Appendix 4. Rayleigh stability criteria

Brief historical overview of double-diffusive convection (DDC)

Convective flows are driven by density variations within fluid. DDC is mentioned when the density variations are caused by two factors which have different rates of diffusion. The archetypal example is heat and dissolved salt in water often referred to as thermohaline convection.

First expererimental investigations on convective currents and stability were carried out in viscous liquids by Jevons (1857), Rayleigh (1883) and Ekman (1906) over the period 1856-1906. These pioneers of convection motions generally investigated the stability of stratified fluid layers with density either increasing or decreasing by means of different observational scenarios.

As shown in the schematic picture in Fig.1, we can distinguish two main groups of instability:

- A diffusive regime favoring oscillatory instability (Fig.1A), where the destabilizing potential results from the component with larger diffusivity (i.e., colder at the top, warmer at the bottom) while the concentration gradient is stabilizing (i.e., fresher at the top, saltier at the bottom).
- A finger regime favoring monotonic instability (Fig.1B), where the driving destabilizing forces are caused by the more slowly diffusing component (i.e., saltier at the top, fresher at the bottom) while the temperature gradient is stable (warmer at the top, colder at the bottom).
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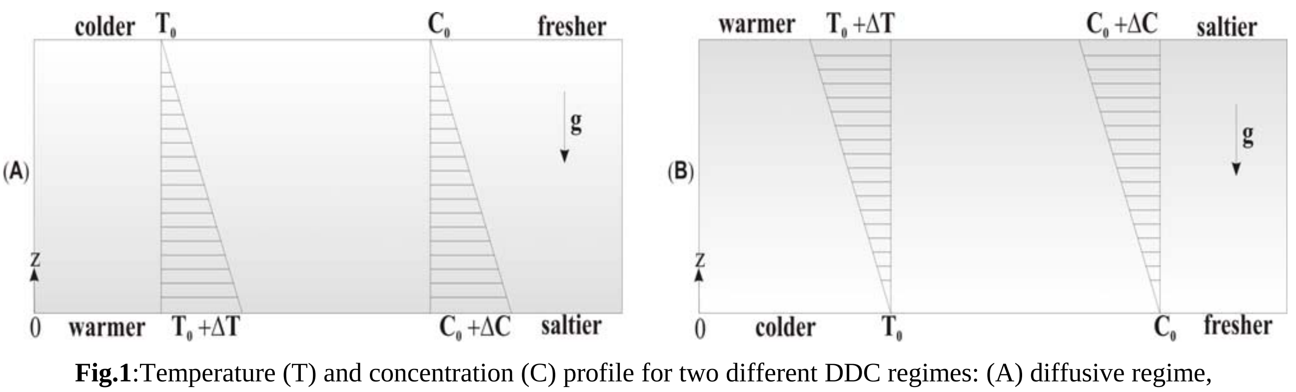

Fig.1:Temperature (T) and concentration (C) profile for two different DDC regimes: (A) diffusive regime, (B) finger regime. From Nield and Bejan 1999

From these stability studies, in 1916 Lord Rayleigh provided a criterion to predict the occurrence of instability in an adverse linear temperature gradient in a fluid layer (Rayleigh 1916). Such condition is described by a dimensionless number named in his honor, the Rayleigh number.

Nevertheless, DDC was “rediscovered” and understood only in 1960 as an oceanographic curiosity by Stommel (1956) and Stern (1960). Stern studied the long fingers of rising and sinking water which are produced when hot salty water lies over cold fresh water. A blob of hot salty water which finds itself surrounded by cold water rapidly loses its heat while retaining its salt due to the very different rates of diffusion of heat and salt (the first diffusing much more faster than the latter). Consequently, the salty blob loses part of its heat while it keeps more or less the same salt concentration hence becoming denser than the surrounding fluid. This tends to make the blob sink further, drawing down more hot salty water from above which gives rise to sinking fingers of salty fluid. At the same time, the surrounding fluid gains heat from these descending fingers and in turn it becomes lighter and moves upward. Eventually, the region becomes filled with fingers of salty and fresh water protruding in alternating downwards and upwards directions.

