

# Chapter 2

## Linear elasticity

This chapter introduces the theoretical background and summarizes the most important constitutive equations of the stress sensitivity approach for arbitrary anisotropic media under arbitrary load. Therefore, it starts with some general remarks the principles of linear elasticity for anisotropic media as defined by Hooke's Law. This will include the definitions of the important elastic parameters, e.g. the compliances of anisotropic rocks, bulk modulus, shear modulus, and Young's modulus, and the definition of the stress and strain tensor.

Section (2.1.2) to (2.1.4) explain the elastic properties of isotropic, transversely isotropic and orthorhombic media in more detail. Among the different symmetry classes these types of anisotropy represent the most important symmetry classes for geophysical applications. The detailed description of these media covers their general symmetry properties and the most common geological features responsible for the anisotropy. Moreover, aspects of plane wave propagation through these media are considered. Thereby, the well known Thomsen's parameters for vertical transversely isotropic media as well as their equivalents for orthorhombic media, the Tsvankin's parameters, will be introduced.

From section (2.3) on the media under consideration are no longer treated as monophasic materials only, as done in classical elasticity and seismics. The description of the media approaches the more realistic situation of a porous rock. This leads to the fundamental concept of poroelasticity. However, it is beyond the scope of this thesis to consider all aspects of the mechanics of and wave propagation through poroelastic media. Hence, the considerations made here are limited to aspects concerning the deformation of porous media.

From section (3.2) to section (3.2.5) a summary of the derivation of the stress sensitivity approach for anisotropic media under non-isostatic load is given. The complete derivation of the approach is given in Appendix (B) which reflects the paper of Shapiro & Kaselow (2003).

## 2.1 Basics of elasticity

When a force, either internal or external, is applied to a continuum every point of this continuum is influenced by this force. It is common to denote internal forces as *body forces* and external forces as *contact forces*. The most common body force results from the acceleration due to gravity. Body forces are proportional to the volume of the medium and to its density and have the unit force per volume. Contact forces depend on the surface they are acting on and have the unit force per area.

Imagine external forces acting on a continuum. In general, these forces will lead to a deformation of the medium resulting in changes of size and shape. Internal forces acting within the medium try to resist this deformation. As a consequence the medium will return to its initial shape and volume when the external forces are removed. If this recovery of the original shape is perfect, the medium is called *elastic*.

The constitutive law relating the applied force to the resulting deformation is Hooke's Law. It is defined in terms of stress and strain. The exact form of the stress, the *state of stress*, at an arbitrary point P of the continuum depends on the orientation of the force acting on P and the orientation of the reference plane with respect to a reference coordinate system. To quantify the state of stress at a point P resulting from the force  $\mathbf{F}$ , P is imagined as an infinitesimal small cube. The stress acting on each of the six sides of the cube can be resolved into components normal to the face and within it. This situation is illustrated in Fig. (2.1) for the plane normal to the 2 axes. In the following, a plane oriented normal to an axes  $i$  is called the  $i$ -plane.

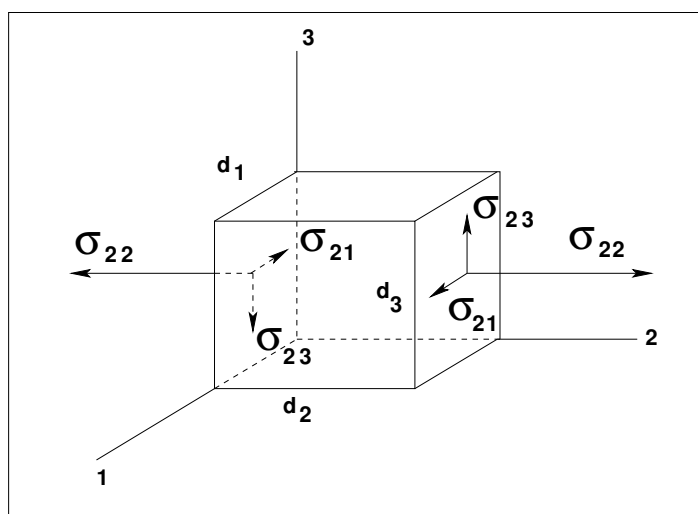


Figure 2.1: Stress components acting on the 2-plane.

A stress  $\sigma_{ij}$  is defined as acting on the  $i$ -plane and being oriented in the  $j$  direction. Components of the stress tensor with repeating indices, e.g.  $\sigma_{11}$ , are denoted as normal stress while a stress component with different indices is called a shear stress. Consequently, this gives six shear and three normal stress components acting on the cube. If the medium is in static equilibrium the sum of all stress components acting in the 1, 2, and 3 direction and the total moment is zero. This means:

$$\sigma_{ij} = \sigma_{ji}.$$

Thus, the stress tensor  $\sigma_{ij}$  completely describes the state of stress at any point P of the continuum.

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{32} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}, \quad \text{with } i, j = 1, 2, 3 \quad (2.1)$$

Normal stresses with positive values directed outward from faces are called *tensional* stress, and negative values correspond to *compressional* stress. The SI unit for stress is Pa. In geoscientific practice, stresses are usually given in mega pascal (1 MPa =  $10^6$ P).<sup>1</sup> A special state of stress is found when all normal stresses are equal, i.e.,  $\sigma_{11} = \sigma_{22} = \sigma_{33}$  and all shear stresses are zero. Then, the stress tensor is independent of the reference coordinate system, and the stress can be understood as a scalar, thus, as a pressure. This pressure is given as  $P = -\sigma_{ii}$ . Such a stress state is often denoted as *hydrostatic*, because it is similar to the pressure in a fluid, which is always equal in all directions. However, this state of stress in a solid material depends on the orientation and magnitude of the externally or internally applied forces. In a fluid, it results from the general property of fluids that they can not resist shear stress. Therefore, this state of stress in a solid material should be correctly denoted as *isostatic* and *hydrostatic* should refer to pressure in a fluid.

As mentioned above, when an elastic body is subjected to stress, changes in size and shape occur and these deformations are called strain. Per definition, strain is the relative (fractional) change of a dimension of a body. In the three dimensional case the strain at point P is determined by the *strain tensor*  $\epsilon_{ij}$ , assuming the deformations to be sufficiently small:

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{32} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix}, \quad \text{with } i, j = 1, 2, 3 \quad (2.2)$$

The elements of the strain tensor with repeating indices are denoted as normal strain, all others as shear strain. Just as the stress tensor the strain tensor has six independent components, e.g.,

$$\epsilon_{ij} = \epsilon_{ji}.$$

The volume change of a body is given by the diagonal elements  $\epsilon_{ii}$  of the strain tensor only. Normalizing this volume change to a unit volume defines the *dilatation*  $\Delta$  of a body:

$$\Delta = \epsilon_{ii}, \quad (2.3)$$

where summation over repeated indices is assumed.

### 2.1.1 Hooke's Law

Stress and strain are related to each other by Hooke's Law where the strain is assumed to be sufficient small that stress and strain depend linearly on each other. Such a medium is called *linear* elastic. In its general form Hooke's law reads:

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}, \quad \text{with } i, j, k, l = 1, 2, 3 \quad (2.4)$$

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<sup>1</sup>Especially in the hydrocarbon industry, different units are very common, e.g., *psi* or *lbs*. For more details on pressure terminology and conversion factors, see section 3.1 and appendix G, respectively.

## 2.1. Basics of elasticity

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The fourth-rank tensor  $C_{ijkl}$  is called the *stiffness tensor* and consists of 81 entries. It holds the elastic constants of a medium. This tensor actually links the deformation of a medium to an applied stress. In general, Hooke's law leads to complicated relations, but simplifies remarkably, especially in the case of isotropic media.

