Appendix C

Expected value of the apparent resistivity

This part of the appendix handles estimation of the variance of the apparent resisistivity ρ_a and expected value $E(\rho_a)$ by assuming normal distributed tensor elements.

The variance $Var(\rho_a)$ of the apparent resistivity as a function of an arbitrary tensor element **Z**:

$$\rho_a = 0.2 \cdot T(ReZ^2 + ImZ^2),$$

will depend on their respective impedance data errors ΔReZ and ΔImZ with known statistical distribution. In the majority of the data processing procedures, real and imaginary parts have equal errors, which implies that they are assumed to be uncorrelated since the variable is a complex number. This means that $\Delta ReZ = \Delta ImZ = \Delta|Z|$.
If the impedance tensor elements are considered as random variables (r.v.'s), particularly real and imaginary parts as independent normally distributed r.v.'s, each part of mean $u_r = Re\hat{Z}$ and $u_i = Im\hat{Z}$ and equal standard deviation $\sigma = \Delta Z$ $(N(a_{r,i}, \sigma)$, then the expected values of the squared real and imaginary part of the tensor element are:

$$E(ReZ^2) = Re\hat{Z}^2 + \sigma^2 \tag{C.1}$$

$$E(ImZ^2) = Im\hat{Z}^2 + \sigma^2$$

where $Re\hat{Z}^2$ and $Im\hat{Z}^2$ are the measured data assumed to be the respective expected values both with equally variance σ^2 (i.e. the square of the measured data error assumed to be the 68% confidence limit or its standard deviation). Thus, we are able to determine the apparent resistivity expected value $E(\rho_a)$, which is subject to the ρ_a probability function distribution (p.f.d.). Following the property of independent r.v.'s summation, then:

$$E(\rho_a) = E\left(0.2T|Z|^2\right) = 0.2T \times E\left(ReZ^2 + ImZ^2\right) = 0.2T \times \left[E\left(ReZ^2\right) + E\left(ImZ^2\right)\right]$$

and by using Eq.(C.1) the final expression is obtained:

$$E(\rho_a) = 0.2T(2\Delta Z^2 + \operatorname{Re}\hat{Z}^2 + \operatorname{Im}\hat{Z}^2) = 2 \times 0.2T\Delta Z^2 + \hat{\rho}_a$$

This means that the measured data $\hat{\rho}_a = 0.2T \times |\hat{Z}|^2$ is up-biased from the expected value by $2 \times 0.2T\Delta Z^2$. This fact relies on the p.d.f. characteristic of ρ_a.

Confidence limit of ρ_a

Subject to the complex tensor element −assuming its real and imaginary parts as independent gaussian random variables (r.v.)− it follows that ρ_a has a non-central two degrees of freedom $\chi 2$ p.f.d (e.g., Dudewicz and Mishra [1988]) with expected value

$$E(\rho_a) = 2 \times 0.2T \cdot \Delta Z^2 + \hat{\rho}_a \tag{C.2}$$

Its variance is obtained by using Eq.(C.1) and by knowing that the derivative of a normal distributed $N(u, \sigma)$ characteristic function $\vartheta(t) = \exp(it \cdot u) \cdot \exp(-\frac{1}{2}\sigma t)$ of a r.v. X accomplishes the general relation for the expected value

$$E(X^n) = i^n \frac{\partial^n \vartheta(0)}{\partial t^n} \quad \Rightarrow \quad E(X^2) = i^2 \frac{\partial \vartheta^2(0)}{\partial t^2} = u^2 + \sigma^2$$

and $E(X^4) = 6u^2\sigma^2 + u^4 + 3\sigma^4$. The variance of a r.v. $Y = X^2$ is by definition:

$$Var(Y) = E(Y^2) - (E(Y))^2$$

Then following the property of variance summation one obtains:
$Var(\rho_a) = Var(cY) = c(Var\{\mathrm{Re}\, Z\} + Var\{\mathrm{Im}\, Z\}) = (0.2T) \times [E\{(\mathrm{Re}\, Z)\} + E\{(\mathrm{Im}\, Z)\} - (E\{\mathrm{Re}\, Z\}) - (E\{\mathrm{Re}\, Z\})]$

