Energy Decay Law in n-Dimensional Gowdy Spacetimes with Torus Topology

Dissertation

vorgelegt von

Aneta E. Barbos

eingereicht beim

Fachbereich Mathematik und Informatik
der Freien Universität Berlin

durchgeführt am

Max-Planck-Institut für Gravitationsphysik
Albert-Einstein-Institut

unter der Betreuung von

Prof. Dr. Alan D. Rendall

Berlin, Mai 2010

Erstgutachter: Prof. Dr. Alan D. Rendall

Zweitgutachter: Prof. Dr. Jan Metzger

Tag der Disputation: 26. Juli 2010

Abstract

The present thesis is concerned with the construction of Gowdy spacetimes with torus topology in n dimensions and the study of their asymptotics as $t \to \infty$. Already much is known about these spaces in four dimensions, in particular, the solution to Einstein's vacuum equations can be represented as a wave-map from a Minkowski-like space to the hyperbolic plane whose energy decays as t^{-1} as $t \to \infty$. The behaviour of the solution at large times is governed by a quantity which is invariant under isometries of the target-space. The present thesis contains a derivation of the target-space metric in the general case, i. e., the standard metric on $SL(n-2)/SO(n-2)$ in certain coordinates, and shows that, just as in four dimensions, the energy of the wave-map which describes the solution decays as t^{-1} as $t \to \infty$. A quantity analogous to the invariant in four dimensions is constructed from time-conserved quantities. For homogeneous data, this invariant is, at fixed time, proportional to the kinetic energy of the wave-map.

1 Introduction

Gowdy spacetimes were first studied by Robert H. Gowdy in an article from 1974 [1]. In this seminal paper, Gowdy analysed spaces with a two-parameter space-like abelian isometry group and showed that there are three possible topologies for such spaces, namely $\mathbb{T}^3$, $\mathbb{S}^2 \times \mathbb{S}^1$ and $\mathbb{S}^3$. It later turned out that the $\mathbb{T}^3$ case is the easiest one to handle and its study is relevant to physical applications, namely the modelling of gravitational waves. The $\mathbb{S}^2 \times \mathbb{S}^1$ case is, in turn, relevant for Kerr black holes, since the interior of rotating black holes can be described by a model with two space-like Killing vectors and $\mathbb{S}^2 \times \mathbb{S}^1$ topology. More precisely, Gowdy space-times are time-orientable, globally hyperbolic vacuum Lorentz manifolds with compact Cauchy surfaces and vanishing twist constants that possess two commuting space-like Killing vectors ($\mathbb{T}^2$ symmetry). The requirement that the twist functions (which are constants in the case of vacuum) vanish is a crucial one. Indeed, if we drop this requirement, the spaces we are left with are much more difficult to analyse and so far little is known about them. Insofar as Gowdy space-times represent an intermediate level of complexity for vacuum Einstein equations, they are also important as a first step in studying and understanding more complicated solutions.

Already much is known about Gowdy spacetimes with torus topology in four dimensions. Global existence of solutions was shown in [2] while [3] resolved issues regarding asymptotics of the solutions at large times and showed future causal geodesic completeness of the spacetime. In the latter paper, it was pointed out that Einstein's vacuum equations have a wave-map character and this was used to investigate the late-time dynamics of the solution. Surprisingly, it turns out that nonhomogeneous solutions look like homogeneous ones, in the sense that, as $t \to \infty$, spatial inhomogeneities vanish, however, they do not necessarily behave like homogeneous solutions. In fact, this behaviour is determined by the sign of a certain invariant quantity which, when negative, causes a novel type of behaviour to emerge. A first step in the proof of this result was showing that the energy of the wave-map which describes the solution decays asymptotically at large time as t^{-1}. This, in turn, was used to prove that the length of the corresponding loop in the target space goes to 0 as $t^{-\frac{1}{2}}$ at infinity. This energy decay law was also found by Ringström to hold in the case of the $\mathbb{T}^3$-Gowdy symmetric Einstein-Maxwell model [4]. In this model, provided that a certain term is put to zero, one obtains a wave-map whose target space is essentially the same as the target space of the 5-dimensional Gowdy spacetime derived here in Section 3. This is not entirely surprising, since they are both $\mathbb{T}^3$-Gowdy symmetric.

The goal of the present thesis was to extend the results in [3] concerning the asymptotics at $t \to \infty$ to the case of an arbitrary number of spacetime dimensions, i. e, for Gowdy spacetimes with $\mathbb{T}^N$ symmetry and $\mathbb{T}^{N+1}$ topology. I have succeded to prove that, in n dimensions, the energy also decays as t^{-1} and this automatically implies that the length of the loop decays to 0 as $t^{-\frac{1}{2}}$. However, I could not establish whether the type of behaviour present in the 4-dimensional case is characteristic to all solutions in higher dimensions. The difficulty in establishing this result resides in the computational complexity of the problem in higher dimensions. Thus, the conserved quantities which are crucial to the proof in four dimensions have a much more complicated expression in higher dimensions, which makes them cumbersome to work with.

In four dimensions, the proof of the energy decay law is divided in two parts: first one shows that the energy decays as t^{-1} if the initial energy is sufficiently small; it is then proven that the energy converges to 0 as $t \to \infty$, so that the small data result can be employed. A brief overview of Ringström's proof is presented in Section 2, for details the reader can consult the original paper [3]. After deriving the spacetime metric and the target-space metric in n dimensions in Section 3, we proceed to prove the small data result in Section 4, while Section 5 is concerned with showing that the energy converges to 0 as $t \to \infty$. The proof is a generalization of the one in four dimensions ([3]). Finally, Section 6 shows a derivation of the time-conserved quantities in n dimensions, as well as the construction of a quantity, related to the Casimir operator of the Lie algebra $\mathrm{sl}(N)$, which is invariant under isometries of the target space. In the 4-dimensional case, this invariant plays an important role in the investigation of late time dynamics, however the analogue of this analysis could not be completed in n dimensions. It is, however, interesting to note that, although the physical meaning of the invariant is unclear in the general case, for homogeneous data this quantity is, at fixed time, proportional to the kinetic energy of the solution.

2 Previous Results

This section presents some results for the 4-dimensional Gowdy spacetime due to Hans Ringström published in [3]. The spacetime metric in four dimensions has the following form:

$$\mathrm{d}s^2_{(4)} = t^{-\frac{1}{2}} e^{\frac{\lambda}{2}} \left(-\mathrm{d}t^2 + \mathrm{d}\theta^2\right) + t\left[e^{-P}\mathrm{d}x_1^2 + e^{P}\left(\mathrm{d}x_2 + Q\mathrm{d}x_1\right)^2\right]$$

where λ, P and Q are functions depending on t and θ, periodic in θ. This spacetime obviously has two commuting spacelike Killing vectors, namely $\dfrac{\partial}{\partial x_1}$ and $\dfrac{\partial}{\partial x_2}$ and there is an additional symmetry given by: $x_1 \mapsto -x_1$ and $x_2 \mapsto -x_2$. The coordinate t is defined such that it is equal to area of the orbits of $\mathbb{T}^2$. Physically, the metric describes a gravitational wave propagating in the θ direction, P and Q being the two polarizations of the wave; for $Q = 0$ we speak of a polarized Gowdy spacetime.

Writing down Einstein's vacuum equations for this spacetime, we obtain two sets of differential equations: the evolution equations given by

$$\begin{cases} \partial_t^2 P - \partial_\theta^2 P + \frac{1}{t}\partial_t P & = e^{2P}\left[\left(\partial_t Q\right)^2 - \left(\partial_\theta Q\right)^2\right] \\ \partial_t^2 Q - \partial_\theta^2 Q + \frac{1}{t}\partial_t Q & = -2\left(\partial_t P \partial_t Q - \partial_\theta P \partial_\theta Q\right) \end{cases} \tag{1}$$

and the constraint equations for λ,

$$\begin{cases} \partial_t \lambda & = t\left\{\left(\partial_t P\right)^2 + \left(\partial_\theta P\right)^2 + e^{2P}\left[\left(\partial_\theta Q\right)^2 + \left(\partial_\theta Q\right)^2\right]\right\} \\ \partial_\theta \lambda & = 2t\left(\partial_t P \partial_\theta P + e^{2P}\partial_t Q \partial_\theta Q\right). \end{cases}$$

The evolution equations can be solved globally for $t \in (0, \infty)$ for smooth data [2] and the constraint equations can be integrated to give λ once P and Q are known. It was pointed out in [3] that the evolution equations can be regarded as the wave-map equations of a map (P, Q) from the Minkowski-like space with metric

$$\mathrm{d}s_0^2 = -\mathrm{d}t^2 + \mathrm{d}\theta^2 + t^2\mathrm{d}\varphi^2$$

to a target space (the hyperbolic plane) with metric

$$\mathrm{d}s_1^2 = \mathrm{d}P^2 + \mathrm{e}^{2P}\mathrm{d}Q^2.$$

(For an exposition on wave-maps in general relativity, see [5].) The map does not depend on the φ coordinate which is only introduced in order to obtain the terms containing $\dfrac{1}{t}$ in (1). Note that the wave-map equations can be regarded as the Euler-Lagrange equations of the lagrangian

$$\mathcal{L} = \frac{t}{2}\left[-P_t^2 + P_\theta^2 + \mathrm{e}^{2P}\left(-Q_t^2 + Q_\theta^2\right)\right]. \tag{2}$$

We can also define an energy-like quantity for the wave-map as follows:

$$H = \frac{1}{2}\int\limits_{\mathbb{S}^1}\left[P_t^2 + P_\theta^2 + \mathrm{e}^{2P}\left(Q_t^2 + Q_\theta^2\right)\right]\mathrm{d}\theta \tag{3}$$

It was shown in [3] that the energy decays asymptotically at $t \to \infty$,

$$H(t) \leq \frac{C}{t}, \text{ for all } t \geq T,$$

where C and T are constants. The proof procedes in two steps: first it is shown that the energy decays in this manner if the initial energy is sufficiently small; it is then proven that the energy goes to zero at large times for arbitrary initial data. We will carry out the proof in its full generality in Sections 4 and 5. For now, it suffices to indicate the underlying idea of the method used, as presented in Ringström's paper. The point of the method is obtaining energy estimates for a system, without solving the associated differential equation. This goal is achieved by introducing so-called correction terms. The main idea can be illustrated with a simple example.

Let us consider the ordinary differential equation

$$\ddot{x} + 2a\dot{x} + b^2 x = 0$$

with $a > 0$ and $b^2 > a^2$. We are interested in the behaviour of the energy

$$H = \frac{1}{2}\left(\dot{x}^2 + b^2 x^2\right).$$

We know that the energy decays, since

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -2a\dot{x}^2$$

and we wish to investigate this decay qualitatively in more detail. The crucial trick is the introduction of a correction term,

$$\Gamma := ax\dot{x},$$

which satisfies

$$|\Gamma| = \left|\frac{a}{b}\right| |bx\dot{x}| \leq \left|\frac{a}{b}\right| \frac{1}{2}\left(\dot{x}^2 + b^2 x^2\right) = \left|\frac{a}{b}\right| H.$$

We note that since $\left|\frac{a}{b}\right| < 1$, there exist positive constants c_1, c_2 such that

$$c_1 H \leq H + \Gamma \leq c_2 H$$

In addition, we have:

$$\frac{\mathrm{d}(H + \Gamma)}{\mathrm{d}t} = -2a(H + \Gamma).$$

It then follows that

$$H \leq C \exp(-2at)$$

and we have obtained our desired decay.

In [3] the method is further illustrated with the case of the 4-dimensional polarized Gowdy spacetime. There, the field Q vanishes and the evolution equation of the spacetime becomes

$$P_{tt}+\frac{1}{t}P_t-P_{\theta\theta}=0$$

We can also write the evolution equation in the form

$$-\partial_t\left(tP_t\right)+\partial_\theta\left(tP_\theta\right)=0$$

which is manifestly the Euler-Lagrange equation of (2) for the polarized Gowdy case. The energy, defined by

$$H=\frac{1}{2}\int\limits_{\mathbb{S}^1}\left(P_t^2+P_\theta^2\right)\mathrm{d}\theta,$$

satisfies the equation

$$\frac{\mathrm{d}H}{\mathrm{d}t}=-\frac{1}{t}\int\limits_{\mathbb{S}^1}P_t^2\mathrm{d}\theta.$$

The first obvious choice for the correction term

$$\Gamma'=\frac{1}{2t}\int\limits_{\mathbb{S}^1}PP_t\mathrm{d}\theta$$

turns out to be incorrect, since this quantity cannot be bounded in term of H, which is essential to the proof. Instead, one defines

$$\Gamma=\frac{1}{2t}\int\limits_{\mathbb{S}^1}\left(P-<P>\right)P_t\mathrm{d}\theta$$

where for a function $f:\mathbb{S}^1\to\mathbb{R}$, $<f>$ denotes the average of f on $\mathbb{S}^1$,

$$<f>:=\frac{1}{2\pi}\int\limits_{\mathbb{S}^1}f\mathrm{d}\theta$$

The use of averages is crucial because one is interested in correction terms which can be bounded in terms of H. To see this, let us estimate the quantity

$$\int\limits_{\mathbb{S}^1}\left(P-<P>\right)^2\mathrm{d}\theta=2\pi\sum_{n\in\mathbb{Z}}\left|a_n\right|^2\leq\sum_{n\in\mathbb{Z}}n^2\left|a_n\right|^2=\int\limits_{\mathbb{S}^1}\left(\partial_\theta P\right)^2\mathrm{d}\theta\leq H \qquad (4)$$

where a_n are the Fourier coefficients of the function P. We note that by subtracting the average $<P>$, the coefficent a_0 gets cancelled, which allows us to obtain a useful estimate for the l.h.s. Using Hölder's inequality and (4), we obtain

$$|\Gamma|\leq\frac{1}{2t}H$$

which, as before, implies that, for large enough t, there exist positive constants c_1, c_2 such that

$$c_1 H \leq H + \Gamma \leq c_2 H$$

Using the evolution equation, we can compute

$$\begin{aligned}\frac{\mathrm{d}\Gamma}{\mathrm{d}t} &= -\frac{2}{t}\Gamma + \frac{1}{2t}\int_{\mathbb{S}^1} P_t^2 \mathrm{d}\theta - \frac{\pi}{t} < P_t >^2 + \frac{1}{2t^2}\int_{\mathbb{S}^1} (P - < P >)\, \partial_\theta \left(t P_\theta\right) \mathrm{d}\theta \\ &\leq -\frac{2}{t}\Gamma + \frac{1}{2t}\int_{\mathbb{S}^1} \left(P_t^2 - P_\theta^2\right) \mathrm{d}\theta.\end{aligned}$$

Then

$$\begin{aligned}\frac{\mathrm{d}\left(H + \Gamma\right)}{\mathrm{d}t} &\leq -\frac{2}{t}\Gamma + \frac{1}{2t}\int_{\mathbb{S}^1} \left(P_t^2 - P_\theta^2\right) \mathrm{d}\theta - \frac{1}{t}\int_{\mathbb{S}^1} P_t^2 \mathrm{d}\theta = -\frac{2}{t}\Gamma - \frac{1}{t}H \\ &\leq -\frac{1}{t}\left(\Gamma + H\right) + \frac{1}{t}\left|\Gamma\right| \leq -\frac{1}{t}\left(\Gamma + H\right) + \frac{1}{2t^2}H \\ &\leq -\frac{1}{t}\left(H + \Gamma\right) + \frac{C}{t^2}\left(H + \Gamma\right).\end{aligned}$$

We can infer that

$$H \leq \frac{C}{t}.$$

Here and in the following, C denotes some positive constant which may depend on H, but decreases with decreasing H.

In the case of the nonpolarized 4-dimensional Gowdy spacetime with the energy defined by (3), the correction term has two parts:

$$\begin{aligned}\Gamma &:= \Gamma^P + \Gamma^Q, \text{ with} \\ \Gamma^P &:= \frac{1}{2t}\int_{\mathbb{S}^1} (P - < P >)\, P_t \mathrm{d}\theta \text{ and} \\ \Gamma^Q &:= \frac{1}{2t}\int_{\mathbb{S}^1} \mathrm{e}^{2<P>} (Q - < Q >)\, Q_t \mathrm{d}\theta.\end{aligned}$$

For small data, ($H \leq \epsilon$ for some suitable positive ϵ), the correction terms satisfy ([3])

$$\left|\Gamma^P\right| \leq \frac{C}{t}H \text{ and } \left|\Gamma^Q\right| \leq \frac{C}{t}H.$$

Additionally, we have

$$\begin{aligned}\frac{\mathrm{d}\Gamma^P}{\mathrm{d}t} &\leq -\frac{2}{t}\Gamma^P + \frac{1}{2t}\int_{\mathbb{S}^1} \left(P_t^2 - P_\theta^2\right) \mathrm{d}\theta + \frac{C}{t}H^{\frac{3}{2}} \text{ and} \\ \frac{\mathrm{d}\Gamma^Q}{\mathrm{d}t} &\leq -\frac{2}{t}\Gamma^Q + \frac{1}{2t}\int_{\mathbb{S}^1} \mathrm{e}^{2P}\left(Q_t^2 - Q_\theta^2\right) \mathrm{d}\theta + \frac{C}{t}H^{\frac{3}{2}}.\end{aligned}$$

Since
$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{1}{t}\int\limits_{\mathbb{S}^1} \left(P_t^2 + \mathrm{e}^{2P} Q_t^2\right) \mathrm{d}\theta,$$
it follows that
$$\frac{\mathrm{d}\left(H+\Gamma\right)}{\mathrm{d}t} \leq -\frac{1}{t}\left(H+\Gamma\right) - \frac{1}{t}\Gamma + \frac{C}{t}H^{\frac{3}{2}}.$$
Using an argument similar to the one above, we obtain the result for small data:
$$H \leq \frac{C}{t} \text{ for all } t \geq T,$$
where C and T are certain positive constants.

To prove the result for general initial data, it suffices to show that $H\left(t\right)$ converges to 0 as $t \to \infty$. If this is the case, then $H\left(t\right) \leq \epsilon$ for $t \geq t_0$ for some positive t_0 and we can use the small data result. We note that, in order to prove convergence to 0, it suffices to show that
$$\frac{1}{t}H\left(t\right) \in L^1\left(\left[t_0, \infty\right)\right), \text{ for any } t_0 > 0.$$
Denoting by H_K and H_P the kinetic and potential part of H, respectively, i. e. $H = H_K + H_P$, we can write
$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{2}{t}H_K.$$
It then follows that
$$\frac{1}{t}H_K\left(t\right) \in L^1\left(\left[t_0, \infty\right)\right). \tag{5}$$
It remains to show that
$$\frac{1}{t}H_P\left(t\right) \in L^1\left(\left[t_0, \infty\right)\right). \tag{6}$$
We will not present here the details of the argument, rather we will carry out the proof for n dimensions in Section 5, of which the 4-dimensional case is a special case. It suffices to note that we consider the quantities
$$\int\limits_{t_0}^{t} \frac{1}{s}\int\limits_{\mathbb{S}^1} \left(P_t^2 - P_\theta^2\right) \mathrm{d}\theta \mathrm{d}s$$
and
$$\int\limits_{t_0}^{t} \frac{1}{s}\int\limits_{\mathbb{S}^1} \mathrm{e}^{2P}\left(Q_t^2 - Q_\theta^2\right) \mathrm{d}\theta \mathrm{d}s.$$
Using the Euler-Lagrange equations of (2) (i.e. the evolution equations (1)) and certain estimates, it can be shown that these two quantities are finite. Since (5) holds, (6) follows, hence H converges to 0 as $t \to \infty$.

As mentioned above, the target space of the wave-map describing the solution is the hyperbolic plane which is isomorphic to the upper half plane. Thus, because of the periodicity in θ, at fixed time, a solution can be represented as a loop in the upper half plane. As a consequence of the result concerning the energy decay, it follows that the length of the loop which is defined by

$$l = \int\limits_{\mathbb{S}^1} \left(P_\theta^2 + \mathrm{e}^{2P} Q_\theta^2\right)^{\frac{1}{2}} \mathrm{d}\theta,$$

goes to 0 as $t^{-\frac{1}{2}}$ as $t \to \infty$. Thus, since the θ dependence vanishes, an inhomogeneous solution will asymptotically look homogeneous.

An interesting result in four dimensions concerns the behaviour of the solution as $t \to \infty$. In [3], it was shown that, although the solution becomes spatially homogeneous at large t, it does not necessarily behave as such. The analysis relies on the existence of conserved quantities of which, in four dimensions, there are three:

$$\begin{aligned}
A &= \int\limits_{\mathbb{S}^1} t\left(2\mathrm{e}^{2P} Q_t - 2P_t\right) \mathrm{d}\theta \\
B &= \int\limits_{\mathbb{S}^1} t\mathrm{e}^{2P} Q_t \mathrm{d}\theta \\
C &= \int\limits_{\mathbb{S}^1} t\left[\left(1 - \mathrm{e}^{2P} Q^2\right) Q_t + 2P_t Q\right] \mathrm{d}\theta
\end{aligned}$$

In Section 6, we will present a derivation of these conserved quantities in the general n-dimensional case. In addition to the time-conserved quantities, there also exists a quantity which is invariant under isometries of the target space, given by

$$D = A^2 + 4BC \tag{7}$$

which controls the dynamics of the solution at large times. Thus, for $D > 0$, the solution behaves, indeed, like a homogeneous solution at large time, more precisely, the loop "moves" in the upper half plane along straight lines and circles which meet the boundary transversally. However, for $D < 0$, there emerges a new type of behaviour, namely, the solution oscillates along circles inside the upper half plane. If $D = 0$, but not all constants (A, B and C) vanish, the solution moves either along a curve $y = \text{const}$, or along a circle touching the boundary. If A, B and C all vanish, the solution goes to a point. It is interesting to note, that, in the case of spatially homogeneous data, the quantity D has the form

$$D = 16\pi t^2 H_K$$

where H_K is the kinetic energy of the solution; thus, assuming homogeneity, this quantity is always non-negative. In the general case, D can also assume negative values and its physical significance is unclear.

The starting point for the proof of these statements (which is very technical) is deriving certain ODEs for the averages $< P >$ and $< Q >$ of the parameters describing the solution:

$$t < P_t > = \frac{B}{2\pi} < Q > - \frac{A}{4\pi} + \frac{1}{2\pi}\int\limits_{\mathbb{S}^1} t\mathrm{e}^{2P}\left(Q - < Q >\right) Q_t \mathrm{d}\theta$$
$$t\mathrm{e}^{<P>} < Q_t > = \frac{B}{2\pi}\mathrm{e}^{-<P>} - \frac{1}{2\pi}\mathrm{e}^{<P>}\int\limits_{\mathbb{S}^1} t\left(\mathrm{e}^{2P-2<P>} - 1\right) Q_t \mathrm{d}\theta$$

$$t < Q_t > = \frac{C}{2\pi} + \frac{A}{2\pi} < Q > - \frac{B}{2\pi} < Q >^2 + \frac{t}{\pi}\int\limits_{\mathbb{S}^1}\left(< Q > - Q\right) P_t \mathrm{d}\theta$$
$$+ \frac{t}{2\pi}\int\limits_{\mathbb{S}^1}\mathrm{e}^{2P}\left(Q - < Q >\right)^2 Q_t \mathrm{d}\theta$$

In the derivation of these ODEs, it is important that the integrals appearing on the r.h.s are terms which can be estimated in terms of the wave-map energy, H. These equations, in turn, are used to derive an algebraic inequality:

$$\left| B\left\{\mathrm{e}^{-<P>} + \mathrm{e}^{<P>}\left[\left(< Q > - \frac{A}{2B}\right)^2 - \frac{A^2 + 4BC}{4B^2}\right]\right\}\right| \le K$$

where K is a constant. Using this inequality, certain crucial estimates can be obtained. For the proof, the reader is referred to Ringström's paper, [3]. Let us just point out that the conserved quantities have a much more complicated expression in $n > 4$ dimensions and there are $(n-2)^2 - 1$ of them. This makes it computationally difficult to derive suitable ODEs and an algebraic inequality for the averages.

