Line Bundles on Complexity-One T-Varieties and Beyond

Lars Petersen

Dissertation

eingereicht am
Fachbereich Mathematik und Informatik
der Freien Universität Berlin

2010

Die vorliegende Dissertation wurde von Prof. Dr. Klaus Altmann betreut.

Ich versichere, diese Dissertation selbständig verfasst und alle verwendeten Hilfsmittel sowie Hilfen angegeben zu haben. Ferner habe ich diese Arbeit nicht in einem früheren Promotionsverfahren eingereicht.

Berlin, den Lars Petersen

1. Gutachter: Prof. Dr. Klaus Altmann
2. Gutachter: Prof. Dr. Jürgen Hausen

Datum der Disputation: 11.03.2011

Danksagung

An erster Stelle möchte ich mich bei meinem Betreuer Klaus Altmann für die immerwährende und über mathematische Fragestellungen hinausgehende Unterstützung und Hilfe bedanken, die er mir in den letzten Jahren zuteil werden ließ. Zudem möchte ich zahlreichen weiteren Mathematikern, die immer ein offenes Ohr für meine Fragen hatten, meinen Dank ausspechen. Dies sind Sergej Galkin, Christian Haase, Jürgen Hausen, Georg Hein, Norbert Hoffmann, Priska Jahnke, Andreas Paffenholz, Alexander Schmitt und Jarek Wiśniewski.

Nathan Ilten und Hendrik Süß danke ich nicht nur für das Korrekturlesen von Teilen dieser Arbeit, sondern auch für die vielen fruchtbaren Anregungen, Diskussionen und ihr Interesse an meiner Forschung. Darüber hinaus möchte ich Andreas Hochenegger und Robert Vollmert an dieser Stelle als immer aufgeschlossene und impulsgebende Diskussionspartner besonders erwähnen.

Aber auch allen anderen derzeitigen sowie früheren Mitgliedern der Arbeitsgruppe Algebra, insbesondere jedoch Mary Metzler-Kliegl, gebührt mein herzlicher Dank, da sie mir stets ein Umfeld geboten haben, in dem ich mich wohlfühlen durfte.

Ein außergewöhnliches Dankeschön möchte ich ebenfalls Sophie und Marianne Merz aussprechen, deren Kekse mich immer verzückt und in mancher harten Stunde auch über Wasser gehalten haben.

Nicht bemessen kann ich die Wärme und Liebe, die mir Laura auf meinen Wegen und Irrwegen gespendet hat. Ihre Anwesenheit und ihr Mitgefühl wurden zu einer tragenden, wenn auch unsichtbaren Säule dieser Arbeit.

Schließlich möchte ich mich bei meiner Familie bedanken, die mir in Nortorf immer ein liebevolles Zuhause und festen Halt geboten hat. Ihre Ermunterung, Unterstützung und ihr unbedingtes Vertrauen möchte ich nicht missen.

Abstract

In this dissertation, we focus on the description of equivariant line bundles on complexity-one T-varieties and two applications thereof.

Using the language of polyhedral divisors and divisorial fans developed by Klaus Altmann, Jürgen Hausen and Hendrik Süß, we describe equivariant line bundles in terms of so-called Cartier support functions on the underlying divisorial fan $\mathcal{S}$. Furthermore, we give a precise description of their global sections and provide a vanishing result for cohomology groups of nef line bundles on certain complete rational complexity-one T-varieties. These results are then applied in two different ways.

First, given a Mori dream space $\mathrm{TV}(\mathcal{S})$ with free divisor class group we construct a polyhedral divisor on $\mathbb{P}^1$ which corresponds to the Cox ring of $\mathrm{TV}(\mathcal{S})$. This polyhedral divisor not only allows for a detailed study of torus orbits and deformations but, in special cases, also for a downgrade to another polyhedral divisor previously constructed with different means by Klaus Altmann and Jarek Wiśniewski in the same setting.

The second application lies within the realm of Okounkov bodies. We present a construction of two types of invariant flags and use these to compute Okounkov bodies of rational projective complexity-one T-varieties. In particular, we show that these are rational polytopes. Moreover, using results of Dave Anderson and Nathan Ilten, we exhibit explicit links to degenerations and T-deformations. Finally, we prove that the global Okounkov body of a rational projective complexity-one T-variety with respect to these two types of flags is rational polyhedral. This generalizes an analogous result previously obtained by José González for projectivized rank two toric vector bundles over smooth projective toric varieties.

Contents

List of Figures

Introduction

Toric geometry is a well established branch of algebraic geometry, see e.g. [KKMSD73, Ful93, Oda88, Dan78] for some introductory literature. Among others, its popularity is due to the fact that toric varieties can be described in purely combinatorial terms and that they provide a very fruitful testing ground for general theories. Undoubtedly, they form the best understood and most prominent subclass of the class of T-varieties, i.e. normal varieties that come with an effective algebraic torus action. Within the realm of T-varieties, toric varieties are those of complexity zero, i.e. the dimension of a generic torus orbit is of codimension zero. The following results from toric geometry not only serve as a motivation but also as a guideline for this dissertation.

Equivariant line bundles on toric varieties correspond to continuous piecewise linear functions on the underlying polyhedral fan. Given such a bundle, these data also provide for a finite complex of vector spaces whose cohomology groups are equal to those of the line bundle in question, cf. [Kly90] for the more general setting of equivariant vector bundles.

In the non-degenerate case, i.e. if the toric variety does not have any torus factors, one can use a particular exact sequence to not only present its divisor class group but, furthermore, to give a construction of its Cox ring, cf. [Cox95]. Although the latter "simply" is a polynomial ring, it comes with an unusual grading which is induced by the divisor class group. Since the Cox ring together with the irrelevant ideal also captures the geometry of the underlying toric variety one obtains further (global) insights from its description, see e.g. [Mav] and references therein for an approach towards toric deformation theory.

Okounkov bodies of smooth projective toric varieties with respect to invariant admissible flags were computed by Robert Lazarsfeld and Mircea Mustaţă in [LM]. There, the authors recover the correspondence between line bundles and the associated polytopes of their global sections. In addition, they prove that, up to a linear isomorphism, the global Okounkov body is equal to the positive orthant associated to the Cox ring of the underlying toric variety.

The fundamental objects of study in this thesis are T-varieties of complexity one. First results towards a description of the latter were obtained by David Mumford, cf. [KKMSD73]. However, the full picture was only presented two decades later by Dmitri A. Timashev as a special case within the much more general framework of reductive group actions, cf. [Tim97]. Nonetheless, instead of using the language of hypercones and hyperfans from loc. cit., we will apply the language of polyhedral divisors and divisorial fans which was recently introduced by Klaus Altmann, Jürgen Hausen and Hendrik Süß for the description of T-varieties of arbitrary complexity, cf. [AH06, AHS08].

We now give an overview of the structure and the main results of this thesis. Chapter 1 provides an introduction to the theory of T-varieties in terms of polyhedral divisors and divisorial fans where an emphasis is placed on the complexity-one case. In particular, it recalls the notions of marked fansy divisor and divisorial polytope as introduced in [ISb].

The second chapter addresses the description of equivariant line bundles on complexity-one T-varieties $\mathrm{TV}(\mathcal{S})$. Introducing so-called support-functions, i.e. continuous piecewise affine-linear functions over the polyhedral subdivisions $\mathcal{S}_P$ that arise from the divisorial fan $\mathcal{S}$, we can state our first main result.

Theorem. The group of T-invariant Cartier divisors on $\mathrm{TV}(\mathcal{S})$ is isomorphic the the group of Cartier support functions on the divisorial fan $\mathcal{S}$.

This statement is as close to the toric analogue as one could hope for. Furthermore, we give a description of the global sections of an equivariant line bundle and provide a vanishing result on higher cohomology groups of nef line bundles on complete rational complexity-one T-varieties that come with a quotient morphism to $\mathbb{P}^1$. As an application of the description of equivariant line bundles we give a presentation of the divisor class group for complete rational complexity-one T-varieties.

In Chapter 3, we consider Mori dream spaces $\mathrm{TV}(\mathcal{S})$ with free divisor class group and use the presentation of the latter to construct a polyhedral divisor $\mathcal{D}_{\mathrm{Cox}}$ on $\mathbb{P}^1$ which has the following nice property.

Theorem. The algebra $A(\mathcal{D}_{\mathrm{Cox}})$ and the Cox ring of $\mathrm{TV}(\mathcal{S})$ are isomorphic as $\mathrm{Cl}\left(\mathrm{TV}(\mathcal{S})\right)$-graded algebras.

The construction of $\mathcal{D}_{\mathrm{Cox}}$ is combinatorial in nature. The compact part of each coefficient $(\mathcal{D}_{\mathrm{Cox}})_P$ can be chosen as a simplex which "resolves" the affine linear dependencies of the vertices in the slice $\mathcal{S}_P$. This feature again is a natural generalization of the toric result. In addition, we can apply it to study a particular degeneration and provide examples which strongly indicates that the polyhedral divisor constructed by Klaus Altmann and Jarek Wiśniewski in [AW] may be obtained as a downgrade of $\mathcal{D}_{\mathrm{Cox}}$ in the case that $\mathrm{TV}(\mathcal{S})$ is toroidal.

The final chapter of this thesis is devoted to the computation of Okounkov bodies of rational projective complexity-one T-varieties. To this end, we present two types of invariant admissible flags and show that the Okounkov body of a big line bundle with respect to any of these flags is rational polyhedral. In addition, using these new types of flags, we obtain new results for Okounkov bodies of toric varieties. Furthermore, we link our computations to Dave Anderson's results on toric degenerations (see [And]) and Nathan Ilten's construction of T-deformations by decompositions of divisorial polytopes (see [Ilt10]). Finally, we focus on the global Okounkov body of rational projective complexity-one T-varieties. Again, it is possible to extend a "toric" statement to the complexity-one world:

Theorem. The global Okounkov body of a rational projective complexity-one T-variety with respect to the flags from above is rational polyhedral.

Chapter 1

T-Varieties

This chapter fixes some notation and introduces the language of polyhedral divisors and divisorial fans with a special focus upon complexity-one T-varieties. Several examples which will also reappear in later chapters are presented. Finally, we conclude the chapter with an outlook on (dappled) toric bouquets.

Conventions and General Notation

We adopt the following conventions and notation. If not stated otherwise

- $\mathbb{K}$ denotes an algebraically closed field of characteristic zero.
- a *variety* means an integral, separated scheme of finite type over the ground field $\mathbb{K}$.
- N denotes a lattice, i.e. a free abelian group of finite rank. Its dual $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ is usually denoted by M. Given a lattice L, we set $L_{\mathbb{Q}} := L \otimes_{\mathbb{Z}} \mathbb{Q}$ and $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$.
- a *cone* is supposed to be pointed and polyhedral.
- we call a real-valued function $f : M \to \mathbb{R}$ defined over some convex subset $M \subset \mathbb{R}^k$ *concave* if $f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2)$ for all $x_1, x_2 \in M$ and $0 \leq t \leq 1$.

1.1 Polyhedral Divisors and Divisorial Fans

The fundamental geometric objects of interest in this thesis are varieties that come with a torus action.

Definition 1.1. A *T-variety* is a normal variety X together with an effective algebraic torus action $T \times X \to X$. Its *complexity* is defined as the codimension of a generic T-orbit.

The most prominent and best understood fraction of the class of T-varieties is the subclass of complexity-zero T-varieties. Their elements can be described via the combinatorial language of so-called polyhedral fans and are much better known under the name of toric varieties.

Recently, Klaus Altmann, Jürgen Hausen and Hendrik Süß developed a language which extends the well known description in complexity zero to arbitrary complexity [AH06, AHS08]. Leaving toric geometry, the picture does no longer stay purely combinatorial but also comprises a geometric base space whose dimension is equal to the complexity of the torus action.

We will now briefly review the fundamental notions used by the above authors in their partially combinatorial and partially geometric description of T-varieties.

Definition 1.2. A polyhedron $\Delta \subset N_{\mathbb{Q}}$ which may be written as a Minkowski sum $\Delta = \Delta^c + \sigma$ of a compact polyhedron $\Delta^c \subset N_{\mathbb{Q}}$ and a cone $\sigma \subset N_{\mathbb{Q}}$ is called a *σ-polyhedron*. Furthermore, σ is referred to as the *tailcone* of Δ which is also denoted by $\operatorname{tail}\Delta$.

Note that the compact part of such a decomposition may not be unique. Yet, its tailcone is unique, which guarantees that the latter notion is well-defined.

Denote by Y a normal *semiprojective* variety over the ground field $\mathbb{K}$, meaning that Y is projective over $\operatorname{Spec}\Gamma(Y, \mathcal{O}_Y)$.

Definition 1.3. A *polyhedral divisor* $\mathcal{D}$ on Y with tailcone $\sigma \subset N_{\mathbb{Q}}$ is a formal sum

$$\mathcal{D} = \sum_Z \mathcal{D}_Z \otimes Z\,,$$

running over all prime divisors Z on Y such that

1. $\mathcal{D}_Z \subset N_{\mathbb{Q}}$ is either a polyhedron with tailcone σ or the empty set.
2. For all but finitely many prime divisors Z the polyhedral coefficient $\mathcal{D}_Z$ is equal to the tailcone σ.

The *locus* of a polyhedral divisor $\mathcal{D}$ on Y is defined as

$$\operatorname{Loc}\mathcal{D} \;:=\; Y \setminus \big(\bigcup_{\mathcal{D}_Z = \emptyset} Z \big).$$

The following evaluation map with image inside the free abelian group of rational Weil divisors on Y will be crucial for many discussions coming up later.

$$\sigma^\vee \cap M \to \operatorname{Div}_{\mathbb{Q}}(Y), \quad u \mapsto \mathcal{D}(u) \;:=\; \sum \min_{v \in \mathcal{D}_Z} \langle u, v\rangle Z\,.$$

Our main focus, however, does not lie on polyhedral divisors as defined above, but on the subclass of those objects which have the property that the evaluations from above fullfill certain positivity criteria.

Recall that a divisor D on Y is called *big* if some positive integral multiple admits a section with affine complement. On the other hand, the divisor D is called *semiample* if some positive integral multiple is globally generated.

With the help of these notions we are now ready to introduce the most prominent object of this dissertation.

Definition 1.4. A *proper polyhedral divisor* on Y is a polyhedral divisor $\mathcal{D}$ on Y with the following properties:

1. $\mathcal{D}(u)$ is $\mathbb{Q}$-Cartier for all $u \in \sigma^\vee \cap M$.

2. $\mathcal{D}(u)$ is semiample for all $u \in \sigma^\vee \cap M$.

3. $\mathcal{D}(u)$ is big for all $u \in (\operatorname{relint} \sigma^\vee) \cap M$.

Notation 1.5. If not stated otherwise, a polyhedral divisor will always mean a proper polyhedral divisor.

The evaluation map from above can be used to construct the following M-graded sheaf of $\mathcal{O}_Y$-algebras

$$\mathcal{A}(\mathcal{D}) \;:=\; \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}_{\operatorname{Loc} \mathcal{D}}(\mathcal{D}(u)) \cdot \chi^u.$$

Its global sections form an M-graded $\mathbb{K}$-algebra which we denote by

$$A(\mathcal{D}) \;:=\; \Gamma(\mathcal{A}(\mathcal{D})) = \bigoplus_{u \in \sigma^\vee \cap M} \Gamma\big(\operatorname{Loc} \mathcal{D}, \mathcal{O}_{\operatorname{Loc} \mathcal{D}}(\mathcal{D}(u))\big) \cdot \chi^u.$$

We set

$$\widetilde{\mathrm{TV}}(\mathcal{D}) \;:=\; \mathbf{Spec}_{\operatorname{Loc} \mathcal{D}} \mathcal{A}(\mathcal{D}) \quad \text{and} \quad \mathrm{TV}(\mathcal{D}) \;:=\; \operatorname{Spec} A(\mathcal{D})$$

Note that $\widetilde{\mathrm{TV}}(\mathcal{D}) \cong \mathrm{TV}(\mathcal{D})$ in the case that $\operatorname{Loc} \mathcal{D}$ is affine. Adding up the statements of Theorem 3.1 and Theorem 3.4 from [AH06], we have the following.

Theorem 1.6. Let $\mathcal{D}$ be a polyhedral divisor on a normal semiprojective variety Y over $\mathbb{K}$, and set $T := \operatorname{Spec} \mathbb{K}[M]$, i.e. T is an algebraic torus whose dimension is equal to the rank of the lattice M.

