

Chapter 2

Confidence limit of the magnetotelluric phase sensitive skew

When determining a parameter function of the impedance tensor, its corresponding error is normally not taken into account if it is a complicated non-linear function of the elements. One specific example treated here is the phase sensitive skew defined by Bahr (1991), which is a rotationally invariant parameter of the impedance tensor.

The skew parameter is based on the hypothesis that the impedance tensor is affected by telluric distortion, produced by shallow 3-D anomalies overlaying a regional 2-D structure (i.e., a superposition 2-D model). A 2-D model affected by telluric distortion implies equal phases between each pair of column tensor elements (in the regional coordinate system). Skew measures these impedance phase differences and thus indicates the departure from two dimensionality. It would be zero if the telluric distortion hypothesis is valid for noise-free data, whereas values over 0.3 can be considered as an indicator of 3-D inductive effects (Bahr, 1991). However, with the addition of noise to the tensor elements the skew values can suffer significant bias, leading to a false interpretation of dimensionality. A way to avoid this problem is to estimate the probability function of skew, because the tensor elements with errors can be considered analogous to random variables. Thus, instead of the skew value itself, its probability threshold can provide a more plausible information on dimensionality.

In this paper (Lezaeta [2001]), the confidence limits of the regional skew are derived by expressing its distribution function in terms of the tensor element density functions. The Jacobi-matrix transformation of random variables is used in the derivation (e.g., Fisz [1976]). This procedure is valid for functions which are continuous and continuously differentiable. Skew fulfills these conditions. The transformation was considered by assuming the tensor elements as normally distributed random variables. The result is tested for synthetic data with random Gaussian noise added to the tensor. An example applied to measured data is also shown (section 2.2). These experimental data were processed with a robust procedure (Egbert and Booker [1986]), which estimates an error covariance matrix for the impedance

tensor, assumed to approach asymptotically a Gaussian distribution (Egbert, pers.comm.).

The scope of this paper is to show that the mathematical procedure performed in the derivation of the probability function distribution of the skew results in a feasible confidence limit to analyze dimensionality. Examples with synthetic and measured data demonstrates this. In a similar manner, the confidence limits of any other non-linear function of the tensor elements and hence the measured data can be estimated.

The variable transformation for skew

Regional skew (η) is a continuous function of the tensor elements. It has the following form:

$$\eta = \frac{\sqrt{2|x_1x_7 - x_4x_6 + x_2x_8 - x_3x_5|}}{\sqrt{(x_2 - x_3)^2 + (x_6 - x_7)^2}} \quad (2.1)$$

where,

$$\begin{aligned} x_1 &= \text{Re}\{Z_{xx}\} & x_2 &= \text{Re}\{Z_{xy}\} & x_3 &= \text{Re}\{Z_{yx}\} & x_4 &= \text{Re}\{Z_{yy}\} \\ x_5 &= \text{Im}\{Z_{xx}\} & x_6 &= \text{Im}\{Z_{xy}\} & x_7 &= \text{Im}\{Z_{yx}\} & x_8 &= \text{Im}\{Z_{yy}\} \end{aligned} \quad (2.2)$$

These variables correspond to the real and imaginary part of the impedance tensor (\mathbf{Z}) elements defined in magnetotellurics:

$$\mathbf{Z} = \begin{bmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{bmatrix}$$

Skew (η) is rotationally invariant and it vanishes if the response is equivalent to that from the ideal two dimensional superposition model (i.e., electrostatic distortion without magnetic effect). This means that each pair Z_{xx} , Z_{yx} and Z_{xy} , Z_{yy} have equal phases.

Assuming a known density function distribution (d.f.) f for the tensor elements $X = (x_1, \dots, x_8)$ (eq. 2.2) –assumed as random variables (r.v.)– we can derive the function distribution (f.d.) of η in terms of X by using the Jacobi-matrix J for the transformation of variables (e.g., Fisz, 1976).

