Part II

Graphical Recursion Relations for Feynman Diagrams

CHAPTER 5

Quantum Statistics

The free energy of a quantum statistical system with polynomial interaction can be considered as a functional of the free correlation function (3.182). As such it obeys a nonlinear functional differential equation which can be turned into a recursion relation [21–23]. This is solved order by order in the coupling constant of the interaction to find all connected vacuum diagrams with their proper multiplicities. The procedure is applied here to a system with quartic interaction as it occurs for the anharmonic oscillator or the double well. The results obtained with this method are, of course, the same as for a scalar field theory with a ϕ^4 self interaction.

All Feynman diagrams with external lines are obtained from functional derivatives of the connected vacuum diagrams with respect to the free correlation function. The recursive graphical construction can efficiently be automatized by computer algebra with the help of a unique matrix notation for the Feynman diagrams [23].

5.1 Introduction

Within the path integral (3.178) for the partition function Z we have expanded the Boltzmann factor of the action into a Taylor series with respect to the potential and obtained the general perturbative expansion (3.188) for the free energy F of a quantum statistical system. The perturbative coefficients are mainly determined by the time integrals over the connected correlation functions of the potential. As known from the "ordinary" Wick rule, presented for the density matrix in Section 3.4.1, these correlation functions can easily be decomposed into products of two-point functions, if the potential is of polynomial type. For the expansion of the free energy (3.188), the two-point correlation functions (3.182)–(3.185) must be used.

Since the number of contributions to the perturbative coefficients rapidly increases from order to order, it is troublesome to write them down for high orders. Moreover, many contributions are identical. The number of such repetitions is called the *multiplicity* of this contribution. It was a main simplification, when Feynman introduced his pictorial representation. The two-point correlation functions were displayed by lines with ends representing the time arguments of these two-point functions. Lines with joint end points are connected. The joint point is integrated over and is called *vertex*. In diagrams for interacting quantum fields, particles hit each other in these interaction points. Examples for the decomposition of a second-order perturbation contribution for a quartic potential into Feynman diagrams are given in Eqs. (3.140) and (3.141).

In order to circumvent the use of the lengthy analytic description of perturbative contributions for the construction of Feynman diagrams, one can approach from a topological point of view. In any order, which is in our case characterized by the number of vertices, all topologically different diagrams having the appropriate number of vertices and obeying the condition that the number of legs connected by a vertex is identical with the polynomial degree of the potential contribute to the perturbative coefficient of this order. The multiplicity of each Feynman diagram follows combinatorially from its symmetry interchanging its lines and vertices. Although the topological point of view is a main progress in comparison with the naive analytic description of perturbative coefficients, it remains a tedious task to determine all possible topologically different diagrams and their correct multiplicities of a high-order perturbative coefficient. There exist various convenient computer programs, for instance *FeynArts* [24,25] or *QGRAF* [26], for constructing and counting Feynman graphs in different field theories. These programs are based on a combinatorial enumeration of all possible ways of connecting vertices by lines according to Feynman's rules.

We develop an alternative systematic approach to construct all topologically different Feynman diagrams with their multiplicities. It relies on considering a Feynman diagram as a functional of its graphical elements, i.e. its lines and vertices. Functional derivatives with respect to these elements are represented by graphical operations which remove lines or vertices of a Feynman diagram in all possible ways. With these operations, our approach proceeds in two steps. First the connected vacuum diagrams are constructed as solutions of a graphical recursion relation derived from a nonlinear functional differential equation. In a second step, all connected diagrams with external lines are obtained from functional derivatives of the connected vacuum diagrams with respect to the free correlation function. The recursion relation enables one to automatize the process of constructing Feynman diagrams and to count the multiplicity with the help of an efficient computer algorithm which is based on a practical matrix notation for these diagrams [23].

In the following, the graphical recursion relation for the free energy of a system with x^4 potential is derived and graphically solved.

5.2 Systematic Construction of Feynman Diagrams for the Quartic Oscillator Free Energy

In order to illustrate the power of the recursive graphical construction for Feynman diagrams for quantum statistical systems with polynomial interaction, we consider the quartic oscillator in one dimension, whose thermal fluctuations are controlled by a path integral

$$Z = \oint \mathcal{D}x \, e^{-\mathcal{A}[x]} \tag{5.1}$$

over the Boltzmann factor containing the action

$$\mathcal{A}[x] = \frac{1}{2} \int_{12} x_1 G^{-1}_{12} x_2 + \frac{g}{4!} \int_{1234} V_{1234} \, x_1 x_2 x_3 x_4 \tag{5.2}$$

with some coupling constant g. In this short-hand notation, where we have also used natural units ($\hbar = k_B = M = 1$), the argument of the coordinate x, the bilocal kernel G^{-1}, and the quartic interaction V are indicated by simple number indices, i.e.

$$x_i \equiv x(\tau_i), \qquad \int_i \equiv \int_0^1 d\tau_i, \qquad G^{-1}_{12} \equiv G^{-1}(\tau_1, \tau_2), \qquad V_{1234} \equiv V(\tau_1, \tau_2, \tau_3, \tau_4). \tag{5.3}$$

The kernel is a functional matrix G^{-1}, while V is a functional tensor, both being symmetric in their indices.

In the following we shall leave G^{-1} and V completely general, except for the symmetry with respect to their indices, and insert the physical values

$$G^{-1}(\tau_1, \tau_2) = \left(-\frac{\partial^2}{\partial \tau_1^2} + \omega^2 \right) \delta(\tau_1 - \tau_2), \quad V(\tau_1, \tau_2, \tau_3, \tau_4) = \delta(\tau_1 - \tau_2)\delta(\tau_1 - \tau_3)\delta(\tau_1 - \tau_4) \tag{5.4}$$

at the end.