1. $\widetilde{\mathrm{TV}}(\mathcal{D})$ is a T-variety whose complexity is equal to $\dim Y$, and its dimension equals $\dim Y + \dim T$. The effective torus action gives rise to a good quotient map $\pi : \widetilde{\mathrm{TV}}(\mathcal{D}) \to Y$. Furthermore, $\mathrm{TV}(\mathcal{D})$ is an affine T-variety of the same complexity and there is a proper, birational T-equivariant contraction morphism $r : \widetilde{\mathrm{TV}}(\mathcal{D}) \to \mathrm{TV}(\mathcal{D})$.

2. Conversely, any affine T-variety X gives rise to a pair $(Y, \mathcal{D})$ where Y is normal and semiprojective over $\mathbb{K}$, and $\mathcal{D}$ a proper polyhedral divisor on Y such that X and $\mathrm{TV}(\mathcal{D})$ are equivariantly isomorphic.

The next definition will provide the necessary notation for the gluing procedure, which will enable us to proceed from the affine to the non-affine case.

Definition 1.7. Let $\mathcal{D} = \sum_Z \mathcal{D}_Z \otimes Z$, and $\mathcal{D}' = \sum_Z \mathcal{D}'_Z \otimes Z$ be two not necessarily proper polyhedral divisors on Y.

1. We say that $\mathcal{D}'$ is *contained* in $\mathcal{D}$, i.e. $\mathcal{D}' \subset \mathcal{D}$ if $\mathcal{D}'_Z \subset \mathcal{D}_Z$ holds for every prime divisor Z.

2. The *intersection* of $\mathcal{D}$ and $\mathcal{D}'$ is defined as

$$\mathcal{D} \cap \mathcal{D}' \;:=\; \sum_Z (\mathcal{D}'_Z \cap \mathcal{D}_Z) \otimes Z.$$

3. We define the *degree* of a polyhedral divisor $\mathcal{D}$ on a curve Y as

$$\deg \mathcal{D} \;:=\; \sum_P \mathcal{D}_P,$$

Here, we follow the convention that $\emptyset + \Delta = \emptyset$ for any polyhedron Δ. Hence, if $\mathcal{D}$ carries $\emptyset$-coefficients we automatically get that $\deg \mathcal{D} = \emptyset$.

4. For any open subset $U \subset Y$ we set

$$\mathcal{D}|_U \;:=\; \mathcal{D} + \sum_{Z \cap U = \emptyset} \emptyset \otimes Z\,.$$

Definition 1.8. Let $\mathcal{D}' \subset \mathcal{D}$ be polyhedral divisors. Then we have an equivariant dominant morphism $\mathrm{TV}(\mathcal{D}') \to \mathrm{TV}(\mathcal{D})$, which corresponds to the following inclusion

$$\bigoplus_{u \in \sigma^\vee \cap M} \Gamma(\operatorname{Loc}\mathcal{D}, \mathcal{D}(u)) \subset \bigoplus_{u \in (\sigma')^\vee \cap M} \Gamma(\operatorname{Loc}\mathcal{D}', \mathcal{D}'(u))\,.$$

If this is an open embedding, then we say that $\mathcal{D}'$ is a *face* of $\mathcal{D}$ and denote this relation by $\mathcal{D}' \prec \mathcal{D}$.

Definition 1.9. Let $\mathcal{S} = \{\mathcal{D}_i \mid i \in I\}$ be a finite set of polyhedral divisors over a fixed base Y and a fixed lattice N.

1. $\mathcal{S}$ is called a *divisorial fan* if $\mathcal{D}_1 \cap \mathcal{D}_2 \in \mathcal{S}$, and this intersection is a face of both $\mathcal{D}_1$ and $\mathcal{D}_2$ for any elements $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{S}$.
2. The polyhedral complex $\mathcal{S}_y$ for a not necessarily closed point $y \in Y$ which is defined by the polyhedra $\mathcal{D}_y$ for $\mathcal{D} \in \mathcal{S}$ is called a *slice* of the divisorial fan $\mathcal{S}$.

The face relations from above guarantee that we can glue the affine T-varieties $\mathrm{TV}(\mathcal{D}_i),\ i = 1, 2$ along common intersections

$$\mathrm{TV}(\mathcal{D}_1) \leftarrow \mathrm{TV}(\mathcal{D}_{i_1} \cap \mathcal{D}_{i_2}) \to \mathrm{TV}(\mathcal{D}_2).$$

Indeed, Theorem 5.3 from [AHS08] guarantees that the cocycle condition is fulfilled. Thus, the gluing gives rise to a not necessarily separated T-prevariety which we denote by $\mathrm{TV}(\mathcal{S})$. Note that there are valuative criteria which yield necessary and sufficient conditions for $\mathrm{TV}(\mathcal{S})$ to be separated and complete. Since we will later on restrict to the complexity-one case, where these conditions become empty or intuitive (see Section 1.2), we refrain from a comprehensive introduction of those criteria.

Remark 1.10. By definition, a divisorial fan $\mathcal{S}$ comes with an induced open affine covering of $\mathrm{TV}(\mathcal{S})$, namely by the T-invariant subvarieties $\mathrm{TV}(\mathcal{D})$ that are associated to the elements $\mathcal{D} \in \mathcal{S}$. However, it is crucial to observe that this open affine invariant covering of $\mathrm{TV}(\mathcal{S})$ is not unique because we may easily switch to another by appropriately changing $\mathcal{S}$, see Example 1.16.

Finally, we present a little T-equivariant diagram which comprises all the natural maps between the T-varieties considered so far:

$$\begin{tikzcd} \widetilde{\mathrm{TV}}(\mathcal{S}) \arrow[r, "r"] \arrow[d, "\pi"'] & \mathrm{TV}(\mathcal{S}) \arrow[d, "\overline{\pi}"] \\ Y \arrow[r] & \operatorname{Spec}\Gamma(Y, \mathcal{O}_Y)\,. \end{tikzcd}$$

Here, T is supposed to act trivially on the objects in the lower row. The right hand side of the diagram corresponds to the categorical quotient of $\mathrm{TV}(\mathcal{S})$ whereas the morphism π on the left hand side arises as a gluing of good quotients $\pi_{\mathcal{D}} : \widetilde{\mathrm{TV}}(\mathcal{D}) \to Y$ for $\mathcal{D} \in \mathcal{S}$. As in the affine case, r is a proper and birational morphism. Moreover, π is even flat, cf. [Vol11].

Definition 1.11. A T-variety X is called *toroidal* if there exists a divisorial fan $\mathcal{S}$ on a semiprojective variety Y such that $X \cong \mathrm{TV}(\mathcal{S}) \cong \widetilde{\mathrm{TV}}(\mathcal{S})$ are equivariantly isomorphic.

Remark 1.12. The reader may not confuse this notion with the one introduced by David Mumford in [KKMSD73] which comprises ours but not vice versa.

Remark 1.13. Since $\widetilde{X} := \widetilde{\mathrm{TV}}(\mathcal{S})$ and $X := \mathrm{TV}(\mathcal{S})$ are birationally equivalent we have $\mathbb{K}(\widetilde{X}) = \mathbb{K}(X)$. Moreover, we may identify $\mathbb{K}(\widetilde{X})^T = \mathbb{K}(X)^T$ with $\mathbb{K}(Y)$. Hence, a semi-invariant function of weight $u \in M$ on X will be denoted by $f\chi^u$, where $f \in \mathbb{K}(Y)$.

As in the affine case, we also have the notion of *degree* for a divisorial fan $\mathcal{S}$. It will turn out to be very handy when discussing T-varieties of complexity-one.

Definition 1.14. Let $\mathcal{S}$ be a divisorial fan over Y. We set $\operatorname{Loc}\mathcal{S} := \bigcup_{\mathcal{D}\in\mathcal{S}} \operatorname{Loc}\mathcal{D}$ and, if Y is a curve, $\deg\mathcal{S} := \bigcup_{\mathcal{D}\in\mathcal{S}} \deg\mathcal{D} \subset N_{\mathbb{Q}}$.

We finish by recalling a small but significant result on open affine invariant coverings of an affine T-variety $\mathrm{TV}(\mathcal{D})$, cf. [Süß10, Lemma 2.22].

Lemma 1.15. Let $\mathrm{TV}(\mathcal{D})$ be an affine T-variety and $\{U_i\}_{i\in I}$ an open invariant covering. Then there exists a refinement which is induced by a covering of $Y_0 = \operatorname{Spec}\Gamma(\operatorname{Loc}\mathcal{D}, \mathcal{O}_{\operatorname{Loc}\mathcal{D}})$.

In particular, this implies that open affine invariant coverings of affine T-varieties $\mathrm{TV}(\mathcal{D})$ for which $\operatorname{Loc}\mathcal{D}$ is complete must always contain $\mathrm{TV}(\mathcal{D})$ itself since their categorical quotient Y_0 is just a point.

1.1.1 Toric Downgrades

A very fruitful technique to generate instructive examples is to consider a toric variety with an effective subtorus action. We will use this section to shortly present a method how to obtain a divisorial fan from the polyhedral fan of a given toric variety after choosing an effective subtorus action, see also [AHS08, Section 5].

Let us consider a complete d-dimensional toric variety $\mathrm{TV}(\Sigma)$ with torus T_N and subtorus $T_{N'} \hookrightarrow T_N$ such that $N'' := \operatorname{coker}(N' \hookrightarrow N)$ is a lattice. Hence,

$$0 \longrightarrow N' \xrightarrow{F} N \xrightarrow{P} N'' \longrightarrow 0$$

is an exact sequence of lattices. By choosing a cosection $s : N \to N'$, we induce a splitting $N \cong N' \oplus N''$ with projections

$$s : N \to N', \quad P : N \to N''.$$

Define $Y := \mathrm{TV}(\Sigma')$, where Σ' is an arbitrary smooth projective fan Σ' refining the images $P(\delta)$ of all cones $\delta \in \Sigma$. Then every cone $\sigma \in \Sigma(d)$ gives rise to a polyhedral divisor $\mathcal{D}^\sigma$: For each ray $\rho' \in \Sigma'(1)$, let $n_{\rho'}$ denote its primitive generator and set

$$\mathcal{D}_{\rho'}(\sigma) = s_{\mathbb{Q}}\big(P_{\mathbb{Q}}^{-1}(n_{\rho'}) \cap \sigma\big) \quad \text{and} \quad \mathcal{D}^\sigma = \sum_{\rho' \in \Sigma'(1)} \mathcal{D}_{\rho'}(\sigma) \otimes D_{\rho'}.$$

Finally, $\{\mathcal{D}^\sigma\}_{\sigma \in \Sigma(d)}$ is a divisorial fan. Observe that for certain polyhedral divisors $\mathcal{D}^\sigma$ and rays $\rho' \in \Sigma'(1)$ the intersection $P_{\mathbb{Q}}^{-1}(n_{\rho'}) \cap \sigma$ may be empty. In this case we have that $\mathcal{D}_{\rho'}(\sigma) = \emptyset$.

Example 1.16. We consider the Hirzebruch surface $\mathbb{F}_n$ as a $\mathbb{K}^*$-surface via the following maps of lattices

$$F = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 1 \end{pmatrix}, \qquad s = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

The slices of the divisorial fan $\mathcal{S}$ arising from this downgrade are displayed in Figure 1.1.

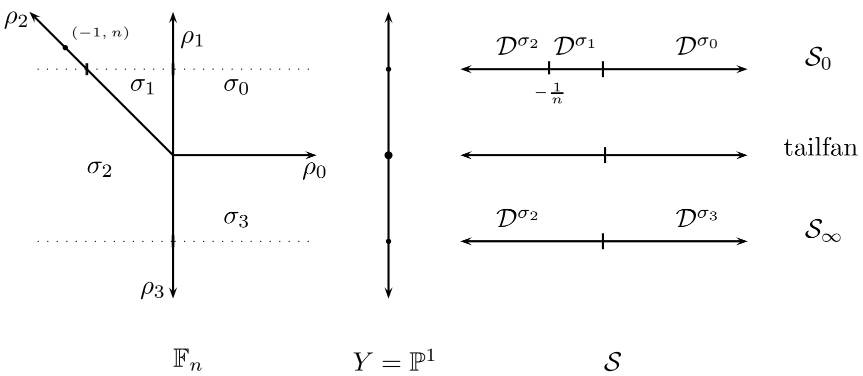


Figure 1.1: Divisorial fan associated to $\mathbb{F}_n$, cf. Example 1.16.

In more detail, we have that

$$\mathcal{D}^{\sigma_0} = [0, \infty) \otimes [0] + \emptyset \otimes [\infty], \quad \mathcal{D}^{\sigma_1} = [-1/n \;\; 0] \otimes [0] + \emptyset \otimes [\infty],$$
$$\mathcal{D}^{\sigma_2} = (-\infty \;\; -1/n] \otimes [0], \qquad \mathcal{D}^{\sigma_3} = \emptyset \otimes [0] + [0 \;\; \infty) \otimes [\infty].$$

To give another and in a sense "finer" divisorial fan $\mathcal{S}' = \{\mathcal{D}'_1, \ldots, \mathcal{D}'_6\}$ such that $\mathrm{TV}(\mathcal{S}') \cong \mathrm{TV}(\mathcal{S})$, we set $P_1 \neq P_2 \in \mathbb{P}^1 \setminus \{0, \infty\}$ and define

$$\mathcal{D}'_1 = [0, \infty) \otimes [0] + \emptyset \otimes [\infty] + \emptyset \otimes [P_1], \quad \mathcal{D}'_2 = \emptyset \otimes [0] + \emptyset \otimes [\infty] + [0 \;\; \infty) \otimes [P_1],$$
$$\mathcal{D}'_3 = \emptyset \otimes [0] + [0 \;\; \infty) \otimes [\infty] + \emptyset \otimes [P_1], \quad \mathcal{D}'_4 = [-1/n \;\; 0] \otimes [0] + \emptyset \otimes [\infty] + \emptyset \otimes [P_2],$$
$$\mathcal{D}'_5 = \emptyset \otimes [0] + \emptyset \otimes [\infty] + \{0\} \otimes [P_2], \qquad \mathcal{D}'_6 = (-\infty \;\; -1/n] \otimes [0].$$

It is then easy to check that $\mathrm{TV}(\mathcal{S}) \cong \mathrm{TV}(\mathcal{S}')$.

1.2 Complexity-One T-Varieties

Having established the theory of T-varieties for arbitrary complexity in the previous section, we will from now on restrict to the complexity-one case. This restriction comes with a lot of simplifications for certain technical details. For example, it means that the underlying variety Y is a smooth projective curve. Hence, the locus of a polyhedral divisor $\mathcal{D}$ on Y is either affine or complete. Moreover, every divisorial fan $\mathcal{S}$ on a smooth projective curve Y gives rise to a variety, i.e. $\mathrm{TV}(\mathcal{S})$ is automatically separated, cf. [AHS08, Section 7]. Furthermore, $\mathrm{TV}(\mathcal{S})$ is complete if and only if every slice $\mathcal{S}_P$ for a closed point $P \in Y$ covers the whole vector space $N_{\mathbb{Q}}$, cf. loc. cit.

We already pointed out in Remark 1.10 that different divisorial fans $\mathcal{S}_1 \neq \mathcal{S}_2$ may yield equivariantly isomorphic T-varieties $\mathrm{TV}(\mathcal{S}_1) \cong \mathrm{TV}(\mathcal{S}_2)$. On the other hand, one may be tempted to assume that divisorial fans with identical slices yield the same T-varieties. This is not true either. Nonetheless, there is an elegant notational remedy for (complete) complexity-one T-varieties which fixes the latter ambiguity, cf. [ISb, Section 1].

Before introducing the notion of marked fansy divisor, let us conclude this section with some definitions and notation which shall prove very useful in later chapters.

Definition 1.17. Let $\mathcal{S}$ be a divisorial fan on the curve Y. A slice $\mathcal{S}_P$ for $P \in Y$ is called *trivial* if it is equal to the tailfan. The set of all points with non-trivial slices is denoted by $\mathcal{P} := \mathcal{P}(\mathcal{S}) \subset Y$, whereas the set of all vertices $\{v \in \mathcal{S}_P \mid P \in \mathcal{P}\}$ is denoted by $\mathcal{V} := \mathcal{V}(\mathcal{S})$.

We will also single out specific rays in the tailfan of a divisorial fan $\mathcal{S}$ on a smooth curve Y.

Definition 1.18. Let $\mathcal{D} \in \mathcal{S}$ be a polyhedral divisor with tailcone σ. A ray $\rho \in \sigma(1)$ with $\deg \mathcal{D} \cap \rho = \emptyset$ is called an *extremal ray.* The set of extremal rays is denoted by $\mathcal{R} := \mathcal{R}(\mathcal{S}) := \{\mathcal{R}(\mathcal{D}) \,|\, \mathcal{D} \in \mathcal{S}\}$.