The transformation of X into η is given by a space Y , which contains again the r.v.'s of X , except for one r.v. x_p which is replaced by η :

$$X = (\dots, x_{p-1}, x_p, x_{p+1}, \dots) \rightarrow Y = (\dots, x_{p-1}, \eta, x_{p+1}, \dots) \quad (2.3)$$

The Jacobi transformation is valid if η is continuously differentiable in X . This property is valid in the space of X where the sum of variables contained in the modulus in the numerator of η (Eq. 2.1) is either a negative or a positive real number. Thus η is continuously differentiable in X except at $\eta(X) = 0$, i.e., when the numerator of η does not vanish. In order to fulfill this property for the further steps of the d.f. derivation, η will be regarded statistically as a non-zero positive real number. The lowest limit of η will be assigned as 0^+ . It is worth pointing out that a zero skew value is feasible for noise-free data (for which

no statistical error estimation of η is required), thus describing the validity of a "perfect" regional 2-D model, likewise obtainable from numerical modeling.

It is also required for the derivation of the skew d.f. that η with regards to the r.v. x_p should satisfy the following conditions required by the Jacobi-transformation:

- (1) η is monotonic with respect to x_p , i.e., for a given $x_p^a < x_p^b$ in the range $(-\infty, \infty)$, η is either monotonically increasing if $\eta(x_p^a) < \eta(x_p^b)$, or monotonically decreasing if $\eta(x_p^a) > \eta(x_p^b)$
- (2) The partial derivative is equivalent with the inversion $\frac{\partial x_p}{\partial \eta} = \left(\frac{\partial \eta}{\partial x_p}\right)^{-1}$

Condition (1) is not completely satisfied because η is an absolute value as function of the tensor elements. A further analysis is required to account for this (section 2.1). The second condition is true for the variables x_1, x_4, x_5 and x_8 having absolute partial derivatives of the form:

$$\left|\frac{\partial x_p}{\partial \eta}\right| = \frac{\eta \cdot \left[(x_2 - x_3)^2 + (x_6 - x_7)^2\right]}{|x_i|} \quad (2.4)$$

for $(p, i) = (1, 7), (4, 6), (5, 3), (8, 2)$.

In consequence, the choice of one of these x_p 's is arbitrary in the transformation due to the symmetry of $\left|\frac{\partial x_p}{\partial \eta}\right|$. Note that these variables are the diagonal impedance elements Z_{xx}, Z_{yy} , where η encounters a local minimum in them and is symmetrical with respect to this minimum (Fig.2.1).

The off-diagonal elements Z_{xy}, Z_{yx} do not fulfill condition (2) and therefore η is not symmetrical about a minimum and a maximum value in the off-diagonal elements (fig.2.2).

Having satisfied these conditions, the density function $g(\eta)$ of skew takes the form (e.g., Fisz, 1976):

$$g(\eta) = |\det[J(X/Y)]| \cdot f(X); \quad X = (x_1, \dots, x_8) \quad (2.5)$$

The matrix J is of dimension 8x8 determined by the number of r.v.'s contained in X , and have the partial derivatives of X at Y (eq. 2.3). $\text{Det}[J(X/Y)]$ is its determinant $\left|\frac{\partial x_p}{\partial \eta}\right|$. The expression for the multi-variate probability function of skew in terms of eq. 2.5 is:

$$G(\eta) = \int_{0^+}^{\eta_0} g(\eta) d\eta = \int_{0^+}^{\eta_0} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left| \frac{\partial x_p(\eta, \dots, x_{p-1}, x_{p+1}, \dots)}{\partial \eta} \right| \cdot f(\dots, x_{p-1}, x_{p+1}, \dots) \dots dx_{p-1} dx_{p+1} \dots \right\} d\eta \quad (2.6)$$

Solving this multi-variate integration is complicated; to make further progress we simplify the problem to a uni-variate system. This implies determination of a conditional probability function for η in terms of one r.v. x_p , while the other variables of X are kept fixed at their respective mean values u_i ($i = 1, \dots, 8$ with $i \neq p$). As mentioned above in regards to conditions 1 and 2 to validate the variable transformation, the r.v. x_p should be one of the diagonal tensor elements.