3 The Spacetime and Target-Space Metrics

To derive the spacetime metric in an arbitrary number of dimensions, we will use Kaluza-Klein reduction on the circle ([6]). Namely, given the metric in n-dimensions $\mathrm{d}s^2_{(n)}$, we add another coordinate $x_{n+1} \in \mathbb{S}^1$, by using the following ansatz for the higher dimensional metric:

$$\mathrm{d}s_{(n+1)} = \mathrm{e}^{2\alpha\varphi}\mathrm{d}s^2_{(n)} + \mathrm{e}^{2\beta\varphi}\left(\mathrm{d}x^{n+1} + \mathcal{A}\right)^2,$$

where $\mathcal{A} = \sum\limits_{i=1}^{n} A_i \mathrm{d}x^i$ and φ are fields independent of x_{n+1}, while α and β are constants. It now turns out that the dimensionally reduced lagrangian is of the form $\mathrm{e}^{[\beta+(n-2)\alpha]\varphi}\sqrt{-g}R + \ldots$, where g and R are, respectively, the determinant and curvature of the metric in n-dimensions. Requiring that the dimensionally reduced lagrangian be of the usual form, $\sqrt{-g}R + \ldots$, corresponding to the Einstein-Hilbert action, we obtain the constraint $\beta = -(n-2)\,\alpha$.

With this information, we can now construct the Gowdy spacetime metric in arbitrary dimensions. Namely, start by defining the 3-dimensional metric

$$\mathrm{d}s^2_{(3)} = \mathrm{e}^{\frac{\lambda}{2}}\left(-\mathrm{d}t^2 + \mathrm{d}\theta^2\right) + t^2\left(\mathrm{d}x^1\right)^2,$$

where $(\theta,\, x_1)$ are coordinates on $\mathbb{T}^2$ and λ is a function of $(t,\, \theta)$ periodic in θ. If we want to add another coordinate $x_2 \in \mathbb{S}^1$, we can use Kaluza-Klein decomposition to derive the metric in 4-dimensions:

$$\mathrm{d}s^2_{(4)} = \mathrm{e}^{2\alpha\varphi}\mathrm{d}s^2_{(3)} + \mathrm{e}^{2\beta\varphi}\left(\mathrm{d}x^2 + A^1_2\mathrm{d}x^1\right)^2,$$

where φ and A^1_2 are functions of $(t,\, \theta)$ and α, β are constants with $\beta = -\alpha$ required for Lorentz invariance of the reduced action. Note that A^1_2 does *not* denote a component of a (1,1) tensor. We can further write

$$\mathrm{d}s^2_{(4)} = \mathrm{e}^{2\alpha\varphi}\left[\mathrm{e}^{\frac{\lambda}{2}}\left(-\mathrm{d}t^2 + \mathrm{d}\theta^2\right) + t^2\left(\mathrm{d}x^1\right)^2\right] + \mathrm{e}^{-2\alpha\varphi}\left(\mathrm{d}x^2 + A^1_2\mathrm{d}x^1\right)^2.$$

If we denote $p_1 := -2\alpha\varphi$, we can express the metric in the form

$$\mathrm{d}s^2_{(4)} = \mathrm{e}^{-p_1+\frac{\lambda}{2}}\left(-\mathrm{d}t^2 + \mathrm{d}\theta^2\right) + t^2\mathrm{e}^{-p_1}\left(\mathrm{d}x^1\right)^2 + \mathrm{e}^{p_1}\left(\mathrm{d}x^2 + A^1_2\mathrm{d}x^1\right)^2.$$

If we now finally transform $\lambda \to \lambda - 2p_1 - \ln t$ and $p_1 \to p_1 - \ln t$, we obtain the desired 4-dimensional Gowdy spacetime metric

$$\mathrm{d}s^2_{(4)} = t^{-\frac{1}{2}}\mathrm{e}^{\frac{\lambda}{2}}\left(-\mathrm{d}t^2 + \mathrm{d}\theta^2\right) + t\left[\mathrm{e}^{-p_1}\left(\mathrm{d}x^1\right)^2 + \mathrm{e}^{p_1}\left(\mathrm{d}x^2 + A^1_2\mathrm{d}x^1\right)^2\right].$$

The conformal factor is chosen to have this form in order to simplify the expression of Einstein's vacuum equations. Repeating the procedure, we obtain the metric in five dimensions,

$$\begin{aligned}\mathrm{d}s^2_{(5)} = t^{-\frac{2}{3}}\mathrm{e}^{\frac{\lambda}{2}}\left(-\mathrm{d}t^2 + \mathrm{d}\theta^2\right) + t^{\frac{2}{3}}\Big[&\mathrm{e}^{-p_1-\frac{p_2}{2}}\left(\mathrm{d}x^1\right)^2 + \mathrm{e}^{p_1-\frac{p_2}{2}}\left(\mathrm{d}x^2 + A^1_2\mathrm{d}x^1\right)^2 \\ &+ \mathrm{e}^{p_2}\left(\mathrm{d}x^3 + A^1_3\mathrm{d}x^1 + A^2_3\mathrm{d}x^2\right)^2\Big],\end{aligned}$$

where we have absorbed a factor proportional to $\ln t$ in the definition of λ.

We can now carry out this procedure to obtain the metric in arbitrary n dimensions. We will show, by induction, that the metric has the following expression:

$$\begin{aligned}\mathrm{d}s^2_{(n)} = t^{-\frac{N-1}{N}}\mathrm{e}^{\frac{\lambda}{2}}\left(-\mathrm{d}t^2 + \mathrm{d}\theta^2\right) + t^{\frac{2}{N}}\Bigg[&\exp\left(-p_1 - \sum_{i=2}^{N-1}\frac{p_i}{i}\right)\left(\mathrm{d}x^1\right)^2 \\ &+ \sum_{m=2}^{N}\exp\left(p_{m-1} - \sum_{i=m}^{N-1}\frac{p_i}{i}\right)\left(\mathrm{d}x^m + \sum_{i=1}^{m-1}A^i_m\mathrm{d}x^i\right)^2\Bigg],\end{aligned}$$

where $N := n-2$, and we have used the convention that, whenever in a sum the lower limit is greater than the upper one, the term is equal to zero. To show that this formula holds, let us compute $\mathrm{d}s^2_{(n+1)}$ using Kaluza-Klein decomposition:

$$\mathrm{d}s^2_{(n+1)} = \mathrm{e}^{2\alpha\varphi}\mathrm{d}s^2_{(n)} + \mathrm{e}^{2\beta\varphi}\left(\mathrm{d}x^{N+1} + \sum_{i=1}^{N} A^i_{N+1}\mathrm{d}x^i\right)^2,$$

with $\beta = -(n-2)\,\alpha = -N\alpha$. Denoting $p_N := 2\beta\varphi$, we can write

$$\begin{aligned}\mathrm{d}s^2_{(n+1)} = {} & t^{-\frac{N-1}{N}}\mathrm{e}^{-\frac{p_N}{N}+\frac{\lambda}{2}}\left(-\mathrm{d}t^2+\mathrm{d}\theta^2\right) + t^{\frac{2}{N}}\left[\exp\left(-p_1-\sum_{i=2}^{N}\frac{p_i}{i}\right)\left(\mathrm{d}x^1\right)^2\right. \\ & \left. + \sum_{m=2}^{N}\exp\left(p_{m-1}-\sum_{i=m}^{N}\frac{p_i}{i}\right)\left(\mathrm{d}x^m+\sum_{i=1}^{m-1}A^i_m\mathrm{d}x^i\right)^2\right] \\ & + \mathrm{e}^{p_N}\left(\mathrm{d}x^{N+1}+\sum_{i=1}^{N}A^i_{N+1}\mathrm{d}x^i\right)^2.\end{aligned}$$

Making the transformations

$$\begin{aligned}\lambda &\to \lambda - \frac{2p_N}{N} + \frac{1}{N\,(N+1)}\ln t, \\ p_N &\to p_N - \frac{2}{N+1}\ln t,\end{aligned}$$

we obtain the desired result,

$$\begin{aligned}\mathrm{d}s^2_{(n+1)} = {} & t^{-\frac{N}{N+1}}\mathrm{e}^{\frac{\lambda}{2}}\left(-\mathrm{d}t^2+\mathrm{d}\theta^2\right) + t^{\frac{2}{N+1}}\left[\exp\left(-p_1-\sum_{i=2}^{N}\frac{p_i}{i}\right)\left(\mathrm{d}x^1\right)^2\right. \\ & \left. + \sum_{m=2}^{N+1}\exp\left(p_{m-1}-\sum_{i=m}^{N}\frac{p_i}{i}\right)\left(\mathrm{d}x^m+\sum_{i=1}^{m-1}A^i_m\mathrm{d}x^i\right)^2\right],\end{aligned}$$

which proves the claim. We note that $\dfrac{\partial}{\partial x^i}$ are Killing vectors and the space has $\mathbb{T}^{N+1}$ topology. To simplify notation, let us define the following quantities in n-dimensions, with $N = n-2$:

$$\begin{aligned}\alpha_1 &:= \frac{1}{2}\left(-p_1-\sum_{i=2}^{N-1}\frac{p_i}{i}\right) \\ \alpha_m &:= \frac{1}{2}\left(p_{m-1}-\sum_{i=m}^{N-1}\frac{p_i}{i}\right),\ \text{for } 2\le m\le N.\end{aligned} \tag{8}$$

Note that $\sum_{m=1}^{N} \alpha_m = 0$. With these definitions, we can write

$$\begin{aligned}
\mathrm{d}s^2_{(n)} &= t^{-\frac{N-1}{N}} \mathrm{e}^{\frac{\lambda}{2}} \left(-\mathrm{d}t^2 + \mathrm{d}\theta^2\right) \\
&\quad + t^{\frac{2}{N}} \left[\mathrm{e}^{2\alpha_1} \left(\mathrm{d}x^1\right)^2 + \sum_{m=2}^{N} \mathrm{e}^{2\alpha_m} \left(\mathrm{d}x^m + \sum_{i=1}^{m-1} A^i_m \mathrm{d}x^i \right)^2 \right] \\
&= t^{-\frac{N-1}{N}} \mathrm{e}^{\frac{\lambda}{2}} \left(-\mathrm{d}t^2 + \mathrm{d}\theta^2\right) + t^{\frac{2}{N}} \tilde{g}_{ab} \mathrm{d}x^a \mathrm{d}x^b,
\end{aligned} \tag{9}$$

where $\tilde{g}$ is the metric on the orbits of the symmetry group $\mathbb{T}^N$, with $\det \tilde{g} = \prod_{m=1}^{N} \mathrm{e}^{\alpha_m} = \exp\left(\sum_{m=1}^{N} \alpha_m\right) = 1$. Let us now define another local 1-form basis as follows:

$$\omega^m = \mathrm{e}^{\alpha_m} \left(\mathrm{d}x^m + \sum_{i=1}^{m-1} A^i_m \mathrm{d}x^i \right),$$

where we use the aforementioned convention that sums where the upper limit is smaller than the lower one give zero. In this basis, the metric $\tilde{g}$ on $\mathbb{T}^N$ is equal to the Kronecker delta: $\tilde{g}_{ab} = \delta_{ab}$. Let M be the change of basis matrix from $\mathrm{d}x$ to ω. Then

$$M_{ij} = \begin{cases} \mathrm{e}^{\alpha_i}, & \text{if } i = j \\ \mathrm{e}^{\alpha_i} A^j_i, & \text{if } i > j \\ 0, & \text{otherwise,} \end{cases} \tag{10}$$

and the inverse M^{-1} is given by

$$M^{-1}_{ij} = \begin{cases} \mathrm{e}^{-\alpha_i}, & \text{if } i = j \\ \mathrm{e}^{-\alpha_j} \sum_{l=1}^{i-j} (-1)^l \sum_{i=k_0>k_1>\ldots>k_l=j} A^{k_1}_{k_0} \ldots A^{k_l}_{k_{l-1}}, & \text{if } i > j \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

Let us check that, indeed, $\sum_j M_{ij} M^{-1}_{jk} = \delta_{ik}$. Using the notation

$$S^k_j := \sum_{l=1}^{j-k} (-1)^l \sum_{j=k_0>\ldots>k_l=k} A^{k_1}_{k_0} \ldots A^{k_l}_{k_{l-1}}, \quad \text{for } N \geq j > k \geq 1, \tag{12}$$

for $i > k$, we can write,

$$\begin{aligned}
\sum_j M_{ij} M^{-1}_{jk} &= M_{ii} M^{-1}_{ik} + M_{ik} M^{-1}_{kk} + \sum_{i>j>k} M_{ij} M^{-1}_{jk} \\
&= S_i^k + A_i^k + \sum_{i>j>k} A_i^j S_j^k \\
&= S_i^k - S_i^k \\
&= 0.
\end{aligned}$$

For $i = k$, we have $\sum_j M_{ij} M^{-1}_{ji} = M_{ii} M^{-1}_{ii} = 1$, as claimed. We note, for later use, that the metric $\tilde{g}_{ab}$ and its inverse $\tilde{g}^{ab}$ in the x-coordinates are given by

$$\begin{aligned}
\tilde{g}_{ab} &= \sum_{m,n} M_{ma} M_{nb} \delta_{mn} = \sum_m M_{ma} M_{mb} \\
\tilde{g}^{ab} &= \sum_{m,n} M^{-1}_{am} M^{-1}_{bn} \delta_{mn} = \sum_m M^{-1}_{am} M^{-1}_{bm}.
\end{aligned} \tag{13}$$

We now wish to derive the form of the target space metric in terms of the coordinates p_i and A_i^j. To do that, we will use dimensional reduction from n to $n-2$ dimensions of the lagrangian of the n-dimensional gravitational theory, as performed in [7]. We start by defining the n-bein

$$E_B^A = \begin{pmatrix} \lambda \hat{e}^a_b & 0 \\ 0 & \bar{e}^\alpha_\beta \end{pmatrix},$$

in $n = N+2$ dimensional coordinates $X^A = (\hat{x}^a, \bar{x}^\alpha)$, with $\hat{x}^a = (t, \theta)$ labeling the orbits of the symmetry group $\mathbb{T}^N$ and $\bar{x}^\alpha = (x^1, \ldots, x^N)$ coordinates on $\mathbb{T}^N$. $\hat{e}^a_b$ is a 2-bein on the orbit space, while $\bar{e}^\alpha_\beta$ is an N-bein on the orbits, so $\bar{g} = \bar{e}^T \delta \bar{e}$, with δ being the Kronecker-delta, is the metric on the orbits of $\mathbb{T}^N$; λ is a conformal factor and, just as $\bar{g}$, it depends only on the coordinates (t, θ). We now want to compute the n-dimensional gravitational lagrangian in terms of the lower dimensional fields

$$\mathcal{L}^{(n)} = -\frac{1}{2} E R = -\frac{1}{2} \det\left(E_B^A\right) R = -\frac{1}{2} \lambda^2 \hat{e} \rho R$$

where $\rho = \det(\bar{e})$ and R is the scalar curvature of E_B^A. To compute R, we need to know the Christoffel symbols associated to E_B^A. If we denote $\hat{h}_{mn} = \eta_{ab} \hat{e}^a_m \hat{e}^b_n$ and $\hat{\Gamma}^m_{kl}$ are the Christoffel symbols associated to $\hat{e}^a_b$, the only nonvanishing connection coefficients can be written as

$$\begin{aligned}
\Gamma^m_{kl} &= \hat{\Gamma}^m_{kl} + \lambda^{-1} \left(\delta^m_k \partial_l \lambda + \delta^m_l \partial_k \lambda - \hat{h}^{mn} \hat{h}_{kl} \partial_n \lambda \right), \\
\Gamma^m_{\alpha\beta} &= -\frac{1}{2} \lambda^{-2} \hat{h}^{mn} \bar{g}_{\alpha\beta,m}, \\
\Gamma^\alpha_{m\beta} &= \frac{1}{2} \bar{g}^{\alpha\gamma} \bar{g}_{\beta\gamma,m}.
\end{aligned}$$

Since

$$\begin{aligned}R = \lambda^{-2}\hat{h}^{kl}&\left(\Gamma^m_{kl,m} - \Gamma^m_{km,l} + \Gamma^m_{kl}\Gamma^n_{mn} - \Gamma^n_{km}\Gamma^m_{ln}\right)\\ &+ \bar{g}^{\alpha\beta}\left(\Gamma^m_{\alpha\beta,m} + \Gamma^m_{\alpha\beta}\Gamma^n_{mn} + \Gamma^m_{\alpha\beta}\Gamma^\gamma_{m\gamma} - \Gamma^\gamma_{m\alpha}\Gamma^m_{\gamma\beta} - \Gamma^m_{\gamma\alpha}\Gamma^\gamma_{m\beta}\right)\\ &+ \lambda^{-2}\hat{h}^{kl}\left(-\Gamma^\alpha_{k\alpha,l} - \Gamma^\beta_{k\alpha}\Gamma^\alpha_{l\beta} + \Gamma^m_{kl}\Gamma^\alpha_{\alpha m}\right),\end{aligned}$$

we get, after some calculations,

$$\begin{aligned}R = \lambda^{-2}&\left[\hat{R} - \frac{1}{2}\hat{h}^{mn}\mathrm{Tr}\left(\bar{g}^{-1}\partial_m\bar{g}\right)\mathrm{Tr}\left(\hat{h}^{-1}\partial_n\hat{h}\right) - \frac{1}{4}\hat{h}^{mn}\mathrm{Tr}\left(\bar{g}^{-1}\partial_m\bar{g}\right)\mathrm{Tr}\left(\bar{g}^{-1}\partial_n\bar{g}\right)\right.\\ &\left.+\frac{3}{8}\hat{h}^{mn}\mathrm{Tr}\left(\bar{g}^{-1}\partial_m\bar{g}\bar{g}^{-1}\partial_n\bar{g}\right) - \hat{h}^{mn}\mathrm{Tr}\left(\bar{g}^{-1}\partial_{mn}\bar{g}\right) - \partial_m\hat{h}^{mn}\mathrm{Tr}\left(\bar{g}^{-1}\partial_n\bar{g}\right)\right]\\ &- \lambda^{-3}\left\{\partial_n\lambda\left[2\partial_m\hat{h}^{mn} + \hat{h}^{mn}\mathrm{Tr}\left(\hat{h}^{-1}\partial_m\hat{h}\right)\right] + 2\hat{h}^{mn}\partial_{mn}\lambda\right\}\\ &+ 2\lambda^{-4}\hat{h}^{mn}\partial_m\lambda\partial_n\lambda,\end{aligned}$$

where $\hat{R}$ is the scalar curvature of $\hat{h}$. Hence, the lagrangian can be written as

$$\begin{aligned}\mathcal{L}^{(n)} = -\frac{1}{2}\lambda^2\hat{e}\rho = &-\frac{1}{2}\hat{e}\rho\left[\hat{R} - 2\partial_m\left(\hat{h}^{mn}\lambda^{-1}\partial_n\lambda\right) - \hat{h}^{mn}\lambda^{-1}\partial_n\lambda\mathrm{Tr}\left(\hat{h}^{-1}\partial_m\hat{h}\right)\right.\\ &-\frac{1}{2}\hat{h}^{mn}\mathrm{Tr}\left(\bar{g}^{-1}\partial_m\bar{g}\right)\mathrm{Tr}\left(\hat{h}^{-1}\partial_n\hat{h}\right)\\ &-\frac{1}{4}\hat{h}^{mn}\mathrm{Tr}\left(\bar{g}^{-1}\partial_m\bar{g}\right)\mathrm{Tr}\left(\bar{g}^{-1}\partial_n\bar{g}\right)\\ &+\frac{3}{8}\hat{h}^{mn}\mathrm{Tr}\left(\bar{g}^{-1}\partial_m\bar{g}\bar{g}^{-1}\partial_n\bar{g}\right) - \hat{h}^{mn}\mathrm{Tr}\left(\bar{g}^{-1}\partial_{mn}\bar{g}\right)\\ &\left.-\partial_m\hat{h}^{mn}\mathrm{Tr}\left(\bar{g}^{-1}\partial_n\bar{g}\right)\right].\end{aligned}$$

Since for a metric g, we have

$$\partial_m\left(\sqrt{\det g}\right) = \frac{1}{2}\sqrt{\det g}\,\mathrm{Tr}\left(g^{-1}\partial_m g\right),$$

it follows that

$$\begin{aligned}\hat{e}\rho\partial_m\left(\hat{h}^{mn}\lambda^{-1}\partial_n\lambda\right) = \partial_m\left(\hat{e}\rho\hat{h}^{mn}\lambda^{-1}\partial_n\lambda\right) - \frac{1}{2}\hat{e}\rho\hat{h}^{mn}\lambda^{-1}\partial_n\lambda\mathrm{Tr}\left(\hat{h}^{-1}\partial_m\hat{h}\right)\\ - \hat{e}\hat{h}^{mn}\lambda^{-1}\partial_n\lambda\partial_m\rho\end{aligned}$$

and

$$\begin{aligned}\hat{e}\rho\mathrm{Tr}\left(\bar{g}^{-1}\partial_n\bar{g}\right)\partial_m\hat{h}^{mn} = \partial_m&\left[\hat{e}\rho\mathrm{Tr}\left(\bar{g}^{-1}\partial_n\bar{g}\right)\hat{h}^{mn}\right]\\ &-\frac{1}{2}\hat{e}\rho\hat{h}^{mn}\mathrm{Tr}\left(\bar{g}^{-1}\partial_n\bar{g}\right)\mathrm{Tr}\left(\hat{h}^{-1}\partial_m\hat{h}\right)\\ &-\hat{e}\hat{h}^{mn}\mathrm{Tr}\left(\bar{g}^{-1}\partial_n\bar{g}\right)\partial_m\rho + \hat{e}\rho\hat{h}^{mn}\mathrm{Tr}\left(\bar{g}^{-1}\partial_m\bar{g}\bar{g}^{-1}\partial_n\bar{g}\right)\\ &-\hat{e}\rho\hat{h}^{mn}\mathrm{Tr}\left(\bar{g}^{-1}\partial_{mn}\bar{g}\right).\end{aligned}$$

Thus, up to total derivatives, the lagrangian is given by

$$\mathcal{L}^{(n)} = \hat{e}\rho \left\{ -\frac{1}{2}\hat{R} \right. \\ \left. + \frac{1}{8}\hat{h}^{mn}\left[\mathrm{Tr}\left(\bar{g}^{-1}\partial_m\bar{g}\bar{g}^{-1}\partial_n\bar{g}\right) - 4\rho^{-2}\partial_m\rho\partial_n\rho - 8\lambda^{-1}\partial_n\lambda\rho^{-1}\partial_m\rho\right] \right\}.$$

