CLOSED HYPERSURFACES DRIVEN BY THEIR MEAN CURVATURE AND INNER PRESSURE

Dissertation von

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Hiermit versichere ich, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Hilfsmittel verwendet habe.

Berlin, 15. April 2010

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Zusammenfassung

Die vorliegende Dissertation untersucht eine neue geometrische Flussgleichung, die die Bewegung geschlossener Hyperflächen in Riemannschen Mannigfaltigkeiten beschreibt. Ist die Hyperfläche sphärisch, kann diese Bewegungsgleichung als ein idealisiertes mathematisches Modell für die Bewegung einer Seifenblase betrachtet werden. Sie wird als Euler-Lagrange-Gleichung eines Wirkungsintegrals hergeleitet. Dieses enthält neben der kinetischen Energie auch Terme für die Oberflächenspannung und den Innendruck abhängig vom eingeschlossenen Volumen. Die resultierende Euler-Lagrange-Gleichung ist eine quasilineare entartete hyperbolische partielle Differentialgleichung zweiter Ordnung, die extrinsisch die Bewegung einer Fläche beschreibt.

Dieser Typ der Gleichung begründet das mathematische Interesse an der Untersuchung. Während die Einsteingleichungen zwar eine ähnliche Struktur haben, beschreiben diese jedoch die Evolution der Geometrie nur durch intrinsische Größen. Im Unterschied zu Wave-Maps ist die vorgestellte Gleichung nicht mehr semilinear, sondern quasilinear und entartet. Eine der wenigen mathematisch exakten Untersuchungen von Gleichungen dieses Typs ist die Arbeit von Smoczyk und LeFloch [**LS08**].

Einführend leiten wir die Gleichung her, um anschließend grundlegende Eigenschaften wie Energie- und Impulserhaltung zu untersuchen. Als spezielle Lösungen dieser Gleichung finden wir Sphären mit oszillierendem Radius sowie Sphären, die zusätzlich mit konstanter Geschwindigkeit translatieren.

Die Frage der Existenz einer Lösung für kurze Zeit wird in Kapitel 2 wie folgt beantwortet: Zu einer gegebenen glatten Immersion der Ausgangsmannigfaltigkeit und einer gegebenen glatten Anfangsgeschwindigkeit existiert für kurze Zeit eine glatte Lösung mit diesen Anfangsdaten. Der Beweis dieses Kurzzeitexistenzsatzes wird mit Hilfe des Satzes über inverse Funktionen von Nash und Moser geführt.

In Kapitel 3 beweisen wir ein Fortsetzungskriterium (Theorem 3.1), das eine hinreichende Bedingung angibt, unter der eine Lösung auf ein größeres Zeitintervall fortgesetzt werden kann. Die Bedingung ist, dass die Familie der Parametrisierungen der Flächen sowie deren Zeitableitung in der räumlichen C^4-Norm beschränkt sind. Anders ausgedrückt: Ist das maximale Existenzintervall endlich, werden diese C^4-Normem zum Ende des Intervalls unbeschränkt. Darüber hinaus beweisen wir, dass der Abstand zweier Lösungen nicht schneller als exponentiell wächst, wenn die Anfangsdaten dicht beieinander liegen (Theorem 3.7). Aus dieser Abschätzung folgt die Eindeutigkeit von Lösungen der Gleichung und eine untere Schranke an die maximale Existenzzeit. Wenn eine der beiden Lösungen für unendliche Zeit existiert, dann wächst die Existenzzeit der anderen Lösung mindestens wie der negative Logarithmus des Abstands der Anfangsdaten, wenn dieser nach Null geht. Eine analoges Wachstum der Existenzzeit erhalten wir auch, wenn sich die Metrik des umgebenden Raumes der euklidischen Metrik annähert (Theorem 3.10).

Contents

Preface

This thesis is devoted to the study of a new geometric flow equation, which describes the motion of closed hypersurfaces in Riemannian manifolds. If the surface is spherical, this equation can be considered as an idealised mathematical model of a moving soap bubble. It will be obtained as an Euler-Lagrange equation of a suitable action integral. In addition to the kinetic energy this action integral contains terms for the surface tension and the inner pressure, which depends on the enclosed volume. The resulting Euler-Lagrange equation is a quasilinear degenerate hyperbolic partial differential equation of second order, which describes the motion of the surface extrinsically.

The structure of this equation generates interest from a mathematical point of view. Although Einstein's equations have a similar structure they describe the evolution of the geometry via intrinsic quantities. In contrast to wave maps our equation is not semilinear, but rather quasilinear and degenerate. One of the few mathematically rigorous studies of equations in this category is the paper of Smoczyk and LeFloch [**LS08**].

In the introduction we derive the equation. We then study the basic properties of solutions of this equation, like energy and momentum conservation, and find special solutions of the equation such as oscillating and translating spheres.

In Chapter 2 we answer the question of short time existence in the following way: given a smooth immersion of a closed hypersurface and a smooth initial velocity, there exists a smooth solution for a short time attaining these initial data. The proof relies on the Nash-Moser inverse function theorem.

Finally in Chapter 3 we prove a continuation criterion (Theorem 3.1) which gives a sufficient condition under which the solution can be extended to a larger time interval. The condition is that the family of parametrisations of the surface and its time derivative are bounded in the spatial C^4-norm. To state it differently: at the end of the maximal time interval these C^4-norms become unbounded if the interval is finite. Furthermore in that chapter we prove that the distance between two solutions grows at most exponentially if they are close to each other initially (Theorem 3.7). This estimate implies the uniqueness of solutions and gives a lower bound on the maximal time of existence. If one of the two solutions exists for all future times then the maximal time of existence of the other solution goes to infinity at least as fast as the negative logarithm of the initial distance between the solutions if this initial distance goes to zero. A similar stability estimate holds if the metric of the ambient manifold is close to the Euclidean metric (Theorem 3.10).

CHAPTER 1

Introduction

This chapter is structured as follows. In Section 1.1 we set up the notation for closed hypersurfaces moving in an ambient manifold and we state some formulas that we will use frequently. In Section 1.2 we define the action integral and derive the equation [**EQ**] that we will study in this thesis. Section 1.3 contains the derivation of conservation laws and its impliciations. Special solutions of our equation are given in Section 1.4.

1.1. Notation and Preliminaries

Let $\mathcal{N}$ be a smooth closed oriented manifold of dimension n. By $\mathcal{S}^n$ we will denote the n-dimensional sphere. Let $(\mathcal{M}^{n+1}, \overline{g})$ be a smooth complete oriented $n+1$-dimensional Riemannian manifold. We will mostly write $\langle \cdot, \cdot \rangle$ for $\overline{g}(\cdot, \cdot)$ and $\mathcal{M}$ for $(\mathcal{M}^{n+1}, \overline{g})$.

We represent the evolving surfaces by a smooth family of immersions $u : [0, T) \times \mathcal{N} \to \mathcal{M}$ and for the surface at time t we write $\Sigma_t = u(t, \mathcal{N})$. The induced metric on $\mathcal{N}$ at time t is $g(t) = u(t)^* \overline{g}$. Now suppose we have local coordinates x^i on $\mathcal{N}$ and y^α on $\mathcal{M}$. Here latin indices run from 1 to n and greek indices run from 0 to n. Then the canonical tangent vectors associated to x^i and y^α are $\partial_1, \dots, \partial_n$ and $\bar\partial_0, \dots, \bar\partial_n$ respectively. The Levi-Civita connection of g is denoted by $\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k$ and that of $\overline{g}$ by $\overline{\nabla}_{\bar\partial_\alpha} \bar\partial_\beta = \overline{\Gamma}^\gamma_{\alpha\beta} \bar\partial_\gamma$. In local coordinates the inverse of the metric g_{ij} is denoted g^{ij}, i. e. $g^{ij} g_{jk} = \delta^i_k$. We use the summation convention, i. e. we sum over repeated indices if they have a position of different height and the sum goes over the whole range of values the index can take.

The map u induces a Riemannian vector bundle $u^*T\mathcal{M}$ over $[0, T) \times \mathcal{N}$ whose fibre at (t, x) is $T_{u(t,x)}\mathcal{M}$ with metric $\overline{g}_{u(t,x)}$. The Levi-Civita connection $\overline{\nabla}$ of $\overline{g}$ gives rise to a connection $u^*\overline{\nabla}$ on $u^*T\mathcal{M}$. This is the unique connection with the property that for $X \in \Gamma(T([0, T) \times \mathcal{N}))$ and $Z \in \Gamma(T\mathcal{M})$ we have

$$(u^*\overline{\nabla})_X (Z \circ u) = \overline{\nabla}_{u_* X} Z.$$

By $\Gamma(\mathcal{V})$ we denote the space of smooth sections in a vector bundle $\mathcal{V}$. Mostly we denote $u^*\overline{\nabla}$ again by $\overline{\nabla}$ as long as it is not necessary to distinguish between these two. The connection $u^*\overline{\nabla}$ is called the covariant derivative along the map u (cf. [**Fer08**, Satz 21, Satz 24] and also [**Jos08**, Ch. 4.1], [**Spi79**, Ch. 6]). It is metric compatible and torsion free in the sense that for $X, Y \in \Gamma(T([0, T) \times \mathcal{N}))$ and $Z_1, Z_2 \in \Gamma(u^*T\mathcal{M})$ we have

$$X(\overline{g}_u(Z_1, Z_2)) = \overline{g}_u((u^*\overline{\nabla})_X Z_1, Z_2) + \overline{g}_u(Z_1, (u^*\overline{\nabla})_X Z_2)$$

and

$$(u^*\overline{\nabla})_X u_* Y - (u^*\overline{\nabla})_Y u_* X = u_*[X,Y]. \qquad [\mathbf{1.1}]$$

In local coordinates we have e. g.

$$\begin{aligned}\overline{\nabla}_{\partial_t}\partial_t u^\alpha =&(u^*\overline{\nabla})_{\partial_t}\partial_t u^\alpha = \partial_t^2 u^\alpha + \overline{\Gamma}^\alpha_{\beta\gamma}(u)\partial_t u^\beta \partial_t u^\gamma \\ \overline{\nabla}_{\partial_i}\partial_t u^\alpha =&(u^*\overline{\nabla})_{\partial_i}\partial_t u^\alpha = \partial_i\partial_t u^\alpha + \overline{\Gamma}^\alpha_{\beta\gamma}(u)\partial_i u^\beta \partial_t u^\gamma.\end{aligned}$$

Relation [**1.1**] implies for example

$$\overline{\nabla}_{\partial_i}\partial_t u = (u^*\overline{\nabla})_{\partial_i}\partial_t u = (u^*\overline{\nabla})_{\partial_t}\partial_i u = \overline{\nabla}_{\partial_t}\partial_i u$$

since $[\partial_i, \partial_t] = 0$. The same considerations hold for variations $u : (-\varepsilon_0, \varepsilon_0)\times[0,T]\times\mathcal{N} \to \mathcal{M}$. We have

$$\overline{\nabla}_{\partial_\varepsilon}\partial_t u = \overline{\nabla}_{\partial_t}\partial_\varepsilon u$$

where in local coordinates

$$\overline{\nabla}_{\partial_\varepsilon}\partial_t u^\alpha = \partial_\varepsilon\partial_t u^\alpha + \overline{\Gamma}^\alpha_{\beta\gamma}(u)\partial_\varepsilon u^\beta \partial_t u^\gamma.$$

Here ∂_ε is the derivative with respect to the first coordinate. Another important identity that can be checked for $X, Y \in \Gamma(T([0,T)\times\mathcal{N}))$, $Z \in \Gamma(u^*T\mathcal{M})$ by direct calculation is

$$\overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z = \overline{\mathrm{R}}(u_*X, u_*Y)Z.$$

Here $\overline{\mathrm{R}}$ is the Riemann tensor on $\mathcal{M}$.

Let ν denote the outer unit normal to Σ_t. The second fundamental form is given by

$$-h_{ij}\nu^\alpha = \partial_i\partial_j u^\alpha - \Gamma^k_{ij}\partial_k u^\alpha + \overline{\Gamma}^\alpha_{\beta\gamma}\partial_i u^\beta \partial_j u^\gamma$$

and the mean curvature vector is

$$-H\nu^\alpha = -g^{ij}h_{ij}\nu^\alpha = \Delta u = g^{ij}\left(\partial_i\partial_j u^\alpha - \Gamma^k_{ij}\partial_k u^\alpha + \overline{\Gamma}^\alpha_{\beta\gamma}\partial_i u^\beta \partial_j u^\gamma\right).$$

The norm of the second fundamental form is $|h|^2 = h^{ij}h_{ij}$. For a section $f_i^\alpha dx^i \otimes \bar{\partial}_\alpha$ of the bundle $T^*\mathcal{N}\otimes u^*T\mathcal{M}$ we define the connection

$$\nabla_{\partial_j} f_i^\alpha = \partial_j f_i^\alpha - \Gamma^k_{ij} f_k^\alpha + \overline{\Gamma}^\alpha_{\beta\gamma}\partial_j u^\beta f_i^\gamma.$$

With this definition we have that $\nabla_{\partial_i}\partial_j u = -h_{ij}\nu$ and

$$\overline{\nabla}_{\partial_i}\partial_j u = -h_{ij}\nu + \Gamma^k_{ij}\partial_k u. \qquad [\mathbf{1.2}]$$

Of course this last expression is not a tensor.

The induced surface measure of $g(t)$ is denoted by $d\mu_t$. We will fix a reference surface measure $d\mu_0$ on $\mathcal{N}$ with a smooth density function. For notational convenience we assume that $d\mu_0$ is the surface measure of a fixed Riemannian metric g_0 on $\mathcal{N}$. We note that only the measure $d\mu_0$ will play a role in the evolution equation. The ratio of $d\mu_t$ and $d\mu_0$ is a scalar function which in local coordinates can be computed as

$$\frac{d\mu_t}{d\mu_0} = \frac{\sqrt{\det(g_{ij})}}{\sqrt{\det(g_{0ij})}}.$$

For any smooth function f on $\mathcal{N}$ we have

$$\int_{\mathcal{N}} f d\mu_t = \int_{\mathcal{N}} f \frac{d\mu_t}{d\mu_0} d\mu_0.$$

By $\mathring{\Gamma}^k_{ij}$ we will denote the Christoffel symbols of the metric g_0.

The enclosed volume of $\Sigma_t = u(t, \mathcal{N})$ will be denoted by $\mathrm{Vol}(u)$. If u_ε is a variation of u with $u_0 = u$ then we have from [**BdCE88**, 2.1 Lemma] that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathrm{Vol}(u_\varepsilon) = \int_{\mathcal{N}} \langle \frac{\partial u}{\partial \varepsilon}\Big|_{\varepsilon=0}, \nu \rangle d\mu_t \qquad [\mathbf{1.3}]$$

and so

$$\partial_t \mathrm{Vol}(u) = \int_{\mathcal{N}} \langle \nu, \partial_t u \rangle d\mu_t.$$

Setting $\mathrm{Vol}_0 = \mathrm{Vol}(u(0))$ we can write

$$\mathrm{Vol}(u) = \mathrm{Vol}_0 + \int_0^t \int_{\mathcal{N}} \langle \partial_t u, \nu \rangle d\mu_t dt'.$$

We can take this expression as a definition if it is not clear how to define the volume. We will of course always assume that it is possible to define the enclosed volume for the initial surface.

We denote by C any universal constant appearing in estimates. This constant may depend on fixed quantities such as the dimension of the manifold and derivatives of coordinate changes of a fixed atlas. The dependence on other quantities will be stated explicitly. If we want to point out that C depends on some other constant, say K, then we use a subscript C_K.

We note the well known variation formulas for g_{ij} and $d\mu_t$ when tangential variations are also allowed.

LEMMA 1.1. *Let u_ε be a smooth variation of an immersion $u : \mathcal{N} \to \mathcal{M}$ with $u_0 = u$ and $\frac{\partial u_\varepsilon}{\partial \varepsilon}\big|_{\varepsilon=0} = X$. Then*

$$\frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} g_{ij}(u_\varepsilon) = \nabla_{\partial_i} \langle X, \partial_j u \rangle + \nabla_{\partial_j} \langle X, \partial_i u \rangle + 2\langle X, \nu \rangle h_{ij} = \mathscr{L}_{X^\top} g_{ij} + 2\langle X, \nu \rangle h_{ij}.$$

and

$$\frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} d\mu(u_\varepsilon) = (\langle X, \nu \rangle H + \mathrm{div}\, X^\top) d\mu \qquad [\mathbf{1.4}]$$

where $X^\top = \langle X, \partial_i u \rangle g^{ij} \partial_j u$ is the tangential part of X and $\mathscr{L}$ is the Lie derivative.

Proof. We write $\partial_\varepsilon = \frac{\partial}{\partial\varepsilon}$ and omit the index ε from u_ε. Start with

$$\begin{aligned}\partial_\varepsilon g_{ij} =&\langle\overline{\nabla}_{\partial_\varepsilon}\partial_i u, \partial_j u\rangle + \langle\partial_i u, \overline{\nabla}_{\partial_\varepsilon}\partial_j u\rangle = \langle\overline{\nabla}_{\partial_i}\partial_\varepsilon u, \partial_j u\rangle + \langle\partial_i u, \overline{\nabla}_{\partial_j}\partial_\varepsilon u\rangle\\ =&\partial_i\langle\partial_\varepsilon u, \partial_j u\rangle - \langle\partial_\varepsilon u, \overline{\nabla}_{\partial_i}\partial_j u\rangle + \partial_j\langle\partial_\varepsilon u, \partial_i u\rangle - \langle\partial_\varepsilon u, \overline{\nabla}_{\partial_j}\partial_i u\rangle\\ =&\partial_i\langle\partial_\varepsilon u, \partial_j u\rangle - \Gamma^k_{ij}\langle\partial_\varepsilon u, \partial_k u\rangle + h_{ij}\langle\partial_\varepsilon u, \nu\rangle\\ &+ \partial_j\langle\partial_\varepsilon u, \partial_i u\rangle - \Gamma^k_{ij}\langle\partial_\varepsilon u, \partial_k u\rangle + h_{ij}\langle\partial_\varepsilon u, \nu\rangle.\end{aligned}$$

We used that $\overline{\nabla}_{\partial_i}\partial_j u = -h_{ij}\nu + \Gamma^k_{ij}\partial_k u$. Setting $\varepsilon = 0$ we have

$$\partial_\varepsilon|_{\varepsilon=0} g_{ij} = \nabla_{\partial_i}\langle X, \partial_j u\rangle + \nabla_{\partial_j}\langle X, \partial_i u\rangle + 2\langle X, \nu\rangle h_{ij}.$$

Hence

$$\partial_\varepsilon|_{\varepsilon=0} d\mu = \frac{1}{2} g^{ij}\partial_\varepsilon|_{\varepsilon=0} g_{ij} d\mu = (\langle X, \nu\rangle H + \operatorname{div} X^\top) d\mu. \qquad \square$$

Definition 1.2. Let $u : [0, T) \times \mathcal{N} \to \mathcal{M}$ be a smooth family of immersions. Define

$$\begin{aligned}\sigma =&\langle\partial_t u, \nu\rangle, & S_i =&\langle\partial_t u, \partial_i u\rangle,\\ \alpha =&\langle\overline{\nabla}_{\partial_t}\partial_t u, \nu\rangle, & A_i =&\langle\overline{\nabla}_{\partial_t}\partial_t u, \partial_i u\rangle.\end{aligned}$$

Hence we have

$$\partial_t u = \sigma\nu + S^i\partial_i u \quad \text{and} \quad \overline{\nabla}_{\partial_t}\partial_t u = \alpha\nu + A^i\partial_i u.$$

1.2. The Equation

For a smooth family of immersions $u : [0, T] \times \mathcal{N} \to \mathcal{M}$ we want to define an action integral of the form

$$\mathcal{A}(u) = \int_0^T \mathcal{K}(u) - \mathcal{I}(u) - \mathcal{J}(u) dt$$

where $\mathcal{K}$ is the kinetic energy and $\mathcal{I}$, $\mathcal{J}$ contribute to the potential energy. As the driving forces should be surface tension and inner pressure we define $\mathcal{I}$ to be the energy of the surface tension, i. e. the surface area

$$\mathcal{I}(u) = \int_{\mathcal{N}} d\mu_t.$$

The inner pressure is motivated by that of an ideal gas with constant temperature, i. e. proportional to $\mathrm{Vol}(u)^{-1}$. Therefore we define for a parameter $\varrho > 0$

$$\mathcal{J}(u) = -\varrho \log\left(\frac{\mathrm{Vol}(u)}{\mathrm{Vol}_0}\right).$$

The initial volume Vol_0 does not contribute to the Euler-Lagrange equation but we include it to make the expression in the logarithm scale free. The constant $\varrho > 0$ determines the strength of the influence of the inner pressure compared to the surface tension. Another reason to include this constant is to compensate for the different scaling of the energies. Of course other functions of the enclosed volume could be considered if they lead to a lower volume bound as in Corollary 1.6 below.

We define the kinetic energy as

$$\mathcal{K}(u) = \int_{\mathcal{N}} \frac{1}{2}|\partial_t u|^2 d\mu_0$$

which can be thought of as being the "sum" of the kinetic energies $\frac{1}{2}|\partial_t u|^2 d\mu_0$ of each point of the surface. This then describes the physical energy of the point particles making up the surface. In Appendix D we discuss another choice of kinetic energy. We could also introduce another constant in front of the kinetic energy but this can be set to one by scaling in the time variable or by including it in $d\mu_0$.

Altogether the action integral is

$$\mathcal{A}(u) = \int_0^T \int_{\mathcal{N}} \frac{1}{2}|\partial_t u|^2 d\mu_0 dt - \int_0^T \int_{\mathcal{N}} d\mu_t dt + \varrho \int_0^T \log\left(\frac{\mathrm{Vol}(u)}{\mathrm{Vol}_0}\right) dt.$$

PROPOSITION 1.3. *Let u_ε be a variation of u with $u_0 = u$ and $\frac{\partial u_\varepsilon}{\partial \varepsilon}\big|_{\varepsilon=0} = X$. Then*

$$\begin{aligned} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{A}(u_\varepsilon) = &\int_{\mathcal{N}} \langle X, \partial_t u\rangle d\mu_0 \Big|_{t=T} - \int_{\mathcal{N}} \langle X, \partial_t u\rangle d\mu_0 \Big|_{t=0} \\ &- \int_0^T \int_{\mathcal{N}} \langle X, \overline{\nabla}_{\partial_t} \partial_t u\rangle d\mu_0 dt - \int_0^T \int_{\mathcal{N}} H\langle \nu, X\rangle - \frac{\varrho}{\mathrm{Vol}(u)}\langle \nu, X\rangle d\mu_t dt. \end{aligned} \qquad [\mathbf{1.5}]$$

PROOF. We first compute

$$\begin{aligned} \frac{d}{d\varepsilon} \int_0^T \int_{\mathcal{N}} \frac{1}{2}|\partial_t u_\varepsilon|^2 d\mu_0 dt &= \int_0^T \int_{\mathcal{N}} \langle \overline{\nabla}_{\partial_\varepsilon} \partial_t u_\varepsilon, \partial_t u_\varepsilon\rangle d\mu_0 dt = \int_0^T \int_{\mathcal{N}} \langle \overline{\nabla}_{\partial_t} \frac{\partial u_\varepsilon}{\partial \varepsilon}, \partial_t u_\varepsilon\rangle d\mu_0 dt \\ &= \int_0^T \int_{\mathcal{N}} \partial_t \langle \frac{\partial u_\varepsilon}{\partial \varepsilon}, \partial_t u_\varepsilon\rangle - \langle \frac{\partial u_\varepsilon}{\partial \varepsilon}, \overline{\nabla}_{\partial_t} \partial_t u_\varepsilon\rangle d\mu_0 dt. \end{aligned}$$

From the variation of $d\mu_t$ [**1.4**] and the divergence theorem

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_0^T \int_{\mathcal{N}} d\mu_t(u_\varepsilon) dt = \int_0^T \int_{\mathcal{N}} H\langle \nu, X\rangle d\mu_t dt$$

and by [**1.3**] we have that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_0^T \log(\mathrm{Vol}(u_\varepsilon)) dt = \int_0^T \frac{1}{\mathrm{Vol}(u)} \int_{\mathcal{N}} \langle \nu, X\rangle d\mu_t dt.$$

Adding these together and setting $\varepsilon = 0$ where necessary yields the statement. □

If the variation vector field X vanishes at the boundary i.e. $X(0) = 0$ and $X(T) = 0$ we obtain the Euler-Lagrange equation.

COROLLARY 1.4. *The Euler-Lagrange equation of $\mathcal{A}$ is*

$$\overline{\nabla}_{\partial_t} \partial_t u = \frac{d\mu_t}{d\mu_0}\left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)\nu. \qquad [\mathbf{EQ}]$$

1.3. Conservation Laws

1.3.1. Energy Conservation. Define the energy

$$\mathcal{E}(u(t,\cdot)) = \int_{\mathcal{N}} \frac{1}{2}|\partial_t u|^2 d\mu_0 + \int_{\mathcal{N}} d\mu_t - \varrho \log\left(\frac{\mathrm{Vol}(u)}{\mathrm{Vol}_0}\right).$$

Let $u : [0,T) \times \mathcal{N} \to \mathcal{M}$ solve [**EQ**] with $\mathcal{E}_0 = \mathcal{E}(u(0,\cdot))$.

Proposition 1.5. *We have $\mathcal{E}(u(t,\cdot)) = \mathcal{E}_0$ for all $t \in [0,T)$.*

Proof. Simply compute

$$\frac{d}{dt}\mathcal{E}(u(t,\cdot)) = \int_{\mathcal{N}} \langle \overline{\nabla}_{\partial_t}\partial_t u, \partial_t u\rangle d\mu_0 + \int_{\mathcal{N}} H\langle \nu, \partial_t u\rangle d\mu_t - \frac{\varrho}{\mathrm{Vol}(u)}\int_{\mathcal{N}} \langle \nu, \partial_t u\rangle d\mu_t = 0$$

if u solves the equation. □

Corollary 1.6. (1) *The enclosed volume is bounded from below by*

$$\mathrm{Vol}(u) \geq \mathrm{Vol}_0\, e^{-\frac{\mathcal{E}_0}{\varrho}}.$$

(2) *Assume that an isoperimetric inequality holds on $\mathcal{M}$, namely that there is a constant $c_{\mathrm{iso}} > 0$ such that*

$$\int_{\mathcal{N}} d\mu_t \geq c_{\mathrm{iso}}\, \mathrm{Vol}(u)^{\frac{n}{n+1}}. \qquad [\mathbf{1.6}]$$

Then there is a constant K depending only on c_{iso}, ϱ, $\mathcal{E}_0$ and Vol_0 *such that* $\mathrm{Vol}(u) \leq K$ *and consequently*

$$\int_{\mathcal{N}} \frac{1}{2}|\partial_t u|^2 d\mu_0 + \int_{\mathcal{N}} d\mu_t \leq \mathcal{E}_0 + \varrho \log\left(\frac{K}{\mathrm{Vol}_0}\right).$$

Proof. The lower volume bound is immediate from the energy conservation and the nonnegativity of $\int_{\mathcal{N}} \frac{1}{2}|\partial_t u|^2 d\mu_0 + \int_{\mathcal{N}} d\mu_t$.

Using the energy conservation and the isoperimetric inequality [**1.6**] it follows that

$$\begin{aligned} 0 \leq & \mathcal{E}_0 + \varrho \log(\mathrm{Vol}(u)) - \int_{\mathcal{N}} d\mu_t - \varrho \log(\mathrm{Vol}_0) \\ \leq & \mathcal{E}_0 + \varrho \log(\mathrm{Vol}(u)) - c_{\mathrm{iso}}\, (\mathrm{Vol}(u))^{\frac{n}{n+1}} - \varrho \log(\mathrm{Vol}_0). \end{aligned}$$

Since the function $f(x) = \varrho \log x - c_{\mathrm{iso}} x^{\frac{n}{n+1}} + \mathcal{E}_0 - \varrho \log(\mathrm{Vol}_0)$ becomes negative for large x there must exist a number $K > 0$ such that $x \leq K$ if $f(x) \geq 0$. This means $\mathrm{Vol}(u) \leq K$. □

REMARK 1.7. Another way to formulate the conservation of energy is by a sort of continuity equation. Using the variation of $d\mu_t$ [**1.4**] with $X = \partial_t u$ we compute

$$\begin{aligned}\partial_t\left(\frac{1}{2}|\partial_t u|^2 + \frac{d\mu_t}{d\mu_0}\right) =&\langle \partial_t u, \overline{\nabla}_{\partial_t}\partial_t u\rangle + \operatorname{div} S\frac{d\mu_t}{d\mu_0} + \langle \partial_t u, \nu\rangle H\frac{d\mu_t}{d\mu_0}\\ =&-\langle \partial_t u, H\nu\rangle\frac{d\mu_t}{d\mu_0} + \frac{\varrho}{\operatorname{Vol}(u)}\langle \partial_t u, \nu\rangle\frac{d\mu_t}{d\mu_0} + \operatorname{div} S\frac{d\mu_t}{d\mu_0} + \langle \partial_t u, \nu\rangle H\frac{d\mu_t}{d\mu_0}\\ =&\operatorname{div} S\frac{d\mu_t}{d\mu_0} + \frac{\varrho}{\operatorname{Vol}(u)}\langle \partial_t u, \nu\rangle\frac{d\mu_t}{d\mu_0}.\end{aligned}$$

Integrating this with respect to $d\mu_0$ and dt gives the energy conservation as above.

1.3.2. Momentum Conservation. Let X be a Killing vector field on $\mathcal{M}$ and φ_s its local flow, which by definition is an isometry. Define the momentum with respect to X of a solution u of [**EQ**] by

$$\mathcal{P}_X(u(t,\cdot)) = \int_{\mathcal{N}} \langle \partial_t u, X(u)\rangle d\mu_0.$$

PROPOSITION 1.8. *Let $u : [0,T)\times\mathcal{N}\to\mathcal{M}$ solve* [**EQ**]. *Then $\mathcal{P}_X(u(t,\cdot))$ is constant as a function of t.*

PROOF. Let $u_s = \varphi_s\circ u$. Then $\frac{\partial u_s}{\partial s}\big|_{s=0} = X(u)$. As φ_s is an isometry we have

$$\frac{d}{ds}\bigg|_{s=0}\mathcal{A}(u_s) = 0.$$

From the variation formula [**1.5**] we get that

$$0 = \mathcal{P}_X(u(t,\cdot)) - \mathcal{P}_X(u(0,\cdot)) + 0$$

as u solves the Euler-Lagrange equation. □

REMARK 1.9. We can also formulate the momentum conservation as a continuity equation. Again let $u_s = \varphi_s\circ u$. We have

$$\begin{aligned}\partial_t\langle \partial_t u, X(u)\rangle =&\langle \overline{\nabla}_{\partial_t}\partial_t u, X(u)\rangle + \underbrace{\langle \partial_t u, \overline{\nabla}_{\partial_t} X(u)\rangle}_{=0}\\ =&-\langle H\nu, X(u)\rangle\frac{d\mu_t}{d\mu_0} + \frac{\varrho}{\operatorname{Vol}(u)}\langle \nu, X\rangle\frac{d\mu_t}{d\mu_0}\\ =&\operatorname{div} X^\top\frac{d\mu_t}{d\mu_0} - \frac{\partial}{\partial s}\bigg|_{s=0}\log(d\mu_t(u_s))\frac{d\mu_t}{d\mu_0} + \frac{\varrho}{\operatorname{Vol}(u)}\langle \nu, X\rangle\frac{d\mu_t}{d\mu_0}\\ =&\operatorname{div} X^\top\frac{d\mu_t}{d\mu_0} + \frac{\varrho}{\operatorname{Vol}(u)}\langle \nu, X\rangle\frac{d\mu_t}{d\mu_0}\end{aligned}$$

since $\frac{\partial}{\partial s}\big|_{s=0} d\mu_t(u_s) = 0$ as X is Killing. Integrating with respect to $d\mu_0$ and dt and using that $\int_{\mathcal{N}}\langle \nu, X\rangle d\mu_t = \frac{d}{ds}\Big|_{s=0}\operatorname{Vol}(u_s) = 0$ we get the same momentum conservation as above.

We can obtain a third conservation law by exploiting another symmetry of the action, namely the invariance under diffeomorphisms of $\mathcal{N}$ that leave $d\mu_0$ invariant. So let Y be a vector field on $\mathcal{N}$ with $\operatorname{div}_{d\mu_0} Y = \operatorname{div}_{g_0} Y = 0$. We define the interior momentum with respect to Y as

$$\mathcal{Q}_Y(u(t,\cdot)) = \int_{\mathcal{N}} \langle \partial_t u, u_* Y\rangle d\mu_0.$$

PROPOSITION 1.10. *Let $u : [0,T) \times \mathcal{N} \to \mathcal{M}$ solve* [**EQ**]. *Then $\mathcal{Q}_Y(u(t,\cdot))$ is constant as a function of t. Furthermore we have*

$$\partial_t \langle \partial_t u, u_* Y\rangle = \frac{1}{2} \operatorname{div}_{d\mu_0}(|\partial_t u|^2 Y).$$

PROOF. In local coordinates on $\mathcal{N}$ write $Y = Y^i \partial_i$. We have that $u_* Y = Y^i \partial_i u$ and compute

$$\begin{aligned} \partial_t \langle \partial_t u, u_* Y\rangle =& \underbrace{\langle \overline{\nabla}_{\partial_t} \partial_t u, u_* Y\rangle}_{=0} + \langle \partial_t u, \overline{\nabla}_{\partial_t} \partial_i u\rangle Y^i. \\ =& Y^i \frac{1}{2} \partial_i |\partial_t u|^2 = \frac{1}{2} \operatorname{div}_{d\mu_0}(|\partial_t u|^2 Y) - \frac{1}{2} |\partial_t u|^2 \operatorname{div}_{d\mu_0} Y. \end{aligned}$$

Integrating with respect to $d\mu_0$ and dt using the divergence theorem and that $\operatorname{div}_{d\mu_0} Y = 0$ yields the result. □

1.4. Special Solutions

1.4.1. Rotationally Symmetric Solutions. Assume $u : \mathbb{R} \times \mathbb{S}^n \to \mathbb{R}^{n+1}$ has the form $u(t,x) = r(t)x$ with initial conditions $r(0) = r_0 > 0$ and $\dot r(0) = r_1$. Let $d\mu_0$ be the surface measure of a spherical metric g_0, i. e. $g_0 = \gamma_0^2 g_{\mathbb{S}^n}$ where $g_{\mathbb{S}^n}$ is the standard metric on $\mathbb{S}^n$ and $\gamma_0 > 0$ is a constant. Of course the mean curvature of $u(t, \mathbb{S}^n)$ is given by $H(t) = n/r(t)$. Furthermore $\operatorname{Vol}(u) = \omega_{n+1} r(t)^{n+1}$ and $\frac{d\mu_t}{d\mu_0} = r(t)^n/\gamma_0^n$. So for the radius $r(t)$ we get the ordinary differential equation

$$\ddot r(t) = -\frac{n r(t)^{n-1}}{\gamma_0^n} + \frac{\varrho}{\omega_{n+1}\gamma_0^n r(t)}.$$

This second order ODE can be written as a system of first order ODEs for $(r,z) = (r, \dot r)$

$$\begin{aligned} \dot r =& z \\ \dot z =& -\frac{n r(t)^{n-1}}{\gamma_0^n} + \frac{\varrho}{\omega_{n+1}\gamma_0^n r(t)}. \end{aligned} \qquad [\mathbf{1.6}]$$

Clearly the right hand side is locally Lipschitz and in fact smooth around (r_0, r_1) so there exists a local smooth solution. Using the energy conservation we will see that the solution stays bounded and that its orbits are periodic. For the energy we have

$$\mathcal{E}(u(t,\cdot)) = \frac{1}{2}\dot r^2 (n+1)\omega_{n+1}\gamma_0^n + (n+1)\omega_{n+1} r^n - \varrho(n+1)\log\left(\frac{r}{r_0}\right).$$

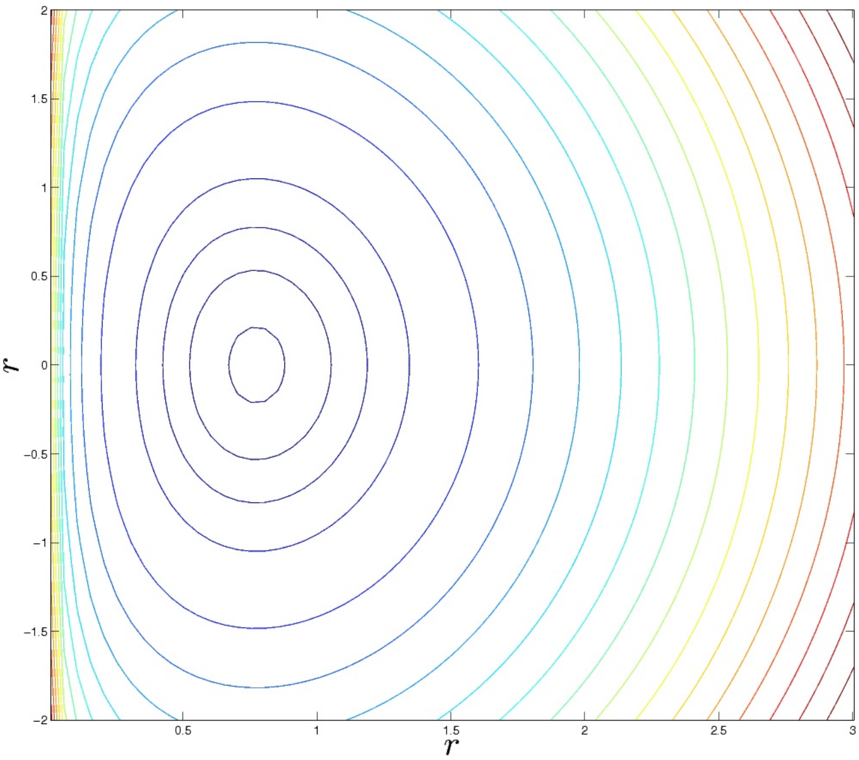


Figure 1.1: Integral Curves for the ODE System (1.6)

Hence

$$\dot{r}^2 = \frac{2\mathcal{E}_0}{(n+1)\omega_{n+1}\gamma_0^n} - 2\frac{r^n}{\gamma_0^n} + \frac{2\varrho}{\omega_{n+1}\gamma_0^n}\log\left(\frac{r}{r_0}\right)$$

which implies that the integral curves can be written as a graph

$$\dot{r} = \pm\sqrt{\frac{2\mathcal{E}_0}{(n+1)\omega_{n+1}\gamma_0^n} - 2\frac{r^n}{\gamma_0^n} + \frac{2\varrho}{\omega_{n+1}\gamma_0^n}\log\left(\frac{r}{r_0}\right)}.$$

To see that these curves are closed, let $f(r)$ be the expression under the root which becomes negative if r gets large and if r gets small. So the curve given by the graphs of $\sqrt{f}$ and $-\sqrt{f}$ consists of two arcs which meet at the r-axis. If initially $r^n = \frac{\varrho}{n\omega_{n+1}}$ and $\dot{r} = 0$ then we are in an equilibrium and this is the only equilibrium. So by the uniqueness of the solution of the ODE we cannot reach an equilibrium if we are not in an equilibrium initially. As the solution has to be on the upper or the lower graph of $\pm\sqrt{f}$ and cannot change its direction of motion the solution is periodic. The smoothness of the curve at the transition between the two graphs is guaranteed by the standard ODE existence and uniqueness theorem. See Figure 1.1 for examples of the integral curves. We summarise this as a proposition.

PROPOSITION 1.11. *Let* $g_0 = \gamma_0^2 g_{\mathbb{S}^n}$ *be a spherical metric with* $\gamma_0 > 0$ *and* $d\mu_0$ *its surface measure. Let* $r_0 > 0$, $r_1 \in \mathbb{R}$. *Then there exists a unique rotationally symmetric periodic solution* $u : \mathbb{R} \times \mathbb{S}^n \to \mathbb{R}^{n+1}$ *of equation* [**EQ**] *centered at the origin with initial conditions* $u(0,x) = r_0 x$, $\partial_t u(0,x) = r_1 x$. *If* $r_0 = \sqrt[n]{\frac{\varrho}{n\omega_{n+1}}}$ *and* $r_1 = 0$ *then the solution is constant in* t.

1.4.2. Translating Solutions. If $u : [0,T) \times \mathcal{N}^n \to \mathbb{R}^{n+1}$ is a solution of [**EQ**] and ξ is a vector in $\mathbb{R}^{n+1}$ then $\tilde{u}(t,\cdot) = u(t,\cdot) + t\xi$ is also a solution of equation [**EQ**] with initial data $\tilde{u}(0,\cdot) = u(0,\cdot)$, $\partial_t \tilde{u}(0,\cdot) = \partial_t u(0,\cdot) + \xi$. This is easy to see since $\frac{d\mu_t}{d\mu_0}(-H + \varrho\,\mathrm{Vol}(u)^{-1})\nu$ is translation invariant and $\partial_t^2 \tilde{u} = \partial_t^2 u$. Together with Proposition 1.11 we obtain translating vibrating solutions.

PROPOSITION 1.12. *Let* $g_0 = \gamma_0^2 g_{\mathbb{S}^n}$ *be a spherical metric with* $\gamma_0 > 0$ *and* $d\mu_0$ *its surface measure. Let* $r_0 > 0$, $r_1 \in \mathbb{R}$, $p, \xi \in \mathbb{R}^{n+1}$. *There exists a unique solution* $u : \mathbb{R} \times \mathbb{S}^n \to \mathbb{R}^{n+1}$ *of* [**EQ**] *having the form* $u(t,x) = p + r(t)x + t\xi$ *with* $u(0,x) = p + r_0 x$ *and* $\partial_t u(0,x) = r_1 x + \xi$. *This solution is the oscillating solution from Proposition 1.11 with initial conditions* r_0, r_1 *translating with velocity* ξ. *At* $t = 0$ *it is centered at* p.

CHAPTER 2

Short Time Existence

2.1. The Strategy

The objective of this chapter is to prove the following theorem.

THEOREM 2.1. *For every smooth immersion* $u_0 : \mathcal{N} \to \mathcal{M}$ *with* $\mathrm{Vol}(u_0) = \mathrm{Vol}_0 > 0$ *and initial velocity* $u_1 \in \Gamma(u_0^* T\mathcal{M})$ *there exists* $\varepsilon > 0$ *and a smooth family of immersions* $u : [0, \varepsilon) \times \mathcal{N} \to \mathcal{M}$ *solving the Cauchy problem*

$$\begin{cases} \overline{\nabla}_{\partial_t} \partial_t u = \frac{d\mu_t}{d\mu_0} \left(-H(u) + \frac{\varrho}{\mathrm{Vol}(u)} \right) \nu, \ \textit{for all } t \in [0, \varepsilon) \\ u(0, \cdot) = u_0 \\ \partial_t u(0, \cdot) = u_1. \end{cases}$$

REMARK 2.2. In this chapter we only prove the existence of a solution for a short time. The uniqueness is a special case of our stability estimate Theorem 3.7 (see Corollary 3.8).

We will first prove Theorem 2.1 for the simpler case $\mathcal{M} = \mathbb{R}^{n+1}$. The modifications necessary to generalise this result to arbitrary target manifolds are indicated in Section 2.6.

Our equation [**EQ**] is a quasilinear second order partial differential equation. As we will see in Section 2.2 the linearisation is not strictly hyperbolic. Due to the diffeomorphism invariance of the mean curvature only the normal part of the linearised operator is a wave operator. For Ricci flow and Mean Curvature Flow a suitable family of reparametrisations has been used to remove such a degeneracy. This procedure is known as DeTurck's trick [**DeT03**]. Here this does not work since, due to the $d\mu_0$-term, the evolution of the reparametrisations does not decouple from our equation, and it is not clear how this degeneracy could be removed. We therefore work directly with the degenerate equation and use an inverse function theorem to obtain short time existence. As previously mentioned, the linearised equation contains a wave equation. By the standard energy estimates for wave equations, we cannot guarantee that the solution of a wave equation has two orders of differentiability more than the right hand side. So we cannot consider the wave operator and hence our linearisation as an invertible operator between fixed Banach spaces and we cannot apply the inverse function theorem for Banach spaces. This phenomenon, usually called "loss of derivatives", suggests that we should work in C^∞ and use the Nash-Moser inverse function theorem instead. Another loss of derivatives comes from the decomposition of the linearisation into normal and tangential parts, since the application of a second order operator to the normal vector ν and tangent vectors $\partial_k u$ gives third derivatives of u. We will now explain how to use the Nash-Moser inverse function theorem to obtain a solution for a short time and after that we carry out the steps necessary for the application of this theorem. The strategy of the proof is similar to the short time existence proof for the Ricci flow given by Hamilton in [**Ham82b**].

Let $\mathbf{F}$ be the Fréchet space $C^\infty([0,T]\times\mathcal{N},\mathbb{R}^{n+1})$ and let $\mathbf{F}_0$ be the Fréchet space $C^\infty(\mathcal{N},\mathbb{R}^{n+1})$. We define the open subsets

$$\begin{aligned}\mathbf{U} =&\{u\in\mathbf{F},\quad \det(g_{ij})>0,\ \text{for all } t\in[0,T]\},\\ \mathbf{U}_0 =&\{u\in\mathbf{F}_0,\quad \det(g_{ij})>0\}\end{aligned}$$

and on a subset $\mathbf{U}'\subset\mathbf{U}$ we define the operator $\mathfrak{P}:\mathbf{U}'\to\mathbf{F}$ by

$$\mathfrak{P}(u)=\overline{\nabla}_{\partial_t}\partial_t u-\frac{d\mu_t}{d\mu_0}\left(-H(u)+\frac{\varrho}{\mathrm{Vol}(u)}\right)\nu. \qquad [\mathbf{2.1}]$$

The subset $\mathbf{U}'$ will be chosen later in Proposition 2.21 such that $\mathrm{Vol}(u)$ is defined for all $u\in\mathbf{U}'$. Of course $\mathfrak{P}(u)$ is a vector field along u but we identify $T_{u(x)}\mathbb{R}^{n+1}$ with $\mathbb{R}^{n+1}$ in the usual way. In the Euclidean standard coordinates we may write $\partial_t^2 u$ instead of $\overline{\nabla}_{\partial_t}\partial_t u$.

The strategy is as follows. Below we shall construct an approximate solution $\bar u:[0,T]\times\mathcal{N}\to\mathbb{R}^{n+1}$ with $\bar u(0)=u_0$, $\partial_t\bar u(0)=u_1$ for which $\mathfrak{P}(\bar u)$ is defined such that $\bar f:=\mathfrak{P}(\bar u)$ satisfies $\partial_t^k\bar f\big|_{t=0}=0$ for all $k=0,1,\ldots$. By making T smaller if necessary we may assume that $\bar u$ is defined for $t\in[0,T+\varepsilon_0]$. We put $\bar f=0$ for $t<0$ which keeps $\bar f$ smooth. Then we define $f_\varepsilon(t)=\bar f(t-\varepsilon)$ for $0\le\varepsilon\le\varepsilon_0$ which satisfies $f_\varepsilon(t,\cdot)=0$ for $0\le t\le\varepsilon$. Now f_ε converges to $\bar f$ in C^∞. So for every neighborhood $\mathcal{U}$ of $\bar f\in\mathbf{F}$ there is an $\varepsilon\in[0,\varepsilon_0]$ such that $f_\varepsilon\in\mathcal{U}$.

We shall use the Nash-Moser inverse function theorem in the form of [**Ham82a**] (see also Appendix B) to show that the operator $\mathscr{P}:\mathbf{U}\to\mathbf{F}\times\mathbf{U}_0\times\mathbf{F}_0$ defined by

$$\mathscr{P}(u)=(\mathfrak{P}(u),u(0,\cdot),\partial_t u(0,\cdot))$$

is locally invertible in a neighborhood of $\bar u$.

This implies that there exists a neighborhood $\mathcal{W}$ of $(\bar u,u_0,u_1)$ such that we can solve $\mathscr{P}(u)=(f,\tilde u_0,\tilde u_1)$ for every $(f,\tilde u_0,\tilde u_1)\in\mathcal{W}$. If ε is small enough such that $(f_\varepsilon,u_0,u_1)\in\mathcal{W}$ then we get a solution of $\mathfrak{P}(u)=f_\varepsilon$ with the right initial conditions. Then in fact $\mathfrak{P}(u)=0$ for $0\le t\le\varepsilon$.

To construct the approximate solution we first compute inductively from the initial data all time derivatives that a solution of the equation must have at $t=0$. Assume for a moment that u solves $\mathfrak{P}(u)=0$, i. e.

$$\partial_t^2 u=\mathfrak{E}(u)$$

with $\mathfrak{E}(u) = \frac{d\mu_t}{d\mu_0}\left(-H(u) + \frac{\varrho}{\mathrm{Vol}(u)}\right)\nu$ and initial conditions $u(0) = u_0$, $\partial_t u(0) = u_1$. Then we can compute

$$\begin{aligned}
\partial_t^2 u =& \mathfrak{E}(u)\\
\partial_t^3 u =& D\mathfrak{E}(u)\{\partial_t u\}\\
\partial_t^4 u =& D\mathfrak{E}(u)\{\partial_t^2 u\} + D^2\mathfrak{E}(u)\{\partial_t u, \partial_t u\}\\
&\vdots
\end{aligned}$$

Since $u(0) = u_0$ and $\partial_t u(0) = u_1$ are given as initial data, we can evaluate the expressions on the right hand side line by line at $t = 0$ to compute the next time derivative of u at $t = 0$. Borel's Lemma (see [**GG73**, p. 98]) based on a formal power series can be used to define a smooth function $\bar{u} : [0, T] \times \mathcal{N} \to \mathbb{R}^{n+1}$ which has these time derivatives at $t = 0$. Of course we do this locally in charts and patch this together using a partition of unity. This is possible due to the compactness of $\mathcal{N}$. If T is small enough then $\bar{u}(t, \cdot)$ is an immersion since $\bar{u}(0, \cdot)$ is an immersion and this condition is open. By making T small enough we can also assume that $\mathrm{Vol}(\bar{u}) > 0$. By construction we have that $\partial_t^k \mathfrak{P}(\bar{u}) = 0$ at $t = 0$. Hence $\bar{u}$ can be used as an approximate solution.