2.1 PROBABILITY FUNCTION OF SKEW

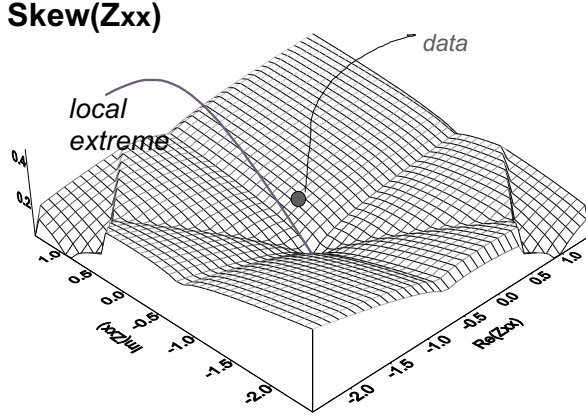


Figure 2.1: Characteristic surface plot from synthetic data of the regional skew parameter (η) as function of the real and imaginary part of the tensor element Z_{xx} (km/s). The gridding interval of the variables is 0.01 centered on the extremal point, while the other tensor elements are kept fixed at their synthetic values. This plot is also characteristic for the other diagonal tensor element Z_{yy} . Any of these 4 diagonal elements can be used in the Jacobi-transformation of random variables.

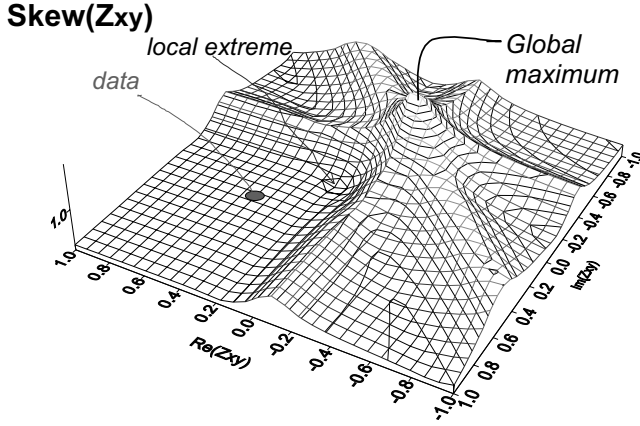


Figure 2.2: Characteristic surface plot from synthetic data of the regional skew parameter (η) as function of the real and imaginary part of the tensor element Z_{xy} (km/s). Details as in Fig.2.1. Skew has global positive & negative extreme. This is also true for the other off-diagonal tensor element Z_{yx} . These 4 elements cannot be used in the Jacobi-transformation of random variables (see text).

2.1 Derivation of the probability function of the regional skew parameter

The conditional probability function P of $\eta(x_p) = \eta_p$, given the known mean values u_i of the tensor elements ($i \neq p$), will be expressed as:

$$G_p(\tilde{\eta}_p) = P(\eta_p < \tilde{\eta}_p) \quad (2.7)$$

The derivation of P should satisfy condition 1 of the variable transformation, i.e, η should be either monotonically increasing or decreasing with respect to x_p . Also, the r.v. x_p should be one of the diagonal tensor elements.

In order to simplify the following equations for the further derivation of G_p , the skew parameter from eq.(2.1) will be expressed with the new term:

$$\eta_p = \sqrt{\frac{2|x_p(s_i u_i) + c|}{d}} \quad (2.8)$$

where the sub-index pairs (p, i) are as in eq.(2.4), and $d = (u_2 - u_3)^2 + (u_6 - u_7)^2$ is in the denominator of η at the corresponding mean values. The parameter c contains 6 conditional

variables at their respective mean values and, $s_i = \pm 1$ is assigned to the addition or subtraction symbol which precedes the respective pair $x_p x_i$ in eq.(2.1).

Analysing for example the conditional probability for the r.v. x_1 (i.e., $Re(Z_{xx})$; eq. 2.2), then $u_i = u_7$, $s_7 = 1$ and $c = -u_4 u_6 + u_2 u_8 - u_3 u_5$.

In the following, the p.f P of η (eq. 2.7) is expressed in terms of eq.(2.8), and after a change of variables a new expression is derived:

$$\begin{aligned} G_p(\tilde{\eta}_p) &= P\left(\sqrt{\frac{2|x_p s_i u_i + c|}{d}} < \tilde{\eta}_p\right) = P\left(|x_p s_i u_i + c| < \frac{\tilde{\eta}_p^2 d}{2}\right) \\ &= P\left(-\frac{\tilde{\eta}_p^2 d}{2} < (x_p s_i u_i + c) < \frac{\tilde{\eta}_p^2 d}{2}\right) \end{aligned} \quad (2.9)$$

Assuming that the tensor elements are normally distributed, the r.v. x_p has a normal d.f. $\phi(x_p)$ with mean u_p and standard deviation σ_p .