Notation 1.19. For a vertex $v \in N_{\mathbb{Q}}$ we denote by $\mu(v)$ the smallest positive integer k such that $kv \in N$. Moreover, we denote by n_ρ the primitive generator of some ray $\rho \subset \sigma(1)$ in some polyhedral cone $\sigma \subset N_{\mathbb{Q}}$.

1.2.1 Marked Fansy Divisors

Definition 1.20. A *marked fansy divisor* on a curve Y is a formal sum

$$\Xi = \sum_{P \in Y} \Xi_P \otimes [P]$$

together with a complete fan $\Sigma \subset N_{\mathbb{Q}}$ and a subset $C \subset \Sigma$ such that

1. For all $P \in Y$, the coefficient Ξ_P is a complete polyhedral subdivision of $N_{\mathbb{Q}}$ with $\mathrm{tail}\,\Xi_P = \Sigma$.

2. For a cone $\sigma \in C$ of full dimension the polyhedral divisor $\mathcal{D}^\sigma = \sum_P \mathcal{D}_P^\sigma \otimes [P]$ is proper where $\mathcal{D}_P^\sigma$ denotes the unique polyhedron in Ξ_P whose tailcone is equal to σ.

3. For a full dimensional cone $\sigma \in C$ and a face $\tau \prec \sigma$ we have that $\tau \in C$ if and only if $\deg \mathcal{D}^\sigma \cap \tau \neq \emptyset$.

4. If τ is a face of σ then $\tau \in C$ implies that $\sigma \in C$.

The elements of $C \subset \Sigma$ are called *marked* cones. They capture the information which orbits are identified via the map $r : \widetilde{X} \to X$.

To a given divisorial fan $\mathcal{S}$ one can easily associate a marked fansy divisor, namely by setting

$$\Xi(\mathcal{S}) := \sum_P \mathcal{S}_P \otimes [P], \quad \text{and} \quad C(\mathcal{S}) := \left\{ \operatorname{tail} \mathcal{D} \mid \mathcal{D} \in \mathcal{S}, \operatorname{Loc} \mathcal{D} = Y \right\}.$$

Conversely, given a marked fansy divisor Ξ on the curve Y Proposition 1.6 from [ISb] tells us how to construct a divisorial fan from Ξ and, furthermore, fixes the above mentioned ambiguity of representations:

Proposition 1.21. For any marked fansy divisor Ξ on Y there exists a complete divisorial fan $\mathcal{S}$ with $\Xi(\mathcal{S}) = \Xi$. Moreover, two divisorial fans $\mathcal{S}_1, \mathcal{S}_2$ with $\Xi(\mathcal{S}_1) = \Xi(\mathcal{S}_2)$ yield the same T-variety $\mathrm{TV}(\mathcal{S}_1) = \mathrm{TV}(\mathcal{S}_2)$.

Example 1.22. The marked fansy divisor for $\mathbb{F}_n$ as depicted in Example 1.16 consists of the following data:

$$\Xi_0 = \mathcal{S}_0, \quad \Xi_\infty = \mathcal{S}_\infty, \quad \Sigma = \operatorname{tail} \mathcal{S}, \quad C = \{(-\infty, 0]\}.$$

Remark 1.23. One can also define marked fansy divisors in the non-complete setting. Indeed, we only have to allow arbitrary polyhedral fans $\Sigma \subset N_{\mathbb{Q}}$ and reformulate the first condition in Definition 1.20 as follows:

1. For all $P \in Y$, the coefficient Ξ_P is a polyhedral complex in $N_{\mathbb{Q}}$ with $\operatorname{tail} \Xi_P \subset \Sigma$. Furthermore, for all cones $\sigma \in \Sigma$ we have that $\sigma \in \operatorname{tail} \Xi_P$ for all but finitely many $P \in Y$.

Leaving the remaining conditions as they are, and using the same arguments as given in [ISb, Section 1], it is then not hard to check that this allows for a description of an arbitrary complexity-one T-variety $\mathrm{TV}(\mathcal{S})$ in terms of a "standard covering".

1.2.2 Divisorial Polytopes

Following [ISb], we also briefly recall the description of polarized complexity-one T-varieties in terms of divisorial polytopes. This correspondence is a generalization of the relation between polarized projective toric varieties and lattice polytopes.

Definition 1.24. A divisorial polytope $(\Psi, \Box, Y)$ consists of a lattice polytope $\Box \subset M_{\mathbb{Q}}$, a smooth projective curve Y, and a map

$$\Psi = \sum_{P \in Y} \Psi_P \otimes [P] : \ \Box \longrightarrow \operatorname{CaDiv}_{\mathbb{Q}} Y,$$

with concave piecewise affine linear "coordinate" functions $\Psi_P : \Box \to \mathbb{Q}$ such that

1. for all but finitely many $P \in Y$ we have that $\Psi_P \equiv 0$.
2. $\deg \Psi(u) > 0$ for u in the interior of $\Box$;
3. for u a vertex of $\Box$, $\deg \Psi(u) > 0$ or $\lambda\Psi(u) \sim 0$ for some $\lambda \in \mathbb{N}$;
4. for all $P \in Y$ the graph of Ψ_P is integral, i.e. its vertices lie in $M \times \mathbb{Z}$.

Before stating the correspondence theorem, we quickly recall the procedure given in [ISb, Section 3] how to construct a marked fansy divisor from a triple $(\Psi, \Box, Y)$. To begin with, one defines for every point $P \in Y$ a piecewise affine concave function

$$\Psi_P^* : N_{\mathbb{Q}} \to \mathbb{Q}, \quad \Psi_P^*(v) \ := \ \min_{u \in \Box} \left\{ \langle u, v\rangle - \Psi_P(u) \right\}.$$

The breaks in its affine linear structure induce a polyhedral subdivision of $N_{\mathbb{Q}}$ which we denote by $\Xi_P \ := \ \Xi_P(\Psi)$. Our first ingredient Ξ now crystallizes as the formal sum

$$\Xi \ := \ \Xi(\Psi) \ := \ \sum_P \Xi_P \otimes [P].$$

The set of marks $C := C(\Psi) \subset \operatorname{tail}(\Xi)$ is obtained in the following way. A cone $\sigma \subset \operatorname{tail}\Xi$ is an element of C if and only if $(\deg \circ \Psi)|_{F_\sigma} \equiv 0$, where F_σ is the face of $\Box$ which minimizes the linear functional $\langle \cdot, v\rangle$ on $\Box$ for all $v \in \sigma$. Moreover, it will become clear from the discussion in Chapter 2 that $\Psi^* = (\Psi_P^*)_{P \in Y}$ gives rise to an (ample) Cartier divisor D_{Ψ^*} on $\operatorname{TV}(\Xi)$ which induces a polarization.

Theorem 1.25. [ISb, Theorem 3.2]. There is a one-to-one correspondence between divisorial polytopes and pairs $(X, \mathcal{L})$ of complexity-one T varieties X with an equivariant ample line bundle $\mathcal{L}$ via the map

$$(\Psi, \Box, Y) \mapsto \big(\operatorname{TV}\big(\Xi(\Psi)\big), \mathcal{O}(D_{\Psi^*})\big).$$

Let us illustrate this correspondence for a toric downgrade, cf. [ISb, Section 3]. We consider a lattice polytope $\Delta \subset M_X \otimes_{\mathbb{Z}} \mathbb{Q}$ together with the following exact sequence of character lattices

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "P^\vee"] & M_X \arrow[r, shift left, "F^\vee"] & M \arrow[l, shift left, "s^\vee"] \arrow[r] & 0 \end{tikzcd}$$

which fixes a complexity-one action of the torus $T_M = \operatorname{Spec} \mathbb{K}[M]$ on the toric variety $\operatorname{TV}(\Delta)$ together with a splitting $M_X \cong M \oplus M'$. The map $\Psi_\Delta : \Box = F^\vee(\Delta) \to \operatorname{CaDiv}_{\mathbb{Q}} \mathbb{P}^1$ is defined by

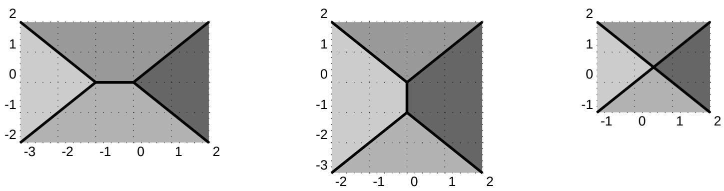
$$\begin{aligned} (\Psi_\Delta)_\infty(u) &= \max\left\{a \in \mathbb{Q} \mid P_{\mathbb{Q}}^\vee(a) + s^\vee(u) \in \Delta \cap (F^\vee)_{\mathbb{Q}}^{-1}(u)\right\}, \\ (\Psi_\Delta)_0(u) &= -\min\left\{a \in \mathbb{Q} \mid P_{\mathbb{Q}}^\vee(a) + s^\vee(u) \in \Delta \cap (F^\vee)_{\mathbb{Q}}^{-1}(u)\right\}, \\ (\Psi_\Delta)_P & \quad \text{vanishes for all } P \in \mathbb{P}^1 \setminus \{0, \infty\}. \end{aligned}$$

It follows that $\big(\Psi_\Delta, F^\vee(\Delta)\big)$ is a divisorial polytope and the construction given in [ISb, Section 3] gives back the toric variety $\operatorname{TV}(\Delta)$ together with the polarization coming from Δ.

Remark 1.26. Assume that the upper splitting of lattices is trivial, i.e. $M_X = M \oplus \mathbb{Z}$. Then the upper maps $(\Psi_\Delta)_0$, $(\Psi_\Delta)_\infty$ are defined on $\square = F^\vee(\Delta)$ and their graphs can be considered as "roof" and "$-$ floor" of the polytope Δ, i.e. what part of the boundary of Δ can be seen from above and below with respect to the projection $F^\vee$. For an illustration, see Example 2.36.

1.2.3 Some Examples

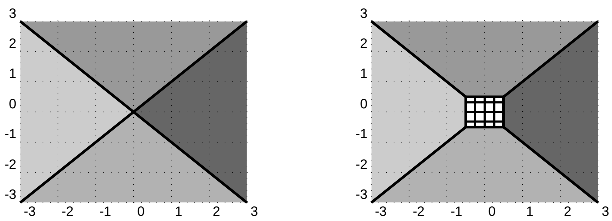
Example 1.27. The divisorial fan associated to the smooth quadric $Q = \mathrm{TV}(\mathcal{S})$ in $\mathbb{P}^4$ as presented in [Süß, Example 1.10] is given in Figure 1.2.



(a) $\mathcal{S}_1$ (b) $\mathcal{S}_0$ (c) $\mathcal{S}_\infty$

Figure 1.2: Non-trivial slices of $\mathcal{S}(Q)$.

Note that Q is Fano, i.e. $-K_Q$ is ample. The associated tailfan Σ and degree $\deg \mathcal{S}$ are given in Figure 1.3. In addition, all maximal polyhedral divisors have complete locus, i.e. $\mathcal{R} = \emptyset$ and all rays in the tailfan are marked.



(a) $\Sigma = \operatorname{tail} \mathcal{S}$ (b) $\deg \mathcal{S}$

Figure 1.3: Tailfan and degree of $\mathcal{S}(Q)$.

Projectivized Cotangent Bundles over Toric Surfaces A very prominent class of complexity-one T-Mori dream spaces (T-MDS) is given by projectivizations of rank two toric vector bundles over smooth projective toric varieties. For their description as T-varieties we follow the notation of [AHS08], which we briefly recall here.

Let $\mathcal{E}$ be a locally free equivariant sheaf of rank r on the toric variety $\mathrm{TV}(\Sigma)$. Denote by $\sigma \in \Sigma$ a cone of the fan, and by ρ_i an element of $\Sigma(1)$. Following Klyachko's description from [Kly90], $\mathcal{E}$ can be given by an r-dimensional $\mathbb{K}$-vector space E together with $\mathbb{Z}$-labeled increasing filtrations $E^\rho(i)$ for every $\rho \in \Sigma(1)$ having the following compatibility property:

For every $\sigma \in \Sigma$ there is a basis $e_1^\sigma, \ldots, e_r^\sigma$ of E, and weights $u_1^\sigma, \ldots, u_r^\sigma \in M$ such that for

$$e_j^\sigma \in E^\rho(i) \Longleftrightarrow \langle u_j^\sigma, \rho\rangle \geq i \quad \text{for all} \quad \rho \in \sigma(1).$$

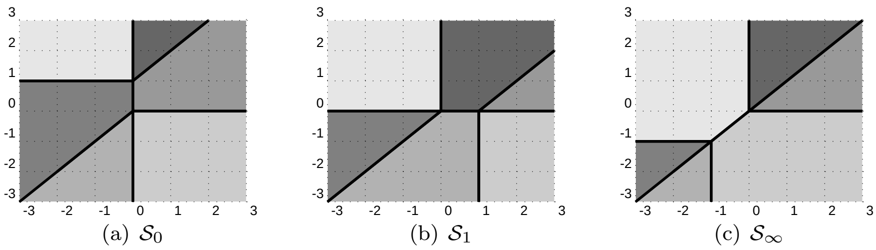


Figure 1.4: Non-trivial slices of $\mathcal{S}(\mathbb{P}(\Omega_{\mathbb{P}^2}))$, cf. Example 1.28.

For an equivariant bundle $\mathcal{E}$ of rank 2 which is given by filtrations $E^\rho(i)$ of $E = \mathbb{K}^2$ the description of the divisorial fan may be given in a very condensed form. Setting $Y = \mathbb{P}(E^\vee) = \mathbb{P}^1$, one obtains two polyhedral divisors for every maximal cone $\sigma \in \Sigma$:

$$\begin{aligned} \mathcal{D}_\sigma^+ &:= \Delta_\sigma^1 \otimes [(e_1^\sigma)^\perp] + \Delta_\sigma^2 \otimes [(e_2^\sigma)^\perp]\,, \\ \mathcal{D}_\sigma^- &:= \nabla_\sigma^1 \otimes [(e_1^\sigma)^\perp] + \nabla_\sigma^2 \otimes [(e_2^\sigma)^\perp]\,. \end{aligned}$$

The polytopes occurring in the definitions from above are given by

$$\begin{aligned} \Delta_\sigma^1 &:= \{v \in N_\mathbb{Q} \mid \langle u_1^\sigma - u_2^\sigma, v\rangle \geq 1\} \cap \sigma, \\ \Delta_\sigma^2 &:= \{v \in N_\mathbb{Q} \mid \langle u_2^\sigma - u_1^\sigma, v\rangle \geq 1\} \cap \sigma, \\ \nabla_\sigma^1 &:= \{v \in N_\mathbb{Q} \mid \langle u_1^\sigma - u_2^\sigma, v\rangle \leq 1\} \cap \sigma, \\ \nabla_\sigma^2 &:= \{v \in N_\mathbb{Q} \mid \langle u_2^\sigma - u_1^\sigma, v\rangle \leq 1\} \cap \sigma\,. \end{aligned}$$

The set of all of these polyhedral divisors finally gives us the divisorial fan which encodes $\mathbb{P}(\mathcal{E})$.

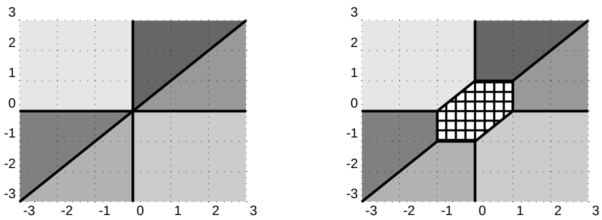
Let us take a closer look at projectivizations of cotangent bundles of smooth projective toric surfaces. It is not hard to see that $\mathcal{R} = \emptyset$ in this case. Furthermore, $\mathcal{P}$ corresponds to the set of one-dimensional subspaces of E which occur in the filtration. The cardinality of the set of vertices in a slice $\mathcal{S}_P$ for $P \in \mathcal{P}$ is either two or three, and there are as many slices $\mathcal{S}_P$ with $|\mathcal{S}_P(0)| = 3$ as there are pairs of rays $(\rho, -\rho)$ in $\Sigma(1)$. Note also that $\mu(v) = 1$ for all $v \in \mathcal{S}_P(0)$.

Example 1.28. We consider the smooth projective complexity-one Fano T-threefold $\mathbb{P}(\Omega_{\mathbb{P}^2})$ from [AHS08, section 8.5]. The non-trivial slices of its divisorial fan $\mathcal{S}$ over $\mathbb{P}^1$ are illustrated in Figure 1.4, whereas $\Sigma = \operatorname{tail} \mathcal{S}$ and $\deg \mathcal{S}$ are given in Figure 1.5.