If we now introduce the unit determinant metric on the orbits, $\tilde{g}_{\alpha\beta} = \rho^{-\frac{2}{N}}\bar{g}_{\alpha\beta}$, we obtain

$$\mathcal{L}^{(n)} = \hat{e}\rho \left\{ -\frac{1}{2}\hat{R} + \frac{1}{8}\hat{h}^{mn}\left[\mathrm{Tr}\left(\tilde{g}^{-1}\partial_m\tilde{g}\tilde{g}^{-1}\partial_n\tilde{g}\right) \right.\right. \\ \left.\left. - \frac{4(N-1)}{N}\rho^{-2}\partial_m\rho\partial_n\rho - 8\lambda^{-1}\partial_n\lambda\rho^{-1}\partial_m\rho\right]\right\}.$$

We can finally rescale the conformal factor ,$\lambda \to \rho^{\frac{N-1}{2N}}\lambda$, to arrive at the result

$$\mathcal{L}^{(n)} = \hat{e}\rho\left[-\frac{1}{2}\hat{R} + \frac{1}{8}\mathrm{Tr}\left(\tilde{g}^{-1}\partial\tilde{g}\tilde{g}^{-1}\partial\tilde{g}\right) - \lambda^{-1}\partial\lambda\rho^{-1}\partial\rho\right] \tag{14}$$

Let us now write down the Euler-Lagrange equations for this lagrangian. For the components of the metric $\hat{h}$ the resulting equations are

$$\hat{R}_{kl} - \frac{1}{2}\hat{h}_{kl}\hat{R} = \frac{1}{4}\mathrm{Tr}\left(\tilde{g}^{-1}\partial_k\tilde{g}\tilde{g}^{-1}\partial_l\tilde{g}\right) - 2\lambda^{-1}\partial_{(k}\lambda\rho^{-1}\partial_{l)}\rho \\ - \hat{h}_{kl}\left[\frac{1}{8}\mathrm{Tr}\left(\tilde{g}^{-1}\partial\tilde{g}\tilde{g}^{-1}\partial\tilde{g}\right) - \lambda^{-1}\partial\lambda\rho^{-1}\partial\rho\right]$$

where we have used the relation

$$\delta\sqrt{-\det g} = -\frac{1}{2}\sqrt{-\det g}g_{kl}\delta g^{kl},$$

valid for a metric g, and the parentheses denote symmetrization over the indices. We note that we can write $\hat{h}_{kl} = \eta_{kl}$ by suitably choosing the coordinates on the orbit space and the conformal factor λ. With this choice and because $\rho = \sqrt{\det\bar{g}} = t$, the Euler-Lagrange equations for $\hat{h}_{kl}$ become

$$\lambda^{-1}\partial_t\lambda = \frac{t}{8}\mathrm{Tr}\left[\left(\tilde{g}^{-1}\partial_t\tilde{g}\right)^2 + \left(\tilde{g}^{-1}\partial_\theta\tilde{g}\right)^2\right],$$
$$\lambda^{-1}\partial_t\lambda = \frac{t}{8}\mathrm{Tr}\left(\tilde{g}^{-1}\partial_\theta\tilde{g}\tilde{g}^{-1}\partial_t\tilde{g}\right).$$

Using the relation

$$\delta g^{\zeta\xi} = -g^{\zeta\alpha}g^{\xi\beta}\delta g_{\alpha\beta},$$

valid for any metric g, we can derive the Euler-Lagrange equations for the components $\tilde{g}_{\alpha\beta}$:

$$\hat{h}_{kl}\partial_k\left(\rho\tilde{g}^{-1}\partial_l\tilde{g}\right) = 0,$$

or, with our particular choices,

$$-\partial_t\left(t\tilde{g}^{-1}\partial_t\tilde{g}\right)+t\partial_\theta\left(\tilde{g}^{-1}\partial_\theta\tilde{g}\right)=0. \tag{15}$$

As was pointed out in [7], (14) represents the lagrangian of a generalized $SL(N)/SO(N)$ nonlinear σ-model. This will become more apparent if we write down the lagrangian in terms of the parametrizing fields, p_i and A_i^j. There are a number of $N-1+\dfrac{N\left(N-1\right)}{2}=\dfrac{N^2+N-2}{2}$ parametrizing fields which is equal to the dimension of $SL(N)/SO(N)$: $N^2-1-\dfrac{N\left(N-1\right)}{2}=\dfrac{N^2+N-2}{2}$. To understand how this happens, let us note that the unit-determinant metric $\tilde{g}$ on the orbits of $\mathbb{T}^N$ is an element of $SL(N)/SO(N)$, since $SO(N)$ corresponds to a rotation of the coordinate basis at a point on the orbits of $\mathbb{T}^N$ in our Gowdy spacetime, which leaves the metric $\tilde{g}$ invariant. We also remark that we have first dimensionally reduced the lagrangian using the symmetries of the model and then wrote down the Euler-Lagrange equations of the resulting lagrangian. That we are allowed to procede in this way is a consequence of the so-called principle of symmetric criticality ([8]). This principle guarantees that the Euler-Lagrange equations of the symmetry-reduced lagrangian are the same as the symmetry-reduced Euler-Lagrange equations of the unreduced lagrangian, provided that the group used to reduce the lagrangian is compact. This is fortunately the case of $SO(N)$.

We note that (15) which are the evolution equations of the metric (9), are the wave-map equations of the map $\tilde{g}$ from the Minkowski-like space $V=\mathbb{R}_+\times\mathbb{T}^1$ with metric

$$\mathrm{d}s_0^2=-\mathrm{d}t^2+\mathrm{d}\theta^2+t^2\mathrm{d}\varphi^2,$$

to the space $\widetilde{M}=SL(N)/SO(N)$ with metric

$$\mathrm{d}s_1^2=\frac{1}{2}\mathrm{Tr}\left(\tilde{g}^{-1}\mathrm{d}\tilde{g}\tilde{g}^{-1}\mathrm{d}\tilde{g}\right) \tag{16}$$

with $\tilde{g}$ independent of the variable φ. To see that, recall that a map u from a Lorentzian manifold V with metric g, to a Riemannian manifold M with metric h is by definition a wave-map, iff it is a critical point of the action:

$$\int\limits_V g^{ij}\partial_i u^A\partial_j u^B h_{AB}\mathrm{d}v.$$

In our case, this translates to the requirement that $\tilde{g}$ be a critical point of the action

$$\int\limits_V\frac{1}{2}\mathrm{Tr}\left[-\left(\tilde{g}^{-1}\partial_t\tilde{g}\right)^2+\left(\tilde{g}^{-1}\partial_\theta\tilde{g}\right)^2\right]2t\mathrm{d}\varphi\mathrm{d}t\mathrm{d}\theta$$
$$=C\int\limits_{\mathbb{S}^1}\frac{t}{4}\mathrm{Tr}\left[-\left(\tilde{g}^{-1}\partial_t\tilde{g}\right)^2+\left(\tilde{g}^{-1}\partial_\theta\tilde{g}\right)^2\right]\mathrm{d}t\mathrm{d}\theta,$$

since the volume element of V is $\mathrm{d}v = 2t\mathrm{d}\varphi\mathrm{d}t\mathrm{d}\theta$. This, in turn, follows immediately from the fact that (15) are the Euler-Lagrange equations of the lagrangian

$$\mathcal{L} = \frac{t}{4}\mathrm{Tr}\left[-\left(\tilde{g}^{-1}\partial_t\tilde{g}\right)^2 + \left(\tilde{g}^{-1}\partial_\theta\tilde{g}\right)^2\right]. \tag{17}$$

In the following, we will only be concerned with the nontrivial part of the lagrangian which, after rescaling, we can write in the form (17). We note that the lagrangian (17) and the target space metric (16) have a very similar form, so it suffices to compute the target-space metric in our particular parametrization. To do that, let us first define the step function $\sigma : \mathbb{R} \to \mathbb{R}$,

$$\sigma(x) := \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Using (12), we can then write (10) and (11) in the following form:

$$\begin{aligned} M_{ij} &= \mathrm{e}^{\alpha_i}\left[\delta_{ij} + (1-\delta_{ij})\,A_i^j\right]\sigma(i-j), \\ M_{ij}^{-1} &= \mathrm{e}^{-\alpha_j}\left[\delta_{ij} + (1-\delta_{ij})\,S_i^j\right]\sigma(i-j). \end{aligned}$$

Using (13), we can now compute the target-space metric as follows (all indices in the sums go from 1 to N, unless otherwise specified):

$$\begin{aligned} \mathrm{d}s_1^2 &= \frac{1}{2}\mathrm{Tr}\left[\left(\tilde{g}^{-1}\mathrm{d}\tilde{g}\right)^2\right] \\ &= \mathrm{Tr}\left\{M^{-1}\left[(\mathrm{d}M)\,M^{-1} + \left(M^{-1}\right)^T\mathrm{d}M^T\right]\mathrm{d}M\right\} \\ &= \sum_{a,b,c,d} M_{ab}^{-1}\left[\mathrm{d}M_{bc}M_{cd}^{-1} + M_{cb}^{-1}\mathrm{d}M_{dc}\right]\mathrm{d}M_{da} \\ &= \sum_{a,b,c,d} \mathrm{e}^{-\alpha_b}\left[\delta_{ab} + (1-\delta_{ab})\,S_a^b\right]\sigma(a-b)\cdot\{\mathrm{d}\left[\mathrm{e}^{\alpha_b}\left(\delta_{bc} + (1-\delta_{bc})\,A_b^c\right)\right]\sigma(b-c) \\ &\quad \cdot\mathrm{e}^{-\alpha_d}\left[\delta_{dc} + (1-\delta_{dc})\,S_c^d\right]\sigma(c-d) + \mathrm{e}^{-\alpha_b}\left[\delta_{cb} + (1-\delta_{cb})\,S_c^b\right]\sigma(c-b) \\ &\quad \cdot\mathrm{d}\left[\mathrm{e}^{\alpha_d}\left(\delta_{dc} + (1-\delta_{dc})\,A_d^c\right)\right]\sigma(d-c)\}\cdot\mathrm{d}\left\{\mathrm{e}^{\alpha_d}\left[\delta_{da} + (1-\delta_{da})\,A_d^a\right]\right\}\sigma(d-a) \\ &= \sum_i \mathrm{d}\alpha_i^2 + \sum_{a,b,c,d}\mathrm{e}^{-\alpha_b}\left[\delta_{ab} + (1-\delta_{ab})\,S_a^b\right]\sigma(a-b)\mathrm{e}^{-\alpha_b}\left[\delta_{cb} + (1-\delta_{cb})\,S_c^b\right] \\ &\quad \cdot\ \sigma(c-b)\mathrm{d}\left\{\mathrm{e}^{\alpha_d}\left[\delta_{dc} + (1-\delta_{dc})\,A_d^c\right]\right\}\sigma(d-c)\mathrm{d}\left\{\mathrm{e}^{\alpha_d}\left[\delta_{da} + (1-\delta_{da})\,A_d^a\right]\right\} \\ &\quad \cdot\ \sigma(d-a) \end{aligned}$$

where, in the last equality, we have used the fact that terms which contain the product $\sigma(a-b)\sigma(b-c)\sigma(c-d)\sigma(d-a)$ vanish, unless $a = b = c = d$. Computing further, we obtain

$$
\begin{aligned}
\mathrm{d}s_1^2 &= 2\sum_i \mathrm{d}\alpha_i^2 + \sum_{d>a}\left[\mathrm{d}\left(\mathrm{e}^{\alpha_d}A_d^a\right) + S_d^a \mathrm{d}\mathrm{e}^{\alpha_d}\right]^2 \mathrm{e}^{-\alpha_a} \\
&\quad + 2\sum_{d>a>b}\left[S_a^b \mathrm{d}\left(\mathrm{e}^{\alpha_d}A_d^a\right)\mathrm{d}\left(\mathrm{e}^{\alpha_d}A_d^b\right) + S_a^b S_d^b \mathrm{d}\left(\mathrm{e}^{\alpha_d}A_d^a\right)\mathrm{d}\mathrm{e}^{\alpha_d}\right]\mathrm{e}^{-2\alpha_b} \\
&\quad + \sum_{d>a,c>b} S_a^b S_c^b \mathrm{d}\left(\mathrm{e}^{\alpha_d}A_d^c\right)\mathrm{d}\left(\mathrm{e}^{\alpha_d}A_d^a\right)\mathrm{e}^{-2\alpha_b} \\
&= 2\sum_i \mathrm{d}\alpha_i^2 + \sum_{d>a}\mathrm{e}^{2(\alpha_d-\alpha_a)}\left(\mathrm{d}A_d^a\right)^2 + 2\sum_{d>a>b}\mathrm{e}^{2(\alpha_d-\alpha_b)}S_a^b \mathrm{d}A_d^a \mathrm{d}A_d^b \\
&\quad + \sum_{d>a,c>b}\mathrm{e}^{2(\alpha_d-\alpha_b)}S_a^b S_c^b \mathrm{d}A_d^c \mathrm{d}A_d^a + \sum_d\left[E_d\left(\mathrm{d}\mathrm{e}^{\alpha_d}\right)^2 + F_d \mathrm{e}^{\alpha_d}\mathrm{d}\mathrm{e}^{\alpha_d}\right],
\end{aligned}
$$

where we have denoted

$$
\begin{aligned}
E_d &:= \sum_{a=1}^{d-1}\mathrm{e}^{-2\alpha_a}\left[\left(A_d^a\right)^2 + 2A_d^a S_d^a + \left(S_d^a\right)^2\right] + 2\sum_{a=1}^{d-1}\sum_{b=1}^{a-1}\mathrm{e}^{-2\alpha_b}A_d^a S_a^b\left(A_d^b + S_d^b\right) \\
&\quad + \sum_{d>a,c>b}\mathrm{e}^{-2\alpha_b}A_d^c A_d^a S_a^b S_c^b \\
&= \sum_{a=1}^{d-1}\mathrm{e}^{-2\alpha_a}\left(S_d^a + A_d^a\right)^2 - 2\sum_{a=1}^{d-1}\mathrm{e}^{-2\alpha_a}\left(S_d^a + A_d^a\right)^2 + \sum_{a=1}^{d-1}\mathrm{e}^{-2\alpha_a}\left(S_d^a + A_d^a\right)^2 = 0
\end{aligned}
$$

and

$$
\begin{aligned}
F_d &= 2\sum_{a=1}^{d-1}\left(S_d^a \mathrm{d}A_d^a + A_d^a \mathrm{d}A_d^a\right) + 2\sum_{a=1}^{d-1}\sum_{b=1}^{a-1}\left(A_d^a S_a^b \mathrm{d}A_d^b + A_d^b S_a^b \mathrm{d}A_d^a + S_a^b S_d^b \mathrm{d}A_d^a\right) \\
&\quad + \sum_{d>a,c>b}\left(A_d^c \mathrm{d}A_d^a + A_d^a \mathrm{d}A_d^c\right)S_a^b S_c^b \\
&= 2\sum_{a=1}^{d-1}\left(S_d^a + A_d^a\right)\mathrm{d}A_d^a - 2\sum_{a=1}^{d-1}\left(S_d^a + A_d^a\right)\mathrm{d}A_d^a + 2\sum_{a=1}^{d-1}\sum_{b=1}^{a-1}\left(S_d^b + A_d^b\right)S_a^b \mathrm{d}A_d^a \\
&\quad - 2\sum_{a=1}^{d-1}\sum_{b=1}^{a-1}\left(S_d^b + A_d^b\right)S_a^b \mathrm{d}A_d^a \;=\; 0.
\end{aligned}
$$

We have used above the fact that

$$
\sum_{a=1}^{d-1}\sum_{b=1}^{a-1}\mathrm{e}^{-2\alpha_b}A_d^a S_a^b = -\sum_{b=1}^{d-2}\mathrm{e}^{-2\alpha_b}\left(S_d^b + A_d^b\right) = -\sum_{b=1}^{d-1}\mathrm{e}^{-2\alpha_b}\left(S_d^b + A_d^b\right),
$$

since $S_d^{d-1} = -A_d^{d-1}$. Hence, we obtain

$$
\begin{aligned}
\mathrm{d}s_1^2 =& 2\sum_i \mathrm{d}\alpha_i^2 + \sum_{d>a} \mathrm{e}^{2(\alpha_d-\alpha_a)}\left(\mathrm{d}A_d^a\right)^2 + 2\sum_{d>a>b} \mathrm{e}^{2(\alpha_d-\alpha_b)} S_a^b \mathrm{d}A_d^a \mathrm{d}A_d^b \\
&+ \sum_{d>a,c>b} \mathrm{e}^{2(\alpha_d-\alpha_b)} S_a^b S_c^b \mathrm{d}A_d^a \mathrm{d}A_d^c
\end{aligned}
$$

which we can compactly write in the following manner:

$$
\mathrm{d}s_1^2 = 2\sum_i \mathrm{d}\alpha_i^2 + \sum_{i>j} \mathrm{e}^{2(\alpha_i-\alpha_j)}\left(\mathrm{d}A_i^j + \sum_{k=j+1}^{i-1} S_k^j \mathrm{d}A_i^k\right)^2
$$

As mentioned before, we use the convention that sums where the lower limit is greater than the upper one vanish. Let us note that, using (8), we can write

$$
\sum_{i=1}^{N} \mathrm{d}\alpha_i^2 = \frac{1}{4}\sum_{i=1}^{N-1} \frac{i+1}{i}\mathrm{d}p_i^2.
$$

If we make the transformation $p_i \to \sqrt{\frac{i+1}{2i}}p_i$, then the αs are accordingly given by the formulae

$$
\begin{aligned}
\alpha_1 &:= \frac{1}{2}\left(-p_1 - \sum_{i=2}^{N-1}\sqrt{\frac{2}{i\,(i+1)}}p_i\right) \\
\alpha_m &:= \frac{1}{2}\left(\sqrt{\frac{2\,(m-1)}{m}}p_{m-1} - \sum_{i=m}^{N-1}\sqrt{\frac{2}{i\,(i+1)}}p_i\right), \text{ for } 2 \leq m \leq N.
\end{aligned} \tag{18}
$$

The target space metric can then be written in the following form

$$
\mathrm{d}s_1^2 = \sum_{i=1}^{N-1} \mathrm{d}p_i^2 + \sum_{i=2}^{N}\sum_{j=1}^{i-1} \mathrm{e}^{2(\alpha_i-\alpha_j)}\left(\mathrm{d}A_i^j + \sum_{k=j+1}^{i-1} S_k^j \mathrm{d}A_i^k\right)^2 \tag{19}
$$

To illustrate this result, let us write down the target space metrics in four, five and six spacetime dimensions. For $n = 4$ ($N = 2$), using (19) and (18), we obtain

$$
\mathrm{d}s_{(2)}^2 = \mathrm{d}p_1^2 + \mathrm{e}^{2p_1}\left(\mathrm{d}A_2^1\right)^2.
$$

For $n = 5$ ($N = 3$), we have

$$
\mathrm{d}s_{(3)}^2 = \mathrm{d}p_1^2 + \mathrm{d}p_2^2 + \mathrm{e}^{2p_1}\left(\mathrm{d}A_2^1\right)^2 + \mathrm{e}^{p_1+\sqrt{3}p_2}\left(\mathrm{d}A_3^1 - A_2^1\mathrm{d}A_3^2\right)^2 + \mathrm{e}^{-p_1+\sqrt{3}p_2}\left(\mathrm{d}A_3^2\right)^2
$$

Finally, for $n = 6$ ($N = 4$), the target-space metric can be written as

$$\begin{aligned}
\mathrm{d}s^2_{(4)} = {} & \mathrm{d}p_1^2 + \mathrm{d}p_2^2 + \mathrm{d}p_3^2 + \mathrm{e}^{2p_1}\left(\mathrm{d}A_2^1\right)^2 + \mathrm{e}^{p_1+\sqrt{3}p_2}\left(\mathrm{d}A_3^1 - A_2^1\mathrm{d}A_3^2\right)^2 \\
& + \mathrm{e}^{-p_1+\sqrt{3}p_2}\left(\mathrm{d}A_3^2\right)^2 \\
& + \mathrm{e}^{p_1+p_2/\sqrt{3}+4p_3/\sqrt{6}}\left[\mathrm{d}A_4^1 - A_2^1\mathrm{d}A_4^2 + \left(-A_3^1 + A_3^2A_2^1\right)\mathrm{d}A_4^3\right]^2 \\
& + \mathrm{e}^{-p_1+p_2/\sqrt{3}+4p_3/\sqrt{6}}\left(\mathrm{d}A_4^2 - A_3^2\mathrm{d}A_4^3\right)^2 \\
& + \mathrm{e}^{-2p_2/\sqrt{3}+4p_3/\sqrt{6}}\left(\mathrm{d}A_4^3\right)^2 .
\end{aligned}$$

As expected, $\mathrm{d}s^2_{(N+1)} = \mathrm{d}s^2_{(N)}$, if one requires that the fields p_{N-1} and A_N^i vanish.

To prove global existence of solutions to (15), one can use the argument in [4], adapted to our problem, with the following notations.

$$\begin{aligned}
\mathcal{A} &:= \frac{t}{4}\mathrm{Tr}\left[\left(\tilde{g}^{-1}\partial_t\tilde{g}^2 + \tilde{g}^{-1}\partial_\theta\tilde{g}\right)^2\right] \\
\mathcal{B} &:= \frac{t}{4}\mathrm{Tr}\left[\left(\tilde{g}^{-1}\partial_t\tilde{g}^2 - \tilde{g}^{-1}\partial_\theta\tilde{g}\right)^2\right]
\end{aligned}$$

Using the Euler-Lagrange equations (15), the cyclicity of trace and the relation $\partial_x\tilde{g}^{-1} = -\tilde{g}^{-1}\partial_x\tilde{g}\tilde{g}^{-1}$, with $x = \theta$ or t, we obtain

$$\left(\partial_t - \partial_\theta\right)\mathcal{A} = \left(\partial_t + \partial_\theta\right)\mathcal{B} = \frac{1}{t}\mathcal{L} \tag{20}$$

The proof in [4] concerns the Einstein-Maxwell $\mathbb{T}^3$-Gowdy symmetric model, however, by setting a certain term to zero, one obtains the target space of the wave-map describing the solution in the 5-dimensional Gowdy spacetime. Thus, one can use the proof there to prove global existence in the case $n = 5$, and the argument can be easily generalized for arbitrary n. The proof uses certain energy estimates to derive bounds for p_i on compact subintervals of $\mathbb{R}_+$ and, consequently, bounds for the quantities $\mathrm{e}^{2(\alpha_i-\alpha_j)}$ on such intervals. This means that the supremum norms of p_i and A_i^{i-1} and their first derivatives are bounded on these intervals. The quantities A_i^j and their first derivatives are then also bounded because of the way the metric (19) is constructed. Using this information, together with energy estimates, we can obtain bounds for higher order derivatives in L^2, whence global existence of smooth solutions follows.

Lemma 1. *For smooth initial data at some* $t_0 \in \mathbb{R}_+$, *there exists a unique smooth solution to* (15) *defined on all of* $\mathbb{R}_+$.

Proof. Very similar to the proof of Theorem 2 in [4].