Each component of stress  $\sigma_{ij}$  is linearly dependent upon every component of strain  $\epsilon_{kl}$  and vice versa. Since all directional indices may assume values 1, 2, and 3 one obtains 9 relations. Each of these relations involves one component of stress and nine components of strain.

Since the stress tensor is symmetrical, i.e.  $\sigma_{ij} = \sigma_{ji}$ , only six of these equations are independent. This is also valid for the strain. Thus, also only six terms of the right side of eq. (2.4) are independent.

Alternatively, one may express the strain as a linear combination of stress

$$\epsilon_{ij} = \sum_{k=1}^3 \sum_{l=1}^3 S_{ijkl} \sigma_{kl}, \quad i, j = 1, 2, 3. \quad (2.5)$$

In this case,  $S_{ijkl}$  is called the *elastic compliance tensor* and its elements are called *compliances*. Both tensors  $C$  and  $S$  have the same symmetry and it is:

$$C_{ijkl} S_{klmn} = I_{ijmn} \quad (2.6)$$

For simplicity, it is useful to apply the Voigt notation to express the 3 x 3 x 3 x 3 stiffness tensor  $C_{ijkl}$  as a 6 x 6 stiffness matrix  $C_{IJ}$ . This means, that each pair of indices ij(kl) is replaced by one index I(J), according to Tab. (2.1).

ij(kl)	I(J)
11	1
22	2
33	3
23,32	4
13,31	5
12,21	6

Table 2.1: Voigt notation: Scheme for index replacement(Thomsen, 2002).

Not all matrices are tensors. A tensor is a matrix with elements that depend upon an assumed coordinate "frame of reference", and that transforms (when referred to a different "frame of reference") according to a certain transformation rule. The 6 x 6 matrix  $c$  is not a tensor. When a tensor transformation is necessary it has to be performed either directly on  $C_{ijkl}$  or, more efficiently, by using the 6 x 6 *Bond transformation matrices* together with  $C_{IJ}$  (Auld, 1990).

As mentioned, in its most general form the stiffness tensor has 81 entries. However, due to the symmetry of stress and strain the number of independent entries reduces to 36:

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk}$$

Moreover, the existence of a unique strain energy potential requires that

$$C_{ijkl} = C_{klij}.$$

Thus, the number of independent entries in the stiffness tensor reduces to 21. A medium characterized by such a stiffness tensor is called triclinic. Crystals are organized due to their macroscopical symmetry in 32 classes which are subdivided into 7 systems. In seismics only three symmetry classes are important: orthorhombic, transversel (or hexagonal), and isotropic (Thomsen, 2002). In some cases, when dealing with rock or mineral properties, also cubic symmetry might be considered (Mavko *et al.*, 1998). The certain stiffness tensors of the mentioned media are given in Appendix (C) in Voigt notation.

### 2.1.2 Isotropic media

In the most simple symmetry case of an isotropic elastic solid, the material has only two independent elastic moduli, called the *Lamé constants*,  $\lambda$  and  $\mu$ . In such a medium the elastic properties at any point P are independent from direction. The Lamé constants are related to the stiffness tensor  $C_{ijkl}$  by

$$C_{ijkl} = [\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]\epsilon_{kl}, \quad (2.7)$$

where  $\delta$  is the Kronecker delta function defined as

$$\delta_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}, \quad i, j = 1, 2, 3. \quad (2.8)$$

The stiffness tensor of isotropic media in Voigt notation explicitly reads:

$$\begin{pmatrix} C_{33} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{33} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{pmatrix},$$

with

$$C_{12} = C_{33} - 2C_{55}$$

and

$$C_{55} = \mu \quad \text{and} \quad C_{33} = K + \frac{4}{3}\mu$$

The *shear modulus*  $\mu$  is a measure of resistance to shear stress, according to:

$$\sigma_{ij} = 2\mu\epsilon_{ij}, \quad \text{with } i \neq j. \quad (2.9)$$

In a fluid the shear modulus always equals zero and the deformation of a solid material is small if  $\mu$  is high. Its physical unit is Pa, and is usually given for rocks in GPa (1GPa =  $10^9$ Pa).

The second Lamé parameter  $\lambda$  is important as a combination with other elastic constants, e.g., the Young's modulus  $E$ , the bulk modulus  $K$ , and Poisson's ratio  $\nu$ .

## 2.1. Basics of elasticity

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$\mu$	$K$	$\lambda$	$E$	$\nu$
$\frac{3(K-\lambda)}{2}$	$\lambda + \frac{2\mu}{3}$	$K - \frac{2\mu}{3}$	$\frac{9K\mu}{3K+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$
$\lambda \left( \frac{1-2\nu}{2\nu} \right)$	$\mu \left[ \frac{2(1+\nu)}{3(1-2\nu)} \right]$	$\frac{2\mu\nu}{1-2\nu}$	$2\mu(1+\nu)$	$\frac{\lambda}{3K-\lambda}$
$3K \left( \frac{1-2\nu}{2+2\nu} \right)$	$\lambda \left( \frac{1+\nu}{3\nu} \right)$	$3K \left( \frac{\nu}{1+\nu} \right)$	$\mu \left( \frac{3\lambda+2\mu}{\lambda+\mu} \right)$	$\frac{3K-2\mu}{2(3K+\mu)}$
$\frac{E}{2(1+\nu)}$	$\frac{E}{3(1-2\nu)}$	$\frac{E\nu}{3(1+\nu)(1-2\nu)}$	$3K(1-2\nu)$	$\frac{3K-E}{6K}$

Table 2.2: Relationships between the different elastic moduli (Thorne & Wallace, 1995).

The bulk modulus  $K$  is defined as the ratio of an applied isostatic stress to the fractional volumetric change. It is also called *incompressibility*, following the *compressibility*  $C = 1/K$ :

$$\frac{1}{3}\sigma_{ii} = K\epsilon_{ii}. \quad (2.10)$$

In a uniaxial state of stress (e.g.,  $\sigma_{11} \neq 0$ ,  $\sigma_{22} = \sigma_{33} = 0$ ), Young's modulus  $E$  relates the stress to the resulting strain in the same direction.

$$\sigma_{ii} = E\epsilon_{ii}. \quad (2.11)$$

The Poisson's ratio  $\nu$  is also defined for an uniaxial stress state and relates the lateral strain (j-direction) to axial strain (i-direction):

$$\nu = -\frac{\epsilon_{jj}}{\epsilon_{ii}}, \quad (2.12)$$

where no summation over repeated indices is implied. In contrast to the other elastic parameters which have the physical unit of a pressure the Poisson's ratio is dimensionless.

The mentioned elastic parameters can be obtained for a given rock sample in the laboratory by strain measurements during uniaxial or triaxial compression tests, dependent on the desired parameter.

A large collection of typical orders of magnitude for the mentioned elastic parameters for the most important rock forming minerals is given in, e.g., Mavko *et al.* (1998).

In the case of an isotropic linear elastic material the bulk modulus and the shear modulus, as well as the density  $\rho$  of the material define the compressional and shear wave velocity  $V_P$  and  $V_S$ , respectively. This dependence reads:

$$V_S = \sqrt{\frac{\mu}{\rho}}, \quad (2.13)$$

$$V_P = \sqrt{\frac{K + \frac{4}{3}\mu}{\rho}}. \quad (2.14)$$

An additional parameter often used in rock physics is the P-wave modulus  $M$ :

$$M = V_P^2 \rho. \quad (2.15)$$

Considering eq. (2.13) and (2.14) it is possible to calculate the elastic parameters of an isotropic medium when P- and S-wave velocity and the medium density are known or can be reasonably assumed. From many observations it is known that the elastic parameters obtained from so-called static laboratory compression test differ from those obtained by inverting the wave velocities. Popp (1994) found the discrepancy between the "static" and "dynamic" moduli of samples from the KTB pilot hole to be on the order of 10 % whereby dynamic moduli are usually higher than static.