As in the previous example, all maximal polyhedral divisors have complete locus, i.e. $\mathcal{R} = \emptyset$ and all rays in the tailfan are marked.

Example 1.29. Finally, we also consider the projectivized cotangent bundle on the first Hirzebruch surface $\mathbb{F}_1$. The non-trivial slices of its divisorial fan $\mathcal{S}$ over $\mathbb{P}^1$ are illustrated in Figure 1.6.

The associated tailfan Σ and degree $\deg \mathcal{S}$ are given in Figure 1.7. Again, all polyhedral divisors have complete locus, i.e. $\mathcal{R} = \emptyset$ and all rays in the tailfan are marked.



(a) $\Sigma = \operatorname{tail} \mathcal{S}$ (b) $\deg \mathcal{S}$

Figure 1.5: Tailfan and degree of $\mathcal{S}\big(\mathbb{P}(\Omega_{\mathbb{P}^2})\big)$, cf. Example 1.28.

1.3 Toric Bouquets

This section will recall the notion of toric bouquet which was introduced in [AH06, Section 7] to study the fibres of the quotient map $\pi : \widetilde{X} \to Y$.

1.3.1 Affine Case

Let $\Delta \subset N_{\mathbb{Q}}$ be a σ-polyhedron and associate to any face $F \prec \Delta$ a cone $\lambda(F) \subset M_{\mathbb{Q}}$ in the following way:

$$F \mapsto \lambda(F) = \big\{ u \in M_{\mathbb{Q}} \,|\, \langle u, v - v' \rangle \geq 0 \text{ for all } \; v \in \Delta, v' \in F \big\}\,,$$

The collection of all cones $\lambda(F)$ then forms a so-called *quasifan* $\Lambda(\Delta)$.

Definition 1.30. Let $\Lambda \subset M_{\mathbb{Q}}$ be a quasifan with convex support $|\Lambda| \subset M_{\mathbb{Q}}$. The *fan ring* associated to Λ is the affine $\mathbb{K}$-algebra given by

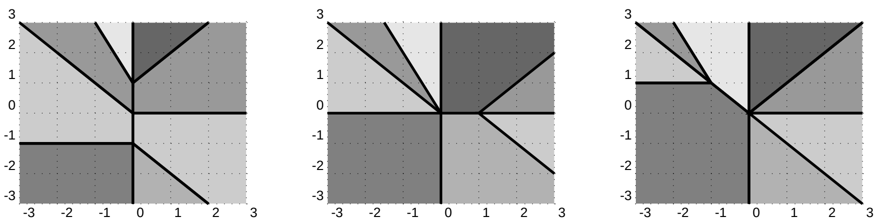
$$\mathbb{K}[\Lambda] \;:=\; \bigoplus_{u \in |\Lambda| \cap M} \mathbb{K}\chi^u, \qquad \chi^u \chi^w \;:=\; \begin{cases} \chi^{u+w} & \text{if } u, w \in \lambda \text{ for some } \lambda \in \Lambda, \\ 0 & \text{else.} \end{cases}$$

The *affine toric bouquet* associated to the σ-polyhedron Δ is defined as

$$\operatorname{TB}(\Delta) = \operatorname{Spec} \mathbb{K}[\Lambda(\Delta)].$$

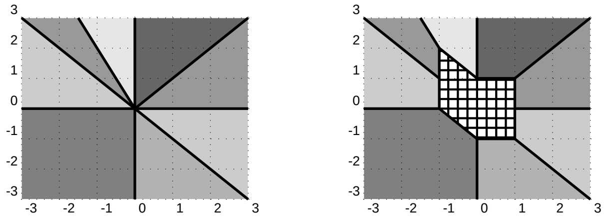
A toric bouquet comes with an effective action of the torus $T = \operatorname{Spec} \mathbb{K}(M)$. Its T-orbits are in dimension reversing one-to-one correspondence with the faces of Δ. An orbit closure associated to a face $F \prec \Delta$ is an affine toric variety whose cone and lattice of one-parameter subgroups are given by the cone

$$\sigma(F) \;:=\; \mathbb{Q}_{\geq 0}(\Delta - F)/\operatorname{lin}(F) \subset N(F) \;:=\; \big(N/(\operatorname{lin}(F) \cap N)\big)_{\mathbb{Q}}\,,$$



(a) $\mathcal{S}_0$ (b) $\mathcal{S}_1$ (c) $\mathcal{S}_\infty$

Figure 1.6: Non-trivial slices of $\mathcal{S}(\mathbb{P}(\Omega_{\mathbb{F}_1}))$, cf. Example 1.29.



(a) $\Sigma = \operatorname{tail} \mathcal{S}$ (b) $\deg \mathcal{S}$

Figure 1.7: Tailfan and degree of $\mathcal{S}\big(\mathbb{P}(\Omega_{\mathbb{F}_1})\big)$, cf. Example 1.29.

which is dual to $\lambda(F)$. Hence, the irreducible components of $\mathrm{TB}(\Delta)$ correspond exactly to the vertices $v \in \Delta(0)$.

Example 1.31. Consider the cone $\sigma \subset \mathbb{Q}^2$ together with the σ-polyhedron Δ and the quasifan $\Lambda(\Delta)$ as depicted in Figure 1.8.
Observe that $\operatorname{Spec} \mathbb{K}[\Lambda(\Delta)]$ is equidimensional and consists of two irreducible components isomorphic to $\mathbb{A}^2$ which are glued along an affine line.

For a comprehensive description of the fibers of $\pi : \widetilde{X} \to Y$ we also have to introduce a conewise varying lattice structure.

Definition 1.32. Let $\Lambda := \Lambda(\Delta)$ be a quasifan of a σ-polyhedron. Let $S \subset |\Lambda| \cap M$ be a subset such that

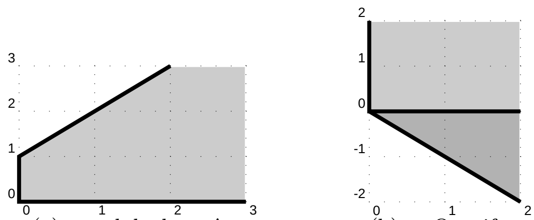
$$S_\lambda := \lambda \cap S = \lambda \cap M_\lambda$$

for each cone $\lambda \subset \Lambda$, where $M_\lambda \subset M \cap \operatorname{lin}(\lambda)$ is a lattice of full rank in $\operatorname{lin}(\lambda)$. Then we obtain a finitely generated subalgebra

$$\mathbb{K}[\Lambda, S] := \bigoplus_{u \in S} \mathbb{K}\chi^u \subset \mathbb{K}[\Lambda]$$

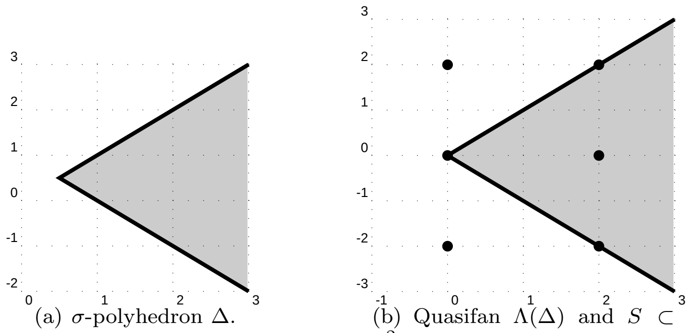
We call $\mathrm{TB}(\Delta, S) := \operatorname{Spec} \mathbb{K}[\Lambda, S]$ the corresponding *dappled affine toric bouquet.*

Passing from $\mathrm{TB}(\Delta)$ to $\mathrm{TB}(\Delta, S)$ means that we take a finite group quotient for every component. Note that the group may of course vary from component to component. The above inclusion of $\mathbb{K}$-algebras then amounts to a finite equivariant morphism $\gamma : \mathrm{TB}(\Delta) \to \mathrm{TB}(\Delta, S)$.



(a) σ-polyhedron Δ. (b) Quasifan $\Lambda(\Delta)$.

Figure 1.8: An affine toric bouquet, cf. Example 1.31.



(a) σ-polyhedron Δ. (b) Quasifan $\Lambda(\Delta)$ and $S \subset \mathbb{Z}^2$.

Figure 1.9: An affine dappled toric bouquet, cf. Example 1.33.

Example 1.33. Consider the cone $\sigma \subset \mathbb{Q}^2$ generated by the rays $\mathbb{Q}_{\geq 0}(1,1)$ and $\mathbb{Q}_{\geq 0}(1,-1) \in \mathbb{Z}^2$, and define $\Delta = (1/2, 1/2) + \sigma$. Furthermore, we set $S := 2(\mathbb{Z}^2)^\vee \subset (\mathbb{Q}^2)^\vee$. Then we obtain the following picture, cf. Figure 1.9. Observe that $\operatorname{Spec} \mathbb{K}[\Lambda(\Delta), S]$ is a two dimensional A_1-singularity and the finite morphism $\operatorname{Spec} \mathbb{K}[\Lambda(\Delta)] \to \operatorname{Spec} \mathbb{K}[\Lambda(\Delta), S]$ is of degree 2.

We return to our original quotient map $\widetilde{X} \to Y$ and intend to give an explicit description of the reduced fiber $\pi^{-1}(P)$ in terms of dappled toric bouquets.

Definition 1.34. Let $\mathcal{D}$ be a polyhedral divisor on Y with tailcone σ, and let Λ_P be the quasifan corresponding to the σ-polyhedron $\mathcal{D}_P$. The *fiber monoid complex* of $P \in Y$ is defined as

$$S_P \ := \ \left\{ u \in \sigma^\vee \cap M \mid \mathcal{D}(u) \text{ is principal at } P \right\}.$$

Furthermore, for a cone $\lambda \subset \Lambda_P$ we denote by $M_{P,\lambda} \subset M$ the sublattice generated by $S_P \cap \lambda$.

Proposition 1.35. Cf. [AH06, Proposition 7.10]. Let $\mathcal{D}$ be a polyhedral divisor on the smooth projective curve Y. Then, for every $P \in Y$, the reduced fiber of $\pi : \widetilde{X}(\mathcal{D}) \to Y$ is T-equivariantly isomorphic to $\mathrm{TB}(\mathcal{D}_P, S_P) = \operatorname{Spec} \mathbb{K}[\Lambda_P, S_P]$.

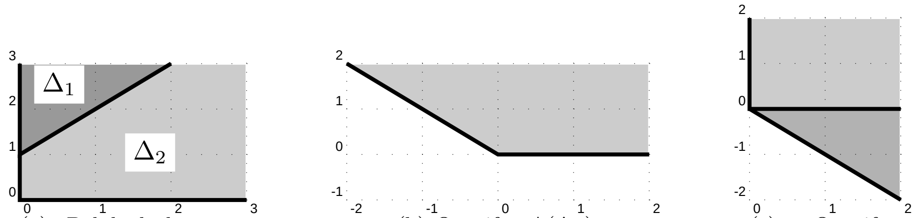
1.3.2 Non-Affine Case

Our next goal is to patch dappled affine toric bouquets together. Let $\boldsymbol{\Delta} = \{\Delta_i\}_i$ denote a polyhedral complex consisting of a finite number of σ_i-polyhedra Δ_i. As in the usual toric case we can glue two affine toric bouquets $\mathrm{TB}(\Delta_1)$ and $\mathrm{TB}(\Delta_2)$ along $\mathrm{TB}(\Delta_1 \cap \Delta_2)$ since $\mathrm{TB}(\Delta_1 \cap \Delta_2) \hookrightarrow \mathrm{TB}(\Delta_i)$ is an open embedding for $i = 1, 2$. Indeed, this can be checked locally on every toric component for which the result is classical.

Definition 1.36. The *toric bouquet* associated to $\boldsymbol{\Delta}$ is denoted by $\mathrm{TB}(\boldsymbol{\Delta})$. It is obtained by gluing the affine toric bouquets $\mathrm{TB}(\Delta_i)$ along the common faces of the polyhedra Δ_i.

Example 1.37. We consider the polyhedral complex $\boldsymbol{\Delta}$ which is pictured in Figure 1.10. There, we have that $\Delta_1 = (0,1) + \langle (0,1), (1,1) \rangle_{\geq 0}$ and $\Delta_2 = \operatorname{conv}\{(0,0),(0,1)\} + \langle (1,1),(1,0) \rangle_{\geq 0}$. Hence, $\operatorname{Spec} \mathbb{K}[\Lambda(\Delta_1)]$ is an ordinary $\mathbb{A}^2$

that is glued along the ray $(0,1) + \mathbb{Q}_{\geq 0}(1,1)$ (i.e. $\mathbb{K}^* \times \mathbb{A}^1$) to the irreducible component of $\operatorname{Spec} \mathbb{K}[\Lambda(\Delta_2)]$ corresponding to the lower cone of the quasifan $\Lambda(\Delta_2)$.



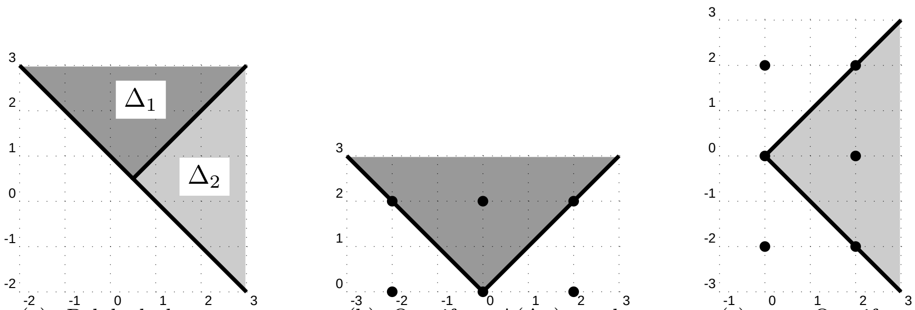
(a) Polyhedral complex $\mathbf{\Delta}$. (b) Quasifan $\Lambda(\Delta_1)$. (c) Quasifan $\Lambda(\Delta_2)$.

Figure 1.10: A non-affine toric bouquet, cf. Example 1.37.

Observe that $\mathrm{TB}(\mathbf{\Delta})$ is equidimensional and consists of two irreducible components. One is isomorphic to $\mathbb{A}^2 = \mathrm{TV}(\delta)$ with $\delta = \langle (1,0),(0,1) \rangle_{\geq 0}$ whereas the second component is isomorphic to $\mathrm{TV}(\Sigma)$ where Σ consists of the two cones $\sigma_1 = \langle (0,1),(1,1) \rangle_{\geq 0}$ and $\sigma_2 = \langle (1,1),(0,-1) \rangle_{\geq 0}$.

We can also glue dappled affine toric bouquets. Denote by $(\mathbf{\Delta}, \mathbf{S}) = \{(\Delta_i, S_i)_i\}$ a polyhedral complex $\mathbf{\Delta}$ all of whose elements Δ_i are dappled in such a way that S_i and S_j induce subsets S_λ which are identified by the gluing of $\mathrm{TB}(\Delta_i)$ and $\mathrm{TB}(\Delta_j)$ along the common face $\Delta_i \succ \lambda \prec \Delta_j$.

Example 1.38. We consider the pair $(\mathbf{\Delta}, \mathbf{S})$ which is pictured in Figure 1.11. More specifically, $\Delta_1 = (1/2,1/2) + \langle (-1,1),(1,1) \rangle_{\geq 0}$ and $\Delta_2 = (1/2,1/2) + \langle (1,1),(1,-1) \rangle_{\geq 0}$ with $S_1 = S_2 = 2\mathbb{Z}^2$.



(a) Polyhedral complex $\mathbf{\Delta}$. (b) Quasifan $\Lambda(\Delta_1)$ and $S_1 \subset \mathbb{Z}^2$. (c) Quasifan $\Lambda(\Delta_2)$ and $S_2 \subset \mathbb{Z}^2$.

Figure 1.11: A non-affine dappled toric bouquet, cf. Example 1.38.

The two (dappled) A_1-singularities $\operatorname{Spec} \mathbb{K}[\Lambda(\Delta_1), S_1]$ and $\operatorname{Spec} \mathbb{K}[\Lambda(\Delta_2), S_2]$ are glued along a dappled $\mathbb{K}^* \times \mathbb{A}^1$ that corresponds to the ray $(1/2,1/2) + \mathbb{Q}_{\geq 0}(1,1)$.

In complete analogy to Proposition 1.35 we have the following result in the non-affine case.