$\square$

4 Small Data

We will now procede to prove the small data result in n dimensions, i. e. we intend to show that the energy of the solution decays in a specified manner, provided that the initial energy is sufficiently small. For purposes of simplicity and compactness, we will, from now on, employ the following notations:

$$S_k^j := \sum_{l=1}^{k-j}(-1)^l \sum_{k=k_0>...>k_l=j} A_{k_0}^{k_1}...A_{k_{l-1}}^{k_l} \tag{21}$$

$$\widehat{S}_k^j := \sum_{l=1}^{k-j}(-1)^l \sum_{k=k_0>...>k_l=j} < A_{k_0}^{k_1} > ... < A_{k_{l-1}}^{k_l} > \tag{22}$$

$$K_i^j := (A_i^j)_t + \sum_{k=j+1}^{i-1} S_k^j (A_i^k)_t \tag{23}$$

$$U_i^j := (A_i^j)_\theta + \sum_{k=j+1}^{i-1} S_k^j (A_i^k)_\theta \tag{24}$$

$$\widehat{U}_i^j := A_i^j - < A_i^j > + \sum_{k=j+1}^{i-1} S_k^j \left(A_i^k - < A_i^k >\right) \tag{25}$$

$$\overline{K}_i^j := (A_i^j)_t + \sum_{k=j+1}^{i-1} \widehat{S}_k^j (A_i^k)_t \tag{26}$$

$$\overline{U}_i^j := (A_i^j)_\theta + \sum_{k=j+1}^{i-1} \widehat{S}_k^j (A_i^k)_\theta \tag{27}$$

Let us first write the target space metric:

$$\begin{aligned} \mathrm{d}s_1^2 &= \frac{1}{2}\mathrm{Tr}\left[\left(\tilde{g}^{-1}\mathrm{d}\tilde{g}\right)^2\right] \\ &= \sum_{i=1}^{N-1}\mathrm{d}p_i^2 + \sum_{i=2}^{N}\sum_{j=1}^{i-1}\left(\mathrm{d}A_i^j + \sum_{k=j+1}^{i-1} S_k^j \mathrm{d}A_i^k\right)^2 \mathrm{e}^{2(\alpha_i-\alpha_j)} \end{aligned}$$

where $N \geq i > j \geq 1$, α_i defined by (8). As discussed in Section 3, the evolution equations for this metric can be written as the Euler-Lagrange equations of the following lagrangian:

$$\begin{aligned} \mathcal{L} &= \frac{t}{4}\mathrm{Tr}\left[-\left(\tilde{g}^{-1}\partial_t\tilde{g}\right)^2 + \left(\tilde{g}^{-1}\partial_\theta\tilde{g}\right)^2\right] \\ &= \frac{t}{2}\left\{\sum_{i=1}^{N-1}\left[-\left(p_i\right)_t^2 + \left(p_i\right)_\theta^2\right] + \sum_{i=2}^{N}\sum_{j=1}^{i-1}\left[-\left(K_i^j\right)^2 + \left(U_i^j\right)^2\right]\mathrm{e}^{2(\alpha_i-\alpha_j)}\right\} \end{aligned} \tag{28}$$

One can also define the energy of the wave-map which has form

$$H = \frac{1}{4}\int_{\mathbb{S}^1} \mathrm{Tr}\left[\left(\tilde{g}^{-1}\partial_t\tilde{g}\right)^2 + \left(\tilde{g}^{-1}\partial_\theta\tilde{g}\right)^2\right] \mathrm{d}\theta, \tag{29}$$

or, in terms of the fields p_i and A_i^j,

$$H = \frac{1}{2}\int_{\mathbb{S}^1} \left\{ \sum_{i=1}^{N-1} \left[-\left(p_i\right)_t^2 + \left(p_i\right)_\theta^2 \right] + \sum_{i=2}^{N}\sum_{j=1}^{i-1} \left[\left(K_i^j\right)^2 + \left(U_i^j\right)^2 \right] \mathrm{e}^{2(\alpha_i-\alpha_j)} \right\} \mathrm{d}\theta \tag{30}$$

Let us now show that the energy H is a monotonically decreasing function of time. This is valid more generally, but we will carry out the proof for our particular wave-map.

Lemma 2.

For a solution to (15) *and H defined by* (29)*, we have:*

$$\frac{dH}{dt} = -\frac{2}{t}H_K, \tag{31}$$

where H_K is the kinetic part of H, i. e.

$$H_K := \frac{1}{4}\int_{\mathbb{S}^1} Tr\left[\left(\tilde{g}^{-1}\partial_t\tilde{g}\right)^2\right] d\theta.$$

Proof.

Let us compute:

$$\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left\{ \frac{1}{4}\int_{\mathbb{S}^1} \mathrm{Tr}\left[\left(\tilde{g}^{-1}\partial_t\tilde{g}\right)^2 + \left(\tilde{g}^{-1}\partial_\theta\tilde{g}\right)^2\right] \mathrm{d}\theta \right\} \\
&= \frac{1}{2}\int_{\mathbb{S}^1} \mathrm{Tr}\left[\tilde{g}^{-1}\partial_t\tilde{g}\partial_t\left(\tilde{g}^{-1}\partial_t\tilde{g}\right) + \tilde{g}^{-1}\partial_\theta\tilde{g}\partial_t\left(\tilde{g}^{-1}\partial_\theta\tilde{g}\right)\right] \mathrm{d}\theta.
\end{aligned}$$

Now, the Euler-Lagrange equations for the lagrangian $\mathcal{L}$ in (17) read

$$-\partial_t\left(t\tilde{g}^{-1}\partial_t\tilde{g}\right) + \partial_\theta\left(t\tilde{g}^{-1}\partial_\theta\tilde{g}\right) = 0$$

Multiplying this equation by $\tilde{g}^{-1}\partial_t\tilde{g}$, we obtain

$$-\left(\tilde{g}^{-1}\partial_t\tilde{g}\right)^2 - t\tilde{g}^{-1}\partial_t\tilde{g}\partial_t\left(\tilde{g}^{-1}\partial_t\tilde{g}\right) + t\tilde{g}^{-1}\partial_t\tilde{g}\partial_\theta\left(\tilde{g}^{-1}\partial_\theta\tilde{g}\right) = 0.$$

Hence, we have

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &= \frac{1}{2t}\int\limits_{\mathbb{S}^1} \mathrm{Tr}\left[-\left(\tilde{g}^{-1}\partial_t\tilde{g}\right)^2 + t\tilde{g}^{-1}\partial_t\tilde{g}\partial_\theta\left(\tilde{g}^{-1}\partial_\theta\tilde{g}\right) + t\tilde{g}^{-1}\partial_\theta\tilde{g}\partial_t\left(\tilde{g}^{-1}\partial_\theta\tilde{g}\right)\right]\mathrm{d}\theta \\ &= -\frac{2}{t}H_K + \int\limits_{\mathbb{S}^1}\mathrm{Tr}\left[-\partial_\theta\left(\tilde{g}^{-1}\partial_t\tilde{g}\right)\tilde{g}^{-1}\partial_\theta\tilde{g} + \tilde{g}^{-1}\partial_\theta\tilde{g}\partial_t\left(\tilde{g}^{-1}\partial_\theta\tilde{g}\right)\right]\mathrm{d}\theta \\ &= -\frac{2}{t}H_K + \int\limits_{\mathbb{S}^1}\mathrm{Tr}\left[-\partial_\theta\tilde{g}^{-1}\partial_t\tilde{g}\tilde{g}^{-1}\partial_\theta\tilde{g} + \tilde{g}^{-1}\partial_\theta\tilde{g}\partial_t\tilde{g}^{-1}\partial_\theta\tilde{g}\right]\mathrm{d}\theta,\end{aligned}$$

where, in the second equality, we have used integration by parts. Since $\partial_x\tilde{g}^{-1} = -\tilde{g}^{-1}\partial_x\tilde{g}\tilde{g}^{-1}$, with $x = \theta$ or t, it follows that

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{2}{t}H_K,$$

as claimed.

□

Claim 3.

For the lagrangian (28)*, the Euler- Lagrange equations read:*

$$-\partial_t\left[t\left(p_m\right)_t\right] + \partial_\theta\left[t\left(p_m\right)_\theta\right] = t\sum_{i=2}^{N}\sum_{j=1}^{i-1}\left(c_i^m - c_j^m\right)\left[-\left(K_i^j\right)^2 + \left(U_i^j\right)^2\right]e^{2(\alpha_i-\alpha_j)} \tag{32}$$

for p_m, and

$$\begin{aligned}&-\partial_t\left[te^{2(\alpha_i-\alpha_j)}K_i^j\right] + \partial_\theta\left[te^{2(\alpha_i-\alpha_j)}U_i^j\right] \\ &\quad+\sum_{m=1}^{j-1}te^{2(\alpha_i-\alpha_m)}\left(K_i^mK_j^m - U_i^mU_j^m\right) - \sum_{m=i+1}^{N}te^{2(\alpha_m-\alpha_j)}\left(K_m^jK_m^i - U_m^jU_m^i\right) \\ &\quad= 0.\end{aligned} \tag{33}$$

for A_i^j.

Proof. The proof of (32) is straightforward. Let us now now show that the Euler-Lagrange equation for A_i^j has indeed the form (33). Writing down the lagrangian (28) in the form

$$\mathcal{L} = \frac{t}{2}\left\{\sum_{i=1}^{N-1}\left[-\left(p_i\right)_t^2 + \left(p_i\right)_\theta^2\right] + \sum_{i=2}^{N}\sum_{j=1}^{i-1}\left[-\left(K_i^j\right)^2 + \left(U_i^j\right)^2\right]\mathrm{e}^{2(\alpha_i-\alpha_j)}\right\},$$

we get, for A_i^j, the following equation:

$$
\begin{aligned}
&-\partial_t \left[t\mathrm{e}^{2(\alpha_i-\alpha_j)}K_i^j\right] + \partial_\theta \left[t\mathrm{e}^{2(\alpha_i-\alpha_j)}U_i^j\right] \\
&\quad -\sum_{m=1}^{j-1} \partial_t \left[t\mathrm{e}^{2(\alpha_i-\alpha_m)}K_i^m S_j^m\right] + \sum_{m=1}^{j-1} \partial_\theta \left[t\mathrm{e}^{2(\alpha_i-\alpha_m)}U_i^m S_j^m\right] \\
&\quad + \sum_{p=i+1}^{N} \sum_{q=1}^{j} t\mathrm{e}^{2(\alpha_p-\alpha_q)} \\
&\quad \cdot \sum_{k=i}^{p-1} \sum_{l=l_{kiqj}}^{k-q} (-1)^l \sum_{k=k_0>\ldots>k_{s-1}=i>k_s=j>\ldots>k_l=q} A_{k_0}^{k_1} \ldots \widehat{A_{k_{s-1}}^{k_s}} \ldots A_{k_{l-1}}^{k_l} \\
&\quad \cdot \left[K_p^q \left(A_p^k\right)_t - U_p^q \left(A_p^k\right)_\theta\right] = 0,
\end{aligned}
\tag{34}
$$

with $l \geq s \geq 1$ and $l_{kiqj} := 3 - \delta_{ki}$-$\delta_{qj}$. The hat over the term $A_{k_{s-1}}^{k_s}$ denotes the fact that the term is missing and there is a factor of 1 instead of it in the product.

Let us now consider the term

$$
\begin{aligned}
&-\sum_{m=1}^{j-1} \partial_t \left[t\mathrm{e}^{2(\alpha_i-\alpha_m)}K_i^m S_j^m\right] \\
&= -\sum_{m=1}^{j-1} \partial_t \left[-t\mathrm{e}^{2(\alpha_i-\alpha_m)}K_i^m A_j^m - t\mathrm{e}^{2(\alpha_i-\alpha_m)}K_i^m \sum_{p=m+1}^{j-1} S_p^m A_j^p\right] \\
&= \sum_{m=1}^{j-1} \partial_t \left[t\mathrm{e}^{2(\alpha_i-\alpha_m)}K_i^m\right] A_j^m + \sum_{m=1}^{j-1} t\mathrm{e}^{2(\alpha_i-\alpha_m)}K_i^m \left(A_j^m\right)_t \\
&\quad + \sum_{m=1}^{j-1} t\mathrm{e}^{2(\alpha_i-\alpha_m)}K_i^m \sum_{p=m+1}^{j-1} S_p^m \left(A_j^p\right)_t + \sum_{m=1}^{j-1} \sum_{p=m+1}^{j-1} A_j^p \partial_t \left[t\mathrm{e}^{2(\alpha_i-\alpha_m)}K_i^m S_p^m\right] \\
&= \sum_{m=1}^{j-1} \partial_t \left[t\mathrm{e}^{2(\alpha_i-\alpha_m)}K_i^m\right] A_j^m + \sum_{m=1}^{j-1} t\mathrm{e}^{2(\alpha_i-\alpha_m)}K_i^m \left[\left(A_j^m\right)_t + \sum_{p=m+1}^{j-1} S_p^m \left(A_j^p\right)_t\right] \\
&\quad + \sum_{m=1}^{j-1} \sum_{p=m+1}^{j-1} A_j^p \partial_t \left[t\mathrm{e}^{2(\alpha_i-\alpha_m)}K_i^m S_p^m\right] \\
&= \sum_{m=1}^{j-1} A_j^m \partial_t \left[t\mathrm{e}^{2(\alpha_i-\alpha_m)}K_i^m\right] + \sum_{m=1}^{j-1} t\mathrm{e}^{2(\alpha_i-\alpha_m)}K_i^m K_j^m \\
&\quad + \sum_{m=1}^{j-1} \sum_{p=1}^{m-1} A_j^m \partial_t \left[t\mathrm{e}^{2(\alpha_i-\alpha_p)}K_i^p S_m^p\right] \\
&\quad - \sum_{m=1}^{j-1} A_j^m \left\{-\partial_t \left[t\mathrm{e}^{2(\alpha_i-\alpha_m)}K_i^m\right] - \sum_{p=1}^{m-1} \partial_t \left[t\mathrm{e}^{2(\alpha_i-\alpha_p)}K_i^p S_m^p\right]\right\} \\
&\quad + \sum_{m=1}^{j-1} t\mathrm{e}^{2(\alpha_i-\alpha_m)}K_i^m K_j^m.
\end{aligned}
$$

Doing a similar calculation for the term $\sum_{m=1}^{j-1} \partial_\theta \left[t e^{2(\alpha_i-\alpha_m)} U_i^m S_j^m\right]$ and using the Euler-Lagrange equation for A_i^m, (34), we get the result

$$
\begin{aligned}
&-\sum_{m=1}^{j-1} \partial_t \left[\mathrm{t}e^{2(\alpha_i-\alpha_m)} K_i^m S_j^m\right] + \sum_{m=1}^{j-1} \partial_\theta \left[\mathrm{t}e^{2(\alpha_i-\alpha_m)} U_i^m S_j^m\right] \\
&\quad= \sum_{m=1}^{j-1} \mathrm{t}e^{2(\alpha_i-\alpha_m)} \left(K_i^m K_j^m - U_i^m U_j^m\right) + \sum_{m=1}^{j-1} A_j^m \Big\{-\partial_t \left[\mathrm{t}e^{2(\alpha_i-\alpha_m)} K_i^m\right] \\
&\qquad + \partial_\theta \left[\mathrm{t}e^{2(\alpha_i-\alpha_m)} U_i^m\right] - \sum_{p=1}^{m-1} \partial_t \left[\mathrm{t}e^{2(\alpha_i-\alpha_p)} K_i^p S_m^p\right] + \sum_{p=1}^{m-1} \partial_\theta \left[\mathrm{t}e^{2(\alpha_i-\alpha_p)} U_i^p S_m^p\right] \Big\} \\
&\quad= \sum_{m=1}^{j-1} \mathrm{t}e^{2(\alpha_i-\alpha_m)} \left(K_i^m K_j^m - U_i^m U_j^m\right) + \sum_{m=1}^{j-1} A_j^m \sum_{p=i+1}^{N} \sum_{q=1}^{m} \mathrm{t}e^{2(\alpha_p-\alpha_q)} \\
&\qquad \cdot \sum_{k=i}^{p-1} \sum_{l=l_{kiqm}}^{k-q} (-1)^l \sum_{k=k_0>\ldots>k_{s-1}=i>k_s=m>\ldots>k_l=q} A_{k_0}^{k_1} \ldots \widehat{A_{k_{s-1}}^{k_s}} \ldots A_{k_{l-1}}^{k_l} \\
&\qquad \cdot \left[K_p^q \left(A_p^k\right)_t - U_p^q \left(A_p^k\right)_\theta\right] \\
&\quad= \sum_{m=1}^{j-1} \mathrm{t}e^{2(\alpha_i-\alpha_m)} \left(K_i^m K_j^m - U_i^m U_j^m\right) - \sum_{p=i+1}^{N} \sum_{q=1}^{j-1} \mathrm{t}e^{2(\alpha_p-\alpha_q)} \\
&\qquad \cdot \sum_{k=i}^{p-1} \sum_{l=l_{kiqm}}^{k-q} (-1)^l \sum_{k=k_0>\ldots>k_{s-1}=i>k_s=j>\ldots>k_l=q} A_{k_0}^{k_1} \ldots \widehat{A_{k_{s-1}}^{k_s}} \ldots A_{k_{l-1}}^{k_l} \\
&\qquad \cdot \left[K_p^q \left(A_p^k\right)_t - U_p^q \left(A_p^k\right)_\theta\right].
\end{aligned}
$$

Hence, we can write the equation for A_i^j as follows:

$$
\begin{aligned}
&-\partial_t \left[\mathrm{t}e^{2(\alpha_i-\alpha_j)} K_i^j\right] + \partial_\theta \left[\mathrm{t}e^{2(\alpha_i-\alpha_j)} U_i^j\right] + \sum_{m=1}^{j-1} \mathrm{t}e^{2(\alpha_i-\alpha_m)} \left(K_i^m K_j^m - U_i^m U_j^m\right) \\
&\quad + \sum_{p=i+1}^{N} \sum_{q=1}^{j-1} \mathrm{t}e^{2(\alpha_p-\alpha_q)} \\
&\quad \cdot \sum_{k=i}^{p-1} \sum_{l=l_{kiqj}}^{k-q} (-1)^l \sum_{k=k_0>\ldots>k_{s-1}=i>k_s=j>\ldots>k_l=q} A_{k_0}^{k_1} \ldots \widehat{A_{k_{s-1}}^{k_s}} \ldots A_{k_{l-1}}^{k_l} \\
&\quad \cdot \left[K_p^q \left(A_p^k\right)_t - U_p^q \left(A_p^k\right)_\theta\right] \\
&\quad + \sum_{p=i+1}^{N} \sum_{q=1}^{j-1} \mathrm{t}e^{2(\alpha_p-\alpha_q)} \\
&\quad \cdot \sum_{k=i}^{p-1} \sum_{l=l_{kiqj}}^{k-q} (-1)^l \sum_{k=k_0>\ldots>k_{s-1}=i>k_s=j>\ldots>k_l=q} A_{k_0}^{k_1} \ldots \widehat{A_{k_{s-1}}^{k_s}} \ldots A_{k_{l-1}}^{k_l} \\
&\quad \cdot \left[K_p^q \left(A_p^k\right)_t - U_p^q \left(A_p^k\right)_\theta\right]
\end{aligned}
$$

$$
\begin{aligned}
=&-\partial_t\left[te^{2(\alpha_i-\alpha_j)}K_i^j\right]+\partial_\theta\left[te^{2(\alpha_i-\alpha_j)}U_i^j\right]\\
&+\sum_{m=1}^{j-1}te^{2(\alpha_i-\alpha_m)}\left(K_i^mK_j^m-U_i^mU_j^m\right)\\
&+\sum_{p=i+1}^{N}te^{2(\alpha_p-\alpha_j)}\sum_{k=i}^{p-1}\sum_{l=l_{kijj}}^{k-i+1}(-1)^l\sum_{k=k_0>\ldots>k_{l-1}=i>k_l=j}A_{k_0}^{k_1}\ldots\widehat{A_{k_{l-1}}^{k_l}}\\
&\cdot\left[K_p^j\left(A_p^k\right)_t-U_p^j\left(A_p^k\right)_\theta\right]\\
=&-\partial_t\left[te^{2(\alpha_i-\alpha_j)}K_i^j\right]+\partial_\theta\left[te^{2(\alpha_i-\alpha_j)}U_i^j\right]\\
&+\sum_{m=1}^{j-1}te^{2(\alpha_i-\alpha_m)}\left(K_i^mK_j^m-U_i^mU_j^m\right)\\
&+\sum_{p=i+1}^{N}te^{2(\alpha_p-\alpha_j)}K_p^j\left[-(A_p^i)_t-\sum_{k=i+1}^{p-1}S_k^i(A_p^k)_t\right]\\
&-\sum_{p=i+1}^{N}te^{2(\alpha_p-\alpha_j)}U_p^j\left[-(A_p^i)_\theta-\sum_{k=i+1}^{p-1}S_k^i(A_p^k)_\theta\right]\\
=&-\partial_t\left[te^{2(\alpha_i-\alpha_j)}K_i^j\right]+\partial_\theta\left[te^{2(\alpha_i-\alpha_j)}U_i^j\right]\\
&+\sum_{m=1}^{j-1}te^{2(\alpha_i-\alpha_m)}\left(K_i^mK_j^m-U_i^mU_j^m\right)-\sum_{p=i+1}^{N}te^{2(\alpha_p-\alpha_j)}\left(K_p^jK_p^i-U_p^jU_p^i\right).
\end{aligned}
$$

This proves the claim. $\square$

To establish the small data result we need to prove some additional lemmas.