We may evaluate the partition function (5.1) perturbatively as a power series in the coupling constant g. From this we obtain the functional $W = \ln Z$, which is related to the free energy F of the system by $W = -\beta F$, as an expansion

$$W = \sum_{p=0}^{\infty} \frac{1}{p!} \left(\frac{-g}{4!} \right)^p W^{(p)} . \tag{5.5}$$

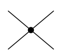
The coefficients $W^{(p)}$ may be displayed as connected vacuum diagrams constructed from lines and vertices. Each line represents a free correlation function

$$1 \;\text{———}\; 2 \quad \equiv G_{12} \, , \tag{5.6}$$

which is the functional inverse of the kernel G^{-1} in the energy functional (5.2), defined by

$$\int_2 G_{12} \, G^{-1}_{23} = \delta_{13} \, . \tag{5.7}$$

The vertices represent an integral over the interaction



$$\equiv \int_{1234} V_{1234} \, . \tag{5.8}$$

To construct all connected vacuum diagrams contributing to $W^{(p)}$ to each order p in perturbation theory, one connects p vertices with $4p$ legs in all possible ways according to Feynman's rules which follow from Wick's expansion of correlation functions into a sum of all pair contractions. This yields an increasing number of Feynman diagrams, each with a certain multiplicity which follows from combinatorics. In total there are $4!^p p!$ ways of ordering the $4p$ legs of the p vertices. This number is reduced by permutations of the legs and the vertices which leave a vacuum diagram invariant. Denoting the number of self, double, triple and fourfold connections with S, D, T, F, there are $2!^S, 2!^D, 3!^T, 4!^F$ leg permutations. An additional reduction arises from the number N of identical vertex permutations, where the vertices remain attached to the lines emerging from them in the same way as before. The resulting multiplicity of a connected vacuum diagram in the ϕ^4 theory is therefore given by the formula [5,27]

$$M^{E=0} = \frac{4!^p \, p!}{2!^{S+D} \, 3!^T \, 4!^F \, N} \, , \tag{5.9}$$

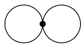
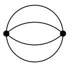
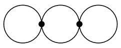
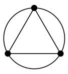
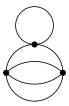
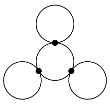
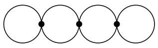
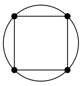
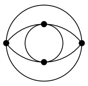
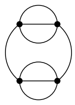
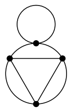
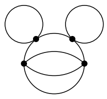
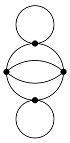
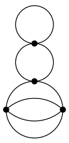
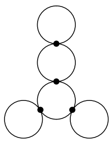
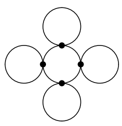
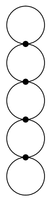
where $E = 0$ records that the number of external legs of vacuum diagrams is zero. The diagrammatic representation of the coefficients $W^{(p)}$ in the expansion (5.5) of the quantity W is displayed in Table 5.1 up to five loops [28–30]. For higher orders, the factorially increasing number of diagrams makes it more and more difficult to construct all topologically different diagrams and to count their multiplicities. In particular, it becomes quite hard to identify by inspection the number N of identical vertex permutations. This identification problem is solved by introducing a unique matrix notation for the graphs [23].

In the following, we shall generate iteratively all connected vacuum diagrams. We start by identifying graphical operations associated with functional derivatives with respect to the kernel G^{-1}, or the propagator G. Then we show that these operations can be applied to the one-loop contribution of the free partition function to generate all perturbative contributions to the partition function (5.1). After deriving a nonlinear functional differential equation for W, its graphical solution yields all connected vacuum diagrams order by order in the coupling strength.

5.2.1 Basic Graphical Operations

Each Feynman diagram is composed of integrals over products of free correlation functions G and may thus be considered as a functional of the kernel G^{-1}. The connected vacuum diagrams satisfy a

TABLE 5.1: Connected vacuum diagrams and their multiplicities of the x^4 theory up to five loops. Each diagram is characterized by the vector $(S, D, T, F; N)$ whose components specify the number of self, double, triple and fourfold connections, and of the identical vertex permutations leaving the vacuum diagram unchanged, respectively.

p	$W^{(p)}$				
1	#1 3 (2,1,0,0;1) 				
2	#2 24 (0,0,0,1;2) 	#3 72 (2,1,0,0;2) 			
3	#4 1728 (0,3,0,0;6) 	#5 3456 (1,0,1,0;2) 	#6 1728 (3,0,0,0;6) 	#7 2592 (2,2,0,0;2) 	
4	#8 62208 (0,4,0,0;8) 	#9 248832 (0,2,0,0;8) 	#10 55296 (0,0,2,0;4) 	#11 497664 (1,2,0,0;2) 	#12 165888 (2,0,1,0;2) 
	#13 248832 (2,1,0,0;4) 	#14 165888 (1,1,1,0;2) 	#15 248832 (3,1,0,0;2) 	#16 62208 (4,0,0,0;8) 	#17 124416 (2,3,0,0;2) 

certain functional differential equation, from which they will be constructed recursively. This will be done by a graphical procedure, for which we set up the necessary graphical rules now. First we observe that functional derivatives with respect to the kernel G^{-1} or to the free propagator G correspond to the graphical prescriptions of cutting or of removing a single line of a diagram in all possible ways, respectively.

Cutting Lines

Since x is a real scalar coordinate, the kernel G^{-1} is a symmetric functional matrix. This property has to be taken into account when performing functional derivatives with respect to the kernel G^{-1}, whose basic rule is

$$\frac{\delta G_{12}^{-1}}{\delta G_{34}^{-1}} = \frac{1}{2} \left\{ \delta_{13}\delta_{42} + \delta_{14}\delta_{32} \right\} . \tag{5.10}$$

From the identity (5.7) and the functional chain rule, we find the effect of this derivative on the free correlation function

$$-2\, \frac{\delta G_{12}}{\delta G_{34}^{-1}} = G_{13}G_{42} + G_{14}G_{32} \tag{5.11}$$

which has the graphical representation

$$-2\,\frac{\delta}{\delta G_{34}^{-1}}\;\; 1 \;\text{———}\; 2 \;=\; 1 \;\text{———}\; 3 \quad 4 \;\text{———}\; 2 \;+\; 1 \;\text{———}\; 4 \quad 3 \;\text{———}\; 2 \;. \tag{5.12}$$

Thus differentiating a propagator with respect to the kernel G^{-1} amounts to cutting the associated line into two pieces. The differentiation rule (5.10) ensures that the spatial indices of the kernel are symmetrically attached to the newly created line ends in the two possible ways. When differentiating a general Feynman integral with respect to G^{-1}, the product rule of functional differentiation leads to a sum of diagrams in which each line is cut once.