The assumption of an isotropic medium is frequently used in geophysical applications. An obvious reason for this is that the constitutive equations are much easier than for anisotropic media and can thus be understood and evaluated more intuitively. Moreover, less parameters have to be determined to describe the model and thereby, the computational costs are remarkably lower, especially when dealing with seismic field data. However, also the limitation to an isotropic model has proven to be sufficient in many field studies there are situations where a more sophisticated approach is required which takes the anisotropy of the medium into account.

### 2.1.3 Transversely isotropic media.

The most simple anisotropic model is that of a transversely isotropic (TI) medium. Such a medium is characterized by the existence of a single plane of isotropy and one single axis of rotational symmetry, the normal to the isotropy plane. Any plane containing the axis of symmetry represents a plane of mirror symmetry. All seismic signatures in such a medium depend only on the angle between the direction of wave propagation and the symmetry axis.

The vast majority of anisotropic field and laboratory studies assume TI symmetry. A reason is that this symmetry is assumed to be dominant for the intrinsic anisotropy of shales, caused by the preferred orientation of clay minerals. Shales represent almost 75% of the clastic fill of sedimentary basins (Tsvankin, 2001), which host many hydrocarbon reservoirs. The second prominent source for TI symmetry is periodic thin layering, also common in sedimentary basins. Thin layering means an interbedding of thin isotropic layers with different elastic properties on a scale small in comparison with the wavelength. This case is illustrated in Fig. (2.2), where the  $x_3$  axis represents the symmetry axis.

Due to the frequent application of this model to sedimentary basins with more or less horizontal layering some special cases of TI symmetry have been established. A TI medium with a vertical axis of symmetry and thus a horizontal plane of isotropy is called a *vertical transversel isotropic* (VTI) medium. However, in many geological settings layers are dipping for numerous reasons, e.g., when approaching a salt dome flank, in overthrust areas or due to block rotation associated with normal faulting. In such cases the symmetry axis may be tilted with respect to the earth's surface. Such a medium shows an effective anisotropy of a *tilted transversel isotropic* (TTI) medium. Tilting the symmetry axis all the way up to the horizontal produces a *horizontal transversel isotropic* (HTI) medium. Such a symmetry is usually assumed to be caused by a set of parallel vertical cracks embedded in an isotropic background medium (Tsvankin, 2001; Bakulin *et al.*, 2000a). Both VTI as well as HTI symmetry can be understood as special cases of the more complex orthorhombic symmetry class (see section 2.1.4 and

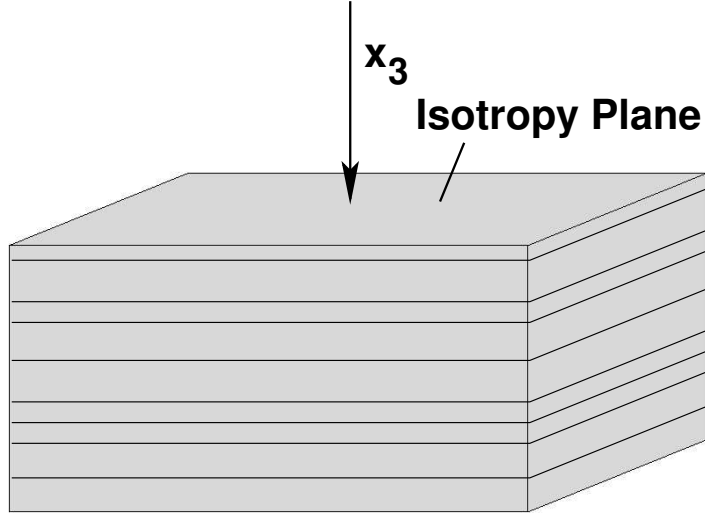


Figure 2.2: TI medium with vertical symmetry axis (VTI medium).

2.2.2).

The stiffness tensor in Voigt notation for VTI media, when we assume that the axis of symmetry is the 3-axis, has the form (e.g., Tsvankin, 2001)

$$C_{IJ}^{\text{VTI}} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix}, \quad (2.16)$$

with

$$C_{12} = C_{11} - 2C_{66}.$$

Note, in many publications, e.g., the fundamental paper of Thomsen (1986),  $C_{44}$  is used in eq. (2.16) instead of  $C_{55}$ . However, this is just a question of notation since  $C_{44} = C_{55}$  in VTI media.

### 2.1.4 Orthorhombic media

Orthorhombic media are characterized by three mutually orthogonal planes of symmetry. If the axes of a Cartesian reference coordinate system are aligned within the symmetry planes, i.e., if the coordinate planes coincide with the symmetry planes, the stiffness matrix of the orthorhombic system has nine independent entries and reads (Tsvankin, 2001):

$$C_{IJ}^{\text{Ortho}} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix}. \quad (2.17)$$



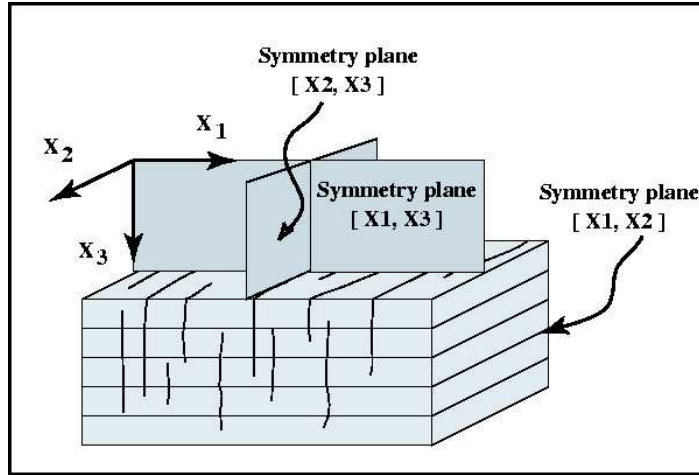


Figure 2.3: Symmetry planes in orthorhombic media where anisotropy arises from a system of parallel vertical fractures embedded in a VTI medium, after Tsvankin (2001).

In sedimentary basins orthorhombic symmetry is commonly caused by parallel vertical fractures embedded within a VTI background medium (see 2.1.3) as illustrated in Fig. (2.3). It may also arise due to two or three mutually orthogonal fracture sets, or due to two identical fracture systems criss-crossing with an arbitrary angle (Tsvankin, 1997). Such sets of fractures are common, e.g., for thick sandstone beds and granites. Bakulin *et al.* (2000b) conclude that orthorhombic symmetry thus might be the most realistic symmetry for many geophysical problems. Despite this conclusion, the application of orthorhombic anisotropy in seismic inversion and processing is obstructed by the large number of nine independent entries of the stiffness matrix. Moreover, if the orientation of the symmetry planes is unknown, as it is usually the case in seismic field experiments the number of unknowns increases to 12, since the angles between the symmetry planes and the coordinate planes of the reference observation system have to be determined additionally.

## 2.2 Plane waves in isotropic and anisotropic media

The considerations concerning waves in anisotropic media presented in this thesis are limited to the case of weak anisotropy only, i.e., media with an anisotropy below 10-20 %. This is consistent with most seismic applications and research, since the algebraic complexity of the exact constitutive relations for all magnitudes of anisotropy is the primary obstacle in analyzing seismic data. For details on these equations, their simplification in the case of weak anisotropy and the resulting benefit in seismic applications see Appendix D and, e.g., Thomsen (1986).