The main difficulty is the application of the Nash-Moser inverse function theorem. We will carry out the necessary steps in the following sections. In Section 2.2 we compute the linearisation of $\mathfrak{P}$. After that, in Section 2.3, we prove estimates for solutions of the linearised equation and more general systems, including a tame estimate that is needed for the application of the Nash-Moser inverse function theorem. Using these estimates we prove in Section 2.4 that the linearised operator is invertible in a neighborhood of the approximate solution. We are then able to conclude the short time existence proof in the Euclidean case in Section 2.5. For the convenience of the reader we included some material concerning the Nash-Moser inverse function theorem in Appendix B.

2.2. The Linearisation

We will compute the derivative of the operator $\mathfrak{P}$ defined in [**2.1**]. The decomposition of the derivative into normal and tangential part given in Corollary 2.4 is crucial for the proof of the short time existence theorem. Differentiating $\mathfrak{P}$ in tangential direction $\psi^k \partial_k u$ gives no derivatives of ψ^k in the tangential part of the derivative of $\mathfrak{P}$. This fact reflects the degeneracy of the mean curvature in tangential direction which comes from the diffeomorphism invariance. As we will see this still allows us to derive useful estimates.

LEMMA 2.3. *Let $V \in \mathbf{F}$ and let $u : [0,T]\times\mathcal{N} \to \mathbb{R}^{n+1}$ be a smooth family of immersions such that* $\mathrm{Vol}(u) > 0$. *We have*

$$\begin{aligned} D\mathfrak{P}(u)\{V\} =&\partial_t^2 V - \frac{d\mu_t}{d\mu_0}\left(\Delta\langle V,\nu\rangle + |h|^2\langle V,\nu\rangle - \langle\nabla H, V^\top\rangle - \frac{\varrho}{\mathrm{Vol}(u)^2}\int_{\mathcal{N}}\langle V,\nu\rangle d\mu_t\right)\nu \\ &- \frac{d\mu_t}{d\mu_0}\left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)\left(\operatorname{div} V^\top + \langle V,\nu\rangle H\right)\nu \\ &+ \frac{d\mu_t}{d\mu_0}\left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)(\nabla\langle V,\nu\rangle - \langle V,\partial_l u\rangle h^{lk}\partial_k u). \end{aligned} \quad [\mathbf{2.2}]$$

PROOF. Let u_ε be a variation of u with $\partial_\varepsilon|_{\varepsilon=0}u_\varepsilon = V$. It is well known that

$$-\partial_\varepsilon|_{\varepsilon=0}H(u_\varepsilon) = \Delta\langle V,\nu\rangle + |h|^2\langle V,\nu\rangle - \langle\nabla H, V^\top\rangle$$

and

$$\begin{aligned} -\partial_\varepsilon|_{\varepsilon=0}\nu(u_\varepsilon) =& -\langle\partial_\varepsilon|_{\varepsilon=0}\nu(u_\varepsilon), \partial_l u\rangle g^{lk}\partial_k u = \langle\partial_l V,\nu\rangle g^{lk}\partial_k u \\ =&\partial_l\langle V,\nu\rangle g^{lk}\partial_k u - \langle V,\partial_l\nu\rangle g^{kl}\partial_k u = \nabla\langle V,\nu\rangle - \langle V,\partial_l u\rangle h^{lk}\partial_k u. \end{aligned}$$

By [**1.3**] the variation of $\mathrm{Vol}(u_\varepsilon)^{-1}$ is given by

$$-\frac{1}{\mathrm{Vol}(u)^2}\int_{\mathcal{N}}\langle V,\nu\rangle d\mu_t$$

and the variation of $d\mu_t$ is given by [**1.4**]. □

COROLLARY 2.4. *Let $V \in \mathbf{F}$ and u be as in Lemma 2.3. Decompose $V = \varphi\nu + \psi^k\partial_k u$. Then we can write $D\mathfrak{P}(u)V = W^0\nu + W^k\partial_k u$ with*

$$\begin{aligned} W^0 =&\partial_t^2\varphi - \frac{d\mu_t}{d\mu_0}\Big\{\Delta\varphi + |h|^2\varphi - \langle\nabla H,\psi\rangle \\ &- \frac{\varrho}{\mathrm{Vol}(u)^2}\int_{\mathcal{N}}\varphi d\mu_t + \left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)(\operatorname{div}\psi + H\varphi)\Big\} \\ &+ \varphi\langle\partial_t^2\nu,\nu\rangle + 2\partial_t\psi^k\langle\partial_t\partial_k u,\nu\rangle + \psi^k\langle\partial_t^2\partial_k u,\nu\rangle \\ W^k =&\partial_t^2\psi^k + \frac{d\mu_t}{d\mu_0}\left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)\left(\nabla^k\varphi - h_i{}^k\psi^i\right) + 2\partial_t\varphi\langle\partial_t\nu,\partial_j u\rangle g^{jk} \\ &+ \varphi\langle\partial_t^2\nu,\partial_j u\rangle g^{jk} + 2\partial_t\psi^l\langle\partial_t\partial_l u,\partial_j u\rangle g^{jk} + \psi^l\langle\partial_t^2\partial_l u,\partial_j u\rangle g^{jk}. \end{aligned} \quad [\mathbf{2.3}]$$

PROOF. The only term which is not yet decomposed in [**2.2**] is $\partial_t^2 V$. We compute

$$\begin{aligned} \partial_t^2 V =&\partial_t^2(\varphi\nu + \psi^k\partial_k u) = \partial_t^2\varphi\nu + 2\partial_t\varphi\partial_t\nu + \varphi\partial_t^2\nu + \partial_t^2\psi^k\partial_k u + 2\partial_t\psi^k\partial_t\partial_k u + \psi^k\partial_t^2\partial_k u \\ =&(\partial_t^2\varphi + \varphi\langle\partial_t^2\nu,\nu\rangle + 2\partial_t\psi^k\langle\partial_t\partial_k u,\nu\rangle + \psi^k\langle\partial_t^2\partial_k u,\nu\rangle)\nu \\ &+ \Big(\partial_t^2\psi^k + 2\partial_t\varphi\langle\partial_t\nu,\partial_j u\rangle g^{jk} + \varphi\langle\partial_t^2\nu,\partial_j u\rangle g^{jk} \\ &+ 2\partial_t\psi^l\langle\partial_t\partial_l u,\partial_j u\rangle g^{jk} + \psi^l\langle\partial_t^2\partial_l u,\partial_j u\rangle g^{jk}\Big)\partial_k u. \end{aligned}$$

□

Clearly the derivative of $\mathscr{P}$ is $D\mathscr{P}(u)\{V\} = (D\mathfrak{P}(u)\{V\}, V|_{t=0}, \partial_t V|_{t=0})$. To prove the invertibility of $D\mathscr{P}(u)$ we have to solve the equation $D\mathscr{P}(u)\{V\} = W$ uniquely for any given W and initial conditions $V(0) = V_0$, $\partial_t V(0) = V_1$. Decomposing $W = W^0\nu + W^k\partial_k u$, $V_0 = \varphi_0\nu + \psi^k\partial_k u$, $V_1 = \varphi_1\nu + \psi_1^k\partial_k u$ we have to solve the system [**2.3**] for φ and ψ^k subject to the initial conditions $\varphi(0) = \varphi_0$, $\psi^k(0) = \psi_0^k$ and

$$\begin{aligned}\partial_t\varphi(0) =&\varphi_1 - \psi_0^j\langle\partial_t\partial_j u(0), \nu(0)\rangle,\\ \partial_t\psi^k(0) =&\psi_1^k - \varphi_0\langle\partial_t\nu(0), \partial_l u(0)\rangle g^{kl}(0) - \psi_0^j\langle\partial_t\partial_j u(0), \partial_l u(0)\rangle g^{lk}(0).\end{aligned}$$

Furthermore we have to prove a tame estimate for the solution operator of this system where on the right hand side norms of u, W, V_0, V_1 may appear. The invertibility and the tame estimate will follow from the separate treatment of more general systems in Sections 2.3 and 2.4.

2.3. Estimates for Weakly Hyperbolic Linear Systems (WHLS)

In this section we will define weakly hyperbolic linear systems (WHLS) and derive estimates for them. These are systems that consist of a number of coupled linear wave equations that are again coupled together with a system of ODEs with suitable conditions on the appearance of the highest order terms. We also allow integrals of the unknowns to appear. These systems generalise the linearised equation [**2.3**] in the Euclidean case. We need this generalisation when we prove short time existence for more general target manifolds in Section 2.6 and when we estimate the maximal time of existence in Chapter 3. Weakly hyperbolic linear systems are a hyperbolic analogue of Hamilton's weakly parabolic linear systems in [**Ham82b**].

We will derive a tame estimate for solutions of these systems in terms of the coefficients, the right hand side, the initial data and a special set of basis vectors used to split the system into a wave and an ODE part. In order to derive this estimate we combine a very simple ODE estimate with the usual energy estimate for wave equations. Along the way we need to prove a version of the standard elliptic regularity estimate that allows us to prove a tame estimate later. In contrast to the usual statement of the elliptic regularity estimate (see e. g. [**Eva98**, Theorem 1, 6.3.2]) we are not allowed to have a nonlinear dependence of high derivatives of the coefficients which is usually hidden in the constant .

In a first step our estimates (Proposition 2.13 and Proposition 2.14) will look very similar to energy estimates for the wave equation which estimate the spatial H^s-norm of the solution at a time. As a grading (see Definition B.1) for $C^\infty([0,T]\times\mathcal{N},\mathcal{V})$ we will choose an H^s-grading in space and time. So in Subsection 2.3.7 we will integrate the estimates of spatial norms in time in order to derive estimates of the H^s-norm in space and time. Therefore we need a kind of Moser inequality that does not mix space and time derivatives in the highest order terms. This inequality is proved in Appendix C.2 where we included also the frequently used Moser inequalities. Norms are defined in Appendix C.1.

2.3.1. Definition of Weakly Hyperbolic Linear Systems. Let $\pi : \mathcal{V} \to \mathcal{N}$ be a d-dimensional Riemannian vector bundle over $\mathcal{N}$. Let $\mathbf{F}$ be the Fréchet space $C^\infty([0,T] \times \mathcal{N}, \mathcal{V})$ of smooth time dependent sections of $\mathcal{V}$. Assume that we have an atlas of coordinate charts (x_α, U_α) as in Appendix C.1 of $\mathcal{N}$, i.e. $\alpha = 1, \ldots, J$, $x_\alpha(U_\alpha) = B_3(0)$ and the sets $x_\alpha^{-1}(B_1(0))$ cover $\mathcal{N}$. Assume also that for each such chart there are smooth time dependent local sections $\nu_A^{(\alpha)}$, $A = 1, \ldots, d'$, and $\tau_k^{(\alpha)}$, $k = 1, \ldots, d''$ of $\mathcal{V}$ $(d' + d'' = d)$ defined on the domain of the chart which together form a basis of the fiber over each point in U_α. For any other chart (x_β, U_β) with $U_\alpha \cap U_\beta \neq \emptyset$ we assume that the $\nu_A^{(\alpha)}(p)$ and $\nu_A^{(\beta)}(p)$, $p \in U_\alpha \cap U_\beta$ span the same space and are bases for this space. Furthermore we assume that the spaces spanned by the $\nu_A^{(\alpha)}$ and the $\tau_k^{(\alpha)}$ are orthogonal. If the specific coordinate chart does not play a role or is fixed, then we will omit the index (α). Let $d\mu_0$ be the volume form of a reference metric g_0. Let $V \in \mathbf{F}$. In each coordinate chart we can decompose

$$V = V_\perp + V_\top := \varphi^A \nu_A + \psi^k \tau_k.$$

We say that V satisfies a weakly hyperbolic linear system if in each coordinate chart (x_α, U_α) we have

$$\begin{aligned} \partial_t^2 \varphi^A - L^A \varphi^A - N^A \psi - Q^A \varphi &= v^A \\ \partial_t^2 \psi^k - M^k \psi - P^k \varphi &= w^k \end{aligned} \qquad \textbf{[2.4]}$$

for some given $W = v^a \nu_A + w^k \tau_k$. The operators are assumed to be of the following form in local coordinates

$$\begin{aligned} L^A \varphi^A &= a^{Aij} \partial_i \partial_j \varphi^A + a^{Ai} \partial_i \varphi^A + a^A \varphi^A \\ N^A \psi &= n_j^{Ai} \partial_i \psi^j + n_i^A \psi^i + n_k^{A0} \partial_t \psi^k + n^{A1} \sum_{\beta=1}^J \int_{\mathcal{N}} b_{(\beta)j}^A \psi_{(\beta)}^j d\mu_0 \\ Q^A \varphi &= q_B^{Ai} \partial_i \varphi^B + q_B^A \varphi^B + q_B^{A0} \partial_t \varphi^B + q^{A1} \sum_{\beta=1}^J \int_{\mathcal{N}} c_{(\beta)B}^A \varphi_{(\beta)}^B d\mu_0 \\ M^k \psi &= m_i^k \psi^i + m_i^{k0} \partial_t \psi^i \\ P^k \varphi &= p_B^{kj} \partial_j \varphi^B + p_B^{k0} \partial_t \varphi^B. \end{aligned}$$

Of course we do not apply the summation convention for the indices A. We assume all coefficients and also v^A and w^k to be smooth functions on $x_\alpha(U_\alpha)$ and $\Lambda \delta^{ij} \xi_i \xi_j \geq a^{Aij} \xi_i \xi_j \geq \lambda \delta^{ij} \xi_i \xi_j$ for all $\xi \in \mathbb{R}^n$ with some fixed $\Lambda, \lambda > 0$. Assume $a^{Aij} = a^{Aji}$. Assume $\operatorname{supp} b_{(\beta)j}^A \subset x_\beta^{-1}(B_2(0))$ and $\operatorname{supp} c_{(\beta)B}^A \subset x_\beta^{-1}(B_2(0))$. Furthermore we want that $W = v^A \nu_A + w^k \tau_k$ is an element of $\mathcal{V}$, i.e. the equations transform appropriately under coordinate transformations on $\mathcal{N}$ and under change of basis between different $(\nu_A^{(\alpha)}, \tau_k^{(\alpha)})$ and $(\nu_A^{(\beta)}, \tau_k^{(\beta)})$.

2.3.2. Preliminary Estimates. Assume that we are given the local bases ν_A, τ_k as in the previous subsection and that $V = V_\perp + V_\top \in \mathbf{F}$. Locally we write $V = \varphi^A \nu_A + \psi^k \tau_k$. We want to use a notation similar to that of Appendix C.1 for norms that measure the components φ^A, ψ^k of $V_\perp$, $V_\top$ with respect to the time dependent frames ν_A, τ_k. We also want to take norms of the frames ν_A, τ_k although these frames are only defined locally.

Therefore we add the respective norms which are defined locally in a coordinate chart over all coordinate charts, e. g.

$$\begin{aligned}\|\varphi\|_s =&\sum_{\alpha=1}^{J}\sum_{A=1}^{d'}\|\varphi^A_{(\alpha)}\|_{H^s(B_2(0))}\\ \|\nu\|_s =&\sum_{\alpha=1}^{J}\sum_{A=1}^{d'}\|\nu^{(\alpha)}_A\circ x_\alpha^{-1}\|_{H^s(B_2(0))}.\end{aligned}$$

We do the same for the other norms $\|\cdot\|_{C^s}$, $|\!|\!|\cdot|\!|\!|_s$, $|\!|\!|\cdot|\!|\!|_{C^s}$ defined in Appendix C.1.

In Appendix C.1 we have defined norms $[\,\cdot\,]_s$, $[\,\cdot\,]_{C^s}$, $[\![\,\cdot\,]\!]_s$, $[\![\,\cdot\,]\!]_{C^s}$ for linear operators which are not the usual operator norms but measure the norms of the coefficients. To apply this notation to the operators N and Q in [**2.4**] we write in a local coordinate chart (x_α, U_α)

$$\begin{aligned}[N]_{s,\alpha} =&\sum_{A,i,j}\|n^{Ai}_j\|_{H^s(B_2(0))}+\sum_{A,i}\|n^A_i\|_{H^s(B_2(0))}+\sum_{A,k}\|n^{A0}_k\|_{H^s(B_2(0))}\\ &+\sum_{A}\|n^{A1}\|_{H^s(B_2(0))}+\sum_{A,j}\|b^A_{(\alpha)j}\|_{H^s(B_2(0))}\end{aligned}$$

and define

$$[N]_s=\sum_{\alpha=1}^{J}[N]_{s,\alpha}\,.$$

Note that this is not a norm for N.

We will occasionally identify ν_A and $\nu_A\circ x_\alpha^{-1}$ and similarly for other quantities.

From the WHLS we will later derive estimates for φ and ψ. To this end we will apply estimates that only allow us to estimate a function on a smaller domain (e. g. the elliptic estimate Lemma 2.11). So we have to estimate norms of φ and ψ by norms on smaller domains. The objective of this subsection is to provide these estimates. Doing this there will of course arise norms of the basis transitions between different $\nu^{(\alpha)}_A$ and $\tau^{(\alpha)}_k$. We will express these as norms of ν_A and τ_k and include them in our estimates. We will not use the system [**2.4**] yet in this subsection. The following notation will be used several times.

Definition 2.5. Define

$$\begin{aligned}\nu_{AB} =&\langle\nu_A,\nu_B\rangle, & \tau_{kl} =&\langle\tau_k,\tau_l\rangle,\\ (\nu^{AB}) =&(\nu_{AB})^{-1}, & (\tau^{kl}) =&(\tau_{kl})^{-1}.\end{aligned}$$

LEMMA 2.6. *Let*

$$\|\nu\|_{C^0} + \|\tau\|_{C^0} \leq K$$

and

$$\det(\nu_{AB}) > \lambda_1, \quad \det(\tau_{kl}) > \lambda_1$$

for some $K, \lambda_1 > 0$. *We can estimate*

$$\begin{aligned}
\|\nu^{AB}\|_{H^s(B_2(0))} \leq & C(1 + \|\nu\|_{H^s(B_2(0))}) \\
\|\tau^{kl}\|_{H^s(B_2(0))} \leq & C(1 + \|\tau\|_{H^s(B_2(0))}) \\
|||\nu^{AB}|||_{H^s([0,T]\times B_2(0))} \leq & C(1 + \|\nu\|_{H^s([0,T]\times B_2(0))}) \\
|||\tau^{AB}|||_{H^s([0,T]\times B_2(0))} \leq & C(1 + \|\tau\|_{H^s([0,T]\times B_2(0))}) \\
\|\nu^{AB}\|_{C^0(B_2(0))} \leq & C \\
\|\tau^{kl}\|_{C^0(B_2(0))} \leq & C
\end{aligned}$$

with C *depending only on* λ_1 *and on* K.

PROOF. We want to use the third Moser inequality (Theorem C.5) to prove these estimates. Therefore we have to prove that ν^{AB} is a smooth function of the ν_A and that this function and its derivatives are bounded on the range of the ν_A. This is easy to see if we use the cofactor representation of the inverse

$$\nu^{CD} = (-1)^{C+D} \frac{\det(N_{DC})}{\det(\nu_{AB})}$$

where N_{CD} is created from (ν_{AB}) by deleting line C and row D. By our assumption $\det(\nu_{AB}) > \lambda_1$ and determinants are only polynomials of the matrix entries. Therefore as claimed ν^{AB} is a smooth function of ν_{AB} and all derivatives are bounded in the range of ν_A. Hence we can apply the third Moser inequality. The estimate for τ^{kl} is exactly the same.

For the C^0 estimate simply take the supremum in the stated expression for ν^{AB} and similarly for τ^{kl}. □

LEMMA 2.7. *Let* $s \geq \lfloor \frac{n}{2} \rfloor + 1$ *and* $V = \varphi^A \nu_A + \psi^k \tau_k$ *locally. Assume further*

$$\|\nu\|_s + \|\partial_t \nu\|_s + \|\tau\|_s + \|\partial_t \tau\|_s \leq K,$$

$$\det(\nu_{AB}) > \lambda_1, \quad \det(\tau_{kl}) > \lambda_1$$

for some $K, \lambda_1 > 0$. *Then*

(1)

$$\|\varphi\|_s \leq C \sum_{\alpha=1}^{J} \sum_{A=1}^{d'} \|\varphi^A_{(\alpha)}\|_{H^s(B_1(0))}$$

(2)

$$\|\partial_t\varphi\|_s \leq C\sum_{\alpha=1}^{J}\sum_{A=1}^{d'}\big(\|\partial_t\varphi^A_{(\alpha)}\|_{H^s(B_1(0))}+\|\varphi^A_{(\alpha)}\|_{H^s(B_1(0))}\big)$$

(3)

$$\|\psi\|_s \leq C\sum_{\alpha=1}^{J}\sum_{k=1}^{d''}\|\psi^k_{(\alpha)}\|_{H^s(B_1(0))}$$

(4)

$$\|\partial_t\psi\|_s \leq C\sum_{\alpha=1}^{J}\sum_{k=1}^{d''}\big(\|\partial_t\psi^k_{(\alpha)}\|_{H^s(B_1(0))}+\|\psi^k_{(\alpha)}\|_{H^s(B_1(0))}\big).$$

If additionally $\|\partial_t^2\nu\|_s \leq K$ *then*

(5)

$$\|\partial_t^2\varphi\|_s \leq C\sum_{\alpha=1}^{J}\sum_{A=1}^{d'}\big(\|\partial_t^2\varphi^A_{(\alpha)}\|_{H^s(B_1(0))}+\|\partial_t\varphi^A_{(\alpha)}\|_{H^s(B_1(0))}+\|\varphi^A_{(\alpha)}\|_{H^s(B_1(0))}\big).$$

The constant C *depends on* K *and on* λ_1*.*

Proof. To prove the first statement write $\varphi^A_{(\alpha)} = \nu^{AB}_{(\alpha)}\langle V_\perp\circ x_\alpha^{-1}, \nu_B^{(\alpha)}\rangle$. By the first Moser inequality (Theorem C.3)

$$\begin{aligned}\|\varphi^A_{(\alpha)}\|_{H^s(B_2(0))} \leq C\big(&\|V_\perp\circ x_\alpha^{-1}\|_{H^s(B_2(0))}\|\nu^{AB}\nu_B\|_{C^0(B_2(0))}\\ &+\|V_\perp\circ x_\alpha^{-1}\|_{C^0(B_2(0))}\|\nu^{AB}\nu_B\|_{H^s(B_2(0))}\big).\end{aligned}$$

Now by the Sobolev embedding theorem since $s\geq\lfloor\frac{n}{2}\rfloor+1$

$$\|V_\perp\circ x_\alpha^{-1}\|_{C^0(B_2(0))}\leq C\|V_\perp\circ x_\alpha^{-1}\|_{H^s(B_2(0))}$$

and

$$\|\nu\|_{C^0}\leq C\|\nu\|_s\leq K.$$

So we can use the first Moser inequality and Lemma 2.6 to estimate the $\|\nu^{AB}\nu_B\|_{H^s(B_2(0))}$-term

$$\begin{aligned}\|\nu^{AB}\nu_B\|_{H^s(B_2(0))} \leq& C(\|\nu^{AB}\|_{C^0(B_2(0))}\|\nu_B\|_{H^s(B_2(0))}+\|\nu^{AB}\|_{H^s(B_2(0))}\|\nu_B\|_{C^0(B_2(0))})\\ \leq& C(1+\|\nu\|_s)\leq C_K.\end{aligned}$$

In view of Remark C.2 we have $\|V_\perp\circ x_\alpha^{-1}\|_{H^s(B_2(0))}\leq C\sum_\beta\|V_\perp\circ x_\beta^{-1}\|_{H^s(B_1(0))}$. So we obtain

$$\|\varphi^A_{(\alpha)}\|_{H^s(B_2(0))}\leq C\sum_\beta\|V_\perp\circ x_\beta^{-1}\|_{H^s(B_1(0))}.$$

Writing locally $V_\perp \circ x_\beta^{-1} = \varphi^A_{(\beta)}\nu^{(\beta)}_A$ and using again the first Moser inequality and the Sobolev embedding theorem we obtain

$$\begin{aligned}\|\varphi^A_{(\alpha)}\|_{H^s(B_2(0))} \leq & C\sum_\beta\sum_A \left(\|\varphi^A_{(\beta)}\|_{H^s(B_1(0))}\|\nu\|_{C^0} + \|\varphi^A_{(\beta)}\|_{C^0(B_1(0))}\|\nu\|_s\right)\\ \leq & C\sum_\beta\sum_A\|\varphi^A_{(\beta)}\|_{H^s(B_1(0))}\end{aligned}$$

Summing over all coordinate charts yields the desired estimate (1). Statement (3) is proved similarly.

Via

$$\partial_t V_\perp = \partial_t\varphi^A\nu_A + \varphi^A\partial_t\nu_A \qquad [\mathbf{2.5}]$$

we obtain

$$\partial_t\varphi^A = \langle\partial_t V_\perp, \nu_B\rangle\nu^{AB} - \varphi^C\langle\partial_t\nu_C, \nu_B\rangle\nu^{AB}.$$

Proceeding similarly as before we can estimate

$$\begin{aligned}\|\partial_t\varphi^A_{(\alpha)}\|_{H^s(B_2(0))} \leq C\Big(&\|\partial_t V_\perp\circ x_\alpha^{-1}\|_{H^s(B_2(0))}\|\nu^{AB}\nu_B\|_{C^0(B_2(0))}\\ &+ \|\partial_t V_\perp\circ x_\alpha^{-1}\|_{C^0(B_2(0))}\|\nu^{AB}\nu_B\|_{H^s(B_2(0))} + \|\varphi^C\|_{H^s(B_2(0))}\|\nu^{AB}\langle\partial_t\nu_C,\nu_B\rangle\|_{C^0(B_2(0))}\\ &+ \|\varphi^C\|_{C^0(B_2(0))}\|\nu^{AB}\langle\partial_t\nu_C,\nu_B\rangle\|_{H^s(B_2(0))}\Big).\end{aligned}$$

Using again the first Moser inequality, Lemma 2.6 and the Sobolev embedding theorem

$$\|\nu^{AB}\langle\partial_t\nu_C,\nu_B\rangle\|_{H^s(B_2(0))} \leq C(1 + \|\partial_t\nu\|_s + \|\nu\|_s) \leq C_K.$$

Using also the previous estimate for $\|\varphi^A_{(\alpha)}\|_{H^s(B_2(0))}$

$$\|\partial_t\varphi^A_{(\alpha)}\|_{H^s(B_2(0))} \leq C\sum_\beta\sum_A\left(\|\partial_t V_\perp\circ x_\beta^{-1}\|_{H^s(B_1(0))} + \|\varphi^A_{(\beta)}\|_{H^s(B_1(0))}\right).$$

Then using [**2.5**], the first Moser inequality and the Sobolev embedding theorem as before we obtain our estimate (2). Estimate (4) is proved in the same way.

Estimate (5) is proved using

$$\partial_t^2 V_\perp = \partial_t^2\varphi^A\nu_A + 2\partial_t\varphi^A\partial_t\nu_A + \varphi^A\partial_t^2\nu_A$$

which implies

$$\partial_t^2\varphi^A = \langle\partial_t^2 V_\perp,\nu_B\rangle\nu^{AB} - 2\partial_t\varphi^C\langle\partial_t\nu_C,\nu_B\rangle\nu^{AB} - \varphi^C\langle\partial_t^2\nu_C,\nu_B\rangle\nu^{AB}.$$

Then we proceed in the same manner as before. □

LEMMA 2.8. *Let $V = \varphi^A\nu_A + \psi^k\tau_k$ locally and assume*

$$\begin{aligned}\|\nu\|_{C^0} + \|\partial_t\nu\|_{C^0} + \|\partial_t^2\nu\|_{C^0} + \|\tau\|_{C^0} + \|\partial_t\tau\|_{C^0} \leq & K,\\ \|\varphi\|_{C^0} + \|\partial_t\varphi\|_{C^0} + \|\partial_t^2\varphi\|_{C^0} + \|\psi\|_{C^0} + \|\partial_t\psi\|_{C^0} \leq & K\end{aligned}$$

and

$$\det(\nu_{AB}) > \lambda_1, \quad \det(\tau_{kl}) > \lambda_1$$

for some $K, \lambda_1 > 0$. *Then*

(1)

$$\|\varphi\|_s \leq C\Big(\|\nu\|_s + \sum_{\alpha=1}^{J}\sum_{A=1}^{d'}\|\varphi^A_{(\alpha)}\|_{H^s(B_1(0))} + 1\Big)$$

(2)

$$\|\partial_t\varphi\|_s \leq C\Big(\|\partial_t\nu\|_s + \|\nu\|_s + \sum_{\alpha=1}^{J}\sum_{A=1}^{d'}(\|\partial_t\varphi^A_{(\alpha)}\|_{H^s(B_1(0))} + \|\varphi^A_{(\alpha)}\|_{H^s(B_1(0))}) + 1\Big)$$

(3)

$$\begin{aligned}\|\partial_t^2\varphi\|_s \leq & C\Big(\|\partial_t^2\nu\|_s + \|\partial_t\nu\|_s + \|\nu\|_s \\ & + \sum_{\alpha=1}^{J}\sum_{A=1}^{d'}(\|\partial_t^2\varphi^A_{(\alpha)}\|_{H^s(B_1(0))} + \|\partial_t\varphi^A_{(\alpha)}\|_{H^s(B_1(0))} + \|\varphi^A_{(\alpha)}\|_{H^s(B_1(0))}) + 1\Big)\end{aligned}$$

(4)

$$\|\psi\|_s \leq C\Big(\|\tau\|_s + \sum_{\alpha=1}^{J}\sum_{k=1}^{d''}\|\psi^k_{(\alpha)}\|_{H^s(B_1(0))} + 1\Big)$$

(5)

$$\|\partial_t\psi\|_s \leq C\Big(\|\partial_t\tau\|_s + \|\tau\|_s + \sum_{\alpha=1}^{J}\sum_{k=1}^{d''}(\|\partial_t\psi^k_{(\alpha)}\|_{H^s(B_1(0))} + \|\psi^k_{(\alpha)}\|_{H^s(B_1(0))}) + 1\Big)$$

where C *depends on* K *and on* λ_1.

Proof. We begin as in the proof of Lemma 2.7

$$\begin{aligned}\|\varphi^A_{(\alpha)}\|_{H^s(B_2(0))} \leq C(&\|V_\perp \circ x_\alpha^{-1}\|_{H^s(B_2(0))}\|\nu^{AB}\nu_B\|_{C^0(B_2(0))} \\ &+ \|V_\perp \circ x_\alpha^{-1}\|_{C^0(B_2(0))}\|\nu^{AB}\nu_B\|_{H^s(B_2(0))}).\end{aligned}$$

We estimate $\|\nu^{AB}\nu_B\|_{C^0(B_2(0))} \leq C$ and $\|\nu^{AB}\nu_B\|_{H^s(B_2(0))} \leq C(1 + \|\nu\|_s)$ which gives

$$\|\varphi^A_{(\alpha)}\|_{H^s(B_2(0))} \leq C(\|V_\perp \circ x_\alpha^{-1}\|_{H^s(B_2(0))} + \|V_\perp \circ x_\alpha^{-1}\|_{C^0(B_2(0))} + \|V_\perp \circ x_\alpha^{-1}\|_{C^0(B_2(0))}\|\nu\|_s).$$

By the assumption $\|V_\perp \circ x_\alpha^{-1}\|_{C^0(B_2(0))} \leq C$ and we can proceed similarly as in the proof of Lemma 2.7 using the assumption instead of the Sobolev embedding theorem. □

2.3.3. Basic L^2-Energy Estimate. The estimate from this subsection is standard but we will need a version that accounts for the finite speed of propagation and uses the more precise Gronwall inequality Lemma C.8. For a constant $\Lambda > 0$ and $(t_0, x_0) \in \mathbb{R}^{n+1}$ we denote $S_t(t_0, x_0) = \{x \in \mathbb{R}^n, |x - x_0| < \sqrt{\Lambda}(t_0 - t)\}$. If there is no confusion about the point (t_0, x_0) we only write S_t.

PROPOSITION 2.9. *Let $\Omega \subset \mathbb{R}^n$ be open and let $\varphi : [0, T] \times \Omega \to \mathbb{R}$ satisfy*

$$\partial_t^2 \varphi(t, x) - a^{ij}(t, x)\partial_i\partial_j\varphi(t, x) - a^i(t, x)\partial_i\varphi(t, x) - a(t, x)\varphi(t, x) = F(t, x)$$

where a^{ij} is symmetric and satisfies

$$\lambda\delta^{ij}\xi_i\xi_j \leq a^{ij}\xi_i\xi_j \leq \Lambda\delta^{ij}\xi_i\xi_j$$

for constants $\lambda, \Lambda > 0$. Furthermore let a^{ij}, a^k, a and F be smooth functions with

$$1 + \sum_{\alpha=0}^{n}\sum_{i,j=1}^{n}\|\partial_\alpha a^{ij}\|_{C^0(\Omega)} + \sum_{k=1}^{n}\|a^k\|_{C^0(\Omega)} + \|a\|_{C^0(\Omega)} \leq K$$

for some $K > 0$. Let $(t_0, x_0) \in [0, T] \times \Omega$ such that $S_0 = S_0(t_0, x_0) \subset \Omega$. Then there is a constant C depending on λ, Λ and K such that

$$\|D\varphi(t,\cdot)\|_{L^2(S_t)} + \|\partial_t\varphi(t,\cdot)\|_{L^2(S_t)} + \|\varphi(t,\cdot)\|_{L^2(S_t)} \leq$$
$$Ce^{Ct}\Big(\|D\varphi(0,\cdot)\|_{L^2(S_0)} + \|\partial_t\varphi(0,\cdot)\|_{L^2(S_0)} + \|\varphi(0,\cdot)\|_{L^2(S_0)} + \int_0^t e^{-Ct'}\|F(t',\cdot)\|_{L^2(S_{t'})}dt'\Big).$$

PROOF. Define

$$e(\varphi) = \frac{1}{2}|\partial_t\varphi|^2 + \frac{1}{2}|D\varphi|^2 + \frac{1}{2}\varphi^2, \qquad \tilde{e}(\varphi) = \frac{1}{2}|\partial_t\varphi|^2 + \frac{1}{2}a^{ij}\partial_i\varphi\partial_j\varphi + \frac{1}{2}\varphi^2$$

and

$$E(t) = \int_{S_t} e(\varphi)dx, \qquad \tilde{E}(t) = \int_{S_t}\tilde{e}(\varphi)dx.$$

It holds that

$$\max(\Lambda, 1)e(\varphi) \geq \tilde{e}(\varphi) \geq \min(\lambda, 1)e(\varphi)$$

and

$$\max(\Lambda, 1)E(t) \geq \tilde{E}(t) \geq \min(\lambda, 1)E(t).$$

We have for a smooth function $f : \Omega \to \mathbb{R}$

$$\int_{S_t} f(x)dx = \int_0^{\sqrt{\Lambda}(t_0-t)}\int_{\partial B_r(x_0)} f d\omega dr$$

and consequently

$$\frac{d}{dt}\int_{S_t} f(x)dx = -\sqrt{\Lambda}\int_{\partial S_t} f d\omega.$$

Using integration by parts we calculate

$$\begin{aligned}\partial_t \tilde{E}(t) = &\int_{S_t} \partial_t\varphi F + \partial_t\varphi a^k\partial_k\varphi + a\partial_t\varphi\varphi + \varphi\partial_t\varphi - \partial_t\varphi\partial_j a^{ij}\partial_i\varphi dx \\ &+ \int_{S_t} \frac{1}{2}\partial_t a^{ij}\partial_i\varphi\partial_j\varphi dx + \int_{\partial S_t} \partial_t\varphi a^{ij}\partial_i\varphi\zeta_j - \frac{1}{2}\sqrt{\Lambda}\left(\partial_t\varphi^2 + a^{ij}\partial_i\varphi\partial_j\varphi + \varphi^2\right) d\omega.\end{aligned} \quad [\mathbf{2.6}]$$

Here ζ is the outer unit normal of $\partial S_t = \partial B_{\sqrt{\Lambda}(t_0-t)}(x_0)$ and $d\omega$ the surface measure of ∂S_t. By the generalised Cauchy-Schwarz inequality [**Eva98**, §B.2] and the Cauchy inequality

$$\begin{aligned}\partial_t\varphi a^{ij}\partial_i\varphi\zeta_j \le |\partial_t\varphi|\left(a^{ij}\partial_i\varphi\partial_j\varphi\right)^{\frac{1}{2}}\left(a^{ij}\zeta_i\zeta_j\right)^{\frac{1}{2}} &\le |\partial_t\varphi|(a^{ij}\partial_i\varphi\partial_j\varphi)^{\frac{1}{2}}(\Lambda\delta^{ij}\zeta_i\zeta_j)^{\frac{1}{2}} \\ &\le \frac{1}{2}\left(|\partial_t\varphi|^2 + a^{ij}\partial_i\varphi\partial_j\varphi\right)\sqrt{\Lambda}.\end{aligned}$$

So the last integral in [**2.6**] is nonpositive and can be discarded in our estimate. Using Hölder's inequality we can estimate

$$\begin{aligned}\partial_t\tilde{E}(t) \le& \|F(t,\cdot)\|_{L^2(S_t)}\|\partial_t\varphi(t,\cdot)\|_{L^2(S_t)} + CK\int_{S_t} e(\varphi)dx \\ \le& \|F(t,\cdot)\|_{L^2(S_t)}\tilde{E}(t)^{\frac{1}{2}} + C(1+\lambda^{-1})K\tilde{E}(t).\end{aligned} \quad [\mathbf{2.7}]$$

Let $\varepsilon > 0$ and $\tilde{E}_\varepsilon = \tilde{E} + \varepsilon$. Since [**2.7**] also holds for $\tilde{E}_\varepsilon$ we can divide by $\tilde{E}_\varepsilon^{\frac{1}{2}} > 0$ to get

$$\partial_t\tilde{E}_\varepsilon(t)^{\frac{1}{2}} \le \|F(t,\cdot)\|_{L^2(S_t)} + C\tilde{E}_\varepsilon(t)^{\frac{1}{2}}.$$

By Gronwall's inequality Lemma C.8 we conclude that

$$\tilde{E}_\varepsilon(t)^{\frac{1}{2}} \le e^{\int_0^t Cdr}\left(\tilde{E}_\varepsilon(0)^{\frac{1}{2}} + \int_0^t e^{-\int_0^{t'} Cds}\|F(t',\cdot)\|_{L^2(S_{t'})}dt'\right).$$

Letting $\varepsilon \to 0$ and comparing $E(t) \le (1+\lambda^{-1})\tilde{E}(t)$ and $\tilde{E}(0) \le (1+\Lambda)E(0)$ we get the stated result. $\square$

2.3.4. ODE Estimate.

LEMMA 2.10. *If $\Omega \subset \mathbb{R}^n$ is open and bounded and $\psi : [0,T]\times\Omega \to \mathbb{R}^d$ is smooth and satisfies $\partial_t^2\psi^k = w^k$ then*

$$\begin{aligned}\|\psi(t,\cdot)\|_{H^1(\Omega)} + \|\partial_t\psi(t,\cdot)\|_{H^1(\Omega)} \le Ce^{Ct}\Big(&\|\psi(0,\cdot)\|_{H^1(\Omega)} + \|\partial_t\psi(0,\cdot)\|_{H^1(\Omega)} \\ &+ \int_0^t e^{-Ct'}\|w(t',\cdot)\|_{H^1(\Omega)}dt'\Big).\end{aligned}$$

PROOF. Define

$$E(t) = \frac{1}{2}\int_\Omega |\psi|^2 + |\partial_t\psi|^2 + |D\psi|^2 + |\partial_t D\psi|^2 dx$$

and compute using Hölder's inequality and Cauchy's inequality

$$\begin{aligned}\partial_t E =& \int_\Omega \delta_{ij}\psi^i\partial_t\psi^j + \delta_{ij}\partial_t\psi^i\partial_t^2\psi^j + \delta_{ij}\delta^{kl}\partial_k\psi^i\partial_t\partial_l\psi^j + \delta_{ij}\delta^{kl}\partial_t\partial_k\psi^i\partial_l\partial_t^2\psi^j dx\\ \leq& C\Bigg(\int_\Omega |\psi|^2 + |\partial_t\psi|^2 + |D\psi|^2 + |\partial_t D\psi|^2 dx\\ &+ \|\partial_t\psi\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} + \|\partial_t D\psi\|_{L^2(\Omega)}\|Dw\|_{L^2(\Omega)}\Bigg)\\ \leq& C(E + E^{\frac{1}{2}}\|w\|_{H^1(\Omega)}). \end{aligned} \qquad [\mathbf{2.8}]$$

Let $\varepsilon > 0$ and $E_\varepsilon = E + \varepsilon$. Inequality [**2.8**] also holds for E_ε and dividing by $E_\varepsilon^{\frac{1}{2}} > 0$ we get

$$\partial_t E_\varepsilon(t)^{\frac{1}{2}} \leq C(E_\varepsilon(t)^{\frac{1}{2}} + \|w(t,\cdot)\|_{H^1(\Omega)}).$$

By Gronwall's inequality Lemma C.8

$$E_\varepsilon(t)^{\frac{1}{2}} \leq Ce^{Ct}\left(E_\varepsilon(0)^{\frac{1}{2}} + \int_0^t e^{-Ct'}\|w(t',\cdot)\|_{H^1(\Omega)}dt'\right).$$

Letting $\varepsilon \to 0$ and using the equivalence of $E(t)^{\frac{1}{2}}$ and $\|\psi(t,\cdot)\|_{H^1(\Omega)} + \|\partial_t\psi(t,\cdot)\|_{H^1(\Omega)}$ this implies the result. ☐

2.3.5. Elliptic Estimate.

Lemma 2.11. *Let $\Omega \subset\subset B_2(0)$ be open and let*

$$L\varphi = a^{ij}\partial_i\partial_j\varphi + a^i\partial_i\varphi + a\varphi = F$$

on $B_2(0)$ with $a^{ij} = a^{ji}$, $a^{ij}\xi_i\xi_j \geq \lambda\delta^{ij}\xi_i\xi_j$ for some $\lambda > 0$ and F, φ and all coefficients be smooth with bounded derivatives.

(1) *If $[L]_{H^s(B_2(0))} \leq K$ for some $s \geq \lfloor\frac{n}{2}\rfloor + 2$ and $K > 0$ then there is a constant C depending only on K, λ, s and on Ω such that*

$$\|\varphi\|_{H^{s+2}(\Omega)} \leq C\left(\|F\|_{H^s(B_2(0))} + \|\varphi\|_{H^{s+1}(B_2(0))}\right).$$

(2) *If $[L]_{C^1(B_2(0))} \leq K$ and $\|\varphi\|_{C^2(B_2(0))} \leq K'$ for some $K, K' > 0$ then for every $s \geq 0$ there is a constant C depending only on K, K', λ, s and on Ω such that*

$$\|\varphi\|_{H^{s+2}(\Omega)} \leq C\left(\|F\|_{H^s(B_2(0))} + [L]_{H^s(B_2(0))} + \|\varphi\|_{H^{s+1}(B_2(0))}\right).$$

Proof. For $s = 0$ part (2) is a simple corollary of the standard elliptic regularity estimate [**Eva98**, Theorem 1, 6.3.1] and would be much easier to prove directly with our smoothness assumptions. Note that by the Sobolev embedding theorem we have $[L]_{C^1(B_2(0))} \leq C\,[L]_{H^{\lfloor\frac{n}{2}\rfloor+2}(B_2(0))}$. So $[L]_{C^1(B_2(0))} \leq C_K$ in both cases. For $s > 0$ let ∂^β be a spatial derivative with $|\beta| \leq s$. Differentiating the equation yields

$$L\partial^\beta\varphi = \partial^\beta F - (\partial^\beta(L\varphi) - L\partial^\beta\varphi) =: \tilde{F}.$$

Applying the estimate [**Eva98**, Theorem 1, 6.3.1] yields

$$\|\partial^\beta \varphi\|_{H^2(\Omega)} \leq C\left(\|\tilde{F}\|_{L^2(B_2(0))} + \|\varphi\|_{H^{s+1}(B_2(0))}\right)$$

with C depending only on K, λ and Ω. We can estimate by the second Moser inequality Theorem C.4

$$\|\partial^\beta(L\varphi) - L\partial^\beta\varphi\|_{L^2(B_2(0))} \leq C\left([L]_{H^s(B_2(0))}\|\varphi\|_{C^2(B_2(0))} + [L]_{C^1(B_2(0))}\|\varphi\|_{H^{s+1}(B_2(0))}\right).$$

Hence for the first assertion we estimate

$$\|\tilde{F}\|_{L^2(B_2(0))} \leq C\left(\|F\|_{H^s(B_2(0))} + \|\varphi\|_{C^2(B_2(0))} + \|\varphi\|_{H^{s+1}(B_2(0))}\right)$$

and use the Sobolev embedding theorem to estimate $\|\varphi\|_{C^2(B_2(0))} \leq C\|\varphi\|_{H^{s+1}(B_2(0))}$. For the second assertion we estimate

$$\|\tilde{F}\|_{L^2(B_2(0))} \leq C\Big(\|F\|_{H^s(B_2(0))} + [L]_{H^s(B_2(0))} + \|\varphi\|_{H^{s+1}(B_2(0))}\Big). \qquad \square$$

REMARK 2.12. If we do not use the assumption that $\|\varphi\|_{C^2(B_2(0))} \leq K'$ in the second part then we get the estimate

$$\|\varphi\|_{H^{s+2}(\Omega)} \leq C\left(\|F\|_{H^s(B_2(0))} + [L]_{H^s(B_2(0))}\|\varphi\|_{C^2(B_2(0))} + \|\varphi\|_{H^{s+1}(B_2(0))}\right).$$

2.3.6. Estimates for the Full System. To combine the estimate from the previous subsections we define the total energy of the system [**2.4**] as

$$E_s(t) = \|\partial_t^2\varphi(t,\cdot)\|_s + \|\partial_t\varphi(t,\cdot)\|_{s+1} + \|\varphi(t,\cdot)\|_{s+1} + \|\partial_t\psi(t,\cdot)\|_{s+1} + \|\psi(t,\cdot)\|_{s+1}. \quad [\mathbf{2.9}]$$

PROPOSITION 2.13. *Assume that φ, ψ satisfy the weakly hyperbolic linear system* [**2.4**] *on a time interval $[0,T]$ and that for some $s \geq \lfloor \frac{n}{2} \rfloor + 2$ and $K_1, K_2, \lambda_1 > 0$*

$$\|\nu\|_{s+1} + \|\partial_t\nu\|_{s+1} + \|\partial_t^2\nu\|_s + \|\tau\|_{s+1} + \|\partial_t\tau\|_{s+1} \leq K_1,$$
$$1 + [L]_s + [\partial_t L]_s + [M]_{s+1} + [Q]_s + [\partial_t Q]_s + [N]_s + [\partial_t N]_s + [P]_{s+1} \leq K_2$$

and

$$\det(\nu_{AB}) > \lambda_1, \quad \det(\tau_{kl}) > \lambda_1.$$

Then we have the estimate

$$E_s(t) \leq Ce^{Ct}E_s(0) + C\int_0^t e^{C(t-t')}\big(\|v(t',\cdot)\|_s + \|\partial_t v(t',\cdot)\|_s + \|w(t',\cdot)\|_{s+1}\big)dt'.$$

where C only depends on K_1, K_2, λ, λ_1, Λ and s.

PROOF. *1.* Set $B_1 = B_1(0) \subset \mathbb{R}^n$ and $B = B_2(0) \subset \mathbb{R}^n$. Choose $(t_0, x_0) = (\frac{3}{2\sqrt{\Lambda}}, 0)$. Recall from Subsection 2.3.3 that $S_t = \{x \in \mathbb{R}^n, |x - x_0| < \sqrt{\Lambda}(t_0 - t)\}$. Then we always have $S_t \subset B$. We also have $B_1(0) \subset S_t$ if $t \leq \frac{1}{2\sqrt{\Lambda}}$ because then $\sqrt{\Lambda}(t_0 - t) \geq \frac{3}{2} - \frac{1}{2} = 1 > |x|$ for $x \in B_1(0)$. We also choose a set $\Omega \subset\subset B$ such that

$S_t \subset\subset \Omega$ in order to avoid constants depending on S_t, e. g. $\Omega = B_{\frac{7}{4}}(0)$. We will prove the estimate first for $t \le t^* := \frac{1}{2\sqrt{\Lambda}}$. When we apply Proposition 2.9 we use that

$$1 + \sum_{\alpha,i,j} \|\partial_\alpha a^{ij}\|_{C^0(\Omega)} + \sum_k \|a^k\|_{C^0(\Omega)} + \|a\|_{C^0(\Omega)} \le 1 + [L]_{C^1} + [\partial_t L]_{C^1} \le K_2$$

by the Sobolev embedding theorem since $s \ge \lfloor \frac{n}{2} \rfloor + 2$.