With this graphical operation, the product of two fields can be rewritten as a derivative of the action with respect to the kernel

$$x_1 x_2 = 2\,\frac{\delta \mathcal{A}[x]}{\delta G_{12}^{-1}}\,, \tag{5.13}$$

as follows directly from (5.2) and (5.10). Applying the substitution rule (5.13) to the functional integral for the fully interacting two-point function

$$\boldsymbol{G}_{12} = \frac{1}{Z}\int \mathcal{D}x\, x_1 x_2\, e^{-\mathcal{A}[x]}\,, \tag{5.14}$$

we obtain the fundamental identity

$$\boldsymbol{G}_{12} = -2\,\frac{\delta W}{\delta G_{12}^{-1}}\,. \tag{5.15}$$

Thus, by cutting a line of the connected vacuum diagrams in all possible ways, we obtain all diagrams of the fully interacting two-point function. Analytically this has a Taylor series expansion in powers of the coupling constant g similar to (5.5)

$$\boldsymbol{G}_{12} = \sum_{p=0}^{\infty} \frac{1}{p!}\left(\frac{-g}{4!}\right)^{p} \boldsymbol{G}_{12}^{(p)} \tag{5.16}$$

with coefficients

$$\boldsymbol{G}_{12}^{(p)} = -2\,\frac{\delta W^{(p)}}{\delta G_{12}^{-1}}. \tag{5.17}$$

The cutting prescription (5.17) converts the vacuum diagrams of pth order in the coefficients $W^{(p)}$ in Table 5.1 to the corresponding ones in the coefficients $\boldsymbol{G}_{12}^{(p)}$ of the two-point function. The results are shown in Table 5.2 up to four loops. The numbering of diagrams used in Table 5.2 reveals from which connected vacuum diagrams they are obtained by cutting a line.

For instance, the diagrams #15.1–#15.5 and their multiplicities in Table 5.2 follow from the connected vacuum diagram #15 in Table 5.1. We observe that the multiplicity of a diagram of a two-point function obeys a formula similar to (5.9):

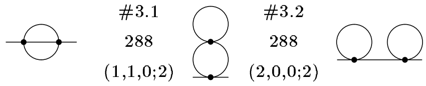
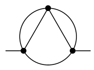
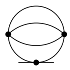
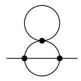
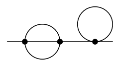
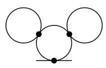
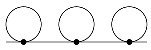
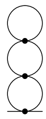
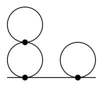
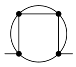
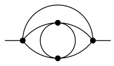
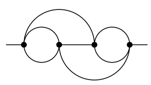
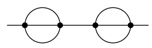
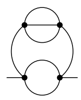
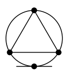
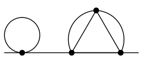
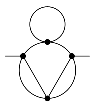
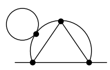
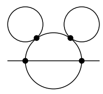
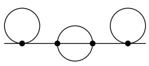
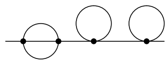
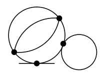
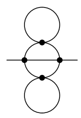
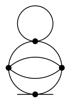
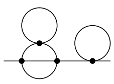
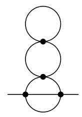
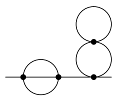
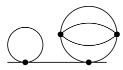
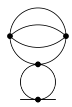
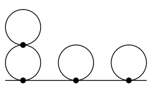
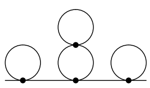
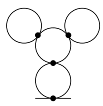
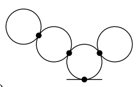
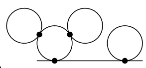
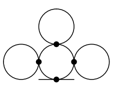
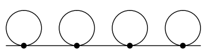
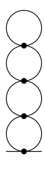
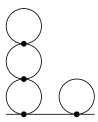
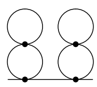
$$M^{E=2} = \frac{4!^{p}\, p!\, 2!}{2!^{S+D}\, 3!^{T}\, N}\,. \tag{5.18}$$

In the numerator, the $4!^{p}\, p!$ permutations of the $4p$ legs of the p vertices are multiplied by a factor 2! for the permutations of the two end points of the two-point function. The number N in the denominator counts the identical permutations of both the p vertices and the two end points.

Performing a differentiation of the two-point function (5.14) with respect to the kernel G^{-1} yields

$$-2\,\frac{\delta \boldsymbol{G}_{12}}{\delta G_{34}^{-1}} = \boldsymbol{G}_{1234} - \boldsymbol{G}_{12}\boldsymbol{G}_{34}\,, \tag{5.19}$$

TABLE 5.2: Connected diagrams of the two-point function and their multiplicities of the x^4 theory up to four loops. Each diagram is characterized by the vector $(S, D, T; N)$ whose components specify the number of self, double, triple connections, and of the combined permutations of vertices and external lines leaving the diagram unchanged, respectively.

p	$\boldsymbol{G}_{12}^{(p)}$
1	#1.1 12 (1,0,0;2) 
2	#2.1 192 (0,0,1;2) #3.1 288 (1,1,0;2) #3.2 288 (2,0,0;2) 
3	#4.1 20736 (0,2,0;2)  #5.1 6912 (0,0,1;4)  #5.2 20736 (1,1,0;2)  #5.3 13824 (1,0,1;1)  #6.1 10368 (2,0,0;4)  #6.2 10368 (3,0,0;2)  #7.1 10368 (1,2,0;2)  #7.2 20736 (2,1,0;1) 
4	#8.1 995328 (0,3,0;2)  #9.1 1990656 (0,1,0;4)  #9.2 1990656 (0,2,0;2)  #10.1 221184 (0,0,2;2)  #10.2 663552 (0,1,1;2)  #11.1 995328 (0,2,0;4)  #11.2 1990656 (1,2,0;1)  #11.3 995328 (1,2,0;2)  #11.4 3981312 (1,1,0;1)  #12.1 995328 (2,1,0;2)  #12.2 331776 (2,0,1;2)  #12.3 663552 (2,0,1;1)  #12.4 663552 (1,0,1;2)  #13.1 995328 (2,0,0;4)  #13.2 995328 (1,1,0;4)  #13.3 1990656 (2,1,0;1)  #14.1 995328 (1,2,0;2)  #14.2 663552 (1,1,1;1)  #14.3 663552 (1,0,1;2)  #14.4 331776 (0,1,1;4)  #15.1 995328 (3,1,0;1)  #15.2 497664 (3,1,0;2)  #15.3 497664 (2,1,0;4)  #15.4 995328 (2,1,0;2)  #15.5 995328 (3,0,0;2)  #16.1 497664 (3,0,0;4)  #16.2 497664 (4,0,0;2)  #17.1 497664 (1,3,0;2)  #17.2 995328 (2,2,0;1)  #17.3 497664 (2,2,0;2) 

where $\boldsymbol{G}_{1234}$ denotes the fully interacting four-point function

$$\boldsymbol{G}_{1234} = \frac{1}{Z} \int \mathcal{D}x \, x_1 x_2 x_3 x_4 \, e^{-\mathcal{A}[x]} \, . \tag{5.20}$$