Proposition 1.39. *Let $\mathcal{S}$ be a divisorial fan on the smooth projective curve Y. Then, for every $P \in Y$, the reduced fiber of $\pi : \widetilde{X}(\mathcal{S}) \to Y$ is T-equivariantly isomorphic to $\mathrm{TB}(\mathbf{\Delta}, \mathbf{S})$.*

Chapter 2

Equivariant Line Bundles

The description of T-invariant divisors on complexity-one T-varieties naturally extends the one which is known in toric geometry. Beginning with Weil divisors, we then focus on Cartier divisors for which we also discuss cohomology computations. We conclude with some examples and a glimpse upon further topics to be investigated.

2.1 Weil Divisors

Let $\mathcal{D}$ be a polyhedral divisor with tailcone σ on a smooth projective curve Y. As usual, we set $d := \dim \mathrm{TV}(\mathcal{D}) = \dim T + 1$. The following diagram briefly recalls our setting, cf. Section 1.1:

$$\begin{tikzcd} \widetilde{\mathrm{TV}}(\mathcal{D}) \arrow[r, "r"] \arrow[d, "\pi"'] & \mathrm{TV}(\mathcal{D}) \\ \mathrm{Loc}\,\mathcal{D} & \end{tikzcd}$$

It follows from the description of the orbit structure of the quotient map π (cf. [AH06, Section 7]) and the contraction map r (cf. [AH06, Section 10]) that one can distinguish between two types of T-invariant prime divisors on $\mathrm{TV}(\mathcal{D})$ and $\widetilde{\mathrm{TV}}(\mathcal{D})$, namely

1. orbit closures of dimension $d-1$.
2. families of orbit closures of dimension $d-2$.

Proposition 2.1. Let $\mathcal{D}$ be as above. Then there are one-to-one correspondences

- between invariant prime divisors of the first type on both, $\widetilde{\mathrm{TV}}(\mathcal{D})$ and $\mathrm{TV}(\mathcal{D})$, and pairs (P, v) with $P \in Y$ a closed point and v a vertex of $\mathcal{D}_P$,
- between invariant prime divisors of the second type on $\widetilde{\mathrm{TV}}(\mathcal{D})$ and rays $\rho \in (\mathrm{tail}\,\mathcal{D})(1)$,
- between invariant prime divisors of the second type on $\mathrm{TV}(\mathcal{D})$ and rays $\rho \in (\mathrm{tail}\,\mathcal{D})(1)$ such that $\deg \mathcal{D} \cap \rho = \emptyset$.

Proof. We know from [AH06, Section 7] that k-dimensional faces of $\mathcal{D}_P$ correspond to T-orbits of codimension k in the fiber $\pi^{-1}(P)$.

First, we consider invariant prime divisors on $\widetilde{\mathrm{TV}}(\mathcal{D})$. The above remark immediately gives us that invariant prime divisors of the first type correspond to pairs (P, v) where $P \in Y$ is a closed point and $v \in \mathcal{D}_P(0)$. Analogously, we may identify invariant prime divisors of the second type with pairs (η, ρ) where $\eta \in Y$ is the generic point and $\rho \in (\operatorname{tail}\mathcal{D})(1)$.

We now come to the description of invariant prime divisors of the first type on $\mathrm{TV}(\mathcal{D})$. To this end, we only have to check which of the invariant prime divisors on $\widetilde{\mathrm{TV}}(\mathcal{D})$ are not contracted via the map $r : \widetilde{\mathrm{TV}}(\mathcal{D}) \to \mathrm{TV}(\mathcal{D})$. In particular, we may assume that $\operatorname{Loc}\mathcal{D} = Y$ (otherwise $\operatorname{Loc}\mathcal{D}$ would be affine and thus $\widetilde{\mathrm{TV}}(\mathcal{D}) \cong \mathrm{TV}(\mathcal{D})$). Theorem 10.1 from [AH06] now tells us that r maps two different T-orbits (P, F_P) and (Q, F_Q) to the same orbit in $\mathrm{TV}(\mathcal{D})$ if and only if

1. their corresponding cones $\lambda(F_P)$ and $\lambda(F_Q)$ in the normal fans of $\mathcal{D}_P$ and $\mathcal{D}_Q$ (which are subdivisions of $\operatorname{tail}\mathcal{D}$) coincide, and

2. if the map
$$\vartheta_u : Y \longrightarrow \operatorname{Proj}\Big(\bigoplus_{l \geq 0} \Gamma(\mathcal{O}_Y(\mathcal{D}(lu)))\Big),$$
sends P and Q to the same point for some $u \in \operatorname{relint}\lambda(F_Q)$.

So let (P, v_P) and (Q, v_Q) be invariant prime divisors of the first type on $\widetilde{\mathrm{TV}}(\mathcal{D})$ and assume that $\lambda(v_P) = \lambda(v_Q)$. Since $\operatorname{relint}\lambda(v_Q) \subset \operatorname{relint}(\operatorname{tail}\mathcal{D})^\vee$ we deduce that $\mathcal{D}(u)$ is big and semiample. Hence, ϑ_u is an isomorphism for $u \in \operatorname{relint}\lambda(v_Q)$ and there are no contractions for invariant prime divisors of the first type.

Let us finally come to invariant prime divisors of the second type on $\mathrm{TV}(\mathcal{D})$. To do so, we consider the pair (η, ρ) for $\rho \in (\operatorname{tail}\mathcal{D})(1)$. Now, the corresponding family of orbits over the generic point $\eta \in Y$ is "contracted" if and only if $\langle u, \deg\mathcal{D}\rangle = \deg\mathcal{D}(u) = 0$ for some $u \in \operatorname{relint}\lambda(\rho) = \operatorname{relint}(\operatorname{tail}\mathcal{D} - \rho)^\vee \subsetneq \operatorname{relint}(\operatorname{tail}\mathcal{D})^\vee$. Since $\deg\mathcal{D}(u) \geq 0$ and $\lambda(\rho)^\perp = \langle\rho\rangle$ we deduce that the previous statement is equivalent to the fact that $\deg\mathcal{D} \cap \rho \neq \emptyset$.

□

Using Propsition 2.1 and Definition 1.18, we establish the following useful notation before we proceed.

Notation 2.2. Prime divisors of the first type which correspond to a point $P \in Y$ and a vertex $v \in \mathcal{D}_P(0)$ are denoted by $D_{(P,v)}$. They may also be referred to as *vertical* prime divisors. A prime divisor of the second type which corresponds to an extremal ray $\rho \in \mathcal{R}$ will be denoted by D_ρ. We also call them *horizontal*.

Before we can get our hands on the divisor class group $\operatorname{Cl}\big(\mathrm{TV}(\mathcal{S})\big)$ we need to know how to describe invariant principal divisors. More concretely, we need to express divisors of the form $\operatorname{div}\big(f\chi^u\big)$ with $f \in \mathbb{K}(Y)$ and $u \in M$ as a sum of prime divisors.

Proposition 2.3. Let $\mathcal{D}$ and $f\chi^u$ be as above, i.e. the latter is a semi-invariant rational function of weight u. Then the corresponding principal divisor is given by

$$\operatorname{div}(f\chi^u) = \sum_{\rho\in\mathcal{R}} \langle u, n_\rho\rangle D_\rho + \sum_{(P,v)} \mu(v)(\langle u, v\rangle + \operatorname{ord}_P f)D_{(P,v)} .$$

Proof. The proof follows [AP, Theorem 2.2]. An alternative but longer proof due to Hendrik Süß is given in [PS, Proposition 3.14].

We only have to verify this formula for $\widetilde{\mathrm{TV}}(\mathcal{D})$. Indeed, when considering $\mathrm{TV}(\mathcal{D})$ we merely have to forget about those prime divisors which are contracted. Since $\widetilde{\mathrm{TV}}(\mathcal{D})$ is toroidal and the equality can be checked locally, we may pass to a formal neighbourhood of $P \in Y$. Hence, we may identify (Y, P) with $(\mathbb{A}^1, 0)$ as formal germs. Moreover, we may assume that our polyhedral divisor is of the form $\mathcal{D} = \mathcal{D}_0 \otimes [0]$. Hence, we are in a purely toric situation, namely

$$\widetilde{\mathrm{TV}}(\mathcal{D}) = \mathrm{TV}\left(\operatorname{cone}(\mathcal{D}_0, 1) \subset N_{\mathbb{Q}} \oplus \mathbb{Q}\right) .$$

A ray $\rho \in \sigma(1)$ translates into the ray $\mathbb{Q}_{\geq 0}(n_\rho, 0) \in N_{\mathbb{Q}} \oplus \mathbb{Q}$, and a vertex $v \in \mathcal{D}_0(0)$ induces the ray with primitive generator $\mu(v)(v, 1)$. Furthermore, our semi-invariant function $f\chi^u$ is identified with $t^{\operatorname{ord}_P f}\chi^u$, which corresponds to the tuple $[u, \operatorname{ord}_P f] \in M \oplus \mathbb{Z}$ in the toric character lattice. The usual pairings $\langle [u, \operatorname{ord}_P f], (n_\rho, 0)\rangle$ and $\langle [u, \operatorname{ord}_P f], \mu(v)(v, 1)\rangle$, respectively, then complete the proof. $\square$

The invariant divisor class group $\text{T-Div}\left(\mathrm{TV}(\mathcal{S})\right)$, which is the quotient of the group of T-invariant Weil divisors modulo the group of T-invariant principal divisors, is isomorphic to the ordinary divisor class group $\mathrm{Cl}\left(\mathrm{TV}(\mathcal{S})\right)$. Indeed, this is a special case of Theorem 1 from [FMSS95]. As an immediate consequence of Proposition 2.3 we have the following

Corollary 2.4. The divisor class group of $\mathrm{TV}(\mathcal{S})$ is given by

$$\mathrm{Cl}\left(\mathrm{TV}(\mathcal{S})\right) = \frac{\bigoplus_{\rho\in\mathcal{R}} \mathbb{Z}\, D_\rho \oplus \bigoplus_{(P,v)} \mathbb{Z}\, D_{(P,v)}}{\left\langle \sum_{\rho\in\mathcal{R}} \langle u, n_\rho\rangle D_\rho + \sum_{(P,v)} \mu(v)(\langle u, v\rangle + a_P)D_{(P,v)} \right\rangle} ,$$

where u runs over all elements of M and $\sum_P a_P P$ over all principal divisors on Y. Equivalently, it is isomorphic to

$$\operatorname{Pic}(Y) \oplus \bigoplus_{\rho\in\mathcal{R}} \mathbb{Z}\, D_\rho \oplus \bigoplus_{(P,v)} \mathbb{Z}\, D_{(P,v)}$$

modulo the relations

$$\begin{aligned} [P] &= \sum_{v\in\mathcal{S}_P} \mu(v)D_{(P,v)} , \\ 0 &= \sum_{\rho\in\mathcal{R}} \langle u, n_\rho\rangle D_\rho + \sum_{(P,v)} \mu(v)\langle u, v\rangle D_{(P,v)} . \end{aligned}$$

Remark 2.5. Assuming $\mathrm{TV}(\mathcal{S})$ to be rational and complete, one can derive an exact sequence presenting the divisor class group $\mathrm{Cl}(\mathrm{TV}(\mathcal{S}))$ which is very close to the purely toric setting (cf. Section 3.3.3 for an alternative proof):

$$0 \longrightarrow M \oplus (\mathbb{Z}^{\mathcal{P}}/\mathbb{Z})^\vee \xrightarrow{\phi^\vee \oplus Q^\vee} (\mathbb{Z}^{\mathcal{V}\cup\mathcal{R}})^\vee \longrightarrow \mathrm{Cl}\left(\mathrm{TV}(\mathcal{S})\right) \longrightarrow 0 .$$

The above exact sequence also makes it easy to determine the Picard rank ρ_X of a rational $\mathbb{Q}$-factorial complexity-one T-variety X.

Corollary 2.6. Let $\mathrm{TV}(\mathcal{S})$ be complete and $\mathbb{Q}$-factorial. Then we have that

$$\rho_{\mathrm{TV}(\mathcal{S})} = 2 + \#\mathcal{R} + \#\mathcal{V} - \#\mathcal{P} - d\,.$$

Proof. The Picard rank is exactly equal to

$$\operatorname{rank}\left(\mathbb{Z}^{\mathcal{V}\cup\mathcal{R}}\right)^{\vee} - \operatorname{rank} M - \operatorname{rank}\left(\mathbb{Z}^{\mathcal{P}}/\mathbb{Z}\right)^{\vee}.$$

□

For the sake of completeness and some examples which will follow later on, we also provide a description of the canonical class $K_{\mathrm{TV}(\mathcal{S})}$ which is due to Hendrik Süß, cf. [PS, Theorem 3.21].

Proposition 2.7. The canonical class $K_{\mathrm{TV}(\mathcal{S})}$ of $\mathrm{TV}(\mathcal{S})$ can be represented as

$$K_{\mathrm{TV}(\mathcal{S})} = -\sum_{\rho\in\mathcal{R}} D_\rho + \sum_{(P,v)} \big(\mu(v)\operatorname{coeff}_P(K_Y) + \mu(v) - 1\big) D_{(P,v)}\,,$$

where K_Y is a fixed representative of the canonical divisor of the curve Y.

2.2 Cartier Divisors

The goal of this section is to describe invariant Cartier divisors on a complexity-one T-variety $\mathrm{TV}(\mathcal{S})$ in terms of continuous piecewise affine linear functions on the slices of $\mathcal{S}$. There have already been other approaches in this direction, e.g. for toroidal complexity-one T-varieties in [KKMSD73, Theorem 9*] and, much more generally, for G-varieties in [Tim00].

2.2.1 Divisorial Support Functions

Definition 2.8. Let $\Sigma \subset N_{\mathbb{Q}}$ be a polyhedral complex. A continuous function $h : |\Sigma| \to \mathbb{Q}$ which is affine linear on every polyhedron $\sigma \in \Sigma$ is called a $\mathbb{Q}$-*support function* or simply *support function* on Σ if it has integral slope and integral translation, i.e. $\mu(v)h(v) \in \mathbb{Z}$ for $v \in |\Sigma|$.

The set of support functions on a fixed polyhedral complex Σ clearly forms an abelian group under addition. We will denote this group by $\mathrm{SF}(\Sigma)$.

Definition 2.9. Let h be a support function on a polyhedral complex $\Sigma \subset N_{\mathbb{Q}}$. The linear part of the restriction of h to an element $\sigma \in \Sigma$ then defines a linear function on the tailcone of σ. We denote this function by $\underline{h}^\sigma$.

The free abelian group of integral linear functions on a fixed cone $\delta \in N_{\mathbb{Q}}$ is isomorphic to $M/M(\delta)$ with $M(\delta) := M \cap \delta^{\perp}$. Given a polyhedral complex $\Sigma \subset N_{\mathbb{Q}}$ we denote the set of all maximal elements in the tailfan of Σ by $(\operatorname{tail}\Sigma)^{\max}$.

Definition 2.10. Let $\Sigma \subset N_{\mathbb{Q}}$ be a polyhedral complex. Then we denote by $\underline{h} := \underline{h}(\Sigma)$ the set of elements $[\underline{h}^\sigma] \in M/M(\operatorname{tail}\sigma)$ such that $\operatorname{tail}\sigma \in (\operatorname{tail}\Sigma)^{\max}$.

We now come to the crucial definition of this section. As before, let $\mathcal{S}$ denote a divisorial fan on a smooth projective curve Y.

Definition 2.11. By $\mathrm{SF}(\mathcal{S})$ we denote the set of all collections

$$(h_P)_{P\in Y} \in \prod_{P\in Y} \mathrm{SF}(\mathcal{S}_P) \quad \text{such that}$$

1. All support functions h_P have the same linear part $\underline{h}$.
2. The set of points $P \in Y$ for which the support function h_P differs from its linear part $\underline{h}$ is finite.

Clearly, $\mathrm{SF}(\mathcal{S})$ forms an abelian group under addition. Its elements are called *divisorial support functions* on $\mathcal{S}$.

Observe that we may restrict an element $h_P \in \mathrm{SF}(\mathcal{S}_P)$ to a subcomplex of $\mathcal{S}_P$. More generally, we may restrict a divisorial support function $h \in \mathrm{SF}(\mathcal{S})$ to a polyhedral divisor $\mathcal{D} \in \mathcal{S}$. The latter restriction will be denoted by $h|_{\mathcal{D}}$.

In addition, we can associate a divisorial support function $\mathrm{SF}(D)$ to any Cartier divisor $D \in \mathrm{CaDiv}\, Y$ by setting $\mathrm{SF}(D)_P \equiv \mathrm{coeff}_P(D)$. Moreover, we can consider any element $u \in M$ as a divisorial support function by setting $\mathrm{SF}(u)_P \equiv u$.