2. We write the system [**2.4**] as

$$\partial_t^2 \varphi^A - L^A \varphi^A = v^A + N^A \psi + Q^A \varphi \qquad [\mathbf{2.10a}]$$

$$\partial_t^2 \psi^k = w^k + P^k \varphi + M^k \psi. \qquad [\mathbf{2.10b}]$$

Since $s \ge \lfloor \frac{n}{2} \rfloor + 2$ we can use the Sobolev embedding theorem to estimate the C^0- and C^1-norms of the coefficients of the operators by CK_2. Clearly $\|q^{A1} \sum_\beta \int c^A_{(\beta)B} \varphi^B_{(\alpha)} d\mu_0\|_{L^2(S_{t'})} \le C\|\varphi\|_{L^2}$ by Hölder's inequality and $\|Q^A\varphi\|_{L^2(B)} \le C(\|\varphi\|_1 + \|\partial_t \varphi\|_0)$. Whenever we estimate $Q\varphi$, we need a "global" term including $\|\varphi\|_{L^2}$ or $\|\varphi\|_{C^0}$ because of the integrals. The same applies for estimates of $N\psi$. If we do not intend to apply the elliptic regularity estimate we will carry out the estimates on the larger domain Ω instead of S_t. Hence we can estimate the $L^2(S_{t'})$-norm of the right hand side of the wave part [**2.10a**] by

$$C(\|v\|_{L^2(B)} + \|\psi\|_1 + \|\partial_t \psi\|_0 + \|\varphi\|_1 + \|\partial_t \varphi\|_0)$$

and the $H^1(\Omega)$-norm of the right hand side of the ODE part [**2.10b**] by

$$C\left(\|w\|_{H^1(B)} + \|\psi\|_{H^1(B)} + \|\partial_t \psi\|_{H^1(B)} + \|\varphi\|_{H^2(\Omega)} + \|\partial_t \varphi\|_{H^1(B)}\right).$$

By the elliptic regularity estimate [**Eva98**, Theorem 1, 6.3.1] and the equation [**2.10a**] for $L\varphi$ we have

$$\begin{aligned} \|\varphi\|_{H^2(\Omega)} \le & C\left(\|L\varphi\|_{L^2(B)} + \|\varphi\|_{H^1(B)}\right) \\ \le & C\left(\|\partial_t^2 \varphi\|_{L^2(B)} + \|v\|_{L^2(B)} + \|\psi\|_1 + \|\partial_t \psi\|_0 + \|\varphi\|_1 + \|\partial_t \varphi\|_0\right). \end{aligned} \qquad [\mathbf{2.11}]$$

Applying the energy estimate Proposition 2.9 and the ODE estimate Lemma 2.10 to equations [**2.10a**] and [**2.10b**] we get

$$\begin{aligned} &\|\partial_t \varphi^A(t,\cdot)\|_{L^2(B_1)} + \|D\varphi^A(t,\cdot)\|_{L^2(B_1)} + \|\varphi^A(t,\cdot)\|_{L^2(B_1)} \\ &\quad \le Ce^{Ct}\Big(\|\partial_t \varphi^A(0,\cdot)\|_{L^2(B)} + \|D\varphi^A(0,\cdot)\|_{L^2(B)} + \|\varphi^A(0,\cdot)\|_{L^2(B)}\Big) \\ &\qquad + C\int_0^t e^{C(t-t')} \left(\|v\|_0 + \|\psi\|_1 + \|\partial_t \psi\|_1 + \|\varphi\|_1 + \|\partial_t \varphi\|_0\right) dt' \end{aligned} \qquad [\mathbf{2.12}]$$

and

$$\begin{aligned} \|\psi^k(t,\cdot)\|_{H^1(B_1)} + \|\partial_t \psi^k(t,\cdot)\|_{H^1(B_1)} \le Ce^{Ct}\left(\|\psi^k(0,\cdot)\|_{H^1(B)} + \|\partial_t \psi^k(0,\cdot)\|_{H^1(B)}\right) \\ + C\int_0^t e^{C(t-t')}\left(\|w\|_1 + \|v\|_0 + \|\partial_t^2 \varphi\|_0 + \|\partial_t \varphi\|_1 + \|\varphi\|_1 + \|\psi\|_1 + \|\partial_t \psi\|_1\right) dt'. \end{aligned} \qquad [\mathbf{2.13}]$$

3. In order to use Gronwall's inequality later we also need the terms $\|\partial_t^2\varphi\|_{L^2(B_1)} + \|\partial_t\varphi\|_{H^1(B_1)}$ on the left hand side of the estimate. Therefore we differentiate the wave part [**2.10a**] with respect to time

$$\partial_t^2\partial_t\varphi^A - L^A\partial_t\varphi^A = \partial_t v^A + \partial_t L^A\varphi^A + \partial_t N^A\psi + N^A\partial_t\psi + \partial_t Q^A\varphi + Q^A\partial_t\varphi.$$

For the application of the basic energy estimate we estimate the $L^2(S_{t'})$-norm of the right hand side by

$$C\left(\|\partial_t v\|_{L^2(B)} + \|\varphi\|_{H^2(\Omega)} + \|\psi\|_1 + \|\partial_t\psi\|_1 + \|\partial_t^2\psi\|_{L^2(B)} + \|\varphi\|_1 + \|\partial_t\varphi\|_1 + \|\partial_t^2\varphi\|_0\right).$$

The term $\|\varphi\|_{H^2(\Omega)}$ is estimated as in [**2.11**]. By the equation for $\partial_t^2\psi$ [**2.10b**] we have

$$\|\partial_t^2\psi\|_{L^2(B)} \le C\left(\|\partial_t\psi\|_{L^2(B)} + \|\psi\|_{L^2(B)} + \|\varphi\|_{H^1(B)} + \|\partial_t\varphi\|_{L^2(B)} + \|w\|_{L^2(B)}\right).$$

An application of the basic energy estimate Proposition 2.9 yields

$$\begin{aligned}
&\|\partial_t^2\varphi^A(t,\cdot)\|_{L^2(B_1)} + \|\partial_t D\varphi^A(t,\cdot)\|_{L^2(B_1)} + \|\partial_t\varphi(t,\cdot)\|_{L^2(B_1)} \\
&\qquad \le Ce^{Ct}\left(\|\partial_t^2\varphi^A(0,\cdot)\|_{L^2(B)} + \|\partial_t D\varphi^A(0,\cdot)\|_{L^2(B)} + \|\partial_t\varphi^A(0,\cdot)\|_{L^2(B)}\right) \\
&\qquad\qquad + C\int_0^t e^{C(t-t')}\big(\|\partial_t v\|_0 + \|v\|_0 + \|w\|_0 + \|\partial_t\psi\|_1 + \|\psi\|_1 \\
&\qquad\qquad\qquad + \|\partial_t^2\varphi\|_0 + \|\varphi\|_1 + \|\partial_t\varphi\|_1\big)dt'. \qquad [\mathbf{2.14}]
\end{aligned}$$

4. Let β be a multiindex with $1 \le |\beta| \le s$ and ∂^β a be spatial derivative. Note that by the Sobolev embedding theorem we have $\|\cdot\|_{C^k} \le C\|\cdot\|_{\lfloor\frac{n}{2}\rfloor+1+k}$. Differentiating the system [**2.10**] in a coordinate chart yields

$$\begin{aligned}
\partial_t^2\partial^\beta\varphi^A - L^A\partial^\beta\varphi^A &= \partial^\beta v^A + \partial^\beta(L^A\varphi^A) - L^A\partial^\beta\varphi^A + \partial^\beta(N^A\psi) + \partial^\beta(Q^A\varphi) =: \tilde{v}^A \\
\partial_t^2\partial^\beta\psi^k &= \partial^\beta w^k + \partial^\beta(M^k\psi) + \partial^\beta(P^k\varphi) =: \tilde{w}^k.
\end{aligned} \qquad [\mathbf{2.15}]$$

We want to apply again the basic energy estimate Proposition 2.9 and the ODE estimate Lemma 2.10 to this system and hence we must estimate the terms $\|\tilde{v}\|_{L^2(S_{t'})}$ and $\|\tilde{w}\|_{H^1(\Omega)}$. By the second Moser inequality Theorem C.4 and the Sobolev embedding theorem we have

$$\begin{aligned}
\|\partial^\beta(L\varphi) - L\partial^\beta\varphi\|_{L^2(B)} \le& C\left([L]_{H^s(B)}\|\varphi\|_{C^2(B)} + [L]_{C^1(B)}\|\varphi\|_{H^{s+1}(B)}\right) \\
\le& C\|\varphi\|_{H^{s+1}(B)}. \qquad [\mathbf{2.16}]
\end{aligned}$$

By the first Moser inequality C.3 and the Sobolev embedding theorem we have

$$\begin{aligned}
\|\partial^\beta(N\psi)\|_{L^2(B)} \le& C\big([N]_s(\|\psi\|_{C^1} + \|\partial_t\psi\|_{C^0}) + [N]_{C^0}(\|\psi\|_{s+1} + \|\partial_t\psi\|_s)\big) \\
\le& C(\|\psi\|_{s+1} + \|\partial_t\psi\|_{s+1}). \qquad [\mathbf{2.17}]
\end{aligned}$$

Note that we could estimate the integrals by

$$\|\partial^\beta(n^{A1}\sum_{\beta=1}^J\int_{\mathcal{N}} b^A_{(\beta)j}\psi^j_{(\beta)}d\mu_0)\|_{L^2(B)} = \|\partial^\beta(n^{A1})\sum_{\beta=1}^J\int_{\mathcal{N}} b^A_{(\beta)j}\psi^j_{(\beta)}d\mu_0\|_{L^2(B)} \le C\|\psi\|_{C^0}[N]_s.$$

Similarly

$$\|\partial^\beta(Q\varphi)\|_{L^2(B)} \leq C\big(\|\varphi\|_{s+1} + \|\partial_t\varphi\|_s\big) \qquad [\mathbf{2.18}]$$

and trivially $\|\partial^\beta v\|_{L^2(B)} \leq \|v\|_{H^s(B)}$. Hence

$$\|\tilde v\|_{L^2(S_{t'})} \leq C\left(\|v\|_{H^s(B)} + \|\varphi\|_{s+1} + \|\partial_t\varphi\|_{s+1} + \|\psi\|_{s+1} + \|\partial_t\psi\|_{s+1}\right).$$

Application of Proposition 2.9 implies

$$\begin{aligned}
&\|\partial_t\partial^\beta\varphi^A(t,\cdot)\|_{L^2(B_1)} + \|D\partial^\beta\varphi^A(t,\cdot)\|_{L^2(B_1)} + \|\partial^\beta\varphi^A(t,\cdot)\|_{L^2(B_1)} \\
&\qquad \leq Ce^{Ct}\big(\|\partial_t\partial^\beta\varphi^A(0,\cdot)\|_{L^2(B)} + \|D\partial^\beta\varphi^A(0,\cdot)\|_{L^2(B)} + \|\partial^\beta\varphi^A(0,\cdot)\|_{L^2(B)}\big) \\
&\qquad\qquad + C\int_0^t e^{C(t-t')}\big(\|v\|_s + \|\varphi\|_{s+1} + \|\partial_t\varphi\|_{s+1} + \|\psi\|_{s+1} + \|\partial_t\psi\|_{s+1}\big)dt'. \qquad [\mathbf{2.19}]
\end{aligned}$$

We estimate similarly

$$\begin{aligned}
\|\partial^\beta(M\psi)\|_{H^1(B)} \leq& C\big(\,[M]_{H^{s+1}(B)}\,(\|\partial_t\psi\|_{C^0(B)} + \|\psi\|_{C^0(B)}) \\
&\quad + [M]_{C^0(B)}\,(\|\partial_t\psi\|_{H^{s+1}(B)} + \|\psi\|_{H^{s+1}(B)})\big) \\
\leq& C\left(\|\psi\|_{H^{s+1}(B)} + \|\partial_t\psi\|_{H^{s+1}(B)}\right) \qquad [\mathbf{2.20}]
\end{aligned}$$

and

$$\begin{aligned}
\|\partial^\beta(P\varphi)\|_{H^1(\Omega)} \leq& C\big(\,[P]_{H^{s+1}(\Omega)}\,(\|\partial_t\varphi\|_{C^0(\Omega)} + \|\varphi\|_{C^1(\Omega)}) \\
&\quad + [P]_{C^0(\Omega)}\,(\|\varphi\|_{H^{s+2}(\Omega)} + \|\partial_t\varphi\|_{H^{s+1}(\Omega)})\big) \\
\leq& C\left(\|\varphi\|_{H^{s+2}(\Omega)} + \|\partial_t\varphi\|_{H^{s+1}(\Omega)}\right). \qquad [\mathbf{2.21}]
\end{aligned}$$

By the elliptic estimate Lemma 2.11 part (1) and equation [**2.10a**]

$$\begin{aligned}
\|\varphi\|_{H^{s+2}(\Omega)} \leq& C\left(\|L\varphi\|_{H^s(B)} + \|\varphi\|_{H^{s+1}(B)}\right) \\
\leq& C\left(\|\partial_t^2\varphi\|_{H^s(B)} + \|v\|_{H^s(B)} + \|N\psi\|_{H^s(B)} + \|Q\varphi\|_{H^s(B)} + \|\varphi\|_{H^{s+1}(B)}\right) \\
\leq& C\big(\|v\|_s + \|\partial_t^2\varphi\|_s + \|\psi\|_{s+1} + \|\partial_t\psi\|_{s+1} + \|\varphi\|_{s+1} + \|\partial_t\varphi\|_s\big) \qquad [\mathbf{2.22}]
\end{aligned}$$

where $\|N\psi\|_{H^s(B)}$ and $\|Q\varphi\|_{H^s(B)}$ were estimated as in [**2.17**] and [**2.18**]. Hence

$$\begin{aligned}
\|\tilde w\|_{H^1(\Omega)} \leq C\big(&\|w\|_{H^{s+1}(B)} + \|v\|_{H^s(B)} + \|\partial_t^2\varphi\|_s + \|\partial_t\varphi\|_{s+1} \\
&+ \|\varphi\|_{s+1} + \|\psi\|_{s+1} + \|\partial_t\psi\|_{s+1}\big).
\end{aligned}$$

Thus the ODE estimate Lemma 2.10 implies

$$\begin{aligned}
&\|\partial^\beta\psi^k(t,\cdot)\|_{H^1(B_1)} + \|\partial^\beta\partial_t\psi^k(t,\cdot)\|_{H^1(B_1)} \leq Ce^{Ct}\big(\|\partial^\beta\psi^k(0,\cdot)\|_{H^1(B)} + \|\partial^\beta\partial_t\psi^k(0,\cdot)\|_{H^1(B)}\big) \\
&\qquad + C\int_0^t e^{C(t-t')}\big(\|w\|_{s+1} + \|v\|_s + \|\partial_t^2\varphi\|_s + \|\partial_t\varphi\|_{s+1} + \|\varphi\|_{s+1} \\
&\qquad\qquad + \|\psi\|_{s+1} + \|\partial_t\psi\|_{s+1}\big)dt'. \qquad [\mathbf{2.23}]
\end{aligned}$$

5. As in step 3 we need an additional time derivative and so we differentiate the wave part of the system [**2.10a**] using $\partial_t\partial^\beta$ in order to get an estimate for $\partial_t^2\varphi$. We calculate

$$\begin{aligned}\partial_t^2\partial_t\partial^\beta\varphi^A - L^A\partial_t\partial^\beta\varphi^A =&\partial_t\partial^\beta v^A + \partial^\beta(\partial_t L^A\varphi^A) + \partial^\beta(L^A\partial_t\varphi^A) - L^A\partial_t\partial^\beta\varphi^A \\ &+ \partial^\beta(\partial_t Q^A\varphi) + \partial^\beta(Q^A\partial_t\varphi) + \partial^\beta(\partial_t N^A\psi) + \partial^\beta(N^A\partial_t\psi) =: \tilde{\tilde{v}}^A\end{aligned}$$

and we will estimate $\|\tilde{\tilde{v}}\|_{L^2(S_{t'})}$. To this end

$$\begin{aligned}\|\partial^\beta(\partial_t L\varphi)\|_{L^2(\Omega)} \leq& C\left([\partial_t L]_{H^s(B)}\|\varphi\|_{C^2(B)} + [\partial_t L]_{C^0(B)}\|\varphi\|_{H^{s+2}(\Omega)}\right) \\ \leq& C\big(\|v\|_s + \|\partial_t^2\varphi\|_s + \|\partial_t\varphi\|_s + \|\psi\|_{s+1} + \|\partial_t\psi\|_{s+1} + \|\varphi\|_{s+1}\big) \quad [\mathbf{2.24}]\end{aligned}$$

using the first Moser inequality and [**2.22**]. Futhermore by the first and second Moser inequality and the Sobolev embedding theorem

$$\begin{aligned}\|\partial^\beta(L\partial_t\varphi) - L\partial^\beta\partial_t\varphi\|_{L^2(B)} \leq& C\left([L]_{H^s(B)}\|\partial_t\varphi\|_{C^2(B)} + [L]_{C^1(B)}\|\partial_t\varphi\|_{H^{s+1}(B)}\right) \\ \leq& C\|\partial_t\varphi\|_{H^{s+1}(B)} \quad [\mathbf{2.25}] \\ \|\partial^\beta(\partial_t N\psi)\|_{L^2(B)} \leq& C\big([\partial_t N]_s(\|\psi\|_{C^1} + \|\partial_t\psi\|_{C^0}) + [\partial_t N]_{C^0}(\|\psi\|_{s+1} + \|\partial_t\psi\|_s)\big) \\ \leq& C(\|\psi\|_{s+1} + \|\partial_t\psi\|_{s+1}) \quad [\mathbf{2.26}] \\ \|\partial^\beta(N\partial_t\psi)\|_{L^2(B)} \leq& C\big([N]_s(\|\partial_t\psi\|_{C^1} + \|\partial_t^2\psi\|_{C^0(B)}) \\ &+ [N]_{C^0}(\|\partial_t\psi\|_{s+1} + \|\partial_t^2\psi\|_{H^s(B)})\big) \\ \leq& C\left(\|\partial_t\psi\|_{s+1} + \|\partial_t^2\psi\|_{H^s(B)}\right) \quad [\mathbf{2.27}]\end{aligned}$$

and analogously

$$\begin{aligned}\|\partial^\beta(\partial_t Q\varphi)\|_{L^2(B)} \leq& C(\|\partial_t\varphi\|_s + \|\varphi\|_{s+1}) \quad [\mathbf{2.28}] \\ \|\partial^\beta(Q\partial_t\varphi)\|_{L^2(B)} \leq& C(\|\partial_t\varphi\|_{s+1} + \|\partial_t^2\varphi\|_s). \quad [\mathbf{2.29}]\end{aligned}$$

We use the ODE part [**2.10b**] to estimate

$$\begin{aligned}\|\partial_t^2\psi\|_{H^s(B)} \leq& C\left(\|w\|_{H^s(B)} + \|M\psi\|_{H^s(B)} + \|P\varphi\|_{H^s(B)}\right). \\ \leq& C\big(\|w\|_{H^s(B)} + \|\partial_t\psi\|_{H^{s+1}(B)} + \|\psi\|_{H^{s+1}(B)} \\ &+ \|\varphi\|_{H^{s+1}(B)} + \|\partial_t\varphi\|_{H^s(B)}\big). \quad [\mathbf{2.30}]\end{aligned}$$

Consequently

$$\begin{aligned}\|\tilde{\tilde{v}}\|_{L^2(B)} \leq C\big(&\|\partial_t v\|_{H^s(B)} + \|v\|_{H^s(B)} + \|w\|_{H^s(B)} + \|\psi\|_{s+1} \\ &+ \|\partial_t\psi\|_{s+1} + \|\varphi\|_{s+1} + \|\partial_t\varphi\|_{s+1} + \|\partial_t^2\varphi\|_s\big).\end{aligned}$$

We apply again the basic energy estimate Proposition 2.9 to get

$$\begin{aligned}
&\|\partial_t^2\partial^\beta\varphi(t,\cdot)\|_{L^2(B_1)} + \|D\partial_t\partial^\beta\varphi(t,\cdot)\|_{L^2(B_1)} + \|\partial_t\partial^\beta\varphi(t,\cdot)\|_{L^2(B_1)} \\
&\qquad \leq Ce^{Ct}\big(\|\partial_t^2\partial^\beta\varphi(0,\cdot)\|_{L^2(B)} + \|D\partial_t\partial^\beta\varphi(0,\cdot)\|_{L^2(B)} + \|\partial_t\partial^\beta\varphi(0,\cdot)\|_{L^2(B)}\big) \\
&\qquad\quad + C\int_0^t e^{C(t-t')}\big(\|\partial_t v\|_s + \|v\|_s + \|w\|_s + \|\psi\|_{s+1} + \|\partial_t\psi\|_{s+1} \\
&\qquad\qquad + \|\varphi\|_{s+1} + \|\partial_t\varphi\|_{s+1} + \|\partial_t^2\varphi\|_s\big)dt'. \quad [\mathbf{2.31}]
\end{aligned}$$

6. We sum the estimates [**2.12**],[**2.13**],[**2.14**],[**2.19**],[**2.23**] and [**2.31**] over all coordinate charts and over all β with $1 \leq |\beta| \leq s$ and we use Lemma 2.7 to obtain

$$E_s(t) \leq Ce^{Ct}E_s(0) + C\int_0^t e^{C(t-t')}\big(\|v\|_s + \|\partial_t v\|_s + \|w\|_{s+1}\big)dt' + C\int_0^t e^{C(t-t')}E_s(t')dt'.$$

We will apply the Gronwall type inequality from Lemma C.10 with $A(t) = E_s(t)$, $g(t) = e^{Ct}$, $h(t') = Ce^{-Ct'}$ and

$$B(t) = CE_s(0) + C\int_0^t e^{-Ct'}\big(\|v\|_s + \|\partial_t v\|_s + \|w\|_{s+1}\big)dt'.$$

We have that $B'(t') = Ce^{-Ct'}\big(\|v\|_s + \|\partial_t v\|_s + \|w\|_{s+1}\big)$ and $h(r)g(r) = C$. Consequently Lemma C.10 implies

$$E_s(t) \leq Ce^{Ct}E_s(0) + e^{Ct}\int_0^t e^{C(t-t')}Ce^{-Ct'}\big(\|v\|_s + \|\partial_t v\|_s + \|w\|_{s+1}\big)dt'$$

which is after adapting the constant C the result for $t \leq \frac{1}{2\sqrt{\Lambda}} = t^*$.

7. Now let $t \leq T$ be arbitrary. We write $t = kt^* + \tilde{t}$ with $t^* = \frac{1}{2\sqrt{\Lambda}}$ and $\tilde{t} \leq t^*$, $k \in \mathbb{N}$. We shall iterate the estimate finitely many times to estimate $E_s(t)$. For a moment we will denote the constant C which has been used in the estimate for $t \leq t^*$ by C_1. Clearly $C_1 \geq 1$. We estimate

$$\begin{aligned}
E_s(t) =& E_s(kt^* + \tilde{t}) \leq C_1e^{C(t-kt^*)}E_s(kt^*) + C_1\int_{kt^*}^{kt^*+\tilde{t}} e^{C(t-t')}\dots dt' \\
\leq& C_1^2e^{C(t-(k-1)t^*)}E_s((k-1)t^*) + C_1^2\int_{(k-1)t^*}^{kt^*} e^{C(t-t')}\dots dt' + C_1\int_{kt^*}^{kt^*+\tilde{t}} e^{C(t-t')}\dots dt' \\
\leq& \dots \leq C_1^{k+1}e^{Ct}E_s(0) + \sum_{i=0}^{k-1} C_1^{k+1-i}\int_{it^*}^{(i+1)t^*} e^{C(t-t')}\dots dt' + C_1\int_{kt^*}^{kt^*+\tilde{t}} e^{C(t-t')}\dots dt'.
\end{aligned}$$

Now by definition $k \leq \frac{t}{t^*} \leq k+1$ and $it^* \leq t' \leq (i+1)t^*$ in every integral. Hence we can estimate

$$k - i \leq \frac{1}{t^*}(t - (i+1)t^*) + 1 \leq \frac{1}{t^*}(t - t') + 1$$

and consequently in every summand it holds $C_1^{k+1-i} \leq C_1^2 C_1^{(t-t')/t^*}$ and in the first term $C_1^{k+1} \leq C_1 C_1^{t/t^*}$. Writing $C_1^x = e^{x \log C_1}$ we obtain the estimate in the desired form. □

PROPOSITION 2.14. *Assume that φ, ψ satisfy the weakly hyperbolic system* [**2.4**] *on a time interval $[0,T]$ and that for some $K_1, K_2, \lambda_1 > 0$*

$$
\begin{gathered}
\|\nu\|_{C^0} + \|\partial_t \nu\|_{C^0} + \|\partial_t^2 \nu\|_{C^0} + \|\tau\|_{C^0} + \|\partial_t \tau\|_{C^0} \leq K_1, \\
1 + [L]_{C^1} + [\partial_t L]_{C^0} + [M]_{C^1} + [Q]_{C^0} + [\partial_t Q]_{C^0} + [N]_{C^0} + [\partial_t N]_{C^0} + [P]_{C^1} \leq K_2
\end{gathered}
$$

and

$$
\det(\nu_{AB}) > \lambda_1, \quad \det(\tau_{kl}) > \lambda_1.
$$

Suppose further

$$
\|\varphi\|_{C^2} + \|\partial_t \varphi\|_{C^2} + \|\psi\|_{C^1} + \|\partial_t \psi\|_{C^1} \leq K_3
$$

and

$$
\|v\|_{C^0} + \|w\|_{C^0} \leq K_4
$$

for some $K_3, K_4 > 0$. Then for any $s \geq 0$ we have the estimate

$$
\begin{aligned}
E_s(t) \leq C e^{Ct} \sup_{[0,t]} \big(\|\partial_t^2 \nu\|_s + \|\partial_t \nu\|_{s+1} + \|\nu\|_{s+1} + \|\tau\|_{s+1} + \|\partial_t \tau\|_{s+1} + 1\big) + C e^{Ct} E_s(0) \\
+ C \int_0^t e^{C(t-t')} \Big(\|v\|_s + \|\partial_t v\|_s + \|w\|_{s+1} + [L]_s + [\partial_t L]_s \\
+ [Q]_s + [\partial_t Q]_s + [N]_s + [\partial_t N]_s + [M]_{s+1} + [P]_{s+1} \Big) dt'
\end{aligned}
$$

where C only depends on K_1, K_2, K_3, K_4, λ, λ_1, Λ and s.

PROOF. *1.* The strategy of the proof is very similar to the proof of Proposition 2.13. For the low order terms or if $s = 0$ we simply reuse the estimates [**2.12**],[**2.13**] and [**2.14**]. Then we consider the differentiated equations [**2.15**] again. We replace the estimates [**2.16**],[**2.17**] and [**2.18**] by

$$
\begin{aligned}
\|\partial^\beta (L\varphi) - L \partial^\beta \varphi\|_{L^2(B)} &\leq C\big([L]_{H^s(B)} \|\varphi\|_{C^2(B)} + [L]_{C^1(B)} \|\varphi\|_{H^{s+1}(B)} \big) \\
&\leq C([L]_{H^s(B)} + \|\varphi\|_{H^{s+1}(B)}) \qquad [\mathbf{2.32}] \\
\|\partial^\beta (N\psi)\|_{L^2(B)} &\leq C\big([N]_s (\|\psi\|_{C^1} + \|\partial_t \psi\|_{C^0}) + [N]_{C^0} (\|\psi\|_{s+1} + \|\partial_t \psi\|_s) \big) \\
&\leq C([N]_s + \|\psi\|_{s+1} + \|\partial_t \psi\|_{s+1}) \\
\|\partial^\beta (Q\varphi)\|_{L^2(B)} &\leq C\big([Q]_s (\|\varphi\|_{C^1} + \|\partial_t \varphi\|_{C^0}) + [Q]_{C^0} (\|\varphi\|_{s+1} + \|\partial_t \varphi\|_s) \big) \\
&\leq C\big([Q]_s + \|\varphi\|_{s+1} + \|\partial_t \varphi\|_s \big).
\end{aligned}
$$

We replace [**2.20**],[**2.21**] by

$$\begin{aligned}
\|\partial^\beta(M\psi)\|_{H^1(B)} \leq & C\big([M]_{H^{s+1}(B)} (\|\partial_t\psi\|_{C^0(B)} + \|\psi\|_{C^0(B)}) \\
& \quad + [M]_{C^0(B)} (\|\partial_t\psi\|_{H^{s+1}(B)} + \|\psi\|_{H^{s+1}(B)})\big) \\
\leq & C\big([M]_{H^{s+1}(B)} + \|\psi\|_{H^{s+1}(B)} + \|\partial_t\psi\|_{H^{s+1}(B)}\big) \\
\|\partial^\beta(P\varphi)\|_{H^1(\Omega)} \leq & C\big([P]_{H^{s+1}(\Omega)} (\|\partial_t\varphi\|_{C^0(\Omega)} + \|\varphi\|_{C^1(\Omega)}) \\
& \quad + [P]_{C^0(\Omega)} (\|\varphi\|_{H^{s+2}(\Omega)} + \|\partial_t\varphi\|_{H^{s+1}(\Omega)})\big) \\
\leq & C\big([P]_{H^{s+1}(\Omega)} + \|\varphi\|_{H^{s+2}(\Omega)} + \|\partial_t\varphi\|_{H^{s+1}(\Omega)}\big).
\end{aligned}$$

We use the elliptic estimate Lemma 2.11 part (2)

$$\|\varphi\|_{H^{s+2}(\Omega)} \leq C\big(\|L\varphi\|_{H^s(B)} + \|\varphi\|_{H^{s+1}(B)} + [L]_{H^s(B)}\big)$$

to obtain

$$\begin{aligned}
\|\varphi\|_{H^{s+2}(\Omega)} \leq C\big(&\|v\|_{H^s(B)} + \|\partial_t^2\varphi\|_s + \|\partial_t\varphi\|_s + \|\varphi\|_{s+1} + \|\psi\|_{s+1} + \|\partial_t\psi\|_{s+1} \\
&+ [L]_s + [N]_s + [Q]_s\big) \qquad [\mathbf{2.33}]
\end{aligned}$$

which replaces [**2.22**].

2. After differentiating the wave part [**2.10a**] additionally in time we replace the estimates [**2.24**],[**2.25**],[**2.26**],[**2.27**],[**2.28**],[**2.29**] by

$$\begin{aligned}
\|\partial^\beta(\partial_t L\varphi)\|_{L^2(\Omega)} \leq & C\left([\partial_t L]_{H^s(B)} \|\varphi\|_{C^2(B)} + [\partial_t L]_{C^0(B)} \|\varphi\|_{H^{s+2}(\Omega)}\right) \\
\leq & C\left(\|\varphi\|_{H^{s+2}(\Omega)} + [\partial_t L]_{H^s(B)}\right) \qquad [\mathbf{2.34}] \\
\|\partial^\beta(L\partial_t\varphi) - L\partial^\beta\partial_t\varphi\|_{L^2(B)} \leq & C\left([L]_{H^s(B)} \|\partial_t\varphi\|_{C^2(B)} + [L]_{C^1(B)} \|\partial_t\varphi\|_{H^{s+1}(B)}\right) \\
\leq & C\left(\|\partial_t\varphi\|_{H^{s+1}(B)} + [L]_{H^s(B)}\right) \qquad [\mathbf{2.35}] \\
\|\partial^\beta(\partial_t N\psi)\|_{L^2(B)} \leq & C\big([\partial_t N]_s (\|\psi\|_{C^1} + \|\partial_t\psi\|_{C^0}) + [\partial_t N]_{C^0} (\|\psi\|_{s+1} + \|\partial_t\psi\|_s)\big) \\
\leq & C\left([\partial_t N]_s + \|\psi\|_{s+1} + \|\partial_t\psi\|_{s+1}\right) \\
\|\partial^\beta(N\partial_t\psi)\|_{L^2(B)} \leq & C\big([N]_s (\|\partial_t\psi\|_{C^1} + \|\partial_t^2\psi\|_{C^0}) \\
& \quad + [N]_{C^0} (\|\partial_t\psi\|_{s+1} + \|\partial_t^2\psi\|_{H^s(B)})\big) \\
\leq & C\left([N]_s + \|\partial_t\psi\|_{s+1} + \|\partial_t^2\psi\|_{H^s(B)}\right) \\
\|\partial^\beta(\partial_t Q\varphi)\|_{L^2(B)} \leq & C\big([\partial_t Q]_s (\|\varphi\|_{C^1} + \|\partial_t\varphi\|_{C^0}) + [\partial_t Q]_{C^0} (\|\varphi\|_{s+1} + \|\partial_t\varphi\|_s)\big) \\
\leq & C\left([\partial_t Q]_s + \|\varphi\|_{s+1} + \|\partial_t\varphi\|_{s+1}\right) \\
\|\partial^\beta(Q\partial_t\varphi)\|_{L^2(B)} \leq & C\big([Q]_s (\|\partial_t\varphi\|_{C^1} + \|\partial_t^2\varphi\|_{C^0}) + [Q]_{C^0} (\|\partial_t\varphi\|_{s+1} + \|\partial_t^2\varphi\|_s)\big) \\
\leq & C\left([Q]_s + \|\partial_t\varphi\|_{s+1} + \|\partial_t^2\varphi\|_s\right).
\end{aligned}$$

We used that $\|\partial_t^2\varphi\|_{C^0} \le C$ and $\|\partial_t^2\psi\|_{C^0} \le C$ by the equations and the assumptions. We use [**2.33**] to estimate $\|\varphi\|_{H^{s+2}(\Omega)}$ and the ODE part [**2.10b**] to estimate

$$\begin{aligned}\|\partial_t^2\psi\|_{H^s(B)} \le& C(\|M\psi\|_{H^s(B)} + \|P\varphi\|_{H^s(B)} + \|w\|_{H^s(B)})\\ \le& C([M]_{H^s(B)} + \|\partial_t\psi\|_{H^s(B)} + \|\psi\|_{H^s(B)} + [P]_{H^s(B)}\\ &+ \|\varphi\|_{H^{s+1}(B)} + \|\partial_t\varphi\|_{H^s(B)} + \|w\|_{H^s(B)}).\end{aligned}$$

3. Now define $\tilde{E}_s$ to be the energy taken with norms on B_1, i. e.

$$\begin{aligned}\tilde{E}_s(t) =& \sum_{\alpha=1}^{J}\sum_{A=1}^{d'}\big(\|\varphi^A_{(\alpha)}\|_{H^{s+1}(B_1)} + \|\partial_t\varphi^A_{(\alpha)}\|_{H^{s+1}(B_1)} + \|\partial_t^2\varphi^A_{(\alpha)}\|_{H^s(B_1)}\big)\\ &+ \sum_{\alpha=1}^{J}\sum_{k=1}^{d''}\big(\|\psi^k_{(\alpha)}\|_{H^{s+1}(B_1)} + \|\partial_t\psi^k_{(\alpha)}\|_{H^{s+1}(B_1)}\big).\end{aligned}$$

Altogether we get, similarly to the proof of Proposition 2.13, that

$$\begin{aligned}\tilde{E}_s(t) \le& Ce^{Ct}E_s(0) + C\int_0^t e^{C(t-t')}E_s(t')dt'\\ &+ C\int_0^t e^{C(t-t')}\big(\|v\|_s + \|\partial_t v\|_s + \|w\|_{s+1} + [L]_s + [\partial_t L]_s + [Q]_s\\ &\qquad + [\partial_t Q]_s + [N]_s + [\partial_t N]_s + [M]_{s+1} + [P]_{s+1}\big)dt'\\ \le& Ce^{Ct}E_s(0) + C\int_0^t e^{C(t-t')}\tilde{E}_s(t')dt'\\ &+ C\int_0^t e^{C(t-t')}\big(\|v\|_s + \|\partial_t v\|_s + \|w\|_{s+1} + [L]_s + [\partial_t L]_s + [Q]_s\\ &\qquad + [\partial_t Q]_s + [N]_s + [\partial_t N]_s + [M]_{s+1} + [P]_{s+1}\\ &\qquad + \|\partial_t^2\nu\|_s + \|\partial_t\nu\|_{s+1} + \|\nu\|_{s+1} + \|\tau\|_{s+1} + \|\partial_t\tau\|_{s+1} + 1\big)dt'\end{aligned}$$

where we used Lemma 2.8 to estimate E_s against $\tilde{E}_s$. Now we apply Lemma C.10 and use again Lemma 2.8 to estimate E_s against $\tilde{E}_s$. Then iterate the estimate for $t > t^*$ similarly to the proof of Proposition 2.13. □

REMARK 2.15. The norms of ν_A and τ_k only arise in the last step from the application of Lemma 2.8 and also the additional +1. If

$$\|\nu\|_s + \|\partial_t\nu\|_s + \|\partial_t^2\nu\|_s + \|\tau\|_s + \|\partial_t\tau\|_s \le C_s.$$

for all $s > 0$ and constants $C_s > 0$ we could apply Lemma 2.7 instead of Lemma 2.8 and these terms would not appear.

We can use the modification of the elliptic estimate in Remark 2.12 and we do not need to use the assumption $\|\varphi\|_{C^2} + \|\partial_t\varphi\|_{C^2} \le K_3$ in [**2.32**],[**2.34**] and [**2.35**]. If we also assume that $P = 0$, $Q = 0$, $M = 0$ then we obtain the estimate

$$E_s(t) \le Ce^{Ct}E_s(0) + C\int_0^t e^{C(t-t')}\big(\|v\|_s + \|\partial_t v\|_s + \|w\|_{s+1} \\ + [L]_s\,\|\varphi\|_{C^2} + [\partial_t L]_s\,\|\varphi\|_{C^2} + [L]_s\,\|\partial_t\varphi\|_{C^2}\big)dt'.$$

We will apply this modified estimate to estimate the time of existence in Chapter 3. The idea behind this is that $[L]_s + [\partial_t L]_s$ might not be small and so it needs a factor that is small if φ is small.

2.3.7. Tame Estimate for Weakly Hyperbolic Linear Systems.

THEOREM 2.16. *Let the assumptions of Proposition 2.13 be satisfied with some $s_0 \ge \lfloor\frac{n}{2}\rfloor + 2$ and initial conditions $\varphi^A(0) = \varphi_0^A$, $\psi^k(0) = \psi_0^k$, $\partial_t\varphi^A(0) = \varphi_1^A$, $\partial_t\psi^k(0) = \psi_1^k$ with the bounds*

$$\|\varphi_0\|_{s_0+2} + \|\varphi_1\|_{s_0+1} + \|\psi_0\|_{s_0+1} + \|\psi_1\|_{s_0+1} \le K_3, \\ \|v\|_{s_0} + \|\partial_t v\|_{s_0} + \|w\|_{s_0+1} \le K_4$$

for some $K_3, K_4 > 0$. Then for any $s \ge 1$ we have the estimate

$$|||\varphi|||_s + |||\psi|||_s \le C\Big(\|\varphi_0\|_{s+1} + \|\psi_0\|_s + \|\varphi_1\|_s + \|\psi_1\|_s + |||v|||_{\lfloor\frac{n}{2}\rfloor+s} + |||w|||_s \\ + |||\nu|||_{\lfloor\frac{n}{2}\rfloor+s+2} + |||\tau|||_{\lfloor\frac{n}{2}\rfloor+s+2} + |[L]|_{\lfloor\frac{n}{2}\rfloor+s} + |[M]|_s + |[N]|_{\lfloor\frac{n}{2}\rfloor+s} + |[P]|_s + |[Q]|_{\lfloor\frac{n}{2}\rfloor+s} + 1\Big) \qquad \textbf{[2.36]}$$

with C depending on K_1, K_2, K_3, K_4, λ, λ_1, Λ, s and T.

PROOF. *1.* First we want to apply the estimate from Proposition 2.13 in order to get pointwise bounds for up to second derivatives of φ and first derivatives of ψ in terms of the data. By the Sobolev embedding theorem and the assumptions we have

$$|[L]|_{C^1} + |[\partial_t L]|_{C^1} + |[N]|_{C^1} + |[\partial_t N]|_{C^1} + |[Q]|_{C^1} + |[\partial_t Q]|_{C^1} + |[M]|_{C^1} + |[P]|_{C^1} \le C.$$

We can estimate the $\|\partial_t^2\varphi\|_{s_0}$-term in $E_{s_0}(0)$ using the equation, the first Moser inequality and the Sobolev embedding theorem by

$$C(\|\varphi_0\|_{s_0+2} + \|\varphi_1\|_{s_0} + \|\psi_0\|_{s_0+1} + \|\psi_1\|_{s_0} + \|v(0)\|_{s_0}) \le C.$$

So by the Sobolev embedding theorem and Proposition 2.13 we have the estimate

$$\|\varphi\|_{C^2} + \|\partial_t\varphi\|_{C^2} + \|\partial_t^2\varphi\|_{C^1} + \|\partial_t\psi\|_{C^1} + \|\psi\|_{C^1} \le C \qquad \textbf{[2.37]}$$

with C depending on K_1, K_2, K_3, K_4, λ, Λ, λ_1, s and T. Also by the Sobolev embedding theorem we have

$$\|\nu\|_{C^0} + \|\partial_t\nu\|_{C^0} + \|\partial_t^2\nu\|_{C^0} + \|\tau\|_{C^0} + \|\partial_t\tau\|_{C^0} \le CK_1$$

and

$$\|v\|_{C^0} + \|w\|_{C^0} \le CK_4.$$

Hence also the assumptions of Proposition 2.14 are satisfied.

2. We will obtain the result by induction on s. The statement for $s = 1$ is trivial since by [**2.37**] we have

$$|||\varphi|||_1 + |||\psi|||_1 \leq C.$$

The induction hypothesis is that inequality [**2.36**] holds true for some $s \in \mathbb{N}$, i. e.

$$|||\varphi|||_s + |||\psi|||_s \leq CR_s$$

with

$$\begin{aligned} R_s := {} & \|\varphi_0\|_{s+1} + \|\psi_0\|_s + \|\varphi_1\|_s + \|\psi_1\|_s + |||v|||_{\lfloor\frac{n}{2}\rfloor+s} + |||w|||_s \\ & + |||\nu|||_{\lfloor\frac{n}{2}\rfloor+s+2} + |||\tau|||_{\lfloor\frac{n}{2}\rfloor+s+2} + \|[L]\|_{\lfloor\frac{n}{2}\rfloor+s} + \|[M]\|_s + \|[N]\|_{\lfloor\frac{n}{2}\rfloor+s} + \|[P]\|_s + \|[Q]\|_{\lfloor\frac{n}{2}\rfloor+s} + 1. \end{aligned}$$

We will prove estimate [**2.36**] for $s + 1$ under this assumption.

3. We have to prove the estimate

$$\int_0^T \|\partial_t^j \varphi\|^2_{s+1-j} dt + \int_0^T \|\partial_t^j \psi\|^2_{s+1-j} dt \leq CR^2_{s+1} \qquad [\mathbf{2.38}]$$

for $j = 0, \dots, s+1$. To this end we do an induction on j as long as $j \leq s+1$. For the base cases $j = 0$ and $j = 1$ we apply Proposition 2.14

$$\begin{aligned} \|\partial_t^2\varphi(t,\cdot)\|_s &+ \|\partial_t\varphi(t,\cdot)\|_{s+1} + \|\varphi(t,\cdot)\|_{s+1} + \|\partial_t\psi(t,\cdot)\|_{s+1} + \|\psi(t,\cdot)\|_{s+1} \\ &\leq C \sup_{[0,T]} \big(\|\partial_t^2\nu\|_s + \|\partial_t\nu\|_{s+1} + \|\nu\|_{s+1} + \|\tau\|_{s+1} + \|\partial_t\tau\|_{s+1} + 1\big) \\ &\quad + CE_s(0) + C\int_0^t \|v\|_s + \|\partial_t v\|_s + \|w\|_{s+1} + [L]_s + [\partial_t L]_s \\ &\qquad + [Q]_s + [\partial_t Q]_s + [N]_s + [\partial_t N]_s + [M]_{s+1} + [P]_{s+1}\, dt'. \end{aligned}$$

Now we neglect every term from the left that contains more than $s+1$ derivatives in space or time. Using $\sum_i a_i^2 \leq C(\sum_i a_i)^2 \leq C\sum_i a_i^2$ for a finite number of nonnegative a_i and $(\int_0^T f dt)^2 \leq T\int_0^T f^2 dt$ by Hölder's inequality, we can square all norms. Then we integrate the estimate in t. Therefore we use that for $f \geq 0$

$$\int_0^T \int_0^t f(t')dt'dt \leq C \int_0^T f(t)dt$$

which is easy to see using an integration by parts

$$\int_0^T \int_0^t f(t')dt'dt = -\int_0^T \partial_t(T-t)\int_0^t f(t')dt'dt = \int_0^T (T-t)f(t)dt.$$

We can estimate $\|\partial_t^2\varphi(0,\cdot)\|_s$ using the equation by

$$\|\partial_t^2\varphi(0)\|_s \leq \|L(0)\varphi(0)\|_s + \|N(0)\psi(0)\|_s + \|Q(0)\varphi(0)\|_s + \|v(0)\|_s.$$

Then we estimate

$$\begin{aligned}\|L(0)\varphi(0)\|_s \leq & C([L(0)]_s \|\varphi_0\|_{C^2} + [L(0)]_{C^0} \|\varphi_0\|_{s+2}) \\ \leq & C([L(0)]_{C^s} + \|\varphi_0\|_{s+2}) \\ \leq & C(|[L]|_{C^s} + \|\varphi_0\|_{s+2}) \\ \leq & C(|[L]|_{\lfloor \frac{n}{2} \rfloor + s+1} + \|\varphi_0\|_{s+2}).\end{aligned}$$

The other terms can be estimated similarly and hence

$$\begin{aligned}\|\partial_t^2 \varphi(0)\|_s \leq & C(\|\varphi_0\|_{s+2} + \|\varphi_1\|_s + \|\psi_0\|_{s+1} + \|\psi_1\|_s + |||v|||_{\lfloor \frac{n}{2} \rfloor + s+1} \\ & + |[L]|_{\lfloor \frac{n}{2} \rfloor + s+1} + |[N]|_{\lfloor \frac{n}{2} \rfloor + s+1} + |[Q]|_{\lfloor \frac{n}{2} \rfloor + s+1}).\end{aligned}$$

This estimate for $\|\partial_t^2 \varphi(0, \cdot)\|_s$ is the reason that we need $\|\varphi_0\|_{s+1}$ on the right of the statement and the high derivatives of the operators L, N and Q.

We can estimate

$$\begin{aligned}\sup_{[0,T]} \left(\|\partial_t^2 \nu\|_s + \|\partial_t \nu\|_{s+1} + \|\nu\|_{s+1} + \|\tau\|_{s+1} + \|\partial_t \tau\|_{s+1}\right) \leq & C(|||\nu|||_{C^{s+2}} + |||\tau|||_{C^{s+2}}) \\ \leq & C(|||\nu|||_{\lfloor \frac{n}{2} \rfloor + s+3} + |||\tau|||_{\lfloor \frac{n}{2} \rfloor + s+3}).\end{aligned}$$

This yields the base cases for the induction on j.

