Appendix A

Derivation of the probability function of the regional skew for normal distributed tensor elements

The conditional probability function (p.f.) of the regional skew parameter $G_p(\tilde{\eta}_p)$ derived from the transformation of variables (Chapter 2; eq. 2.9) is:

$$G_p(\tilde{\eta}_p) = P\left(-\frac{\tilde{\eta}_p^2 d}{2} < (x_p s_i u_i + c) < \frac{\tilde{\eta}_p^2 d}{2}\right) \tag{A.1a}$$

where x_p is the conditional random variable (r.v.) valid for the transformation of spaces (diagonal tensor element; Fig. 2.1), u_i is the mean value of the variable x_i, s_i and c as defined in eq. 2.8.
We make the variable transformation:

$$y(x_p) = y_p = (x_p s_i u_i + c) \quad \rightarrow \quad x_p = \frac{y_p - c}{s_i u_i} \tag{A.1b}$$

$$y_p = \frac{\eta_p^2 d}{2} \tag{A.1c}$$

to treat the upper limit of p.f. written in eq. A.1a in terms of the r.v. x_p. We refer to this as the p.f. $F(\tilde{y}(x_p))$:

$$F\left(\tilde{y}(x_p)\right) = P\left(y_p < \frac{\tilde{\eta}_p^2 d}{2} = \tilde{y}_p\right) = P\left(x_p < \frac{\tilde{y}_p - c}{s_i u_i} = x_p(\tilde{\eta}_p)\right) \tag{A.2a}$$

The transformation of variable from y_p to x_p is valid since they fulfill the required properties for a valid transformation of spaces. The p.f. F (eq. A.2a) is transformed to the space of x_p, thus F can be determined given a known p.f. for x_p.
The r.v. x_p is assumed normally distributed with d.f. $\phi(x_p)$, mean value u_p and standard deviation σ_p. Considering eq. A.2a, the p.f. F as function of $\tilde{y}(x_p) = \tilde{\eta}_p^2 d/2$ (eqs. A.1b,

A.1c) takes the form:

$$F(\frac{\tilde{\eta}_p^2 d}{2}) = \begin{cases} \int\limits_{-\infty}^{x_p(\tilde{\eta}_p)} \phi(x_p)dx_p = \psi_o\left(\frac{x_p(\tilde{\eta}_p)-u_p}{\sigma_p}\right) \;\; if(s_iu_i) > 0 \\ \int\limits_{x_p(\tilde{\eta}_p)}^{\infty} \phi(x_p)dx_p = 1-\psi_o\left(\frac{x_p(\tilde{\eta}_p)-u_p}{\sigma_p}\right) \;\; if(s_iu_i) < 0 \end{cases} \tag{A.2b}$$

where ψ_o is a Gaussian distribution with unit variance and zero mean. The two relations in the right side of eq. A.2b come from the first condition of a valid transformation of spaces, i.e., y_p $(= \eta_p^2\ d/2)$ is monotonic in x_p. For example, if $s_iu_i < 0$, the r.v.'s defined in eq.(A.1b) approach $y_p(x_p^b) < y_p(x_p^a)$ if $x_p^b > x_p^a$. This implies reversing the integration limits in eq. A.2b.

The p.f of η written in eq.(A.1a) corresponds to a folded distribution, which is related to the p.f. F defined in eq.(A.2b) by the form (e.g., Dudewicz & Mishira, 1987):

$$F\left(\frac{\eta_p^2 d}{2}\right) - F\left(\frac{-\eta_p^2 d}{2}\right) = \Psi_o\left(\frac{(\frac{\eta_p^2 d}{2s_iu_i} - \frac{c}{s_iu_i}) - u_p}{\sigma_p}\right) - \Psi_o\left(\frac{(\frac{-\eta_p^2 d}{2s_iu_i} - \frac{c}{s_iu_i}) - u_p}{\sigma_p}\right) \tag{A.3}$$

after expressing x_p in terms of $\eta_p = \sqrt{\frac{2|x_p(s_iu_i)+c|}{d}}$ (eq. 2.8). The right term is obtained after standardizing F, valid for $s_iu_i > 0$.
With the variable transformations

$$x_p^+(\eta) = \frac{\eta^2 d}{2s_iu_i} - \frac{c}{s_iu_i} \tag{A.4}$$

$$x_p^-(\eta) = \frac{-\eta^2 d}{2s_iu_i} - \frac{c}{s_iu_i}$$

a final expression is defined for the conditional p.f. of η, by introducing these variables (eq. A.4) in eq. A.3:

$$G_p(\eta) = \begin{cases} \Psi_o\left(\frac{x_p^+(\eta)-u_p}{\sigma_p}\right) - \Psi_o\left(\frac{x_p^-(\eta)-u_p}{\sigma_p}\right) \;\; if\; s_iu_i > 0 \\ \Psi_o\left(\frac{x_p^-(\eta)-u_p}{\sigma_p}\right) - \Psi_o\left(\frac{x_p^+(\eta)-u_p}{\sigma_p}\right) \;\; if\; s_iu_i < 0 \end{cases} \tag{A.5}$$

This p.f. is related to the standardized folded normal distribution function (e.g., Dudewicz & Mishira, 1988). The two relations on the right come from the monotonic condition for a valid transformation of spaces as mentioned above (eq. A.2b).