The term $\boldsymbol{G}_{12}\boldsymbol{G}_{34}$ in (5.19) subtracts a certain set of disconnected diagrams from $\boldsymbol{G}_{1234}$. By subtracting *all* disconnected diagrams from $\boldsymbol{G}_{1234}$, we obtain the connected four-point function

$$\boldsymbol{G}^{\rm c}_{1234} \equiv \boldsymbol{G}_{1234} - \boldsymbol{G}_{12}\boldsymbol{G}_{34} - \boldsymbol{G}_{13}\boldsymbol{G}_{24} - \boldsymbol{G}_{14}\boldsymbol{G}_{23} \tag{5.21}$$

in the form

$$\boldsymbol{G}^{\rm c}_{1234} = -2 \, \frac{\delta \boldsymbol{G}_{12}}{\delta G^{-1}_{34}} - \boldsymbol{G}_{13}\boldsymbol{G}_{24} - \boldsymbol{G}_{14}\boldsymbol{G}_{23} \, . \tag{5.22}$$

The first term contains all diagrams obtained by cutting a line in the diagrams of the two-point-function $\boldsymbol{G}_{12}$. The second and third terms remove from these the disconnected diagrams. In this way we obtain the perturbative expansion

$$\boldsymbol{G}^{\rm c}_{1234} = \sum_{p=1}^{\infty} \frac{1}{p!} \left(\frac{-g}{4!} \right)^p \boldsymbol{G}^{{\rm c},(p)}_{1234} \tag{5.23}$$

with coefficients

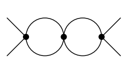
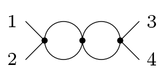
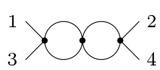
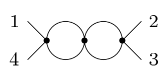
$$\boldsymbol{G}^{c,(p)}_{1234} = -2 \, \frac{\delta \boldsymbol{G}^{(p)}_{12}}{\delta G^{-1}_{34}} - \sum_{q=0}^{p} \begin{pmatrix} p \\ q \end{pmatrix} \left(\boldsymbol{G}^{(p-q)}_{13} \boldsymbol{G}^{(q)}_{24} + \boldsymbol{G}^{(p-q)}_{14} \boldsymbol{G}^{(q)}_{23} \right) . \tag{5.24}$$

They are listed diagrammatically in Table 5.3 up to three loops. As before in Table 5.2, the multiple numbering in Table 5.3 indicates the origin of each diagram of the connected four-point function. For instance, the diagram #11.2.2, #11.4.3, #14.1.2, #14.3.3 in Table 5.3 stems together with its multiplicity from the diagrams #11.2, #11.4, #14.1, #14.3 in Table 5.2.

The multiplicity of each diagram of a connected four-point function obeys a formula similar to (5.18):

$$M^{E=4} = \frac{4!^p \, p! \, 4!}{2!^{S+D} \, 3!^T \, N} \, . \tag{5.25}$$

This multiplicity decomposes into equal parts if the spatial indices $1, 2, 3, 4$ are assigned to the four end points of the connected four-point function, for instance:

62208  $\equiv$ 20736  + 20736  + 20736  . (5.26)

Generalizing the multiplicities (5.9), (5.18), and (5.25) for connected vacuum diagrams, two- and four-point functions to an arbitrary connected correlation function with an even number E of end points, we see that

$$M^{E} = \frac{4!^p \, p! \, E!}{2!^{S+D} \, 3!^T \, 4!^F \, N} \, , \tag{5.27}$$

where N counts the number of permutations of vertices and external lines which leave the diagram unchanged.

TABLE 5.3: Connected diagrams of the four-point function and their multiplicities of the x^4 theory up to four loops. Each diagram is characterized by the vector $(S, D, T; N)$ whose components specify the number of self, double, triple connections, and of the combined permutations of vertices and external lines leaving the diagram unchanged, respectively.