4. The hypothesis for the induction on j is that inequality [**2.38**] holds true for some $j > 1$ and for $j - 1$, i. e.

$$\begin{aligned}\int_0^T \|\partial_t^j \varphi\|_{s+1-j}^2 dt + \int_0^T \|\partial_t^j \psi\|_{s+1-j}^2 dt \leq & C R_{s+1}^2 \\ \int_0^T \|\partial_t^{j-1} \varphi\|_{s+2-j}^2 dt + \int_0^T \|\partial_t^{j-1} \psi\|_{s+2-j}^2 dt \leq & C R_{s+1}^2.\end{aligned}$$

If $j = s + 1$ we are done. Otherwise we prove the estimate for $j + 1$. Using the equations

$$\begin{aligned}\int_0^T \|\partial_t^{j+1} \varphi\|_{s-j}^2 dt = & \int_0^T \|\partial_t^{j-1}(L\varphi + Q\varphi + N\psi + v)\|_{s-j}^2 dt \\ \int_0^T \|\partial_t^{j+1} \psi\|_{s-j}^2 dt = & \int_0^T \|\partial_t^{j-1}(M\psi + P\varphi + w)\|_{s-j}^2 dt.\end{aligned}$$

We estimate the right hand side term by term. Trivially

$$\int_0^T \|\partial_t^{j-1} v\|_{s-j}^2 dt \leq C R_{s+1}^2, \qquad \int_0^T \|\partial_t^{j-1} w\|_{s-j}^2 dt \leq C R_{s+1}^2.$$

We use Lemma C.7 (replace s by $s-1$ and $k=j-1$), the induction hypotheses and the observations from the first step to estimate

$$\begin{aligned}\int_0^T \|\partial_t^{j-1}(L\varphi)\|_{s-j}^2 dt \leq & C\Big(\|[L]\|_{C^0}^2 \int_0^T \|\partial_t^{j-1} D^2\varphi\|_{s-j}^2 + \|\partial_t^{j-1} D\varphi\|_{s-j}^2 + \|\partial_t^{j-1}\varphi\|_{s-j}^2 dt \\ & + \|[L]\|_{C^1}^2 (|||D^2\varphi|||_{s-2}^2 + |||D\varphi|||_{s-2}^2 + |||\varphi|||_{s-2}^2) + |||\varphi|||_{C^2}^2 \|[L]\|_{s-1}^2 \Big) \\ \leq & C\left(\int_0^T \|\partial_t^{j-1}\varphi\|_{s-j+2}^2 dt + |||\varphi|||_s^2 + \|[L]\|_s^2 \right) \leq CR_{s+1}^2.\end{aligned}$$

We used that we can estimate all terms with less then s derivatives by $R_s^2 \leq R_{s+1}^2$ due to the hypothesis from the induction on s. For the following terms we use the first Moser inequality

$$\begin{aligned}\int_0^T \|\partial_t^{j-1}(M\psi)\|_{s-j}^2 dt \leq & |||M\psi|||_{s-1}^2 \leq C(\|[M]\|_{s-1} |||\psi|||_{C^1} + \|[M]\|_{C^0} |||\psi|||_s)^2 \leq CR_{s+1}^2 \\ \int_0^T \|\partial_t^{j-1}(P\varphi)\|_{s-j}^2 dt \leq & |||P\varphi|||_{s-1}^2 \leq C(\|[P]\|_{s-1} |||\varphi|||_{C^1} + \|[P]\|_{C^0} |||\varphi|||_s)^2 \leq CR_{s+1}^2.\end{aligned}$$

To estimate the integrals occuring in $N\psi$ and $Q\varphi$ we use Hölder's inequality and the first Moser inequality

$$\begin{aligned}\sum_{\beta=1}^J |||\int_{\mathcal{N}} b^A_{(\beta)j}\psi^j_{(\beta)} d\mu_0|||_{H^{s-1}([0,T]\times B_2(0))} \leq & C\sum_{\beta=1}^J |||b^A_{(\beta)j}\psi^j_{(\beta)}|||_{H^{s-1}([0,T]\times B_2(0))} \\ \leq C\sum_{\beta=1}^J \Big(& |||b^A_{(\beta)j}|||_{H^{s-1}([0,T]\times B_2(0))} |||\psi^j_{(\beta)}|||_{C^0([0,T]\times B_2(0))} \\ & + |||b^A_{(\beta)j}|||_{C^0([0,T]\times B_2(0))} |||\psi^j_{(\beta)}|||_{H^{s-1}([0,T]\times B_2(0))}\Big) \\ \leq C(\|[N]\|_{s-1} & |||\psi|||_{C^0} + \|[N]\|_{C^0} |||\psi|||_{s-1})\end{aligned}$$

and hence

$$\begin{aligned}|||n^{A1} & \sum_{\beta=1}^J \int_{\mathcal{N}} b^A_{(\beta)j}\psi^j_{(\beta)} d\mu_0|||_{H^{s-1}([0,T]\times B_2(0))} \\ \leq & C\sum_{\beta=1}^J \Big(|||n^{A1}|||_{H^{s-1}([0,T]\times B_2(0))} |||\int_{\mathcal{N}} b^A_{(\beta)j}\psi^j_{(\beta)} d\mu_0|||_{C^0([0,T]\times B_2(0))} \\ & + |||n^{A1}|||_{C^0([0,T]\times B_2(0))} |||\int_{\mathcal{N}} b^A_{(\beta)j}\psi^j_{(\beta)} d\mu_0|||_{H^{s-1}([0,T]\times B_2(0))}\Big) \\ \leq & C(\|[N]\|_{s-1} + |||\psi|||_{s-1}).\end{aligned}$$

Consequently we can estimate

$$\int_0^T \|\partial_t^{j-1}(N\psi)\|_{s-j}^2 dt \leq \||N\psi\||_{s-1}^2 \leq CR_{s+1}^2$$

and similarly

$$\int_0^T \|\partial_t^{j-1}(Q\varphi)\|_{s-j}^2 dt \leq \||Q\varphi\||_{s-1}^2 \leq CR_{s+1}^2.$$

This proves the theorem. □

Corollary 2.17. *Let the assumptions of Proposition 2.13 be satisfied with* $s_0 \geq \lfloor \frac{n}{2} \rfloor + 2$. *Let locally* $V = \varphi^A \nu_A + \psi^k \tau_k$ *and* $W = v^A \nu_A + w^k \tau_k$. *Let* $V(0,\cdot) = V_0$, $\partial_t V(0,\cdot) = V_1$ *with*

$$\|V_0\|_{s_0+2} + \|V_1\|_{s_0+1} \leq K_3$$

for some $K_3 > 0$. *Suppose further*

$$\|\nu\|_{s_0+2} \leq K_2'$$

and

$$\|W\|_{s_0+1} + \|\partial_t W\|_{s_0} \leq K_4$$

for some $K_2', K_4 > 0$. *Then for any* $s \geq 1$ *we have the estimate*

$$\begin{aligned}\||V\||_s \leq C\Big(&\|V_0\|_{s+1} + \|V_1\|_s + \||W\||_{\lfloor \frac{n}{2} \rfloor + s} + \|[L]\|_{\lfloor \frac{n}{2} \rfloor + s} + \|[M]\|_s + \|[N]\|_{\lfloor \frac{n}{2} \rfloor + s} \\ &+ \|[P]\|_s + \|[Q]\|_{\lfloor \frac{n}{2} \rfloor + s} + \||\nu\||_{s+\lfloor \frac{n}{2} \rfloor + 2} + \||\tau\||_{s+\lfloor \frac{n}{2} \rfloor + 2} + 1\Big)\end{aligned}$$

with C *depending on* K_1, K_2, K_3, K_4, λ, λ_1, Λ, s *and* T.

Proof. We only have to rewrite the estimate from Theorem 2.16 in terms of V and W. Therefore we use that locally $V = \varphi^A \nu_A + \psi^k \tau_k$. Applying the first Moser inequality in space and time (see Section C.2) we can estimate

$$\begin{aligned}\||V\||_s \leq& C\Big(\||\varphi\||_s \||\nu\||_{C^0} + \||\varphi\||_{C^0} \||\nu\||_s + \||\psi\||_s \||\tau\||_{C^0} + \||\psi\||_{C^0} \||\tau\||_s\Big) \\ \leq& C\Big(\||\varphi\||_s + \||\nu\||_s + \||\psi\||_s + \||\tau\||_s\Big)\end{aligned}$$

where we estimated the $\||\cdot\||_{C^0}$-terms as in the first step of the proof of Theorem 2.16.

Since

$$\partial_t V(0) = \partial_t \varphi^A(0)\nu_A(0) + \partial_t \psi^k(0)\tau_k(0) + \varphi^A(0)\partial_t \nu_A(0) + \psi^k(0)\partial_t \tau_k(0)$$

our initial conditions in terms of $\varphi_0^A := \varphi^A(0)$, $\psi_0^k := \psi^k(0)$, $\varphi_1^A := \partial_t \varphi^A(0)$, $\psi_1^k := \partial_t \psi^k(0)$ are $\varphi_0^A = \langle V_0, \nu_B(0)\rangle \nu^{BA}(0)$, $\psi_0^k = \langle V_0, \tau_l(0)\rangle \tau^{kl}(0)$ and

$$\begin{aligned}\varphi_1^A =& \langle V_1, \nu_B(0)\rangle \nu^{BA}(0) - \varphi_0^C \langle \partial_t \nu_C(0), \nu_B(0)\rangle \nu^{BA}(0) - \psi_0^k \langle \partial_t \tau_k(0), \nu_B(0)\rangle \nu^{BA}(0) \\ \psi_1^k =& \langle V_1, \tau_l(0)\rangle \tau^{lk}(0) - \varphi_0^A \langle \partial_t \nu_A(0), \tau_l(0)\rangle \tau^{lk}(0) - \psi_0^m \langle \partial_t \tau_m(0), \tau_l(0)\rangle \tau^{lk}(0).\end{aligned}$$

By the same methods as in the proof of Lemma 2.7 since $\|\nu\|_{s_0+2} \le K_2'$

$$\|\varphi_0\|_{s_0+2} + \|\varphi_1\|_{s_0+1} + \|\psi_0\|_{s_0+1} + \|\psi_1\|_{s_0+1} \le C(\|V_0\|_{s_0+2} + \|V_1\|_{s_0+1}) \le C$$

and

$$\|v\|_{s_0} + \|\partial_t v\|_{s_0} + \|w\|_{s_0+1} \le C(\|W\|_{s_0+1} + \|\partial_t W\|_{s_0}) \le C.$$

In fact this follows as in the proof of Lemma 2.7 from the first Moser inequality and the Sobolev embedding theorem. Having these bounds we can apply Theorem 2.16.

We want to estimate the initial data by V_0 and V_1 for any $s > 0$. Using the same methods as in the proof of Lemma 2.8 we can estimate

$$\begin{aligned}\|\varphi_0\|_{s+1} + \|\varphi_1\|_s + \|\psi_0\|_s + \|\psi_1\|_s \le C(&\|V_0\|_{s+1} + \|V_1\|_s + \|\nu(0)\|_{s+1} + \|\tau(0)\|_s \\ &+ \|\partial_t\nu(0)\|_s + \|\partial_t\tau(0)\|_s + 1).\end{aligned}$$

Now we do not want norms of $\nu(0)$ and $\tau(0)$ to appear. So we estimate by the Sobolev embedding theorem

$$\begin{aligned}\|\nu(0)\|_{s+1} + \|\partial_t\nu(0)\|_s \le& C\,|||\nu|||_{C^{s+1}} \le C|||\nu|||_{s+\lfloor\frac{n}{2}\rfloor+2} \\ \|\tau(0)\|_s + \|\partial_t\tau(0)\|_s \le& C\,|||\tau|||_{C^{s+1}} \le C|||\tau|||_{s+\lfloor\frac{n}{2}\rfloor+2}.\end{aligned}$$

We have locally $v^A = \nu^{AB}\langle W, \nu_B\rangle$ and $w^k = \tau^{kl}\langle W, \tau_l\rangle$. We can estimate similarly

$$|||v|||_{\lfloor\frac{n}{2}\rfloor+s} + |||w|||_s \le C\Big(|||W|||_{\lfloor\frac{n}{2}\rfloor+s} + |||\nu|||_{\lfloor\frac{n}{2}\rfloor+s} + |||\tau|||_{\lfloor\frac{n}{2}\rfloor+s} + 1\Big). \qquad \square$$

2.4. Solvability of WHLS

In this section we use the estimates from the previous section to prove the existence and uniqueness of a solution to the WHLS.

PROPOSITION 2.18. *Let $V_0, V_1 \in C^\infty(\mathcal{N}, \mathcal{V})$ and $W \in C^\infty([0,T]\times\mathcal{N}, \mathcal{V})$ be given. Then the system [**2.4**] has a unique smooth solution V on $[0,T]\times\mathcal{N}$ with $V(0) = V_0$ and $\partial_t V(0) = V_1$.*

PROOF. Write locally $V = \varphi^A\nu_A + \psi^k\tau_k$ and $W = v^A\nu_A + w^k\tau_k$. Define $\varphi_0^A = \langle V_0, \nu_B(0)\rangle\nu^{BA}(0)$, $\psi_0^k = \langle V_0, \tau_l(0)\rangle\tau^{lk}(0)$ and

$$\begin{aligned}\varphi_1^A =& \langle V_1, \nu_B(0)\rangle\nu^{BA}(0) - \varphi_0^C\langle\partial_t\nu_C(0), \nu_B(0)\rangle\nu^{BA}(0) - \psi_0^k\langle\partial_t\tau_k(0), \nu_B(0)\rangle\nu^{BA}(0) \\ \psi_1^k =& \langle V_1, \tau_l(0)\rangle\tau^{lk}(0) - \varphi_0^A\langle\partial_t\nu_A(0), \tau_l(0)\rangle\tau^{lk}(0) - \psi_0^j\langle\partial_t\tau_j(0), \tau_l(0)\rangle\tau^{lk}(0).\end{aligned}$$

We will solve the system [**2.4**] for φ^A, ψ^k by a simple fixed point iteration. Start with $\varphi_{(0)}^A = 0$, $\psi_{(0)}^k = 0$. Then we solve inductively

$$\begin{aligned}\partial_t^2\varphi_{(m+1)}^A - L^A\varphi_{(m+1)}^A =& v^A + N^A\psi_{(m)} + Q^A\varphi_{(m)} \\ \partial_t^2\psi_{(m+1)}^k =& w^k + M^k\psi_{(m)} + P^k\varphi_{(m)}\end{aligned} \qquad [\mathbf{2.39}]$$

with initial conditions

$$\varphi^A_{(m+1)}(0) = \varphi^A_0, \qquad \psi^k_{(m+1)}(0) = \psi^k_0,$$
$$\partial_t \varphi^A_{(m+1)}(0) = \varphi^A_1, \qquad \partial_t \psi^k_{(m+1)}(0) = \psi^k_1.$$

The system [**2.39**] only consists of linear wave equations for φ^A and linear ODEs for ψ^k. The ODEs have a unique smooth solution on $[0,T]$. The wave equations can be solved locally in space and for a short time due to finite speed of propagation. The coordinate invariance of the system implies that $V_{(m+1)} = \varphi^A_{(m+1)}\nu_A + \psi^k_{(m+1)}\tau_k$ is well defined for small t. This can be iterated such that we get a solution on $[0,T]$.

The differences $\tilde\varphi^A_{(m+1)} = \varphi^A_{(m+1)} - \varphi^A_{(m)}$ and $\tilde\psi^k_{(m+1)} = \psi^k_{(m+1)} - \psi^k_{(m)}$ satisfy the system

$$\partial_t^2 \tilde\varphi^A_{(m+1)} - L^A \tilde\varphi^A_{(m+1)} = N^A \tilde\psi_{(m)} + Q^A \tilde\varphi_{(m)} =: \tilde v^A_{(m)}$$
$$\partial_t^2 \tilde\psi^k_{(m+1)} = M^k \tilde\psi_{(m)} + P^k \tilde\varphi_{(m)} =: \tilde w^k_{(m)}$$

which is a WHLS. Smoothness and compactness imply bounds on all necessary norms of the operators and of ν, τ and that $\det(\nu_{AB}) > \lambda_1$, $\det(\tau_{kl}) > \lambda_1$ uniformly for some $\lambda_1 > 0$. We want to apply Proposition 2.13 for any $s \geq \lfloor \frac{n}{2} \rfloor + 2$ and so we have to estimate $\|\tilde v\|_s$, $\|\partial_t \tilde v\|_s$ and $\|\tilde w\|_{s+1}$. Define

$$c_m(t) = \|\tilde\varphi_{(m)}(t,\cdot)\|_{s+1} + \|\partial_t\tilde\varphi_{(m)}(t,\cdot)\|_{s+1} + \|\partial_t^2\tilde\varphi_{(m)}(t,\cdot)\|_s + \|\tilde\psi_{(m)}(t,\cdot)\|_{s+1} + \|\partial_t\tilde\psi_{(m)}(t,\cdot)\|_{s+1}.$$

As in [**2.17**],[**2.18**] we can estimate

$$\|\tilde v(t,\cdot)\|_s \leq C c_m(t).$$

Repeating the arguments of [**2.20**],[**2.21**],[**2.22**] using Lemma 2.7 we can also estimate

$$\|\tilde w(t,\cdot)\|_{s+1} \leq C c_m(t).$$

Repeating the estimates [**2.26**],[**2.27**],[**2.28**],[**2.29**],[**2.30**] we obtain

$$\|\partial_t \tilde v(t,\cdot)\|_s \leq C c_m(t).$$

Hence Proposition 2.13 implies

$$c_{m+1}(t) \leq C \int_0^t c_m(t')dt'.$$

Inductively we get

$$c_{m+1}(t) \leq C^m \int\limits_{0\leq t_1 \leq t_2 \leq \dots \leq t_m \leq t} \cdots \int c_1(t_1)dt_1 \dots dt_m \leq \frac{(Ct)^m}{m!} \sup_{0\leq t'\leq T} c_1(t'). \qquad [\mathbf{2.40}]$$

This shows that for every $t \in [0,T]$ the functions $\varphi^A_{(m)}(t,\cdot)$, $\partial_t\varphi^A_{(m)}(t,\cdot)$, $\psi^k_{(m)}(t,\cdot)$ and $\partial_t\psi^k_{(m)}(t,\cdot)$ are C^∞-Cauchy sequences and have limits $\varphi^A_{(\infty)}(t,\cdot)$, $\hat\varphi^A_{(\infty)}(t,\cdot)$, $\psi^k_{(\infty)}(t,\cdot)$ and $\hat\psi^k_{(\infty)}(t,\cdot)$. As we can also take the supremum over $t \in [0,T]$ in [**2.40**] we get uniform convergence in t and the limits are continuously differentiable in t. This implies that we can also interchange first and second time derivatives with the limit. Passing to the limit in [**2.39**] shows that $\varphi_{(\infty)}$ and $\psi_{(\infty)}$ satisfy the WHLS [**2.4**]. Smoothness in time follows from the equation by an induction.

To show uniqueness we assume there exist two solutions $\varphi^A_{(1)}$, $\psi^k_{(1)}$ and $\varphi^A_{(2)}$, $\psi^k_{(2)}$. The differences $\tilde{\varphi}^A = \varphi^A_{(1)} - \varphi^A_{(2)}$ and $\tilde{\psi}^k = \psi^k_{(1)} - \varphi^A_{(2)}$ satisfy the system

$$\begin{aligned} \partial_t^2 \tilde{\varphi}^A - L^A \tilde{\varphi}^A - N^A \tilde{\psi} - Q^A \tilde{\varphi} =& 0 \\ \partial_t^2 \tilde{\psi}^k - M^k \tilde{\psi} - P^k \tilde{\varphi} =& 0 \end{aligned}$$

with vanishing initial data. Hence by the estimate from Proposition 2.13 for any $s \geq \lfloor \frac{n}{2} \rfloor + 2$

$$\|\tilde{\varphi}\|_{s+1} + \|\partial_t \tilde{\varphi}\|_{s+1} + \|\partial_t^2 \tilde{\varphi}\|_s + \|\tilde{\psi}\|_{s+1} + \|\partial_t \tilde{\psi}\|_{s+1} \leq 0.$$

So $\tilde{\varphi}^A = 0$ and $\tilde{\psi}^k = 0$. □

2.5. Conclusion of the Short Time Existence Proof

We wish to write the linearisation [**2.3**] as a WHLS and derive a tame estimate for solutions of this system. We will need to estimate the coefficients of the operators L, M, N, P, Q in terms of u to derive our tame estimate. This can be done by the third Moser inequality but we have to be aware of the terms that could possibly blow up. These are only $\|\nu\|_s$ and $\|g^{-1}\|_s$. In the following Lemma we show that we can indeed apply the third Moser inequality to these terms if we assume that $\det(g_{ij}) \geq \lambda_1 > 0$ for some λ_1. This assumption will be satisfied in Proposition 2.21 due to the choice of neighborhood. For later applications the proof of the following Lemma will be valid also on a general oriented manifold $\mathcal{M}$.

LEMMA 2.19. *If $\|u\|_{C^1} \leq K$ and $\det(g_{ij}) \geq \lambda_1$ for some $K, \lambda_1 > 0$ then*

$$\begin{aligned} \|\nu\|_s \leq& C(1 + \|u\|_{s+1}) \\ |||\nu|||_s \leq& C(1 + |||u|||_{s+1}) \end{aligned}$$

and

$$\begin{aligned} \|g^{-1}\|_s \leq& C(1 + \|u\|_{s+1}) \\ |||g^{-1}|||_s \leq& C(1 + |||u|||_{s+1}) \end{aligned}$$

with C depending on K, λ_1 and s. If also $\sum_{l=0}^i \|\partial_t^i u\|_{C^1} \leq K$ for some $i \geq 1$ then

$$\|\partial_t^i \nu\|_s \leq C(1 + \sum_{l=0}^i \|\partial_t^i u\|_{s+1})$$

and

$$\|\partial_t^i g^{-1}\|_s \leq C(1 + \sum_{l=0}^i \|\partial_t^i u\|_{s+1})$$

with C depending on K, λ_1, s and i.

PROOF. To estimate $\|\nu\|_s$ we will use the third Moser inequality. We have to show that ν is a smooth function of u and Du with all derivatives bounded. Now using the cross product on manifolds we can write up to sign

$$\nu = \frac{\partial_1 u \times \cdots \times \partial_n u}{|\partial_1 u \times \cdots \times \partial_n u|}.$$

By definition

$$V_1 \times \cdots \times V_n = \sigma_{or}\sqrt{\det(\overline{g}_{\alpha\beta})}\varepsilon_{\alpha_1\ldots\alpha_n\gamma}V_1^{\alpha_1}\ldots V_n^{\alpha_n}\overline{g}^{\gamma\delta}\bar{\partial}_\delta$$

where $V_1, \ldots, V_n \in T_p\mathcal{M}$ are vectors, $\varepsilon_{\alpha_1\ldots\alpha_{n+1}}$ is the Levi-Civita symbol and σ_{or} is a sign determined by the orientation of $\mathcal{M}$. Let $\overline{V} = (V_i^\alpha)$. Let $\overline{V}_\gamma$ be the $n \times n$ matrix created from $\overline{V}$ when line γ is deleted. Similarly let $(\overline{g})_{\gamma\delta}$ be the matrix $\overline{g}$ where line γ and row δ are deleted. By the multiplication theorem for determinants (see [**Fis79**] also known as Cauchy-Binet formula) we have that

$$\begin{aligned}
\det(\langle V_i, V_j\rangle) &= \det(\overline{V}^T\overline{g}\overline{V}) = \sum_{\gamma=1}^{n+1}\det\overline{V}_\gamma^T\det(\overline{g}\overline{V})_\gamma \\
&= \sum_{\gamma=1}^{n+1}\sum_{\delta=1}^{n+1}\det\overline{V}_\gamma^T\det((\overline{g})_{\gamma\delta})\det\overline{V}_\delta \\
&= \sum_\gamma\sum_\delta(-1)^{\delta+\gamma}\det(\overline{g})\overline{g}^{\delta\gamma}\det\overline{V}_\gamma\det\overline{V}_\delta \\
&= \det(\overline{g})\overline{g}^{\gamma\delta}\varepsilon_{\alpha_1\ldots\alpha_n\gamma}\varepsilon_{\beta_1\ldots\beta_n\delta}V_1^{\alpha_1}\ldots V_n^{\alpha_n}V_1^{\beta_1}\ldots V_n^{\beta_n} \\
&= |V_1 \times \cdots \times V_n|^2.
\end{aligned}$$

We used the cofactor representation of $\overline{g}^{\gamma\delta}$ and that $\det\overline{V}_\gamma = (-1)^\gamma\varepsilon_{\alpha_1\ldots\alpha_n\gamma}V_1^{\alpha_1}\ldots V_n^{\alpha_n}$ by cofactor expansion (no sum over γ). Hence

$$|\partial_1 u \times \cdots \times \partial_n u| = \sqrt{\det(g_{ij})}.$$

This is the denominator in the expression for ν and it is bounded below by assumption. So ν is a function of u and $\partial_i u$ and the denominator is bounded in the range of u, Du. This also holds for all derivatives as they can only contain powers of this denominator. Clearly this also works for time derivatives.

The estimate for g^{ij} works exactly in the same way as the estimates in Lemma 2.6 and this clearly also works for time derivatives. □

REMARK 2.20. For later reference we note the essential fact of the preceeding proof: If $\|u\|_{C^1} \leq K$ and $\det(g_{ij}) \geq \lambda_1 > 0$ then we can write $g^{ij} = F^{ij}(u, Du)$ and $\nu = G(u, Du)$ with smooth functions F and G that are bounded with bounded derivatives in the range of u and Du.

From Corollary 2.17 we get the promised tame estimate for $D\mathscr{P}^{-1}$ on a neighborhood of the approximate solution $\bar{u}$.

PROPOSITION 2.21. *Let $\bar{u} : [0,T] \times \mathcal{N} \to \mathbb{R}^{n+1}$ be the approximate solution constructed in Section 2.1. There is a neighborhood $\mathbf{U}' \subset \mathbf{F}$ of $\bar{u}$ on which the operator*

$$\mathfrak{P}(u) = \overline{\nabla}_{\partial_t}\partial_t u - \frac{d\mu_t}{d\mu_0}\left(-H(u) + \frac{\varrho}{\operatorname{Vol}(u)}\right)\nu$$

can be defined for all $u \in \mathbf{U}'$. Furthermore $\mathscr{P} : \mathbf{U}' \to \mathbf{F} \times \mathbf{F}_0 \times \mathbf{F}_0$ defined by $\mathscr{P}(u) = (\mathfrak{P}(u), u(0,\cdot), \partial_t u(0,\cdot))$ is a smooth tame map and $D\mathscr{P}^{-1} : \mathbf{U}' \times \mathbf{F} \times \mathbf{F}_0 \times \mathbf{F}_0 \to \mathbf{F}$ exists and is a smooth tame map.

PROOF. *1. Choice of* $\mathbf{U}'$. In order to apply Corollary 2.17 we choose $\mathbf{U}'$ such that the following properties are satisfied for all $u \in \mathbf{U}'$

(1) $\Lambda\delta^{ij} \geq g^{ij} \geq \lambda\delta^{ij}$ for some $\Lambda, \lambda > 0$,
(2) $\det(g_{ij}) > \lambda_1$ for some $\lambda_1 > 0$,
(3) $\|u\|_{\lfloor\frac{n}{2}\rfloor+5} + \|\partial_t u\|_{\lfloor\frac{n}{2}\rfloor+5} + \|\partial_t^2 u\|_{\lfloor\frac{n}{2}\rfloor+4} + \|\partial_t^3 u\|_{\lfloor\frac{n}{2}\rfloor+3} < K_1^*$ for some $K_1^* > 0$,
(4) $\operatorname{Vol}(u)$ is defined and $\Lambda_2 > \operatorname{Vol}(u) > \lambda_2$ for some $\Lambda_2, \lambda_2 > 0$.

Therefore define for some $0 < \delta < 1$

$$\mathbf{U}' = \{u \in \mathbf{F} \mid \||u - \bar{u}\||_{C^{\lfloor\frac{n}{2}\rfloor+6}} \leq \delta\}.$$

We will now see that if δ is small enough then $\mathbf{U}'$ will have the desired properties. For any $u \in \mathbf{U}$ we have

$$g_{ij}(u) = g_{ij}(\bar{u}) + \langle\partial_i\bar{u}, \partial_j(u-\bar{u})\rangle + \langle\partial_j\bar{u}, \partial_i(u-\bar{u})\rangle + \langle\partial_i(u-\bar{u}), \partial_j(u-\bar{u})\rangle$$

and hence

$$|g_{ij}(u) - g_{ij}(\bar{u})| \leq C(\|u-\bar{u}\|_{C^1} + \|u-\bar{u}\|_{C^1}^2) \leq C\delta.$$

So if δ is small enough we get properties (1) and (2). For $s \in [0,1]$ define $u_s = \bar{u} + s(u-\bar{u})$. We have the same estimate for u_s

$$|g_{ij}(u_s) - g_{ij}(\bar{u})| \leq C(\|u-\bar{u}\|_{C^1} + \|u-\bar{u}\|_{C^1}^2). \qquad [\mathbf{2.41}]$$

Hence properties (1) and (2) also hold for u_s. We can compare $d\mu_t(u_s) \leq C d\mu_0$ using [**2.41**] independently of $\delta < 1$. We can then define $\operatorname{Vol}(u)$ in a neighborhood of $\bar{u}$ and we have

$$\begin{aligned}\operatorname{Vol}(u) =& \operatorname{Vol}(\bar{u}) + \int_0^1 \partial_s \operatorname{Vol}(u_s)ds \\ =& \operatorname{Vol}(\bar{u}) + \int_0^1 \int_{\mathcal{N}} \langle\nu(u_s), u-\bar{u}\rangle d\mu_t(u_s)ds. \qquad [\mathbf{2.42}]\end{aligned}$$

We use this to estimate

$$\begin{aligned}|\operatorname{Vol}(u) - \operatorname{Vol}(\bar{u})| \leq& \int_0^1 \int_{\mathcal{N}} |\langle\nu(u_s), u-\bar{u}\rangle| d\mu_t(u_s)ds \\ \leq& C\|u-\bar{u}\|_{C^0} \leq C\delta.\end{aligned}$$

Since property (4) holds for $\bar{u}$, we can get this property for all $u \in \mathbf{U}'$ if we just make δ small enough. Property (3) is immediate from the definition of $\mathbf{U}'$.

2. Tameness of $\mathfrak{P}$. From [**Ham82a**, Corollary II.2.2.7] we know that a nonlinear partial differential operator is a smooth tame map. But $\mathfrak{P}$ is not a differential operator in this sense because it includes the $\mathrm{Vol}(u)$-term. But we can write $\mathfrak{P}(u) = \hat{\mathfrak{P}}(u, \mathrm{Vol}(u))$ where $\hat{\mathfrak{P}}$ is a differential operator of second order in u and zeroth order in $\mathrm{Vol}(u)$. Since the composition of tame maps is tame we only have to show that the map $u \mapsto \mathrm{Vol}(u), \mathbf{U}' \to C^\infty([0,T], \mathbb{R})$ is a smooth tame map. Therefore we write

$$\mathrm{Vol}(u) = \mathrm{Vol}(u_0) + \int_0^t \int_{\mathcal{N}} \langle \partial_t u, \nu \rangle d\mu_t dt.$$

Let $\omega(u) := \langle \partial_t u, \nu \rangle \frac{d\mu_t}{d\mu_0}$. This is a nonlinear partial differential operator that assigns to u a smooth function on $\mathcal{N}$, i. e. $\omega : \mathbf{U}' \to C^\infty([0,T] \times \mathcal{N}, \mathbb{R})$. Since this is a smooth tame map we only have to show that the map f assigning to ω the function $f(\omega)(t) = \int_0^t \int_{\mathcal{N}} \omega d\mu_0 dt$ is a smooth tame map. Clearly f is continuous. Now $Df(\omega)\{\tilde{\omega}\}(t) = \int_0^t \int_{\mathcal{N}} \tilde{\omega} d\mu_0 dt$ and $D^2 f(\omega)$ vanishes. So f is smooth and we only have to prove a tame estimate for f and Df. We estimate for $k > 1$

$$\begin{aligned}
\|\partial_t^k f(\omega)\|^2_{L^2([0,T])} = &\int_0^T (\partial_t^k f(\omega)(t))^2 dt = \int_0^T \left(\int_{\mathcal{N}} \partial_t^{k-1} \omega d\mu_0 \right)^2 dt \\
\leq &C \int_0^T \int_{\mathcal{N}} (\partial_t^{k-1} \omega)^2 d\mu_0 dt \\
\leq &C |||\omega|||^2_{k-1} \leq C |||\omega|||^2_k.
\end{aligned}$$

Taking the square root this yields a tame estimate. The estimate for Df is the same. Since $\mathrm{Vol}(u)$ is bounded by the choice of neighborhood we see that the map $u \mapsto \mathrm{Vol}(u)$ is tame.

3. The Linearisation as a WHLS. We write the system [**2.3**] as a WHLS. For the bundle $\mathcal{V}$ take $\mathcal{N} \times \mathbb{R}^{n+1}$, $d' = 1$, $d'' = n$. Take $\nu_1 = \nu$ and $\tau_k = \partial_k u$. Let η_α be a partition of unity subordinate to the sets $x_\alpha^{-1}(B_2(0))$. Then [**2.3**] is a WHLS with the operators

$$\begin{aligned}
L\varphi = &\frac{d\mu_t}{d\mu_0} \left\{ \Delta\varphi + |h|^2 \varphi + \left(-H + \frac{\varrho}{\mathrm{Vol}(u)} \right) H\varphi \right\} - \langle \partial_t^2 \nu, \nu \rangle \varphi \\
N\psi = &\frac{d\mu_t}{d\mu_0} \{ -\partial_l H \psi^l + \left(-H + \frac{\varrho}{\mathrm{Vol}(u)} \right) (\partial_i \psi^i + \Gamma^i_{il} \psi^l) \} \\
&- 2\langle \partial_t \partial_k u, \nu \rangle \partial_t \psi^k - \langle \partial_t^2 \partial_k u, \nu \rangle \psi^k \\
Q\varphi = &- \frac{d\mu_t}{d\mu_0} \frac{\varrho}{\mathrm{Vol}(u)^2} \sum_{\alpha=1}^J \int_{\mathcal{N}} \eta_\alpha \varphi \frac{d\mu_t}{d\mu_0} d\mu_0 \\
M^k\psi = &\frac{d\mu_t}{d\mu_0} \left(-H + \frac{\varrho}{\mathrm{Vol}(u)} \right) h_i{}^k \psi^i - 2\langle \partial_t \partial_l u, \partial_j u \rangle g^{jk} \partial_t \psi^l - \langle \partial_t^2 \partial_l u, \partial_j u \rangle g^{jk} \psi^l \\
P^k\varphi = &- \frac{d\mu_t}{d\mu_0} \left(-H + \frac{\varrho}{\mathrm{Vol}(u)} \right) \nabla^k \varphi - \langle \partial_t \nu, \partial_j u \rangle g^{jk} \partial_t \varphi - \langle \partial_t^2 \nu, \partial_j u \rangle g^{jk} \varphi.
\end{aligned}$$

4. Tameness of $D\mathscr{P}^{-1}$. Let $s_0 = \lfloor \frac{n}{2} \rfloor + 2$ and $(\bar{W}, \bar{V}_0, \bar{V}_1) \in \mathbf{F} \times \mathbf{F}_0 \times \mathbf{F}_0$. Choose constants K_2^*, K_3^*, K_4^* such that $\|\bar{W}\|_{s_0+1} + \|\partial_t \bar{W}\|_{s_0} < K_2^*$, $\|\bar{V}_0\|_{s_0+2} < K_3^*$ and $\|\bar{V}_1\|_{s_0+1} < K_4^*$. Choose a neighborhood $\mathbf{V} \subset \mathbf{F}$ of $\bar{W}$ such that $\|W\|_{s_0+1} + \|\partial_t W\|_{s_0} \leq K_2^*$ for all $W \in \mathbf{V}$. Choose also neighborhoods $\mathbf{V}_0, \mathbf{V}_1 \subset \mathbf{F}_0$ around $\bar{V}_0$ and $\bar{V}_1$ such that $\|V_0\|_{s_0+2} \leq K_3^*$ and $\|V_1\|_{s_0+1} \leq K_4^*$ for all $V_0 \in \mathbf{V}_0$, $V_1 \in \mathbf{V}_1$. For any $(u, W, V_0, V_1) \in \mathbf{U}' \times \mathbf{V} \times \mathbf{V}_0 \times \mathbf{V}_1$ let $V = D\mathscr{P}^{-1}(u, W, V_0, V_1)$ be the unique solution to $D\mathfrak{P}(u)\{V\} = W$ with initial conditions $V(0) = V_0$, $\partial_t V(0) = V_1$ whose existence is assured by Proposition 2.18. Using Lemma 2.19 and the Moser inequalities we can estimate the operators by

$$\begin{aligned}
[L]_{s_0} &\leq C(1 + \|u\|_{s_0+2} + \|\partial_t u\|_{s_0+1} + \|\partial_t^2 u\|_{s_0+1}) \leq C_{K_1^*} \\
[\partial_t L]_{s_0} &\leq C(1 + \|u\|_{s_0+2} + \|\partial_t u\|_{s_0+2} + \|\partial_t^2 u\|_{s_0+1} + \|\partial_t^3 u\|_{s_0+1}) \leq C_{K_1^*} \\
[N]_{s_0} &\leq C(1 + \|u\|_{s_0+3} + \|\partial_t u\|_{s_0+1} + \|\partial_t^2 u\|_{s_0+1}) \leq C_{K_1^*} \\
[\partial_t N]_{s_0} &\leq C(1 + \|u\|_{s_0+3} + \|\partial_t u\|_{s_0+3} + \|\partial_t^2 u\|_{s_0+1} + \|\partial_t^3 u\|_{s_0+1}) \leq C_{K_1^*} \\
[Q]_{s_0} &\leq C(1 + \|u\|_{s_0+1}) \leq C_{K_1^*} \\
[\partial_t Q]_{s_0} &\leq C(1 + \|u\|_{s_0+1} + \|\partial_t u\|_{s_0+1}) \leq C_{K_1^*} \\
[M]_{s_0+1} &\leq C(1 + \|u\|_{s_0+3} + \|\partial_t u\|_{s_0+2} + \|\partial_t^2 u\|_{s_0+2}) \leq C_{K_1^*} \\
[P]_{s_0+1} &\leq C(1 + \|u\|_{s_0+3} + \|\partial_t u\|_{s_0+2} + \|\partial_t^2 u\|_{s_0+2}) \leq C_{K_1^*}.
\end{aligned}$$

We can also estimate $\|[E]\|_{\lfloor \frac{n}{2} \rfloor + s} \leq C(1 + |||u|||_{\lfloor \frac{n}{2} \rfloor + s + 3})$ for $E \in \{L, M, N, P, Q\}$ using Lemma 2.19 and the Moser inequalities since at most third derivatives of u occur in the operators. We apply again Lemma 2.19 to estimate

$$\begin{aligned}
\|\nu\|_{s_0+2} + \|\partial_t \nu\|_{s_0+1} + \|\partial_t^2 \nu\|_{s_0} + \|\tau\|_{s_0+1} + \|\partial_t \tau\|_{s_0+1}& \\
\leq C(1 + \|u\|_{s_0+3} + \|\partial_t u\|_{s_0+2} + \|\partial_t^2 u\|_{s_0+1}) \leq C_{K_1^*}&.
\end{aligned}$$

Also by Lemma 2.19

$$|||\nu|||_{s+\lfloor \frac{n}{2} \rfloor+2} \leq C(1 + |||u|||_{s+\lfloor \frac{n}{2} \rfloor+3}) \text{ and } |||\tau|||_{s+\lfloor \frac{n}{2} \rfloor+2} \leq C(1 + |||u|||_{s+\lfloor \frac{n}{2} \rfloor+3}).$$

This implies that the assumptions of Proposition 2.13 and Corollary 2.17 are satisfied and in view of Corollary 2.17 we can estimate for any $s \geq 1$

$$|||V|||_s \leq C\left(1 + \|V_0\|_{s+1} + \|V_1\|_s + |||W|||_{\lfloor \frac{n}{2} \rfloor+s} + |||u|||_{s+\lfloor \frac{n}{2} \rfloor+3}\right).$$

The constant C only depends on K_1^*, K_2^*, K_3^*, K_4^*, Λ, λ, λ_1, T and s. This clearly is a tame estimate for $D\mathscr{P}^{-1}$.

5. Continuity of $D\mathscr{P}^{-1}$. We wish to check the continuity of $D\mathscr{P}^{-1}$. Therefore let $(u_k, W_k, V_{0k}, V_{1k})$ be a sequence in $\mathbf{U}' \times \mathbf{F} \times \mathbf{F}_0 \times \mathbf{F}_0$ converging to $(u, W, V_0, V_1) \in \mathbf{U}' \times \mathbf{F} \times \mathbf{F}_0 \times \mathbf{F}_0$ in C^∞. As this sequence is bounded, we can choose constants K_2', K_3', K_4' such that the bounds $\|W_k\|_{s_0+1} + \|\partial_t W_k\|_{s_0} < K_2'$, $\|V_{0k}\|_{s_0+2} < K_3'$ and $\|V_{1k}\|_{s_0+1} < K_4'$ hold uniformly in k. As in step 4 we get the estimate

$$|||V_k|||_s \leq C\left(1 + \|V_{0k}\|_{s+1} + \|V_{1k}\|_s + |||W_k|||_{\lfloor \frac{n}{2} \rfloor+s} + |||u_k|||_{s+\lfloor \frac{n}{2} \rfloor+3}\right)$$

for $V_k = D\mathscr{P}^{-1}(u_k, W_k, V_{0k}, V_{1k})$ and consequently the sequence (V_k) is also bounded.

As every bounded sequence in C^∞ has a convergent subsequence there exists a subsequence V_{k_i} and converging to some $V \in \mathbf{F}$. The convergence of the data and of the V_{k_i} in C^∞ implies that we can take the limit in the equations for V_{k_i} and so V solves $D\mathfrak{P}(u)\{V\} = W$ with initial data $V(0) = V_0$, $\partial_t V(0) = V_1$. The solution of this equation is unique and so we can conclude convergence of the whole sequence V_k to V. By definition $V = D\mathscr{P}^{-1}(u, W, V_0, V_1)$. Hence $D\mathscr{P}^{-1}$ is continuous.

6. Conclusion. We have shown that $D\mathscr{P}^{-1}$ exists, is continuous and satisfies a tame estimate. By [**Ham82a**, Theorem II.3.1.1] it follows that $D\mathscr{P}^{-1}$ is a smooth tame map. □

This means that we can apply the Nash-Moser inverse function theorem as described in Section 2.1. This concludes the short time existence proof for the Euclidean case $\mathcal{M} = \mathbb{R}^{n+1}$.

2.6. Generalisation to Manifolds

The space $\mathbf{E} \subset C^\infty([0,T]\times\mathcal{N}, \mathcal{M})$ of time dependent immersions from $\mathcal{N}$ to the manifold $\mathcal{M}$ is a Fréchet manifold. For $u \in \mathbf{E}$ the operator

$$\mathfrak{P}(u) = \overline{\nabla}_{\partial_t}\partial_t u - \frac{d\mu_t}{d\mu_0}\left(-H(u) + \frac{\varrho}{\mathrm{Vol}(u)}\right)\nu$$

is a vectorfield along u. The space of vectorfields along a map $u \in \mathbf{E}$ can be identified with the tangent space $T_u\mathbf{E}$. This means that $\mathfrak{P}$ is a vectorfield on the manifold $\mathbf{E}$. In order to replicate the short time existence proof and apply the Nash-Moser argument in this setting one would have to choose a local coordinate chart for $\mathbf{E}$ around u and consider the operator $\mathfrak{P}$ in such a chart. Then we are in the situation that $\mathfrak{P}$ can be considered as a map between Fréchet spaces. However it seems to be very complicated to carry this out in detail. To avoid these complications we will in the following translate our problem to an equivalent problem for maps in the Fréchet space $C^\infty([0,T] \times \mathcal{N}, \mathbb{R}^d)$.

2.6.1. Extrinsic Formulation of the Problem. By the Nash embedding theorem we can suppose that the ambient manifold $\mathcal{M}$ is isometrically embedded into $\mathbb{R}^d$ by $\iota : \mathcal{M} \to \mathbb{R}^d$ for some d. We derive an extrinsic form of the Euler-Lagrange equation [**EQ**] which is similar to the extrinsic form of wave maps (see e. g. [**SS98**]) and the extrinsic form of the evolution equation for magnetic geodesics [**Koh09**]. Let $\pi_\mathcal{M}$ be the closest point projection to $\iota(\mathcal{M})$ which can be defined on a neighborhood

$$\tilde{\mathcal{M}} = \{x + v \mid x \in \iota(\mathcal{M}), \quad v \in (T_x\iota(\mathcal{M}))^\perp, \quad |v| < \delta(x)\}$$

of $\iota(\mathcal{M})$ and is smooth there. Here δ is a positive smooth function on $\iota(\mathcal{M})$. Now the second fundamental form of $\mathcal{M}$ is given by

$$\bar{h}_{\alpha\beta} = \bar{\partial}_\alpha\bar{\partial}_\beta\iota - \overline{\Gamma}^\gamma_{\alpha\beta}\bar{\partial}_\gamma\iota \qquad [\mathbf{2.43}]$$

and this is normal to $\iota(\mathcal{M})$ and so $D\pi_\mathcal{M}(\iota(p))(\bar{h}_{\alpha\beta}) = 0$ for $p \in \mathcal{M}$. Since $\iota = \pi_\mathcal{M} \circ \iota$ we have

$$\bar{\partial}_\alpha\bar{\partial}_\beta\iota - \overline{\Gamma}^\gamma_{\alpha\beta}\bar{\partial}_\gamma\iota = D_A D_B\pi_\mathcal{M}\bar{\partial}_\beta\iota^A\bar{\partial}_\alpha\iota^B + D\pi_\mathcal{M}(\bar{\partial}_\alpha\bar{\partial}_\beta\iota - \overline{\Gamma}^\gamma_{\alpha\beta}\bar{\partial}_\gamma\iota) = D_A D_B\pi_\mathcal{M}\bar{\partial}_\beta\iota^A\bar{\partial}_\alpha\iota^B. \qquad [\mathbf{2.44}]$$

Here D_A is the derivative in the direction of the canonical basis vector e_A in $\mathbb{R}^d$.

Now if $u : [0,T] \times \mathcal{N} \to \iota(\mathcal{M})$ we can write $u = \iota \circ \hat{u}$ with $\hat{u} : [0,T] \times \mathcal{N} \to \mathcal{M}$. We compute

$$\begin{aligned}\partial_t^2 u =& \bar{\partial}_\alpha \iota \partial_t^2 \hat{u}^\alpha + \bar{\partial}_\beta \bar{\partial}_\gamma \iota \partial_t \hat{u}^\beta \partial_t \hat{u}^\gamma \\ =& \bar{\partial}_\alpha \iota (\partial_t^2 \hat{u}^\alpha + \overline{\Gamma}^\alpha_{\beta\gamma} \partial_t \hat{u}^\beta \partial_t \hat{u}^\gamma) + \partial_t \hat{u}^\beta \partial_t \hat{u}^\gamma (\bar{\partial}_\beta \bar{\partial}_\gamma \iota - \overline{\Gamma}^\alpha_{\beta\gamma} \bar{\partial}_\alpha \iota) \\ =& D\iota(\overline{\nabla}_{\partial_t} \partial_t \hat{u}) + D_A D_B \pi_{\mathcal{M}}(u) \partial_t u^A \partial_t u^B\end{aligned}$$

and similarly

$$\begin{aligned}g^{ij}(\partial_i \partial_j u - \Gamma^k_{ij} \partial_k u) =& \bar{\partial}_\alpha \iota (g^{ij} \partial_i \partial_j \hat{u}^\alpha - \Gamma^k_{ij} \partial_k \hat{u}^\alpha + \overline{\Gamma}^\alpha_{\beta\gamma} \partial_i \hat{u}^\beta \partial_j \hat{u}^\gamma) + g^{ij} D_A D_B \pi_{\mathcal{M}}(u) \partial_i u^A \partial_j u^B \\ =& D\iota(-H(\hat{u})\nu(\hat{u})) + g^{ij} D_A D_B \pi_{\mathcal{M}}(u) \partial_i u^A \partial_j u^B.\end{aligned}$$

Clearly the normal vector $\nu(u)$ and the volume $\mathrm{Vol}(u)$ of u in $\iota(\mathcal{M})$ are given by $\nu(u) = D\iota(\nu(\hat{u}))$ and $\mathrm{Vol}(u) = \mathrm{Vol}(\hat{u})$ since ι is an isometric embedding. We write

$$\tilde{\Box} u = \partial_t^2 u - \frac{d\mu_t}{d\mu_0} g^{ij} \left(\partial_i \partial_j u - \Gamma^k_{ij} \partial_k u\right).$$

So

$$\begin{aligned}\tilde{\Box} u - \frac{d\mu_t}{d\mu_0}(u) \frac{\varrho}{\mathrm{Vol}(u)} \nu(u) - D_A D_B \pi_{\mathcal{M}}(u)(\partial_t u^A \partial_t u^B - \frac{d\mu_t}{d\mu_0}(u) g^{ij} \partial_i u^A \partial_j u^B) \\ = D\iota\left(\overline{\nabla}_{\partial_t} \partial_t \hat{u} - \frac{d\mu_t}{d\mu_0}(\hat{u}) \left(-H(\hat{u}) + \frac{\varrho}{\mathrm{Vol}(\hat{u})}\right) \nu(\hat{u})\right). \quad [\mathbf{2.45}]\end{aligned}$$

Hence $\hat{u} : [0,T] \times \mathcal{N} \to \mathcal{M}$ solves [**EQ**] if and only if $u : [0,T] \times \mathcal{N} \to \iota(\mathcal{M})$ solves

$$0 = \tilde{\Box} u - \frac{d\mu_t}{d\mu_0}(u) \frac{\varrho}{\mathrm{Vol}(u)} \nu(u) - D_A D_B \pi_{\mathcal{M}}(u)(\partial_t u^A \partial_t u^B - \frac{d\mu_t}{d\mu_0}(u) g^{ij} \partial_i u^A \partial_j u^B). [\mathbf{2.46}]$$

In order to solve [**2.46**] we will formulate this equation for functions $u : [0,T] \times \mathcal{N} \to \tilde{\mathcal{M}} \subset \mathbb{R}^d$ which do not necessarily map to $\iota(\mathcal{M})$. We will do this in such a way that the linearisation is a WHLS in order to apply the Nash-Moser argument for this new equation. The quantities $\nu(u)$ and $\mathrm{Vol}(u)$ have to be replaced to make sense for such maps. Let $\pi_{\Sigma_t^\perp}(u)$ be the projection onto the normal space of $\Sigma_t = u(t, \mathcal{N})$, i. e. $\pi_{\Sigma_t^\perp}(u) V = V - g^{ij} \langle V, \partial_j u \rangle \partial_i u$ for $V \in \mathbb{R}^d$. We replace $\nu(u)$ by $\tilde{\nu}(u) = \pi_{\Sigma_t^\perp}(u) \nu(\pi_{\mathcal{M}} \circ u)$, i. e. the projection onto the normal space of Σ_t of the normal vector $\nu(\pi_{\mathcal{M}} \circ u)$ in $\iota(\mathcal{M})$ of the map $\pi_{\mathcal{M}} \circ u : [0,T] \times \mathcal{N} \to \iota(\mathcal{M})$. If u is close enough in C^1 to a family of immersions that map to $\iota(\mathcal{M})$ then $\pi_{\mathcal{M}} \circ u$ is also a family of immersions and $\nu(\pi_{\mathcal{M}} \circ u)$ is defined. We can assume this as we just need to make our neighborhood smaller in the Nash-Moser argument. By definition $\tilde{\nu}(u)$ is normal to $\partial_i u$ and $\nu(\pi_{\mathcal{M}} \circ u)$ is an element of $T_{\pi_{\mathcal{M}}(u)} \iota(\mathcal{M})$. We define $\widetilde{\mathrm{Vol}}(u) := \mathrm{Vol}(\pi_{\mathcal{M}} \circ u)$. Clearly if u maps to $\iota(\mathcal{M})$ then $\nu(u) = \tilde{\nu}(u)$ and $\mathrm{Vol}(u) = \widetilde{\mathrm{Vol}}(u)$.

We then define $\Pi_u(\partial_\alpha u, \partial_\beta u) = \pi_{\Sigma_t^\perp}(u) D_A D_B \pi_{\mathcal{M}}(u) \partial_\alpha u^A \partial_\beta u^B$. If u maps to $\iota(\mathcal{M})$ then $\pi_{\Sigma_t^\perp}(u) D_A D_B \pi_{\mathcal{M}}(u) \partial_\alpha u^A \partial_\beta u^B = D_A D_B \pi_{\mathcal{M}}(u) \partial_\alpha u^A \partial_\beta u^B$ because it follows from [**2.43**] and [**2.44**] that $D_A D_B \pi_{\mathcal{M}}(u) \partial_\alpha u^A \partial_\beta u^B$ is orthogonal to $\iota(\mathcal{M})$ and hence to Σ_t.