Lemma 4. *For a solution to* (32)-(33) *and with the notations* (25)*, the following equalities hold:*

$$\overline{K}_i^j-K_i^j=\sum_{m=j+1}^{i-1}K_i^m\widehat{U}_m^j \tag{35}$$

and

$$\overline{U}_i^j-U_i^j=\sum_{m=j+1}^{i-1}U_i^m\widehat{U}_m^j \tag{36}$$

Proof. Let us compute the right hand side of the first formula:

$$
\begin{aligned}
\sum_{m=j+1}^{i-1}K_i^m\widehat{U}_m^j=&\sum_{m=j+1}^{i-1}\left[(A_i^m)_t+\sum_{k=m+1}^{i-1}S_k^m\left(A_i^k\right)_t\right]\\
&\cdot\left[A_m^j-<A_m^j>+\sum_{k=j+1}^{m-1}\widehat{S}_k^j\left(A_m^k-<A_m^k>\right)\right]
\end{aligned}
$$

$$= \sum_{m=j+1}^{i-1} (A_i^m)_t \left[- < A_m^j > - \sum_{k=j+1}^{m-1} \widehat{S}_k^j < A_m^k > \right]$$

$$+ \sum_{m=j+1}^{i-1} \left[\sum_{k=m+1}^{i-1} S_k^m \left(A_i^k\right)_t A_m^j + (A_i^m)_t A_m^j \right] + T_{(ij)}$$

where we have defined the quantity

$$T_{(ij)} := \sum_{m=j+1}^{i-1} \left\{ (A_i^m)_t \sum_{k=j+1}^{m-1} \widehat{S}_k^j A_m^k - \sum_{m=j+1}^{i-2} \sum_{k=m+1}^{i-1} S_k^m \left(A_i^k\right)_t < A_m^j > \right.$$

$$\left. + \sum_{m=j+2}^{i-2} \sum_{k=m+1}^{i-1} S_k^m \left(A_i^k\right)_t \sum_{k'=j+1}^{m-1} \widehat{S}_{k'}^j \left(A_m^{k'} - < A_m^{k'} > \right) \right\}$$

We will now show that $T_{(ij)} = 0$:

$$T_{(ij)} = \sum_{m=j+2}^{i-1} (A_i^m)_t \sum_{k=j+1}^{m-1} \widehat{S}_k^j A_m^k - \sum_{m=j+1}^{i-2} \sum_{k=m+1}^{i-1} S_k^m \left(A_i^k\right)_t < A_m^j >$$

$$+ \sum_{m=j+2}^{i-2} \sum_{k=m+1}^{i-1} S_k^m \left(A_i^k\right)_t \sum_{k'=j+1}^{m-1} \widehat{S}_{k'}^j \left(A_m^{k'} - < A_m^{k'} > \right)$$

$$= \sum_{m=j+2}^{i-1} A_m^k (A_i^m)_t \sum_{k=j+1}^{m-1} \widehat{S}_k^j - \sum_{m=j+1}^{i-2} < A_m^j > \sum_{k=m+1}^{i-1} S_k^m \left(A_i^k\right)_t$$

$$- \sum_{k=j+3}^{i-1} \sum_{k'=j+1}^{k-2} \sum_{l=2}^{k-k'} (-1)^l \sum_{k=k_0 > \ldots > k_l = k'} A_{k_0}^{k_1} \ldots A_{k_{l-1}}^{k_l} (A_i^k)_t \widehat{S}_{k'}^j$$

$$+ \sum_{m=j+2}^{i-2} \sum_{k=m+1}^{i-1} S_k^m \left(A_i^k\right)_t \sum_{l=2}^{m-j} (-1)^l \sum_{m=k_0 > \ldots > k_l = j} < A_{k_0}^{k_1} > \ldots < A_{k_{l-1}}^{k_l} >$$

$$= - \sum_{k=j+2}^{i-1} \sum_{k'=j+1}^{k-1} S_k^{k'} \left(A_i^k\right)_t \widehat{S}_{k'}^j + \sum_{k=j+2}^{i-1} \sum_{k'=j+1}^{k-1} S_k^{k'} \left(A_i^k\right)_t \widehat{S}_{k'}^j \ = \ 0.$$

Let us now continue our calculation:

$$\sum_{m=j+1}^{i-1} K_i^m \widehat{U}_m^j = \sum_{m=j+1}^{i-1} (A_i^m)_t \left[- < A_m^j > - \sum_{k=j+1}^{m-1} \widehat{S}_k^j < A_m^k > \right]$$

$$+ \sum_{m=j+1}^{i-1} \left[\sum_{k=m+1}^{i-1} S_k^m \left(A_i^k\right)_t A_m^j + (A_i^m)_t A_m^j \right]$$

$$= \sum_{m=j+1}^{i-1} \left(A_m^j - < A_m^j > \right) (A_i^m)_t$$

$$
\begin{aligned}
&+\sum_{m=j+2}^{i-1}\left[\sum_{l=2}^{m-j}(-1)^l \sum_{m=k_0>\ldots>k_l=j} < A_{k_0}^{k_1} > \ldots < A_{k_{l-1}}^{k_l} > \right. \\
&\left. -\sum_{l=2}^{m-j}(-1)^l \sum_{m=k_0>\ldots>k_l=j} A_{k_0}^{k_1}\ldots A_{k_{l-1}}^{k_l}\right](A_i^m)_t \\
&= \sum_{m=j+1}^{i-1}\left(\widehat{S}_m^j - S_m^j\right)(A_i^m)_t \\
&= \overline{K}_i^j - K_i^j,
\end{aligned}
$$

which proves the claim (the calculation for $\overline{U}_i^j - U_i^j$ is similar). □

Lemma 5. *For a solution to* (32)-(33) *and with the notations* (25)*, the following equality holds:*

$$
\partial_t \widehat{U}_i^j = \overline{K}_i^j - < \overline{K}_i^j > - \sum_{m=j+1}^{i-1} < \overline{K}_m^j > \widehat{U}_i^m \tag{37}
$$

Proof. Let us compute the left hand side of the expression:

$$
\begin{aligned}
\partial_t \widehat{U}_i^j &= \left(A_i^j\right)_t - < A_i^j >_t + \sum_{k=j+1}^{i-1} \widehat{S}_k^j \left(A_i^k\right)_t - \sum_{k=j+1}^{i-1} \widehat{S}_k^j < A_i^k >_t \\
&\quad + \sum_{k=j+1}^{i-1}\sum_{l=1}^{k-j}(-1)^l \sum_{s=0}^{l-1} \\
&\quad \cdot \sum_{k=k_0>\ldots>k_s>\ldots>k_l=j} < A_{k_0}^{k_1} > \ldots < A_{k_s}^{k_{s+1}} >_t \ldots < A_{k_{l-1}}^{k_l} > \left(A_i^k - < A_i^k >\right) \\
&= \overline{K}_i^j - < \overline{K}_i^j > + \sum_{k=j+1}^{i-1}\left(A_i^k - < A_i^k >\right) \\
&\quad \cdot \left[- < A_k^j >_t - \sum_{p=j+1}^{k-1} \widehat{S}_p^j < A_k^p >_t - \sum_{m=j+2}^{k-1}\sum_{n=j+1}^{m-1} \widehat{S}_k^m \widehat{S}_n^j < A_m^n >_t \right. \\
&\quad \left. - \sum_{m=j+1}^{k-1} \widehat{S}_k^m < A_m^j >_t \right] \\
&= \overline{K}_i^j - < \overline{K}_i^j > - \sum_{k=j+1}^{i-1}\left(A_i^k - < A_i^k >\right) \\
&\quad \cdot \left\{ < \overline{K}_k^j > - \sum_{m=j+1}^{k-1} \widehat{S}_k^m \left[< A_m^j >_t + \sum_{n=j+1}^{m-1} \widehat{S}_n^j < A_m^n >_t \right] \right\}
\end{aligned}
$$

$$= \overline{K}_i^j - < \overline{K}_i^j > - \sum_{k=j+1}^{i-1} \left(A_i^k - < A_i^k > \right) < \overline{K}_k^j >$$

$$- \sum_{k=j+2}^{i-1} \left(A_i^k - < A_i^k > \right) \sum_{m=j+1}^{k-1} \widehat{S}_k^m < \overline{K}_m^j >$$

$$= \overline{K}_i^j - < \overline{K}_i^j > - \sum_{k=j+1}^{i-1} \left(A_i^k - < A_i^k > \right) < \overline{K}_k^j >$$

$$- \sum_{k=j+1}^{i-1} \sum_{m=k+1}^{i-1} \widehat{S}_m^k < \overline{K}_k^j > (A_i^m - < A_i^m >)$$

$$= \overline{K}_i^j - < \overline{K}_i^j > - \sum_{k=j+1}^{i-1} < \overline{K}_k^j > \left[A_i^k - < A_i^k > - \sum_{m=k+1}^{i-1} \widehat{S}_m^k \left(A_i^m - < A_i^m > \right) \right]$$

$$= \overline{K}_i^j - < \overline{K}_i^j > - \sum_{m=j+1}^{i-1} < \overline{K}_m^j > \widehat{U}_i^m,$$

as claimed. □

We will now state the small data result we wish to prove, namely, that the energy decays asymptotically as t^{-1}, provided that the initial energy is sufficiently small.

Theorem 6. *For the metric* (19)*, there exists an $\epsilon > 0$ such that if $H(t_0) \leq \epsilon$, for a solution of* (32)-(33)*, with $t_0 > 0$ and H as defined in* (30)*, there exist T and C such that:*

$$H(t) \leq \frac{C}{t}, \text{ for all } t \geq T.$$

To show that Theorem 6 holds, we will first prove some lemmas which provide crucial estimates.

Lemma 7. *For a solution of* (32)-(33) *with the notations* (25) *and for $t \geq t_0$, we have:*

a)

$$\| p_i - < p_i > \|_{C^0(\mathbb{S}^1, \mathbb{R})} \leq C H^{\frac{1}{2}} \tag{38}$$

b)

$$\left\| 1 - e^{\alpha - <\alpha>} \right\|_{C^0(\mathbb{S}^1, \mathbb{R})} \leq C H^{\frac{1}{2}} \text{ and } \left\| 1 - e^{-\alpha + <\alpha>} \right\|_{C^0(\mathbb{S}^1, \mathbb{R})} \leq C H^{\frac{1}{2}} \tag{39}$$

c) for $\alpha = \sum_{i=1}^{N-1} c_i p_i$ with c_i some constants, there exist constants a_1 and a_2 such that

$$a_1 \mathrm{e}^{<\alpha>} \leq \mathrm{e}^{\alpha} \leq a_2 \mathrm{e}^{<\alpha>} \tag{40}$$

d)

$$\int\limits_{\mathbb{S}^1} e^{2(\alpha_i-\alpha_j)}\left|\overline{U}_i^j\right|^2 d\theta \leq \int\limits_{\mathbb{S}^1} e^{2(\alpha_i-\alpha_j)}\left|U_i^j\right|^2 d\theta + CH^{\frac{3}{2}} \leq CH \tag{41}$$

e)

$$\int\limits_{\mathbb{S}^1} e^{2(\alpha_i-\alpha_j)}\left|\overline{K}_i^j\right|^2 d\theta \leq \int\limits_{\mathbb{S}^1} e^{2(\alpha_i-\alpha_j)}\left|K_i^j\right|^2 d\theta + CH^{\frac{1}{2}}\cdot H_K \leq CH_K \tag{42}$$

f)

$$\left\| e^{\alpha_i-\alpha_j}\widehat{U}_i^j\right\|_{C^0(\mathbb{S}^1,\mathbb{R})} \leq CH^{\frac{1}{2}} \tag{43}$$

Proof. a) We have:

$$\|p_i-<p_i>\|_{C^0(\mathbb{S}^1,\mathbb{R})} \leq C\left(\int\limits_{\mathbb{S}^1} |\partial_\theta p_i|^2 \,\mathrm{d}\theta\right)^{\frac{1}{2}} \leq CH^{\frac{1}{2}}.$$

Note: As discussed in [3], the constants in the following estimates depend on $\|p_i-<p_i>\|_{C^0(\mathbb{S}^1,\mathbb{R})}$, but they decrease with the supremum norm. Since $H(t) \leq \epsilon$, for all $t \geq t_0$, if we take $\epsilon \leq 1$, we can take the same constants for all solutions with $H(t) \leq \epsilon$.

b) We have:

$$\begin{aligned}\left\|1-\mathrm{e}^{\alpha-<\alpha>}\right\|_{C^0(\mathbb{S}^1,\mathbb{R})} &\leq \|\alpha-<\alpha>\|_{C^0(\mathbb{S}^1,\mathbb{R})} + \frac{1}{2!}\|\alpha-<\alpha>\|^2_{C^0(\mathbb{S}^1,\mathbb{R})} + \dots \\ &\leq CH^{\frac{1}{2}} + \frac{1}{2!}\left(CH^{\frac{1}{2}}\right)^2 + \dots \leq \ CH^{\frac{1}{2}}\end{aligned}$$

for sufficiently small ϵ. The other calculation is similar.

c) Since $\left\|1-\mathrm{e}^{\alpha-<\alpha>}\right\|_{C^0(\mathbb{S}^1,\mathbb{R})} \leq CH^{\frac{1}{2}}$, it follows that $\mathrm{e}^{\alpha-<\alpha>} \leq 1+CH^{\frac{1}{2}} \leq 1+C\epsilon^{\frac{1}{2}} \leq a_2$, where a_2 is a constant, hence $\mathrm{e}^{\alpha} \leq a_2\mathrm{e}^{<\alpha>}$. Similarly, from

$$\left\|1-\mathrm{e}^{<\alpha>-\alpha}\right\|_{C^0(\mathbb{S}^1,\mathbb{R})} \leq CH^{\frac{1}{2}},$$

it follows that $a_1\mathrm{e}^{<\alpha>} \leq \mathrm{e}^{\alpha}$, for some constant a_1. The conclusion is straightforward.

d) We will prove the statement by induction over i, keeping j fixed. Let us show that the statement is true for $i=j+1$:

$$\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_{j+1}-\alpha_j)}\left|\overline{U}_{j+1}^j\right|^2 \mathrm{d}\theta = \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_{j+1}-\alpha_j)}\left(\partial_\theta A_{j+1}^j\right)^2 \mathrm{d}\theta \leq CH.$$

Assume the statement is true for $\overline{U}_m^j$, for all m, with $i > m > j$, i. e.,

$$\int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i - \alpha_j)} \left|\overline{U}_m^j\right|^2 \mathrm{d}\theta \leq CH$$

Let us now show that the statement is true for $\overline{U}_i^j$. Using (36), we obtain

$$\begin{aligned}
\int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i - \alpha_j)} \left|\overline{U}_i^j\right|^2 \mathrm{d}\theta =& \int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i - \alpha_j)} \left(U_i^j + \sum_{m=j+1}^{i-1} U_i^m \widehat{U}_m^j\right)^2 \mathrm{d}\theta \\
=& \int_{\mathbb{S}^1} \mathrm{d}\theta \mathrm{e}^{2(\alpha_i - \alpha_j)} \left[\left(U_i^j\right)^2 + 2\sum_{m=j+1}^{i-1} U_i^m \widehat{U}_m^j U_i^j \right. \\
& \left. + \left(\sum_{m=j+1}^{i-1} U_i^m \widehat{U}_m^j\right)^2\right] \\
\leq& \int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i - \alpha_j)} \left(U_i^j\right)^2 \mathrm{d}\theta \\
& + 2\sum_{m=j+1}^{i-1} \int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i - \alpha_j)} \left|U_i^m\right| \left|\widehat{U}_m^j\right| \left|U_i^j\right| \mathrm{d}\theta \\
& + \int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i - \alpha_j)} \left(\sum_{m=j+1}^{i-1} \left|U_i^m\right| \left|\widehat{U}_m^j\right|\right)^2 \mathrm{d}\theta.
\end{aligned}$$

Let us show that $\left\|\mathrm{e}^{\alpha_m - \alpha_j} \widehat{U}_m^j\right\|_{C^0(\mathbb{S}^1,\mathbb{R})} \leq CH^{\frac{1}{2}}$, for $i > m > j$. Since $\partial_\theta \widehat{U}_i^j = \overline{U}_i^j$, we have:

$$\begin{aligned}
\left\|\mathrm{e}^{\alpha_m - \alpha_j} \widehat{U}_m^j\right\|_{C^0(\mathbb{S}^1,\mathbb{R})} &\leq C \left\|\mathrm{e}^{<\alpha_m> - <\alpha_j>} \widehat{U}_m^j\right\|_{C^0(\mathbb{S}^1,\mathbb{R})} \\
&\leq C \int_{\mathbb{S}^1} \left|\mathrm{e}^{<\alpha_m> - <\alpha_j>} \overline{U}_m^j\right| \mathrm{d}\theta \\
&\leq C \left[\int_{\mathbb{S}^1} \mathrm{e}^{\alpha_m - \alpha_j} \left(\overline{U}_m^j\right)^2 \mathrm{d}\theta\right]^{\frac{1}{2}} \leq CH^{\frac{1}{2}},
\end{aligned}$$

where we have used (40), Hölder's inequality and the induction hypothesis.

Hence, we obtain:

$$
\begin{aligned}
\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} \left|\overline{U}_i^j\right|^2 \mathrm{d}\theta \leq & \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} \left(U_i^j\right)^2 \mathrm{d}\theta + 2\sum_{m=j+1}^{i-1} \left\|\mathrm{e}^{\alpha_m-\alpha_j}\widehat{U}_m^j\right\|_{C^0(\mathbb{S}^1,\mathbb{R})} \\
& \cdot \left[\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_m)} \left(U_i^m\right)^2 \mathrm{d}\theta\right]^{\frac{1}{2}} \left[\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} \left(U_i^j\right)^2 \mathrm{d}\theta\right]^{\frac{1}{2}} \\
& + \ C \sum_{m=j+1}^{i-1} \left\|\mathrm{e}^{\alpha_m-\alpha_j}\widehat{U}_m^j\right\|^2_{C^0(\mathbb{S}^1,\mathbb{R})} \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_m)} \left(U_i^m\right)^2 \mathrm{d}\theta \\
\leq & \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} \left(U_i^j\right)^2 \mathrm{d}\theta + CH^{\frac{3}{2}} + CH^2 \\
\leq & \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} \left(U_i^j\right)^2 \mathrm{d}\theta + CH^{\frac{3}{2}} \leq \ CH.
\end{aligned}
$$

e) The proof is very similar to that in d). Using the same arguments as above, we obtain:

$$
\begin{aligned}
\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} \left|\overline{K}_i^j\right|^2 \mathrm{d}\theta \leq & \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} \left(K_i^j\right)^2 \mathrm{d}\theta + 2\sum_{m=j+1}^{i-1} \left\|\mathrm{e}^{\alpha_m-\alpha_j}\widehat{U}_m^j\right\|_{C^0(\mathbb{S}^1,\mathbb{R})} \\
& \cdot \left[\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_m)} \left(K_i^m\right)^2 \mathrm{d}\theta\right]^{\frac{1}{2}} \left[\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} \left(K_i^j\right)^2 \mathrm{d}\theta\right]^{\frac{1}{2}} \\
& + \ C \sum_{m=j+1}^{i-1} \left\|\mathrm{e}^{\alpha_m-\alpha_j}\widehat{U}_m^j\right\|^2_{C^0(\mathbb{S}^1,\mathbb{R})} \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_m)} \left(K_i^m\right)^2 \mathrm{d}\theta \\
\leq & \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} \left(K_i^j\right)^2 \mathrm{d}\theta + CH^{\frac{1}{2}}H_K + CHH_K \\
\leq & \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} \left(K_i^j\right)^2 \mathrm{d}\theta + CH_K \leq \ CH_K.
\end{aligned}
$$

f) As before,

$$
\begin{aligned}
\left\|\mathrm{e}^{\alpha_m-\alpha_j}\widehat{U}_m^j\right\|_{C^0(\mathbb{S}^1,\mathbb{R})} & \leq C\left\|\mathrm{e}^{<\alpha_m>-<\alpha_j>}\widehat{U}_m^j\right\|_{C^0(\mathbb{S}^1,\mathbb{R})} \\
& \leq C\int\limits_{\mathbb{S}^1} \mathrm{e}^{<\alpha_i>-<\alpha_j>} \left|\overline{U}_i^j\right| \mathrm{d}\theta \leq \ C\int\limits_{\mathbb{S}^1} \mathrm{e}^{\alpha_i-\alpha_j} \left|\overline{U}_i^j\right| \mathrm{d}\theta \\
& \leq C\left[\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} \left(\overline{U}_i^j\right)^2 \mathrm{d}\theta\right]^{\frac{1}{2}} \leq \ CH^{\frac{1}{2}}.
\end{aligned}
$$

□

As in the 4-dimensional case ([3]), the proof of Theorem 6 relies on introducing certain correction terms which can be bounded in terms of H. These correction terms were defined as generalizations of the ones in four dimensions (they reduce to these for $N=2$) and give similar bounds. More precisely, the correction terms are given by

$$\begin{aligned}\Gamma^0 \quad &:= \frac{1}{2t^2}\sum_{m=1}^{N-1}\int_{\mathbb{S}^1}(p_m - <p_m>)\, t\partial_t p_m \mathrm{d}\theta \\ \Gamma^i \quad &:= \frac{1}{2t^2}\sum_{m=1}^{N-i}\int_{\mathbb{S}^1}\mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)}\widehat{U}^m_{m+i} t\overline{K}^m_{m+i}\mathrm{d}\theta,\end{aligned} \tag{44}$$

for $N > i > 0$. The crucial results concerning these terms are stated below, in Lemma 8 and Lemma 9.

Lemma 8. *For a solution of* (32)-(33) *with* Γ^i, $N > i \geq 0$, *as defined above and for* $H \leq \epsilon$, *we have:*

$$\left|\Gamma^i\right| \leq \frac{C}{t}H \tag{45}$$

Proof. We wil provide the proof separately for Γ^0 and for Γ^i, with $N > i > 0$. In the case of Γ^0, we have:

$$\left|\Gamma^0\right| \leq \frac{C}{t}\sum_{m=1}^{N-i}\|p_m - <p_m>\|_{C^0(\mathbb{S}^1,\mathbb{R})}\left[\int_{\mathbb{S}^1}(\partial_t p_m)^2\,\mathrm{d}\theta\right]^{\frac{1}{2}} \leq \frac{C}{t}H^{\frac{1}{2}}\cdot H^{\frac{1}{2}} \leq \frac{C}{t}H,$$

while for Γ^i, the argument is as follows:

$$\begin{aligned}\left|\Gamma^i\right| &\leq \frac{1}{2t}\sum_{m=1}^{N-i}\int_{\mathbb{S}^1}\mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)}\left|\widehat{U}^{m+i}_m\right|\left|\overline{K}^{m+i}_m\right|\mathrm{d}\theta \\ &\leq \frac{C}{t}\sum_{m=1}^{N-i}\int_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_{m+i}-\alpha_m)}\left|\widehat{U}^{m+i}_m\right|\left|\overline{K}^{m+i}_m\right|\mathrm{d}\theta \\ &\leq \frac{C}{t}\sum_{m=1}^{N-i}\left\|\mathrm{e}^{\alpha_{m+i}-\alpha_m}\widehat{U}^{m+i}_m\right\|_{C^0(\mathbb{S}^1,\mathbb{R})}\left[\int_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_{m+i}-\alpha_m)}\left|\overline{K}^{m+i}_m\right|^2\mathrm{d}\theta\right]^{\frac{1}{2}} \\ &\leq \frac{C}{t}H^{\frac{1}{2}}\cdot H_K^{\frac{1}{2}} \leq \frac{C}{t}H,\end{aligned}$$

where we have used (38), (40), (43) and Hölder's inequality. □

Having established the manner in which the absolute values are bounded by H, we now wish to derive some estimates for the derivatives of the correction terms.

Lemma 9. *For a solution of* (32)-(33) *with* Γ^i *as defined in* (44) *and for* $H \leq \epsilon$, *we have:*

$$\frac{d\Gamma^0}{dt} \leq -\frac{2}{t}\Gamma^0 + \frac{1}{2t}\sum_{i=1}^{N-1}\int_{\mathbb{S}^1}\left[(\partial_t p_i)^2 - (\partial_\theta p_i)^2\right] d\theta + \frac{C}{t}H^{\frac{3}{2}} \tag{46}$$

and

$$\frac{d\Gamma^i}{dt} \leq -\frac{2}{t}\Gamma^i + \frac{1}{2t}\sum_{m=1}^{N-i}\int_{\mathbb{S}^1} e^{2(\alpha_{m+i}-\alpha_m)}\left[\left(K^m_{m+i}\right)^2 - \left(U^m_{m+i}\right)^2\right] d\theta + \frac{C}{t}H^{\frac{3}{2}} \tag{47}$$

for $N > i > 0$.