In contrast to isotropic media the seismic velocities in anisotropic media vary depending on the direction of propagation with respect to the symmetry properties of the medium in anisotropic media. An analytical description of plane waves in arbitrary anisotropic media can be derived using the elastodynamic wave equation (eq. 2.18) as, e.g., given by Tsvankin (2001).

$$\rho \frac{\partial^2 u_i}{\partial t^2} - C_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l} = 0 \quad (2.18)$$

## 2.2. Plane waves in isotropic and anisotropic media

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Here,  $\rho$  is the density of the medium,  $u_i$  is the displacement vector,  $t$  is time and  $x_i$  are the Cartesian coordinates and summation over repeated indices is implied. We see that the anisotropy of the medium enters equation (2.18) via the stiffness tensor  $C_{ijkl}$ . In this context, talking about anisotropic media includes isotropic media as well, since also the stiffness tensor of an isotropic can be inserted into equation (2.18).

Inserting a harmonic plane wave representation like

$$u_k = U_k \exp(i\omega(n_j x_j / V - t)) \quad (2.19)$$

into equation 2.18 leads to the Christoffel equation (Musgrave, 1970):

$$\begin{bmatrix} G_{11} - \rho V^2 & G_{12} & G_{13} \\ G_{21} & G_{22} - \rho V^2 & G_{23} \\ G_{31} & G_{32} & G_{33} - \rho V^2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = 0, \quad (2.20)$$

where  $G_{ij}$  is the Christoffel matrix, which is a function of the material properties and the direction of wave propagation:

$$G_{ij} = C_{ijkl} n_j n_i. \quad (2.21)$$

Here,  $n_i$  is the direction vector of wave propagation. In a more compact way, the Christoffel equation can be written as:

$$[G_{ik} - \rho V^2 \delta_{ik}] U_k = 0 \quad (2.22)$$

Alternatively to the velocity of a wave the concept of slowness is frequently used in wave theory. However, the slowness is simply defined as the inverse of the velocity  $V$ . Thus, the slowness vector is given by:

$$p_i = \frac{n_i}{V} \quad (2.23)$$

The Christoffel equation describes a standard eigenvalue( $\rho V^2$ )-eigenvector( $U_i$ ) problem with the eigenvalues determined from

$$\det [G_{ij} - \rho V^2 \delta_{ij}] = 0. \quad (2.24)$$

For any specific direction  $n_i$  in an anisotropic media, the solution of cubic equation (2.24) yields three possible values of the squared velocity  $V$ , namely, one P-wave velocity and two S-wave velocities. Thus, in any anisotropic medium the shear wave is split into two modes with different velocities, except in some special directions where both S-wave velocities coincide. This leads to so-called shear wave singularities. In this context, isotropy can be understood as a special case of anisotropy where both S-wave velocities always coincide in all directions.

Since the Christoffel matrix is real and symmetric all polarization vectors of the three modes are mutually orthogonal. However, none of them is necessarily parallel or perpendicular to the phase direction. As a consequence, there are no pure longitudinal or transverse waves in anisotropic media, except for some special directions. Therefore, the velocities are usually denoted as „quasi-P“-wave and „quasi-S1“- and „quasi S2“-wave.

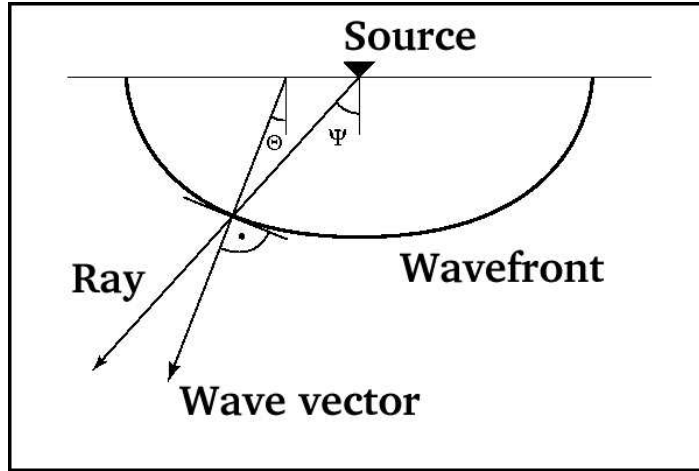


Figure 2.4: Graphical illustration of the difference between phase  $\Theta$  and group angle  $\Psi$  (after Thomsen (1986)).

For any particular phase the *phase velocity surface* can be constructed by plotting the phase velocity as the radius vector, for a given point source, in all directions of propagation. In the same way the corresponding *slowness surface* can be obtained. The shape of the slowness surface is directly related to the shape of the wavefront from point sources and to the presence from shear wave singularities. In homogeneous isotropic media, both surfaces as well as the wavefront are spherical.

However, the energy of a wave propagates with the *group* (or *ray*) velocity. This velocity describes rays used in seismic ray theory and is thus important for seismic traveltimes and inversion methods. Group and ray velocity may differ due to dispersion or anisotropy. In homogeneous isotropic media the ray and phase vector point into the same direction. In homogeneous but anisotropic media their directions can differ since the wavefront is no longer spherical.

Let the phase vector being denoted as  $k_i$ . It is locally always normal to the slowness wavefront. The phase velocity is also called the wavefront velocity, since it is the propagation velocity of the wavefront along the phase vector. In contrast, the ray vector points always from the source to the considered point on the wave front. The difference between phase and ray angle is illustrated in Fig. (2.4).

In contrast to the phase velocity the group velocity can not be obtained directly from the Christoffel equation. In its most general form the group velocity vector in arbitrary anisotropic media can be written as (e.g., Tsvankin, 2001)

$$\mathbf{V}_G = \text{grad}^{\mathbf{k}}(kV) = \frac{\partial(kV)}{\partial k_1} \mathbf{i}_1 + \frac{\partial(kV)}{\partial k_2} \mathbf{i}_2 + \frac{\partial(kV)}{\partial k_3} \mathbf{i}_3, \quad (2.25)$$

where  $k_i$  is the wave vector (see Fig. 2.4),  $V$  is the phase velocity and  $i_i$  is the unit coordinate vector. The magnitude of  $\mathbf{k}$  is  $k = \omega/V$  where  $\omega$  is the angular frequency. Using an auxiliary coordinate system  $[x,y,z]$  which is rotated by the azimuthal phase angle  $\phi$  around the  $x_3$  axes of the original  $[x_1, x_2, x_3]$  coordinate system, the components

## 2.2. Plane waves in isotropic and anisotropic media

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of the group velocity vector are given by (Thomsen, 1986):

$$V_{Gx} = V \sin \theta + \left. \frac{\partial V}{\partial \theta} \right|_{\phi=\text{const}} \cos \theta, \quad (2.26)$$

$$V_{Gz} = V \cos \theta - \left. \frac{\partial V}{\partial \theta} \right|_{\phi=\text{const}} \sin \theta, \quad (2.27)$$

$$V_{Gy} = \frac{1}{\sin \theta} \left. \frac{\partial V}{\partial \phi} \right|_{\theta=\text{const}}. \quad (2.28)$$

The y-axis in eq. (2.28) points in the direction of increasing azimuthal angle  $\phi$  counter-clockwise from the  $x_1$ -direction of the original coordinate system. The group velocity vector in vertical symmetry planes of any anisotropic media is completely determined by equations (2.26) and (2.27).