Then we want to solve the equation

$$\tilde{\Box} u = \frac{d\mu_t}{d\mu_0}\frac{\varrho}{\widetilde{\mathrm{Vol}}(u)}\tilde{\nu} + \Pi_u(\partial_t u, \partial_t u) - \frac{d\mu_t}{d\mu_0} g^{ij}\Pi_u(\partial_i u, \partial_j u) \qquad [\mathbf{2.47}]$$

subject to given initial conditions. We will see in Subsection 2.6.2 that this is possible.

Assume u solves [**2.47**]. It is clear from our considerations that if u maps to $\iota(\mathcal{M})$ then we have solved [**2.46**]. We will prove in the following lemma that if u maps to $\iota(\mathcal{M})$ initially and the initial velocity is tangent to $\iota(\mathcal{M})$ then u maps to $\iota(\mathcal{M})$ for all time.

LEMMA 2.22. *Let $u : [0,T] \times \mathcal{N} \to \tilde{\mathcal{M}} \subset \mathbb{R}^d$ be a smooth solution of equation* [**2.47**] *with $u(0,x) \in \iota(\mathcal{M})$ and $\partial_t u(0,x) \in T_{u(0,x)}\iota(\mathcal{M})$ for all $x \in \mathcal{N}$. Then $u(t,x) \in \iota(\mathcal{M})$ for all $(t,x) \in [0,T] \times \mathcal{N}$ and $\hat{u} = \iota^{-1} \circ u$ solves* [**EQ**] *with $\hat{u}(0,\cdot) = \iota^{-1} \circ u(0,\cdot)$ and $\partial_t \hat{u}(0,\cdot) = D\iota^{-1}(\partial_t u(0,\cdot))$.*

PROOF. Define $\pi^\perp_{\mathcal{M}}(x) = x - \pi_{\mathcal{M}}(x)$. Then clearly $D\pi^\perp_{\mathcal{M}} = \mathbf{1} - D\pi_{\mathcal{M}}$ and $D_A D_B \pi^\perp_{\mathcal{M}} = -D_A D_B \pi_{\mathcal{M}}$. We compute

$$\begin{aligned}
&\partial_t^2(\pi^\perp_{\mathcal{M}} \circ u) - \frac{d\mu_t}{d\mu_0} g^{ij}(\partial_i\partial_j(\pi^\perp_{\mathcal{M}} \circ u) - \Gamma^k_{ij}\partial_k(\pi^\perp_{\mathcal{M}} \circ u)) \\
&= D_C\pi^\perp_{\mathcal{M}}(u)\tilde{\Box} u^C + D_A D_B \pi^\perp_{\mathcal{M}}(u)(\partial_t u^A \partial_t u^B - \frac{d\mu_t}{d\mu_0} g^{ij}\partial_i u^A \partial_j u^B) \\
&= \frac{\varrho}{\widetilde{\mathrm{Vol}}(u)}\frac{d\mu_t}{d\mu_0} D_C\pi^\perp_{\mathcal{M}}(u)\tilde{\nu}^C + D_C\pi^\perp_{\mathcal{M}}(u) D_A D_B \pi^C_{\mathcal{M}}(u)(\partial_t u^A \partial_t u^B - \frac{d\mu_t}{d\mu_0} g^{ij}\partial_i u^A \partial_j u^B) \\
&\quad - D_C\pi^\perp_{\mathcal{M}}(u)\partial_k u^C \langle D_A D_B \pi_{\mathcal{M}}(u), \partial_l u\rangle g^{kl}(\partial_t u^A \partial_t u^B - \frac{d\mu_t}{d\mu_0} g^{ij}\partial_i u^A \partial_j u^B) \\
&\quad + \underbrace{D_A D_B \pi^\perp_{\mathcal{M}}(u)}_{=-D_A D_B \pi_{\mathcal{M}}(u)}(\partial_t u^A \partial_t u^B - \frac{d\mu_t}{d\mu_0} g^{ij}\partial_i u^A \partial_j u^B) \\
&= \frac{\varrho}{\widetilde{\mathrm{Vol}}(u)}\frac{d\mu_t}{d\mu_0} D_C\pi^\perp_{\mathcal{M}}(u)\nu^C(\pi_{\mathcal{M}} \circ u) - \frac{\varrho}{\widetilde{\mathrm{Vol}}(u)}\frac{d\mu_t}{d\mu_0}\langle \nu(\pi_{\mathcal{M}} \circ u), \partial_l u\rangle g^{kl}\partial_k(\pi^\perp_{\mathcal{M}} \circ u) \\
&\quad - D_C\pi_{\mathcal{M}}(u) D_A D_B \pi^C_{\mathcal{M}}(u)(\partial_t u^A \partial_t u^B - \frac{d\mu_t}{d\mu_0} g^{ij}\partial_i u^A \partial_j u^B) \\
&\quad - \langle D_A D_B \pi_{\mathcal{M}}(u), \partial_l u\rangle g^{kl}(\partial_t u^A \partial_t u^B - \frac{d\mu_t}{d\mu_0} g^{ij}\partial_i u^A \partial_j u^B)\partial_k(\pi^\perp_{\mathcal{M}} \circ u). \qquad [\mathbf{2.48}]
\end{aligned}$$

If u maps to $\iota(\mathcal{M})$ then [**2.48**] is a linear wave equation for $\pi^\perp_{\mathcal{M}} \circ u$ because then we know that $D_C\pi^\perp_{\mathcal{M}}(u)\nu^C(\pi_{\mathcal{M}} \circ u) = 0$ and $D_C\pi_{\mathcal{M}}(u) D_A D_B \pi^C_{\mathcal{M}}(u)\partial_\alpha u^A \partial_\beta u^B = 0$ since then $D_A D_B \pi^C_{\mathcal{M}}(u)\partial_\alpha u^A \partial_\beta u^B$ is normal to $\iota(\mathcal{M})$. As we will see in the following energy estimate this is approximately true even if u does not map to $\iota(\mathcal{M})$ and we can estimate the remaining terms in terms of the distance $|\pi^\perp_{\mathcal{M}}(u)|$ of u to $\iota(\mathcal{M})$. Define

$$e(t) = \frac{1}{2}\int_{\mathcal{N}} |\partial_t(\pi^\perp_{\mathcal{M}} \circ u)|^2 d\mu_0 + \frac{1}{2}\int_{\mathcal{N}} |\nabla(\pi^\perp_{\mathcal{M}} \circ u)|^2 d\mu_t + \frac{1}{2}\int_{\mathcal{N}} |\pi^\perp_{\mathcal{M}} \circ u|^2 d\mu_0$$

where of course in local coordinates $|\nabla(\pi^{\perp}_{\mathcal{M}} \circ u)|^2 = g^{ij}\langle \partial_i(\pi^{\perp}_{\mathcal{M}} \circ u), \partial_j(\pi^{\perp}_{\mathcal{M}} \circ u)\rangle$. In the following we compute in local coordinates under the integral which is easy to make rigorous using a partition of unity.

$$\begin{aligned}
\partial_t e(t) = & \int_{\mathcal{N}} \langle \partial_t(\pi^{\perp}_{\mathcal{M}} \circ u), \partial_t^2(\pi^{\perp}_{\mathcal{M}} \circ u)\rangle d\mu_0 + \int_{\mathcal{N}} g^{ij}\langle \partial_i(\pi^{\perp}_{\mathcal{M}} \circ u), \partial_t\partial_j(\pi^{\perp}_{\mathcal{M}} \circ u)\rangle d\mu_t \\
& + \frac{1}{2}\int_{\mathcal{N}} \partial_t g^{ij}\langle \partial_i(\pi^{\perp}_{\mathcal{M}} \circ u), \partial_j(\pi^{\perp}_{\mathcal{M}} \circ u)\rangle d\mu_t + \frac{1}{4}\int_{\mathcal{N}} |\nabla(\pi^{\perp}_{\mathcal{M}} \circ u)|^2 g^{kl}\partial_t g_{kl} d\mu_t \\
& + \int_{\mathcal{N}} \langle \partial_t(\pi^{\perp}_{\mathcal{M}} \circ u), \pi^{\perp}_{\mathcal{M}} \circ u\rangle d\mu_0 \\
\leq & \int_{\mathcal{N}} \langle \partial_t(\pi^{\perp}_{\mathcal{M}} \circ u), \partial_t^2(\pi^{\perp}_{\mathcal{M}} \circ u) - \frac{d\mu_t}{d\mu_0} g^{ij}(\partial_i\partial_j(\pi^{\perp}_{\mathcal{M}} \circ u) - \Gamma^k_{ij}\partial_k(\pi^{\perp}_{\mathcal{M}} \circ u))\rangle d\mu_0 + Ce(t) \\
\overset{[\mathbf{2.48}]}{\leq} & \; Ce(t) + \frac{\varrho}{\widetilde{\mathrm{Vol}}(u)}\int_{\mathcal{N}} \langle D_C\pi^{\perp}_{\mathcal{M}}(u)\nu^C(\pi_{\mathcal{M}} \circ u), \partial_t(\pi^{\perp}_{\mathcal{M}} \circ u)\rangle d\mu_t \\
& - \int_{\mathcal{N}} \langle \partial_t(\pi^{\perp}_{\mathcal{M}} \circ u), D_C\pi_{\mathcal{M}}(u) D_A D_B \pi^C_{\mathcal{M}}(u)(\partial_t u^A \partial_t u^B - \frac{d\mu_t}{d\mu_0} g^{ij}\partial_i u^A \partial_j u^B)\rangle d\mu_0 \\
& - \frac{\varrho}{\widetilde{\mathrm{Vol}}(u)}\int_{\mathcal{N}} \langle \nu(\pi_{\mathcal{M}} \circ u), \partial_l u\rangle g^{kl}\langle \partial_k(\pi^{\perp}_{\mathcal{M}} \circ u), \partial_t(\pi^{\perp}_{\mathcal{M}} \circ u)\rangle d\mu_t \\
& - \int_{\mathcal{N}} (\partial_t u^A \partial_t u^B - \frac{d\mu_t}{d\mu_0} g^{ij}\partial_i u^A \partial_j u^B)\langle D_A D_B \pi_{\mathcal{M}}(u), \partial_l u\rangle g^{kl}\langle \partial_t(\pi^{\perp}_{\mathcal{M}} \circ u), \partial_k(\pi^{\perp}_{\mathcal{M}} \circ u)\rangle d\mu_0.
\end{aligned}$$

Now we show that we can estimate all terms by $Ce(t)$. For the last two integrals this is clear using the Cauchy-Schwarz inequality and $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$. Since we have a fixed solution u we can use that all quantities are bounded depending on u. We can also compare $d\mu_0$ and $d\mu_t$ uniformly. In $p \in \iota(\mathcal{M})$ it is true that $D\pi_{\mathcal{M}}(p)V = V$ for $V \in T_p\iota(\mathcal{M})$ and hence

$$D_C\pi^{\perp}_{\mathcal{M}}(\pi_{\mathcal{M}}(u))\nu^C(\pi_{\mathcal{M}} \circ u) = \nu(\pi_{\mathcal{M}} \circ u) - D_C\pi_{\mathcal{M}}(\pi_{\mathcal{M}}(u))\nu^C(\pi_{\mathcal{M}} \circ u) = 0.$$

Furthermore $\partial_\alpha(\pi_{\mathcal{M}} \circ u)$ is tangent to $\iota(\mathcal{M})$ and hence

$$D_A D_B \pi_{\mathcal{M}}(\pi_{\mathcal{M}}(u))\partial_\alpha(\pi_{\mathcal{M}} \circ u)^A \partial_\beta(\pi_{\mathcal{M}} \circ u)^B$$

is normal to $\iota(\mathcal{M})$ in view of [**2.43**] and [**2.44**]. Since $D\pi_{\mathcal{M}}(\pi_{\mathcal{M}}(u))$ vanishes on normal vectors we have

$$D_C\pi_{\mathcal{M}}(\pi_{\mathcal{M}}(u)) D_A D_B \pi^C_{\mathcal{M}}(\pi_{\mathcal{M}}(u))\partial_\alpha(\pi_{\mathcal{M}} \circ u)^A \partial_\beta(\pi_{\mathcal{M}} \circ u)^B = 0.$$

For $s \in [0,1]$ write $u_s = (1-s)\pi_{\mathcal{M}}(u) + su$. We can estimate

$$\begin{aligned}
\int_{\mathcal{N}} \langle \partial_t(\pi_{\mathcal{M}}^{\perp} \circ u), D_C \pi_{\mathcal{M}}^{\perp}(u)\nu^C(\pi_{\mathcal{M}} \circ u)\rangle d\mu_t &= \int_{\mathcal{N}} \langle \partial_t(\pi_{\mathcal{M}}^{\perp} \circ u), D_C \pi_{\mathcal{M}}^{\perp}(\pi_{\mathcal{M}}(u))\nu^C(\pi_{\mathcal{M}} \circ u)\rangle d\mu_t \\
&\quad + \int_{\mathcal{N}} \langle \partial_t(\pi_{\mathcal{M}}^{\perp} \circ u), \nu^C(\pi_{\mathcal{M}} \circ u) \int_0^1 \partial_s D_C \pi_{\mathcal{M}}^{\perp}(u_s) ds\rangle d\mu_t \\
&= \int_{\mathcal{N}} \langle \partial_t(\pi_{\mathcal{M}}^{\perp} \circ u), \nu^C(\pi_{\mathcal{M}} \circ u) \underbrace{(u - \pi_{\mathcal{M}}(u))}_{=\pi_{\mathcal{M}}^{\perp}(u)}{}^E \int_0^1 D_E D_C \pi_{\mathcal{M}}^{\perp}(u_s) ds\rangle d\mu_t \\
&\leq Ce(t).
\end{aligned}$$

The appereance of $\pi_{\mathcal{M}}^{\perp}(u) = u - \pi_{\mathcal{M}}(u)$ here is the reason to include $|\pi_{\mathcal{M}}^{\perp}(u)|^2$ in the energy $e(t)$. Similarly we use

$$\begin{aligned}
&D_C \pi_{\mathcal{M}}(u) D_A D_B \pi_{\mathcal{M}}^C(u) \partial_\alpha u^A \partial_\beta u^B \\
&\quad = D_C \pi_{\mathcal{M}}(\pi_{\mathcal{M}}(u)) D_A D_B \pi_{\mathcal{M}}^C(\pi_{\mathcal{M}}(u)) \partial_\alpha(\pi_{\mathcal{M}} \circ u)^A \partial_\beta(\pi_{\mathcal{M}} \circ u)^B \\
&\qquad + \int_0^1 \partial_s (D_C \pi_{\mathcal{M}}(u_s) D_A D_B \pi_{\mathcal{M}}^C(u_s) \partial_\alpha u_s^A \partial_\beta u_s^B) ds \\
&\quad = (u - \pi_{\mathcal{M}}(u))^E \int_0^1 D_E D_C \pi_{\mathcal{M}}(u_s) D_A D_B \pi_{\mathcal{M}}^C(u_s) \partial_\alpha u_s^A \partial_\beta u_s^B ds \\
&\qquad + (u - \pi_{\mathcal{M}}(u))^E \int_0^1 D_C \pi_{\mathcal{M}}(u_s) D_E D_A D_B \pi_{\mathcal{M}}^C(u_s) \partial_\alpha u_s^A \partial_\beta u_s^B ds \\
&\qquad + \partial_\alpha (u - \pi_{\mathcal{M}}(u))^E \int_0^1 D_C \pi_{\mathcal{M}}(u_s) D_A D_B \pi_{\mathcal{M}}^C(u_s) \partial_\beta u_s^B ds \\
&\qquad + \partial_\beta (u - \pi_{\mathcal{M}}(u))^E \int_0^1 D_C \pi_{\mathcal{M}}(u_s) D_A D_B \pi_{\mathcal{M}}^C(u_s) \partial_\alpha u_s^A ds
\end{aligned}$$

to estimate

$$\int_{\mathcal{N}} \langle \partial_t(\pi_{\mathcal{M}}^{\perp} \circ u), D_C \pi_{\mathcal{M}}(u) D_A D_B \pi_{\mathcal{M}}^C(u)(\partial_t u^A \partial_t u^B - \frac{d\mu_t}{d\mu_0} g^{ij} \partial_i u^A \partial_j u^B)\rangle d\mu_0 \leq Ce(t).$$

Due to the initial conditions $e(0) = 0$. By Gronwall's Lemma we conclude that $e(t) = 0$ for all $t \in [0,T]$. This implies $(\pi_{\mathcal{M}}^{\perp} \circ u)(t,x) = 0$ for all $(t,x) \in [0,T] \times \mathcal{N}$. This is equivalent to $u(t,x) \in \iota(\mathcal{M})$ for all $(t,x) \in [0,T] \times \mathcal{N}$. Since $\iota : \mathcal{M} \to \iota(\mathcal{M})$ is an isometry we get from [**2.45**] that $\hat{u}$ solves [**EQ**]. □

2.6.2. The Linearisation in the Extrinsic Formulation. To conclude the short time existence proof in the general case, we have to prove that the linearisation of [**2.47**] is a weakly hyperbolic linear system. Now we will take $\mathbf{F}$ to be the Fréchet space $C^\infty([0,T] \times \mathcal{N}, \mathbb{R}^d)$ and $\mathbf{F}_0 = C^\infty(\mathcal{N}, \mathbb{R}^d)$. Define similarly as in Section 2.1

$$\begin{aligned}
\mathbf{U} &= \{u \in \mathbf{F}, \quad \det(g_{ij}) > 0, \text{ for all } t \in [0,T]\}, \\
\mathbf{U}_0 &= \{u \in \mathbf{F}_0, \quad \det(g_{ij}) > 0\}.
\end{aligned}$$

We only need to apply the inverse function theorem on a neighborhood $\mathbf{U}' \subset \mathbf{U}$ of the approximate solution $\bar{u}$ which can be constructed as in Section 2.1. Compared to Proposition 2.21 we have to impose additional conditions on $\mathbf{U}'$ and the time interval. We need $\pi_{\mathcal{M}}(\bar{u})$ to be an immersion in order to define $\tilde{\nu}(\bar{u})$ in [**2.47**]. This will be true if T is small enough since $\pi_{\mathcal{M}}(\bar{u}(0)) = u_0$ is an immersion and this condition is open. If $\mathbf{U}'$ is small enough then this will also be true for all $u \in \mathbf{U}'$.

For the bundle $\mathcal{V}$ we take $\mathcal{V} = \mathcal{N} \times \mathbb{R}^d$. Given a map $u \in \mathbf{U}'$ we have to define the vectors ν_A and τ_k. Choose a local orthonormal frame $\bar{\nu}_A$, $A = 2, \dots, d-(n+1)$ on a neighborhood of a point $u_0(x)$ that spans $\ker D\pi_{\mathcal{M}}$. We then want to define $\bar{\nu}_A^{(\alpha)} = \bar{\nu}_A \circ u$ on a chart domain U_α with $x \in U_\alpha$. By making the coordinate charts and the time interval smaller this can be done for u_0 and $\bar{u}$. By making the neighborhood $\mathbf{U}'$ smaller in C^0 this can be achieved for all $u \in \mathbf{U}'$. Define $\nu_1 = \tilde{\nu} = \pi_{\Sigma_t^\perp}(u)\nu(\pi_{\mathcal{M}} \circ u)$ and $\nu_A = \pi_{\Sigma^\perp}(u)(\bar{\nu}_A)$ for $A = 2, \dots, d-(n+1)$. This will then give a basis for the space orthogonal to $T_{u(x)}\Sigma$ if u is close to $\pi_{\mathcal{M}}(\bar{u})$ in C^1. As the projection $\pi_{\Sigma^\perp}(u)$ and $\tilde{\nu}$ depend smoothly on u and Du also the ν_A will depend smoothly on u and Du. The vector $\tilde{\nu}$ can be estimated as the ν in Lemma 2.19 if we assume $\det(g_{ij}(\pi_{\mathcal{M}} \circ u)) > \lambda_1' > 0$. We also get similar estimates for the other ν_A since the only term that could blow up in the projection is g^{ij} which we can also estimate as in Lemma 2.19. For τ_k we simply take $\partial_k u$.

To estimate $\mathrm{Vol}(u)$ define $u_s = \pi_{\mathcal{M}}(\bar{u} + s(u - \bar{u}))$ for $s \in [0,1]$ and use

$$\mathrm{Vol}(u) = \mathrm{Vol}(\bar{u}) + \int_0^1 \partial_s \mathrm{Vol}(u_s) ds = \mathrm{Vol}(\bar{u}) + \int_0^1 \int_{\mathcal{N}} \langle \nu(u_s), D\pi_{\mathcal{M}}(u - \bar{u})\rangle d\mu_t$$

instead of [**2.42**]. Suitable estimates on $\mathrm{Vol}(u)$ can then be obtained using conditions (3) and (5) below. Then we choose $\mathbf{U}'$ such that the following properties are satisfied

(1) $u(x,t) \in \tilde{\mathcal{M}}$ for all $(t,x) \in [0,T] \times \mathcal{N}$,
(2) $\Lambda\delta^{ij} \geq g^{ij}(u) \geq \lambda\delta^{ij}$ for some $\Lambda, \lambda > 0$,
(3) $\Lambda'\delta^{ij} \geq g^{ij}(u_s) \geq \lambda'\delta^{ij}$ for some $\Lambda', \lambda' > 0$ and all $s \in [0,1]$,
(4) $\det(g_{ij}) > \lambda_1$ for some $\lambda_1 > 0$,
(5) $\det(g_{ij}(u_s)) > \lambda_1'$ for some $\lambda_1' > 0$ and for all $s \in [0,1]$,
(6) $\pi_{\mathcal{M}}(u(t,\cdot))$ is an immersion,
(7) ν_A can be defined as above,
(8) $\det(\nu_{AB}) > \lambda_1''$ for some $\lambda_1'' > 0$,
(9) $\|u\|_{\lfloor\frac{n}{2}\rfloor+5} + \|\partial_t u\|_{\lfloor\frac{n}{2}\rfloor+5} + \|\partial_t^2 u\|_{\lfloor\frac{n}{2}\rfloor+4} + \|\partial_t^3 u\|_{\lfloor\frac{n}{2}\rfloor+3} < K_1^*$ for some $K_1^* > 0$,
(10) $\mathrm{Vol}(\pi_{\mathcal{M}}(u))$ is defined and $\Lambda_2 > \mathrm{Vol}(\pi_{\mathcal{M}}(u)) > \lambda_2$ for some $\Lambda_2, \lambda_2 > 0$.

Clearly all these properties, which are in parts redundant, can be achieved on an open neighborhood $\mathbf{U}'$ of $\bar{u}$.

Define

$$\mathfrak{Q}(u) = \tilde{\Box} u - \frac{d\mu_t}{d\mu_0} \frac{\varrho}{\widetilde{\mathrm{Vol}}(u)} \tilde{\nu} - \Pi_u(\partial_t u, \partial_t u) + \frac{d\mu_t}{d\mu_0} g^{ij} \Pi_u(\partial_i u, \partial_j u).$$

Proposition 2.23. *Let $V, W \in C^\infty([0,T] \times \mathcal{N}, \mathbb{R}^d)$. Then for $u \in \mathbf{U}'$ the equation $D\mathfrak{Q}(u)\{V\} = W$ is a WHLS with respect to ν_A, τ_k and the bundle $\mathcal{V}$ defined above.*

PROOF. Let u_ε be a variation of u with $\partial_\varepsilon|_{\varepsilon=0}u_\varepsilon = V$. If $V = \psi^k\tau_k$ is tangential then the variation of $d\mu_t$ gives a derivative of V. But this term always accompanies a normal term. This is the reason for defining Π and $\tilde{\nu}$ with the additional projection to the normal space of Σ_t. All the other terms are diffeomorphism invariant and thus no spatial derivative of V occurs in the tangential part of $D\mathfrak{Q}(u)\{V\}$.

Now let V be arbitrary. The decomposition into normal and tangential part of the variation of the second time derivatives $\partial_\varepsilon|_{\varepsilon=0}\partial_t^2 u = \partial_t^2 V$ is

$$\begin{aligned}\langle\partial_t^2 V, \nu_B\rangle\nu^{BA} =&\partial_t^2\varphi^A + 2\partial_t\varphi^C\langle\partial_t\nu_C, \nu_B\rangle\nu^{BA} + \varphi^C\langle\partial_t^2\nu_C, \nu_B\rangle\nu^{BA}\\ &+ 2\partial_t\psi^k\langle\partial_t\tau_k, \nu_B\rangle\nu^{BA} + \psi^k\langle\partial_t^2\tau_k, \nu_B\rangle\nu^{BA}\\ \langle\partial_t^2 V, \tau_l\rangle\tau^{lk} =&\partial_t^2\psi^k + 2\partial_t\psi^m\langle\partial_t\tau_m, \tau_l\rangle\tau^{lk} + \psi^m\langle\partial_t^2\tau_m, \tau_l\rangle\tau^{lk}\\ &+ 2\partial_t\varphi^A\langle\partial_t\nu_A, \tau_l\rangle\tau^{lk} + \varphi^A\langle\partial_t^2\nu_A, \tau_l\rangle\tau^{lk}.\end{aligned}$$

Since the operator $\mathfrak{Q}$ is of second order, $D\mathfrak{Q}(u)V$ is also of second order in V. The only term generating second derivatives in the variation is $\Delta u = g^{ij}(\partial_i\partial_j u - \langle\partial_i\partial_j u, \partial_l u\rangle g^{lk}\partial_k u)$. If $V = \varphi^A\nu_A$ then

$$\begin{aligned}\partial_\varepsilon|_{\varepsilon=0}\Delta u =&g^{ij}(\partial_i\partial_j V - \langle\partial_i\partial_j V, \partial_l u\rangle g^{kl}\partial_k u) + \text{ lower order terms}\\ =&g^{ij}\partial_i\partial_j\varphi^A\nu_A + \text{ lower order terms}\end{aligned}$$

and so

$$\begin{aligned}\langle\partial_\varepsilon|_{\varepsilon=0}\Delta u, \nu_B\rangle\nu^{BA} =&g^{ij}\partial_i\partial_j\varphi^A + \text{ lower order terms}\\ \langle\partial_\varepsilon|_{\varepsilon=0}\Delta u, \tau_l\rangle\tau^{lk} =&0 + \text{ lower order terms.}\end{aligned}$$

By "lower order terms" we mean terms with at most first derivatives of φ^A and at most third derivatives of u.

Now

$$\begin{aligned}\partial_\varepsilon\widetilde{\mathrm{Vol}}(u) =& \int_{\mathcal{N}}\langle\nu(\pi_{\mathcal{M}}\circ u), V\rangle d\mu_t\\ =&\sum_\alpha\int_{\mathcal{N}}\eta_\alpha\langle\nu(\pi_{\mathcal{M}}\circ u), \nu_B\rangle\varphi^B_{(\alpha)}\frac{d\mu_t}{d\mu_0}d\mu_0 + \sum_\alpha\int_{\mathcal{N}}\eta_\alpha\langle\nu(\pi_{\mathcal{M}}\circ u), \tau_l\rangle\psi^l_{(\alpha)}\frac{d\mu_t}{d\mu_0}d\mu_0\end{aligned}$$

where η_α is a partition of unity subordinate to the sets $x_\alpha^{-1}(B_2(0))$. The variation of the volume only occurs in the $\nu_1 = \tilde{\nu}$-part of the system.

We have checked the wave equation structure for the φ^A, the ODE structure for the ψ^k and the integral terms which will subsumed into the operators N and Q. This implies that we indeed have a WHLS. □

The Nash-Moser argument applies and this concludes the proof of Theorem 2.1 in the general case.

CHAPTER 3

A Continuation Criterion and Stability Estimates

In the first part of this chapter we will prove a continuation criterion (Theorem 3.1) for a solution u of [**EQ**] which can also be interpreted as a singularity criterion. If $u : [0,T) \times \mathcal{N} \to \mathcal{M}$ then this will be a condition under which there is no singularity at time T, i. e. the solution can be extended uniquely to $[0, T + \delta]$ for some $\delta > 0$.

For quasilinear symmetric hyperbolic systems the standard continuation criterion is the condition that first derivatives of the solution stay bounded [**Tay97**, Ch. 16, Prop. 1.5]. For second order quasilinear wave equations this corresponds to the condition that second derivatives have to be bounded in order to extend the solution. As our equation is not strictly hyperbolic we do not get the same condition. Our condition will be a bound on fourth derivatives of the parametrisation and its time derivative.

Other well known examples of continuation/singularity criteria come from Ricci-flow and Mean Curvature Flow. If the curvature tensor stays bounded then the solution of Ricci flow can be extended [**Ham82b**], while if the second fundamental form is bounded, then a solution to Mean Curvature Flow can be extended [**Hui84**].

The difficulty is that, as our equation is not strictly hyperbolic, the solution u itself does not satisfy an equation for which we have any useful estimates in this context so far. In Chapter 2 we have only obtained estimates for the linearised operator. But if we differentiate the equation in space or time, the leading order term will always be the linearised operator. So we could use this and try to estimate derivatives of u. As our equation is of second order, the decomposition into normal and tangential parts gives us terms involving third derivatives no matter how often we differentiate the equation. So differentiating the equation once is not enough to close the loop in the Gronwall argument. But if we compute enough higher derivatives then it is possible to close the loop and obtain estimates for the solution. We do most of the necessary computations in Appendix A.

We use the same method to estimate the distance from u to another solution $\tilde{u}$ in Theorem 3.2. We thereby obtain the estimate that the distance between the two solutions only grows exponentially if they are close to each other initially. As a special case we get the uniqueness of solutions. The maximal time of existence can be estimated from below by the negative logarithm of the initial distance between u and $\tilde{u}$. A similar estimate holds if the ambient manifold is a perturbation of Euclidean space. We can then estimate the maximal time of existence in terms of the distance from the ambient metric to the Euclidean metric, see Theorem 3.10.

Throughout this chapter we assume that the metric $\overline{g}$ of the ambient manifold $\mathcal{M}$ and all its derivatives are uniformly bounded in local coordinates. By the Nash embedding theorem we can assume that $\mathcal{M}$ is isometrically embedded into $\mathbb{R}^d$. We assume that the second fundamental form of $\mathcal{M}$ and its derivatives are uniformly bounded.

3.1. The Continuation Criterion

THEOREM 3.1. *Let $u : [0,T) \times \mathcal{N} \to \mathcal{M}$ be a solution of* [**EQ**]. *Assume that for all $t \in [0,T)$*

$$\|u(t,\cdot)\|_{C^4} + \|\partial_t u(t,\cdot)\|_{C^4} \le K$$

for some $K > 0$. Then there exists $\delta > 0$ such that u can be extended to a solution $\tilde{u} : [0, T+\delta] \times \mathcal{N} \to \mathcal{M}$ of [**EQ**].

REMARK 3.2. This statement can be formulated as a singularity criterion: If the solution u cannot be extended beyond time T then $\|u\|_{C^4} + \|\partial_t u\|_{C^4}$ becomes unbounded as $t \to T$.

DEFINITION 3.3. Let $u : [0,T) \times \mathcal{N} \to \mathcal{M}$. Define

$$\beta_i = \langle \overline{\nabla}_{\partial_i} \partial_t u, \nu \rangle \text{ and } B_{im} = \langle \overline{\nabla}_{\partial_i} \partial_t u, \partial_m u \rangle$$

i. e. $\overline{\nabla}_{\partial_i} \partial_t u = \beta_i \nu + B_i{}^k \partial_k u$.

Recall from Definition 1.2 that $\partial_t u = \sigma\nu + S^k \partial_k u$ and $\overline{\nabla}_{\partial_t} \partial_t u = \alpha\nu$ if u solves [**EQ**].

REMARK 3.4. As in Section 2.6 we assume that $\iota : \mathcal{M} \to \mathbb{R}^d$ is an isometric embedding and if we identify $\mathcal{M}$ and $\iota(\mathcal{M})$ we can assume $\mathcal{M} \subset \mathbb{R}^d$. When we consider norms of u we will take the norm of $u : [0,T) \times \mathcal{N} \to \mathbb{R}^d$ as a map into $\mathbb{R}^d$. If we identify u and $\iota \circ u$ we can also identify $\nu(u)$ and $\nu(\iota \circ u)$ and $\partial_k u$ and $\partial_k(\iota \circ u)$. Let $\bar{h}$ be the vector valued second fundamental form of $\mathcal{M}$. With our identifications we can write

$$\begin{aligned} \partial_i \partial_j u =& \overline{\nabla}_{\partial_i} \partial_j u + \bar{h}(\partial_i u, \partial_j u) \\ =& - h_{ij}\nu + \Gamma_{ij}^k \partial_k u + \bar{h}(\partial_i u, \partial_j u) \end{aligned}$$

and similarly

$$\begin{aligned} \partial_t \partial_i u =& \beta_i \nu + B_i{}^k \partial_k u + \bar{h}(\partial_t u, \partial_i u) \\ \partial_t \partial_t u =& \alpha\nu + \bar{h}(\partial_t u, \partial_t u) \\ =& \left(-H + \frac{\varrho}{\mathrm{Vol}(u)} \right) \frac{d\mu_t}{d\mu_0} \nu + \sigma^2 \bar{h}(\nu,\nu) + 2\sigma S^l \bar{h}(\nu, \partial_l u) + S^l S^k \bar{h}(\partial_l u, \partial_k u) \end{aligned} \quad [\mathbf{3.1}]$$

and

$$\begin{aligned}\partial_t\partial_t\partial_i u =& \partial_t(\overline{\nabla}_{\partial_t}\partial_i u) + \partial_t(\bar{h}(\partial_t u, \partial_i u)) = \overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t}\partial_i u + \bar{h}(\partial_t u, \overline{\nabla}_{\partial_t}\partial_i u) + \partial_t(\bar{h}(\partial_t u, \partial_i u))\\ =& \partial_t\beta_i\nu + \beta_i\overline{\nabla}_{\partial_t}\nu + \partial_t B_i{}^k\partial_k u + B_i{}^k\overline{\nabla}_{\partial_t}\partial_k u + \bar{h}(\partial_t u, \overline{\nabla}_{\partial_t}\partial_i u)\\ &+ D\bar{h}(\partial_t u, \partial_i u, \partial_t u) + \bar{h}(\overline{\nabla}_{\partial_t}\partial_t u, \partial_i u) + \bar{h}(\partial_t u, \overline{\nabla}_{\partial_t}\partial_i u)\\ =& (\partial_t\beta_i + B_i{}^k\beta_k)\nu + (\partial_t B_i{}^k - \beta_i\beta^k + B_i{}^j B_j{}^k)\partial_k u + 2\sigma\beta_i\bar{h}(\nu,\nu)\\ &+ 2\sigma B_i{}^k\bar{h}(\nu, \partial_k u) + 2S^l\beta_i\bar{h}(\partial_l u, \nu) + 2S^l B_i{}^k\bar{h}(\partial_l u, \partial_k u) + \alpha\bar{h}(\nu, \partial_i u)\\ &+ \sigma^2 D\bar{h}(\nu, \partial_i u, \nu) + \sigma S^l D\bar{h}(\nu, \partial_i u, \partial_l u)\\ &+ \sigma S^l D\bar{h}(\partial_l u, \partial_i u, \nu) + S^k S^l D\bar{h}(\partial_l u, \partial_i u, \partial_k u). \end{aligned} \qquad [\mathbf{3.2}]$$

LEMMA 3.5. *Let u be a solution of* [**EQ**]. *Let*

$$\begin{aligned}\left(\varphi^A\right)_{A=1,\dots,1+n^2} =& \Big(\sigma, (h_{ij})_{i,j=1,\dots,n}\Big)\\ \left(\psi^k\right)_{k=1,\dots,d+nd+n+n^2+n^3} =& \Bigg(\left(u^A\right)_{A=1,\dots,d}, \left(\partial_i u^A\right)_{\substack{i=1,\dots,n\\ A=1,\dots,d}}, \left(S^j\right)_{j=1,\dots,n},\\ & \qquad (B_l{}^m)_{l,m=1,\dots,n}, \left(\Gamma^k_{ij} - \mathring{\Gamma}^k_{ij}\right)_{i,j,k=1,\dots,n}\Bigg).\end{aligned}$$

Then (φ^A) *and* (ψ^k) *satisfy*

$$\begin{aligned}\partial_t^2\varphi^A(t,x) - L^A\varphi^A(t,x) =& F^A(x, \varphi, D\varphi, \partial_t\varphi, \psi, D\psi, \partial_t\psi, \mathrm{Vol}(u))\\ \partial_t^2\psi^k =& G^k(x, \varphi, D\varphi, \partial_t\varphi, \psi, \partial_t\psi, \mathrm{Vol}(u))\end{aligned}$$

where F^A and G^k are smooth functions in all their arguments such that if $\|\varphi\|_{C^1} + \|\partial_t\varphi\|_{C^0} + \|\psi\|_{C^1} + \|\partial_t\psi\|_{C^0} \le K$ and $\det(g_{ij}) > \lambda_1$ for some $K, \lambda_1 > 0$ then F^A and G^A and all their derivatives stay bounded. The operator L is given by $L^A\varphi^A = \frac{d\mu_t}{d\mu_0}\Delta\varphi^A$.

PROOF. The evolution equations for u^A and $\partial_i u^A$ are [**3.1**] and [**3.2**] if we use [**A.2**] and [**A.4**] to replace β_i and $\partial_t\beta_i$. The calculations for the other evolution equations are postponed to Appendix A. The evolution equations for σ, S^k, h_{ij}, Γ^k_{ij}, and $B_i{}^k$ are equations [**A.9**],[**A.10**],[**A.11**],[**A.12**] and [**A.13**], respectively. We have written $\Gamma^k_{ij} - \mathring{\Gamma}^k_{ij}$ instead of only Γ^k_{ij} as we want ψ^k to be a tensor. All evolution equations can be written with a dependence on $\Gamma^k_{ij} - \mathring{\Gamma}^k_{ij}$ instead on Γ^k_{ij}. We only have to check that the functions F and G and their derivatives do not blow up. The only terms that could possibly blow up in the evolution equations are g^{ij}, ν and $\mathrm{Vol}(u)^{-1}$. The latter is bounded in view of the energy conservation. For the other ones we use Remark 2.20. □

LEMMA 3.6. *Let $u : [0,T) \times \mathcal{N} \to \mathcal{M}$ with $\|\partial_t u\|_{C^1} + \|u\|_{C^1} \le K$ for some $K > 0$. Then the metrics $g_{ij}(t)$ for all different times are equivalent, and they converge as $t \to T$ uniformly to a positive definite metric $g_{ij}(T)$ which is continuous and also equivalent.*

Proof. We have that

$$\partial_t g_{ij} = \langle \overline{\nabla}_{\partial_t}\partial_i u, \partial_j u\rangle + \langle \partial_i u, \overline{\nabla}_{\partial_t}\partial_j u\rangle.$$

By the assumptions all these terms are bounded and so $\partial_t g_{ij}$ is bounded uniformly for $0 \le t < T$. Now we apply [**Ham82b**, 14.2 Lemma]. □

Proof of Theorem 3.1. Let $s \ge \lfloor \frac{n}{2} \rfloor + 2$.

1. Estimating the operator L. The coefficients of $L = \frac{d\mu_t}{d\mu_0}\Delta$ contain g^{ij}, Christoffel symbols and derivatives of Christoffel symbols. Lemma 3.6 implies $\det(g_{ij}) \ge \lambda_1$ for some $\lambda_1 > 0$. By the Moser inequalities and Lemma 2.19 we can estimate

$$[L]_s \le C(1 + \|u\|_{s+3})$$

with C depending on K since L contains no more than third derivatives of u. Clearly also by the assumption $[L]_{C^1} \le C_K$. Similarly $[\partial_t L]_s \le C(1 + \|u\|_{s+3} + \|\partial_t u\|_{s+3})$ and $[\partial_t L]_{C^1} \le C_K$.

2. Application of the estimate for WHLS. In order to apply Proposition 2.14 we have to specify the vector bundle $\mathcal{V}$ and the basis in which the system from Lemma 3.5 is a WHLS if we consider the right hand side $F^A(\dots)$, $G^k(\dots)$ as fixed functions. For the bundle we take

$$\begin{aligned}\mathcal{V} = (\mathcal{N} \times \mathbb{R}) \oplus (T^*\mathcal{N} \otimes T^*\mathcal{N}) \oplus (\mathcal{N} \times \mathbb{R}^d) \oplus ((\mathcal{N} \times \mathbb{R}^d) \otimes T^*\mathcal{N})& \\ \oplus T\mathcal{N} \oplus (T^*\mathcal{N} \otimes T\mathcal{N}) \oplus (T^*\mathcal{N} \otimes T^*\mathcal{N} \otimes T\mathcal{N})& \quad [\mathbf{3.3}]\end{aligned}$$

with metric induced by g_0 on $T\mathcal{N}$ and $T^*\mathcal{N}$ and the Euclidean metric on $\mathcal{N} \times \mathbb{R}^d$. Let (x_α, U_α) be a local coordinate chart with canonical tangent vectors ∂_k and dx^i its dual covectors. For ν_A we take

$$\{\nu_A\}_{A=1,\dots,1+n^2} = \{(1,0,0,0,0,0,0)(0, dx^i \otimes dx^j, 0,0,0,0,0)_{i,j=1,\dots,n}\} \qquad [\mathbf{3.4}]$$

and for τ_k we take

$$\begin{aligned}\{\tau_k\}_{k=1,\dots,d+nd+n+n^2+n^3} = \{&(0,0,e_A,0,0,0,0)_{A=1,\dots,d}, (0,0,0,e_A \otimes dx^i,0,0,0)_{\substack{i=1,\dots,n\\ A=1,\dots,d}} \\ &(0,0,0,0,\partial_k,0,0)_{k=1,\dots,n}, (0,0,0,0,0,dx^i \otimes \partial_k, 0)_{i,k=1,\dots,n}, \\ &(0,0,0,0,0,0,dx^i \otimes dx^j \otimes \partial_k)_{i,j,k=1,\dots,n}\}. \quad [\mathbf{3.5}]\end{aligned}$$

Here ν_A and τ_k are independent of the solution and $\det(\nu_{AB}) \ge \lambda_1$ and $\det(\tau_{kl}) \ge \lambda_1$ for some $\lambda_1 > 0$. The system for φ, ψ from Lemma 3.5 is a WHLS with $M = N = P = Q = 0$ and right hand side F^A, G^k. By assumption and Lemma 3.5 the right hand side is bounded. In order to apply Proposition 2.14 we have to check that

$$\|\varphi\|_{C^2} + \|\partial_t\varphi\|_{C^2} + \|\psi\|_{C^1} + \|\partial_t\psi\|_{C^1} \le C_K. \qquad [\mathbf{3.6}]$$

By assumption it is clear that σ, h_{ij} and $\partial_t h_{ij}$ are bounded in C^2. To bound $\partial_t\sigma$ in C^2 we compute using [**A.3**]

$$\partial_t\sigma = \langle \overline{\nabla}_{\partial_t}\partial_t u, \nu\rangle + \langle \partial_t u, \overline{\nabla}_{\partial_t}\nu\rangle = \alpha - \langle \nabla\sigma, S\rangle + h_{ij}S^i S^j.$$

Hence also $\partial_t\sigma$ is bounded in C^2 by assumption. Clearly u^A, $\partial_i u^A$, S^j, ${B_i}^k$, Γ_{ij}^k, $\partial_t u^A$, $\partial_t\partial_i u^A$, $\partial_t\Gamma_{ij}^k$ are bounded in C^1 by assumption. For $\partial_t S^j$ and $\partial_t {B_i}^k$ a C^1-bound is readily obtained using $\partial_t S_i = \sigma\beta_i + S^k B_{ik}$ and [**A.6**]. As in [**2.9**] define

$$E_s(t) = \|\partial_t^2\varphi(t,\cdot)\|_s + \|\partial_t\varphi(t,\cdot)\|_{s+1} + \|\varphi(t,\cdot)\|_{s+1} + \|\partial_t\psi(t,\cdot)\|_{s+1} + \|\psi(t,\cdot)\|_{s+1}.$$

Hence we can apply Proposition 2.14 to obtain

$$E_s(t) \le Ce^{Ct}(1 + E_s(0) + \int_0^t \|F(\dots)\|_s + \|\partial_t(F(\dots))\|_s + \|G(\dots)\|_{s+1} + [L]_s + [\partial_t L]_s\, dt').$$

3. Estimating the right hand side. We have to estimate $\|F(\dots)\|_s$, $\|\partial_t(F(\dots))\|_s$ and $\|G(\dots)\|_{s+1}$. By assumption

$$|\mathrm{Vol}(u)| = |\mathrm{Vol}_0 + \int_0^t \int_{\mathcal{N}} \langle\partial_t u, \nu\rangle d\mu_t dt'| \le C$$

and

$$|\partial_t \mathrm{Vol}(u)| = |\int_{\mathcal{N}} \langle\partial_t u, \nu\rangle d\mu_t| \le C. \qquad [\mathbf{3.7}]$$

By assumption and Lemma 3.5 we can apply the third Moser inequality Theorem C.5 to estimate

$$\begin{aligned}
\|F(\dots)\|_s \le& C(1 + \|\varphi\|_{s+1} + \|\partial_t\varphi\|_s + \|\psi\|_{s+1} + \|\partial_t\psi\|_s + \|\mathrm{Vol}(u)\|_0)\\
\le& C(1 + E_s)\\
\|G(\dots)\|_{s+1} \le& C(1 + \|\varphi\|_{s+2} + \|\partial_t\varphi\|_{s+1} + \|\psi\|_{s+1} + \|\partial_t\psi\|_{s+1} + \|\mathrm{Vol}(u))\|_0)\\
\le& C(1 + E_s + \|\varphi\|_{s+2}).
\end{aligned}$$

We use the elliptic estimate Lemma 2.11 part (2) since $\|\varphi\|_{C^2} \le K$

$$\begin{aligned}
\|\varphi\|_{s+2} \le& C(\|L\varphi\|_s + \|\varphi\|_{s+1} + [L]_s)\\
\le& C(\|\partial_t^2\varphi\|_s + \|F(\dots)\|_s + \|\varphi\|_{s+1} + \|u\|_{s+3} + 1)\\
\le& C(1 + E_s).
\end{aligned}$$

Note that an application of Lemma 2.7 was used here to estimate norms on large domains against norms on small domains.

Now by the chain rule

$$\partial_t(F(\dots)) = \tilde{F}(x, \varphi, D\varphi, \partial_t\varphi, \partial_t^2\varphi, \partial_t D\varphi, \psi, D\psi, \partial_t\psi, \partial_t^2\psi, \partial_t D\psi, \mathrm{Vol}(u), \partial_t \mathrm{Vol}(u))$$

and $\tilde{F}$ is also smooth and with bounded derivatives as long as the arguments are bounded because the partial derivatives of F are bounded as long as the arguments are bounded. In fact, the arguments are bounded by [**3.6**],[**3.7**] and the evolution equations. So we can use the third Moser inequality to estimate

$$\begin{aligned}
\|\partial_t(F(\dots))\|_s \le& C(1 + \|\varphi\|_{s+1} + \|\partial_t\varphi\|_{s+1} + \|\partial_t^2\varphi\|_s + \|\psi\|_{s+1} + \|\partial_t\psi\|_{s+1} + \|\partial_t^2\psi\|_s\\
&+ \|\mathrm{Vol}(u)\|_0 + \|\partial_t \mathrm{Vol}(u)\|_0)\\
\le& C(1 + E_s + \|\partial_t^2\psi\|_s).
\end{aligned}$$

By the equation for $\partial_t^2\psi$ we can estimate

$$\|\partial_t^2\psi\|_s \le C\|G(\dots)\|_s \le C\|G(\dots)\|_{s+1}$$

and we have already estimated the last term. Altogether we find that

$$\|F(\dots)\|_s + \|\partial_t(F(\dots))\|_s + \|G(\dots)\|_{s+1} \le C(1+E_s).$$

The initial energy $E_s(0)$ is fixed. So we obtain the estimate

$$E_s(t) \le C_{T,K}(1 + \int_0^t 1 + E_s(t')dt').$$

We use Gronwall's Lemma to conclude

$$\|\varphi(t,\cdot)\|_{s+1} + \|\partial_t\varphi(t,\cdot)\|_{s+1} + \|\partial_t^2\varphi(t,\cdot)\|_s + \|\psi(t,\cdot)\|_{s+1} + \|\partial_t\psi(t,\cdot)\|_{s+1} \le C$$

for all $0 \le t < T$. This can be done for any $s \ge \lfloor \frac{n}{2} \rfloor + 2$. This implies that all derivatives of u are bounded uniformly in t.

4. Convergence for $t \to T$. The sequence $u(t_i)$ for $t_i \to T$, $t_i < T$ is bounded in C^∞ and therefore has a convergent subsequence to a smooth limit function $u(T)$. Since

$$|u(t_i) - u(t_k)| \le C \int_{t_i}^{t_k} |\partial_t u| dt' \le C(t_i - t_k)$$

we have uniform convergence in C^0. This implies that the limit is unique and that the whole sequence converges. The same argument applies for $\partial_t u(t_i)$ and the limit $\partial_t u(T)$ is the time derivative of $u(t)$ at $t = T$. Smoothness in time on $[0,T]$ follows by an induction using the equation. As we have seen before in Lemma 3.6, $u(T)$ is an immersion. Then we can apply the short time existence result Theorem 2.1 with initial conditions $u(T)$ and $\partial_t u(T)$ to extend the solution. □

3.2. Stability Estimates

Assume that we have a solution $\tilde{u} : [0,\tilde{T}) \times \mathcal{N} \to \mathcal{M}$ of [**EQ**]. For instance this could be one of the special solutions in Section 1.4 with $\tilde{T} = \infty$. All quantities with ˜ will refer to $\tilde{u}$, e. g. $\tilde{h}_{ij}$ is the second fundamental form of $\tilde{u}$. We intend to prove the following theorem.