Proof. Let us first prove the inequality for $\dfrac{\mathrm{d}\Gamma^0}{\mathrm{d}t}$:

$$\begin{aligned}
\frac{\mathrm{d}\Gamma^0}{\mathrm{d}t} =& -\frac{2}{t}\Gamma^0 \\
&+ \frac{1}{2t^2}\sum_{m=1}^{N-1}\int_{\mathbb{S}^1}\mathrm{d}\theta\left\{\left[(p_m)_t - < p_m >_t\right] t\,(p_m)_t + (p_m - < p_m >)\,\partial_t\left[t\,(p_m)_t\right]\right\} \\
=& -\frac{2}{t}\Gamma^0 + \frac{1}{2t}\sum_{m=1}^{N-1}\int_{\mathbb{S}^1}(p_m)_t^2\,\mathrm{d}\theta - \frac{1}{2t}\sum_{m=1}^{N-1}\left(< p_m >_t\right)^2 \\
&+ \frac{1}{2t^2}\sum_{m=1}^{N-1}\int_{\mathbb{S}^1}(p_m - < p_m >)\left\{\partial_\theta\left[t\,(p_m)_\theta\right]\right. \\
&\left. - t\sum_{i=2}^{N}\sum_{j=1}^{i-1}\left(c_i^m - c_j^m\right)\left[-\left(K_i^j\right)^2 + \left(U_i^j\right)^2\right]\mathrm{e}^{2(\alpha_i-\alpha_j)}\right\}\mathrm{d}\theta \\
\leq& -\frac{2}{t}\Gamma^0 + \frac{1}{2t}\sum_{m=1}^{N-1}\int_{\mathbb{S}^1}\left[(p_m)_t^2 - (p_m)_\theta^2\right]\mathrm{d}\theta + \frac{1}{2t}\sum_{m=1}^{N-1}\|p_m - < p_m >\|_{C^0(\mathbb{S}^1,\mathbb{R})} \\
&\cdot\sum_{i=2}^{N}\sum_{j=1}^{i-1}\left(|c_i^m| + |c_j^m|\right)\int_{\mathbb{S}^1}\left[\left(K_i^j\right)^2 + \left(U_i^j\right)^2\right]\mathrm{e}^{2(\alpha_i-\alpha_j)}\mathrm{d}\theta \\
\leq& -\frac{2}{t}\Gamma^0 + \frac{1}{2t}\sum_{m=1}^{N-1}\int_{\mathbb{S}^1}\left[(p_m)_t^2 - (p_m)_\theta^2\right]\mathrm{d}\theta + \frac{C}{t}H^{\frac{3}{2}}
\end{aligned}$$

where we have used (32) and (38). In the case of Γ^i, with $0 < i < N$, we can write as follows:

$$\frac{\mathrm{d}\Gamma^i}{\mathrm{d}t} = -\frac{2}{t}\Gamma^i + \frac{1}{t}\sum_{m=1}^{N-i}\int_{\mathbb{S}^1} \mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)}\left(<\alpha_{m+i}>_t - <\alpha_m>_t\right)\widehat{U}^m_{m+i}\overline{K}^m_{m+i}\mathrm{d}\theta$$

$$+ \frac{1}{2t}\sum_{m=1}^{N-i}\int_{\mathbb{S}^1} \mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)}\partial_t\widehat{U}^m_{m+i}\overline{K}^m_{m+i}\mathrm{d}\theta$$

$$+ \frac{1}{2t^2}\sum_{m=1}^{N-i}\int_{\mathbb{S}^1} \mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)}\widehat{U}^m_{m+i}\partial_t\left(t\overline{K}^m_{m+i}\right)\mathrm{d}\theta.$$

Let us call the three integrals appearing in the expression above I_1, I_2 and I_3, respectively. We have:

$$\begin{aligned}
|I_1| &\leq \frac{C}{t}\sum_{m=1}^{N-i}\left\|\mathrm{e}^{\alpha_{m+i}-\alpha_m}\widehat{U}^m_{m+i}\right\|_{C^0(\mathbb{S}^1,\mathbb{R})} \\
&\quad \cdot \int_{\mathbb{S}^1}\mathrm{e}^{\alpha_{m+i}-\alpha_m}\left(|<\alpha_{m+i}>_t| + |<\alpha_m>_t|\right)\left|\overline{K}^m_{m+i}\right|\mathrm{d}\theta \\
&\leq \frac{C}{t}H^{\frac{1}{2}}H = \frac{C}{t}H^{\frac{3}{2}},
\end{aligned}$$

where we have used (40), (42), (43), Hölder's inequality and the fact that the αs are linear combinations of the p_is.

Let us now estimate I_2. Using Lemma 5, we can write:

$$\begin{aligned}
I_2 &= \frac{1}{2t}\sum_{m=1}^{N-i} \\
&\quad \int_{\mathbb{S}^1}\mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)}\left(\overline{K}^m_{m+i} - <\overline{K}^m_{m+i}> - \sum_{k=m+1}^{m+i-1} <\overline{K}^j_k>\widehat{U}^k_{m+i}\right)\overline{K}^m_{m+i}\mathrm{d}\theta \\
&\leq \frac{1}{2t}\sum_{m=1}^{N-i}\left\{\int_{\mathbb{S}^1}\mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)}\left(\overline{K}^m_{m+i}\right)^2\mathrm{d}\theta\right. \\
&\quad -\int_{\mathbb{S}^1}\mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)} <\overline{K}^m_{m+i}>^2 . \\
&\quad \left. + \left|\int_{\mathbb{S}^1}\mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)}\sum_{k=m+1}^{m+i-1} <\overline{K}^j_k>\widehat{U}^k_{m+i}\overline{K}^m_{m+i}\right|\right\}
\end{aligned}$$

Let us estimate the term:

$$\begin{aligned}
&\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)} \left(\overline{K}^m_{m+i}\right)^2 \mathrm{d}\theta \\
&= \int\limits_{\mathbb{S}^1} \left[\mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>-\alpha_{m+i}+\alpha_m)} - 1 + 1\right] \mathrm{e}^{2(\alpha_{m+i}-\alpha_m)} \left(\overline{K}^m_{m+i}\right)^2 \mathrm{d}\theta \\
&\leq \left\|\mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>-\alpha_{m+i}+\alpha_m)} - 1\right\|_{C^0(\mathbb{S}^1,\mathbb{R})} \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_{m+i}-\alpha_m)} \left(\overline{K}^m_{m+i}\right)^2 \mathrm{d}\theta \\
&\quad + \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_{m+i}-\alpha_m)} \left(\overline{K}^m_{m+i}\right)^2 \mathrm{d}\theta \\
&\leq CH^{\frac{1}{2}} H + \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_{m+i}-\alpha_m)} \left(K^m_{m+i}\right)^2 \mathrm{d}\theta + CH^{\frac{3}{2}} \\
&\leq \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_{m+i}-\alpha_m)} \left(K^m_{m+i}\right)^2 + CH^{\frac{3}{2}}
\end{aligned}$$

where we have used (39), (40) and (42). Furthermore, using (42), (43) and Hölder's inequality, we can write

$$\begin{aligned}
&\left|\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)} \sum_{k=m+1}^{m+i-1} < \overline{K}^j_k > \widehat{U}^k_{m+i} \overline{K}^m_{m+i}\right| \\
&\leq C \sum_{k=m+1}^{m+i-1} \left\|\mathrm{e}^{<\alpha_{m+i}>-<\alpha_k>} \widehat{U}^k_{m+i}\right\|_{C^0(\mathbb{S}^1,\mathbb{R})} \\
&\quad \cdot \left[\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)} \left(\overline{K}^m_{m+i}\right)^2 \mathrm{d}\theta\right]^{\frac{1}{2}} \left[\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(<\alpha_k>-<\alpha_m>)} \left(\overline{K}^m_k\right)^2 \mathrm{d}\theta\right]^{\frac{1}{2}} \\
&\leq CH^{\frac{3}{2}}.
\end{aligned}$$

Hence, we obtain

$$I_2 \leq \frac{1}{2t} \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_{m+i}-\alpha_m)} \left(K^m_{m+i}\right)^2 \mathrm{d}\theta + \frac{C}{t} H^{\frac{3}{2}}.$$

Let us now consider I_3:

$$I_3 = \frac{1}{2t^2} \sum_{m=1}^{N-i} \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)} \widehat{U}^m_{m+i} \partial_t \left(t\overline{K}^m_{m+i}\right) \mathrm{d}\theta.$$

We have shown before (Lemma 4) that

$$\overline{K}^m_{m+i} - K^m_{m+i} = \sum_{k=m+1}^{m+i-1} K^k_{m+i} \widehat{U}^m_k.$$

It follows that

$$\partial_t\left(t\overline{K}^m_{m+i}\right)=\partial_t\left(tK^m_{m+i}\right)+\sum_{k=m+1}^{m+i-1}\left[\partial_t\left(tK^k_{m+i}\right)\widehat{U}^m_k+tK^k_{m+i}\partial_t\widehat{U}^m_k\right].$$

Hence, we have

$$\begin{aligned}I_3\leq\frac{1}{2t^2}\sum_{m=1}^{N-i}&\Bigg\{\int_{\mathbb{S}^1}\mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)}\widehat{U}^m_{m+i}\partial_t\left(tK^m_{m+i}\right)\mathrm{d}\theta\\&+\sum_{k=m+1}^{m+i-1}\left[\left|\int_{\mathbb{S}^1}\mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)}\widehat{U}^m_{m+i}\partial_t\left(tK^k_{m+i}\right)\widehat{U}^m_k\mathrm{d}\theta\right|\right.\\&\left.+\left|\int_{\mathbb{S}^1}\mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)}\widehat{U}^m_{m+i}tK^k_{m+i}\partial_t\widehat{U}^m_k\mathrm{d}\theta\right|\right]\Bigg\}\end{aligned}\tag{48}$$

Let us consider the first term in the expression (48). Using the Euler-Lagrange equations for A^m_{m+i}, we get:

$$\begin{aligned}&\int_{\mathbb{S}^1}\mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)}\widehat{U}^m_{m+i}\partial_t\left(tK^m_{m+i}\right)\mathrm{d}\theta\\&\quad=\int_{\mathbb{S}^1}\mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)}\widehat{U}^m_{m+i}\left\{\partial_t\left[t\mathrm{e}^{2(\alpha_{m+i}-\alpha_m)}K^m_{m+i}\right]\mathrm{e}^{-2(\alpha_{m+i}-\alpha_m)}\right.\\&\quad\left.-2\partial_t\left(\alpha_{m+i}-\alpha_m\right)tK^m_{m+i}\right\}\mathrm{d}\theta\\&\quad\leq\int_{\mathbb{S}^1}\mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)}\widehat{U}^m_{m+i}\left\{\partial_\theta\left[t\mathrm{e}^{2(\alpha_{m+i}-\alpha_m)}U^m_{m+i}\right]\right.\\&\quad+\sum_{p=1}^{m-1}t\mathrm{e}^{2(\alpha_{m+i}-\alpha_p)}\left(K^p_{m+i}K^p_m-U^p_{m+i}U^p_m\right)\\&\quad\left.-\sum_{p=m+i-1}^{N}t\mathrm{e}^{2(\alpha_p-\alpha_m)}\left(K^m_pK^{m+i}_p-U^m_pU^{m+i}_p\right)\right\}\mathrm{e}^{-2(\alpha_{m+i}-\alpha_m)}\mathrm{d}\theta\\&\quad+\ 2t\left\|\mathrm{e}^{<\alpha_{m+i}>-<\alpha_m>}\widehat{U}^m_{m+i}\right\|_{C^0(\mathbb{S}^1,\mathbb{R})}\\&\quad\cdot\int_{\mathbb{S}^1}\mathrm{e}^{<\alpha_{m+i}>-<\alpha_m>}\left[\left|\left(\alpha_{m+i}\right)_t\right|+\left|\left(\alpha_m\right)_t\right|\right]\left|K^m_{m+i}\right|\mathrm{d}\theta\\&\quad\leq-t\int_{\mathbb{S}^1}\mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)}\left(U^m_{m+i}\right)^2\mathrm{d}\theta+t\left\|\mathrm{e}^{<\alpha_{m+i}>-<\alpha_m>}\widehat{U}^m_{m+i}\right\|_{C^0(\mathbb{S}^1,\mathbb{R})}\end{aligned}$$

$$
\begin{aligned}
&\cdot\left[\sum_{p=1}^{m-1}\int_{\mathbb{S}^1}\mathrm{e}^{<\alpha_{m+i}>-<\alpha_m>}\mathrm{e}^{2(\alpha_m-\alpha_p)}\left(\left|K^p_{m+i}\right|\left|K^p_m\right|+\left|U^p_{m+i}\right|\left|U^p_m\right|\right)\mathrm{d}\theta\right.\\
&\left.+\sum_{p=m+i+1}^{N}\int_{\mathbb{S}^1}\mathrm{e}^{<\alpha_{m+i}>-<\alpha_m>}\mathrm{e}^{2(\alpha_p-\alpha_{m+i})}\left(\left|K^m_p\right|\left|K^{m+i}_p\right|+\left|U^m_p\right|\left|U^{m+i}_p\right|\right)\mathrm{d}\theta\right]\\
&+\ CtH^{\frac{3}{2}}\\
&\leq -t\int_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_{m+i}-\alpha_m)}\left(U^m_{m+i}\right)^2\mathrm{d}\theta+Ct\left\|\mathrm{e}^{<\alpha_{m+i>}>-<\alpha_m>}\widehat{U}^m_{m+i}\right\|_{C^0(\mathbb{S}^1,\mathbb{R})}\\
&\cdot\left[\sum_{p=1}^{m-1}\int_{\mathbb{S}^1}\mathrm{e}^{\alpha_{m+i}+\alpha_m-2\alpha_p}\left(\left|K^p_{m+i}\right|\left|K^p_m\right|+\left|U^p_{m+i}\right|\left|U^p_m\right|\right)\mathrm{d}\theta+\right.\\
&\left.+\sum_{p=m+i+1}^{N}\int_{\mathbb{S}^1}\mathrm{e}^{-\alpha_{m+i}-\alpha_m+2\alpha_p}\left(\left|K^m_p\right|\left|K^{m+i}_p\right|+\left|U^m_p\right|\left|U^{m+i}_p\right|\right)\mathrm{d}\theta\right]+CtH^{\frac{3}{2}}\\
&\leq -t\int_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_{m+i}-\alpha_m)}\left(U^m_{m+i}\right)^2\mathrm{d}\theta+CtH^{\frac{3}{2}},
\end{aligned}
$$

where we have used arguments similar to the ones previously employed.

Let us now estimate the second term in (48). Using the Euler-Lagrange equations for A^k_{m+i} we can write:

$$
\begin{aligned}
&\left|\int_{\mathbb{S}^1}\mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)}\widehat{U}^m_{m+i}\partial_t\left(tK^k_{m+i}\right)\widehat{U}^m_k\mathrm{d}\theta\right|\\
&=\left|\int_{\mathbb{S}^1}\mathrm{d}\theta\mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)}\widehat{U}^m_{m+i}\widehat{U}^m_k\cdot\right.\\
&\cdot\left\{\left[\partial_\theta\left(tU^k_{m+i}\right)+t\sum_{p=1}^{k-1}\int_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_{m+i}-\alpha_p)}\left(K^p_{m+i}K^p_k-U^p_{m+i}U^p_k\right)\right.\right.\\
&\left.-t\sum_{p=m+i+1}^{N}\mathrm{e}^{2(\alpha_p-\alpha_k)}\left(K^{m+i}_pK^k_p-U^{m+i}_pU^k_p\right)\right]\mathrm{e}^{-2(\alpha_{m+i}-\alpha_k)}\\
&\left.-2\partial_t\left(\alpha_{m+i}-\alpha_k\right)tK^k_{m+i}\right\}\\
&\leq\left|-t\int_{\mathbb{S}^1}\mathrm{d}\theta\mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)}\left(\overline{U}^m_{m+i}\widehat{U}^m_kU^k_{m+i}-\widehat{U}^m_{m+i}\overline{U}^m_kU^k_{m+i}\right)\right|\\
&+\ Ct\left\|\mathrm{e}^{\alpha_{m+i}-\alpha_m}\widehat{U}^m_{m+i}\right\|_{C^0(\mathbb{S}^1,\mathbb{R})}\left\|\mathrm{e}^{\alpha_k-\alpha_m}\widehat{U}^m_k\right\|_{C^0(\mathbb{S}^1,\mathbb{R})}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\left\{\sum_{p=1}^{k-1}\left[\left(\int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_{m+i}-\alpha_p)}\left(K^p_{m+i}\right)^2 \mathrm{d}\theta\right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_k-\alpha_p)}\left(K^p_k\right)^2 \mathrm{d}\theta\right)^{\frac{1}{2}}\right.\right.\\
&+\left.\left(\int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_{m+i}-\alpha_p)}\left(U^p_{m+i}\right)^2 \mathrm{d}\theta\right)^{\frac{1}{2}} \left(\int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_k-\alpha_p)}\left(U^p_k\right)^2 \mathrm{d}\theta\right)^{\frac{1}{2}}\right]\\
&+\sum_{p=m+i+1}^{N}\left[\left(\int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_p-\alpha_{m+i})}\left(K^{m+i}_p\right)^2 \mathrm{d}\theta\right)^{\frac{1}{2}} \left(\int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_p-\alpha_k)}\left(K^k_p\right)^2 \mathrm{d}\theta\right)^{\frac{1}{2}}\right.\\
&+\left.\left(\int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_p-\alpha_{m+i})}\left(U^{m+i}_p\right)^2 \mathrm{d}\theta\right)^{\frac{1}{2}} \left(\int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_p-\alpha_k)}\left(U^k_p\right)^2 \mathrm{d}\theta\right)^{\frac{1}{2}}\right]\\
&\left.+2\sum_{s=1}^{N-1}\left(\left|c^s_{m+i}\right|+\left|c^s_k\right|\right)\left[\int_{\mathbb{S}^1} (p_s)_t^2\,\mathrm{d}\theta\right]^{\frac{1}{2}}\left[\int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_{m+i}-\alpha_k)}\left(K^k_{m+i}\right)^2 \mathrm{d}\theta\right]^{\frac{1}{2}}\right\}\\
&\le\; CtH^{\frac{3}{2}}.
\end{aligned}
$$

Let us now consider the third term in (48). Using (37), we obtain:

$$
\begin{aligned}
&\left|\int_{\mathbb{S}^1} \mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)}\widehat{U}^m_{m+i} t K^k_{m+i}\partial_t \widehat{U}^m_k \mathrm{d}\theta\right|\\
&= t\left|\int_{\mathbb{S}^1} \mathrm{e}^{2(<\alpha_{m+i}>-<\alpha_m>)}\widehat{U}^m_{m+i} K^k_{m+i}\left(\overline{K}^m_k - <\overline{K}^m_k> - \sum_{j=m+1}^{k-1} <\overline{K}^m_j>\widehat{U}^j_k\right)\mathrm{d}\theta\right|\\
&\le Ct\left\|\mathrm{e}^{\alpha_{m+i}-\alpha_m}\widehat{U}^m_{m+i}\right\|_{C^0(\mathbb{S}^1,\mathbb{R})}\left[\int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_{m+i}-\alpha_k)}\left(K^k_{m+i}\right)^2 \mathrm{d}\theta\right]^{\frac{1}{2}}\\
&\cdot\left\{\left[\int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_k-\alpha_m)}\left(\overline{K}^m_k\right)^2 \mathrm{d}\theta\right]^{\frac{1}{2}}\right.\\
&\left.+\left\|\mathrm{e}^{<\alpha_k>-<\alpha_j>}\widehat{U}^j_k\right\|_{C^0(\mathbb{S}^1,\mathbb{R})}\left[\int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_j-\alpha_m)}\left(\overline{K}^m_j\right)^2 \mathrm{d}\theta\right]^{\frac{1}{2}}\right\}\\
&\le\; CtH^{\frac{3}{2}}.
\end{aligned}
$$

In conclusion, we obtain:

$$I_3 \leq -\frac{1}{2t}\sum_{m=1}^{N-i}\int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_{m+i}-\alpha_m)}\left(U^m_{m+i}\right)^2 \mathrm{d}\theta + \frac{C}{t}H^{\frac{3}{2}}.$$

Adding up all the contributions to $\frac{\mathrm{d}\Gamma_i}{\mathrm{d}t}$, we arrive at the desired formula, (47). $\square$

Having established these preliminary results, we are now ready to prove the main statement concerning small initial data.

Proof of Theorem 6. The proof is almost identical to that in four dimensions carried out in [3]. We first define the total correction term

$$\Gamma := \sum_{i=0}^{N-1}\Gamma^i.$$

Because of Lemma 8 and Lemma 9, we have:

$$|\Gamma| \leq \frac{C}{t}H \tag{49}$$

and

$$\begin{aligned}\frac{\mathrm{d}\Gamma}{\mathrm{d}t} \leq & -\frac{2}{t}\Gamma + \frac{1}{2t}\int_{\mathbb{S}^1}\mathrm{d}\theta\sum_{i=1}^{N-1}\Big\{\Big[\left(\partial_t p_i\right)^2 - \left(\partial_\theta p_i\right)^2\Big] \\ & + \sum_{m=1}^{N-i}\mathrm{e}^{2(\alpha_{m+i}-\alpha_m)}\Big[\left(K^m_{m+i}\right)^2 - \left(U^m_{m+i}\right)^2\Big]\Big\} + \frac{C}{t}H^{\frac{3}{2}}.\end{aligned} \tag{50}$$

From (49), it follows that there exists a $T \geq t_0$, such that

$$\frac{1}{2}H \leq H + \Gamma \leq 2H \text{ for } t \geq T. \tag{51}$$

Define now the quantity:

$$\mathcal{E} := H + \Gamma.$$

and let us find an estimate for $\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}$. We have:

$$\begin{aligned}\frac{\mathrm{d}\left(H+\Gamma\right)}{\mathrm{d}t} &\leq \frac{\mathrm{d}H}{\mathrm{d}t} - \frac{2}{t}\Gamma + \frac{1}{t}\left(H_K - H_P\right) + \frac{C}{t}H^{\frac{3}{2}} \\ &= -\frac{2}{t}H_K - \frac{2}{t}\Gamma + \frac{1}{t}\left(H_K - H_P\right) + \frac{C}{t}H^{\frac{3}{2}} \\ &= -\frac{2}{t}\Gamma - \frac{1}{t}\left(H_K + H_P\right) + \frac{C}{t}H^{\frac{3}{2}} \\ &\leq -\frac{1}{t}\Gamma + \frac{1}{t}|\Gamma| - \frac{1}{t}H + \frac{C}{t}H^{\frac{3}{2}} \\ &\leq -\frac{1}{t}\left(H+\Gamma\right) + \frac{C}{t^2}H + \frac{C}{t}H^{\frac{3}{2}}\end{aligned}$$

where we have used (31), (50) and (49). Using (51), it follows that:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} \leq -\frac{1}{t}\mathcal{E} + \frac{C}{t^2}\mathcal{E} + \frac{C}{t}\mathcal{E}^{\frac{3}{2}}$$

Now, for small enough ϵ, with $H \leq \epsilon$, we have:

$$C\mathcal{E}^{\frac{1}{2}}(t) \leq \frac{1}{2} \text{ for } t \geq T.$$

Hence, we can write:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} \leq -\frac{1}{t}\mathcal{E} + \frac{C}{t^2}\mathcal{E} + \frac{1}{2t}\mathcal{E} = \left(-\frac{1}{2t} + \frac{C}{t^2}\right)\mathcal{E}$$

It follows that:

$$\mathcal{E}(t) \leq Ct^{-\frac{1}{2}},$$

and therefore,

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} \leq \left(-\frac{1}{t} + \frac{C}{t^2} + \frac{C}{t^{\frac{5}{4}}}\right)\mathcal{E}. \tag{52}$$