In the late 1960s, DDC has been demonstrated to be also possible in porous media (Bird et al. 1958; Nield 1968; Nield 1991). Since then a multitude of stability analysis based on laboratory experiments were carried out on saturated porous media with vertical gradients of temperature and salinity (Trevisan and Bejan 1987; Qin et al. 1995; Cooper et al. 1997; Tan et al. 2003; Kubitschek and Weidman 2003). Owing to the presence of the solid matrix, the theory of convective motions in porous media introduces essential differences from the one in viscous fluids. For example, according to the term of heat accumulation in equation (2.55), the contribution due to thermal absorption by the solid grains is taken into account, a feature totally absent in thermohaline convection in viscous fluids. The solid grains play a primary role on the dynamic of mass transport as already seen in the paragraph 2.5 dedicated to hydrodynamic dispersion. In addition, from equation (2.49) it results that advection is governed by Darcy’s law rather than by the Navier-Stokes equations applicable for viscous flow. Darcy’s law introduces a new fundamental parameter, the hydraulic permeability, characterizing unequivocally the porous media in which the flow takes place and controls its velocity. The equation of motion expressed in the form of (2.50) shows further that the flow is caused by two driving force: one resulting from piezometric head differences and the other resulting form a buoyancy force.

Convection stability

Here the convection stability criteria is derived for a porous media in diffusive regime, i.e. where heat and salt are respectively the destabilizing and stabilizing buoyancy forces, such as in the case of a infinite horizontal porous layer heated from below. This problem, also known as the Horton-Rogers-Lapwood problem (Horton and Rogers 1945), was studied in its double-diffusive generalization by Nield (Nield 1968). Since there are two sources of buoyancy (heat and dissolved salt), there are two Rayleigh numbers that characterize the convection processes: the thermal Rayleigh number Ra and the solutal Rayleigh number Ra_D. From a dimensional analysis of the governing balance equations in paragraph 2.5 the following definition of Rayleigh number for solutes and thermal energy are given (Nield and Bejan 1999):

$$Ra = \frac{K\beta\Delta Td}{\lambda} \tag{4.1}$$

$$Ra_D = \frac{K\alpha\Delta Cd}{D} \tag{4.2}$$

where K is the hydraulic conductivity as defined in Eq.(2.51) α introduces the effect of a density change due to the concentration the solute at temperature and pressure β is the coefficient of thermal expansion at constant pressure and concentration, λ is the coefficient of thermal conductivity, ΔC is the concentration variation, ΔT is the temperature variation, d is a characteristic length of the porous media (e.g., the layer thickness), D is the coefficient of molecular diffusion.

The solutal and thermal Rayleigh numbers are related by:

$$Ra_D = N \cdot Le \cdot Ra \tag{4.3}$$

where the dimensionless numbers in connection with heat and mass transport are:

$$\begin{aligned} N &= \frac{\alpha\Delta C}{\beta\Delta T} \\ Le &= \frac{\lambda}{D} \end{aligned} \tag{4.4}$$

N is called the buoyancy ratio (or Turner number) and Le is the Lewis number.

The Rayleigh numbers as expressed in equation (4.1) and (4.2) are the ratio of driving forces of buoyancy and gravitation to the dissipative mechanisms of viscous drag and heat conduction (Ra), and hydrodynamic dispersion (Ra_D).

In the diffusive regime considered here, Ra promotes convection while Ra_D inhibits it. If the top and bottom surface of the porous layer are impermeable, isothermal and isosolutal (C constant) boundary conditions, a criteria for the onset of DDC is summarized as follows (Nield 1991, Diersch and Kolditz 2002 and Trevisan and Bejan 1987):