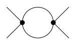
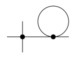
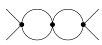
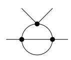
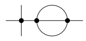
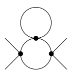
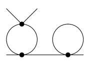
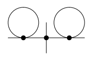
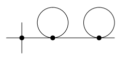
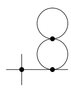
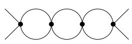
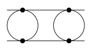
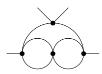
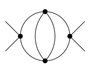
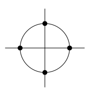
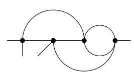
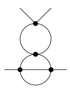
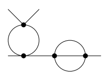
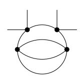
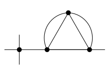
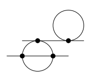
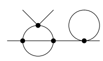
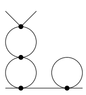
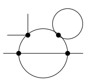
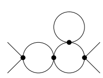
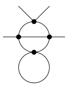
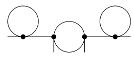
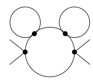
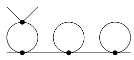
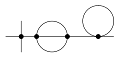
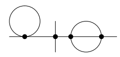
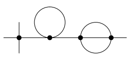
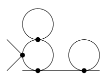
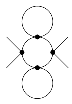
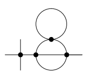
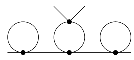
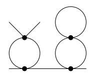
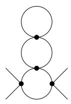
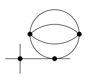
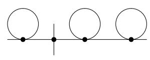
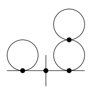
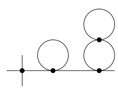
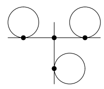
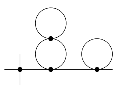
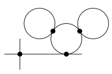
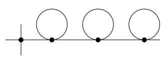
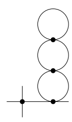
p	$G^{c,(p)}_{1234}$	
1	#1.1.1 24 (0,0,0;24)	
2	#2.1.1, #3.1.1 1152,576 1728 (0,1,0;8)	
	#3.1.2, #3.2.1 1152,1152 2304 (1,0,0;6)	
3	#4.1.1, #7.1.1 41472,20736 62208 (0,2,0;8)	
	#4.1.2, #5.1.1, #5.2.1 165888,41472,41472 248832 (0,1,0;4)	
	#5.1.2, #5.3.2 27648,27648 55296 (0,0,1;6)	
	#5.2.2, #6.1.1 82944,41472 124416 (1,0,0;8)	
	#5.2.3, #5.3.1, #7.1.2, #7.2.1 82944,82944,41472,41472 248832 (1,1,0;2)	
	#6.1.2, #6.2.2, #7.2.2 20736,20736,82944 124416 (2,0,0;4)	
	#6.1.3, #6.2.1 41472,41472 82944 (2,0,0;6)	
	#7.1.3, #7.2.3 41472,41472 82944 (1,1,0;6)	
4	#8.1.1, #17.1.1 1990656,995328 2985984 (0,3,0;8)	
	#8.1.2, #9.2.1, #10.2.1 3981312,3981312,3981312 11943936 (0,2,0;4)	
	#8.1.3, #11.1.2, #11.3.1 7962624,1990656,1990656 11943936 (0,2,0;4)	
	#9.1.1, #13.2.1 3981312,1990656 5971968 (0,1,0;16)	
	#9.1.2 7962624 (0,0,0;24)	
	#9.1.3, #9.2.3, #11.1.1, #11.4.1 15925248,15925248,7962624,7962624 47775744 (0,1,0;2)	
	#9.2.2, #14.1.1, #14.4.3 7962624,1990656,1990656 11943936 (0,2,0;4)	
	#10.1.1, #10.2.3, # 14.2.1, #14.4.2 2654208,2654208,1327104,1327104 7962624 (0,1,1;2)	
	#10.2.2, #12.4.1 2654208,1327104 3981312 (0,0,1;8)	
	#11.1.3, #11.2.1 3981312,3981312 7962624 (0,2,0;6)	
	#11.2.2, #11.4.3, #14.1.2, #14.3.3 7962624,7962624,3981312,3981312 23887872 (1,1,0;2)	
	#11.2.3, #11.4.2, #13.2.2, #13.3.1 7962624,7962624,3981312,3981312 23887872 (1,1,0;2)	
	#11.2.4, #11.3.2, #17.1.2, #17.2.1 3981312,3981312,1990656,1990656 11943936 (1,2,0;2)	
	#11.3.3, #11.4.4, #12.1.1, #12.4.5 7962624,7962624,3981312,3981312 23887872 (1,1,0;2)	
	#11.4.5, #15.3.1, #15.4.1 7962624,1990656,1990656 11943936 (1,1,0;4)	

Table 5.3 (*Continued*)

4	#11.4.6, #13.1.1, #13.2.3 15925248,3981312,3981312 23887872 (1,0,0;4)	
	#12.1.2, #12.2.2, #13.3.3, #17.2.2 1990656,1990656,3981312,3981312 11943936 (2,1,0;2)	
	#12.1.3, #16.1.2 3981312,1990656 5971968 (2,0,0;8)	
	#12.1.4, #12.3.3, #15.1.1, #15.3.2 3981312,3981312,1990656,1990656 11943936 (2,1,0;2)	
	#12.2.1, #12.4.2 1327104,1327104 2654208 (1,0,1;6)	
	#12.3.2, #12.4.3, #14.2.2, #14.3.2 1327104,1327104,2654208,2654208 7962624 (1,0,1;2)	
	#12.3.1, #12.4.4 1327104,1327104 2654208 (1,0,1;6)	
	#13.1.2, #13.3.4, #15.4.2, #15.5.1 7962624,7962624,3981312,3981312 23887872 (2,0,0;2)	
	#13.1.3, #16.1.1 1990656,995328 2985984 (2,0,0;16)	
	#13.2.4, #13.3.5 3981312,3981312 7962624 (1,1,0;6)	
	#13.3.2, #15.2.1, #15.3.3 3981312,995328,995328 5971968 (2,1,0;4)	
	#14.1.3, #14.2.3, #17.1.3, #17.3.1 3981312,3981312,1990656,1990656 11943936 (1,2,0;2)	
	#14.1.4, #15.4.4 3981312,1990656 5971968 (1,1,0;8)	
	#14.3.1, #14.4.1 1327104,1327104 2654208 (0,0,1;12)	
	#15.1.2, #15.5.3, #16.1.3, #16.2.2 3981312,3981312,1990656,1990656 11943936 (3,0,0;2)	
	#15.1.3, #15.4.3, #17.2.3, #17.3.2 1990656,1990656,3981312,3981312 11943936 (2,1,0;2)	
	#15.1.4, #15.4.5 1990656,1990656 3981312 (2,1,0;6)	
	#15.2.2, #15.5.2 1990656,1990656 3981312 (3,0,0;6)	
	#15.2.3, #15.4.6 1990656,1990656 3981312 (2,1,0;6)	
	#15.3.4, #15.5.4 1990656,1990656 3981312 (2,0,0;12)	
	#16.1.4, #16.2.1 1990656,1990656 3981312 (3,0,0;6)	
	#17.1.4, #17.2.4 1990656,1990656 3981312 (1,2,0;6)	

Removing Lines

We now study the graphical effect of functional derivatives with respect to the free propagator G, where the basic differentiation rule (5.10) becomes

$$\frac{\delta G_{12}}{\delta G_{34}} = \frac{1}{2}\left\{\delta_{13}\delta_{42} + \delta_{14}\delta_{32}\right\} . \tag{5.28}$$

We represent this graphically by extending the elements of Feynman diagrams by an open dot with two labeled line ends representing the δ function:

$$1 -\!\circ\!- 2 \;=\; \delta_{12} . \tag{5.29}$$

Thus we can write the differentiation (5.28) graphically as follows:

$$\frac{\delta}{\delta\, 3\!-\!\!-\!4}\;\; 1 \;\text{———}\; 2 \;=\; \frac{1}{2}\left\{ 1 -\!\circ\!- 3 \quad 4 -\!\circ\!- 2 \;+\; 1 -\!\circ\!- 4 \quad 3 -\!\circ\!- 2 \right\} . \tag{5.30}$$

Differentiating a line with respect to the free correlation function removes the line, leaving in a symmetrized way the spatial indices of the free correlation function on the vertices to which the line was connected.