Consequently, we have:

$$\mathcal{E}(t) \leq C\frac{t_0}{t}\mathcal{E}(t_0)$$

and because of (51), we obtain the desired estimate:

$$H \leq \frac{C}{t} \text{ for all } t \geq T. \tag{53}$$

□

5 Large Data

As discussed in Section 2, the second step in proving that the energy decays asymptotically as t^{-1} is to show that $H(t)$ converges to 0 as $t \to \infty$. This result, combined with Theorem 6, will then provide us with the desired decay law. Thus, we need to show that the following theorem holds:

Theorem 10. *For a solution of* (32)-(33) *with* H *defined by* (30)*, we have*

$$H \to 0 \textit{ as } t \to \infty.$$

Proof. Since $H(t)$ is a monotonically decreasing function, it suffices to show that

$$\frac{1}{t}H(t) \in L^1([t_0, \infty))$$

for $t_0 > 0$. First, we note that, since

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{2}{t} H_K,$$

and H is a decreasing function, it follows that

$$\frac{1}{t} H_K(t) \in L^1([t_0, \infty)). \tag{54}$$

Consider now the quantity

$$\int_{t_0}^{t} \frac{1}{s} \int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i - \alpha_j)} \left[\left(K_i^j \right)^2 - \left(U_i^j \right)^2 \right] \mathrm{d}\theta \mathrm{d}s, \text{ for } 1 \leq j < i \leq N. \tag{55}$$

Using the results of Lemma 4 and Lemma 5 which we rewrite here for convenience,

$$\overline{U}_i^j - U_i^j = \sum_{m=j+1}^{i-1} U_i^m \widehat{U}_m^j,$$

$$\overline{K}_i^j - K_i^j = \sum_{m=j+1}^{i-1} K_i^m \widehat{U}_m^j,$$

$$\partial_t \widehat{U}_i^j = \overline{K}_i^j - < \overline{K}_i^j > - \sum_{m=j+1}^{i-1} < \overline{K}_m^j > \widehat{U}_i^m,$$

we obtain

$$K_i^j = \partial_t \widehat{U}_i^j + < \overline{K}_i^j > + \sum_{m=j+1}^{i-1} < \overline{K}_m^j > \widehat{U}_i^m - \sum_{m=j+1}^{i-1} K_i^m \widehat{U}_m^j \tag{56}$$

and

$$U_i^j = \partial_\theta \widehat{U}_i^j - \sum_{m=j+1}^{i-1} U_i^m \widehat{U}_m^j \tag{57}$$

Let us now consider the quantity

$$\begin{aligned}
\left| \frac{1}{t} \int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i - \alpha_j)} K_i^j < \overline{K}_i^j > \mathrm{d}\theta \right| &\leq \frac{C}{t} \left| \mathrm{e}^{<\alpha_i> - <\alpha_j>} < \overline{K}_i^j > \right| \int_{\mathbb{S}^1} \mathrm{e}^{\alpha_i - \alpha_j} \left| K_i^j \right| \mathrm{d}\theta \\
&\leq \frac{C}{t} \int_{\mathbb{S}^1} \mathrm{e}^{<\alpha_i> - <\alpha_j>} \left| \overline{K}_i^j \right| \mathrm{d}\theta \int_{\mathbb{S}^1} \mathrm{e}^{\alpha_i - \alpha_j} \left| K_i^j \right| \mathrm{d}\theta \\
&\leq \frac{C}{t} \left[\int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i - \alpha_j)} \left(\overline{K}_i^j \right)^2 \mathrm{d}\theta \right]^{\frac{1}{2}} \left[\int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i - \alpha_j)} \left(K_i^j \right)^2 \mathrm{d}\theta \right]^{\frac{1}{2}} \leq \frac{C}{t} H_K,
\end{aligned}$$

and therefore,

$$\frac{1}{t}\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} K_i^j < \overline{K}_i^j > \mathrm{d}\theta \in L^1\left([t_0,\infty)\right).$$

Furthermore,

$$\begin{aligned}
&\left|\frac{1}{t}\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} K_i^j < \overline{K}_m^j > \widehat{U}_i^m \mathrm{d}\theta\right| \\
&\quad\le \frac{C}{t}\left|\mathrm{e}^{<\alpha_m>-<\alpha_j>} < \overline{K}_m^j >\right| \left\|\mathrm{e}^{\alpha_i-\alpha_m}\widehat{U}_i^m\right\|_{C^0(\mathbb{S}^1,\mathbb{R})} \int\limits_{\mathbb{S}^1} \mathrm{e}^{\alpha_i-\alpha_j}\left|K_i^j\right| \mathrm{d}\theta \\
&\quad\le \frac{C}{t} H^{\frac{1}{2}} H_K \le \frac{C}{t} H_K
\end{aligned}$$

so,

$$\frac{1}{t}\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} K_i^j < \overline{K}_m^j > \widehat{U}_i^m \mathrm{d}\theta \in L^1\left([t_0,\infty)\right).$$

In addition, we have

$$\begin{aligned}
&\left|\frac{1}{t}\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} K_i^j K_i^m \widehat{U}_m^j \mathrm{d}\theta\right| \le \frac{1}{t}\left\|\mathrm{e}^{\alpha_m-\alpha_j}\widehat{U}_m^j\right\|_{C^0(\mathbb{S}^1,\mathbb{R})} \int\limits_{\mathbb{S}^1} \mathrm{e}^{2\alpha_i-\alpha_j-\alpha_m}\left|K_i^j\right|\left|K_i^m\right| \mathrm{d}\theta \\
&\le \frac{C}{t} H^{\frac{1}{2}}\left[\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)}\left(K_i^j\right)^2 \mathrm{d}\theta\right]^{\frac{1}{2}}\left[\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_m)}\left(K_i^m\right)^2 \mathrm{d}\theta\right]^{\frac{1}{2}} \\
&\le \frac{C}{t} H^{\frac{1}{2}} H_K \le \frac{C}{t} H_K \in L^1\left([t_0,\infty)\right).
\end{aligned}$$

Note: In the following, we will use the convention from [3] that dots denote terms in $L^1\left([t_0,\infty)\right)$.

Using the results above, we obtain

$$\begin{aligned}
\int\limits_{t_0}^{t}\frac{1}{s}\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)}\left(K_i^j\right)^2 \mathrm{d}\theta\mathrm{d}s &= \int\limits_{t_0}^{t}\frac{1}{s}\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} K_i^j \partial_t \widehat{U}_i^j \mathrm{d}\theta\mathrm{d}s + \ldots \\
&= \frac{1}{s}\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} K_i^j \widehat{U}_i^j \mathrm{d}\theta\Bigg|_{t_0}^{t} - \int\limits_{t_0}^{t}\frac{1}{s^2}\int\limits_{\mathbb{S}^1} \partial_t\left[s\mathrm{e}^{2(\alpha_i-\alpha_j)} K_i^j\right]\widehat{U}_i^j \mathrm{d}\theta\mathrm{d}s + \ldots \\
&= -\int\limits_{t_0}^{t}\frac{1}{s^2}\int\limits_{\mathbb{S}^1} \partial_t\left[s\mathrm{e}^{2(\alpha_i-\alpha_j)} K_i^j\right]\widehat{U}_i^j \mathrm{d}\theta\mathrm{d}s + \ldots
\end{aligned}$$

because

$$\begin{aligned}
\left|\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} K_i^j \widehat{U}_i^j \mathrm{d}\theta\right| &\le C\left\|\mathrm{e}^{\alpha_i-\alpha_j}\widehat{U}_i^j\right\|_{C^0(\mathbb{S}^1,\mathbb{R})}\left[\int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)}\left(K_i^j\right)^2 \mathrm{d}\theta\right]^{\frac{1}{2}} \\
&\le C H^{\frac{1}{2}} H_K^{\frac{1}{2}} \le CH < \infty.
\end{aligned}$$

Now, using (57) and integration by parts, we can compute

$$-\int_{t_0}^{t} \frac{1}{s} \int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} \left(U_i^j\right)^2 \mathrm{d}\theta \mathrm{d}s = \int_{t_0}^{t} \frac{1}{s^2} \int_{\mathbb{S}^1} \partial_\theta \left[s\mathrm{e}^{2(\alpha_i-\alpha_j)} U_i^j\right] \widehat{U}_i^j \mathrm{d}\theta \mathrm{d}s$$
$$+ \sum_{m=j+1}^{i-1} \int_{t_0}^{t} \frac{1}{s} \int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} U_i^j U_i^m \widehat{U}_m^j \mathrm{d}\theta \mathrm{d}s + \dots$$

Putting the previous results together and using the Euler-Lagrange equations (33), we obtain

$$\int_{t_0}^{t} \frac{1}{s} \int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} \left[\left(K_i^j\right)^2 - \left(U_i^j\right)^2\right] \mathrm{d}\theta \mathrm{d}s = \int_{t_0}^{t} \frac{1}{s^2} \int_{\mathbb{S}^1} \Big\{-\partial_t \left[s\mathrm{e}^{2(\alpha_i-\alpha_j)} K_i^j\right]$$
$$+ \partial_\theta \left[s\mathrm{e}^{2(\alpha_i-\alpha_j)} U_i^j\right]\Big\} \widehat{U}_i^j \mathrm{d}\theta \mathrm{d}s + \sum_{m=j+1}^{i-1} \int_{t_0}^{t} \frac{1}{s} \int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} U_i^j U_i^m \widehat{U}_m^j \mathrm{d}\theta \mathrm{d}s + \dots$$
$$= \int_{t_0}^{t} \frac{1}{s^2} \int_{\mathbb{S}^1} \Bigg\{ \sum_{m=i+1}^{N} s\mathrm{e}^{2(\alpha_m-\alpha_j)} \left(K_m^j K_m^i - U_m^j U_m^i\right)$$
$$- \sum_{m=1}^{j-1} s\mathrm{e}^{2(\alpha_i-\alpha_m)} \left(K_i^m K_j^m - U_i^m U_j^m\right) \Bigg\} \widehat{U}_i^j \mathrm{d}\theta \mathrm{d}s$$
$$+ \sum_{m=j+1}^{i-1} \int_{t_0}^{t} \frac{1}{s} \int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} U_i^j U_i^m \widehat{U}_m^j \mathrm{d}\theta \mathrm{d}s + \dots$$

Since all the "kinetic" terms belong to $L^1([t_0, \infty))$, we have

$$\int_{t_0}^{t} \frac{1}{s} \int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} \left(U_i^j\right)^2 \mathrm{d}\theta \mathrm{d}s = \sum_{m=i+1}^{N} \int_{t_0}^{t} \frac{1}{s} \int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_m-\alpha_j)} U_m^j U_m^i \widehat{U}_i^j \mathrm{d}\theta \mathrm{d}s$$
$$- \sum_{m=1}^{j-1} \int_{t_0}^{t} \frac{1}{s} \int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_m)} U_i^m U_j^m \widehat{U}_i^j \mathrm{d}\theta \mathrm{d}s \tag{58}$$
$$- \sum_{m=j+1}^{i-1} \int_{t_0}^{t} \frac{1}{s} \int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} U_i^j U_i^m \widehat{U}_m^j \mathrm{d}\theta \mathrm{d}s + \dots$$

To prove that

$$\frac{1}{t} \int_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i-\alpha_j)} \left(U_i^j\right)^2 \mathrm{d}\theta \in L^1([t_0, \infty)),$$

we will use induction over j. Let us first fix $j=1$ and sum over i:

$$\sum_{i=2}^{N}\int_{t_0}^{t}\frac{1}{s}\int_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_i-\alpha_1)}\left(U_i^1\right)^2\mathrm{d}\theta\mathrm{d}s=\sum_{i=2}^{N}\sum_{m=i+1}^{N}\int_{t_0}^{t}\frac{1}{s}\int_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_m-\alpha_1)}U_m^1U_m^i\widehat{U}_i^1\mathrm{d}\theta\mathrm{d}s$$
$$-\sum_{i=2}^{N}\sum_{m=2}^{i-1}\int_{t_0}^{t}\frac{1}{s}\int_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_i-\alpha_1)}U_i^1U_i^m\widehat{U}_m^1\mathrm{d}\theta\mathrm{d}s+\dots$$
$$=\sum_{m=3}^{N}\sum_{i=2}^{m-1}\int_{t_0}^{t}\frac{1}{s}\int_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_m-\alpha_1)}U_m^1U_m^i\widehat{U}_i^1\mathrm{d}\theta\mathrm{d}s$$
$$-\sum_{i=3}^{N}\sum_{m=2}^{i-1}\int_{t_0}^{t}\frac{1}{s}\int_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_i-\alpha_1)}U_i^1U_i^m\widehat{U}_m^1\mathrm{d}\theta\mathrm{d}s+\dots=...$$

because the terms in the last expression differ only by an interchange of i and m. So,

$$\sum_{i=2}^{N}\int_{t_0}^{t}\frac{1}{s}\int_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_i-\alpha_1)}\left(U_i^1\right)^2\mathrm{d}\theta\mathrm{d}s<\infty.$$

Assume now that

$$\sum_{i=j+1}^{N}\int_{t_0}^{t}\frac{1}{s}\int_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_i-\alpha_j)}\left(U_i^j\right)^2\mathrm{d}\theta\mathrm{d}s<\infty.$$

We want to prove that

$$\sum_{i=j+2}^{N}\int_{t_0}^{t}\frac{1}{s}\int_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_i-\alpha_{j+1})}\left(U_i^{j+1}\right)^2\mathrm{d}\theta\mathrm{d}s<\infty.$$

Using the previously derived relation (58), we obtain

$$\int_{t_0}^{t}\frac{1}{s}\int_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_i-\alpha_{j+1})}\left(U_i^{j+1}\right)^2\mathrm{d}\theta\mathrm{d}s$$
$$=\sum_{m=i+1}^{N}\int_{t_0}^{t}\frac{1}{s}\int_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_m-\alpha_{j+1})}U_m^{j+1}U_m^i\widehat{U}_i^{j+1}\mathrm{d}\theta\mathrm{d}s$$
$$-\sum_{m=1}^{j}\int_{t_0}^{t}\frac{1}{s}\int_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_i-\alpha_m)}U_i^mU_{j+1}^m\widehat{U}_i^{j+1}\mathrm{d}\theta\mathrm{d}s$$
$$-\sum_{m=j+2}^{i-1}\int_{t_0}^{t}\frac{1}{s}\int_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_i-\alpha_{j+1})}U_i^{j+1}U_i^m\widehat{U}_m^{j+1}\mathrm{d}\theta\mathrm{d}s+\dots=$$

Summing over i, we arrive at

$$\begin{aligned}
&\sum_{i=j+2}^{N}\int\limits_{t_0}^{t}\frac{1}{s}\int\limits_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_i-\alpha_{j+1})}\left(U_i^{j+1}\right)^2\mathrm{d}\theta\mathrm{d}s\\
&=\sum_{i=j+2}^{N-1}\sum_{m=i+1}^{N}\int\limits_{t_0}^{t}\frac{1}{s}\int\limits_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_m-\alpha_{j+1})}U_m^{j+1}U_m^{i}\widehat{U}_i^{j+1}\mathrm{d}\theta\mathrm{d}s\\
&\quad-\sum_{i=j+2}^{N}\sum_{m=1}^{j}\int\limits_{t_0}^{t}\frac{1}{s}\int\limits_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_i-\alpha_m)}U_i^{m}U_{j+1}^{m}\widehat{U}_i^{j+1}\mathrm{d}\theta\mathrm{d}s\\
&\quad-\sum_{i=j+3}^{N}\sum_{m=j+2}^{i-1}\int\limits_{t_0}^{t}\frac{1}{s}\int\limits_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_i-\alpha_{j+1})}U_i^{j+1}U_i^{m}\widehat{U}_m^{j+1}\mathrm{d}\theta\mathrm{d}s+\dots\\
&=\sum_{m=j+3}^{N}\sum_{i=j+2}^{m-1}\int\limits_{t_0}^{t}\tfrac{1}{s}\int\limits_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_m-\alpha_{j+1})}U_m^{j+1}U_m^{i}\widehat{U}_i^{j+1}\mathrm{d}\theta\mathrm{d}s\\
&\quad-\sum_{i=j+2}^{N}\sum_{m=1}^{j}\int\limits_{t_0}^{t}\frac{1}{s}\int\limits_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_i-\alpha_m)}U_i^{m}U_{j+1}^{m}\widehat{U}_i^{j+1}\mathrm{d}\theta\mathrm{d}s\\
&-\sum_{m=j+3}^{N}\sum_{i=j+2}^{m-1}\int\limits_{t_0}^{t}\frac{1}{s}\int\limits_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_m-\alpha_{j+1})}U_m^{j+1}U_m^{i}\widehat{U}_i^{j+1}\mathrm{d}\theta\mathrm{d}s+\dots\\
&=-\sum_{i=j+2}^{N}\sum_{m=1}^{j}\int\limits_{t_0}^{t}\frac{1}{s}\int\limits_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_i-\alpha_m)}U_i^{m}U_{j+1}^{m}\widehat{U}_i^{j+1}\mathrm{d}\theta\mathrm{d}s+\dots
\end{aligned}$$

because the first and third term cancel. Consider now the quantity

$$\begin{aligned}
&\left|\frac{1}{t}\int\limits_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_i-\alpha_m)}U_i^{m}U_{j+1}^{m}\widehat{U}_i^{j+1}\mathrm{d}\theta\right|\\
&\leq\frac{C}{t}\left\|\mathrm{e}^{\alpha_i-\alpha_j}\widehat{U}_i^{j}\right\|_{C^0(\mathbb{S}^1,\mathbb{R})}\int\limits_{\mathbb{S}^1}\mathrm{e}^{-2\alpha_m+\alpha_i+\alpha_{j+1}}\left|U_i^m\right|\left|U_{j+1}^m\right|\mathrm{d}\theta\\
&\leq\frac{C}{t}H^{\frac{1}{2}}\left[\int\limits_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_i-\alpha_m)}\left(U_i^m\right)^2\mathrm{d}\theta\right]^{\frac{1}{2}}\left[\int\limits_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_{j+1}-\alpha_m)}\left(U_{j+1}^m\right)^2\mathrm{d}\theta\right]^{\frac{1}{2}}\\
&\leq\frac{C}{t}\left[\sum_{i=m+1}^{N}\int\limits_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_i-\alpha_m)}\left(U_i^m\right)^2\mathrm{d}\theta\right]^{\frac{1}{2}}\left[\sum_{i=m+1}^{N}\int\limits_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_i-\alpha_m)}\left(U_i^m\right)^2\mathrm{d}\theta\right]^{\frac{1}{2}}\\
&\leq\frac{C}{t}\sum_{i=m+1}^{N}\int\limits_{\mathbb{S}^1}\mathrm{e}^{2(\alpha_i-\alpha_m)}\left(U_i^m\right)^2\mathrm{d}\theta<\infty,
\end{aligned}$$

because of the induction hypothesis and the fact that $1 \leq m \leq j$. Thus,

$$\sum_{i=j+2}^{N} \int\limits_{t_0}^{t} \frac{1}{s} \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i - \alpha_{j+1})} \left(U_i^{j+1} \right)^2 \mathrm{d}\theta \mathrm{d}s < \infty,$$

which completes the inductive argument. Thus, we obtain the desired result:

$$\frac{1}{t} \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i - \alpha_j)} \left(U_i^j \right)^2 \mathrm{d}\theta \in L^1 \left([t_0, \infty) \right), \text{ for } 1 \leq j < i \leq N. \tag{59}$$

In order to prove that $\frac{1}{t} H_P(t) \in L^1 \left([t_0, \infty) \right)$, we finally need to show that

$$\frac{1}{t} \int\limits_{\mathbb{S}^1} (p_m)_\theta^2 \, \mathrm{d}\theta \in L^1 \left([t_0, \infty) \right), \text{ for } 1 \leq m \leq N-1.$$

To that end, we need to estimate the quantity

$$\int\limits_{t_0}^{t} \frac{1}{s} \int\limits_{\mathbb{S}^1} \left[(p_m)_t^2 - (p_m)_\theta^2 \right] \mathrm{d}\theta \mathrm{d}s.$$

We first note the following facts:

$$\frac{1}{t} \int\limits_{\mathbb{S}^1} (p_m)_t < p_m >_t \mathrm{d}\theta \in L^1 \left([t_0, \infty) \right), \tag{60}$$

$$\frac{1}{t} \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i - \alpha_j)} \left[\left(K_i^j \right)^2 - \left(U_i^j \right)^2 \right] (p_m - < p_m >) \, \mathrm{d}\theta \in L^1 \left([t_0, \infty) \right). \tag{61}$$

The first relation follows from

$$\left| \frac{1}{t} \int\limits_{\mathbb{S}^1} (p_m)_t < p_m >_t \mathrm{d}\theta \right| \leq \left| \frac{1}{t} \left(< p_m >_t \right)^2 \right| \leq \frac{C}{t} H_K \in L^1 \left([t_0, \infty) \right).$$

That (61) holds, can be easily seen from

$$\begin{aligned}
&\left| \frac{1}{t} \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i - \alpha_j)} \left[\left(K_i^j \right)^2 - \left(U_i^j \right)^2 \right] (p_m - < p_m >) \, \mathrm{d}\theta \right| \\
&\quad \leq \frac{1}{t} \left\| p_m - < p_m > \right\|_{C^0(\mathbb{S}^1, \mathbb{R})} \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i - \alpha_j)} \left[\left(K_i^j \right)^2 - \left(U_i^j \right)^2 \right] \mathrm{d}\theta \\
&\quad \leq \frac{C}{t} H_K^{\frac{1}{2}} \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i - \alpha_j)} \left[\left(K_i^j \right)^2 - \left(U_i^j \right)^2 \right] \mathrm{d}\theta \\
&\quad \leq \frac{C}{t} \int\limits_{\mathbb{S}^1} \mathrm{e}^{2(\alpha_i - \alpha_j)} \left[\left(K_i^j \right)^2 - \left(U_i^j \right)^2 \right] \mathrm{d}\theta \in L^1 \left([t_0, \infty) \right),
\end{aligned}$$

where we have used (39), (54) and (59).

Using (60) and (61) together with the Euler-Lagrange equations for p_m, (32), and integration by parts, we can calculate as follows:

$$\begin{aligned}
&\int_{t_0}^{t} \frac{1}{s} \int_{\mathbb{S}^1} \left[(p_m)_t^2 - (p_m)_\theta^2 \right] \mathrm{d}\theta \mathrm{d}s \\
&= - \int_{t_0}^{t} \frac{1}{s} \int_{\mathbb{S}^1} \left\{ \partial_t \left[s \left(p_m \right)_t \right] - \partial_\theta \left[s \left(p_m \right)_\theta \right] \right\} \left(p_m - < p_m > \right) \mathrm{d}\theta \mathrm{d}s + \dots \\
&= \int_{t_0}^{t} \frac{1}{s} \int_{\mathbb{S}^1} \sum_{i=2}^{N} \sum_{j=1}^{i-1} \left(-c_i^m + c_j^m \right) \mathrm{e}^{2(\alpha_i - \alpha_j)} \left[\left(K_i^j \right)^2 - \left(U_i^j \right)^2 \right] \left(p_m - < p_m > \right) \mathrm{d}\theta \mathrm{d}s + \dots \\
&= \dots
\end{aligned}$$

Due to (54), this implies that

$$\frac{1}{t} \int_{\mathbb{S}^1} (p_m)_\theta^2 \, \mathrm{d}\theta \in L^1 \left([t_0, \infty) \right)$$

which together with (59) gives the desired result:

$$\frac{1}{t} H_P \left(t \right) \in L^1 \left([t_0, \infty) \right) \tag{62}$$

Combining (54) and (62), it follows that

$$\frac{1}{t} H \left(t \right) \in L^1 \left([t_0, \infty) \right),$$

so H converges to 0 as $t \to \infty$. □

We now finally have all necessary ingredients to prove the main result of this thesis.