### 2.2.1 Transversely isotropic media

Phase velocity and polarization of waves in transversely isotropic media can be obtained from the Christoffel equation (2.22) and the stiffness tensor for TI media (eq. 2.16). Here, the stiffness tensor for VTI media is used since this is the most important case in seismics and rock physics. However, the Christoffel matrix for VTI media reads (Tsvankin, 2001):

$$G_{11} = C_{11}n_1^2 + C_{66}n_2^2 + C_{55}n_3^2 \quad (2.29)$$

$$G_{22} = C_{66}n_1^2 + C_{11}n_2^2 + C_{55}n_3^2 \quad (2.30)$$

$$G_{33} = C_{55}(n_1^2 + n_2^2) + C_{33}n_3^2 \quad (2.31)$$

$$G_{12} = (C_{11} + C_{66})n_1n_2 \quad (2.32)$$

$$G_{13} = (C_{13} + C_{55})n_1n_3 \quad (2.33)$$

$$G_{23} = (C_{13} + C_{55})n_2n_3 \quad (2.34)$$

Due to rotational symmetry in VTI media it is sufficient to analyze only one vertical plane containing the axis of symmetry. Taking, e.g., the  $[x_1, x_3]$  plane into account,  $n_2 = 0$  and inserting eq. (2.29) to (2.34) into the Christoffel equation (2.22) gives:

$$\begin{bmatrix} C_{11}n_1^2 + C_{55}n_3^2 - \rho V^2 & 0 & (C_{13} + C_{55})n_1n_3 \\ 0 & C_{66}n_1^2 + C_{55}n_3^2 - \rho V^2 & 0 \\ (C_{13} + C_{55})n_1n_3 & 0 & C_{55}n_1^2 + C_{33}n_3^2 - \rho V^2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = 0 \quad (2.35)$$

Equation (2.35) clearly states that the three wave modes split into a set of in-plane polarized waves, i.e., waves where particles move within the  $[x_1, x_3]$  plane ( $U_2 = 0$ ), and one pure transversely polarized wave. If we express the direction unit vector  $n_i$  in terms of the phase angle  $\Theta$  with the symmetry axis, i.e.,

$$n_1 = \sin \Theta \quad \text{and} \quad n_3 = \cos \Theta$$

and setting the determinant of the Christoffel equation to zero gives for the pure transversely polarized mode, the so-called SH-wave:

$$V_{\text{SH}} = \sqrt{\frac{C_{66} \sin^2 \Theta + C_{55} \cos^2 \Theta}{\rho}} \quad (2.36)$$

This shear wave is called SH-wave to emphasize its purely horizontal polarisation. For the in-plane waves we obtain in a similar manner:

$$2\rho V_P^2(\Theta) = (C_{11} + C_{55}) \sin^2 \Theta + (C_{33} + C_{55}) \cos^2 \Theta + M, \quad (2.37)$$

$$2\rho V_P^2(\Theta) = (C_{11} + C_{55}) \sin^2 \Theta + (C_{33} + C_{55}) \cos^2 \Theta - M \quad (2.38)$$

with

$$M = \sqrt{((C_{11} + C_{55}) \sin^2 - (C_{33} + C_{55}) \cos^2 \Theta)^2 + 4(C_{13} + C_{55})^2 \sin^2 \Theta \cos^2 \Theta}.$$

If we consider a wave propagating into the three direction, i.e.,  $\Theta = 0$ , we obtain for the three modes:

$$V_P(0) = V_{33} = \sqrt{\frac{C_{33}}{\rho}}, \quad (2.39)$$

$$V_{SV}(0) = V_{31} = \sqrt{\frac{C_{55}}{\rho}}, \quad (2.40)$$

$$V_{SH}(0) = V_{32} = \sqrt{\frac{C_{55}}{\rho}}, \quad (2.41)$$

where notation  $V_{ij}$  means propagation in the  $i$ -direction while polarization is in the  $j$ -direction. Obviously, both S-wave velocities coincide in the case of vertical propagation, creating a shear wave singularity in the direction of symmetry. Thus, the above mention separation of the Christoffel equation into P-SV and SH-waves is unique in this special case, since any combination of polarization directions  $U_1$  and  $U_2$  can form the corresponding eigenvector. Moreover, taking the rotational symmetry of VTI media into account, it is clear that the polarization of both shear waves can lie anywhere in the horizontal isotropy plane. Thus, the actual polarization of both S-waves depends on the source. Consequently, the notation  $V_{SV}(0) = V_{31}$  and  $V_{SH}(0) = V_{32}$  is only valid in the case of a properly oriented source.

Now, let us consider a plane wave propagating in the 1-direction, that means,  $\Theta = 90^\circ$ . We obtain:

$$V_P(90) = V_{11} = \sqrt{\frac{C_{11}}{\rho}}, \quad (2.42)$$

$$V_{SV}(90) = V_{13} = \sqrt{\frac{C_{55}}{\rho}}, \quad (2.43)$$

$$V_{SH}(90) = V_{12} = \sqrt{\frac{C_{66}}{\rho}}. \quad (2.44)$$

Due to the rotational symmetry, we can write immediately the corresponding velocities

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## 2.2. Plane waves in isotropic and anisotropic media

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for propagation in the 2-direction:

$$V_P(90) = V_{22} = \sqrt{\frac{C_{11}}{\rho}}, \quad (2.45)$$

$$V_{SV}(90) = V_{23} = \sqrt{\frac{C_{55}}{\rho}}, \quad (2.46)$$

$$V_{SH}(90) = V_{21} = \sqrt{\frac{C_{66}}{\rho}}. \quad (2.47)$$

We have formulated nine velocities which would be observed if velocities of a VTI sample would be measured in an orthogonal coordinate system and the axis of symmetry would be aligned parallel to the 3-axis of the measurement coordinate system:

$$\begin{pmatrix} V_{11} & V_{12} & V_{31} \\ V_{21} & V_{22} & V_{32} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{C_{11}}{\rho}} & \sqrt{\frac{C_{66}}{\rho}} & \sqrt{\frac{C_{55}}{\rho}} \\ \sqrt{\frac{C_{66}}{\rho}} & \sqrt{\frac{C_{11}}{\rho}} & \sqrt{\frac{C_{55}}{\rho}} \\ \sqrt{\frac{C_{55}}{\rho}} & \sqrt{\frac{C_{55}}{\rho}} & \sqrt{\frac{C_{33}}{\rho}} \end{pmatrix}$$

Obviously, such a measurement arrangement lacks a fifth independent velocity, necessary for a complete inversion of the stiffness tensor. Therefore, an additional P-wave velocity ( $V_{P45}$ ) is measured at an angle of  $45^\circ$  with the symmetry axis. In this case, the five independent entries of the stiffness tensor can be determined from the following equations (e.g., Lo *et al.*, 1986):

$$C_{11} = \rho V_{11}^2 \quad (2.48)$$

$$C_{12} = C_{11} - 2\rho V_{11}^2 \quad (2.49)$$

$$C_{33} = \rho V_{33}^2 \quad (2.50)$$

$$C_{55} = \rho V_{12}^2 \quad (2.51)$$

$$C_{13} = -C_{55} + (4\rho^2 V_{P45} - 2\rho V_{P45} (C_{11} + C_{33} + 2C_{55}) + (C_{11} + C_{55})(C_{33} + C_{55}))^{1/2} \quad (2.52)$$

As mentioned before phase and ray velocity may differ in anisotropic media. However, analytical relationships among them have already been derived in detail (e.g. Thomsen, 1986; Berryman, 1979). Here, the relation will be given for TI media.

Although the previously introduced equations allow for a calculation of elastic velocities in VTI media and the inversion of the stiffness tensor, their formalism does not provide an intuitive insight. It is almost impossible to get an assumption about the degree of, e.g., quasi P-wave anisotropy, from looking at the stiffness tensor and the constitutive equation.