THEOREM 3.7. *Let $s \ge \lfloor \frac{n}{2} \rfloor + 2$, $\tilde{T} \in \mathbb{R}^+ \cup \{\infty\}$ and $\tilde{u} : [0,\tilde{T}) \times \mathcal{N} \to \mathcal{M}$ be a solution of* [**EQ**]. *Assume for all $t \in [0,\tilde{T})$*

$$\|\tilde{u}(t,\cdot)\|_{s+4} + \|\partial_t\tilde{u}(t,\cdot)\|_{s+3} \le K \text{ and } \det(\tilde{g}_{ij}) \ge \lambda_1 \text{ and } \tilde{g}^{ij} \ge \lambda$$

for some constants $K, \lambda_1, \lambda > 0$. There exist constants $c_1, c_2 > 0$ and $\varepsilon_0 > 0$ such that if $u_0 : \mathcal{N} \to \mathcal{M}$ is an immersion and $u_1 : \mathcal{N} \to u_0^*T\mathcal{M}$ is a vector field along u_0 with

$$\|u_0 - \tilde{u}(0)\|_{s+4} + \|u_1 - \partial_t\tilde{u}(0)\|_{s+3} \le \varepsilon$$

for some $0 < \varepsilon \le \varepsilon_0$ *then there exists* $T \ge \min\{\tilde{T}, c_1 \log\left(\frac{c_2}{\varepsilon}\right)\}$ *and* $u : [0,T] \times \mathcal{N} \to \mathcal{M}$ *that solves*

$$\begin{cases} \overline{\nabla}_{\partial_t} \partial_t u = \frac{d\mu_t}{d\mu_0}\left(-H(u) + \frac{\varrho}{\mathrm{Vol}(u)}\right)\nu, \text{ for all } t \in [0,T] \\ u(0,\cdot) = u_0 \\ \partial_t u(0,\cdot) = u_1. \end{cases}$$

For all $t \in [0,T]$ *we have the estimate*

$$\|u(t,\cdot) - \tilde{u}(t,\cdot)\|_{s+4} + \|\partial_t u(t,\cdot) - \partial_t \tilde{u}(t,\cdot)\|_{s+3} \le Ce^{Ct}\varepsilon$$

with C *depending on* s *and* $\tilde{u}$.

Corollary 3.8 (Uniqueness). *Let* $u : [0,T] \times \mathcal{N} \to \mathcal{M}$ *and* $\tilde{u} : [0,T] \times \mathcal{N} \to \mathcal{M}$ *be solutions of* [**EQ**] *with* $u(0,\cdot) = \tilde{u}(0,\cdot)$ *and* $\partial_t u(0,\cdot) = \partial_t \tilde{u}(0,\cdot)$. *Then* $u(t,\cdot) = \tilde{u}(t,\cdot)$ *for all* $t \in [0,T]$.

Lemma 3.9. *Let* u *be a solution of* [**EQ**] *and* $\tilde{u}$ *as in Theorem 3.7. Let*

$$\begin{aligned} \left(\varphi^A\right)_{A=1,\dots,1+n^2} =& \bigg((\sigma - \tilde{\sigma}), \left(h_{ij} - \tilde{h}_{ij}\right)_{i,j=1,\dots,n} \bigg) \\ \left(\psi^k\right)_{k=1,\dots,d+nd+n+n^2+n^3} =& \bigg(\left(u^A - \tilde{u}^A\right)_{A=1,\dots,d}, \left(\partial_i u^A - \partial_i \tilde{u}^A\right)_{\substack{i=1,\dots,n \\ A=1,\dots,d}}, \left(S^j - \tilde{S}^j\right)_{j=1,\dots,n}, \\ & \left(B_l{}^m - \tilde{B}_l{}^m\right)_{l,m=1,\dots,n}, \left(\Gamma^k_{ij} - \tilde{\Gamma}^k_{ij}\right)_{i,j,k=1,\dots,n} \bigg). \end{aligned}$$

Then (φ^A) *and* (ψ^k) *satisfy*

$$\begin{aligned} \partial_t^2 \varphi^A(t,x) - L^A \varphi^A(t,x) =& F^A(x, \varphi, D\varphi, \partial_t\varphi, \psi, D\psi, \partial_t\psi, \mathrm{Vol}(u) - \mathrm{Vol}(\tilde{u})) \\ \partial_t^2 \psi^k =& G^k(x, \varphi, D\varphi, \partial_t\varphi, \psi, \partial_t\psi, \mathrm{Vol}(u) - \mathrm{Vol}(\tilde{u})) \end{aligned}$$

where F^A *and* G^k *are smooth functions in all their arguments such that if* $\|\varphi\|_{C^1} + \|\partial_t\varphi\|_{C^0} + \|\psi\|_{C^1} + \|\partial_t\psi\|_{C^0} \le K$ *and* $\det(g_{ij}) > \lambda_1$ *then* F^A *and* G^A *and all its derivatives stay bounded. Furthermore* $F(x,0,0,0,0,0,0,0) = 0$ *and* $G(x,0,0,0,0,0,0) = 0$ *and the operator* L *is given by* $L^A\varphi^A = \frac{d\mu_t}{d\mu_0}\Delta\varphi^A$.

Proof. Simply substract the evolution equations for all the quantities corresponding to $\tilde{u}$ from the evolution equations for the quantities corresponding to u. Replace everywhere e. g. $\sigma = \tilde{\sigma} + (\sigma - \tilde{\sigma})$ to write the equations as equations for the differences. To get the term including L write e. g.

$$\frac{d\mu_t}{d\mu_0}\Delta\sigma - \frac{\widetilde{d\mu_t}}{d\mu_0}\tilde{\Delta}\tilde{\sigma} = \frac{d\mu_t}{d\mu_0}\Delta(\sigma - \tilde{\sigma}) + \left(\frac{d\mu_t}{d\mu_0}\Delta - \frac{\widetilde{d\mu_t}}{d\mu_0}\tilde{\Delta}\right)\tilde{\sigma}.$$

It is clear that F and G vanish if $u = \tilde{u}$. The only terms that could possibly blow up in the evolution equations are g^{ij}, $\tilde{g}^{ij}$, ν, $\tilde{\nu}$, $\mathrm{Vol}(u)^{-1}$ and $\mathrm{Vol}(\tilde{u})^{-1}$. For g^{ij}, $\tilde{g}^{ij}$, ν, $\tilde{\nu}$ we use Remark 2.20 while $\mathrm{Vol}(u)^{-1}$ and $\mathrm{Vol}(\tilde{u})^{-1}$ are bounded in view of the energy conservation. $\square$

PROOF OF THEOREM 3.7. *1. The Bootstrap Argument.* We will prove the theorem by a bootstrap argument. As in [**2.9**] define

$$E_s(t) = \|\partial_t^2\varphi(t,\cdot)\|_s + \|\partial_t\varphi(t,\cdot)\|_{s+1} + \|\varphi(t,\cdot)\|_{s+1} + \|\partial_t\psi(t,\cdot)\|_{s+1} + \|\psi(t,\cdot)\|_{s+1}$$

with φ and ψ from Lemma 3.9. Define

$$\tilde{E}_s(t) = E_s(t) + |\mathrm{Vol}(u(t)) - \mathrm{Vol}(\tilde{u}(t))|.$$

We make the following hypothesis with respect to a fixed constant $\kappa > 0$.

$\mathbf{H}(t)$: There is a solution u defined on $[0,t)$ satisfying the initial conditions and $\tilde{E}_s(t') \leq \kappa$ holds for $t' \in [0,t)$.

From this hypothesis as long as $t \leq \min\{\tilde{T}, c_1 \log\left(\frac{c_2}{\varepsilon}\right)\}$, we will derive the conclusion

$\mathbf{C}(t)$: There is a solution u defined on $[0,t)$ satisfying the initial conditions and $\tilde{E}_s(t') \leq \frac{1}{2}\kappa$ holds for $t' \in [0,t)$.

The constant κ will be chosen in step 2 below. Then we choose $\varepsilon_0 < 1$ so small such that the hypothesis holds for a small $t > 0$. Clearly by Theorem 2.1 there exists a solution for a short time. We can estimate $E_s(0) \leq C\varepsilon_0$. To estimate $|\mathrm{Vol}(u_0) - \mathrm{Vol}(\tilde{u}(0))|$ we choose ε_0 so small such that if $\pi_{\mathcal{M}}$ is the closest point projection to $\mathcal{M}$ then $v_s := \pi_{\mathcal{M}}(\tilde{u}(0) + s(u_0 - \tilde{u}(0)))$ is an immersion for all $s \in [0,1]$. Then

$$|\mathrm{Vol}(u_0) - \mathrm{Vol}(\tilde{u}(0))| = |\int_0^1 \partial_s \mathrm{Vol}(v_s) ds| \tag{3.8}$$

$$= |\int_0^1 \int_{\mathcal{N}} \langle \nu(v_s), D\pi_{\mathcal{M}}(\tilde{u}(0) + s(u_0 - \tilde{u}(0)))(u_0 - \tilde{u}(0))\rangle d\mu_t(v_s) ds|$$

$$\leq C_\kappa \|u_0 - \tilde{u}(0)\|_{C^0} \leq C_\kappa \varepsilon. \tag{3.9}$$

Note that $D\pi_{\mathcal{M}}(\tilde{u}(0) + s(u_0 - \tilde{u}(0)))$ can be controlled since $\|u_0 - \tilde{u}(0)\|_{s+4} \leq \varepsilon_0$. Hence if ε_0 is small enough then $\tilde{E}_s(0) < \kappa$ and $\mathbf{H}(t)$ will be true for small t.

The conclusion is stronger than the hypothesis, i. e. if $\mathbf{C}(t)$ is true then also $\mathbf{H}(t')$ is true for all t' in a neighborhood of t. This can be seen by the continuity of $E_s(t)$ and the continuation criterion since the C^4-norms of u and $\partial_t u$ can be bounded in terms of E_s.

It is clear that the conclusion is closed, i. e. if $\mathbf{C}(t_i)$ holds for a sequence (t_i) which converges to another time t with $t_i, t \leq \min\{\tilde{T}, c_1 \log\left(\frac{c_2}{\varepsilon}\right)\}$ then also $\mathbf{C}(t)$ is true.

Then the abstract bootstrap principle [**Tao06**, Proposition 1.21] will imply that $\mathbf{C}(t)$ is true for all $0 \leq t \leq \min\{\tilde{T}, c_1 \log\left(\frac{c_2}{\varepsilon}\right)\}$ provided we have assured the implication $\mathbf{H}(t) \Rightarrow \mathbf{C}(t)$ for all $0 \leq t \leq \min\{\tilde{T}, c_1 \log\left(\frac{c_2}{\varepsilon}\right)\}$.

So assume in the following that $\mathbf{H}(t_0)$ is true for some $t_0 \in [0, \min\{\tilde{T}, c_1 \log\left(\frac{c_2}{\varepsilon}\right)\}]$ and let $t < t_0$.

2. Choice of κ. We want the operator L to be elliptic and we need $\det(g_{ij}) \geq \lambda_2$ for some $\lambda_2 > 0$ in order to apply Lemma 2.19. Now $u = \tilde{u} + (u - \tilde{u})$. This implies that

$$g_{ij} = \tilde{g}_{ij} + \langle \partial_i \tilde{u}, \partial_j(u - \tilde{u})\rangle + \langle \partial_j \tilde{u}, \partial_i(u - \tilde{u})\rangle + \langle \partial_i(u - \tilde{u}), \partial_j(u - \tilde{u})\rangle$$

and hence

$$|g_{ij} - \tilde{g}_{ij}| \leq C(\|u - \tilde{u}\|_{C^1} + \|u - \tilde{u}\|_{C^1}^2).$$

Since $\|u - \tilde{u}\|_{C^1} \leq CE_s \leq C\kappa$ by the Sobolev embedding theorem and the bootstrap hypothesis, we can choose κ such that $\det(g_{ij}) \geq \frac{1}{2}\lambda_1$ and $g^{ij} \geq \frac{1}{2}\lambda$ uniformly in t as long as $\mathbf{H}(t)$ holds.

3. Estimating the operator L. The coefficients of $L = \frac{d\mu_t}{d\mu_0}\Delta$ contain g^{ij}, Christoffel symbols and derivatives of Christoffel symbols. By Lemma 2.19 and the Moser inequalities we can estimate

$$[L]_s \leq C_\kappa(1 + \|u - \tilde{u}\|_{s+3}) \leq C_\kappa(1 + E_s) \qquad [\mathbf{3.10}]$$

since L contains no more than third derivatives of u. Clearly also by the hypothesis and the Sobolev embedding theorem $[L]_{C^1} \leq C_\kappa$. Similarly

$$[\partial_t L]_s \leq C_\kappa(1 + \|u - \tilde{u}\|_{s+3} + \|\partial_t u - \partial_t \tilde{u}\|_{s+3}) \leq C_\kappa(1 + E_s) \qquad [\mathbf{3.11}]$$

and $[\partial_t L]_{C^1} \leq C_\kappa$.

4. Application of the estimate for WHLS. We take the same bundle $\mathcal{V}$ as in [**3.3**] and ν_A and τ_k can be taken as in [**3.4**],[**3.5**].

By the Sobolev embedding theorem we have

$$\|\varphi\|_{C^2} + \|\partial_t\varphi\|_{C^2} + \|\psi\|_{C^1} + \|\partial_t\psi\|_{C^1} \leq CE_s \leq C\kappa. \qquad [\mathbf{3.12}]$$

Hence we can apply Proposition 2.14 with the modifications of Remark 2.15 to obtain

$$\begin{aligned} E_s(t) \leq Ce^{Ct}E_s(0) + C\int_0^t e^{C(t-t')}\big(&\|F(\dots)\|_s + \|\partial_t(F(\dots))\|_s + \|G(\dots)\|_{s+1} + [L]_s\,\|\varphi\|_{C^2} \\ &+ [\partial_t L]_s\,\|\varphi\|_{C^2} + [L]_s\,\|\partial_t\varphi\|_{C^2}\big)dt'. \qquad [\mathbf{3.13}] \end{aligned}$$

5. Estimating the right hand side. We have to estimate $\|F(\dots)\|_s$, $\|\partial_t(F(\dots))\|_s$ and $\|G(\dots)\|_{s+1}$. By the hypothesis and Lemma 3.9 we can apply the third Moser inequality in the form of Corollary C.6 with $F(x,0) = 0$ and $G(x,0) = 0$ to estimate

$$\begin{aligned} \|F(\dots)\|_s \leq& C(\|\varphi\|_{s+1} + \|\partial_t\varphi\|_s + \|\psi\|_{s+1} + \|\partial_t\psi\|_s + \|\mathrm{Vol}(u) - \mathrm{Vol}(\tilde{u})\|_0) \\ \leq& C(E_s + \|\mathrm{Vol}(u) - \mathrm{Vol}(\tilde{u})\|_0) \\ \|G(\dots)\|_{s+1} \leq& C(\|\varphi\|_{s+2} + \|\partial_t\varphi\|_{s+1} + \|\psi\|_{s+1} + \|\partial_t\psi\|_{s+1} + \|\mathrm{Vol}(u) - \mathrm{Vol}(\tilde{u})\|_0) \\ \leq& C(E_s + \|\varphi\|_{s+2} + \|\mathrm{Vol}(u) - \mathrm{Vol}(\tilde{u})\|_0). \end{aligned}$$

We use the elliptic estimate Lemma 2.11 part (2) since $\|\varphi\|_{C^2} \leq C_\kappa$ with the modification of Remark 2.12 and obtain

$$\begin{aligned} \|\varphi\|_{s+2} \leq& C(\|L\varphi\|_s + \|\varphi\|_{s+1} + [L]_s\,\|\varphi\|_{C^2}) \\ \leq& C(\|\partial_t^2\varphi\|_s + \|F(\dots)\|_s + \|\varphi\|_{s+1} + (\|u - \tilde{u}\|_{s+3} + 1)\|\varphi\|_{C^2}) \\ \leq& C(E_s + \|\mathrm{Vol}(u) - \mathrm{Vol}(\tilde{u})\|_0). \qquad [\mathbf{3.14}] \end{aligned}$$

Note that we have also used $\|\varphi\|_{C^2} \leq C\|\varphi\|_{s+1}$ by the Sobolev embedding theorem.

Before we estimate$\|\partial_t(F(\dots))\|_s$ we have to estimate $|\partial_t \operatorname{Vol}(u) - \partial_t \operatorname{Vol}(\tilde{u})|$.

$$\begin{aligned}|\partial_t \operatorname{Vol}(u) - \partial_t \operatorname{Vol}(\tilde{u})| =& |\int_{\mathcal{N}} \langle \nu, \partial_t u - \partial_t \tilde{u}\rangle d\mu_t + \int_{\mathcal{N}} \langle \nu \frac{d\mu_t}{\widetilde{d\mu_t}} - \tilde{\nu}, \partial_t \tilde{u}\rangle \widetilde{d\mu_t}| \qquad [\mathbf{3.15}]\\ \leq& C_\kappa \|\partial_t u - \partial_t \tilde{u}\|_0 + C_\kappa \|\nu \frac{d\mu_t}{\widetilde{d\mu_t}} - \tilde{\nu}\|_0\\ \leq& C_\kappa(\|\varphi\|_0 + \|\psi\|_0) \leq C_\kappa E_s. \qquad [\mathbf{3.16}]\end{aligned}$$

We use the fundamental theorem of calculus to estimate

$$\begin{aligned}|\operatorname{Vol}(u) - \operatorname{Vol}(\tilde{u})| \leq& |\operatorname{Vol}(u_0) - \operatorname{Vol}(\tilde{u}(0))| + \int_0^t |\partial_t \operatorname{Vol}(u(t')) - \partial_t \operatorname{Vol}(\tilde{u}(t'))| dt'\\ \leq& Ce^{Ct}|\operatorname{Vol}(u_0) - \operatorname{Vol}(\tilde{u}(0))| + C\int_0^t e^{C(t-t')}E_s(t')dt'. \qquad [\mathbf{3.17}]\end{aligned}$$

Now by the chain rule

$$\begin{aligned}\partial_t(F(\dots)) = \tilde{F}(&x, \varphi, D\varphi, \partial_t\varphi, \partial_t^2\varphi, \partial_t D\varphi,\\ &\psi, D\psi, \partial_t\psi, \partial_t^2\psi, \partial_t D\psi, \operatorname{Vol}(u) - \operatorname{Vol}(\tilde{u}), \partial_t \operatorname{Vol}(u) - \partial_t \operatorname{Vol}(\tilde{u}))\end{aligned}$$

and $\tilde{F}$ is also smooth and with bounded derivatives as long as the arguments are bounded because the partial derivatives of F are bounded as long as the arguments are bounded. In fact, the arguments are bounded by [**3.12**],[**3.16**] and the evolution equations. Furthermore $\tilde{F}(x,0) = 0$. So we can use again Corollary C.6 to estimate

$$\begin{aligned}\|\partial_t(F(\dots))\|_s \leq& C(\|\varphi\|_{s+1} + \|\partial_t\varphi\|_{s+1} + \|\partial_t^2\varphi\|_s + \|\psi\|_{s+1} + \|\partial_t\psi\|_{s+1} + \|\partial_t^2\psi\|_s\\ &+ \|\operatorname{Vol}(u) - \operatorname{Vol}(\tilde{u})\|_0 + \|\partial_t \operatorname{Vol}(u) - \partial_t \operatorname{Vol}(\tilde{u})\|_0)\\ \leq& C(E_s + \|\partial_t^2\psi\|_s + \|\operatorname{Vol}(u) - \operatorname{Vol}(\tilde{u})\|_0 + \|\partial_t \operatorname{Vol}(u) - \partial_t \operatorname{Vol}(\tilde{u})\|_0).\end{aligned}$$

By the equation for $\partial_t^2\psi$ we can estimate

$$\|\partial_t^2\psi\|_s \leq C\|G(\dots)\|_s \leq C\|G(\dots)\|_{s+1}$$

and we have already estimated the last term. Altogether we have that

$$\begin{aligned}\|F(\dots)\|_s + \|\partial_t(F(\dots))\|_s + \|G(\dots)\|_{s+1} \leq C_\kappa(&E_s + \|\operatorname{Vol}(u) - \operatorname{Vol}(\tilde{u})\|_0\\ &+ \|\partial_t \operatorname{Vol}(u) - \partial_t \operatorname{Vol}(\tilde{u})\|_0).\end{aligned}$$

Using [**3.10**],[**3.11**] and the Sobolev embedding theorem the remaining terms in [**3.13**] are estimated by

$$[L]_s \|\varphi\|_{C^2} + [\partial_t L]_s \|\varphi\|_{C^2} + [L]_s \|\partial_t\varphi\|_{C^2} \leq C_\kappa (1 + E_s) E_s \leq C_\kappa E_s.$$

So we obtain the estimate

$$E_s(t) \leq Ce^{Ct}E_s(0) + C\int_0^t e^{C(t-t')}(E_s(t') + |\operatorname{Vol}(u(t')) - \operatorname{Vol}(\tilde{u}(t'))|)dt'. \qquad [\mathbf{3.18}]$$

6. Conclusion. We add [**3.17**] and [**3.18**]

$$\tilde{E}_s(t) \leq Ce^{Ct}\tilde{E}_s(0) + C\int_0^t e^{C(t-t')}\tilde{E}_s(t')dt'$$

and apply the Gronwall type inequality from Lemma C.10 with $A(t) = \tilde{E}_s(t)$, $B(t) = C\tilde{E}_s(0)$, $g(t) = e^{Ct}$, $h(t') = Ce^{-Ct'}$. This yields

$$\tilde{E}_s(t) \leq Ce^{Ct}\tilde{E}_s(0).$$

Now using [**3.9**]

$$\tilde{E}_s(0) \leq C_\kappa\big(\|u_0 - \tilde{u}(0)\|_{s+4} + \|u_1 - \partial_t\tilde{u}(0)\|_{s+3} + |\mathrm{Vol}(u_0) - \mathrm{Vol}(\tilde{u}(0))|\big) \leq C_\kappa\varepsilon$$

and hence

$$\tilde{E}_s(t) \leq Ce^{Ct}\varepsilon.$$

Hence if $t \leq C^{-1}\log\left(\frac{\kappa}{2C\varepsilon}\right)$ then $\tilde{E}_s(t) \leq \frac{1}{2}\kappa$. Define $c_1 = C^{-1}$ and $c_2 = \frac{\kappa}{2C}$. We need $t > 0$ and hence $\varepsilon < c_2$. So the last condition for ε_0 is $\varepsilon_0 < c_2$.

The estimate for the norms is true since using [**3.14**] and the Moser inequalities we have

$$\|u - \tilde{u}\|_{s+4} + \|\partial_t u - \partial_t\tilde{u}\|_{s+3} \leq C_\kappa(E_s + |\mathrm{Vol}(u) - \mathrm{Vol}(\tilde{u})|) \leq Ce^{Ct}\varepsilon. \qquad \square$$

THEOREM 3.10. *Let $\mathcal{M} = (\mathbb{R}^{n+1}, \overline{g})$ and let $\tilde{\mathcal{M}} = (\mathbb{R}^{n+1}, \delta)$ with the Euclidean metric δ on $\mathbb{R}^{n+1}$. Let $s \geq \lfloor\frac{n}{2}\rfloor + 2$ and $\tilde{u} : [0,\infty) \times \mathcal{N} \to \tilde{\mathcal{M}}$ be a solution of* [**EQ**] *and let for all $t \in [0,\infty)$*

$$\|\tilde{u}(t,\cdot)\|_{s+4} + \|\partial_t\tilde{u}(t,\cdot)\|_{s+3} \leq K \text{ and } \det(\tilde{g}_{ij}) \geq \lambda_1 \text{ and } \tilde{g}^{ij} \geq \lambda$$

for some constants $K, \lambda_1, \lambda > 0$. There exist constants $c_1, c_2 > 0$ and $\varepsilon_0, \varepsilon_1 > 0$ such that if $u_0 : \mathcal{N} \to \mathcal{M}$ is an immersion and $u_1 : \mathcal{N} \to u_0^*T\mathcal{M}$ is a vector field along u_0 with

$$\|u_0 - \tilde{u}(0)\|_{s+4} + \|u_1 - \partial_t\tilde{u}(0)\|_{s+3} \leq \varepsilon$$

and

$$\|\overline{g} - \delta\|_{C^{s+4}} \leq \varepsilon'$$

for some $0 < \varepsilon \leq \varepsilon_0$ and $0 < \varepsilon' \leq \varepsilon_1$ then there exists $T \geq c_1\log\left(\frac{c_2}{\varepsilon+\varepsilon'}\right)$ and $u : [0,T]\times\mathcal{N} \to \mathcal{M}$ that solves

$$\begin{cases} \overline{\nabla}_{\partial_t}\partial_t u = \frac{d\mu_t}{d\mu_0}\left(-H(u) + \frac{\varrho}{\mathrm{Vol}(u)}\right)\nu, \text{ for all } t \in [0,T] \\ u(0,\cdot) = u_0 \\ \partial_t u(0,\cdot) = u_1. \end{cases}$$

For all $t \in [0,T]$ we have the estimate

$$\|u(t,\cdot) - \tilde{u}(t,\cdot)\|_{s+4} + \|\partial_t u(t,\cdot) - \partial_t\tilde{u}(t,\cdot)\|_{s+3} \leq Ce^{Ct}(\varepsilon + \varepsilon')$$

with C depending on s and $\tilde{u}$.

PROOF. We will assume that $\varepsilon_1 \leq 1$ in order to get bounds on $\overline{g}$ that do not depend on ε' or ε_1. Only a few modifications of the proof of Theorem 3.7 are necessary. We employ the same bootstrap argument except that here we only consider $E_s(t)$ instead of $\tilde{E}_s(t)$. Again we have to choose the constant κ. We calculate

$$g_{ij} - \tilde{g}_{ij} = (\overline{g} - \delta)(\partial_i \tilde{u}, \partial_j \tilde{u}) + \overline{g}(\partial_j \tilde{u}, \partial_i (u - \tilde{u})) + \overline{g}(\partial_i \tilde{u}, \partial_j (u - \tilde{u})) \\ + \overline{g}(\partial_i (u - \tilde{u}), \partial_j (u - \tilde{u}))$$

and estimate

$$|g_{ij} - \tilde{g}_{ij}| \leq C(\|\overline{g} - \delta\|_{C^0} + \|u - \tilde{u}\|_{C^1} + \|u - \tilde{u}\|_{C^1}^2).$$

Then if ε_1 is small enough we can choose κ similarly as in step 2 of the proof of Theorem 3.7 such that $\det(g_{ij}) \geq \frac{1}{2}\lambda_1$ and $g^{ij} \geq \frac{1}{2}\lambda$ uniformly in t.

We will work in standard coordinates for $\mathbb{R}^{n+1}$. If $\varepsilon_1 \leq 1$ the operator $L = \frac{d\mu_t}{d\mu_0}\Delta$ can be estimated by

$$\begin{aligned} [L]_s \leq & C_\kappa (1 + \|u - \tilde{u}\|_{s+3} + \|\overline{g} \circ u - \delta \circ \tilde{u}\|_{s+4}) \\ \leq & C_\kappa (1 + \|u - \tilde{u}\|_{s+3}) \end{aligned}$$

and similarly

$$[\partial_t L]_s \leq C_\kappa (1 + \|u - \tilde{u}\|_{s+3} + \|\partial_t u - \partial_t \tilde{u}\|_{s+3}).$$

We include the dependence on $\overline{g} \circ u - \delta \circ \tilde{u}$ in the functions F and G from Lemma 3.9 and write the system in the form

$$\begin{aligned} \partial_t^2 \varphi^A(t,x) - L^A \varphi^A(t,x) = & F^A(x, \varphi, D\varphi, \partial_t \varphi, \psi, D\psi, \partial_t \psi, \mathrm{Vol}(u) - \mathrm{Vol}(\tilde{u}), \\ & \overline{g} \circ u - \delta \circ \tilde{u}, (D\overline{g}) \circ u, (D^2 \overline{g}) \circ u, (D^3 \overline{g}) \circ u) \\ \partial_t^2 \psi^k = & G^k(x, \varphi, D\varphi, \partial_t \varphi, \psi, \partial_t \psi, \mathrm{Vol}(u) - \mathrm{Vol}(\tilde{u}), \\ & \overline{g} \circ u - \delta \circ \tilde{u}, (D\overline{g}) \circ u, (D^2 \overline{g}) \circ u, (D^3 \overline{g}) \circ u). \quad [\mathbf{3.19}] \end{aligned}$$

Here $D^i \overline{g}$ stands for the derivatives of order i of $\overline{g}$ in the standard coordinates. In these coordinates the derivatives of δ vanish and so we do not have to include them. All curvature terms occuring in the evolution equations can be expressed by derivatives of $\overline{g}$ up to order three.

We now need to estimate the difference of the volumes. We write $\mathrm{Vol}^{\overline{g}}$ and Vol^δ for the volume taken with respect to the metric $\overline{g}$ and δ respectively. We also mark other quantities with $\overline{g}$ or δ if they are taken with respect to the respective metric. Write

$$\mathrm{Vol}^{\overline{g}}(u) - \mathrm{Vol}^\delta(\tilde{u}) = (\mathrm{Vol}^{\overline{g}}(u) - \mathrm{Vol}^{\overline{g}}(\tilde{u})) + (\mathrm{Vol}^{\overline{g}}(\tilde{u}) - \mathrm{Vol}^\delta(\tilde{u})) =: D_1 + D_2.$$

The difference D_1 can be estimtated similarly as in [**3.8**] using

$$\begin{aligned}\mathrm{Vol}^{\overline{g}}(u) - \mathrm{Vol}^{\overline{g}}(\tilde{u}) &= \int_0^1 \partial_s \mathrm{Vol}^{\overline{g}}(u_s) ds \\ &= \int_0^1 \int_{\mathcal{N}} \overline{g}(\nu^{\overline{g}}(u_s), u - \tilde{u}) d\mu_t^{\overline{g}}(u_s) ds\end{aligned}$$

where $u_s = \tilde{u} + s(u - \tilde{u})$. Using also the bounds on $\overline{g}$ we get

$$|D_1| = |\mathrm{Vol}^{\overline{g}}(u) - \mathrm{Vol}^{\overline{g}}(\tilde{u})| \leq C_\kappa \|u - \tilde{u}\|_{C^0} \leq C_\kappa(\|\varphi\|_{s+1} + \|\psi\|_{s+1}).$$

Let $\mathcal{L}$ denote the Lebesgue measure on $\mathbb{R}^{n+1}$ and let $\mathcal{L}^{\overline{g}} = \sqrt{\det(\overline{g}_{\alpha\beta})}\mathcal{L}$. Let $\Omega \subset \mathbb{R}^{n+1}$ be the interior of the surface $\tilde{u}(t, \mathcal{N})$, i. e. $\mathrm{Vol}^{\delta}(\tilde{u}) = \mathcal{L}(\Omega)$ and $\mathrm{Vol}^{\overline{g}}(\tilde{u}) = \mathcal{L}^{\overline{g}}(\Omega)$. If we apply the mean value theorem to the function $s \mapsto \sqrt{\det(\delta_{\alpha\beta} + s(\overline{g}_{\alpha\beta} - \delta_{\alpha\beta}))}$ and use the assumed bounds on $\overline{g}$ we can estimate

$$|\sqrt{\det(\overline{g}_{\alpha\beta})} - 1| = |\sqrt{\det(\overline{g}_{\alpha\beta})} - \sqrt{\det(\delta_{\alpha\beta})}| \leq C\|\overline{g} - \delta\|_{C^0}.$$

Clearly we need ε_1 small enough such that $\det(\delta_{\alpha\beta} + s(\overline{g}_{\alpha\beta} - \delta_{\alpha\beta})) > \lambda_3$ uniformly for $s \in [0, 1]$ for some $\lambda_3 > 0$. Hence we can estimate

$$\begin{aligned}|D_2| =& |\int_\Omega d\mathcal{L}^{\overline{g}} - \int_\Omega d\mathcal{L}| = |\int_\Omega \sqrt{\det(\overline{g}_{\alpha\beta})} d\mathcal{L} - \int_\Omega d\mathcal{L}| \\ \leq& \int_\Omega |\sqrt{\det(\overline{g}_{\alpha\beta})} - 1| d\mathcal{L} \leq C\|\overline{g} - \delta\|_{C^0} \mathrm{Vol}^{\delta}(\tilde{u}) \leq C\|\overline{g} - \delta\|_{C^0}.\end{aligned}$$

We used the a priori volume bound for $\mathrm{Vol}^{\delta}(\tilde{u})$ from Corollary 1.6. Combining the estimates for D_1 and D_2 we arrive at

$$|\mathrm{Vol}^{\overline{g}}(u) - \mathrm{Vol}^{\delta}(\tilde{u})| \leq C_\kappa(\|\varphi\|_{s+1} + \|\psi\|_{s+1} + \|\overline{g} - \delta\|_{C^0}).$$

Including the different metrics in [**3.15**] we can estimate in a similar way

$$|\partial_t \mathrm{Vol}^{\overline{g}}(u) - \partial_t \mathrm{Vol}^{\delta}(\tilde{u})| \leq C_\kappa(\|\varphi\|_{s+1} + \|\psi\|_{s+1} + \|\overline{g} - \delta\|_{C^0}).$$

If $\overline{g} = \delta$ and $u = \tilde{u}$ then clearly the right hand side of the system [**3.19**] vanishes. So we can apply the third Moser inequality Corollary C.6 to obtain the estimates

$$\begin{aligned}\|F(\dots)\|_s \leq& C(\|\varphi\|_{s+1} + \|\partial_t\varphi\|_s + \|\psi\|_{s+1} + \|\partial_t\psi\|_s + \|\mathrm{Vol}(u) - \mathrm{Vol}(\tilde{u})\|_0 \\ &+ \|\overline{g} \circ u - \delta \circ \tilde{u}\|_{s+3}) \\ \|G(\dots)\|_{s+1} \leq& C(\|\varphi\|_{s+2} + \|\partial_t\varphi\|_{s+1} + \|\psi\|_{s+1} + \|\partial_t\psi\|_{s+1} + \|\mathrm{Vol}(u) - \mathrm{Vol}(\tilde{u})\|_0 \\ &+ \|\overline{g} \circ u - \delta \circ \tilde{u}\|_{s+3}) \\ \|\partial_t(F(\dots))\|_s \leq& C(\|\varphi\|_{s+1} + \|\partial_t\varphi\|_{s+1} + \|\partial_t^2\varphi\|_s + \|\psi\|_{s+1} + \|\partial_t\psi\|_{s+1} + \|\partial_t^2\psi\|_s \\ &+ \|\partial_t u - \partial_t\tilde{u}\|_s + \|\mathrm{Vol}(u) - \mathrm{Vol}(\tilde{u})\|_0 + \|\partial_t \mathrm{Vol}(u) - \partial_t \mathrm{Vol}(\tilde{u})\|_0 \\ &+ \|\overline{g} \circ u - \delta \circ \tilde{u}\|_{s+4}).\end{aligned}$$

Now $\|\overline{g} \circ u - \delta \circ \tilde{u}\|_0 \leq C\|\overline{g} - \delta\|_{C^0}$ and $\|(D^i\overline{g}) \circ u\|_s \leq C\|D^i\overline{g}\|_{C^s}$. Hence

$$\|\overline{g} \circ u - \delta \circ \tilde{u}\|_{s+4} \leq C\|\overline{g} - \delta\|_{C^{s+4}}.$$

Similarly as in the proof of Theorem 3.7 we arrive at the estimate

$$E_s(t) \leq Ce^{Ct}E_s(0) + C\int_0^t e^{C(t-t')}E_s(t')dt' + \int_0^t Ce^{C(t-t')}\|\overline{g} - \delta\|_{C^{s+4}}dt'.$$

An application of Gronwall's inequality yields

$$E_s(t) \leq Ce^{Ct}(E_s(0) + \int_0^t e^{-Ct'}\|\overline{g} - \delta\|_{C^{s+4}}dt').$$

Now by assumption $\|\overline{g} - \delta\|_{C^{s+4}} \leq \varepsilon'$ and (use the third Moser inequality)

$$E_s(0) \leq C_\kappa(\|u_0 - \tilde{u}(0)\|_{s+4} + \|u_1 - \partial_t\tilde{u}(0)\|_{s+3} + \|\overline{g} - \delta\|_{C^{s+4}}) \leq C_\kappa(\varepsilon + \varepsilon').$$

Since $\int_0^t e^{-Ct'}dt' \leq C^{-1}$ we have the estimate

$$E_s(t) \leq Ce^{Ct}(\varepsilon + \varepsilon').$$

As in the last step of the proof of Theorem 3.7 we can use this estimate to apply the bootstrap argument provided that $C\log(\frac{\kappa}{2C(\varepsilon+\varepsilon')}) > 0$. This can be accomplished by demanding that $\varepsilon_0 < \frac{\kappa}{4C}$ and $\varepsilon_1 < \frac{\kappa}{4C}$.

This concludes the proof of Theorem 3.10. $\square$

APPENDIX A

The Evolution Equations

In this section we will compute the evolution equations for σ and S^k, for h_{ij} and Γ^k_{ij}, and for β_i and ${B_i}^k$ for a solution u of the equation $\overline{\nabla}_{\partial_t}\partial_t u = \alpha\nu$ where we write $\alpha = \frac{d\mu_t}{d\mu_0}\big(- H + \frac{\varrho}{\mathrm{Vol}(u)}\big)$ according to Definition 1.2. Recall that by [**1.2**] and Definitions 1.2 and 3.3 each of these quantities can be written as the normal or tangential part of a derivative of u, i. e. $\partial_t u = \sigma\nu + S^k\partial_k u$, $\overline{\nabla}_i\partial_j u = -h_{ij}\nu + \Gamma^k_{ij}\partial_k u$, and $\overline{\nabla}_i\partial_t u = \beta_i\nu + {B_i}^k\partial_k u$. To compute the evolution equations for h_{ij} and Γ^k_{ij}, for example, we will differentiate our differential equation $\overline{\nabla}_{\partial_t}\partial_t u = \alpha\nu$ with $\overline{\nabla}_{\partial_i}\overline{\nabla}_{\partial_j}$. Then we interchange the derivatives to obtain an equation for $\overline{\nabla}_{\partial_i}\partial_j u$ and decompose this into normal and tangential parts. The decomposition of second time derivatives is carried out abstractly in Lemma A.1.

The calculation of each evolution equation only consists of these simple steps. Nevertheless due to the interchange of derivatives many curvature terms will arise which we decompose into their normal and tangential parts as well. In order to suitably express $\nabla_i\nabla_j H$ we use Simons' identity which also contains an interchange of derivatives. We need an analogous identity for $\partial_t\partial_i H$ which will be derived in Lemma A.2. Although the evolution equation for β_i is not needed in the main text, we include the computation for the sake of completeness.

For the Riemann tensor we use the notation

$$\overline{\mathrm{R}}_{0ijk} = \overline{\mathrm{R}}_{\alpha\beta\gamma\delta}\nu^\alpha\partial_i u^\beta\partial_j u^\gamma\partial_k u^\delta = \langle\overline{\mathrm{R}}(\nu,\partial_i u)\partial_j u, \partial_k u\rangle,$$

for example.

A.1. Decomposition and Interchange Identities

We first compute the evolution of the canonical tangent vectors $\partial_k u$ and of the normal vector. We have that

$$\begin{aligned}\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t}\partial_k u =&\overline{\nabla}_{\partial_k}\overline{\nabla}_{\partial_t}\partial_t u + \overline{\mathrm{R}}(\partial_t u,\partial_k u)\partial_t u\\ =&\partial_k\alpha\nu + \alpha {h_k}^l\partial_l u + \sigma^2\overline{\mathrm{R}}(\nu,\partial_k u)\nu + \sigma\overline{\mathrm{R}}(S,\partial_k u)\nu\\ &+ \sigma\overline{\mathrm{R}}(\nu,\partial_k u)S + \overline{\mathrm{R}}(S,\partial_k u)S\\ =&(\partial_k\alpha + \sigma S^p\overline{\mathrm{R}}_{0kp0} + S^q S^p\overline{\mathrm{R}}_{qkp0})\nu\\ &+ (\alpha {h_k}^l + \sigma^2\overline{\mathrm{R}}_{0k0m}g^{ml} + \sigma S^p\overline{\mathrm{R}}_{pk0m}g^{ml}\\ &+ \sigma S^p\overline{\mathrm{R}}_{0kpm}g^{ml} + S^pS^q\overline{\mathrm{R}}_{pkqm}g^{ml})\partial_l u.\end{aligned} \qquad [\mathbf{A.1}]$$

We have the following alternative expression for β_i

$$\beta_i = \langle \overline{\nabla}_{\partial_i}(\sigma\nu + S^k\partial_k u), \nu\rangle = \partial_i\sigma - S^k h_{ik}. \qquad [\mathbf{A.2}]$$

We have

$$\overline{\nabla}_{\partial_t}\nu = \langle \overline{\nabla}_{\partial_t}\nu, \partial_k u\rangle g^{kl}\partial_l u = -\langle \overline{\nabla}_{\partial_t}\partial_k u, \nu\rangle g^{kl}\partial_l u = -\beta^k\partial_k u = -\nabla\sigma + S^l {h_l}^k\partial_k u. \qquad [\mathbf{A.3}]$$

We compute using [**A.1**]

$$\begin{aligned}\partial_t\beta_i =& \langle \overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t}\partial_i u, \nu\rangle + \langle \overline{\nabla}_{\partial_i}\partial_t u, \overline{\nabla}_{\partial_t}\nu\rangle \\ =& \partial_i\alpha - {B_i}^k\beta_k + \sigma S^p\overline{\mathrm{R}}_{0ip0} + S^pS^q\overline{\mathrm{R}}_{piq0}. \end{aligned} \qquad [\mathbf{A.4}]$$

Since $\partial_t g^{ik} = -(B^{ik} + B^{ki})$ we can compute the evolution of ν

$$\begin{aligned}\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t}\nu =& -\partial_t\beta^k\partial_k u - \beta^k\langle \overline{\nabla}_{\partial_t}\partial_k u, \nu\rangle\nu - \beta^k\langle \overline{\nabla}_{\partial_t}\partial_k u, \partial_l u\rangle g^{lm}\partial_m u \\ =& -\nabla\alpha + {B^i}_k\beta^k\partial_i u + (B^{ik} + B^{ki})\beta_i\partial_k u - \sigma S^p\overline{\mathrm{R}}_{0ip0}g^{ik}\partial_k u \\ & - S^pS^q\overline{\mathrm{R}}_{piq0}g^{ik}\partial_k u - \beta^k\beta_k\nu - \beta^k {B_k}^m\partial_m u \\ =& -|\beta|^2\nu - \nabla\alpha + 2\beta^i {B^k}_i\partial_k u \\ & - \sigma S^p\overline{\mathrm{R}}_{0ip0}g^{ik}\partial_k u - S^pS^q\overline{\mathrm{R}}_{piq0}g^{ik}\partial_k u. \end{aligned} \qquad [\mathbf{A.5}]$$

Later we will insert

$$\nabla_k\alpha = -\frac{d\mu_t}{d\mu_0}\nabla_k H + \left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)(\Gamma^i_{ik} - \mathring{\Gamma}^i_{ik})\frac{d\mu_t}{d\mu_0}$$

and

$$|\beta|^2 = |\nabla\sigma|^2 + S^kS^l {h^j}_k h_{jl} - 2\partial_i\sigma {h^i}_k S^k.$$

Note that

$$\partial_k\frac{d\mu_t}{d\mu_0} = \left(\frac{1}{2}g^{ij}\partial_k g_{ij} - \frac{1}{2}g_0^{ij}\partial_k g_{0ij}\right)\frac{d\mu_t}{d\mu_0} = (\Gamma^i_{ik} - \mathring{\Gamma}^i_{ik})\frac{d\mu_t}{d\mu_0}.$$

For reference we also note

$$\begin{aligned}\partial_t {B_i}^k =& \partial_t g^{jk}B_{ij} + g^{jk}\langle \overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t}\partial_i u, \partial_j u\rangle + g^{jk}\langle \overline{\nabla}_{\partial_t}\partial_i u, \overline{\nabla}_{\partial_t}\partial_j u\rangle \\ =& -(B^{kj} + B^{jk})B_{ij} + {B_i}^l B_{jl}g^{jk} + \beta_i\beta_j g^{jk} + \alpha {h_i}^k \\ & + (\sigma^2\overline{\mathrm{R}}_{0i0m} + \sigma S^p\overline{\mathrm{R}}_{pi0m} + \sigma S^p\overline{\mathrm{R}}_{0ipm} + S^qS^p\overline{\mathrm{R}}_{piqm})g^{mk}. \end{aligned} \qquad [\mathbf{A.6}]$$

LEMMA A.1. *Let $V = \varphi\nu + \psi^k\partial_k u$. Then*

$$\langle\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t}V,\nu\rangle = \partial_t^2\varphi - \varphi|\beta|^2 + 2\partial_t\psi^k\beta_k + \psi^k\partial_k\alpha + \psi^k\sigma S^l\overline{\mathrm{R}}_{0kl0} + \psi^k S^l S^p\overline{\mathrm{R}}_{lkp0}$$

and

$$\begin{aligned}\langle\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t}V,\partial_k u\rangle g^{kl} =& \partial_t^2\psi^l + 2\partial_t\psi^k B_k{}^l - 2\partial_t\varphi\beta^l + \alpha\psi^k h_k{}^l - \varphi\nabla^l\alpha + 2\varphi\beta^k B^l{}_k \\ &+ \psi^k\sigma^2\overline{\mathrm{R}}_{0k0m}g^{ml} + \psi^k\sigma S^p\overline{\mathrm{R}}_{pk0m}g^{ml} \\ &+ \psi^k S^p\sigma\overline{\mathrm{R}}_{0kpm}g^{ml} + S^pS^q\psi^k\overline{\mathrm{R}}_{pkqm}g^{ml} \\ &- \varphi\sigma S^p\overline{\mathrm{R}}_{0kp0}g^{kl} - \varphi S^pS^q\overline{\mathrm{R}}_{pkq0}g^{kl}.\end{aligned}$$

PROOF. We have

$$\begin{aligned}\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t}V =& \partial_t^2\varphi\nu + 2\partial_t\varphi\overline{\nabla}_{\partial_t}\nu + \varphi\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t}\nu + \partial_t^2\psi^l\partial_l u + 2\partial_t\psi^k\overline{\nabla}_{\partial_t}\partial_k u + \psi^k\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t}\partial_k u \\ =& (\partial_t^2\varphi + \varphi\langle\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t}\nu,\nu\rangle + 2\partial_t\psi^k\langle\overline{\nabla}_{\partial_t}\partial_k u,\nu\rangle + \psi^k\langle\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t}\partial_k u,\nu\rangle)\nu \\ &+ (\partial_t^2\psi^l + 2\partial_t\psi^k\langle\overline{\nabla}_{\partial_t}\partial_k u,\partial_m u\rangle g^{ml} + \psi^k\langle\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t}\partial_k u,\partial_m u\rangle g^{ml} \\ &+ 2\partial_t\varphi\langle\overline{\nabla}_{\partial_t}\nu,\partial_m u\rangle g^{ml} + \varphi\langle\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t}\nu,\partial_m u\rangle g^{ml})\partial_l u\end{aligned}$$

Using [**A.1**] and [**A.5**] we get the stated result. □

LEMMA A.2. *We have the identities*

$$\begin{aligned}\nabla_i\nabla_j H =& \Delta h_{ij} - Hh_{il}h^l{}_j + |h|^2h_{ij} + H\overline{\mathrm{R}}_{0i0j} - h_{ij}g^{lm}\overline{\mathrm{R}}_{0l0m} \\ &+ h_j{}^l g^{km}\overline{\mathrm{R}}_{lkim} + h_i{}^l g^{km}\overline{\mathrm{R}}_{lkjm} - 2h^{lm}\overline{\mathrm{R}}_{limj} + g^{lm}\overline{\nabla}_j\overline{\mathrm{R}}_{0lim} + g^{lm}\overline{\nabla}_l\overline{\mathrm{R}}_{0ijm}\end{aligned}$$

and

$$\begin{aligned}-\partial_t\partial_i H =& \Delta\beta_i + (B^{kl} + B^{lk})\nabla_i h_{kl} + h^{kl}\nabla_i(B_{lk} + B_{kl}) \\ &+ g^{kl}\Big\{\partial_t\Gamma^p_{lk}h_{ip} - \partial_t\Gamma^p_{il}h_{kp} + \sigma\overline{\nabla}_0\overline{\mathrm{R}}_{0lki} + S^p\overline{\nabla}_p\overline{\mathrm{R}}_{0lki} - \beta^p\overline{\mathrm{R}}_{plki} \\ &+ 2B_l{}^p\overline{\mathrm{R}}_{0pki} + \beta_i\overline{\mathrm{R}}_{0lk0} + B_i{}^p\overline{\mathrm{R}}_{0lkp} \\ &- \nabla_k B_i{}^p h_{pl} - B_i{}^p\nabla_k h_{pl} - \sigma\overline{\nabla}_k\overline{\mathrm{R}}_{0l0i} - S^p\overline{\nabla}_k\overline{\mathrm{R}}_{pl0i} \\ &+ h_{kl}S^p\overline{\mathrm{R}}_{p00i} - h_k{}^p\sigma\overline{\mathrm{R}}_{0lpi} - h_k{}^pS^q\overline{\mathrm{R}}_{qlpi}\Big\}.\end{aligned} \qquad [\mathbf{A.7}]$$