Definition 2.12. A divisorial support function $h \in \mathrm{SF}(\mathcal{S})$ is called *principal* if $h = \mathrm{SF}(u) + \mathrm{SF}(D)$ for some $u \in M$ and some principal divisor D on Y. It is called *Cartier* if its restriction $h|_{\mathcal{D}}$ is principal for every $\mathcal{D} \in \mathcal{S}$ with $\mathrm{Loc}\,\mathcal{D} = Y$. The set of divisorial Cartier support functions is a free abelian group which we denote by $\mathrm{CaSF}(\mathcal{S})$.

2.2.2 A Correspondence

Let $\mathrm{TV}(\mathcal{S})$ be a complexity-one T-variety and $\text{T-CaDiv}(\mathrm{TV}(\mathcal{S}))$ the free abelian group of T-invariant Cartier divisors on $\mathrm{TV}(\mathcal{S})$. As we are going to relate divisorial support functions on $\mathcal{S}$ to T-invariant Cartier divisors on $\mathrm{TV}(\mathcal{S})$ in the upcoming Theorem 2.14 we recall that $\mathbb{K}(\mathrm{TV}(\mathcal{S}))^{\mathrm{hom}} \cong \bigoplus_{u\in M} \mathbb{K}(Y)\chi^u$. Thus, any semi-invariant rational function on $\mathrm{TV}(\mathcal{S})$ has a representation of the form $f\chi^u$ with $f \in \mathbb{K}(Y)$.

Before turning to the general case, let us make an important observation which generalizes the fact that the Picard group of an affine toric variety is trivial.

Proposition 2.13. The Picard group $\mathrm{Pic}\big(\mathrm{TV}(\mathcal{D})\big)$ of a polyhedral divisor $\mathcal{D}$ with complete locus is trivial. In particular, every invariant Cartier divisor is of the form $\mathrm{div}(f\chi^u)$.

Proof. This is a direct consequence of Lemma 1.15 and the fact that any Cartier divisor is linear equivalent to an invariant one. □

Theorem 2.14. For a complexity-one T-variety $\mathrm{TV}(\mathcal{S})$ we have that

$$\text{T-CaDiv}(\mathrm{TV}(\mathcal{S})) \cong \mathrm{CaSF}(\mathcal{S})$$

as free abelian groups.

Proof. First, we consider an element $h = (h_P)_{P \in Y} \in \mathrm{CaSF}(\mathcal{S})$. For every $\mathcal{D} \in \mathcal{S}$ there exist a weight $u_{\mathcal{D}} \in M$ and an integer $a_P^{\mathcal{D}}$ such that $h_P|_{\mathcal{D}}(v) = u_{\mathcal{D}}(v) + a_P^{\mathcal{D}}$. We may now cover Y by open subsets Y_i such that $\sum a_P^{\mathcal{D}} P|_{Y_i}$ becomes principal, i.e. $\sum a_P^{\mathcal{D}} P|_{Y_i} = \mathrm{div}(f_i^{\mathcal{D}})|_{Y_i}$. Note that this divisor is already globally principal for a polyhedral divisor with complete locus. Then, $f_i^{\mathcal{D}} \chi^{u_{\mathcal{D}}}$ defines an invariant principal divisor on $\mathrm{TV}(\mathcal{D}|_{Y_i})$ and all of these principal divisors clearly patch together to a Cartier divisor on $\mathrm{TV}(\mathcal{D})$ with local data $\{(\mathrm{TV}(\mathcal{D}|_{Y_i}), (f_i^{\mathcal{D}})^{-1}\chi^{-u_{\mathcal{D}}})\}$. Doing this for all (maximal) $\mathcal{D} \in \mathcal{S}$ finally gives us an invariant Cartier divisor on $\mathrm{TV}(\mathcal{S})$.

Let us now consider an invariant Cartier divisor $D = \{(U_i, \widetilde{f}_i)\}_{i \in I}$ on $\mathrm{TV}(\mathcal{S})$. We may assume that the open sets U_i are invariant since D is invariant. Using the very same argument, we may suppose that the rational functions $\widetilde{f}_i$ are semi-invariant functions, i.e. $\widetilde{f}_i = f_i \chi^{u_i}$. Intersecting the covering $(U_i)_{i \in I}$ with the affine open invariant covering $\{\mathrm{TV}(\mathcal{D})\}_{\mathcal{D} \in \mathcal{S}}$, we may furthermore suppose that the induced covering of each open affine set $\mathrm{TV}(\mathcal{D})$ comes from a covering $(V_j^{\mathcal{D}})_j$ of $\mathrm{Loc}\,\mathcal{D}$, see Lemma 1.15.

To construct $\underline{h}$, we pick a pair $(U_j^{\mathcal{D}}, f_j^{\mathcal{D}} \chi^{u_j^{\mathcal{D}}})$ with $U_j^{\mathcal{D}} = \mathrm{TV}(\mathcal{D}|_{V_j^{\mathcal{D}}})$ for every maximal cone $\sigma \in \mathrm{tail}\,\mathcal{S}$ such that $\mathrm{tail}\,\mathcal{D} = \sigma$. We set $\underline{h}_\sigma \equiv -u_j^{\mathcal{D}}$ and observe that this construction does not depend on the choices we made, because the quotient of two different weights would give us an element in $\sigma^\perp$ and thus does not affect $\underline{h}_\sigma$.

In the next step we construct a support function $h_{\mathcal{D}}$ for every maximal element $\mathcal{D} \in \mathcal{S}$. Recall that $\mathrm{TV}(\mathcal{D}) = \cup_j \mathrm{TV}(\mathcal{D}|_{V_j^{\mathcal{D}}})$. For a point $P \in V_j^{\mathcal{D}} \subset \mathrm{Loc}\,Y$ we set $a_P^{\mathcal{D}} := -\operatorname{coeff}_P(\mathrm{div}\, f_j^{\mathcal{D}})$. It is clear that this definition does not depend on j. We are left to check that the $h_{\mathcal{D}}$ glue together to become continuous piecewise affine functions on every slice $\mathcal{S}_P$ for $P \in Y$. Since we have already shown their compatibility on the tailfan it is enough to prove that two different support functions $h_{\mathcal{D}_i}$ and $h_{\mathcal{D}_j}$ agree on vertices $v \in \mathcal{S}_P(0)$ which lie in the common support. But this fact follows directly from the definition since both support functions have the same coefficient at $D_{(P,v)}$.

It is not hard to see that both constructions are inverse to each other and respect the group structure. □

The T-invariant Cartier divisor which is induced by an element $h \in \mathrm{CaSF}(\mathcal{S})$ is denoted by D_h. By abuse of notation we will often identify both of them. As an immediate consequence of Theorem 2.14 and Theorem 1 from [FMSS95] we now obtain the following representation of the Picard group:

Corollary 2.15. The Picard group of $\mathrm{TV}(\mathcal{S})$ is given by

$$\mathrm{Pic}\big(\mathrm{TV}(\mathcal{S})\big) \cong \frac{\mathrm{CaSF}(\mathcal{S})}{\langle \mathrm{SF}(u) + \mathrm{SF}(D) \,|\, u \in M, D \sim 0 \rangle}.$$

Remark 2.16. The computation of $\mathrm{Pic}\big(\mathrm{TV}(\mathcal{S})\big)$ does not come without difficulties. Note, however, that it is free abelian for a complete and rational T-variety of complexity one. Indeed, this is a consequence of Theorem 2.14, since $\mathrm{Pic}(\mathbb{P}^1)$ and $\mathrm{Pic}\big(\mathrm{TV}(\mathrm{tail}(\mathcal{S}))\big)$ are free abelian.

We have also seen before how to calculate the Picard rank of a $\mathbb{Q}$-factorial complexity-one T-variety, cf. Corollary 2.6. But it is not clear how to determine the Picard rank if this bound on the type of singularities is removed.

For some further remarks on CaDiv(TV($\mathcal{S}$)) see Section 2.2.3.

Corollary 2.17. Let $h = (h_P)_P$ be a Cartier divisor on TV($\mathcal{S}$). Then the corresponding Weil divisor is given by

$$-\sum_{\rho} \underline{h}(n_\rho) D_\rho - \sum_{(P,v)} \mu(v) h_P(v) D_{(P,v)}.$$

Proof. Using the local equations for invariant Cartier divisors as constructed in the proof of Theorem 2.14 and Proposition 2.3 immediately yields the result. □

2.2.3 Further remarks on T-CaDiv

Let us recall a well known result which describes the free abelian group of torus invariant Cartier divisors on a toric variety TV(Σ), cf. [CLS, Proposition 4.2.9]:

$$\text{T-CaDiv}\left(\mathrm{TV}(\Sigma)\right) = \varprojlim_{\sigma \in \Sigma} M/M(\sigma),$$

where $M(\sigma) = M \cap \sigma^\perp$. Note that the inverse limit is taken over the directed set $(\Sigma, \preceq)$, where $\preceq$ denotes the face relation.

We would like to prove an analogous statement for complexity-one T-varieties. Clearly, given such a variety TV($\mathcal{S}$), we also have a face relation $\preceq$ for the polyhedral divisors $\mathcal{D}$ that are contained in the divisorial fan $\mathcal{S}$. As before, we distinguish between two cases to describe the free abelian group of invariant Cartier divisors on $\mathrm{TV}(\mathcal{D}) \subset \mathrm{TV}(\mathcal{S})$.

1. For a polyhedral divisor $\mathcal{D}$ with complete locus we already know that

$$\text{T-CaDiv}\left(\mathrm{TV}(\mathcal{D})\right) = \mathrm{Princ}(\mathbb{P}^1) \oplus M/M(\operatorname{tail}\mathcal{D}).$$

2. For a polyhedral divisor $\mathcal{D}$ with affine locus we first have to introduce further notation. Let us denote by $\mathrm{Princ}(\operatorname{Loc}\mathcal{D})$ the free abelian group of principal divisors on $\operatorname{Loc}\mathcal{D}$. Furthermore, we denote by $M(\mathcal{D}_P)$ the sublattice of weights in M whose elements are constant along the affine subspace spanned by $\mathcal{D}_P \subset N_\mathbb{Q}$. Since $M(\mathcal{D}_P)$ is a sublattice of $M(\operatorname{tail}\mathcal{D})$ we have a projection $M/M(\mathcal{D}_P) \to M/M(\operatorname{tail}\mathcal{D})$.

   This gives us that

$$\text{T-CaDiv}\left(\mathrm{TV}(\mathcal{D})\right) = \mathrm{Princ}(\operatorname{Loc}\mathcal{D}) \oplus \mathbf{M}_\mathcal{D},$$

   where

$$\mathbf{M}_\mathcal{D} \subset \bigoplus_{P \in \mathcal{P}(\mathcal{D})} M/M(\mathcal{D}_P)$$

   denotes the sublattice consisting of those $\mathbf{u} = ([u]_P)_{P \in \mathcal{P}(\mathcal{D})}$ with $[u]_P \in M/M(\mathcal{D}_P)$ such that all $[u]_P$ have the same image in $M/M(\operatorname{tail}\mathcal{D})$.

Let us now discuss the construction of the inverse limit

$$\varprojlim_{\mathcal{D} \in \mathcal{S}} \text{T-CaDiv}\left(\mathrm{TV}(\mathcal{D})\right).$$

To this end, consider a face $\mathcal{E} \preceq \mathcal{D} \in \mathcal{S}$ together with the induced map

$$\text{T-CaDiv}\left(\operatorname{TV}(\mathcal{D})\right) \to \text{T-CaDiv}\left(\operatorname{TV}(\mathcal{E})\right).$$

On the first summand, this map is equal to the restriction of principal divisors on $\operatorname{Loc}\mathcal{D}$ to $\operatorname{Loc}\mathcal{E}$. On the second summand, it is induced by the map

$$M/M(\operatorname{tail}\mathcal{D}) \to M/M(\operatorname{tail}\mathcal{E}) \quad \text{or} \quad M/M(\mathcal{D}_P) \to M/M(\mathcal{E}_P), \text{ respectively.}$$

Hence, we have proved the following

Proposition 2.18.

$$\text{T-CaDiv}\left(\operatorname{TV}(\mathcal{S})\right) \cong \varprojlim_{\mathcal{D}\in\mathcal{S}} \text{T-CaDiv}\left(\operatorname{TV}(\mathcal{D})\right).$$

Remark 2.19. Without too much effort, the main results of this section could equally well be formulated for marked fansy divisors and thus provide us with statements for "standard coverings", cf. Remark 1.23.

2.3 Global Sections

Consider an invariant Cartier divisor D_h together with its associated equivariant line bundle $\mathcal{O}(D_h)$ on $\operatorname{TV}(\mathcal{S})$. Due to the torus action we have an M-module structure on the $\mathbb{K}$-vector space of global sections. Hence, the latter decomposes into homogeneous summands with respect to the elements of the character lattice, i.e.

$$\Gamma\big(\operatorname{TV}(\mathcal{S}), \mathcal{O}(D_h)\big) = \bigoplus_{u\in M} \Gamma\big(\operatorname{TV}(\mathcal{S}), \mathcal{O}(D_h)\big)_u .$$

We define the *set of weights* of D_h as

$$W(h) := W(D_h) := \{u \in M \mid \Gamma\big(\operatorname{TV}(\mathcal{S}), \mathcal{O}(D_h)\big)_u \neq 0\} .$$

Our next aim is to bound $W(h)$ by a polyhedron which is defined via $h \in \operatorname{CaSF}(\mathcal{S})$ and describe the homogeneous sections of a fixed weight $u \in M$ in terms of rational functions on the curve Y.

Definition 2.20. Given a Cartier support function $h = (h_P)_P$ on $\mathcal{S}$ with linear part $\underline{h}$ we define its associated *weight polyhedron* as

$$\Box_h := \{u \in M \mid \langle u, v\rangle \geq \underline{h}(v) \quad \text{for all} \quad v \in N\} .$$

In addition, we define the map $h^* : \Box_h \to \operatorname{Div}_{\mathbb{Q}} Y$ by

$$h^*(u) := \sum_P h_P^*(u)P := \sum_P \min\nolimits_{\mathrm{v}\in\mathcal{S}_\mathrm{P}}(u - h_P)P,$$

where $\min_{\mathrm{v}\in\mathcal{S}_\mathrm{P}}(u - h_P)$ denotes the minimal value of the continuous piecewise linear function $u - h_P$ along the vertices of $\mathcal{S}_P$.

Remark 2.21. The weight polyhedron captures the restriction of h to the generic fiber $\operatorname{TV}(\operatorname{tail}\mathcal{S})$. It is compact if and only if $\mathcal{S}$ is complete, and its tail cone is given as the intersection of the dual cones of the elements in $\operatorname{tail}\mathcal{S}$.

Moreover, going back to the setting of Section 1.2.2, we may identify $h = \psi^*$ and $h^* = \psi$. In this case, $h \mapsto h^*$ and $\psi \mapsto \psi^*$ can thus be considered as inverse maps.

The following result is due to Hendrik Süß, cf. [PS, Proposition 3.23].

Proposition 2.22. Let $D_h \in \text{T-CaDiv}(\mathcal{S})$ be a Cartier divisor with linear part $\underline{h}$. Then we have the following description of its global sections:

1. $W(h)$ is a subset of $\Box_h$.
2. For a character $u \in \Box_h$ we have that

$$\Gamma\big(\,\mathrm{TV}(\mathcal{S}), \mathcal{O}(D_h)\big)_u = \Gamma\big(\,\mathrm{Loc}\,\mathcal{S}, \mathcal{O}_{\mathrm{Loc}\,\mathcal{S}}(h^*(u))\big).$$

Proof. We use Corollary 2.17 to deduce that $\Gamma\big(\,\mathrm{TV}(\mathcal{S}), \mathcal{O}(D_h)\big)^{\mathrm{hom}}$ is equal to

$$\Big\{ f\chi^u \;\Big|\; \mathrm{div}(f\chi^u) - \sum_{\rho\in\mathcal{R}} \underline{h}(n_\rho)D_\rho - \sum_{(P,v)} \mu(v)h_P(v)D_{(P,v)} \geq 0 \Big\}.$$

Recalling that

$$\mathrm{div}(f\chi^u) = \sum_{\rho\in\mathcal{R}} \langle u, n_\rho\rangle D_\rho + \sum_{(P,v)} \mu(v)\big(\langle u,v\rangle + \mathrm{ord}_P(f)\big)D_{(P,v)}$$

and comparing coefficients, we easily derive the following two criteria for the semi-invariant rational function $f\chi^u$ to lie in $\Gamma\big(\,\mathrm{TV}(\mathcal{S}), \mathcal{O}(D_h)\big)$:

1. $\forall\, \rho \in \mathcal{R}: \qquad \langle u, n_\rho\rangle \geq \underline{h}(n_\rho)$,
2. $\forall\, (P,v): \qquad \mathrm{ord}_P(f) + \langle u, v\rangle \geq h_P(v)$.