To overcome this problem Thomsen (1986) introduced three anisotropy parameters  $\epsilon$ ,  $\delta$ , and  $\gamma$ , which represent combinations of certain elements of the stiffness tensor. The use of these *Thomsen parameters* has a couple of striking advantages:

- They are nondimensional, i.e., they allow for a statement like „anisotropy is X%“.

- In the special case where polar anisotropy reduces to isotropy, the particular combinations become zero.
- If these parameters are less than 0.1, the medium can be assumed to be „weakly anisotropic“.
- In the case of weak anisotropy exact equations for seismic velocities and group and polarization angles can be linearized in Thomsen’s parameters and thus remarkably simplified.

In combination with two velocities  $V_{P0}$  and  $V_{S0}$  transversal anisotropy can be described in terms of these parameters. Note, despite the frequent association of Thomsen’s parameters with weak anisotropic media they are basically derived for any degree of anisotropy. In fact, only the definition of  $\delta$  differs in the „strong“ and weak anisotropy case (see Appendix D for details). However, in the weak anisotropy approximation the Thomsen’s parameters are given as:

$$V_{P0} \equiv \sqrt{\frac{C_{33}}{\rho}}, \quad (2.53)$$

$$V_{S0} \equiv \sqrt{\frac{C_{44}}{\rho}}, \quad (2.54)$$

$$\epsilon \equiv \frac{C_{11} - C_{33}}{2C_{33}}, \quad (2.55)$$

$$\gamma \equiv \frac{C_{66} - C_{44}}{2C_{44}}, \quad (2.56)$$

$$\delta \equiv \frac{(C_{13} + C_{44})^2 - (C_{33} - C_{44})^2}{2C_{33}(C_{33} - C_{44})}. \quad (2.57)$$

$V_{P0}$  and  $V_{S0}$  are P- and S-wave velocity, respectively, in the direction of the symmetry axis, in VTI media the 3 or z-axis. Remember, both S-waves coincide in this special direction. The physical meaning of the weak anisotropy approximation is that of an isotropic background medium, represented by  $V_{P0}$  and  $V_{S0}$ , perturbed by small anisotropy represented in terms of  $\epsilon$ ,  $\gamma$ , and  $\delta$ . If a rock is weakly anisotropic the three phase velocities can be approximated by (Thomsen, 1986):

$$V_P(\Theta) = V_{P0} (1 + \delta \sin^2 \Theta \cos^2 \Theta + \epsilon \sin^4 \Theta) \quad (2.58)$$

$$V_{SV}(\Theta) = V_{S0} \left( 1 + \frac{V_{P0}^2}{V_{S0}^2} (\epsilon - \delta) \sin^2 \Theta \cos^2 \Theta \right) \quad (2.59)$$

$$V_{SH}(\Theta) = V_{S0} (1 + \gamma \sin^2 \Theta), \quad (2.60)$$

Considering a plane P-wave propagating in the 3-direction ( $\Theta = 0$ ), eq. (2.58) gives:

$$V_P(\Theta = 0) = V_{P0}. \quad (2.61)$$

Now considering a plane P-wave propagating in the 1-direction ( $\Theta = 90^\circ$ ). In this case, eq. (2.58) gives:

$$V_P(\Theta = 90) = V_{P0} (1 + \epsilon). \quad (2.62)$$

Thus, it is clear from eq. (2.61) and (2.62) that  $\epsilon$  is close to the fractional difference between horizontal and vertical P-wave velocity and is thus often simplistically denoted

## 2.2. Plane waves in isotropic and anisotropic media

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as the *P-wave anisotropy*. An equivalent measure for the SH-velocity is  $\gamma$ , as shown in eq. (2.56) and (2.60).

The physical meaning of  $\delta$  is not as obvious as in the case of  $\epsilon$  and  $\gamma$ . It determines the second derivative of the P-wave phase function at vertical incidence (Tsvankin, 2001):

$$\left. \frac{d^2 V_P}{d\Theta^2} \right|_{\Theta=0} = 2V_{P0}\delta. \quad (2.63)$$

The first derivative of the phase velocity at vertical incidence ( $\Theta = 0$ ) is equal to zero. Thus,  $\delta$  controls the dependence of  $V_P$  in the vicinity of the symmetry axis. Equation (2.63) shows that the P-wave velocity increases with increasing  $\Theta$  if  $\delta$  is positive and decreases if  $\delta$  is negative (see also eq. 2.58).

### 2.2.2 Orthorhombic media

In equivalence with TI media phase velocities and polarization in orthorhombic media can be derived from the Christoffel equation, which reads (Tsvankin, 2001):

$$G_{11} = C_{11}n_1^2 + C_{66}n_2^2 + C_{55}n_3^2, \quad (2.64)$$

$$G_{22} = C_{66}n_1^2 + C_{22}n_2^2 + C_{44}n_3^2, \quad (2.65)$$

$$G_{33} = C_{55}n_1^2 + C_{44}n_2^2 + C_{33}n_3^2, \quad (2.66)$$

$$G_{12} = (C_{12} + C_{66})n_1n_2, \quad (2.67)$$

$$G_{13} = (C_{13} + C_{55})n_1n_3, \quad (2.68)$$

$$G_{23} = (C_{23} + C_{44})n_2n_3, \quad (2.69)$$

where  $n_i$  is the unit vector in the phase (slowness) direction. Considering the  $[x_1, x_3]$ -plane, the Christoffel equation (2.22) in terms of the phase angle  $\Theta$  with the  $x_3$ -axes ( $n_1 = n_2 = \sin \Theta$ ,  $n_3 = \cos \Theta$ ) becomes:

$$\begin{bmatrix} C_{11}n_1^2\Theta + C_{55}n_3^2\Theta - \rho V^2 & 0 & (C_{13} + C_{55})\sin \Theta \cos \Theta \\ 0 & C_{66}n_1^2\Theta + C_{44}n_3\Theta^2 - \rho V^2 & 0 \\ (C_{13} + C_{55})n_1n_3 & 0 & C_{55}n_1^2\Theta + C_{33}n_3^2\Theta - \rho V^2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = 0 \quad (2.70)$$

Comparing eq. (2.70) with the corresponding equation for VTI media (eq. 2.35) shows that they are identical for the in-plane polarized wave modes and also for the pure transversely polarized mode if  $C_{44} = C_{55}$ . Thus, in-plane polarized waves in the  $[x_1, x_3]$ -plane of an orthorhombic medium depend on the same elastic stiffnesses as in a VTI medium. Moreover, as in VTI media, equation 2.70 splits into two independent equations for the in-plane polarized  $P$  and  $V_{SV}$  waves on the one hand side and the pure transverse motion  $V_{SH}$  on the other hand side.

Considering, for example, a plane wave propagating in the 3 direction, i.e.,  $\Theta = 0$ , gives for the P-wave ( $U_1 = U_2 = 0$ ), the SV-wave ( $U_3 = U_2 = 0$ ), and the SH-wave



( $U_1 = U_3 = 0$ ), respectively:

$$V_P = \sqrt{\frac{C_{33}}{\rho}} \quad (2.71)$$

$$V_{SV} = \sqrt{\frac{C_{55}}{\rho}} \quad (2.72)$$

$$V_{SH} = \sqrt{\frac{C_{44}}{\rho}}. \quad (2.73)$$

For a wave propagating in the same plane but in the 1 direction equation (2.70) gives:

$$V_P = \sqrt{\frac{C_{11}}{\rho}} \quad (2.74)$$

$$V_{SV} = \sqrt{\frac{C_{55}}{\rho}} \quad (2.75)$$

$$V_{SH} = \sqrt{\frac{C_{66}}{\rho}}. \quad (2.76)$$

The phase velocity functions for each wave mode in the  $[x_1, x_3]$ -plane are sufficient for the determination of group velocity and group angle and thus for all kinematic signatures. This means, as a consequence, that the kinematics of wave propagation in the  $[x_1, x_3]$ -plane can be described completely by well known VTI equations. In the same way, phase velocities in the remaining  $[x_1, x_2]$ - and  $[x_2, x_3]$ -plane can be described in a similar way by using the appropriate components of the stiffness tensor.

