\bar{a}^{ij})|_e^2 \;+\; mC_2^{-2}$$

Together with $|(\bar{a}^{ij})|_e \le m^{1/2}C_2^{-1}$ this yields the claim. □

The next lemma establishes a result similar to Lemma 2.4 for differentiable matrices.

Lemma B.3. *Let $A = (a_{\mu\nu})$ be an $(m+1)\times(m+1)$ matrix-valued function defined on $\mathbb{R}^m$ such that $a_{00} \le -\lambda$ and $a_{ij} \ge \mu\delta_{ij}$. Then it holds*

$$|D^kA^{-1}|_e \le c\,\delta^{-1}\big(1 + (\delta^{-1}\|A\|_{e,C^k})^k\big)$$

where δ^{-1} denotes the bound for A^{-1} as constructed in Lemma B.2.

Proof. The proof is due to [Kat75]. From the generic term

$$\partial^\alpha A^{-1} = \textstyle\sum_{\alpha_1+\dots+\alpha_k=\alpha} A^{-1}\partial^{\alpha_1}A\,A^{-1}\cdots\partial^{\alpha_k}A\,A^{-1}$$

for a multi-index $\alpha \in N_0^m$ we derive with $u_j = \partial^{\alpha_j}A\,A^{-1}$ and $|A^{-1}| \le \delta^{-1}$ that

$$|\partial^\alpha A^{-1}|_e \le \delta^{-1}\textstyle\sum|u_1\cdot\dots\cdot u_k|_e \le \delta^{-1}\sum|D^{|\alpha_1|}A|_e\delta^{-1}\cdots|D^{|\alpha_k|}A|_e\delta^{-1}.$$

Assuming $|D^\ell A|_e \le C_\ell$ yields

$$|\partial^\alpha A^{-1}|_e \le \delta^{-1}\sum C_{|\alpha_1|}\delta^{-1}\cdots C_{|\alpha_k|}\delta^{-1} = \delta^{-1}\sum C_{|\alpha_1|}\delta^{-1}\cdots C_{|\alpha_k|}\delta^{-1}.$$

With the norm $\|\,.\,\|_{C^s}$ defined in (2.1.1) we get $\|A\|^2_{e,C^k} \le \sum_{\ell\le k}(C_\ell)^2$. This yields

$$|\partial^\alpha A^{-1}|_e \le c\,\delta^{-1}(1 + B + \cdots + B^{|\alpha|})$$

where c is a constant dependent on the dimension and k and $B = \delta^{-1}\|A\|_{e,C^k}$. Summing over $|\alpha| = k$ and using the inequality $(a+b)^p \le 2^p(a^p + b^p)$ gives the desired estimate. $\square$

The proof can also be applied to a positive definite matrix-valued function. We state the result for reference.

Corollary B.4. *Let $A = (a_{ij})$ be an $n\times n$ matrix-valued function defined on $\mathbb{R}^m$ such that $a_{ij} \ge \mu\delta_{ij}$ for a constant $\mu > 0$. Then it holds*

$$|D^k A^{-1}|_e \le c\,\delta^{-1}\big(1 + (\delta^{-1}\|A\|_{e,C^k})^k\big)$$

with $\delta^{-2} = \frac{n}{\mu}$.

Bibliography

[AAI06] Paul Allen, Lars Andersson, and James Isenberg, *Timelike minimal submanifolds of general co-dimension in Minkowski space time*, J. Hyperbolic Differ. Equ. **3** (2006), no. 4, 691–700.

[Bar84] Robert Bartnik, *Existence of maximal surfaces in asymptotically flat spacetimes*, Comm. Math. Phys. **94** (1984), no. 2, 155–175.

[Bre02] Simon Brendle, *Hypersurfaces in Minkowski space with vanishing mean curvature*, Comm. Pure Appl. Math. **55** (2002), no. 10, 1249–1279.

[CB52] Yvonne Choquet-Bruhat, *Theorème d'existence pour certains systémes d'équations aux dérivées partielles non linéaires*, Acta Math. **88** (1952), 141–225.

[CB99] ———, *Wave maps in general relativity*, On Einstein's path (New York, 1996), Springer, New York, 1999, pp. 147–169.

[CBG69] Yvonne Choquet-Bruhat and Robert Geroch, *Global aspects of the Cauchy problem in general relativity*, Comm. Math. Phys. **14** (1969), 329–335.

[Chr86] Demetrios Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Comm. Pure Appl. Math. **39** (1986), no. 2, 267–282.

[CK93] D. Christodoulou and S. Klainerman, *The Global Nonlinear Stability of the Minkowski Space*, Princeton Mathematical Series, vol. 41, Princeton University Press, Princeton, New Jersey, 1993.