The conditional p.f. $G_p(\eta)$ (eq. 2.9) for normally distributed tensor elements, expressed in terms of the standard distribution having a variance of 1 and mean 0 (ψ_o) is:

$$G_p(\eta) = \begin{cases} \Psi_o\left(\frac{x_p^+(\eta) - u_p}{\sigma_p}\right) - \Psi_o\left(\frac{x_p^-(\eta) - u_p}{\sigma_p}\right) & \text{if } s_i u_i > 0 \\ \Psi_o\left(\frac{x_p^-(\eta) - u_p}{\sigma_p}\right) - \Psi_o\left(\frac{x_p^+(\eta) - u_p}{\sigma_p}\right) & \text{if } s_i u_i < 0 \end{cases} \quad (2.10)$$

where the variables:

$$\begin{aligned} x_p^+(\eta) &= \frac{\eta^2 d}{2s_i u_i} - \frac{c}{s_i u_i} \\ x_p^-(\eta) &= \frac{-\eta^2 d}{2s_i u_i} - \frac{c}{s_i u_i}. \end{aligned} \quad (2.11)$$

The p.f. of η (eq. 2.10) is related to the standardized folded normal distribution function (e.g., Dudewicz and Mishra [1988]). The two relations on the right comes from the monotonic condition for a valid transformation of spaces. The derivation of $G_p(\eta)$ is given in the appendix (A).

Confidence limit

The confidence limit (C.L.) of the skew (η) is defined as the probability (P) that its true value η_o has to lie within a certain range $[\eta_a, \eta_b]$. We use the conditional probability function $G_p(\eta)$ of skew (eq. 2.10) to derive the confidence limit C.L., expressed as:

$$P(\eta_a < \eta_o < \eta_b) = C.L. = G_p(\eta_b) - G_p(\eta_a)$$

Since the p.f. G_p depends on the standardized normal distribution Ψ_o , which is symmetrical around the expected value, the desired confidence limit will be given by the following upper and lower limits:

$$G_p(\eta_b) = \psi_o\left(\frac{x_p^+(\eta_b) - u_p}{\sigma_p}\right) - \psi_o\left(\frac{x_p^-(\eta_b) - u_p}{\sigma_p}\right) = \frac{1 + C.L.}{2}$$

2.1 PROBABILITY FUNCTION OF SKEW

$$G_p(\eta_a) = \psi_o \left(\frac{x_p^+(\eta_a) - u_p}{\sigma_p} \right) - \psi_o \left(\frac{x_p^-(\eta_a) - u_p}{\sigma_p} \right) = \frac{1 - C.L.}{2} \quad (2.12)$$

if $s_i u_i > 0$, otherwise the indexes a with b and b with a should be exchanged, as indicated in eq.(2.10).

The confidence limit (η_a, η_b) can be determined numerically with some iterative algorithm, since this cannot be solved directly by simply inverting the folded standard function $\Psi_o^+ - \Psi_o^-$. The variables η_a, η_b are the quantiles of the d.f. G_p at the values $\frac{1-C.L.}{2}, \frac{1+C.L.}{2}$, respectively (provided that $s_i u_i > 0$, otherwise the limits are reversed). Public function libraries written in Fortran as well as in C language can be used to calculate the quantile of a desired distribution function (e.g., Brandt [1999]). The algorithm to find $C.L.$ consists of minimizing the function:

$$\min \left\{ \left(G_p(n_{a,b}^j) - \frac{1 \pm C.L.}{2} \right) \right\} \quad (2.13)$$

with $\eta_{a,b}^j$ ($j = 0, 1, 2, \dots$) chosen iteratively in order to take appropriate values for the minimization function.

The result is dependent on the variable x_p chosen, which can be either the element ReZ_{xx} , ImZ_{xx} , ReZ_{yy} or ImZ_{yy} . It is however advisable to select the variable which brings the largest confidence limit (section 2.2).