This equivalence between kinematic aspects of wave propagation in orthorhombic and VTI media is limited by some dynamic aspects and cuspidal S-wave group velocity-surfaces near shear-wave point singularities in the symmetry planes of orthorhombic media. Such cuspidal features cannot exist in VTI media (see, e.g., Tsvankin, 1997, 2001, for details).

However, it is reasonable to assume, that the equivalence between orthorhombic and VTI media can be used to simplify the constitutive relations for orthorhombic media and, thus, to obtain equations, applicable in practice. In fact, Tsvankin (1997) introduced Thomsen-style anisotropy parameters for weak anisotropic, providing the full advantage of the anisotropy parameters of TI media for orthorhombic media. In analogy to VTI media, two vertical velocities,  $V_{P0}$  and  $V_{S0}$ , are defined as the body wave velocities of an isotropic reference (or background) velocity model. Although both split shear waves at vertical incidence can be chosen, preference is usually given to the S-wave polarized in the  $x_1$ -direction. This ensures that the new notation for the in-plane polarized waves in the  $[x_1, x_3]$ -plane are identical to Thomsen's notation in the VTI case (Tsvankin, 1997). Thus, we obtain per definition:

$$V_{P0} \equiv \sqrt{\frac{C_{33}}{\rho}} \quad (2.77)$$

$$V_{S0} \equiv \sqrt{\frac{C_{55}}{\rho}} \quad (2.78)$$

## 2.2. Plane waves in isotropic and anisotropic media

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In the following, a superscript will be assigned to Tsvankin's parameters indicating the symmetry plane, for which the parameters are defined. The superscript corresponds to the normal direction of the plane under consideration, i.e, (2) corresponds to the  $[x_1, x_3]$ -plane. In analogy to VTI media, we can define:

$$\epsilon^{(2)} \equiv \frac{C_{11} - C_{33}}{2C_{33}} \quad (2.79)$$

$$\delta^{(2)} \equiv \frac{(C_{13} + C_{55})^2 - (C_{33} - C_{55})^2}{2C_{33}(C_{33} - C_{55})} \quad (2.80)$$

The analogy with VTI media can be used to derive the kinematic signatures and polarizations of the in-plane polarized waves in the  $[x_1, x_3]$ -plane by substituting the Thomsen parameters in the *exact* equation for VTI media with the above defined  $V_{P0}$ ,  $V_{S0}$ ,  $\epsilon^{(2)}$ , and  $\delta^{(2)}$ . Thus (Tsvankin, 1997),

$$V_P^2(\Theta) = V_{P0}^2 \left( 1 + \epsilon^{(2)} \sin^2 \Theta - \frac{f}{2} \pm \frac{f}{2} \sqrt{\left( 1 + \frac{2\epsilon^{(2)} \sin^2 \Theta}{f} \right)^2 - \frac{2(\epsilon^{(2)} - \delta^{(2)}) \sin^2 2\Theta}{f}} \right), \quad (2.81)$$

where

$$f \equiv 1 - \frac{V_{S0}^2}{V_{P0}^2}. \quad (2.82)$$

Equation (2.81) corresponds to the P-wave if the '+' sign is chosen in front of the radical, otherwise to the SV-wave.

Hence, this gives in the weak anisotropy limit (Tsvankin, 1997):

$$V_P(\Theta) = V_{P0} (1 + \delta^{(2)} \sin^2 \Theta \cos^2 \Theta + \epsilon^{(2)} \sin^4 \Theta) \quad (2.83)$$

In the same way,  $\gamma^{(2)}$  and all parameters for the remaining planes can be defined as:

$$\gamma^{(2)} \equiv \frac{C_{66} - C_{44}}{2C_{44}} \quad (2.84)$$

$$\epsilon^{(1)} \equiv \frac{C_{22} - C_{33}}{2C_{33}} \quad (2.85)$$

$$\delta^{(1)} \equiv \frac{(C_{23} + C_{44})^2 - (C_{33} - C_{44})^2}{2C_{33}(C_{33} - C_{44})} \quad (2.86)$$

$$\gamma^{(1)} \equiv \frac{C_{66} - C_{55}}{2C_{55}} \quad (2.87)$$

The phase velocity functions for the in-plane polarized waves in the  $[x_2, x_3]$ -plane can thus be obtained by substituting the appropriate parameters in eq. (2.81). Moreover, in eq. (2.82)  $V_{S0}$  should be replaced by  $V_{S1}$ , the vertical velocity of the in-plane polarized S-wave, given as

$$V_{S1} = \sqrt{\frac{C_{44}}{\rho}} \quad (2.88)$$

Comparing the yet introduced 6 Tsvankin's parameters plus the two vertical velocities shows that they can be used instead of the 8 stiffness coefficients  $C_{11}$ ,  $C_{22}$ ,  $C_{33}$ ,  $C_{44}$ ,  $C_{55}$ ,  $C_{66}$ ,  $C_{23}$ ,  $C_{13}$ , of the orthorhombic stiffness matrix. The remaining element  $C_{12}$  can be replaced by:

$$\delta^{(3)} \equiv \frac{(C_{12} + C_{66})^2 - (C_{11} - C_{66})^2}{2C_{11}(C_{11} - C_{66})}. \quad (2.89)$$

In summary, the Tsvankin's parameters introduced above are:

- $V_{P0}$ : the vertical P-wave velocity;
- $V_{S0}$ : the velocity of the vertically traveling S-wave polarized in the  $x_1$ -direction;
- $\epsilon^{(1)}$ : the VTI parameter  $\epsilon$  in the  $[x_2, x_3]$ -plane (approximately the fractional difference between the P-wave velocities in the  $x_2$  and  $x_2$  direction;)
- $\delta^{(1)}$ : the VTI parameter  $\delta$  in the  $[x_2, x_3]$ -plane, responsible for near-vertical in-plane P-wave variations. It also influences the SV-wave velocity.
- $\gamma^{(1)}$ : the VTI parameter  $\gamma$  in the  $[x_2, x_3]$ -plane (approximately the fractional difference between the SH-wave velocities in the  $x_2$  and  $x_2$  direction;)
- $\epsilon^{(2)}$ : the VTI parameter  $\epsilon$  in the  $[x_1, x_3]$ -plane;
- $\delta^{(2)}$ : the VTI parameter  $\delta$  in the  $[x_1, x_3]$ -plane;
- $\gamma^{(2)}$ : the VTI parameter  $\gamma$  in the  $[x_1, x_3]$ -plane;
- $\delta^{(3)}$ : the VTI parameter  $\delta$  in the  $[x_1, x_2]$ -plane, where the  $x_1$  axis is used as the symmetry axis.

Here, it is important to emphasize that the introduced Tsvankin notation can be used in orthorhombic media of arbitrary strength. Moreover, it is straightforward to use them for obtaining weak anisotropy approximations as in the case of Thomsen's notation for VTI media.