[Dec94] Thomas Deck, *A geometric Cauchy problem for timelike minimal surfaces*, Ann. Global Anal. Geom. **12** (1994), no. 4, 305–312.

[Dou39] Jesse Douglas, *The most general form of the problem of Plateau*, Amer. J. Math. **61** (1939), 590–608.

[ER93] James Eells and Andrea Ratto, *Harmonic maps and minimal immersions with symmetries*, Annals of Mathematics Studies, vol. 130, Princeton University Press, Princeton, NJ, 1993, Methods of ordinary differential equations applied to elliptic variational problems.

[Eva98] L. C. Evans, *Partial differential equations*, Graduate studies in mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998.

[FM72] Arthur E. Fischer and Jerrold E. Marsden, *The Einstein evolution equations as a first-order quasi-linear symmetric hyperbolic system. I*, Comm. Math. Phys. **28** (1972), 1–38.

[FR00] Helmut Friedrich and Alan Rendall, *The Cauchy problem for the Einstein equations*, Einstein's field equations and their physical implications, Lecture Notes in Phys., vol. 540, Springer, Berlin, 2000, pp. 127–223.

[GHL04] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine, *Riemannian geometry*, third ed., Universitext, Springer-Verlag, Berlin, 2004.

[Gu80] C. H. Gu, *On the Cauchy problem for harmonic maps defined on two-dimensional minkowski space*, Commun. Pure Appl. Math. **33** (1980), 727–737.

[Gu83] ______, *On the motion of a string in a curved space-time*, Proc. 3rd Grossmann Meeting on General Relativity, Beijing, Ning, Hu, 1983, pp. 139–142.

[Ham82] R. S. Hamilton, *Three-manifolds with positive ricci curvature*, Journal of Differential Geometry **17** (1982), 255–306.

[HKM76] Thomas J. R. Hughes, Tosio Kato, and Jerrold E. Marsden, *Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity*, Arch. Rational Mech. Anal. **63** (1976), no. 3, 273–294 (1977).

[HN87] Jens Hoppe and Hermann Nicolai, *Relativistic minimal surfaces*, Phys. Lett. B **196** (1987), no. 4, 451–455.

[HP96] G. Huisken and A. Polden, *Geometric evolution equations for hypersurfaces*, Calculus of Variations and geometric evoultion problems (Cetraro,1996), Lecture Notes in Math., vol. 1713, Springer, Berlin, 1996, pp. 45–84.

[Kat75] Tosio Kato, *The Cauchy problem for quasi-linear symmetric hyperbolic systems*, Arch. Rational Mech. Anal. **58** (1975), no. 3, 181–205.

[Kla86] S. Klainerman, *The null condition and global existence to nonlinear wave equations*, Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984), Lectures in Appl. Math., vol. 23, Amer. Math. Soc., Providence, RI, 1986, pp. 293–326.

[Lin04] Hans Lindblad, *A remark on global existence for small initial data of the minimal surface equation in Minkowskian space time*, Proc. Amer. Math. Soc. **132** (2004), no. 4, 1095–1102 (electronic).

[Mor48] Charles B. Morrey, Jr., *The problem of Plateau on a Riemannian manifold*, Ann. of Math. (2) **49** (1948), 807–851.

[Mül03] Olaf Müller, *Natural geometric quantization of first-order field theories*, Ph.D. thesis, University Leipzig, Leipzig, 2003.

[Nes99] Alexander I. Nesterov, *Riemann normal coordinates, Fermi reference system and the geodesic deviation equation*, Classical Quantum Gravity **16** (1999), no. 2, 465–477.

[O'N83] Barrett O'Neill, *Semi-riemannian geometry*, Academic Press, New York, 1983.

[Sog95] Christopher D. Sogge, *Lectures on nonlinear wave equations*, Monographs in Analysis, II, International Press, Boston, MA, 1995.

[Spi65] Michael Spivak, *Calculus on manifolds. A modern approach to classical theorems of advanced calculus*, W. A. Benjamin, Inc., New York-Amsterdam, 1965.

[SS98] Jalal Shatah and Michael Struwe, *Geometric wave equations*, Courant Lecture Notes in Mathematics, vol. 2, New York University Courant Institute of Mathematical Sciences, New York, 1998.

[Sta96] Axel Stahl, *Regularity estimates for solutions to the mean curvature flow with a Neumann boundary condition*, Calc. Var. Partial Differential Equations **4** (1996), no. 4, 385–407.

[Tay96] Michael E. Taylor, *Partial differential equations*, Applied Mathematical Sciences, vol. 117, Springer, New York, 1996.

[Wal84] R. M. Wald, *General relativity*, The University of Chicago Press, Chicago and London, 1984.

[Wil93] T. J. Willmore, *Riemannian geometry*, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1993.