The effect of this derivative is illustrated by studying the diagrammatic effect of the operator

$$\hat{L} = \int_{12} G_{12} \frac{\delta}{\delta G_{12}} \,. \tag{5.31}$$

Applying $\hat{L}$ to a connected vacuum diagram in $W^{(p)}$, the functional derivative $\delta/\delta G_{12}$ generates diagrams in each of which one of the $2p$ lines of the original vacuum diagram is removed. Subsequently, the removed lines are again reinserted, so that the connected vacuum diagrams $W^{(p)}$ are eigenfunctions of $\hat{L}$, whose eigenvalues $2p$ count the lines of the diagrams:

$$\hat{L}\, W^{(p)} = 2p\, W^{(p)} \,. \tag{5.32}$$

As an example, take the explicit first-order expression for the vacuum diagrams, i.e.

$$W^{(1)} = 3 \int_{1234} V_{1234}\, G_{12} G_{34} \,, \tag{5.33}$$

and apply the basic rule (5.28), leading to the desired eigenvalue 2.

5.2.2 Perturbation Theory

We introduce an external current J into the functional (5.2) which is linearly coupled to the coordinate x. Thus the partition function (5.1) becomes the generating functional $Z[J]$ which allows us to find all free n-point functions from functional derivatives with respect to this external current J. Due to the shape of the functional (5.2) the expectation value of the coordinate x is zero and only correlation functions of an even number of coordinates are nonzero. To calculate all of these, it is possible to substitute two functional derivatives with respect to the current J by one functional derivative with respect to the kernel G^{-1}. This reduces the number of functional derivatives in each order of perturbation theory by one half and has the additional advantage that the introduction of the current J becomes superfluous.

Current Approach

Recall briefly the standard perturbative treatment, in which the energy functional (5.2) is artificially extended by a source term

$$\mathcal{A}[x, J] = \mathcal{A}[x] - \int_1 J_1 x_1 \,. \tag{5.34}$$

The functional integral for the generating functional

$$Z[J] = \int \mathcal{D}x\, e^{-\mathcal{A}[x,J]} \tag{5.35}$$

is first explicitly calculated for a vanishing coupling constant g, yielding

$$Z^{(0)}[J] = \exp\left\{ -\frac{1}{2} \operatorname{Tr}\ln G^{-1} + \frac{1}{2} \int_{12} J_1\, G_{12}\, J_2 \right\} , \tag{5.36}$$

where the trace of the logarithm of the kernel is defined by the series [31]

$$\operatorname{Tr}\ln G^{-1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int_{1\ldots n} \left\{ G_{12}^{-1} - \delta_{12} \right\} \cdots \left\{ G_{n1}^{-1} - \delta_{n1} \right\} \,. \tag{5.37}$$

If the coupling constant g does not vanish, one expands the generating functional $Z[J]$ in powers of the quartic interaction V, and re-expresses the resulting powers of the coordinate within the functional integral (5.35) as functional derivatives with respect to the current J. The original partition function (5.1) can thus be obtained from the free generating functional (5.36) by the formula

$$Z = \exp\left\{-\frac{g}{4!}\int_{1234} V_{1234}\,\frac{\delta^4}{\delta J_1\delta J_2\delta J_3\delta J_4}\right\} Z^{(0)}[J]\Bigg|_{J=0} . \tag{5.38}$$

Expanding the exponential in a power series, we arrive at the perturbation expansion

$$\begin{aligned} Z \;=\; &\left\{1 - \frac{g}{4!}\int_{1234} V_{1234}\,\frac{\delta^4}{\delta J_1\delta J_2\delta J_3\delta J_4}\right. \\ &\left. + \frac{1}{2}\frac{g^2}{(4!)^2}\int_{12345678} V_{1234}V_{5678}\,\frac{\delta^8}{\delta J_1\delta J_2\delta J_3\delta J_4\delta J_5\delta J_6\delta J_7\delta J_8} + \ldots\right\} Z^{(0)}[J]\Bigg|_{J=0} , \end{aligned} \tag{5.39}$$

in which the pth order contribution for the partition function requires the evaluation of $4p$ functional derivatives with respect to the current J.

Kernel Approach

The derivation of the perturbation expansion simplifies, if we use functional derivatives with respect to the kernel G^{-1} in the action (5.2) rather than with respect to the current J. This allows us to substitute the previous expression (5.38) for the partition function by

$$Z = \exp\left\{-\frac{g}{6}\int_{1234} V_{1234}\,\frac{\delta^2}{\delta G^{-1}_{12}\delta G^{-1}_{34}}\right\} e^{W^{(0)}} , \tag{5.40}$$

where the zeroth order of the negative free energy has the diagrammatic representation

$$W^{(0)} = -\frac{1}{2}\,\mathrm{Tr}\ln G^{-1} \equiv \frac{1}{2}$$  . (5.41)

Expanding again the exponential into a power series, we obtain

$$Z = \left\{1 - \frac{g}{6}\int_{1234} V_{1234}\,\frac{\delta^2}{\delta G^{-1}_{12}\delta G^{-1}_{34}} + \frac{1}{2}\frac{g^2}{36}\int_{12345678} V_{1234}V_{5678}\frac{\delta^4}{\delta G^{-1}_{12}\delta G^{-1}_{34}\delta G^{-1}_{56}\delta G^{-1}_{78}} + \ldots\right\} e^{W^{(0)}} . \tag{5.42}$$

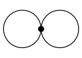
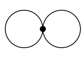
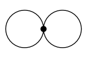
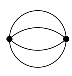
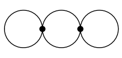
Thus we need only half as many functional derivatives than in (5.39). Taking into account (5.10), (5.11), and (5.37), we obtain

$$\frac{\delta W^{(0)}}{\delta G^{-1}_{12}} = -\frac{1}{2}\,G_{12}\,, \qquad \frac{\delta^2 W^{(0)}}{\delta G^{-1}_{12}\delta G^{-1}_{34}} = \frac{1}{4}\left\{G_{13}G_{24} + G_{14}G_{23}\right\} , \tag{5.43}$$

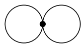
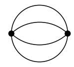
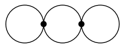
such that the partition function Z becomes

$$\begin{aligned} Z \;=\; &\left\{1 - 3\frac{g}{4!}\int_{1234} V_{1234}\,G_{12}G_{34} + \frac{1}{2}\frac{g^2}{(4!)^2}\int_{12345678} V_{1234}V_{5678}\right. \\ &\left.\times\left[9\,G_{12}G_{34}G_{56}G_{78} + 24\,G_{15}G_{26}G_{37}G_{48} + 72\,G_{12}G_{35}G_{46}G_{78}\right] + \ldots\right\} e^{W^{(0)}} . \end{aligned} \tag{5.44}$$

This has the diagrammatic representation

$Z = \Big\{1 - \frac{g}{4!}\,3$  $+ \frac{1}{2}\frac{g^2}{(4!)^2}\Big[9$   $+ 24$  $+ 72$  $\Big] + \ldots\Big\}\, e^{W^{(0)}}$. (5.45)

All diagrams in this expansion follow directly by successively cutting lines of the basic one-loop vacuum diagram (5.41) according to (5.42). By going to the logarithm of the partition function Z, we find a diagrammatic expansion for W

$W = \frac{1}{2}$  $- \frac{g}{4!}\, 3$  $+ \frac{1}{2}\,\frac{g^2}{(4!)^2} \Big\{ 24$  $+\, 72$  $\Big\} + \dots\,,$ (5.46)

which turns out to contain precisely all connected diagrams in (5.45) with the same multiplicities. In the next section we show that this diagrammatic expansion for W can be derived more efficiently by solving a differential equation.