PROOF. The first statement is Simons' identity (see e. g. [**Hui86**]). For the second statement write

$$\begin{aligned}\partial_t\nabla_i h_{kl} + \nabla_k\nabla_l\beta_i =& (\partial_t\nabla_i h_{kl} - \partial_t\nabla_k h_{il}) + (\partial_t\nabla_k h_{il} - \nabla_k\partial_t h_{il}) \\ &+ (\nabla_k\partial_t h_{il} + \nabla_k\nabla_l\beta_i).\end{aligned} \qquad [\mathbf{A.8}]$$

By the Codazzi equation the first bracket is

$$\begin{aligned}-\partial_t(\overline{\mathrm{R}}(\nu,\partial_l u,\partial_k u,\partial_i u)) =& -\sigma\overline{\nabla}_0\overline{\mathrm{R}}_{0lki}-S^p\overline{\nabla}_p\overline{\mathrm{R}}_{0lki}-\overline{\mathrm{R}}(\overline{\nabla}_{\partial_t}\nu,\partial_l u,\partial_k u,\partial_i u)\\ &-\overline{\mathrm{R}}(\nu,\overline{\nabla}_{\partial_t}\partial_l u,\partial_k u,\partial_i u)-\overline{\mathrm{R}}(\nu,\partial_l u,\overline{\nabla}_{\partial_t}\partial_k u,\partial_i u)\\ &-\overline{\mathrm{R}}(\nu,\partial_l u,\partial_k u,\overline{\nabla}_{\partial_t}\partial_i u)\\ =& -\sigma\overline{\nabla}_0\overline{\mathrm{R}}_{0lki}-S^p\overline{\nabla}_p\overline{\mathrm{R}}_{0lki}+\beta^p\overline{\mathrm{R}}_{plki}\\ &-B_l{}^p\overline{\mathrm{R}}_{0pki}-\beta_k\overline{\mathrm{R}}_{0l0i}-B_k{}^p\overline{\mathrm{R}}_{0lpi}-\beta_i\overline{\mathrm{R}}_{0lk0}-B_i{}^p\overline{\mathrm{R}}_{0lkp}.\end{aligned}$$

For the second bracket in [**A.8**] we compute

$$\partial_t\nabla_k h_{il}=\partial_t(\partial_k h_{il}-\Gamma^p_{ik}h_{pl}-\Gamma^p_{lk}h_{ip})=\nabla_k\partial_t h_{il}-\partial_t\Gamma^p_{ik}h_{pl}-\partial_t\Gamma^p_{lk}h_{ip}.$$

For the third bracket in [**A.8**] we compute

$$\begin{aligned}\partial_t h_{il} =&\langle\overline{\nabla}_{\partial_t}\partial_i u,\overline{\nabla}_{\partial_l}\nu\rangle+\langle\partial_i u,\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_l}\nu\rangle\\ =&B_i{}^p h_{pl}+\langle\partial_i u,\overline{\nabla}_{\partial_l}\overline{\nabla}_{\partial_t}\nu\rangle+\langle\overline{\mathrm{R}}(\partial_t u,\partial_l u)\nu,\partial_i u\rangle\\ =&B_i{}^p h_{pl}+\partial_l\langle\partial_i u,\overline{\nabla}_{\partial_t}\nu\rangle-\langle\overline{\nabla}_{\partial_l}\partial_i u,\overline{\nabla}_{\partial_t}\nu\rangle+\overline{\mathrm{R}}(\partial_t u,\partial_l u,\nu,\partial_i u)\\ =&B_i{}^p h_{pl}-\partial_l\beta_i+\Gamma^p_{li}\beta_p+\overline{\mathrm{R}}(\partial_t u,\partial_l u,\nu,\partial_i u)\\ =&-\nabla_l\beta_i+B_i{}^p h_{pl}+\overline{\mathrm{R}}(\partial_t u,\partial_l u,\nu,\partial_i u).\end{aligned}$$

Hence the third bracket is

$$\begin{aligned}&\nabla_k B_i{}^p h_{pl}+B_i{}^p\nabla_k h_{pl}+\overline{\nabla}_k\overline{\mathrm{R}}(\partial_t u,\partial_l u,\nu,\partial_i u)+\overline{\mathrm{R}}(\overline{\nabla}_{\partial_k}\partial_t u,\partial_l u,\nu,\partial_i u)\\ &+\overline{\mathrm{R}}(\partial_t u,\nabla_k\partial_l u,\nu,\partial_i u)+\overline{\mathrm{R}}(\partial_t u,\partial_l u,\overline{\nabla}_{\partial_k}\nu,\partial_i u)+\overline{\mathrm{R}}(\partial_t u,\partial_l u,\nu,\nabla_k\partial_i u)\\ =&\nabla_k B_i{}^p h_{pl}+B_i{}^p\nabla_k h_{pl}+\sigma\overline{\nabla}_k\overline{\mathrm{R}}_{0l0i}+S^p\overline{\nabla}_k\overline{\mathrm{R}}_{pl0i}\\ &+\beta_k\overline{\mathrm{R}}_{0l0i}+B_k{}^p\overline{\mathrm{R}}_{pl0i}-h_{kl}S^p\overline{\mathrm{R}}_{p00i}\\ &+h_k{}^p\sigma\overline{\mathrm{R}}_{0lpi}+h_k{}^pS^q\overline{\mathrm{R}}_{qlpi}.\end{aligned}$$

Furthermore we compute that

$$\begin{aligned}\partial_t\partial_i H =&g^{kl}\partial_t\partial_i h_{kl}+\partial_i g^{kl}\partial_t h_{kl}+\partial_i h_{kl}\partial_t g^{kl}+h_{kl}\partial_i\partial_t g^{kl}\\ =&g^{kl}\partial_t\partial_i h_{kl}-(\Gamma^l_{ip}g^{kp}+\Gamma^k_{iq}g^{ql})\partial_t h_{kl}-\partial_i h_{kl}(B^{kl}+B^{lk})\\ &+h_{kl}(B_{pq}+B_{qp})(g^{kp}g^{mq}\Gamma^l_{im}+g^{kp}\Gamma^q_{im}g^{ml}+g^{ql}\Gamma^p_{im}g^{mk}+g^{ql}\Gamma^k_{im}g^{mp})\\ &-h_{kl}\partial_i(B_{pq}+B_{qp})g^{kp}g^{ql}\\ =&g^{kl}\nabla_i\partial_t h_{kl}-(B^{kl}+B^{lk})\nabla_i h_{kl}-h^{kl}\nabla_i(B_{lk}+B_{kl})\\ =&g^{kl}\partial_t\nabla_i h_{kl}+g^{kl}\partial_t\Gamma^p_{ik}h_{pl}+g^{kl}\partial_t\Gamma^p_{il}h_{kp}-(B^{kl}+B^{lk})\nabla_i h_{kl}-h^{kl}\nabla_i(B_{lk}+B_{kl}).\end{aligned}$$

Putting things together we get that

$$\begin{aligned}-\partial_t\partial_i H =&\Delta\beta_i - g^{kl}\partial_t\Gamma^p_{ik}h_{pl} - g^{kl}\partial_t\Gamma^p_{il}h_{kp} + (B^{kl}+B^{lk})\nabla_i h_{kl} + h^{kl}\nabla_i(B_{lk}+B_{kl})\\
&+ g^{kl}\Big\{\partial_t\Gamma^p_{ik}h_{pl} + \partial_t\Gamma^p_{lk}h_{ip} + \sigma\overline{\nabla}_0\overline{\mathrm{R}}_{0lki} + S^p\overline{\nabla}_p\overline{\mathrm{R}}_{0lki} - \beta^p\overline{\mathrm{R}}_{plki}\\
&+ B_l{}^p\overline{\mathrm{R}}_{0pki} + \beta_k\overline{\mathrm{R}}_{0l0i} + B_k{}^p\overline{\mathrm{R}}_{0lpi} + \beta_i\overline{\mathrm{R}}_{0lk0} + B_i{}^p\overline{\mathrm{R}}_{0lkp}\\
&- \nabla_k B_i{}^p h_{pl} - B_i{}^p\nabla_k h_{pl} - \sigma\overline{\nabla}_k\overline{\mathrm{R}}_{0l0i} - S^p\overline{\nabla}_k\overline{\mathrm{R}}_{pl0i}\\
&- \beta_k\overline{\mathrm{R}}_{0l0i} - B_k{}^p\overline{\mathrm{R}}_{pl0i} + h_{kl}S^p\overline{\mathrm{R}}_{p00i} - h_k{}^p\sigma\overline{\mathrm{R}}_{0lpi} - h_k{}^pS^q\overline{\mathrm{R}}_{qlpi}\Big\}.\end{aligned}$$

By the Bianchi identity $\overline{\mathrm{R}}_{0lpi} - \overline{\mathrm{R}}_{pl0i} = -\overline{\mathrm{R}}_{p0li}$. Ordering the terms we get the stated result. □

A.2. The Velocity

To compute the evolution of the velocity vector $\partial_t u = \sigma\nu + S^k\partial_k u$ we differentiate the equation $\overline{\nabla}_{\partial_t}\partial_t u = \alpha\nu$ using $\overline{\nabla}_{\partial_t}$. On the right hand side we get

$$\begin{aligned}\overline{\nabla}_{\partial_t}(\alpha\nu) =&\partial_t\alpha\nu + \alpha\overline{\nabla}_{\partial_t}\nu\\
=&\partial_t\alpha\nu + \alpha(S^l h_l{}^k - \nabla^k\sigma)\partial_k u.\end{aligned}$$

We use Lemma A.1 with $V = \partial_t u = \sigma\nu + S^k\partial_k u$ to decompose $\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t}\partial_t u$ and compare the normal parts to obtain

$$\partial_t^2\sigma = \partial_t\alpha + \sigma|\beta|^2 - \langle S, \nabla\alpha\rangle - 2\partial_t S^k\beta_k - \sigma S^k S^l\overline{\mathrm{R}}_{0kl0}.$$

We use the

$$\begin{aligned}\partial_t\alpha =&\frac{d\mu_t}{d\mu_0}\Big\{\Delta\sigma + \sigma(|h|^2 + \overline{\mathrm{Ric}}(\nu,\nu)) - \frac{\varrho}{\mathrm{Vol}(u)^2}\int_{\mathcal{N}}\sigma d\mu_t\\
&- \langle\nabla H, S\rangle + \left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)(\operatorname{div} S + \sigma H)\Big\}\end{aligned}$$

and insert the expressions for α, $\nabla\alpha$ and $|\beta|^2$ to get the evolution equation for σ

$$\begin{aligned}\partial_t^2\sigma =&\frac{d\mu_t}{d\mu_0}\Big\{\Delta\sigma + \sigma(|h|^2 + \overline{\mathrm{Ric}}(\nu,\nu)) - \frac{\varrho}{\mathrm{Vol}(u)^2}\int_{\mathcal{N}}\sigma d\mu_t\\
&+ \left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)(\operatorname{div}_{g_0} S + \sigma H)\Big\} + \sigma(|\nabla\sigma|^2 + S^kS^lh^j{}_kh_{jl} - 2\partial_i\sigma h^i{}_kS^k)\\
&- 2\partial_t S^k\partial_k\sigma + 2\partial_t S^k h_{ik}S^i - \sigma S^kS^l\overline{\mathrm{R}}_{0kl0}.\end{aligned} \quad \textbf{[A.9]}$$

Comparing the tangential parts we get

$$\begin{aligned}\partial_t^2 S^m =&\alpha(S^l h_l{}^m - \nabla^m\sigma - S^k h_k{}^m) + 2\partial_t\sigma\beta^m + \sigma\nabla^m\alpha - 2\partial_t S^k B_k{}^m - 2\sigma\beta^k B^m{}_k\\
&+ \sigma^2 S^l\overline{\mathrm{R}}_{0il0}g^{im} + \sigma S^lS^k\overline{\mathrm{R}}_{lik0}g^{im} - \sigma^2 S^k\overline{\mathrm{R}}_{0k0l}g^{lm} - \sigma S^kS^i\overline{\mathrm{R}}_{0kil}g^{lm}.\end{aligned}$$

We insert the expressions for α, $\nabla\alpha$ and β to get the evolution equation for S^m

$$\begin{aligned}\partial_t^2 S^m =&\frac{d\mu_t}{d\mu_0}\left(-H+\frac{\varrho}{\mathrm{Vol}(u)}\right)(-\nabla^m\sigma+\sigma(\Gamma^i_{ik}-\mathring{\Gamma}^i_{ik})g^{km})\\ &+2\partial_t\sigma(\nabla^m\sigma-h^m{}_kS^k)-\sigma\frac{d\mu_t}{d\mu_0}\nabla^m H-2\sigma(\nabla^i\sigma-h^i{}_jS^j)B^m{}_i-2\partial_tS^kB_k{}^m\\ &+\sigma^2S^l\overline{\mathrm{R}}_{0il0}g^{im}+\sigma S^lS^k\overline{\mathrm{R}}_{lik0}g^{im}-\sigma^2S^k\overline{\mathrm{R}}_{0k0l}g^{lm}-\sigma S^kS^i\overline{\mathrm{R}}_{0kil}g^{lm}.\end{aligned}\qquad[\mathbf{A.10}]$$

A.3. Second Fundamental Form

To compute the evolution equation for the second fundamental form we differentiate the equation $\overline{\nabla}_{\partial_t}\partial_t u=\alpha\nu$ using $\overline{\nabla}_{\partial_i}\overline{\nabla}_{\partial_j}$ and get on the right hand side

$$\begin{aligned}\overline{\nabla}_{\partial_i}\overline{\nabla}_{\partial_j}(\alpha\nu)=&\partial_i\partial_j\alpha\nu+\partial_i\alpha\overline{\nabla}_{\partial_j}\nu+\partial_j\alpha\overline{\nabla}_{\partial_i}\nu+\alpha\overline{\nabla}_{\partial_i}\overline{\nabla}_{\partial_j}\nu\\ =&(\partial_i\partial_j\alpha-\alpha h_i{}^kh_{kj})\nu+(\partial_i\alpha h_j{}^k+\partial_j\alpha h_i{}^k+\alpha\partial_ih_{jm}g^{mk}-\alpha\Gamma^l_{im}h_{lj}g^{mk})\partial_ku.\end{aligned}$$

In the last step we used

$$\langle\overline{\nabla}_{\partial_i}\overline{\nabla}_{\partial_j}\nu,\nu\rangle=\partial_i\underbrace{\langle\overline{\nabla}_{\partial_j}\nu,\nu\rangle}_{=0}-\langle\overline{\nabla}_{\partial_j}\nu,\overline{\nabla}_{\partial_i}\nu\rangle=-h_i{}^kh_{kj}$$

and

$$\begin{aligned}\langle\overline{\nabla}_{\partial_i}\overline{\nabla}_{\partial_j}\nu,\partial_mu\rangle=&\partial_i\langle\overline{\nabla}_{\partial_j}\nu,\partial_mu\rangle-\langle\overline{\nabla}_{\partial_j}\nu,\overline{\nabla}_{\partial_i}\partial_mu\rangle\\ =&\partial_ih_{jm}-\Gamma^l_{im}h_{lj}.\end{aligned}$$

Using Simons' identity we can calculate $\nabla_i\nabla_j\alpha$

$$\begin{aligned}-\nabla_i\nabla_j\alpha=&\frac{d\mu_t}{d\mu_0}\nabla_i\nabla_jH-\left(-H+\frac{\varrho}{\mathrm{Vol}(u)}\right)\nabla_i\nabla_j\frac{d\mu_t}{d\mu_0}+\nabla_i\frac{d\mu_t}{d\mu_0}\nabla_jH+\nabla_j\frac{d\mu_t}{d\mu_0}\nabla_iH\\ =&\frac{d\mu_t}{d\mu_0}\Big\{\Delta h_{ij}-Hh_{il}h^l{}_j+|h|^2h_{ij}+H\overline{\mathrm{R}}_{0i0j}-h_{ij}g^{lm}\overline{\mathrm{R}}_{0l0m}\\ &+h_j{}^lg^{km}\overline{\mathrm{R}}_{lkim}+h_i{}^lg^{km}\overline{\mathrm{R}}_{lkjm}-2h^{lm}\overline{\mathrm{R}}_{limj}+g^{lm}\overline{\nabla}_j\overline{\mathrm{R}}_{0lim}+g^{lm}\overline{\nabla}_l\overline{\mathrm{R}}_{0ijm}\Big\}\\ &-\left(-H+\frac{\varrho}{\mathrm{Vol}(u)}\right)\frac{d\mu_t}{d\mu_0}\Big(\nabla_i(\Gamma^l_{jl}-\mathring{\Gamma}^l_{jl})+(\Gamma^k_{ik}-\mathring{\Gamma}^k_{ik})(\Gamma^l_{jl}-\mathring{\Gamma}^l_{jl})\Big)\\ &+\nabla_jH\frac{d\mu_t}{d\mu_0}(\Gamma^k_{ik}-\mathring{\Gamma}^k_{ik})+\nabla_iH\frac{d\mu_t}{d\mu_0}(\Gamma^l_{jl}-\mathring{\Gamma}^l_{jl})\end{aligned}$$

We also have to interchange derivatives on the other side, whence we obtain

$$\begin{aligned}&\overline{\nabla}_{\partial_i}\overline{\nabla}_{\partial_j}\overline{\nabla}_{\partial_t}\partial_tu-\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_i}\partial_ju=(\overline{\nabla}_{\partial_i}\overline{\nabla}_{\partial_j}\overline{\nabla}_{\partial_t}\partial_tu-\overline{\nabla}_{\partial_i}\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_j}\partial_tu)\\ &\qquad+(\overline{\nabla}_{\partial_i}\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_j}\partial_tu-\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_i}\overline{\nabla}_{\partial_j}\partial_tu)+(\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_i}\overline{\nabla}_{\partial_t}\partial_ju-\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_i}\partial_ju)\\ &\quad=\overline{\nabla}_{\partial_i}(\overline{\mathrm{R}}(\partial_ju,\partial_tu)\partial_tu)+\overline{\mathrm{R}}(\partial_iu,\partial_tu)\overline{\nabla}_{\partial_j}\partial_tu+\overline{\nabla}_{\partial_t}(\overline{\mathrm{R}}(\partial_iu,\partial_tu)\partial_ju).\end{aligned}$$

Using Lemma A.1 with $V = \overline{\nabla}_{\partial_i}\partial_j u = -h_{ij}\nu + \Gamma^k_{ij}\partial_k u$ and comparing the normal parts we get the evolution equation for h_{ij}

$$\begin{aligned}-\partial_t^2 h_{ij} =&\partial_i\partial_j\alpha - \alpha h_{ik}h^k{}_j - h_{ij}|\beta|^2 - 2\partial_t\Gamma^k_{ij}\beta_k - \Gamma^k_{ij}\partial_k\alpha - \Gamma^k_{ij}\sigma S^l\overline{\mathrm{R}}_{0kl0} - \Gamma^k_{ij}S^lS^p\overline{\mathrm{R}}_{lkp0}\\ &- \langle\overline{\nabla}_{\partial_i}(\overline{\mathrm{R}}(\partial_j u,\partial_t u)\partial_t u),\nu\rangle - \langle\overline{\mathrm{R}}(\partial_i u,\partial_t u)\overline{\nabla}_{\partial_j}\partial_t u,\nu\rangle - \langle\overline{\nabla}_{\partial_t}(\overline{\mathrm{R}}(\partial_i u,\partial_t u)\partial_j u),\nu\rangle.\end{aligned}$$

Using the expressions for $-\nabla_i\nabla_j\alpha$, α, $|\beta|^2$ and expanding the curvature terms we get

$$\begin{aligned}\partial_t^2 h_{ij} =&\frac{d\mu_t}{d\mu_0}\Big\{\Delta h_{ij} - Hh_{il}h^l{}_j + |h|^2h_{ij} + H\overline{\mathrm{R}}_{0i0j}\underbrace{-h_{ij}g^{lm}\overline{\mathrm{R}}_{0l0m}}_{=+h_{ij}\overline{\mathrm{Ric}}(\nu,\nu)}\\ &+ h_j{}^lg^{km}\overline{\mathrm{R}}_{lkim} + h_i{}^lg^{km}\overline{\mathrm{R}}_{lkjm} - 2h^{lm}\overline{\mathrm{R}}_{limj} + g^{lm}\overline{\nabla}_j\overline{\mathrm{R}}_{0lim} + g^{lm}\overline{\nabla}_l\overline{\mathrm{R}}_{0ijm}\Big\}\\ &- \left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)\frac{d\mu_t}{d\mu_0}\Big(\nabla_i(\Gamma^l_{jl} - \mathring{\Gamma}^l_{jl}) + (\Gamma^k_{ik} - \mathring{\Gamma}^k_{ik})(\Gamma^l_{jl} - \mathring{\Gamma}^l_{jl})\Big)\\ &+ \nabla_jH\frac{d\mu_t}{d\mu_0}(\Gamma^k_{ik} - \mathring{\Gamma}^k_{ik}) + \nabla_iH\frac{d\mu_t}{d\mu_0}(\Gamma^l_{jl} - \mathring{\Gamma}^l_{jl})\\ &+ \left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)\frac{d\mu_t}{d\mu_0}h_{ik}h^k{}_j + h_{ij}(|\nabla\sigma|^2 + S^kS^lh^j{}_kh_{jl} - 2\partial_i\sigma h^i{}_kS^k)\\ &+ 2\partial_t\Gamma^k_{ij}(\partial_k\sigma - h_{kl}S^l) + \sigma\overline{\nabla}_i\overline{\mathrm{R}}_{j0k0}S^k + \overline{\nabla}_i\overline{\mathrm{R}}_{jkl0}S^kS^l - h_{ij}\overline{\mathrm{R}}_{0kl0}S^kS^l\\ &+ \overline{\mathrm{R}}_{j0k0}(\partial_i\sigma - h_{il}S^l)S^k + \overline{\mathrm{R}}_{jkl0}S^lB_i{}^k + \sigma\overline{\mathrm{R}}_{j0k0}B_i{}^k + S^k\overline{\mathrm{R}}_{jkl0}B_i{}^l\\ &+ \sigma\overline{\mathrm{R}}_{i0k0}B_j{}^k + S^l\overline{\mathrm{R}}_{ilk0}B_j{}^k + \sigma^2\overline{\nabla}_0\overline{\mathrm{R}}_{i0j0} + \sigma S^k\overline{\nabla}_0\overline{\mathrm{R}}_{ikj0} + S^k\sigma\overline{\nabla}_k\overline{\mathrm{R}}_{i0j0}\\ &+ S^kS^l\overline{\nabla}_k\overline{\mathrm{R}}_{ilj0} + (\partial_i\sigma - h_{il}S^l)S^k\overline{\mathrm{R}}_{0kj0} + B_i{}^l\sigma\overline{\mathrm{R}}_{l0j0} + B_i{}^lS^k\overline{\mathrm{R}}_{lkj0}\\ &+ \left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)\frac{d\mu_t}{d\mu_0}\overline{\mathrm{R}}_{i0j0} + B_j{}^l\sigma\overline{\mathrm{R}}_{i0l0} + B_j{}^lS^k\overline{\mathrm{R}}_{ikl0}.\end{aligned} \qquad [\mathbf{A.11}]$$

A.4. Christoffel Symbols

We can then read off the evolution equation for Γ^k_{ij} which is

$$\begin{aligned}\partial_t^2\Gamma^k_{ij} =&\partial_i\alpha h_j{}^k + \partial_j\alpha h_i{}^k + \alpha(\partial_ih_{jm}g^{mk} - \Gamma^l_{im}h_{lj}g^{mk}) - 2\partial_t\Gamma^l_{ij}B_l{}^k\\ &- 2\partial_th_{ij}\beta^k - \alpha h_l{}^k\Gamma^l_{ij} - h_{ij}\nabla^k\alpha + 2h_{ij}\beta^lB_l{}^k + \Big(- \sigma^2\Gamma^l_{ij}\overline{\mathrm{R}}_{0l0m}\\ &- \Gamma^l_{ij}S^p\sigma\overline{\mathrm{R}}_{pl0m} - \Gamma^l_{ij}S^p\sigma\overline{\mathrm{R}}_{0lpm} - \Gamma^l_{ij}S^pS^q\overline{\mathrm{R}}_{plqm} - h_{ij}S^p\sigma\overline{\mathrm{R}}_{0mp0}\\ &- h_{ij}S^pS^q\overline{\mathrm{R}}_{pmq0} - \langle\overline{\nabla}_i(\overline{\mathrm{R}}(\partial_ju,\partial_tu)\partial_tu),\partial_mu\rangle - \langle\overline{\mathrm{R}}(\partial_iu,\partial_tu)\overline{\nabla}_j\partial_tu,\partial_mu\rangle\\ &- \langle\overline{\nabla}_t(\overline{\mathrm{R}}(\partial_iu,\partial_tu)\partial_ju),\partial_mu\rangle\Big)g^{mk}.\end{aligned}$$

Inserting the expressions for α, $\nabla\alpha$, β and expanding the curvature terms this is

$$\begin{aligned}
\partial_t^2\Gamma_{ij}^k =& \left(-\nabla_i H h_j{}^k - \nabla_j H h_i{}^k + \nabla^k H h_{ij}\right)\frac{d\mu_t}{d\mu_0} + \left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)\frac{d\mu_t}{d\mu_0}\nabla_i h_j{}^k \\
&+ \left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)\frac{d\mu_t}{d\mu_0}(h_j{}^k(\Gamma_{il}^l - \mathring{\Gamma}_{il}^l) + h_i{}^k(\Gamma_{jl}^l - \mathring{\Gamma}_{jl}^l) - h_{ij}g^{km}(\Gamma_{ml}^l - \mathring{\Gamma}_{ml}^l)) \\
&- 2\partial_t\Gamma_{ij}^l B_l{}^k - 2\partial_t h_{ij}(\nabla^k\sigma - h^k{}_m S^m) + 2h_{ij}(\nabla^l\sigma - h^l{}_m S^m)B_l{}^k + \Bigg(\underbrace{-\sigma^2\Gamma_{ij}^l\overline{\mathrm{R}}_{0l0m}}_{1} \\
&\underbrace{-\Gamma_{ij}^l S^p\sigma\overline{\mathrm{R}}_{pl0m}}_{2}\underbrace{-\Gamma_{ij}^l S^p\sigma\overline{\mathrm{R}}_{0lpm}}_{3}\underbrace{-\Gamma_{ij}^l S^p S^q\overline{\mathrm{R}}_{plqm}}_{4} - h_{ij}S^p\sigma\overline{\mathrm{R}}_{0mp0} \\
&- h_{ij}S^p S^q\overline{\mathrm{R}}_{pmq0} - \sigma^2\overline{\nabla}_i\overline{\mathrm{R}}_{j00m} - \sigma S^p\overline{\nabla}_i\overline{\mathrm{R}}_{jp0m} - \sigma S^p\overline{\nabla}_i\overline{\mathrm{R}}_{j0pm} - S^p S^l\overline{\nabla}_i\overline{\mathrm{R}}_{jplm} \\
&+ h_{ij}\sigma S^p\overline{\mathrm{R}}_{0p0m} + h_{ij}S^p S^l\overline{\mathrm{R}}_{0plm}\underbrace{-\Gamma_{ij}^l\sigma^2\overline{\mathrm{R}}_{l00m}}_{1}\underbrace{-\Gamma_{ij}^l\sigma S^p\overline{\mathrm{R}}_{lp0m}}_{2} \\
&\underbrace{-\Gamma_{ij}^l\sigma S^p\overline{\mathrm{R}}_{l0pm}}_{3}\underbrace{-\Gamma_{ij}^l S^p S^q\overline{\mathrm{R}}_{lpqm}}_{4} - (\partial_i\sigma - h_{il}S^l)\sigma\overline{\mathrm{R}}_{j00m} - (\partial_i\sigma - h_{il}S^l)S^p\overline{\mathrm{R}}_{j0pm} \\
&- B_i{}^l\sigma\overline{\mathrm{R}}_{jl0m} - B_i{}^l S^p\overline{\mathrm{R}}_{jlpm} - \sigma(\partial_i\sigma - h_{il}S^l)\overline{\mathrm{R}}_{j00m} - S^p(\partial_i\sigma - h_{il}S^l)\overline{\mathrm{R}}_{jp0m} \\
&- \sigma B_i{}^l\overline{\mathrm{R}}_{j0lm} - S^p B_i{}^l\overline{\mathrm{R}}_{jplm} - \sigma(\partial_j\sigma - h_{jl}S^l)\overline{\mathrm{R}}_{i00m} - S^p(\partial_j\sigma - h_{jl}S^l)\overline{\mathrm{R}}_{ip0m} \\
&- \sigma B_j{}^l\overline{\mathrm{R}}_{i0lm} - S^l B_j{}^p\overline{\mathrm{R}}_{ilpm} - \sigma^2\overline{\nabla}_0\overline{\mathrm{R}}_{i0jm} - \sigma S^p\overline{\nabla}_0\overline{\mathrm{R}}_{ipjm} \\
&- \sigma S^p\overline{\nabla}_p\overline{\mathrm{R}}_{i0jm} - S^p S^l\overline{\nabla}_p\overline{\mathrm{R}}_{iljm} - S^p(\partial_i\sigma - h_{il}S^l)\overline{\mathrm{R}}_{0pjm} - B_i{}^l\sigma\overline{\mathrm{R}}_{l0jm} - B_i{}^l S^p\overline{\mathrm{R}}_{lpjm} \\
&- \frac{d\mu_t}{d\mu_0}\left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)\overline{\mathrm{R}}_{i0jm} - \sigma(\partial_j\sigma - h_{jl}S^l)\overline{\mathrm{R}}_{i00m} - S^p(\partial_j\sigma - h_{jl}S^l)\overline{\mathrm{R}}_{ip0m} \\
&- \sigma B_j{}^l\overline{\mathrm{R}}_{i0lm} - S^p B_j{}^l\overline{\mathrm{R}}_{iplm}\Bigg)g^{mk}. \qquad \textbf{[A.12]}
\end{aligned}$$

The terms marked with equal numbers cancel, and this equation is indeed a tensorial equation.

A.5. Mixed Derivatives

Next we differentiate the equation $\overline{\nabla}_{\partial_t}\partial_t u = \alpha\nu$ using $\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_i}$. We have

$$\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_i}\overline{\nabla}_{\partial_t}\partial_t u - \overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_i}\partial_t u = \overline{\nabla}_{\partial_t}(\overline{\mathrm{R}}(\partial_i u, \partial_t u)\partial_t u).$$

We have that

$$\begin{aligned}
\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_i}(\alpha\nu) =& (\partial_t\partial_i\alpha + \alpha\langle\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_i}\nu, \nu\rangle)\nu \\
&+ (-\partial_i\alpha\beta^k + \partial_t\alpha h_i{}^k + \alpha\langle\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_i}\nu, \partial_m u\rangle g^{mk})\partial_k u \\
=& (\partial_t\partial_i\alpha + \alpha h_i{}^k\beta_k)\nu \\
&+ (-\partial_i\alpha\beta^k + \partial_t\alpha h_i{}^k + \alpha\partial_t h_i{}^k + \alpha h_i{}^l B_l{}^k)\partial_k u
\end{aligned}$$

since

$$\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_i}\nu = \partial_t h_i{}^k \partial_k u + h_i{}^k \beta_k \nu + h_i{}^l B_l{}^k \partial_k u.$$

We have computed all that we need for writing down the evolution equation for β_i.

$$\begin{aligned}\partial_t^2\beta_i =& \partial_t\partial_i\alpha + \alpha h_i{}^k\beta_k + \beta_i|\beta|^2 - 2\partial_t B_i{}^l\beta_l - B_i{}^k\partial_k\alpha - B_i{}^l\sigma S^p\overline{\mathrm{R}}_{0kp0} - S^pS^qB_i{}^l\overline{\mathrm{R}}_{plq0}\\ &- \langle\overline{\nabla}_{\partial_t}(\overline{\mathrm{R}}(\partial_i u, \partial_t u)\partial_t u), \nu\rangle\end{aligned}$$

If we expand the last term and use that

$$\begin{aligned}\partial_t\partial_i\alpha =& -\frac{d\mu_t}{d\mu_0}\partial_t\partial_i H - \partial_i H\partial_t\frac{d\mu_t}{d\mu_0} - \partial_t H\partial_i\frac{d\mu_t}{d\mu_0} - \frac{\varrho}{\mathrm{Vol}(u)^2}\int_{\mathcal{N}}\sigma d\mu_t\partial_i\frac{d\mu_t}{d\mu_0}\\ &+ \left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)\partial_i\partial_t\frac{d\mu_t}{d\mu_0}\end{aligned}$$

and if we use [**A.7**] we obtain

$$\begin{aligned}\partial_t^2\beta_i =& \frac{d\mu_t}{d\mu_0}\Bigg(\Delta\beta_i + (B^{kl} + B^{lk})\nabla_i h_{kl} + h^{kl}\nabla_i(B_{lk} + B_{kl})\\ &+ g^{kl}\Big\{\partial_t\Gamma^p_{lk}h_{ip} - \partial_t\Gamma^p_{il}h_{kp} + \sigma\overline{\nabla}_0\overline{\mathrm{R}}_{0lki} + S^p\overline{\nabla}_p\overline{\mathrm{R}}_{0lki} - \beta^p\overline{\mathrm{R}}_{plki}\\ &+ 2B_l{}^p\overline{\mathrm{R}}_{0pki} + \beta_i\overline{\mathrm{R}}_{0lk0} + B_i{}^p\overline{\mathrm{R}}_{0lkp}\\ &- \nabla_k B_i{}^p h_{pl} - B_i{}^p\nabla_k h_{pl} - \sigma\overline{\nabla}_k\overline{\mathrm{R}}_{0l0i} - S^p\overline{\nabla}_k\overline{\mathrm{R}}_{pl0i}\\ &+ h_{kl}S^p\overline{\mathrm{R}}_{p00i} - h_k{}^p\sigma\overline{\mathrm{R}}_{0lpi} - h_k{}^pS^q\overline{\mathrm{R}}_{qlpi}\Big\} - \partial_i H g^{kl}B_{kl}\frac{d\mu_t}{d\mu_0} - \partial_t H(\Gamma^l_{il} - \mathring{\Gamma}^l_{il})\frac{d\mu_t}{d\mu_0}\\ &- \frac{\varrho}{\mathrm{Vol}(u)^2}\int_{\mathcal{N}}\sigma d\mu_t(\Gamma^l_{il} - \mathring{\Gamma}^l_{il})\frac{d\mu_t}{d\mu_0} + \left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)\partial_t(\Gamma^l_{il} - \mathring{\Gamma}^l_{il})\frac{d\mu_t}{d\mu_0}\\ &\left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)(\Gamma^l_{il} - \mathring{\Gamma}^l_{il})g^{kl}B_{kl}\frac{d\mu_t}{d\mu_0} + \left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)\frac{d\mu_t}{d\mu_0}h_i{}^k\beta_k + \beta_i|\beta|^2\\ &- 2\partial_t B_i{}^l\beta_l + B_i{}^k\partial_k H\frac{d\mu_t}{d\mu_0} - \left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)B_i{}^k(\Gamma^l_{kl} - \mathring{\Gamma}^l_{kl})\frac{d\mu_t}{d\mu_0} - B_i{}^l\sigma S^p\overline{\mathrm{R}}_{0kp0}\\ &- S^pS^qB_i{}^l\overline{\mathrm{R}}_{plq0} - \sigma^2S^p\overline{\nabla}_0\overline{\mathrm{R}}_{i0p0} - \sigma S^pS^q\overline{\nabla}_0\overline{\mathrm{R}}_{iqp0} - \sigma S^lS^p\overline{\nabla}_l\overline{\mathrm{R}}_{i0p0}\\ &- S^lS^pS^q\overline{\nabla}_l\overline{\mathrm{R}}_{iqp0} - S^pS^q\beta_i\overline{\mathrm{R}}_{0qp0} - B_i{}^lS^p\sigma\overline{\mathrm{R}}_{l0p0} - B_i{}^lS^pS^q\overline{\mathrm{R}}_{lqp0} - \alpha S^p\overline{\mathrm{R}}_{i0p0}.\end{aligned}$$

We also used that $\partial_t d\mu_t = g^{kl}B_{kl}d\mu_t$ and that $\partial_i\partial_t\frac{d\mu_t}{d\mu_0} = \partial_t(\Gamma^l_{il} - \mathring{\Gamma}^l_{il})\frac{d\mu_t}{d\mu_0} + (\Gamma^l_{il} - \mathring{\Gamma}^l_{il})g^{kl}B_{kl}\frac{d\mu_t}{d\mu_0}$.

We can simply write down the evolution equation for ${B_i}^k$ which is

$$\begin{aligned}\partial_t^2 {B_i}^k =& -\partial_i\alpha\beta^k + \partial_t\alpha {h_i}^k + \alpha\partial_t {h_i}^k + \alpha {h_i}^l B_l^k \\ & - 2\partial_t {B_i}^l {B_l}^k + 2\partial_t\beta_i\beta^k - {B_i}^l {h_l}^k\alpha + \beta_i\nabla^k\alpha - 2\beta_i\beta^l {B^k}_l \\ & + \Big\{ - {B_i}^l\sigma^2\overline{\mathrm{R}}_{0l0m} - {B_i}^l S^p\sigma\overline{\mathrm{R}}_{pl0m} - {B_i}^l S^p\sigma\overline{\mathrm{R}}_{0lpm} - {B_i}^l S^p S^q\overline{\mathrm{R}}_{plqm} \\ & + \beta_i\sigma S^p\overline{\mathrm{R}}_{0mp0} + \beta_i S^p S^q\overline{\mathrm{R}}_{pmq0} - \langle\overline{\nabla}_{\partial_t}(\overline{\mathrm{R}}(\partial_i u, \partial_t u)\partial_t u), \partial_m u\rangle\Big\} g^{mk}.\end{aligned}$$

We expand $\partial_i\alpha$, $\partial_t\alpha$, α, β, $\partial_t\beta$ and the last term to obtain

$$\begin{aligned}\partial_t^2 {B_i}^k =& -\partial_i H\frac{d\mu_t}{d\mu_0}(\nabla^k\sigma - {h^k}_l S^l) + \left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)\frac{d\mu_t}{d\mu_0}(\Gamma^l_{il} - \mathring{\Gamma}^l_{il})(\nabla^k\sigma - {h^k}_m S^m) \\ & + {h_i}^k\Big\{ -\partial_t H\frac{d\mu_t}{d\mu_0} - \frac{\varrho}{\mathrm{Vol}(u)^2}\frac{d\mu_t}{d\mu_0}\int_{\mathcal{N}}\sigma d\mu_t + \left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)\frac{d\mu_t}{d\mu_0} g^{ml} B_{ml}\Big\} \\ & + \left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)\frac{d\mu_t}{d\mu_0}(\partial_t {h_i}^k + {h_i}^l B_l^k) \\ & - 2\partial_t {B_i}^l {B_l}^k + 2(-{B_i}^p(\partial_p\sigma - h_{pm}S^m) + \sigma S^p\overline{\mathrm{R}}_{0ip0} + S^p S^q\overline{\mathrm{R}}_{piq0})(\nabla^k\sigma - {h^k}_l S^l) \\ & - {B_i}^l {h_l}^k\left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)\frac{d\mu_t}{d\mu_0} - (\partial_i\sigma - h_{il}S^l)\nabla^k H\frac{d\mu_t}{d\mu_0} \\ & + (\partial_i\sigma - h_{ip}S^p)\left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)\frac{d\mu_t}{d\mu_0} g^{km}(\Gamma^l_{ml} - \mathring{\Gamma}^l_{ml}) \\ & - 2(\partial_i\sigma - h_{il}S^l)(\nabla^m\sigma - {h^m}_p S^p){B^k}_m \\ & + \Big\{\underline{-{B_i}^l\sigma^2\overline{\mathrm{R}}_{0l0m} - {B_i}^l S^p\sigma\overline{\mathrm{R}}_{pl0m} - {B_i}^l S^p\sigma\overline{\mathrm{R}}_{0lpm} - {B_i}^l S^p S^q\overline{\mathrm{R}}_{plqm}} \\ & \underline{+(\partial_i\sigma - h_{il}S^l)\sigma S^p\overline{\mathrm{R}}_{0mp0} + (\partial_i\sigma - h_{il}S^l)S^p S^q\overline{\mathrm{R}}_{pmq0}} - \sigma^3\overline{\nabla}_0\overline{\mathrm{R}}_{i00m} \\ & - \sigma^2 S^p(\overline{\nabla}_0\overline{\mathrm{R}}_{i0pm} + \overline{\nabla}_p\overline{\mathrm{R}}_{i00m} + \overline{\nabla}_0\overline{\mathrm{R}}_{ip0m}) - \sigma S^p S^q(\overline{\nabla}_0\overline{\mathrm{R}}_{iqpm} + \overline{\nabla}_q\overline{\mathrm{R}}_{i0pm} + \overline{\nabla}_p\overline{\mathrm{R}}_{iq0m}) \\ & - S^p S^q S^l\overline{\nabla}_l\overline{\mathrm{R}}_{ipqm}\underline{-(\partial_i\sigma - h_{il}S^l)S^p\sigma\overline{\mathrm{R}}_{0p0m} - (\partial_i\sigma - h_{il}S^l)S^p S^q\overline{\mathrm{R}}_{0pqm}} \\ & \underline{-{B_i}^l\sigma S^p(\overline{\mathrm{R}}_{l0pm} + \overline{\mathrm{R}}_{lp0m}) - {B_i}^l\sigma^2\overline{\mathrm{R}}_{l00m} - {B_i}^l S^p S^q\overline{\mathrm{R}}_{lpqm}} \\ & + \left(-H + \frac{\varrho}{\mathrm{Vol}(u)}\right)\frac{d\mu_t}{d\mu_0}(-\sigma\overline{\mathrm{R}}_{i00m} - S^p\overline{\mathrm{R}}_{i0pm} - \sigma\overline{\mathrm{R}}_{i00m} - S^p\overline{\mathrm{R}}_{ip0m})\Big\} g^{mk}. \qquad [\mathbf{A.13}]\end{aligned}$$

The underlined terms can be simplified to

$$2(\partial_i\sigma - h_{il}S^l)\sigma S^p\overline{\mathrm{R}}_{0mp0} + 2(\partial_i\sigma - h_{il}S^l)S^p S^q\overline{\mathrm{R}}_{pmq0}.$$

APPENDIX B

The Nash-Moser Inverse Function Theorem

We include some of the necessary definitions, some examples and the statement of the Nash-Moser Theorem. Everything in this Appendix is taken from [**Ham82a**].

DEFINITION B.1. [**Ham82a**, II.1.1.1] A *grading* on a Fréchet space is a collection of seminorms $\{\| \ \|_n : n \in J\}$ indexed by integers $J = \{0, 1, 2, , \dots\}$ which are increasing in strength, so that

$$\|f\|_0 \leq \|f\|_1 \leq \|f\|_2 \leq \dots$$

and which define the topology. A *graded Fréchet space* is one with a choice of grading.

EXAMPLE B.2. [**Ham82a**, II.1.1.2(2)] Let $\Sigma(B)$ denote the space of all sequences $\{f_k\}$ of elements in a Banach space B such that

$$\|\{f_k\}\|_n = \sum_{k=0}^{\infty} e^{nk} \|f_k\|_B < \infty$$

for all $n \geq 0$. Then $\Sigma(B)$ is a graded space with the above norms.

EXAMPLE B.3. [**Ham82a**, II.1.1.2(4)] Let X be a compact manifold. Then $C^\infty(X)$ is a graded space with

$$\|f\|_n = \|f\|_{C^n(X)}$$

where $C^n(X)$ is the Banach space of functions with continuous partial derivatives of degree $\leq n$. If V is a vector bundle over X then the space $C^\infty(X, V)$ of smooth sections of V is also a graded space.

DEFINITION B.4. [**Ham82a**, II.1.1.3] We say that two gradings $\{\| \ \|_n\}$ and $\{\| \ \|'_n\}$ are *tamely equivalent* of degree r and base b if

$$\|f\|_n \leq C\|f\|'_{n+r} \text{ and } \|f\|'_n \leq C\|f\|_{n+r}$$

for all $n \geq b$ (with a constant C which may depend on n).

EXAMPLE B.5. [**Ham82a**, II.1.1.4(3)] If X is a compact manifold, then the following gradings on $C^\infty(X)$ are equivalent

(1) the supremum norms $\|f\|_n = \|f\|_{C^n(X)}$,
(2) the Hölder norms $\|f\|_n = \|f\|_{C^{n+\alpha}}$ for $0 < \alpha < 1$,
(3) the Sobolev norms $\|f\|_n = \|f\|_{W^{n,p}(X)}$ for $1 < p < \infty$,
(4) the Besov norms $\|f\|_n = \|f\|_{B^{n+\alpha}_{p,q}}(X)$ for $0 < \alpha < 1$, $1 < p < \infty$, $1 \leq q \leq \infty$.

For example, by the Sobolev embedding theorem, if $r > \dim X/p$ then

$$\|f\|_{C^n(X)} \leq C\|f\|_{W^{n+r,p}(X)}.$$

Definition B.6. [**Ham82a**, II.1.2.1] We say that a linear map $L : F \to G$ of one graded space into another satisfies a *tame estimate* of degree r and base b if

$$\|Lf\|_n \leq C\|f\|_{n+r}$$

for each $n \geq b$ (with a constant C which may depend on n). We say L is *tame* if it satisfies a tame estimate for some r and b. A tame linear map is automatically continuous in the Fréchet space topologies.

Definition B.7. [**Ham82a**, II.1.3.1] Let F and G be graded spaces. We say that F is a *tame direct summand* of G if we can find tame linear maps $L : F \to G$ and $M : G \to F$ such that the composition $ML : F \to F$ is the identity

$$F \xrightarrow{L} G \xrightarrow{M} F.$$

Definition B.8. [**Ham82a**, II.1.3.2] We say a graded space is *tame* if it is a tame direct summand of a space $\Sigma(B)$ of exponentially decreasing sequences in some Banach space B.

Theorem B.9. *[**Ham82a**, II.1.3.6/II.1.3.7] If X is a compact manifold with or without boundary then $C^\infty(X)$ is tame.*

Remark B.10. Since the product of tame spaces is tame [**Ham82a**, II.1.3.4] this implies that $C^\infty(X, \mathbb{R}^d)$ is tame if X is a compact manifold with or without boundary.

Corollary B.11. *[**Ham82a**, II.1.3.9] If X is a compact manifold and V is a vector bundle over X, then the space $C^\infty(X, V)$ of sections of V over X is tame.*

Definition B.12. [**Ham82a**, II.2.1.1] Let F and G be graded spaces and $P : U \subset F \to G$ a nonlinear map of a subset U of F into G. We say that P satisfies a *tame estimate* of degree r and base b if

$$\|P(f)\|_n \leq C(1 + \|f\|_{n+r})$$

for all $f \in U$ and all $n \geq b$ (with a constant C which may depend on n). We say that P is a *tame* map if P is defined on an open set and is continuous, and satisfies a tame estimate in a neighborhood of each point. (We allow the degree r, base b, and constants C to vary from neighborhood to neighborhood.)

Theorem B.13 (The Nash-Moser Theorem). *[**Ham82a**, III.1.1.1] Let F and G be tame spaces and $P : U \subset F \to G$ a smooth tame map. Suppose that the equation for the derivative $DP(f)h = k$ has a unique solution $h = VP(f)k$ for all f in U and all k, and that the family of inverses $VP : U \times G \to F$ is a smooth tame map. Then P is locally invertible, and each local inverse P^{-1} is a smooth tame map.*

APPENDIX C

Norms and Inequalities

C.1. Norms

We will use an atlas $\mathscr{A} = (x_\alpha, U_\alpha)$ of the compact manifold $\mathcal{N}$ with the properties that $\alpha = 1, \dots, J$ (due to compactness) and $x_\alpha(U_\alpha) = B_3(0)$. Suppose that the sets $x_\alpha^{-1}(B_1(0))$ cover $\mathcal{N}$.

We define the following norms for functions $\varphi : \mathcal{N} \to \mathbb{R}$, $\psi : [0,T] \times \mathcal{N} \to \mathbb{R}$

$$\begin{aligned}
\|\varphi\|_s &= \sum_{\alpha=1}^{J} \|\varphi \circ x_\alpha^{-1}\|_{H^s(B_2(0))} = \sum_{\alpha=1}^{J} \sum_{|\beta| \le s} \|\partial^\beta(\varphi \circ x_\alpha^{-1})\|_{L^2(B_2(0))} \\
\|\varphi\|_{C^s} &= \sum_{\alpha=1}^{J} \|\varphi \circ x_\alpha^{-1}\|_{C^s(B_2(0))} = \sum_{\alpha=1}^{J} \sum_{|\beta| \le s} \sup_{B_2(0)} |\partial^\beta(\varphi \circ x_\alpha^{-1})| \\
|||\psi|||_s^2 &= \sum_{j=0}^{s} \int_0^T \|\partial_t^j \psi(t', \cdot)\|_{s-j}^2 dt' \\
|||\psi|||_{C^s} &= \sum_{\alpha=1}^{J} \sum_{|\beta| \le s} \sum_{j=0}^{|\beta|} \sup_{t \in [0,T]} \sup_{B_2(0)} |\partial_t^{j-|\beta|} \partial^\beta(\psi(t, \cdot) \circ x_\alpha^{-1})|.
\end{aligned}$$

The family of norms $|||\cdot|||_s$, $s = 0, 1, 2, \dots$ defines a grading on $C^\infty([0,T] \times \mathcal{N}, \mathbb{R})$. For functions V in $C^\infty([0,T] \times \mathcal{N}, \mathbb{R}^d)$ we simply add the norms of the components, e. g.

$$|||V|||_s = \sum_{k=1}^{d} |||V^k|||_s.$$

Now let $\pi : \mathcal{V} \to \mathcal{N}$ be a smooth d-dimensional vectorbundle over $\mathcal{N}$. By making the coordinate charts smaller if necessary we can assume that the domains of the local trivialisations Φ_α, $\alpha = 1, \dots, J$, correspond to the domains of the coordinate charts, i. e. $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^d$. Denote the transition function between Φ_α and Φ_β by $\Phi_{\alpha\beta}$, i. e. $\Phi_\beta \circ \Phi_\alpha^{-1}(x, v) = (x, \Phi_{\alpha\beta}(x)v)$ for $(x, v) \in (U_\alpha \cap U_\beta) \times \mathbb{R}^d$. Let $\pi_2 : U_\alpha \times \mathbb{R}^d \to \mathbb{R}^d$ be the projection to the second factor.