The first bound implies that $u \in \Box_{\underline{h}}$, whereas the second says that $\mathrm{ord}_P(f) + (u - h_P)(v) \geq 0$ for all (P,v). ∎

Let us say a few words about the computation of the global sections of an invariant Weil divisor D on $\mathrm{TV}(\mathcal{S})$. As above, we may introduce a weight polyhedron

$$\Box_D \;:=\; \mathrm{conv}(W(D)) \subset M_{\mathbb{Q}}$$

associated to the set of weights $W(D) \subset M$.

Contrary to the case of Cartier divisors, this definition is not constructive, since we do not have an explicit description of $W(D)$. Nevertheless, we can introduce an analogue of h^* for Weil divisors which will be very useful in Chapter 3.

Definition 2.23. Let D be a T-invariant Weil divisor on $\mathrm{TV}(\mathcal{S})$. Then we define $\mathrm{coeff}(D)^* : \Box_D \to \mathrm{Div}_{\mathbb{Q}}\, Y$ by setting

$$\begin{aligned} \mathrm{coeff}(D)^*(u) \;&:=\; \textstyle\sum_P \mathrm{coeff}(D)^*_P(u)P \\ &:=\; \textstyle\sum_P \min_{\mathrm{v}\in\mathcal{S}_{\mathrm{P}}} \left(\langle\, u, \cdot\,\rangle + \frac{\mathrm{coeff}_{D_{(P,\cdot)}} D}{\mu(\cdot)} \right) P\,, \end{aligned}$$

so the coefficient at P is equal to the minimal value of $\langle\, u, \cdot\,\rangle + \frac{\mathrm{coeff}_{D_{(P,\cdot)}} D}{\mu(\cdot)}$ along the vertices $v \in \mathcal{S}_P(0)$.

The following statement is completely analogous to Proposition 2.22.

Proposition 2.24. Let D be an invariant Weil divisor on $\mathrm{TV}(\mathcal{S})$. Then we have the following description of its global sections:

1. By definition, the set of weights $W(D)$ is a subset of $\Box_D$.

2. For a character $u \in \Box_D$ we have that

$$\Gamma\big(\,\mathrm{TV}(\mathcal{S}), \mathcal{O}(D)\big)_u = \Gamma\big(Y, \mathcal{O}_Y\big(\,\mathrm{coeff}(D)^*(u)\big)\big)\,.$$

Remark 2.25. The divisorial support function h of a Cartier divisor $D = D_h$ on a complete complexity-one T-variety $\mathrm{TV}(\mathcal{S})$, which appears as a Weil divisor in disguise, can be "reconstructed" from the given data by solving a linear system of equations for every element $\sigma \in (\mathrm{tail}\,\mathcal{S})^{\max}$. These equations arise from the representation of a principal divisor $\mathrm{div}(f\chi^u)$ as a Weil divisor, cf. Proposition 2.3. Let us briefly sketch the "reconstruction".

If σ is not marked, then the linear part $u_\sigma := \underline{h}|_\sigma \in M$ is determined by the coefficients of D along the horizontal prime divisors D_ρ for $\rho \in \sigma(1)$. In fact, note that $\dim \sigma = d - 1$ and all rays $\rho \in \sigma(1)$ give rise to a horizontal prime divisor, since none of the rays $\rho \in \sigma(1)$ is marked ($\sigma(1) \subset \mathcal{R}$). Shortly, we must have that $\langle u_\sigma, n_\rho \rangle = -\,\mathrm{coeff}_{D_\rho}(D)$. Moreover, the affine translation a_P can be derived from the coefficients of D along the vertices of $\mathcal{D}_P^\sigma$. Indeed, for $v \in \mathcal{D}_P^\sigma(0)$ we must have that $\mu(v)(a_P + \langle u_\sigma, v\rangle) = -\,\mathrm{coeff}_{D_{(P,v)}}(D)$.

If σ is marked, then D is given as $\mathrm{div}(f\chi^{u_\sigma})$ for $f \in \mathbb{K}(Y)$ and $u_\sigma \in M$. The resulting linear system then consists of the following equations:

$$\begin{array}{rcll} \mu(v)(a_P + \langle u_\sigma, v\rangle) & = & -\,\mathrm{coeff}_{D_{(P,v)}}(D) & \text{for } P \in Y, v \in \mathcal{D}_P^\sigma(0)\,, \\ \sum_{P\in Y} a_P & = & 0\,. & \end{array}$$

These relations are sufficient to determine u_σ and the translations a_P. In addition, these functions clearly fit together to a continuous piecewise affine function $h_P : |\mathcal{S}_P| \to \mathbb{Q}$ over each slice $\mathcal{S}_P$ and their linear parts give a continuous piecewise linear function $\underline{h}$ on $\mathrm{tail}\,\mathcal{S}$. Hence, we have constructed an element of $\mathrm{CaSF}(\mathcal{S})$ with the desired properties.

2.4 Higher Cohomology Groups

We now turn to the computation of higher cohomology groups for equivariant line bundles on complexity-one T-varieties. Again, we first recall some well known results from toric geometry (cf. Section 7 in [Dan78]) which, in return, provide us with a rough guideline for the rest of this section.

2.4.1 Toric Varieties

Consider an equivariant line bundle $\mathcal{L}$ on the toric variety $\mathrm{TV}(\Sigma)$. The induced torus action on the cohomology spaces $H^i(\mathrm{TV}(\Sigma), \mathcal{L})$ yields a weight decomposition of the latter

$$H^i(\mathrm{TV}(\Sigma), \mathcal{L}) = \bigoplus_{u\in M} H^i\big(\,\mathrm{TV}(\Sigma), \mathcal{L}\big)_u\,.$$

We denote by $h : |\Sigma| \to \mathbb{Q}$ the continuous piecewise linear function representing the line bundle $\mathcal{L}$. For the computation of the cohomology groups we introduce the closed subsets

$$Z_u \ := \ \{v \in N_{\mathbb{R}} \mid \langle u, v\rangle \geq h(v)\}\,.$$

Then there is a long exact sequence of cohomology groups

$$\cdots \to H^{i-1}\big(\Sigma \setminus Z_u\big) \to H^i\big(\Sigma, Z_u\big) \to H^i(\Sigma) \to \ldots$$

and the following connection to the cohomology of $\mathcal{L}$.

Theorem 2.26. $H^i\big(\,\mathrm{TV}(\Sigma), \mathcal{L}\big)_u = H^i\big(|\Sigma|, Z_u; \mathbb{K}\big)$.

Moreover, we recall a powerful vanishing result.

Corollary 2.27. Suppose that Σ is complete, and h is upper convex, e.g. $\mathcal{L}$ is globally generated. Then $H^i(\mathrm{TV}(\Sigma), \mathcal{L}) = 0$ for $i > 0$.

Note that the properties of being nef and semiample coincide for line bundles on toric varieties, cf. also Section 3.1. The next result will be useful when discussing the cohomology groups of line bundles on dappled toric bouquets, cf. Section 2.4.2.

Proposition 2.28. Let $\mathcal{L}$ be a globally generated line bundle on a complete toric variety $\mathrm{TV}(\Sigma)$ and $\sigma \in \Sigma$. Then the restriction homomorphism

$$\Gamma(\mathrm{TV}(\Sigma), \mathcal{L}) \to \Gamma\big(\overline{\mathrm{orb}(\sigma)}, \mathcal{L}|_{\overline{\mathrm{orb}(\sigma)}}\big)$$

is surjective.

2.4.2 Toric Bouquets

Since we intend to approach higher cohomology group computations for the complexity-one case by a "cohomology and base change" argument, we also have to compute higher cohomology groups for equivariant line bundles on dappled toric bouquets.

In the first step, one should of course develop a suitable description of the latter objects – preferably as close as possible to the known one in toric geometry. We will not do this here but declare it as a project of the near future. Nonetheless, we have the following result.

Proposition 2.29. Let $X = \mathrm{TB}(\mathbf{\Delta})$ be a complete equidimensional toric bouquet and $\mathcal{L}$ a nef line bundle on X. Then $H^i(X, \mathcal{L}) = 0$ for $i > 0$.

Proof. We use induction on the dimension of and the number of irreducible components of X. Let the latter number be denoted by k. There is nothing to prove for $k = 1$ since X then is isomorphic to a complete toric variety and we can apply Corollary 2.27. For $k > 1$ we split off an irreducible component and denote the associated complete toric variety by X_1. Observe that we can find a component such that the rest still is connected. The remaining complete dappled toric bouquet X_2 then has one component less than X itself. Denoting the intersection of X_1 and X_2 by X_{12} and extending the respective restrictions of $\mathcal{L}$ by zero, we obtain an exact sequence of equivariant sheaves on X:

$$0 \to \mathcal{L} \to \mathcal{L}_1 \oplus \mathcal{L}_2 \to \mathcal{L}_{12} \to 0\,.$$

Along with it comes a long exact sequence in cohomology:

$$\begin{aligned}
0 &\to H^0(\mathcal{L}) \to H^0(\mathcal{L}_1) \oplus H^0(\mathcal{L}_2) \to H^0(\mathcal{L}_{12}) \to H^1(\mathcal{L}) \to \\
&\to H^1(\mathcal{L}_1) \oplus H^1(\mathcal{L}_2) \to H^1(\mathcal{L}_{12}) \to H^2(\mathcal{L}) \to H^2(\mathcal{L}_1) \oplus H^2(\mathcal{L}_2) \to \\
&\to H^2(\mathcal{L}_{12}) \to H^3(\mathcal{L}) \to \dots
\end{aligned}$$

Computing the cohomology groups of the respective sheaves on X is equivalent to computing the cohomology groups of the restricted line bundles on the components X_1, X_2 and X_{12}. From the induction argument (number of components and dimension) we infer that the cohomology spaces to the right of $H^1(\mathcal{L})$ vanish. Since X_{12} is a closed subvariety of the toric variety X_1 we may invoke Proposition 2.28. This shows that the map $H^0(\mathcal{L}_1) \oplus H^0(\mathcal{L}_2) \to H^0(\mathcal{L}_{12})$ is surjective since its restriction to the first summand already is surjective. Thus, we conclude that $H^1(\mathcal{L}) = 0$. □

The result of course equally holds for complete equidimensional dappled toric bouquets since the dappling $\mathbf{S}$ only encodes the finite morphism $\mathrm{TB}(\boldsymbol{\Delta}) \to \mathrm{TB}(\boldsymbol{\Delta}, \mathbf{S})$, i.e. it does not affect the "intrinsic" geometry of the latter.

2.4.3 Complexity-One T-Varieties

We suspect that higher cohomology group computations for equivariant line bundles on complexity-one T-varieties require more elaborate means than those we will use below, in particular if one cannot directly make use of a quotient map to the base curve Y. Hence, we restrict to the toroidal case. See Section 2.6 for some remarks on the general case.

So far, exact higher cohomology group computations seem to be out of reach even in the toroidal case. But there are two possible ways to simplify the setting. First, one could consider a broader cohomological invariant, namely the Euler characteristic. Second, one could restrict to special line bundles with nice numerical properties. Finally, one could do both, of course.

The following result by Nathan Ilten and Hendrik Süß is a first step in this direction.

Proposition 2.30. [ISa, Corollary 3.27] Let X be a smooth projective toroidal $\mathbb{K}^*$-surface with base curve Y. For any semiample T-invariant Cartier divisor D_h on X we have

$$\chi(X, \mathcal{O}_X(D_h)) = \sum_{u \in \Box_h \cap M} \chi\big(Y, \mathcal{O}_Y(h^*(u))\big)\,.$$

It is crucial to point out that the proof heavily uses the well known intersection theory for smooth projective surfaces. On the way, they show that $\chi(\mathcal{O}_X(D_h)) = D_h^2 + 1 + \chi(\mathcal{O}_X) - g(C)$ for any curve $C \in |D_h|$.

With only very little effort one can go even a step further. Expressing an arbitrary T-invariant Cartier divisor $D_h = D_{h_1} - D_{h_2}$ as the difference of two semiample ones, we obtain the following

Corollary 2.31. Let D_h be a T-invariant Cartier divisor on a smooth projective toroidal $\mathbb{K}^*$-surface. Then we have that

$$\chi(\mathcal{O}_X(D_h)) = \chi(\mathcal{O}_X(D_{h_1})) - \chi(\mathcal{O}_X(D_{h_2})) + 1 - g(Y) + D_{h_2}^2 - 2D_{h_1}D_{h_2}\,.$$

Proof. Let C_1 and C_2 be arbitrary curves in the linear systems $|D_{h_1}|$, and $|D_{h_2}|$ respectively. Using the Riemann-Roch formula and the adjunction formula for surfaces gives us

$$\begin{aligned}
\chi(\mathcal{O}_X(D_h)) &= 1/2D_h(D_h - K_X) + \chi(\mathcal{O}_X) \\
&= 1/2(D_{h_2}^2 + D_{h_2}K_X) - 1/2(D_{h_1}^2 + D_{h_1}K_X) + D_{h_1}^2 \\
&\quad - D_{h_1}D_{h_2} + \chi(\mathcal{O}_X) \\
&= g(C_2) - 1 - (g(C_1) - 1) + D_{h_1}^2 - D_{h_1}D_{h_2} + \chi(\mathcal{O}_X) \\
&= D_{h_1}^2 + 1 - g(C_1) + \chi(\mathcal{O}_X) - \big(D_{h_2}^2 + 1 - g(C_2) + \chi(\mathcal{O}_X)\big) \\
&\quad + D_{h_2}^2 - D_{h_1}D_{h_2} + \chi(\mathcal{O}_X) \\
&= \chi(\mathcal{O}_X(D_{h_1})) - \chi(\mathcal{O}_X(D_{h_2})) + \chi(\mathcal{O}_X) + D_{h_2}^2 - D_{h_1}D_{h_2}.
\end{aligned}$$

Since $\chi(\mathcal{O}_X) = 1 - g(Y)$ we have completed the proof. □

For the computation of the intersection numbers one may invoke a result due to Hendrik Süß, cf. [PS, Proposition 3.31].

Our principal goal now is to generalize Proposition 2.30 in various directions. Since we have a flat projective morphism $\pi : \widetilde{X} \to Y$ we intend to make use of the theory of "cohomology and base change", cf. [Mum74]. However, our first approach towards the higher direct image sheaves that are associated with π is motivated by a special instance of the Leray spectral sequence.

Proposition 2.32.

$$\chi(\mathcal{O}_{\widetilde{X}}(D_h)) = \sum(-1)^i \chi(R^i\pi_*\mathcal{O}_{\widetilde{X}}(D_h)).$$

Proof. The statement follows from the Leray spectral sequence associated to the map $\pi : \widetilde{X} \to Y$. Indeed, we have that

$$0 \to \Gamma(Y, R^i\pi_*\mathcal{O}_{\widetilde{X}}(D_h)) \to H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(D_h)) \to H^1(Y, R^{i-1}\pi_*\mathcal{O}(D_h)) \to 0$$

for all $i \geq 0$. Summing up then gives us the formula from above. □

Note that these higher direct image sheaves are coherent because $\pi : \widetilde{X} \to Y$ is projective. Focussing on this result and being interested in their Euler characteristics, which are given by

$$\chi(R^i\pi_*\mathcal{O}_{\widetilde{X}}(D_h)) = \deg\, R^i\pi_*\mathcal{O}_{\widetilde{X}}(D_h) + (1-g)\ \mathrm{rank}\, R^i\pi_*\mathcal{O}_{\widetilde{X}}(D_h)\,,$$

we have to compute their degrees and ranks. Although we know that

$$\mathrm{rank}\, R^i\pi_*\mathcal{O}_{\widetilde{X}}(D_h) = h^i(\mathrm{TV}(\mathrm{tail}\,\mathcal{S}), \mathcal{O}(\underline{h}))\,,$$

its degree remains somewhat mysterious and not easily approachable. Hence, we are left with

$$\chi(\mathcal{O}_{\widetilde{X}}(D_h)) = \sum_{i=0}^{d-1}(-1)^i \deg\, R^i\pi_*\mathcal{O}_{\widetilde{X}}(D_h) + (1-g)\chi(\mathrm{TV}(\mathrm{tail}\,\mathcal{S}), \mathcal{O}(\underline{h}))\,.$$

Nontheless, we have control over the direct image sheaf $\pi_*\mathcal{O}_{\widetilde{X}}(D_h)$.

Lemma 2.33. Let D_h be a T-invariant Cartier divisor on $\widetilde{X}$. Then its direct image $\pi_*\mathcal{O}_{\widetilde{X}}(D_h)$ under the map $\pi : \widetilde{X} \to Y$ is locally free and of the form $\bigoplus_{u\in \Box_h \cap M} \mathcal{O}_Y(h^*(u))$.