$$Var(\rho_a) = 4(0.2T)^2 \times (\sigma^4 + \sigma^2 |\hat{Z}|^2) \tag{C.3}$$

This confirms that the true value ρ_a lays within the 86.5% confidence limit (property of the 2 degrees of freedom χ^2 p.d.f.):

$$\left[E(\rho_a) - \sqrt{Var(\rho_a)}\right] < \rho_a < \left[E(\rho_a) + \sqrt{Var(\rho_a)}\right]$$

where the square root of the variance is the deviation of $E(\rho_a)$. By replacing $E(\rho_a)$ with the regular expression of Eq.(C.2) and $Var(\rho_a)$ with that of Eq.(C.3), we thus obtain through simple algebra the lower and upper deviation limits $\Delta\rho_-$ and $\Delta\rho_+$ (i.e., the error bars) of the measured data $\hat{\rho}_a = 0.2T|\hat{Z}|^2$, respectively:
$E(\rho_a) - \sqrt{Var(\rho_a)} \Rightarrow \Delta\rho_- = 2 \times 0.2T \cdot \sigma \cdot (\sigma - \sqrt{\sigma^2 + |\hat{Z}|^2})$ for $\Delta\rho_-$
$E(\rho_a) + \sqrt{Var(\rho_a)} \Rightarrow \Delta\rho_+ = 2 \times 0.2T \cdot \sigma \cdot (\sigma + \sqrt{\sigma^2 + |\hat{Z}|^2})$ for $\Delta\rho_+$.

This implies that the true value ρ_a lies within the following confidence limit with respect to the measured data $\hat{\rho}_a$:

$$[\hat{\rho}_a - |\Delta\rho_-|] < \rho_a < [\hat{\rho}_a + \Delta\rho_+]$$

whose confidence limit is still the 86.5% because the same range has been preserved. This expression brings an asymmetrical error to the measured data, which relies on the asymmetry and non-centrality of its distribution function.
Fig. 1 shows the asymmetrical error of $\hat{\rho}_a$ for a normalized impedance as function of its

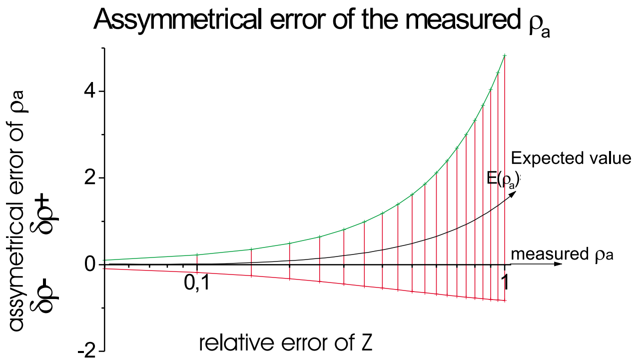

Figure C.1: Graphical representation of the asymmetrical error $(\delta\rho-, \delta\rho+)$ of the measured value ρ_a as function of the tensor element relative error. The values are normalized by $0.2T|\hat{Z}^2|$.

relative error. Above the 50% relative error the asymmetry is more remarkable, abruptly increasing for larger errors. The asymmetry is of course directly proportional to the deviation of the measured data $2 \times 0.2T \cdot \Delta Z^2$ relative to the expected value.
Thereby the χ^2 statistical error −for percentage relative errors not greater than approx 50%− is very similar to the symmetrical linear propagated error[1] bars. For increasing relative error the measured value becomes more shifted from the expected value, reflecting the bias that ρ_a suffers due to noisy tensor elements.
In this thesis the expected value has been used instead of the measured ρ_a, based on the assumption that $E(\rho_a)$ is more confident than the shifted data.

[1] The linear propagated error of apparent resisitivty is: $2 \times 0.2T \cdot \Delta Z \cdot |\hat{Z}|$