For a smooth section V of $\mathcal{V}$ we define its local norm as the norm of the coordinates in the local trivialisation, i. e.

$$\|V \circ x_\alpha^{-1}\|_{H^s(B_2(0))} = \|\pi_2 \circ \Phi_\alpha \circ V \circ x_\alpha^{-1}\|_{H^s(B_2(0))}.$$

To define the norm of V we sum these local norms over all coordinate charts

$$\|V\|_s = \sum_{\alpha=1}^{J} \|V \circ x_\alpha^{-1}\|_{H^s(B_2(0))}$$

with an analogous definition for the other norms $\|\cdot\|_{C^s}$, $|||\cdot|||_s$ and $|||\cdot|||_{C^s}$. The family of norms $|||\cdot|||_s$, $s = 0, 1, 2, \ldots$ then defines a grading on the Fréchet space $C^\infty([0,T]\times\mathcal{N},\mathcal{V})$ of time dependent smooth sections of the vectorbundle $\mathcal{V}$. This grading is tamely equivalent to the usual C^k-grading.

Remark C.1. In these charts we can estimate a smooth function $\varphi : \mathcal{N} \to \mathbb{R}$ via

$$\|\varphi \circ x_\alpha^{-1}\|_{L^2(B_2(0))} \le C \sum_{\beta=1}^{J} \|\varphi \circ x_\beta^{-1}\|_{L^2(B_1(0))},$$

where C depends on the derivatives of the coordinate changes. Furthermore we have

$$\|\varphi \circ x_\alpha^{-1}\|_{H^s(B_2(0))} \le C \sum_{\beta=1}^{J} \|\varphi \circ x_\beta^{-1}\|_{H^s(B_1(0))},$$

where C additionally depends on higher derivatives of the coordinate changes.

Proof. First we show the statement for the L^2-norm. For any other chart x_β let

$$A_\beta = x_\beta\left(x_\alpha^{-1}(B_2(0)) \cap x_\beta^{-1}(B_1(0))\right) \subset B_1(0).$$

We have that

$$x_\alpha\left(x_\alpha^{-1}(B_2(0)) \cap x_\beta^{-1}(B_1(0))\right) = x_\alpha \circ x_\beta^{-1}(A_\beta).$$

Furthermore, by the properties of our atlas

$$\begin{aligned} B_2(0) \subset x_\alpha\Big(\bigcup_\beta \left(x_\alpha^{-1}(B_2(0)) \cap x_\beta^{-1}(B_1(0))\right)\Big) &\subset \bigcup_\beta x_\alpha\left(x_\alpha^{-1}(B_2(0)) \cap x_\beta^{-1}(B_1(0))\right) \\ &= \bigcup_\beta x_\alpha \circ x_\beta^{-1}(A_\beta). \end{aligned}$$

And hence

$$\begin{aligned} \|\varphi \circ x_\alpha^{-1}\|^2_{L^2(B_2(0))} &= \int_{B_2(0)} |\varphi \circ x_\alpha^{-1}|^2 dx \\ &\le \sum_\beta \int_{x_\alpha \circ x_\beta^{-1}(A_\beta)} |\varphi \circ x_\alpha^{-1}|^2 dx \\ &= \sum_\beta \int_{A_\beta} |\varphi \circ x_\alpha^{-1} \circ x_\alpha \circ x_\beta^{-1}|^2 \mathbf{J}(x_\alpha \circ x_\beta^{-1}) dx \\ &\le C \sum_\beta \int_{B_1(0)} |\varphi \circ x_\beta^{-1}|^2 dx. \end{aligned}$$

Here $\mathbf{J}$ denotes the Jacobian. For the H^s-norm we observe that for any multiindex γ with $|\gamma| \le s$

$$|D^\gamma(\varphi \circ x_\alpha^{-1})| = |D^\gamma(\varphi \circ x_\beta^{-1} \circ x_\beta \circ x_\alpha^{-1})|$$

in the set $x_\alpha \circ x_\beta^{-1}(A_\beta)$. Using the chain rule and boundedness of the derivatives of the coordinate changes on $B_2(0)$, this term can be estimated by a sum of terms of the form

$$|D^{\gamma'}(\varphi \circ x_\beta^{-1})(x_\beta \circ x_\alpha^{-1}(p))||D^{\gamma_1}(x_\beta \circ x_\alpha^{-1})(p)| \dots |D^{\gamma_r}(x_\beta \circ x_\alpha^{-1})(p)| \\ \le C|D^{\gamma'}(\varphi \circ x_\beta^{-1})|(x_\beta \circ x_\alpha^{-1}(p))$$

with $|\gamma_1| + \cdots + |\gamma_r| \le s$ and $|\gamma'| \le s$. Then we can do the same as for the L^2-norm. $\square$

We need an analogue of Remark C.1 for sections in a vectorbundle.

Remark C.2. We can estimate a smooth section V of $\mathcal{V}$ via

$$\|V \circ x_\alpha^{-1}\|_{L^2(B_2(0))} \le C \sum_{\beta=1}^{J} \|V \circ x_\beta^{-1}\|_{L^2(B_1(0))},$$

where C depends on derivatives of the coordinate changes and the transition functions $\Phi_{\alpha\beta}$. Furthermore we have

$$\|V \circ x_\alpha^{-1}\|_{H^s(B_2(0))} \le C \sum_{\beta=1}^{J} \|V \circ x_\beta^{-1}\|_{H^s(B_1(0))},$$

where C additionally depends on higher derivatives of the coordinate changes and the transition functions.

Proof. The proof is similar to the proof of Remark C.1. Additionally, one has to express $\pi_2 \circ \Phi_\alpha \circ V \circ x_\beta^{-1} = \Phi_{\beta\alpha} \circ \pi_2 \circ \Phi_\beta \circ V \circ x_\beta^{-1}$ and estimate $\Phi_{\beta\alpha}$ and its derivatives by the supremum. $\square$

For a linear differential operator we always define its "norm" to be the norm of the coefficients in local coordinates. For example if in a local coordinate chart (x_α, U_α) we have $L\varphi = a^{ij}\partial_i\partial_j\varphi + a^i\partial_i\varphi + a\varphi$ then we define the local norm

$$[L]_{s,\alpha} = \sum_{i,j} \|a^{ij}\|_{H^s(B_2(0))} + \sum_i \|a^i\|_{H^s(B_2(0))} + \|a\|_{H^s(B_2(0))}$$

and the full norm

$$[L]_s = \sum_{\alpha=1}^{J} [L]_{s,\alpha} .$$

We define similarly $[L]_{C^s}$ to measure the coefficients in $\|\cdot\|_{C^s}$ and if L depends on time we define $|[L]|_s$ and $|[L]|_{C^s}$ to measure the coefficients in $|||\cdot|||_s$ and $|||\cdot|||_{C^s}$ respectively. Note that these are not the usual operator norms.

C.2. Moser Inequalities

Let $\Omega \subset \mathbb{R}^n$ a bounded domain with Lipschitz boundary. The following theorems follow from the inequalities in [**Tay97**, Ch. 13, §3] and Stein's extension theorem for Sobolev functions [**Ste70**, Ch. VI]. We denote L^p-Sobolev spaces by $W^{k,p}$, where k is the differentiability and put $H^k = W^{k,2}$.

THEOREM C.3 (First Moser inequality). *There exists a constant C such that*

$$\|\partial^\alpha(fg)\|_{L^2(\Omega)} \leq C(\|f\|_{H^s(\Omega)}\|g\|_{L^\infty(\Omega)} + \|g\|_{H^s(\Omega)}\|f\|_{L^\infty(\Omega)})$$

for all $f, g \in H^s(\Omega) \cap L^\infty(\Omega)$ and $|\alpha| = s$.

THEOREM C.4 (Second Moser inequality). *There exists a constant C such that*

$$\|\partial^\alpha(fg) - f\partial^\alpha g\|_{L^2(\Omega)} \leq C(\|f\|_{H^s(\Omega)}\|g\|_{L^\infty(\Omega)} + \|f\|_{W^{1,\infty}(\Omega)}\|g\|_{H^{s-1}(\Omega)})$$

for all $f \in H^s(\Omega) \cap W^{1,\infty}(\Omega)$, $g \in H^{s-1}(\Omega) \cap L^\infty(\Omega)$ and $|\alpha| = s$.

THEOREM C.5 (Third Moser inequality). *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function. There exists a constant C depending only on $\|f\|_{L^\infty(\Omega)}$ such that*

$$\|\partial^\alpha F(f)\|_{L^2(\Omega)} \leq C(1 + \|f\|_{H^s(\Omega)})$$

for all $f \in H^s(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ and $s = |\alpha| > 0$. If $F(0) = 0$ then

$$\|\partial^\alpha F(f)\|_{L^2(\Omega)} \leq C\|f\|_{H^s(\Omega)}.$$

COROLLARY C.6. *Let $F : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function with $F(x, 0) = 0$ for all $x \in \mathbb{R}^m$ and all derivatives bounded. Then there exists a constant C depending only on $\|\varphi\|_{L^\infty(\Omega)}$ such that*

$$\|F(\cdot, f(\cdot))\|_{H^s(\Omega)} \leq C\|f\|_{H^s(\Omega)}.$$

for all $f \in H^s(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$.

PROOF. In order to estimate the L^2-norm write

$$F(x, f(x)) = F(x, 0) + \int_0^1 D_2F(x, sf(x))dsf(x). \qquad [\mathbf{C.1}]$$

To estimate terms where all derivatives fall on the first argument of F, use that $D_x^k F(x, 0) = 0$ and apply [**C.1**] to $D_x^k F(x, f(x))$. For all other terms, the procedure is the same as in the proof of Theorem C.5. □

LEMMA C.7. *For any $s \in \mathbb{N}$ there exists a constant C such that*

$$\int_0^T \|\partial_t^k(fg)\|_{s-k}^2 \leq C\left(|||f|||_{C^0}^2 \int_0^T \|\partial_t^k g\|_{s-k}^2 dt + |||f|||_{C^1}^2\, |||g|||_{s-1}^2 + |||g|||_{C^0}^2\, |||f|||_s^2\right)$$

for all smooth functions $f, g : [0, T] \times \mathcal{N} \to \mathbb{R}$.

Proof. In a local coordinate chart we have to estimate the terms

$$\int_0^T \|\partial_t^k \partial^\beta (fg)\|^2_{L^2(B)} dt$$

where $|\beta| \leq s-k$ and $B = B_2(0)$. By the product rule

$$\int_0^T \|\partial_t^k \partial^\beta (fg)\|^2_{L^2(B)} dt \leq \int_0^T \|f \partial_t^k \partial^\beta g\|^2_{L^2(B)} dt + C \sum_{\substack{k_1+k_2=k,\\ |\beta_1|+|\beta_2|=|\beta|\\ k_1\neq 0 \vee \beta_1 \neq 0}}\sum \int_0^T \|\partial_t^{k_1} \partial^{\beta_1} f \partial_t^{k_2} \partial^{\beta_2} g\|^2_{L^2(B)} dt.$$

The first term on the right is estimated by

$$|||f|||^2_{C^0} \int_0^T \|\partial_t^k g\|^2_{s-k} dt.$$

For the second term we first consider the case $k_1 = 0$. Then $\beta_1 \neq 0$ and we can choose β' with $|\beta'| = |\beta_1| - 1$ and $\partial^{\beta_1} = \partial^{\beta'} \partial_i$. Then

$$\int_0^T \|\partial^{\beta_1} f \partial_t^{k_2} \partial^{\beta_2} g\|^2_{L^2(B)} dt \leq C \int_0^T \|\partial^{\beta'} Df \partial_t^k \partial^{\beta_2} g\|^2_{L^2(B)} dt.$$

Now we apply [**Tay97**, Ch. 13, Prop. 3.6.] on the spacetime domain $[0,T] \times B$ (use again Stein's extension theorem [**Ste70**, Ch. VI]) to obtain the estimate

$$\int_0^T \|\partial^{\beta'} Df \partial_t^k \partial^{\beta_2} g\|^2_{L^2(B)} dt \leq C \left(|||Df|||^2_{C^0} \, |||g|||^2_{s-1} + |||g|||^2_{C^0} \, |||Df|||^2_{s-1}\right)$$

since $|\beta'| + |\beta_2| + k = s-1$. If $k_1 \neq 0$ then choose $k' = k_1 - 1$ and estimate in the same manner

$$\int_0^T \|\partial_t^{k'} \partial^{\beta_1} \partial_t f \partial_t^{k_2} \partial^{\beta_2} g\|^2_{L^2(B)} dt \leq C \left(|||\partial_t f|||^2_{C^0} \, |||g|||^2_{s-1} + |||g|||^2_{C^0} \, |||\partial_t f|||^2_{s-1}\right).$$

As $|||\partial_t f|||_{s-1} \leq |||f|||_s$ and $|||Df|||_{s-1} \leq |||f|||_s$ we obtain the stated result. $\square$

C.3. Gronwall's Inequality

We need modified versions of Gronwall's inequality which are a bit more exact than the inequalities that are stated in most of the literature. The proofs are very similar.

Lemma C.8. *Assume that $\eta, \varphi, \psi \geq 0$ are continuous functions on $[0,T]$ and that η is continuously differentiable on $[0,T]$. If*

$$\eta'(t) \leq \varphi(t)\eta(t) + \psi(t)$$

for all $t \in [0,T]$ then

$$\eta(t) \leq e^{\int_0^t \varphi(r)dr} \eta(0) + \int_0^t e^{\int_s^t \varphi(r)dr} \psi(s) ds.$$

Proof. Compute

$$\frac{d}{ds}\left(\eta(s)e^{-\int_0^s \varphi(r)dr}\right) = e^{-\int_0^s \varphi(r)dr}\left(\eta'(s) - \varphi(s)\eta(s)\right) \leq e^{-\int_0^s \varphi(r)dr}\psi(s).$$

Consequently we have

$$\eta(t)e^{-\int_0^t \varphi(r)dr} \leq \eta(0) + \int_0^t e^{-\int_0^s \varphi(r)dr}\psi(s)ds.$$

which implies the inequality. □

Lemma C.9. *Let $A, B, h \geq 0$ be continuous functions on $[0,T]$ and let B be continuously differentiable on $[0,T]$. If*

$$A(t) \leq B(t) + \int_0^t h(s)A(s)ds$$

for all $t \in [0,T]$ then

$$A(t) \leq e^{\int_0^t h(r)dr}B(0) + \int_0^t e^{\int_s^t h(r)dr}B'(s)ds.$$

Proof. Let

$$H(t) = B(t) + \int_0^t h(s)A(s)ds.$$

Then

$$H'(t) = B'(t) + h(t)A(t) \leq B'(t) + h(t)H(t)$$

and hence

$$\frac{d}{ds}\left(H(s)e^{-\int_0^s h(r)dr}\right) = e^{-\int_0^s h(r)dr}\left(H'(s) - h(s)H(s)\right) \leq e^{-\int_0^s h(r)dr}B'(s).$$

Integrating from 0 to t gives

$$H(t)e^{-\int_0^t h(r)dr} \leq H(0) + \int_0^t e^{-\int_0^s h(r)dr}B'(s)ds$$

and consequently

$$A(t) \leq H(t) \leq e^{\int_0^t h(r)dr}B(0) + \int_0^t e^{\int_s^t h(r)dr}B'(s)ds.$$ □

Lemma C.10. *Let A, B, h be as in Lemma C.9 and let g be a continuous positive function on $[0,T]$. If*

$$A(t) \leq g(t)B(t) + g(t)\int_0^t h(s)A(s)ds$$

for all $t \in [0,T]$ then

$$A(t) \leq g(t)e^{\int_0^t g(r)h(r)dr}B(0) + g(t)\int_0^t e^{\int_s^t g(r)h(r)dr}B'(s)ds \qquad \textbf{[C.2]}$$

for all $t \in [0, T]$.

PROOF. Let $\tilde{A}(t) = A(t)/g(t)$ and $\tilde{h}(s) = h(s)g(s)$. Then

$$\tilde{A}(t) \leq B(t) + \int_0^t \tilde{h}(s)\tilde{A}(s)ds.$$

Lemma C.9 implies

$$\tilde{A}(t) \leq e^{\int_0^t \tilde{h}(r)dr} B(0) + \int_0^t e^{\int_s^t \tilde{h}(r)dr} B'(s)ds$$

which implies inequality [**C.2**]. $\square$

APPENDIX D

Another Choice of Kinetic Energy

In Section 1.2 we defined the kinetic energy with respect to a reference measure $d\mu_0$. In this appendix we discuss another choice, namely if we define the kinetic energy in a more geometric fashion using the induced volume measure $d\mu_t$ on the surface, i. e.

$$\mathcal{K}_2(u) = \int_{\mathcal{N}} \frac{1}{2}|\partial_t u|^2 d\mu_t.$$

In this case the action integral is

$$\begin{aligned}\mathcal{A}_2(u) &= \int_0^T \int_{\mathcal{N}} \frac{1}{2}|\partial_t u|^2 d\mu_t dt - \int_0^T \int_{\mathcal{N}} d\mu_t dt + \varrho \int_0^T \log\left(\frac{\mathrm{Vol}(u)}{\mathrm{Vol}_0}\right) dt \\ &= \int_0^T \int_{\mathcal{N}} \left(\frac{1}{2}|\partial_t u|^2 - 1\right) d\mu_t dt + \varrho \int_0^T \log\left(\frac{\mathrm{Vol}(u)}{\mathrm{Vol}_0}\right) dt.\end{aligned}$$

This kinetic energy is also considered in [**LS08**]. We will first state the resulting Euler-Lagrange equation. Then we discuss the conservation laws and indicate how a short time existence result can be established in a special case. Furthermore in Section D.5, we will see that the behaviour described by this equation does not fit with the physical intuition which is the reason why we chose [**EQ**] for our study.

D.1. The Equation

Proposition D.1. *Let u_ε be a variation of u with $u_0 = u$ and $\frac{\partial u_\varepsilon}{\partial \varepsilon}\big|_{\varepsilon=0}(p) = X(p)$. Then*

$$\begin{aligned}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{A}_2(u_\varepsilon) = &\int_{\mathcal{N}} \langle X, \partial_t u\rangle d\mu_t \Big|_{t=T} - \int_{\mathcal{N}} \langle X, \partial_t u\rangle d\mu_t \Big|_{t=0} \\ &- \int_0^T \int_{\mathcal{N}} \langle X, \overline{\nabla}_{\partial_t} \partial_t u\rangle - \left(\frac{1}{2}|\partial_t u|^2 - 1\right) H\langle \nu, X\rangle + (\sigma H + \operatorname{div} S)\langle \partial_t u, X\rangle \\ &\qquad + \frac{1}{2}\langle \nabla |\partial_t u|^2, X\rangle - \frac{\varrho}{\mathrm{Vol}(u)}\langle \nu, X\rangle d\mu_t dt. \qquad [\mathbf{D.1}]\end{aligned}$$

Proof. The computation is done as in Section 1.2 and [**LS08**]. □

Corollary D.2. *The Euler-Lagrange equation of $\mathcal{A}_2$ is*

$$\begin{aligned}\overline{\nabla}_{\partial_t}\partial_t u &= \left(\frac{1}{2}|\partial_t u|^2 - 1\right) H\nu + \frac{\varrho}{\mathrm{Vol}(u)}\nu - (\sigma H + \operatorname{div} S)\,\partial_t u - \frac{1}{2}\nabla|\partial_t u|^2 \qquad [\mathbf{EQ_2}]\\ &= -\left(\frac{1}{2}|\partial_t u|^2 + 1 - |S|^2\right) H\nu - \sigma \operatorname{div} S\nu + \frac{\varrho}{\mathrm{Vol}(u)}\nu \\ &\quad - (\sigma H + \operatorname{div} S)\, S^k \partial_k u - \frac{1}{2}\nabla|\partial_t u|^2.\end{aligned}$$

Remark D.3. If $S = 0$ (see next section about this assumption) then [$\mathbf{EQ_2}$] reads

$$\overline{\nabla}_{\partial_t}\partial_t u = -\left(\frac{1}{2}|\partial_t u|^2 + 1\right) H\nu + \frac{\varrho}{\mathrm{Vol}(u)}\nu - \frac{1}{2}\nabla(|\partial_t u|^2). \qquad [\mathbf{EQ_2}']$$

We will see by an example in Section D.5 that [$\mathbf{EQ_2}$] and [$\mathbf{EQ_2}'$] are not equivalent, i. e. modifying a solution of [$\mathbf{EQ_2}$] by a diffeomorphism such that the tangential velocity vanishes does not yield a solution of [$\mathbf{EQ_2}'$]. We also give an example for the non-equivalence of the [**HMCF**] and the [**HMCF**$'$] equation from [**LS08**].

D.2. Conservation Laws

The conservation of energy and exterior momentum for [$\mathbf{EQ_2}$] is similar to that of [**EQ**]. The proofs are also very similar and we omit them. The diffeomorphism invariance of $\mathcal{A}_2$ leads to a more general conservation of interior momentum (Proposition D.6) than we had for [**EQ**].

Define the energy

$$\mathcal{E}(u(t,\cdot)) = \int_{\mathcal{N}} \frac{1}{2}|\partial_t u|^2 d\mu_t + \int_{\mathcal{N}} d\mu_t - \varrho \log\left(\frac{\mathrm{Vol}(u)}{\mathrm{Vol}_0}\right).$$

Let $u : [0,T) \times \mathcal{N} \to \mathcal{M}$ solve [$\mathbf{EQ_2}$] with $\mathcal{E}_0 = \mathcal{E}(u(0,\cdot))$.

Proposition D.4. *We have $\mathcal{E}(u(t,\cdot)) = \mathcal{E}_0$ for all $t \in [0,T)$.*

Let X be a Killing vector field on $\mathcal{M}$. Define the exterior momentum with respect to X of a solution u of [$\mathbf{EQ_2}$] by

$$\mathcal{P}_X(u(t,\cdot)) = \int_{\mathcal{N}} \langle \partial_t u, X(u)\rangle d\mu_t.$$

Proposition D.5. *Let $u : [0,T) \times \mathcal{N} \to \mathcal{M}$ solve [$\mathbf{EQ_2}$]. Then $\mathcal{P}_X(u(t,\cdot))$ is constant as a function of t.*

Proposition D.6. *Let Y be an arbitrary vectorfield on $\mathcal{N}$. Define*

$$\mathcal{Q}_Y(u(t,\cdot)) = \int_{\mathcal{N}} \langle \partial_t u, u_* Y\rangle d\mu_t.$$

If u solves [$\mathbf{EQ_2}$] then $\mathcal{Q}_Y(u(t,\cdot))$ is constant as a function of t.

Proof. Let φ_s be the local flow of Y and set $u_s = u \circ \varphi_s$. Since $\mathcal{A}_2$ is diffeomorphism invariant we have $\frac{d}{ds}\big|_{s=0}\mathcal{A}(u_s) = 0$. Using [**D.1**] with $X = u_*Y = \frac{d}{ds}\big|_{s=0}u_s$ we see that

$$0 = \mathcal{Q}_Y(u(T,\cdot)) - \mathcal{Q}_Y(u(0,\cdot)).$$ □

Corollary D.7. *Let $u : [0,T)\times\mathcal{N} \to \mathcal{M}$ solve* [$\mathbf{EQ_2}$] *with $S(0,\cdot) = 0$. Then $S(t,\cdot) = 0$ for all $t \in [0,T)$.*

Proof. We have that $\mathcal{Q}_Y(u(0,\cdot)) = 0$ for any vectorfield Y on $\mathcal{N}$ and hence for all $t \in [0,T)$

$$0 = \mathcal{Q}_Y(u(t,\cdot)) = \int_{\mathcal{N}} \langle \partial_t u, u_*Y\rangle d\mu_t = \int_{\mathcal{N}} \langle S, u_*Y\rangle d\mu_t.$$

Since this holds for all vectorfields Y we conclude that $S = 0$ for all $t \in [0,T)$. □

Corollary D.7 says that if we start with a normal velocity then the velocity stays normal. This is due to the diffeomorphism invariance of the action $\mathcal{A}_2$. In fact, even if $\partial_t u$ is not normal initially, the tangential velocity S follows a simple evolution.

Proposition D.8. *Let $u : [0,T) \times \mathcal{N} \to \mathcal{M}$ solve* [$\mathbf{EQ_2}$]. *Then*

$$S_i(t) = \frac{d\mu_0}{d\mu_t} S_i(0) \qquad [\mathbf{D.2}]$$

where $d\mu_0 = d\mu_t(0)$ is the induced surface measure at $t = 0$.

Proof. Calculate

$$\begin{aligned}\partial_t S_i =& \langle \overline{\nabla}_{\partial_t}\partial_t u, \partial_i u\rangle + \langle \partial_t u, \overline{\nabla}_{\partial_t}\partial_i u\rangle \\ =& -(\sigma H + \operatorname{div} S)S_i - \frac{1}{2}\partial_i|\partial_t u|^2 + \frac{1}{2}\partial_i|\partial_t u|^2 \\ =& - S_i \partial_t \log d\mu_t.\end{aligned}$$

Define $\hat{S}_i = \frac{d\mu_0}{d\mu_t}S_i(0)$. Then

$$\partial_t \hat{S}_i = -\hat{S}_i \partial_t \log d\mu_t$$

and hence S_i and $\hat{S}_i$ satisfy the same ODE and $S_i(0) = \hat{S}_i(0)$. By the standard uniqueness result for ODEs we conclude that $S_i = \hat{S}_i$. □

D.3. A Graphical Formulation

Let $\Sigma_0 = u(0,\mathcal{N})$. We want to write the solution of equation [$\mathbf{EQ_2}$] as a graph over Σ_0. So choose a Gaussian coordinate system (q,x) for a neighborhood of Σ_0 where q is the coordinate orthogonal to Σ_0, i.e. the signed distance to Σ_0. Let $M_\tau = \{q = \tau\}$ and $\sigma_{ij}(\tau)$ be the induced metric on M_τ. Let $\hat{h}_{ij}(\tau)$ be the second fundamental form of M_τ and $\hat{H}(\tau)$ its mean curvature. Write $\tilde{u}(t,x) = (\varphi(t,x),x)$ and $u(t,x) = \tilde{u}(t,\Psi(t,x)) = (\varphi(t,\Psi(t,x)),\Psi(t,x))$ where $\varphi : \Sigma_0 \to \mathbb{R}$ and $\Psi : \Sigma_0 \to \Sigma_0$ is a diffeomorphism. We use $\tilde{u}$ to express geometric quantities. The tangent vectors to $\tilde{u}(\Sigma_0)$ are given by $\partial_i\tilde{u} = (\varphi_i, 0,\ldots,0,1,0,\ldots,0)$. And so the unit normal is given by

$\nu = \sqrt{1+|D\varphi|^2}^{-1}(1, -\varphi^i)$, where $\varphi^j(t,x) = \sigma^{kj}(\varphi(t,x),x)\partial_k\varphi(t,x)$. The induced metric and its inverse are given by

$$g_{ij} = \sigma_{ij} + \partial_i\varphi\partial_j\varphi, \quad g^{ij} = \sigma^{ij} - \frac{\varphi^i\varphi^j}{1+|D\varphi|^2}.$$

We multiply

$$-H\nu^\alpha = g^{ij}\left\{\partial_i\partial_j\tilde{u}^\alpha - \Gamma^k_{ij}\partial_k\tilde{u}^\alpha + \overline{\Gamma}^\alpha_{\beta\gamma}\partial_i\tilde{u}^\beta\partial_j\tilde{u}^\gamma\right\}$$

with ν, to obtain

$$\begin{aligned}-H = \sqrt{1+|D\varphi|^2}^{-1} g^{ij}\Big\{&\partial_i\partial_j\varphi - \Gamma^k_{ij}\partial_k\varphi + \overline{\Gamma}^0_{00}\partial_i\varphi\partial_j\varphi + \overline{\Gamma}^0_{0j}\partial_i\varphi + \overline{\Gamma}^0_{0i}\partial_j\varphi + \overline{\Gamma}^0_{ij} \\ &+ \Gamma^k_{ij}\partial_k\tilde{u}^l\partial_l\varphi - \overline{\Gamma}^k_{\beta\gamma}\partial_i\tilde{u}^\beta\partial_j\tilde{u}^\gamma\partial_k\varphi\Big\}.\end{aligned}$$

Now we have $\partial_k\tilde{u}^l = \delta^l_k$ and

$$\overline{\Gamma}^0_{ij} = -\hat{h}_{ij}, \quad \overline{\Gamma}^0_{00} = \overline{\Gamma}^0_{0i} = \overline{\Gamma}^k_{00} = 0, \quad \overline{\Gamma}^k_{0i} = \hat{h}_{il}\sigma^{lk}, \quad \overline{\Gamma}^k_{ij} = (\Gamma_\sigma)^k_{ij}$$

and consequently

$$-H = \sqrt{1+|D\varphi|^2}^{-1} g^{ij}(\partial_i\partial_j\varphi - (\Gamma_\sigma)^k_{ij}\partial_k\varphi) - \sqrt{1+|D\varphi|^2}^{-1}\hat{H} - \sqrt{1+|D\varphi|^2}^{-3}\hat{h}_{kl}\varphi^k\varphi^l$$

where we used that $g^{ij}\hat{h}_{ij} = \hat{H} - \sqrt{1+|D\varphi|^2}^{-2}\hat{h}_{kl}\varphi^k\varphi^l$ and $1 - \sqrt{1+|D\varphi|^2}^{-2}|D\varphi|^2 = \sqrt{1+|D\varphi|^2}^{-2}$. Write $S = \tilde{S}^k\partial_k\tilde{u} = S^l\partial_l u$. Then

$$\partial_t u^i = \partial_t\Psi^i = \sigma\nu^i + \tilde{S}^k\partial_k\tilde{u}^i = -\frac{\sigma}{\sqrt{1+|D\varphi|^2}}\varphi^i + \tilde{S}^i \qquad \textbf{[D.3]}$$

and

$$\partial_t u^0 = \sigma\nu^0 + \tilde{S}^i\partial_i\tilde{u}^0 = \frac{\sigma}{\sqrt{1+|D\varphi|^2}} + \tilde{S}^k\partial_k\varphi.$$

We have

$$\sigma = \langle\partial_t u, \nu\rangle = \frac{1}{\sqrt{1+|D\varphi|^2}}\left(\partial_t\varphi + \partial_k\varphi\partial_t\Psi^k - \partial_i\varphi\partial_t\Psi^i\right) = \frac{\partial_t\varphi}{\sqrt{1+|D\varphi|^2}}.$$

The 0 component of $\overline{\nabla}_{\partial_t}\partial_t u^\alpha$ is

$$\begin{aligned}\overline{\nabla}_{\partial_t}\partial_t u^0 =& \partial_t^2 u^0 + \overline{\Gamma}^0_{\beta\gamma}\partial_t u^\beta\partial_t u^\gamma \\ =& \partial_t^2\varphi + 2\partial_t\Psi^i\partial_i\partial_t\varphi + \partial_t\Psi^i\partial_t\Psi^j\partial_i\partial_j\varphi + \partial_i\varphi\partial_t^2\Psi^i + \underbrace{\overline{\Gamma}^0_{ij}}_{=-\hat{h}_{ij}}\partial_t\Psi^i\partial_t\Psi^j.\end{aligned}$$

The i components of $\overline{\nabla}_{\partial_t}\partial_t u^\alpha$ are

$$\begin{aligned}\overline{\nabla}_{\partial_t}\partial_t u^i =& \partial_t^2 u^i + \overline{\Gamma}^i_{\beta\gamma}\partial_t u^\beta\partial_t u^\gamma \\ =& \partial_t^2\Psi^i + 2\overline{\Gamma}^i_{0j}\partial_t u^0\partial_t\Psi^j + \overline{\Gamma}^i_{jk}\partial_t\Psi^j\partial_t\Psi^k.\end{aligned}$$

With $\overline{\nabla}_{\partial_t}\partial_t u = \alpha\nu + A^i\partial_i u$ we have

$$\partial_i\varphi\partial_t^2\Psi^i = -\frac{|D\varphi|^2}{\sqrt{1+|D\varphi|^2}}\alpha + A^l\partial_l u^i\partial_i\varphi - 2\hat{h}_{ij}\partial_t u^0\partial_t\Psi^j\varphi^i - (\Gamma_\sigma)_{ij}^k\,\partial_k\varphi\partial_t\Psi^i\partial_t\Psi^j.$$

Note that $A^l\partial_l u^i\partial_i\varphi = A^l\partial_l\Psi^i\partial_i\varphi = A^l\partial_l u^0$. Putting things together

$$\begin{aligned}\partial_t^2\varphi + 2\partial_t\Psi^i\partial_i\partial_t\varphi + \partial_t\Psi^i\partial_t\Psi^j\partial_i\partial_j\varphi - \hat{h}_{ij}\partial_t\Psi^i\partial_t\Psi^j - \frac{|D\varphi|^2}{\sqrt{1+|D\varphi|^2}}\alpha + A^l\partial_l u^i\partial_i\varphi \\ - 2\hat{h}_{ij}\partial_t u^0\partial_t\Psi^j\varphi^i - (\Gamma_\sigma)_{ij}^k\,\partial_k\varphi\partial_t\Psi^i\partial_t\Psi^j = \frac{\alpha}{\sqrt{1+|D\varphi|^2}} + A^l\partial_l u^0.\end{aligned}$$

Inserting

$$\alpha = -\left(\frac{1}{2}\sigma^2 + 1 - \frac{1}{2}|S|^2\right)H - \sigma\operatorname{div}S + \frac{\varrho}{\mathrm{Vol}(u)}$$

and [**D.3**] for $\partial_t\Psi^i$ this finally gives the equation for the graphical evolution

$$\begin{aligned}\partial_t^2\varphi = &\left(\frac{1}{2}\sigma^2 + 1 - \frac{1}{2}|\tilde{S}|^2\right)\left(g^{ij}(\partial_i\partial_j\varphi - (\Gamma_\sigma)_{ij}^k\,\partial_k\varphi) - \hat{H} - \frac{\hat{h}_{kl}\varphi^k\varphi^l}{1+|D\varphi|^2}\right)\\
&- \partial_t\varphi\operatorname{div}\tilde{S} + \sqrt{1+|D\varphi|^2}\frac{\varrho}{\mathrm{Vol}(\varphi)}\\
&- \left(\frac{\sigma\varphi^i}{\sqrt{1+|D\varphi|^2}} - \tilde{S}^i\right)\left(\frac{\sigma\varphi^j}{\sqrt{1+|D\varphi|^2}} - \tilde{S}^j\right)(\partial_i\partial_j\varphi - (\Gamma_\sigma)_{ij}^k\,\partial_k\varphi)\\
&+ 2\left(\frac{\sigma\varphi^i}{\sqrt{1+|D\varphi|^2}} - \tilde{S}^i\right)\partial_i\partial_t\varphi\\
&- 2\hat{h}_{ij}\varphi^i\left(\frac{\sigma\varphi^j}{\sqrt{1+|D\varphi|^2}} - \tilde{S}^j\right)\left(\frac{\sigma}{\sqrt{1+|D\varphi|^2}} + \tilde{S}^k\partial_k\varphi\right)\\
&+ \hat{h}_{ij}\left(\frac{\sigma\varphi^i}{\sqrt{1+|D\varphi|^2}} - \tilde{S}^i\right)\left(\frac{\sigma\varphi^j}{\sqrt{1+|D\varphi|^2}} - \tilde{S}^j\right).\end{aligned} \tag{[D.4]}$$

To obtain a useful equation for φ we have to replace $\tilde{S}^k$ using [**D.2**]. Then [**D.4**] is a scalar equation for φ and $\tilde{S}$ only comes in via its value at time $t = 0$. So using our knowledge [**D.2**] of the tangential velocity we can decouple [$\mathbf{EQ_2}$] into a scalar equation for φ and an ODE [**D.3**] for Ψ. Note that

$$\operatorname{div}\tilde{S} = \partial_i\tilde{S}^i + \Gamma_{ik}^i(\tilde{u})\tilde{S}^k$$

also contains second derivatives of φ which do not appear if $D\varphi = 0$.

In the case that $S = 0$ the scalar equation [**D.4**] simplifies to

$$\begin{aligned}\partial_t^2\varphi = \left(\frac{1}{2}\sigma^2 + 1\right)&\left(g^{ij}(\partial_i\partial_j\varphi - (\Gamma_\sigma)_{ij}^k\,\partial_k\varphi) - \hat{H} - \frac{\hat{h}_{kl}\varphi^k\varphi^l}{1+|D\varphi|^2}\right)\\ &+ \sqrt{1+|D\varphi|^2}\frac{\varrho}{\mathrm{Vol}(\varphi)} - \frac{\sigma^2\varphi^i\varphi^j}{1+|D\varphi|^2}(\partial_i\partial_j\varphi - (\Gamma_\sigma)_{ij}^k\,\partial_k\varphi)\\ &+ \frac{2\sigma\varphi^i}{\sqrt{1+|D\varphi|^2}}\partial_i\partial_t\varphi - \frac{\sigma^2\hat{h}_{ij}\varphi^i\varphi^j}{1+|D\varphi|^2}.\end{aligned}$$

D.4. Hyperbolicity

We want to check in which case the scalar equation [**D.4**] for φ is hyperbolic in order to obtain a short time existence result. Then [**D.4**] can be treated by standard quasilinear hyperbolic theory (the less standard integral terms can be readily dealt with). The ODE for Ψ can subsequently be solved separately. If we only want to get a solution for a short time we only need hyperbolicity for a short time. By continuity it suffices to check this at time $t = 0$. Clearly at time $t = 0$ we have that $\varphi = 0$ and hence $D\varphi = 0$. Furthermore $\tilde{S}^k = S^k$ since the initial condition for Ψ is $\partial_k\Psi^i(0) = \delta_k^i$. The matrix G by which the second derivatives are multiplied is given by

$$G = \begin{pmatrix} -1 & -S^1 & \dots & -S^n \\ -S^1 & & & \\ \vdots & & (\gamma g^{ij} - S^iS^j) & \\ -S^n & & & \end{pmatrix}$$

at $t = 0$ where $\gamma = \left(\frac{1}{2}\sigma^2 + 1 - \frac{1}{2}|S|^2\right)$. Clearly we need that $\gamma \neq 0$ for otherwise $(|S|^2, S_1, \dots, S_n)$ would be a zero eigenvector. The inverse is given by

$$G^{-1} = \begin{pmatrix} -(1-\gamma^{-1}|S|^2) & -\gamma^{-1}S_1 & \dots & -\gamma^{-1}S_n \\ -\gamma^{-1}S_1 & & & \\ \vdots & & (\gamma^{-1}g_{ij}) & \\ -\gamma^{-1}S_n & & & \end{pmatrix}.$$

The matrix G^{-1} defines a Lorentzian metric on $[0,T]\times\mathcal{N}$ if and only if $\gamma > 0$. Hence the hyperbolicity condition is that

$$|S|^2 < 2 + \sigma^2. \qquad [\mathbf{D.5}]$$

Clearly ∂_t is transverse to the spacelike surface $\{0\}\times\mathcal{N}$ and if [**D.5**] holds for the initial data we can solve the scalar equation [**D.4**] for φ subject to the initial conditions $\varphi(0) = 0$, $\partial_t\varphi(0) = \sigma(0)$. We obtain the following short time existence theorem for [$\mathbf{EQ_2}$].

Theorem D.9. *For every smooth immersion $u_0 : \mathcal{N} \to \mathcal{M}$ with $\mathrm{Vol}(u_0) = \mathrm{Vol}_0 > 0$ and initial velocity $u_1 \in \Gamma(u_0^*T\mathcal{M})$ satisfying* [**D.5**] *there exists $\varepsilon > 0$ and a smooth family of* *immersions* $u : [0,\varepsilon) \times \mathcal{N} \to \mathcal{M}$ *solving the Cauchy problem*

$$\begin{cases} \overline{\nabla}_{\partial_t}\partial_t u = \left(\frac{1}{2}|\partial_t u|^2 - 1\right) H\nu + \frac{\varrho}{\mathrm{Vol}(u)}\nu - (\sigma H + \operatorname{div} S)\,\partial_t u - \frac{1}{2}\nabla|\partial_t u|^2 \\ u(0,\cdot) = u_0 \\ \partial_t u(0,\cdot) = u_1. \end{cases}$$

Remark D.10. In [**LS08**] LeFloch and Smoczyk also use the idea of writing the solution as a graph to prove short time existence for their equation if $S = 0$. But we can allow a tangential motion if the equation is hyperbolic. The hyperbolicity condition analogous to [**D.5**] for their equation is $|S|^2 < n + \sigma^2 - \varepsilon$ for some $\varepsilon > 0$.

D.5. Role of Tangential Velocity and Translations

For a physical model we would expect by Newton's law that translating a given solution leads to a new solution since a translation does not change the acceleration, i. e. if $\xi \in \mathbb{R}^{n+1}$ and $u : [0,T] \times \mathcal{N} \to \mathbb{R}^{n+1}$ is a solution of [$\mathbf{EQ_2}$] then $\tilde{u}(t) = u(t) + t\xi$ should again be a solution. This works fine for [**EQ**] as we saw in Subsection 1.4.2. However in [$\mathbf{EQ_2}$] the acceleration depends on $\partial_t u$. So this doesn't work.

To be more specific let $u_r : \mathcal{S}^2 \to \mathbb{R}^3$ be a parametrisation of the sphere with radius r around the origin. Clearly if $r = \sqrt{\frac{\varrho}{2\omega_3}}$ then u_r is an equilibrium solution of [$\mathbf{EQ_2}$]. Let $u = u_r + t\xi$. We have $\partial_t u = \xi$, $(\sigma H + \operatorname{div} S) = \partial_t \log d\mu_t = 0$, $\partial_t^2 u = 0$ and $H = 2/r$. For u to be a solution of [$\mathbf{EQ_2}$] we need that

$$-\left(\frac{1}{2}|\xi|^2 - 1\right) H = \frac{\varrho}{\omega_3 r^3},$$

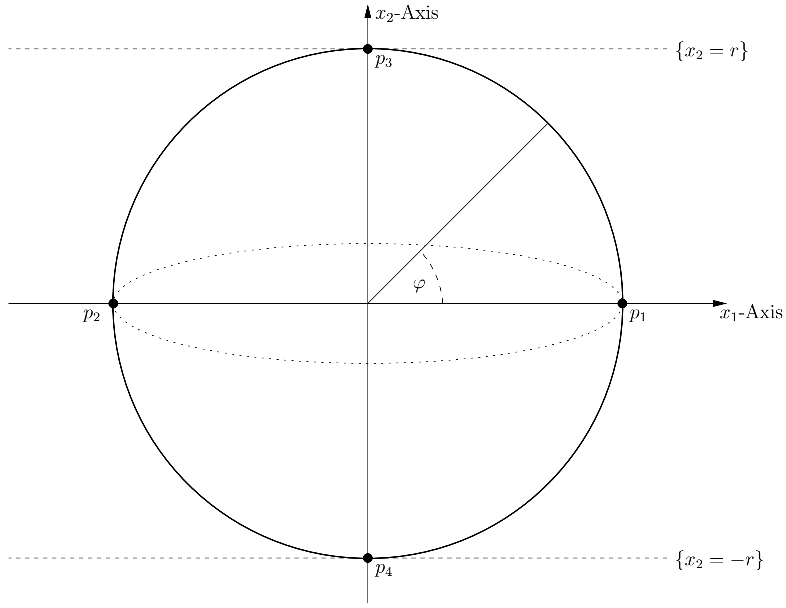
i. e.

$$r^2 = \frac{\varrho}{\omega_3\,(2 - |\xi|^2)}. \qquad [\mathbf{D.6}]$$

So we only get translating spheres with this parametrisation if the translation velocity is small enough, i. e. $|\xi| < \sqrt{2}$, with a radius depending on the velocity that goes to infinity as $|\xi|$ approaches $\sqrt{2}$. This corresponds to the hyperbolicity condition [**D.5**] since there are points where $|S| = |\xi|$ and $\sigma = 0$. Alternatively one could read [**D.6**] as a condition on ϱ saying that we have to adapt the inner pressure to the velocity to obtain a translating sphere with a given radius and velocity ξ with $|\xi| < \sqrt{2}$.

We show in the following that translating spheres cannot be solutions of [$\mathbf{EQ_2}'$]. The normal velocity at $t = 0$ for a sphere translating in direction $\xi = ve_1$, $v > 0$, is $\sigma(0,x) = \langle \nu(x), \xi\rangle = v\cos\varphi(x)$ where $\varphi(x)$ is the angle between x and e_1, i. e. $\partial_t u(0,x) = v\cos\varphi(x)$. Let $p_1 = re_1$ and $p_2 = -re_1$. We have then

$$\alpha(0,p_1) = \alpha(0,p_2) = -\left(\frac{1}{2}v^2 + 1\right) H + \frac{\varrho}{\omega_3 r^3}.$$



Write $u(t,x) = u_r(\Psi(t,x)) + t\xi$ with a diffeomorphism Ψ satisfying $\Psi(0) = \mathrm{id}$. At p_1 and p_2 we have at $t = 0$

$$0 = S_i = \langle \partial_t u, \partial_i u \rangle = \partial_t \Psi^j g_{ij} + \langle \xi, \partial_i u \rangle$$

which implies $\partial_t \Psi(0, p_1) = \partial_t \Psi(0, p_2) = 0$ since ξ is normal at p_1, p_2 and $\partial_i \Psi^l(0) = \delta_i^l$. Now at p_1, p_2 at $t = 0$

$$\alpha = \langle \partial_t^2 u, \nu \rangle = \langle \partial_i \partial_j u_r, \nu \rangle \partial_t \Psi^i \partial_t \Psi^j + \langle \partial_i u_r, \nu \rangle \partial_t^2 \Psi^i = 0.$$

So at p_1, p_2 we get the condition

$$-\left(\frac{1}{2} v^2 + 1 \right) \frac{2}{r} + \frac{\varrho}{\omega_3 r^3} = 0.$$

Assume that $r^2 = (\omega_3 (v^2+2))^{-1} \varrho$ is chosen like that and note that this condition is different from [**D.6**]. Let $p_3 = re_2$ and $p_4 = -re_2$. We have that

$$\alpha(0, p_3) = \alpha(0, p_4) = -\frac{2}{r} + \frac{\varrho}{\omega_3 r^3} > 0.$$

By Taylor expansion for a short time we have that

$$\begin{aligned} \langle u(t, p_3), \nu(0, p_3) \rangle =& \langle u(0, p_3), \nu(0, p_3) \rangle + t \langle \partial_t u(0, p_3), \nu(0, p_3) \rangle + \frac{1}{2} t^2 \alpha(0, p_3) + O(t^3) \\ =& r + \frac{1}{2} t^2 \alpha(0, p_3) + O(t^3) > r \end{aligned}$$

and similarly

$$\langle u(t, p_4), \nu(0, p_4) \rangle > r.$$

But $\nu(0,p_3)=e_2$ and $\nu(0,p_4)=-e_2$. This means that for a short time $u(t,p_3)$ and $u(t,p_4)$ leave the space in between the planes $\{x_2=r\}$ and $\{x_2=-r\}$. But if the sphere were translating in the direction e_1 it must stay in between these planes.

We also saw in this example what we announced in Remark D.3, namely that applying a diffeomorphism to a solution of [**EQ₂**] such that the tangential velocity vanishes does not yield a solution of [**EQ₂′**].

We now give an example of a solution of the [**HMCF**] equation from [**LS08**]. For $u:[0,T]\times\mathcal{N}\to\mathcal{M}$ this equation reads

$$\overline{\nabla}_{\partial_t}\partial_t u=\left(\frac{1}{2}|\partial_t u|^2-\frac{n}{2}\right)H\nu-(\sigma H+\operatorname{div}S)\,\partial_t u-\frac{1}{2}\nabla|\partial_t u|^2. \qquad [\mathbf{HMCF}]$$

If $S=0$ we obtain the [**HMCF′**] equation

$$\overline{\nabla}_{\partial_t}\partial_t u=-\left(\frac{1}{2}|\partial_t u|^2+\frac{n}{2}\right)H\nu-\frac{1}{2}\nabla|\partial_t u|^2. \qquad [\mathbf{HMCF'}]$$

The property $S=0$ is preserved if it is satisfied at time $t=0$.

For some radius $r>0$ and some velocity v define $u:\mathbb{R}\times\mathbb{S}^1\to\mathbb{R}^2$ by

$$u(t,x)=re^{i(x+\frac{v}{r}t)}.$$

This is a parametrisation of a circle rotating with velocity v in the complex plane which we identify with $\mathbb{R}^2$. We have

$$\partial_t u=ive^{i(x+\frac{v}{r}t)},\quad \partial_t^2 u=-\frac{v^2}{r}e^{i(x+\frac{v}{r}t)}=-\frac{v^2}{r}\nu.$$

Hence $\sigma=0$, $|\partial_t u|^2=v^2$ and $\operatorname{div}S=0$ since S is Killing. Furthermore $H=\frac{1}{r}$. Hence u solves [**HMCF**] if

$$-\frac{v^2}{r}=\left(\frac{1}{2}v^2-\frac{1}{2}\right)\frac{1}{r}$$

i.e. $v^2=\frac{1}{3}$. Note that according to the remark at the end of Section D.4 the equation is hyperbolic for this solution since $v^2<1$. Solving the [**HMCF′**] equation with a circle without normal velocity as initial data would yield a circle shrinking to a point in finite time. So these two equations might describe completely different phenomena.

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