Proposition 11. *Consider a solution of* (1). *Then there exist positive C and T, such that for all $t \geq T$, we have*

$$H \left(t \right) \leq \frac{C}{t}.$$

Proof. This result follows from Theorem 6 and Theorem 10. □

6 Conserved Quantities

As mentioned in Section 2, the existence of conserved quantities plays a crucial role in the analysis of the asymptotic behaviour for large t of the solution in the 4-dimensional spacetime. A similar type of analysis was one of the goals of this thesis. Unfortunately, this part could not be completed, however one can show that the n-dimensional Gowdy spacetime also possesses an invariant quantity analogous to (7). The present section is concerned with the construction of this invariant from time-conserved quantities.

The lagrangian (17),

$$\mathcal{L} = \frac{t}{4}\mathrm{Tr}\left[-\left(\tilde{g}^{-1}\partial_t\tilde{g}\right)^2 + \left(\tilde{g}^{-1}\partial_\theta\tilde{g}\right)^2\right].$$

is manifestly $SL(N)$ invariant, consequently, by Noether's theorem, there exist N^2-1 (= dimension of $SL(N)$) conserved quantities of the form

$$\int_{\mathbb{S}^1} t\mathrm{Tr}\left(\tau\tilde{g}^{-1}\partial_t\tilde{g}\right)\mathrm{d}\theta$$

where τ is a generator of the Lie algebra $\mathbf{sl}(N)$. To see that this is the case, consider an element v of $SL(N)$ which acts on $\widetilde{M} = SL(N)/SO(N)$ by the transformation $\tilde{g} \to v^T\tilde{g}v \in \widetilde{M}$ corresponding to a local change of basis on the orbits of $\mathbb{T}^N$ in the n-dimensional Gowdy spacetime. This transformation is a symmetry of the lagrangian: $\mathcal{L} \to \mathcal{L}$. To determine the conserved current corresponding to this symmetry, we need to study the behaviour of our model under infinitesimal transformations. We first note that $v \in SL(N)$ is of the form

$$v = \exp\left(\epsilon\tau\right)$$

for $\epsilon \in \mathbb{R}$ and τ an element of $\mathrm{sl}(N)$, the Lie algebra of $SL(N)$ which has a representation as the algebra of $N \times N$ trace-free matrices. For $\epsilon \to 0$, we have

$$\begin{aligned}\delta\tilde{g}_{\alpha\beta} &= \epsilon\left(\tau^T\tilde{g} + \tilde{g}\tau\right)_{\alpha\beta}\\ \delta\tilde{g}^{\alpha\beta} &= -\epsilon\left(\tau\tilde{g}^{-1} + \tilde{g}^{-1}\tau^T\right)^{\alpha\beta}\end{aligned}$$

Applying now Noether's theorem, we obtain the expression of the conserved current

$$J^a = -\frac{\partial\mathcal{L}}{\partial\left(\partial_a\tilde{g}^{\alpha\beta}\right)}\frac{\partial\tilde{g}^{\alpha\beta}}{\partial\epsilon} = -\frac{t}{2}\mathrm{Tr}\left[\hat{h}^{ab}\partial_b\tilde{g}\left(\tau\tilde{g}^{-1} + \tilde{g}^{-1}\tau^T\right)\right],$$

where $\hat{h}^{ab} = \mathrm{diag}(-1,1)$ and we have used the relations

$$\partial_a\tilde{g}_{\alpha\beta} = -\tilde{g}_{\alpha\mu}\partial_a\tilde{g}^{\mu\nu}\tilde{g}_{\nu\beta},$$

and

$$\frac{\partial\left(\partial_a\tilde{g}_{\alpha\beta}\right)}{\partial\left(\partial_a\tilde{g}^{\mu\nu}\right)} = -\tilde{g}_{\alpha\mu}\tilde{g}_{\nu\beta}.$$

The density of the conserved charge is then given by:

$$Q^0_\tau = \frac{t}{2}\mathrm{Tr}\left[\partial_t\tilde{g}\left(\tau\tilde{g}^{-1} + \tilde{g}^{-1}\tau^T\right)\right] = t\mathrm{Tr}\left(\tau\tilde{g}^{-1}\partial_t\tilde{g}\right),$$

where we have used the cyclicity of the trace and the symmetry of the metric $\tilde{g}$. Thus, we obtain N^2-1 time-conserved quantities corresponding to the N^2-1 generators of $\mathrm{sl}\,(N)$:

$$C_{(a)} = t \int\limits_{\mathbb{S}^1} \mathrm{Tr}\left(\tau_{(a)} \tilde{g}^{-1} \partial_t \tilde{g}\right) \mathrm{d}\theta \tag{63}$$

Let us now construct a basis for the Lie algebra $\mathbf{sl}(N)$ of $N \times N$ trace-free matrices as follows: denote by e_{ij}, with $i, j \in \{1, \ldots, N\}$, the $N \times N$ matrix with all entries equal to 0, except the entry (i, j), which is equal to 1, and define

$$\begin{aligned} \tau_{ij} &:= e_{ij} \text{ if } i \neq j, \\ d_i &:= e_{ii} - e_{NN} \text{ for } i < N. \end{aligned} \tag{64}$$

Then τ_{ij} and d_i form a basis for $\mathbf{sl}(N)$ and satisfy the following commutation relations

$$\begin{aligned} [\tau_{ij}, \tau_{kl}] &= \delta_{jk}\left[\bar{\delta}_{il}\tau_{il} + \delta_{il}\left(\bar{\delta}_{iN} d_i - \bar{\delta}_{jN} d_j\right)\right] - \bar{\delta}_{jk}\delta_{il}\tau_{kj}, \\ [\tau_{kl}, d_i] &= \left(-\delta_{ki} + \delta_{li} + \delta_{kN} - \delta_{lN}\right)\tau_{kl}, \\ [d_i, d_j] &= 0, \end{aligned} \tag{65}$$

where we have defined: $\bar{\delta}_{ij} := 1 - \delta_{ij}$. Let us now denote the conserved quantities in the following manner:

$$\begin{aligned} A_i &:= \int\limits_{\mathbb{S}^1} t\mathrm{Tr}\left(d_i \tilde{g}^{-1} \partial_t \tilde{g}\right) \mathrm{d}\theta \text{ for } i \in \{1, N-1\}, \\ B_{ij} &:= \int\limits_{\mathbb{S}^1} t\mathrm{Tr}\left(\tau_{ij} \tilde{g}^{-1} \partial_t \tilde{g}\right) \mathrm{d}\theta \text{ for } i \neq j \text{ and } i, j \in \{1, N\} \end{aligned}$$

Note that in the 4-dimensional spacetime, i. e. for $N = 2$, with

$$\left(\tilde{g}_{ij}\right)_{i,j \leq 2} = \begin{pmatrix} \mathrm{e}^{-P} + \mathrm{e}^{P} Q^2 & \mathrm{e}^{P} Q \\ \mathrm{e}^{P} Q & \mathrm{e}^{P} \end{pmatrix}$$

the above formulas give, as expected from [3],

$$\begin{aligned} A_1 &= \int\limits_{\mathbb{S}^1} t\left(2\mathrm{e}^{2P} Q_t - 2P_t\right) \mathrm{d}\theta \\ B_{12} &= \int\limits_{\mathbb{S}^1} t\left[\left(1 - \mathrm{e}^{2P} Q^2\right) Q_t + 2P_t Q\right] \mathrm{d}\theta \\ B_{21} &= \int\limits_{\mathbb{S}^1} t\mathrm{e}^{2P} Q_t \mathrm{d}\theta. \end{aligned} \tag{66}$$

We have $\partial_t A_i = \partial_t B_{ij} = 0$, but A_i and B_{ij} are not invariant under the action of $SL(N)$. To find such an invariant, we need to compute the Casimir operator of $\mathbf{sl}(N)$. To understand the connection between the invariant quantity and the Casimir operator, let us note that the conserved quantities A_i and B_{ij} constitute a representation of $\mathbf{sl}(N)$, with the commutators given by Poisson brackets. The Casimir operator of $\mathbf{sl}(N)$.can then be represented as a quadratic expression of A_i and B_{ij}. This expression, by definition of the Casimir operator, commutes with all generators of $\mathbf{sl}(N)$ and, therefore, is invariant under the action of $SL(N)$.

For reasons of simplicity, we will work in the followíng with the matrix representation of $\mathbf{sl}(N)$, as given by (64) and (65). Let us number the generators (64) as follows:

$$\begin{aligned} f_{N(i-1)+j} &:= \tau_{ij} \text{ for } i \neq j, \text{ and } i,j \in \{1,N\}\,, \\ f_{N(i-1)+i} &:= d_i \text{ for } i \in \{1, N-1\} \end{aligned}$$

There are $N(N-1)+N-1 = N^2-1$ such generators, as expected. From the expression of the commutators (65), we can derive the structure constants of $\mathbf{sl}(N)$. For $i \neq j$ and $k \neq l$, the only (possibly) non-vanishing structure constants are of the form

$$\begin{aligned} c^{N(i-1)+l}_{N(i-1)+j,\ N(k-1)+l} &= \delta_{jk}\bar{\delta}_{il}, \\ c^{N(k-1)+j}_{N(i-1)+j,\ N(k-1)+l} &= -\bar{\delta}_{jk}\delta_{il}, \\ c^{N(i-1)+i}_{N(i-1)+j,\ N(k-1)+l} &= \delta_{jk}\delta_{il}\bar{\delta}_{iN}, \\ c^{N(j-1)+j}_{N(i-1)+j,\ N(k-1)+l} &= -\delta_{jk}\delta_{il}\bar{\delta}_{jN}, \\ c^{N(k-1)+l}_{N(i-1)+i,\ N(k-1)+l} &= \left(\delta_{ik} - \delta_{il} - \delta_{kN} + \delta_{lN}\right)\bar{\delta}_{iN}. \end{aligned}$$

with

$$[f_I,\, f_J] = c^K_{IJ} f_K, \text{ for } I,\, J,\, K \in \left\{1, \ldots N^2-1\right\}$$

With this information, we can compute the Killing form of $\mathbf{sl}(N)$ which, by definition, is given by

$$\mathbf{g}_{IJ} = \sum_{K,L} c^L_{IK} c^K_{JL}$$

After some calculations (involving many Kronecker deltas), we can determine the non-vanishing components of $\mathbf{g}$. For $i \neq j$ and $k \neq l$, these can be written as follows:

$$\begin{aligned} \mathbf{g}_{N(i-1)+j,\ N(k-1)+l} &= 2N\delta_{jk}\delta_{il}, \\ \mathbf{g}_{N(i-1)+i,\ N(j-1)+j} &= 2N, \text{ for } i,j \leq N-1, \\ \mathbf{g}_{N(i-1)+i,\ N(i-1)+i} &= 4N, \text{ for } i \leq N-1. \end{aligned}$$

To compute the Casimir operator, we need to determine the inverse $\mathbf{g}^{IJ}$. By inspection, we can tentatively write

$$\begin{aligned}
\mathbf{g}^{N(i-1)+j,\ N(k-1)+l} &= \frac{1}{2N}\delta_{jk}\delta_{il}, \\
\mathbf{g}^{N(i-1)+i,\ N(j-1)+j} &= -\frac{1}{2Nx},\ \text{for } i,j \le N-1, \\
\mathbf{g}^{N(i-1)+i,\ N(j-1)+i} &= \frac{1}{4Ny},\ \text{for } i \le N-1.
\end{aligned}$$

Requiring that $\sum_J \mathbf{g}_{IJ}\mathbf{g}^{JK} = \delta_{IK}$, it follows that $x = \frac{N}{2}$ and $y = \frac{N}{2\,(N-1)}$.

Consequently, we obtain

$$\begin{aligned}
\mathbf{g}^{N(i-1)+j,\ N(k-1)+l} &= \frac{1}{2N}\delta_{jk}\delta_{il}, \\
\mathbf{g}^{N(i-1)+i,\ N(j-1)+j} &= -\frac{1}{2N^2},\ \text{for } i,j \le N-1, \\
\mathbf{g}^{N(i-1)+i,\ N(j-1)+i} &= \frac{N-1}{2N^2},\ \text{for } i \le N-1.
\end{aligned}$$

The Casimir operator, defined by

$$\mathbf{C} = \sum_{I,J} \mathbf{g}^{IJ} f_I f_J,$$

can now be written as

$$\begin{aligned}
\mathbf{C} &= \frac{N-1}{2N^2}\sum_{i<N} d_i^2 - \frac{1}{2N^2}\sum_{N>i\neq j<N} d_i d_j + \frac{1}{2N}\sum_{i\neq j}\tau_{ij}\tau_{ji} \\
&= \frac{N-1}{2N^2}\left(\sum_{i<N} d_i^2 - \frac{1}{N-1}\sum_{N>i\neq j<N} d_i d_j + \frac{N}{N-1}\sum_{i\neq j}\tau_{ij}\tau_{ji}\right)
\end{aligned}$$

Dropping the constant factor in front of the parenthesis, we can write down the expression of the invariant quantity in the A_i, B_{ij} representation as follows:

$$D = \sum_{i=1}^{N-1} A_i^2 - \frac{2}{N-1}\sum_{i=2}^{N-1}\sum_{j=1}^{i-1} A_i A_j + \frac{2N}{N-1}\sum_{i=2}^{N-1}\sum_{j=1}^{i-1} B_{ij}B_{ji} \tag{67}$$

In four dimensions, i. e. for $N = 2$, this quantity has the form

$$D = A_1^2 + 4B_{21}B_{12}$$

with A_1, B_{12} and B_{21} as given in (66). This, again, agrees with the previous result of Ringström presented in [3].

The invariance of (67) can also be seen by studying the action of an element $v \in SL(N)$ on the conserved quantities A_i and B_{ij}. For the following, it is sufficient to take v of the form $v = \exp(\epsilon\tau_{ij})$ or $v = \exp(\epsilon d_i)$. Noting that $\tau_{ij}^2 = 0$, $d_i^2 = d_i + 2e_{NN}$ and $d_i^3 = d_i$, we can write

$$\begin{aligned}
\exp(\epsilon\tau_{ij}) &= \mathbf{1}_N + \epsilon\tau_{ij}, \\
\exp(\epsilon d_i) &= \mathbf{1}_N + (\mathrm{e}^{\epsilon} - 1)\, d_i + \left(\mathrm{e}^{\epsilon} + \mathrm{e}^{-\epsilon} - 2\right) e_{NN}.
\end{aligned}$$

where $\mathbf{1}_N$ is the $N \times N$ identity matrix. Under the action of v, the metric $\tilde{g}$ transforms as: $\tilde{g} \to v^T \tilde{g} v$, hence, if τ is a generator of $\mathbf{sl}(N)$ and A is the corresponding conserved quantity, then A transforms as

$$A = t \int_{\mathbb{S}^1} \mathrm{Tr}\left(\tau\tilde{g}^{-1}\partial_t\tilde{g}\right) \mathrm{d}\theta \to t \int_{\mathbb{S}^1} \mathrm{Tr}\left(v\tau v^{-1}\tilde{g}^{-1}\partial_t\tilde{g}\right) \mathrm{d}\theta$$

Using this information and the commutation relations (65), we can determine how A_i and B_{ij} transform under v. Thus, when $v = \exp(\epsilon d_i)$, we get

$$\begin{aligned}
A_j &\to A_j, \\
B_{jk} &\to B_{jk} \exp\left[(-\delta_{ki} - \delta_{jN} + \delta_{ji} + \delta_{kN})\,\epsilon\right]
\end{aligned}$$

while for $v = \exp(\epsilon\tau_{kl})$ we have

$$\begin{aligned}
A_i &\to A_i \ + \epsilon\,(-\delta_{ki} + \delta_{li} + \delta_{kN} - \delta_{lN})\, B_{kl}, \\
B_{ij} &\to B_{ij} \ - \epsilon\left\{\delta_{jk}\left[\bar{\delta}_{il} B_{il} + \delta_{il}\left(\bar{\delta}_{iN} A_i - \bar{\delta}_{jN} A_j\right)\right] - \bar{\delta}_{jk}\delta_{il} B_{kj}\right\} \\
&\qquad - \ \epsilon^2 \delta_{jk}\delta_{il}\left(\bar{\delta}_{iN}\delta_{kN} B_{Nl} + \bar{\delta}_{jN} B_{jl}\right)
\end{aligned}$$

Applying these transformation laws to D in (67), we obtain, after some lengthy but straightforward calculations: $D \to D$, which is the desired invariance property.

As to the physical significance of this invariant, things are not too clear in the general case. However, for homogeneous data, the invariant quantity turns out to be a multiple of the kinetic energy of the wave map. Recall that the invariant can be written in the form

$$D = \sum_{i<N} A_i^2 - \frac{1}{N-1} \sum_{N>i\neq j<N} A_i A_j + \frac{N}{N-1} \sum_{i\neq j} B_{ij} B_{ji}$$

where

$$\begin{aligned}
A_i &:= \int_{\mathbb{S}^1} t\mathrm{Tr}\left(d_i \tilde{g}^{-1}\partial_t\tilde{g}\right) \mathrm{d}\theta, \ \text{ for } i \in \{1, N-1\}, \\
B_{ij} &:= \int_{\mathbb{S}^1} t\mathrm{Tr}\left(\tau_{ij}\tilde{g}^{-1}\partial_t\tilde{g}\right) \mathrm{d}\theta, \ \text{ for } i \neq j \text{ and } i, j \in \{1, N\}
\end{aligned}$$

Let us denote

$$R := \tilde{g}^{-1}\partial_t\tilde{g}.$$

Since for an invertible matrix A we have

$$\frac{\partial}{\partial t}(\det A) = \det A \operatorname{Tr}\left(A^{-1}\frac{\partial A}{\partial t}\right),$$

it follows that R has the property

$$\operatorname{Tr}(R) = 0,$$

or

$$\sum_{i<N} R_{ii} = -R_{NN}.$$

Note that we have:

$$\begin{aligned}\operatorname{Tr}(d_i R) &= R_{ii} - R_{NN} \text{ for } i < N,\\ \operatorname{Tr}(\tau_{ij} R) &= R_{ji} \text{ for } i \neq j.\end{aligned}$$

In the homogeneous case (no θ-dependence), the integrals over θ give a factor of 2π and we obtain

$$\operatorname{Tr}\left(R^2\right) = \frac{2}{\pi} H_K,$$

where H_K is the kinetic energy of the wave-map $\tilde{g}$.

With this information we can compute the invariant quantity in the homogeneous case as follows:

$$\begin{aligned}
D_h = 4\pi^2 t^2 &\Bigg[\sum_{i<N}(R_{ii} - R_{NN})^2 - \frac{1}{N-1}\sum_{N>i\neq j<N}(R_{ii} - R_{NN})(R_{jj} - R_{NN})\\
&+ \frac{N}{N-1}\sum_{i\neq j} R_{ji} R_{ij}\Bigg]\\
= 4\pi^2 t^2 &\Bigg\{\sum_{i<N} R_{ii}^2 - 2R_{NN}\sum_{i<N} R_{ii} + (N-1) R_{NN}^2\\
&- \frac{1}{N-1}\left[\sum_{N>i\neq j<N} R_{ii} R_{jj} - 2(N-2) R_{NN}\sum_{i<N} R_{ii} + (N-1)(N-2) R_{NN}^2\right]\\
&+ \frac{N}{N-1}\left[\operatorname{Tr}\left(R^2\right) - \sum_{i<N} R_{ii}^2 - R_{NN}^2\right]\Bigg\}\\
= 4\pi^2 t^2 &\Bigg\{\sum_{i<N} R_{ii}^2 + (N+1) R_{NN}^2\\
&- \frac{1}{N-1}\left[\sum_{i<N} R_{ii}(-R_{ii} - R_{NN}) + 2(N-2) R_{NN}^2 + (N-1)(N-2) R_{NN}^2\right]\\
&+ \frac{N}{N-1}\left[\operatorname{Tr}\left(R^2\right) - \sum_{i<N} R_{ii}^2 - R_{NN}^2\right]\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&= 4\pi^2 t^2 \left[\frac{N}{N-1} \mathrm{Tr}\left(R^2\right) + \frac{1}{N-1} R_{NN}^2 + \frac{1}{N-1} R_{NN} \sum_{i<N} R_{ii} \right] \\
&= 4\pi^2 t^2 \frac{N}{N-1} \mathrm{Tr}\left(R^2\right) \; = \; 8\pi t^2 \frac{N}{N-1} H_K.
\end{aligned}
$$

Thus, in the homogeneous case, the invariant is always positive and proportional to $t^2 H_K$. For $N = 2$, we have $\mathrm{Tr}\left(R^2\right) = 2\left(P_t^2 + \mathrm{e}^{2P} Q_t^2\right)$, so the invariant in the case of homogeneous data is given by:

$$
D_h = 16\pi^2 t^2 \left(P_t^2 + \mathrm{e}^{2P} Q_t^2\right),
$$

which agrees with the result in [3].

7 Conclusions

There are many open problems regarding the asymptotic behaviour of higher-dimensional Gowdy spacetimes with torus topology. Although one can derive the same decay law for the energy as in the 4-dimensional case, the dynamics of the solution cannot be readily generalized. This is mainly due to the complexity of the formulas for conserved quantities in higher dimensions. In $n = 4$, these quantities, or rather quadratic expressions of them, determine how the solution behaves as $t \to \infty$; however, in higher dimensions they are, if not intractable, at least very difficult to work with. An interesting issue is the physical meaning of the invariant quantity (67) which, in the 4-dimensional case, governs the asymptotic behaviour. In the case of homogeneous solutions, this quantity is, at fixed time, proportional to the kinetic energy of the wave-map describing the solution. In the general case, its physical meaning is unclear. A further topic of interest is finding a geometric framework for the type of arguments presented here and elsewhere ([3], [4]). In this sense, it is noteworthy that formulas involved here have an algebraically convoluted form, and it is not at all trivial that in the end it all comes out "as it should", indeed, it seems almost magical that it does happen. This hints to the existence of underlying geometric issues which are, however, still a mystery.

References

[1] Gowdy, R. H. Vacuum spacetimes with two-parameter spacelike isometry groups and compact invariant hypersurfaces: topologies and boundary conditions. *Ann. Physics* **83** (1974), 203–241.

[2] Moncrief, V. Global properties of Gowdy spacetimes with $T^3 \times \mathbb{R}$ topology. *Ann. Physics* **132** (1981), no. 1, 87–107.

[3] Ringström, H. On a wave map equation arising in general relativity. *Comm. Pure Appl. Math.* **57** (2004), 657–703.

[4] Ringström, H. On the T^3-Gowdy Symmetric Einstein-Maxwell Equation. *Annales Henri Poincare* **7** (2006), 1–20.

[5] Tataru, D. The wave maps equation. *Bull. Amer. Math. Soc.* **41** (2004), no. 2, 185–204.

[6] Pope, C. Kaluza-Klein theory. http://faculty.physics.tamu.edu/pope/ihplec.pdf

[7] Breitenlohner, P. and Maison, D. On the Geroch group. *Ann. Inst. H. Poincaré, Phys. Théor.* **46** (1987), no. 2, 215–246.

[8] Palais, R. S. The principle of symmetric criticality. *Comm. Math. Phys.* **69** (1979), no. 1, 19–30.