95% confidence limit

To show an example of the 95% confidence limit in terms of an explicit random variable x_p , consider this to be the element $x_1 = ReZ_{xx}$ (eq. 2.2), which, expressed in terms of η (eq. 2.1) is:

$$x_p(\eta, x_{p-1}, \dots) = x_1(\eta, \hat{x}_2, \dots, \hat{x}_8) = \frac{\pm \eta^2}{2\hat{x}_7} \cdot \left[(\hat{x}_2 - \hat{x}_3)^2 + (\hat{x}_6 - \hat{x}_7)^2 \right] + \frac{(\hat{x}_4\hat{x}_6 - \hat{x}_2\hat{x}_8 + \hat{x}_3\hat{x}_5)}{\hat{x}_7}$$

The variables of eq.(2.11), derived from the transformation of limits of the probability function of η ($G_p(\eta_p)$; eq. 2.9), are:

$$x_1^+(\eta) = \frac{\eta^2 d}{2\hat{x}_7} - \frac{c}{\hat{x}_7}, \quad x_1^-(\eta) = \frac{-\eta^2 d}{2\hat{x}_7} - \frac{c}{\hat{x}_7}$$

where

$$\begin{aligned} c &= -\hat{x}_4\hat{x}_6 + \hat{x}_2\hat{x}_8 - \hat{x}_3\hat{x}_5 \\ d &= (\hat{x}_2 - \hat{x}_3)^2 + (\hat{x}_6 - \hat{x}_7)^2 \\ s_i &= s_7 = 1 \quad \text{and} \quad u_i = \hat{x}_7 \end{aligned}$$

Each \hat{x}_i is the measured data considered as the mean value of the respective variable.

To derive the 95% confidence limit of η , i.e., $C.L. = 0.95$, the function to minimize through successive iterations is:

$$\min \left\{ \left(G_1(n_{a,b}^j) - \frac{1 \pm 0.95}{2} \right) \right\} \quad \text{with} \quad \eta_{a,b}^j (j = 0, 1, 2, \dots)$$

until the lower and upper limits η_a, η_b are found for a given tolerance. $G_1(\eta)$ is the probability function of eq.(2.10) at $p = 1$, which approaches the lower and upper limits η_a, η_b (or viceversa) as expressed in eqs. 2.12.

2.2 Examples with synthetic and field data

1. Synthetic data

The confidence limit of η was tested to the responses of a forward 3-D model, calculated with a modified version (Mackie and Booker, 1999) of the algorithm developed by Mackie et al. (1994).

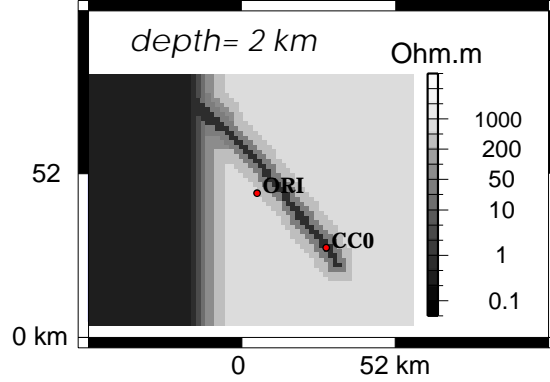


Figure 2.3: Horizontal view of the 3-D model at 2 km depth. The model responses at sites ORI and CC0, located next to the thin conductor, were used in the estimation of the skew confidence limits.

The model consists of a shallow 3-D conductive vertical dike (of $5 \Omega\text{m}$ and 8 km depth) with horizontally finite extensions ($8 \times 40 \text{ km}^2$). It is embedded in a resistive medium ($500 \Omega\text{m}$), and one of its edges is connected to a 2-D conductive block ($1 \Omega\text{m}$), which reaches a depth of 5 km. The dike strikes by 45° with respect to the conductive block. In Fig.2.3, an horizontal view of the 3-D model at 2 km depth is presented. The skew values of two sites (ORI and CC0) were considered in the test. Site ORI is located near the centre of the dike and site CC0 is above and near the edge of the dike (Fig.2.3).

Gaussian noise was added to the tensor elements of the model responses with standard deviation of 2% and 5% of the largest tensor element amplitude. The procedure was repeated 100 times. The mean value of the random sample is the estimate of the noisy tensor element, and its error the estimate of the standard deviation.

The right hand plots of Figs. 2.4 and 2.5 show the 5% noisy tensor elements with their errors, compared against the model responses. The skew parameters calculated from the model responses (ORI, CC0) and the noisy data (ran) are shown on the left hand plots of the figures. The 2% and 5% random noise data are illustrated separately with their respective 95% confidence limits. The latter comes from the conditional probability function of skew ($G_p(\eta)$; eq. 2.10), which minimizes eq.(2.13), for the variable x_p (of one the diagonal tensor elements; eq. 2.2) resulting in the largest confidence limit, since this was seen to cover the region of the model response.