- The monotonic instability (or stationary convection) boundary is a straight line defined by $Ra + Ra_D = Ra_c = 4\pi^2$, where Ra_c is the critical Rayleigh number.
- The region delimited by $Ra + Ra_D < 4\pi^2$ is a stable regime characterized by pure conduction and no convection. The portion of this region for which $\Phi Ra + Ra_D > 4\pi^2(1+\Phi)$, where $\Phi \equiv \dfrac{Le}{R}$, corresponds to oscillatory instability. The boundary lines delimiting stationary convection and oscillatory convection intersects at $Ra = \dfrac{4\pi^2\Phi}{\Phi - 1}$, $Ra_D = \dfrac{4\pi^2}{\Phi - 1}$. In Fig.2 is illustrated the case where $\Phi = 2$
- In a range between $4\pi^2 < Ra < 240\text{-}300$ steady state convective cells develop as two-dimensional rolls rotating in clockwise counter-clockwise direction. A second critical Rayleigh number $Ra_{c2} = 240\text{-}300$ is identified as an upper limit.
- For Ra> Ra_c 2 the convection regime is unstable and characterized by a transition to an oscillatory and transient convection behaviour.

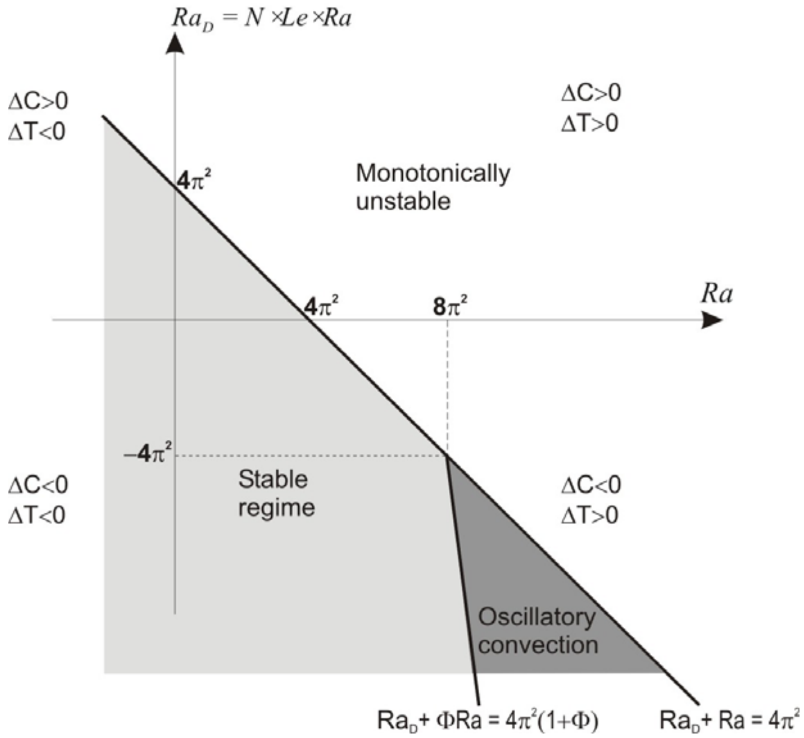

Fig.2 Stability and instability domains for DDC in a horizontal porous layer with identical set of boundary conditions for $\Phi = 2$. Modified from Nield and Bejan 1999

As mentioned by Nield and Bejan (1999), when two settings of boundary conditions are identical thermal and solutal effects are additive otherwise the two effects won't be additive. In the latter, "the coupling between them will be less than perfect and one can expect that the monotonic instability boundary will be concave toward the origin". Critical values of Ra_c for various combinations of boundary conditions are given in Table.1.

Boundary Key: K = ∞ impermeable; K = 0 constant pressure L = ∞ constant temperature; L = 0 constant heat flux **Subscript**: l = lower; u = upper				
K_l	K_u	L_l	L_u	Ra_c
∞	∞	∞	∞	$4\pi^2 \simeq 39.5$
∞	∞	0	0	27.1
∞	∞	0	0	12
∞	0	∞	∞	27.1
∞	0	0	∞	17.65
∞	0	∞	0	$\pi^2 \simeq 9.9$
∞	0	0	0	3
0	0	∞	∞	12
0	0	∞	0	3
0	0	0	0	0

Table 1 Values of the critical Rayleigh number Ra_c for different set of boundary conditions from Nield 1991