5.2.3 Functional Differential Equation for $W = \ln Z$

Regarding the partition function Z as a functional of the kernel G^{-1}, we derive a functional differential equation for Z. We start with the trivial identity

$$\int \mathcal{D}x \frac{\delta}{\delta x_1} \left\{ x_2 \, e^{-\mathcal{A}[x]} \right\} = 0 \,. \tag{5.47}$$

Taking into account the explicit form of the action (5.2), we perform the functional derivative with respect to the coordinate and obtain

$$\int \mathcal{D}x \left\{ \delta_{12} - \int_3 G^{-1}_{13} x_2 x_3 - \frac{g}{6} \int_{345} V_{1345}\, x_2 x_3 x_4 x_5 \right\} e^{-\mathcal{A}[x]} = 0 \,. \tag{5.48}$$

Applying the substitution rule (5.13), this equation can be expressed in terms of the partition function (5.1) and its derivatives with respect to the kernel G^{-1}:

$$\delta_{12}\, Z + 2 \int_3 G^{-1}_{13} \frac{\delta Z}{\delta G^{-1}_{23}} = \frac{2}{3}\, g \int_{345} V_{1345} \frac{\delta^2 Z}{\delta G^{-1}_{23} \delta G^{-1}_{45}} \,. \tag{5.49}$$

Note that this linear functional differential equation for the partition function Z is, indeed, solved by (5.40) due to the commutation relation

$$\begin{aligned} \exp\left\{ -\frac{g}{6} \int_{1234} V_{1234} \frac{\delta^2}{\delta G^{-1}_{12} \delta G^{-1}_{34}} \right\} G^{-1}_{56} - G^{-1}_{56} \exp\left\{ -\frac{g}{6} \int_{1234} V_{1234} \frac{\delta^2}{\delta G^{-1}_{12} \delta G^{-1}_{34}} \right\} \\ = -\frac{g}{3} \int_{78} V_{5678} \frac{\delta}{\delta G^{-1}_{78}} \exp\left\{ -\frac{g}{6} \int_{1234} V_{1234} \frac{\delta^2}{\delta G^{-1}_{12} \delta G^{-1}_{34}} \right\} \end{aligned} \tag{5.50}$$

which follows from the canonical one

$$\frac{\delta}{\delta G^{-1}_{12}} G^{-1}_{34} - G^{-1}_{34} \frac{\delta}{\delta G^{-1}_{12}} = \frac{1}{2} \left\{ \delta_{13}\delta_{24} + \delta_{14}\delta_{23} \right\} \,. \tag{5.51}$$

Going over from Z to $W = \ln Z$, the linear functional differential equation (5.49) turns into a nonlinear one:

$$\delta_{12} + 2 \int_3 G^{-1}_{13} \frac{\delta W}{\delta G^{-1}_{23}} = \frac{2}{3}\, g \int_{345} V_{1345} \left\{ \frac{\delta^2 W}{\delta G^{-1}_{23} \delta G^{-1}_{45}} + \frac{\delta W}{\delta G^{-1}_{23}} \frac{\delta W}{\delta G^{-1}_{45}} \right\} \,. \tag{5.52}$$

If the coupling constant g vanishes, this is immediately solved by (5.41). For a nonvanishing coupling constant g, the right-hand side in (5.52) produces corrections to (5.41) which we shall denote with $W^{(\mathrm{int})}$. Thus the quantity W decomposes according to

$$W = W^{(0)} + W^{(\mathrm{int})} \,. \tag{5.53}$$

Inserting this into (5.52) and taking into account (5.43), we obtain the following functional differential equation for $W^{(\mathrm{int})}$:

$$\int_{12} G_{12}^{-1} \frac{\delta W^{(\mathrm{int})}}{\delta G_{12}^{-1}} = \frac{g}{4}\int_{1234} V_{1234}\, G_{12}G_{34} - \frac{g}{3}\int_{1234} V_{1234}\, G_{12}\, \frac{\delta W^{(\mathrm{int})}}{\delta G_{34}^{-1}} + \frac{g}{3}\int_{1234} V_{1234} \left\{ \frac{\delta^2 W^{(\mathrm{int})}}{\delta G_{12}^{-1}\delta G_{34}^{-1}} + \frac{\delta W^{(\mathrm{int})}}{\delta G_{12}^{-1}} \frac{\delta W^{(\mathrm{int})}}{\delta G_{34}^{-1}} \right\} . \tag{5.54}$$

With the help of the functional chain rule, the first and second derivatives with respect to the kernel G^{-1} are rewritten as

$$\frac{\delta}{\delta G_{12}^{-1}} = -\int_{34} G_{13}G_{24} \frac{\delta}{\delta G_{34}} \tag{5.55}$$

and

$$\frac{\delta^2}{\delta G_{12}^{-1}\delta G_{34}^{-1}} = \int_{5678} G_{15}G_{26}G_{37}G_{48} \frac{\delta^2}{\delta G_{56}\delta G_{78}} + \frac{1}{2}\int_{56} \left\{ G_{13}G_{25}G_{46} + G_{14}G_{25}G_{36} + G_{23}G_{15}G_{46} + G_{24}G_{15}G_{36} \right\} \frac{\delta}{\delta G_{56}} , \tag{5.56}$$

respectively, so that the functional differential equation (5.54) for $W^{(\mathrm{int})}$ takes the form

$$\int_{12} G_{12} \frac{\delta W^{(\mathrm{int})}}{\delta G_{12}} = -\frac{g}{4}\int_{1234} V_{1234}\, G_{12}G_{34} - g\int_{123456} V_{1234}\, G_{12}G_{35}G_{46} \frac{\delta W^{(\mathrm{int})}}{\delta G_{56}} - \frac{g}{3}\int_{12345678} V_{1234}\, G_{15}G_{26}G_{37}G_{48} \left\{ \frac{\delta^2 W^{(\mathrm{int})}}{\delta G_{56}\delta G_{78}} + \frac{\delta W^{(\mathrm{int})}}{\delta G_{56}} \frac{\delta W^{(\mathrm{int})}}{\delta G_{78}} \right\} . \tag{5.57}$$