At site CC0 (located above the end of the dike; Fig.2.3), the skew of the model response indicates that at the period range 100-500 s the departure from the 2-D superposition model is the highest, which means the most significant induction effect at these penetration depths. The noisy skew has been down biased indicating in contrast that the best fit with the 2-D model hypothesis is at this period range. The confidence limit indeed reflects the range of

2.2 EXAMPLES WITH SYNTHETIC AND FIELD DATA

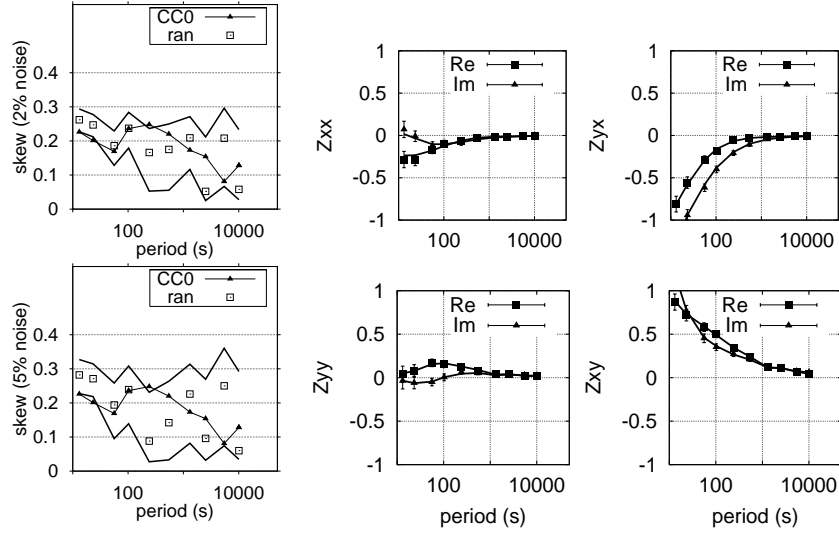


Figure 2.4: An example of skew values using synthetic data (left), estimated from the tensor elements shown in right (units in km/s). The elements of the model response (lines) are shown over the data scattered with 5% Gaussian noise (dots). The skew of the model response (CC0; Fig.2.3) is shown over the noisy skew (ran) within its 95% confidence limit. Above: Skew from the elements with 2% Gaussian noise. Below: Skew from the elements with 5% Gaussian noise.

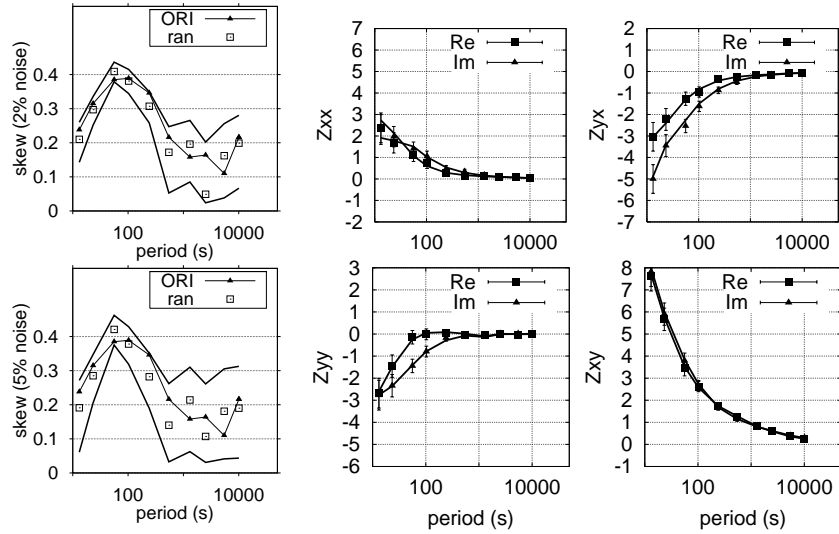


Figure 2.5: As for Fig.2.4, but with station ORI (from 3-D model; Fig.2.3).

the 95% probability in which the true skew value can lie.

By site ORI (close to the centre of the dike; Fig.2.3), the confidence limit of skew also reflects the 95% range probability of the true value. Their thresholds follow generally the trend of the real skew.