In reflection seismic experiments (i.e., near vertically propagating waves) aiming at anisotropic rocks shear wave splitting is a frequently used signature of the medium, describing the time delay between both observed S-waves  $V_{S0}$  and  $V_{S1}$ . This splitting is conveniently described by the fractional difference between  $C_{44}$  and  $C_{55}$ , assuming  $C_{44} > C_{55}$ :

$$\gamma^{(S)} \equiv \frac{C_{44} - C_{55}}{2C_{55}} = \frac{\gamma^{(1)} - \gamma^{(2)}}{1 + 2\gamma^{(2)}} \approx \frac{V_{S1} - V_{S0}}{V_{S0}} \quad (2.90)$$

$\gamma^{(S)}$  is identical to Thomsen's parameter  $\gamma$  in HTI media with  $x_1$  as the symmetry axis.

Characterizing the elastic properties of a material with the introduced elastic constants alone implies that it is either a single phase void free medium or the constants have to be understood "only" as bulk properties, averaged over the size of the sample in a laboratory test or over the wave length in seismic measurements. In seismology and especially in exploration seismics this way of understanding the elastic properties

of rocks, controlling the propagation of seismic waves through the earth, was sufficient when seismic data were primarily interpreted for subsurface structures and velocity models.

However, rocks are obviously always multiphase media, consisting of an assemblage of different minerals and voids of various size and shape, filled with one or more liquid and/or gaseous phase. When it is desired to extract information about, e.g., lithology, porosity, pore fluids, and fluid saturation from seismic data, a more sophisticated description of the seismic parameters of rocks is required.

## 2.3 Poroelasticity

All concepts of elastic media, isotropic as well as anisotropic, and of wave propagation through such media mentioned up to this point assume homogeneous linear elastic bodies. In other words, they are formulated as if the medium would consist of only one single material phase. However, there is obviously no real rock which corresponds to those assumptions. Therefore, these theoretical models can only approximate what is really happening when a wave travels through rocks.

If it is desired to interpret seismic velocities or bulk elastic parameters, e.g., the bulk and shear modulus of a rock, in terms of the elastic properties of the rock forming constituents and their geometrical relationships among each other, a detailed knowledge is required how the mentioned small scale properties influence the bulk properties of the material.

First considerations about the deformation of porous rocks and soils were done by Terzaghi (Terzaghi, 1936, 1943). He found theoretically, that there is an effective stress which controls the changes in bulk volume of a sample and influences its failure conditions. He defined the effective stress  $\sigma^e$  as

$$\sigma^e = \sigma^c + \phi \delta_{ij} P_{\text{fl}}, \quad (2.91)$$

where  $\sigma^c$  is the confining stress,  $P_{\text{fl}}$  is the pore pressure,  $\phi$  is the porosity, and  $\delta_{ij}$  is the Kronecker Delta function. In contrast, he found in experiments with sand, clay, and concrete that the simple difference between  $\sigma_c$  and  $P_{\text{fl}}$  is effective rather than the one theoretically derived, given in eq. (2.91).

The first comprehensive theory of the elasticity of porous media was derived by Maurice A. Biot and published in a series of papers (Biot, 1940, 1941, 1956a,b, 1962). He developed a detailed formalism for the pore and bulk compressibility and derived equations for the consolidation of porous materials as well as for seismic wave propagation in porous rocks. His formalism incorporates some, but not all mechanisms of viscous and inertial interaction between a pore fluid and the rock matrix. A general fact in the elasticity of porous media is that an incremental load, applied to a saturated rock, e.g., by a passing seismic wave, induces an incremental change in pore pressure, which resists the compression and thus stiffens the rock. To what extent the fluid stiffens the rock depends on the rock and fluid properties.

One fundamental result of the Biot theory is that there is a second compressional wave in porous media, the so called *slow wave*. The behavior of this wave depends

strongly on the frequency. Below a critical frequency, which depends on the permeability of the material and the viscosity of the fluid, the slow wave propagates diffusive-like and remarkably slower than the fast P- and S-wave and above the critical frequency as a usual seismic wave. The slow wave is generated when fluid and matrix move out of phase and the fast wave when they are in phase. Due to its very high attenuation it is usually not observable in seismic experiments.

Biot's low frequency limit velocities for the fast P- and S-wave are identically with the results derived by Gassmann (1951). The Gassmann equations allow for a prediction of seismic wave velocities of a porous rock saturated with one fluid from velocities of the rock saturated with another fluid. Gassmann's equations are limited to low frequencies, as usually used in seismic experiments, since it is based on the assumption that the induced incremental pore pressure change can be equilibrated throughout the pore space (Mavko *et al.* (1998)). This means that there is sufficient time for the pore fluid to flow to eliminated wave induced pore pressure gradients. Following Mavko *et al.* (1998) the Gassmann's equations are given as:

$$\frac{K_{\text{sat}}}{K_0 - K_{\text{sat}}} = \frac{K_{\text{dry}}}{K_0 - K_{\text{dry}}} + \frac{K_{\text{fl}}}{\phi(K_0 - K_{\text{fl}})} \quad (2.92)$$

$$\mu_{\text{sat}} = \mu_{\text{dry}}. \quad (2.93)$$

Here,  $K_{\text{sat}}$  is the bulk modulus of a rock with porosity  $\phi$ , and saturated with a fluid having a bulk modulus  $K_{\text{fl}}$ . The pores are assumed to be statistically isotropic but no limiting assumptions are made about the pore geometry.  $K_{\text{dry}}$  is the bulk modulus of the dry rock matrix and  $K_0$  the bulk modulus of the grain material. Obviously, the shear modulus  $\mu_{\text{sat}}$  of the saturated rock is equal to the shear modulus  $\mu_{\text{dry}}$  of the dry matrix, since a fluid has no resistance to shear and, hence, the S-wave propagates through the matrix material only.

The word "dry" in this relation does not mean that the sample is saturated with gas, e.g., air. It further refers to an incremental deformation of the rock matrix by an incremental load but the pore pressure is held constant. This corresponds to a "drained" experiment where pore fluid can flow freely in or out of the sample to insure a constant pore pressure. It also concerns to an "undrained" experiment in which the sample is saturated by a fluid with zero bulk modulus. This is approximately given for an air filled rock at room conditions. Moreover, Mavko *et al.* (1998) emphasize that "dry" does not correspond to very dry rocks, such as those, prepared in a vacuum oven. Adding a first few percent of fluid to such very dry rocks decreases the frame moduli, probably due to disrupting surface forces acting on the pore surfaces.

These fundamental concepts have been modified and extended by many researchers in a broad field of applications in the last decades. A complete revision of all these modification is not possible here. Thus, just some fundamental conceptual extensions are summarized. For instance, Gassmann's equations were extended towards anisotropic media in order to enable the calculation of the effective moduli of an anisotropic rock saturated with a fluid from the elastic moduli of the same but dry rock by Brown & Korringa (1975). Their formulations address the fluid substitution problem in anisotropic media. However, this approach assumes that the matrix is macroscopically homogenous and the rock is completely saturated.

Berryman & Milton (1991) limited their modification to the isotropic case but

### 2.3. Poroelasticity

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extended Gassmann's equation to rocks that are composites of two porous phases. Their approach can be used to calculate, e.g., the effective moduli of a sandstone with embedded shaly patches.

In two publications Dvorkin & Nur (1993) and Dvorkin *et al.* (1994) established a model that unified the Biot "global flow" mechanism and the "local flow" squirt flow mechanism. While the Biot model states a global fluid flow induced by a passing seismic wave between large discontinuities the squirt flow suggests that a passing seismic wave introduces pore pressure fluctuations on all scales of pore space heterogeneities, leading to a fluid flow between small cracks and larger pores. These effects might become important in ultrasonic laboratory experiments but are negligible in the in situ seismic frequency band. However, they must be considered if results from ultrasonic velocity and attenuation data from laboratory observations should be extrapolated to seismic field data.