5.2.4 Recursion Relation and Graphical Solution

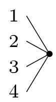
We now convert the functional differential equation (5.57) into a recursion relation by expanding $W^{(\mathrm{int})}$ into a power series in g:

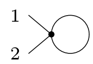
$$W^{(\mathrm{int})} = \sum_{p=1}^{\infty} \frac{1}{p!} \left(\frac{-g}{4!} \right)^p W^{(p)} . \tag{5.58}$$

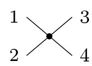
Using the property (5.32) that the coefficient $W^{(p)}$ satisfies the eigenvalue problem of the line numbering operator (5.31), we obtain the recursion relation

$$W^{(p+1)} = 12\int_{123456} V_{1234}\, G_{12}G_{35}G_{46} \frac{\delta W^{(p)}}{\delta G_{56}} + 4\int_{12345678} V_{1234}\, G_{15}G_{26}G_{37}G_{48} \frac{\delta^2 W^{(p)}}{\delta G_{56}\delta G_{78}} + 4\sum_{q=1}^{p-1} \binom{p}{q} \int_{12345678} V_{1234}\, G_{15}G_{26}G_{37}G_{48} \frac{\delta W^{(p-q)}}{\delta G_{56}} \frac{\delta W^{(q)}}{\delta G_{78}} \tag{5.59}$$

and the initial condition (5.33). With the help of the graphical rules of Section 5.2.1, the recursion relation (5.59) can be written diagrammatically as follows

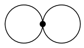
$$W^{(p+1)} = 4 \frac{\delta^2 W^{(p)}}{\delta\, 1 — 2\; \delta\, 3 — 4}$$ 

$$+\; 12\, \frac{\delta W^{(p)}}{\delta\, 1 — 2}$$ 

$$+\; 4 \sum_{q=1}^{p-1} \binom{p}{q} \frac{\delta W^{(p-q)}}{\delta\, 1 — 2}$$ 

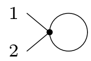
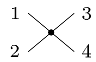
$$\frac{\delta W^{(q)}}{\delta\, 3 — 4} , \qquad p \geq 1 . \tag{5.60}$$

This is iterated starting from

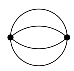
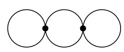
$$W^{(1)} = 3$$  $$. \tag{5.61}$$

The right-hand side of (5.60) contains three different graphical operations. The first two are linear and involve one or two line amputations of the previous perturbative order. The third operation is nonlinear and mixes two different one-line amputations of lower orders.

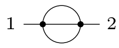
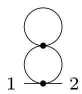
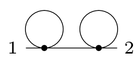
To demonstrate the working of (5.60), we calculate the connected vacuum diagrams up to five loops. Applying the linear operations to (5.59), we obtain immediately

$$\frac{\delta W^{(1)}}{\delta\, 1 \text{——} 2} = 6$$  $$, \quad \frac{\delta^2 W^{(1)}}{\delta\, 1 \text{——} 2\ \delta\, 3 \text{——} 4} = 6$$  $$. \tag{5.62}$$

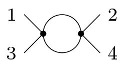
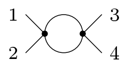
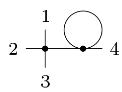
Inserted into (5.60), these lead to the three-loop vacuum diagrams

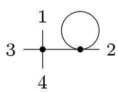
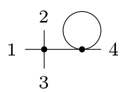
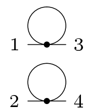
$$W^{(2)} = 24$$  $$+ 72$$  $$. \tag{5.63}$$

Proceeding to the next order, we have to perform one- and two-line amputations on the vacuum graphs in (5.63), leading to

$$\frac{\delta W^{(2)}}{\delta\, 1 \text{——} 2} = 96$$  $$+ 144$$  $$+ 144$$  $$, \tag{5.64}$$

and subsequently to

$$\frac{\delta^2 W^{(2)}}{\delta\, 1 \text{——} 2\ \delta\, 3 \text{——} 4} = 288$$  $$+ 144$$  $$+ 288$$ 

$$+ 144$$  $$+ 144$$  $$+ 144$$  $$. \tag{5.65}$$

Inserting (5.64) and (5.65) into (5.60) and taking into account (5.62), we find the connected vacuum diagrams of order $p = 3$ with their multiplicities as shown in Table 5.1. We observe that the nonlinear operation in (5.60) does not lead to topologically new diagrams. It only corrects the multiplicities of the diagrams generated from the first two operations. This is true also in higher orders. The connected vacuum diagrams of the subsequent order $p = 4$ and their multiplicities are listed in Table 5.1.

As a cross-check we can also determine the total multiplicities $M^{(p)}$ of all connected vacuum diagrams contributing to $W^{(p)}$. To this end we recall that each of the $M^{(p)}$ diagrams in $W^{(p)}$ consists of $2p$ lines. The amputation of one or two lines therefore leads to $2pM^{(p)}$ and $2p(2p-1)M^{(p)}$ diagrams with $2p - 1$ and $2p - 2$ lines, respectively. Considering only the total multiplicities, the graphical recursion relations (5.60) reduce to the form derived before in Ref. [22]

$$M^{(p+1)} = 16p(p+1)M^{(p)} + 16\sum_{q=1}^{p-1} \frac{p!}{(p-q-1)!(q-1)!} M^{(q)} M^{(p-q)}, \qquad p \geq 1\,. \tag{5.66}$$

These are solved starting with the initial value

$$M^{(1)} = 3\,, \tag{5.67}$$

leading to the total multiplicities

$$M^{(2)} = 96\,, \quad M^{(3)} = 9504\,, \quad M^{(4)} = 1880064\,, \tag{5.68}$$

which agree with the results listed in Table 5.1. In addition we note that the next orders would contain

$$M^{(5)} = 616108032\,, \quad M^{(6)} = 301093355520\,, \quad M^{(7)} = 205062331760640 \tag{5.69}$$

connected vacuum diagrams.