2. Field data

As an example, data from two stations (TIQ and GER) obtained from MT measurements carried out in the Southern Central Andes (Chapter 4) are shown. Time series data were processed using the robust technique of Egbert and Booker (1986), performing also a remote reference site to improve the data quality. The impedance tensor is estimated with an error covariance matrix, which is assumed to follow an asymptotically Gaussian distribution. This assumption of course improves with increasing number of sample data recorded.

Figure 2.6 shows the tensor elements of sites TIQ and GER, and the skew values with their confidence limits. The latter was estimated for the diagonal element variable which gave the largest confidence limit, analogous to the synthetic data example. The greatest uncertainties of skew (i.e, a broader confidence limit) are seen at the shortest and longest periods, where the relative errors of the tensor elements are the largest. At long periods, the smallest skew values close to the lower confidence limit could reflect a strong down bias from the true values, since the upper confidence limit is further higher.

For these data set, the upper confidence limit for the skew can be used to analyse dimensionality. Where this upper threshold (i.e., the 97.5% probability) is far greater than 0.3 (f. ex.), while the lower confidence limit is near a zero skew, might indicate to an unreliable skew due to noise rather than an indicator of 3-D inductive effects.

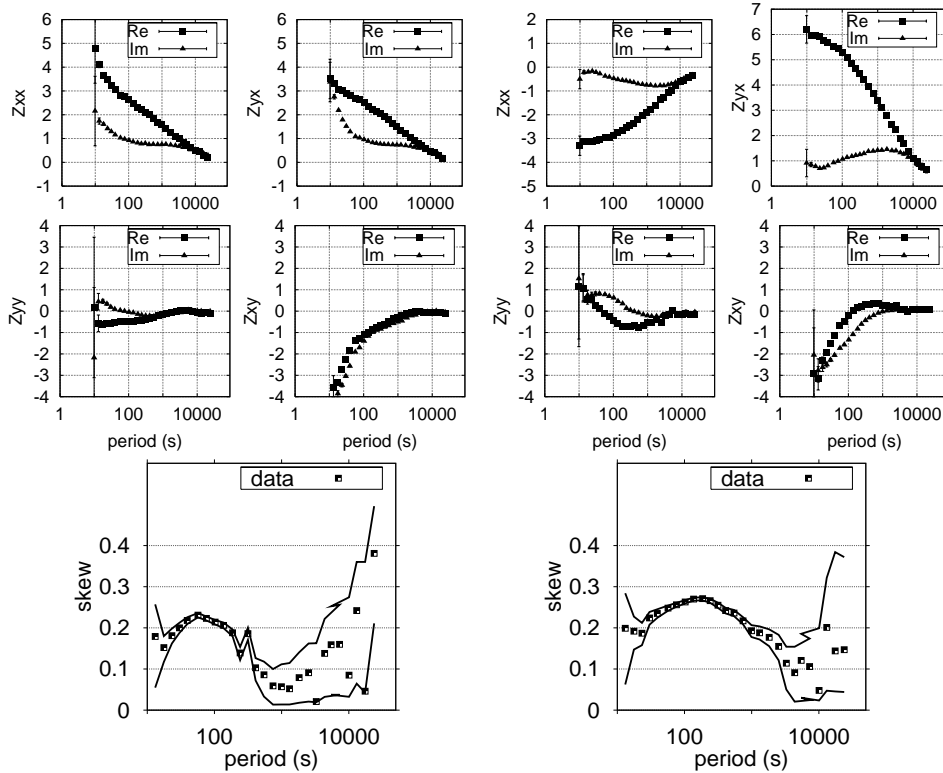


Figure 2.6: Example with field data for site TIQ (left) and GER (right) located in the Southern Central Andes. Above: The tensor elements of the field data processed with a robust procedure (units in km/s). Below: The skew parameters of the data (dots) within the 95% confidence limits (lines).

Conclusions

The derivation of the conditional probability function of the skew parameter allows estimation of a plausible confidence limit of the true value.

To analyse dimensionality on field data, it is advisable to treat the upper 95% confidence limit instead of the skew value itself. It was shown with synthetic data that the skew estimated from tensor elements scattered with 2% and 5% random Gaussian noise could suffer strong bias with regards to the true skew value. Where the confidence limit becomes extreme large ($\gg 0.3$), the data should be discarded from the analysis.

The statistical procedure developed here can also be analogously applied for any other parameter which is a continuous and continuous differentiable non-linear function of the tensor elements.