Proof. All we have to do is to recall the formula for computing global sections, cf. Proposition 2.22:
Let $U \subset Y$ be an open subset. Note that

$$\Gamma(U, \pi_*\mathcal{O}_{\widetilde{X}}(D_h))^{\mathrm{hom}} = \Gamma(\pi^{-1}(U), \mathcal{O}_{\widetilde{X}}(D_h))^{\mathrm{hom}}$$

is equal to

$$\left\{ f\chi^u \;\middle|\; \begin{array}{l} u \in \Box_h \cap M \text{ and } f \in \mathbb{K}(Y) \text{ such that} \\ \mathrm{ord}_P f + (u - h_P)(v) \geq 0 \quad P \in U,\ v \in \mathcal{S}_P(0) \end{array} \right\}.$$

But for a fixed degree $u \in \Box_h$ this set is exactly equal to $\Gamma(U, \mathcal{O}_Y(h^*(u))) \subset \mathbb{K}(Y)$. Hence, we have proved the claim. ☐

Unfortunately, the higher direct image sheaves do not have to be vector bundles anymore as the following example shows.

Example 2.34. Let us consider the smooth projective toric surface $X = \mathrm{TV}(\Sigma)$ whose ordered primitive generators of the rays ρ_i are listed as the columns of the following matrix

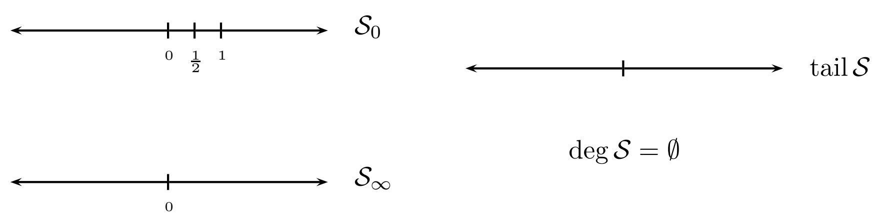
$$\begin{pmatrix} 1 & 2 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \end{pmatrix}.$$

This is $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in two infinitesimally near points. Furthermore, we set $\mathcal{L} := \mathcal{O}(D)$ with $D := -3D_{\rho_1} + D_{\rho_2} - 2D_{\rho_3} + D_{\rho_4} - 2D_{\rho_5} + 3D_{\rho_6}$. The downgrade is defined through the lattice maps

$$F = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad s = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

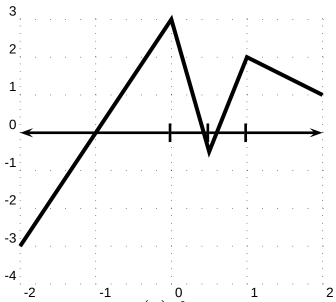
The slices of the divisorial fan arising from this downgrade are displayed in Figure 2.1, whereas the graphs of h_0 and h_∞ can be found in Figure 2.2. Note that $\mathcal{O}_{\mathbb{P}^1}(\underline{h}) \cong \mathcal{O}_{\mathbb{P}^1}(4)$ with $\Box_{\underline{h}} \cap \mathbb{Z} = \{-1, 0, 1, 2, 3\}$. Using the "usual" toric techniques (e.g. [Dan78, Theorem 7.2]), one easily computes that $\chi(\mathcal{O}_X(D)) = -35$. Furthermore, we deduce from Lemma 2.33 that $\pi_*\mathcal{O}_X(D) = \mathcal{O}_{\mathbb{P}^1}(-5)^5$. Hence,

$$\chi(\pi_*\mathcal{O}_X(D)) = \chi(\mathcal{O}_{\mathbb{P}^1}(-5)^5) = -20.$$

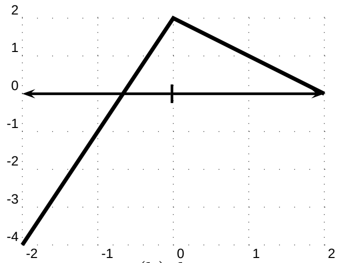


(a) Divisorial fan $\mathcal{S}$ associated to $\mathrm{TV}(\Sigma)$. (b) Tailfan and degree.

Figure 2.1: Divisorial fan associated to $\mathrm{TV}(\Sigma)$, cf. Example 2.34.



(a) h_0



(b) h_∞

Figure 2.2: Graphs of h_0 and h_∞ associated to $(\mathrm{TV}(\Sigma), D)$, cf. Example 2.34.

Since $h^1(\mathcal{O}_{\mathbb{P}^1}(\underline{h})) = 0$ we have that $\operatorname{rank} R^1\pi_*(\mathcal{O}_X(D)) = 0$. Thus, the first direct image has to be purely torsion of length 15 and an easy calculation shows that it is concentrated in $0 \in \mathbb{P}^1$.

Nontheless, we have the following vanishing result which is analogous to Corollary 2.27.

Proposition 2.35. Let X be a complete toroidal complexity-one T-variety over the base curve Y. For any nef T-invariant Cartier divisor D_h on X we have that $R^i\pi_*\mathcal{O}_X(D_h) = 0$ for $i > 0$. Hence,

$$H^i(X, \mathcal{O}_X(D_h)) = \bigoplus_{u \in \Box_h \cap M} H^i\big(Y, \mathcal{O}_Y(h^*(u))\big)\,.$$

In particular, $H^i(X, \mathcal{O}_X(D_h)) = 0$ for $i \geq 2$.

Proof. We would like to show that the higher direct images $R^i\pi_*\mathcal{O}_X(D_h)$ vanish for $i > 0$. To do so, we invoke "cohomology and base change". Thus, it is enough to show that $\dim_{k(P)} H^i(X_P, \mathcal{O}(D_h)_P) = 0$ for every closed point $P \in Y$, where X_P denotes the fiber over the point P. Note that the fiber over a point for which all $\mathcal{D} \in \mathcal{S}$ have trivial coefficients is equal to the toric variety $\mathrm{TV}(\Sigma)$. Restricting $\mathcal{O}_X(D_h)$ to this fiber gives us a nef line bundle on $\mathrm{TV}(\Sigma)$ whose cohomology groups H^i vanish except for $i = 0$.

The fiber over a point $P \in Y$ with non-trivial coefficient is a dappled toric bouquet as described in Section 1.3.2. Again, the restricted line bundle $\mathcal{L}_P$ is nef and we apply Proposition 2.29 to complete the proof. ☐

2.5 Examples

Example 2.36. We consider the downgrade of the second Hirzebruch surface $\mathbb{F}_2$ as described in Example 1.16 together with the line bundle $\mathcal{L} = \mathcal{O}(D)$ given through the following generators of the global sections over the affine charts U_{σ_i}:

$$u_{\sigma_0} = [0\ 0]\,, \quad u_{\sigma_1} = [1\ 0]\,, \quad u_{\sigma_2} = [3\ 1]\,, \quad u_{\sigma_3} = [0\ 1]\,.$$

Note that $\mathcal{L}$ is very ample and defines an embedding into $\mathbb{P}^5$. One can describe the embedding by a polytope $P_D \subset M_{\mathbb{Q}} = \mathbb{Q}^2$ which is the convex hull of the u_{σ_i}. It contains six lattice points which form a basis of the $\mathbb{K}$-vector space $\Gamma(\mathbb{F}_2, \mathcal{L})$, cf. Figure 2.3(a). Using the toric downgrade construction from Section 1.1.1, one finds that $\mathcal{L} = \mathcal{O}(D)$ with

$$D = D_{([0],-1/2)} + D_{([\infty],0)}.$$

By Proposition 2.22, we have $\Box_h = \{u \in \mathbb{Z} \mid 3 \geq u \geq 0\}$, and

$$\begin{aligned} \Gamma(\mathbb{F}_2, D_h)_0 &= \Gamma\big(\mathbb{P}^1, \mathcal{O}([\infty])\big)\,, & \Gamma(\mathbb{F}_2, D_h)_1 &= \Gamma\big(\mathbb{P}^1, \mathcal{O}([\infty])\big)\,, \\ \Gamma(\mathbb{F}_2, D_h)_2 &= \Gamma\big(\mathbb{P}^1, \mathcal{O}([\infty] - 1/2[0])\big)\,, & \Gamma(\mathbb{F}_2, D_h)_3 &= \Gamma\big(\mathbb{P}^1, \mathcal{O}([\infty] - [0])\big)\,. \end{aligned}$$

On the whole, they sum up to a six dimensional vector space as expected from the toric picture. The corresponding graphs of h_0^* and h_∞^* are shown in Figure 2.3(b)+(c).

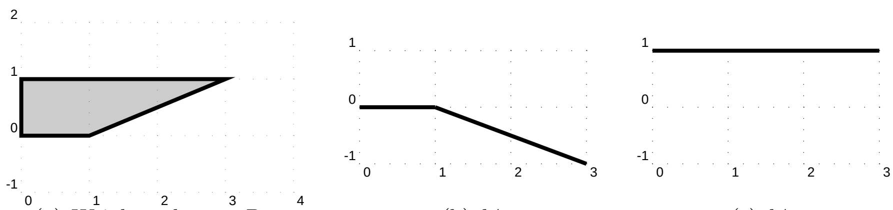
Example 2.37. We return to the smooth quadric Q from Example 1.27 and consider the anti-canonical divisor which may be represented as $3D_{[\infty],(1/2,1/2)}$. Its support function h is given by

$$\begin{aligned} h_1(v) &= \min\Big\{\big\langle\left(\begin{smallmatrix}3\\0\end{smallmatrix}\right), v\big\rangle + 3, \big\langle\left(\begin{smallmatrix}0\\3\end{smallmatrix}\right), v\big\rangle, \big\langle\left(\begin{smallmatrix}-3\\0\end{smallmatrix}\right), v\big\rangle, \big\langle\left(\begin{smallmatrix}0\\-3\end{smallmatrix}\right), v\big\rangle\Big\}\,, \\ h_0(v) &= \min\Big\{\big\langle\left(\begin{smallmatrix}3\\0\end{smallmatrix}\right), v\big\rangle, \big\langle\left(\begin{smallmatrix}0\\3\end{smallmatrix}\right), v\big\rangle + 3, \big\langle\left(\begin{smallmatrix}-3\\0\end{smallmatrix}\right), v\big\rangle, \big\langle\left(\begin{smallmatrix}0\\-3\end{smallmatrix}\right), v\big\rangle\Big\}\,, \\ h_\infty(v) &= \min\Big\{\big\langle\left(\begin{smallmatrix}3\\0\end{smallmatrix}\right), v\big\rangle - 3, \big\langle\left(\begin{smallmatrix}0\\3\end{smallmatrix}\right), v\big\rangle - 3, \big\langle\left(\begin{smallmatrix}-3\\0\end{smallmatrix}\right), v\big\rangle, \big\langle\left(\begin{smallmatrix}0\\-3\end{smallmatrix}\right), v\big\rangle\Big\}\,. \end{aligned}$$

The weight polytope $\Box_h$ is pictured in Figure 2.4(a) and the following list displays the induced divisor $h^*(u)$ on $\mathbb{P}^1$. There every weight $u = (u_1, u_2) \in \Box_h$ yields a triple (a, b, c) which corresponds to the $\mathbb{Q}$-Cartier divisor $D(a, b, c) = a[1] + b[0] + c[\infty]$.

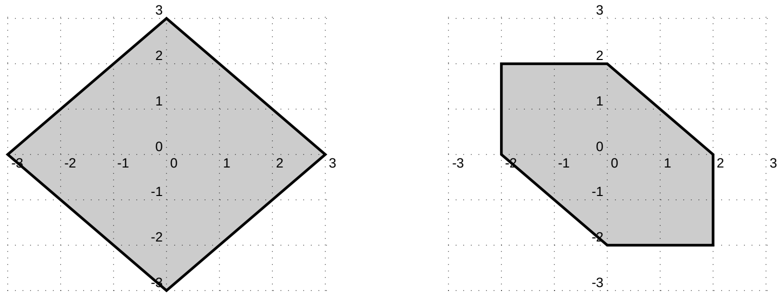
$(0,3)$	$\mapsto$	$(0,-3,3)$	$(-3,0)$	$\mapsto$	$(0,0,0)$	$(0,-1)$	$\mapsto$	$(0,0,1)$
$(-1,2)$	$\mapsto$	$(0,-2,2)$	$(-2,0)$	$\mapsto$	$(0,0,0)$	$(1,-1)$	$\mapsto$	$(-1,0,1)$
$(0,2)$	$\mapsto$	$(0,-2,2)$	$(-1,0)$	$\mapsto$	$(0,0,1)$	$(2,-1)$	$\mapsto$	$(-2,0,2)$
$(1,2)$	$\mapsto$	$(-1,-2,3)$	$(0,0)$	$\mapsto$	$(0,0,1)$	$(-1,-2)$	$\mapsto$	$(0,0,0)$
$(-2,1)$	$\mapsto$	$(0,-1,1)$	$(1,0)$	$\mapsto$	$(-1,0,2)$	$(0,-2)$	$\mapsto$	$(0,0,0)$
$(-1,1)$	$\mapsto$	$(0,-1,1)$	$(2,0)$	$\mapsto$	$(-2,0,2)$	$(1,-2)$	$\mapsto$	$(-1,0,1)$
$(0,1)$	$\mapsto$	$(0,-1,2)$	$(3,0)$	$\mapsto$	$(-3,0,3)$	$(0,-3)$	$\mapsto$	$(0,0,0)$
$(1,1)$	$\mapsto$	$(-1,-1,2)$	$(-2,-1)$	$\mapsto$	$(0,0,0)$			
$(2,1)$	$\mapsto$	$(-2,-1,3)$	$(-1,-1)$	$\mapsto$	$(0,0,0)$			

Summing up the dimensions of the vector spaces $\Gamma\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(D(a,b,c))\big)$ over all degrees yields $\dim \Gamma(Q, -K_Q) = 30$.



(a) Weight polytope P_D. (b) h_0^* (c) h_∞^*

Figure 2.3: Weight polytope and its divisorial analogue of the very ample line bundle $\mathcal{O}(D)$ on $\mathbb{F}_2$, cf. Example 2.36.



(a) The weight polytope of $\mathcal{O}(-K_Q)$.

(b) The weight polytope of $\mathcal{O}(-K_{\mathbb{P}(\Omega_{\mathbb{P}^2})})$.

Figure 2.4: The weight polytopes of $\mathcal{O}(-K_X)$ on two different smooth projective complexity-one T-threefolds, cf. Examples 2.37 and 2.38.

Example 2.38. We come back to Example 1.28 and consider $X = \mathbb{P}(\Omega_{\mathbb{P}^2})$. As in the previous example, want to calculate $\Gamma(X, -K_X)$. To do so, we use $K_{\mathbb{P}^1} = -2[0]$ as a representation of the canonical divisor on $\mathbb{P}^1$. By Proposition 2.7 we have that

$$-K_X = 2([0],(0,0)) + 2([0],(0,1))\,.$$

Using Corollary 2.17, we can construct h explicitly:

$$\begin{aligned}
h_0(v) &= \min\left\{\begin{array}{l} \left\langle\left(\begin{smallmatrix}-2\\0\end{smallmatrix}\right),v\right\rangle - 2, \left\langle\left(\begin{smallmatrix}0\\-2\end{smallmatrix}\right),v\right\rangle, \left\langle\left(\begin{smallmatrix}2\\-2\end{smallmatrix}\right),v\right\rangle, \\ \left\langle\left(\begin{smallmatrix}2\\0\end{smallmatrix}\right),v\right\rangle - 2, \left\langle\left(\begin{smallmatrix}0\\2\end{smallmatrix}\right),v\right\rangle - 2, \left\langle\left(\begin{smallmatrix}-2\\2\end{smallmatrix}\right),v\right\rangle - 2 \end{array}\right\}, \\
h_1(v) &= \min\left\{\begin{array}{l} \left\langle\left(\begin{smallmatrix}-2\\0\end{smallmatrix}\right),v\right\rangle + 2, \left\langle\left(\begin{smallmatrix}0\\-2\end{smallmatrix}\right),v\right\rangle, \left\langle\left(\begin{smallmatrix}2\\-2\end{smallmatrix}\right),v\right\rangle, \\ \left\langle\left(\begin{smallmatrix}2\\0\end{smallmatrix}\right),v\right\rangle, \left\langle\left(\begin{smallmatrix}0\\2\end{smallmatrix}\right),v\right\rangle, \left\langle\left(\begin{smallmatrix}-2\\2\end{smallmatrix}\right),v\right\rangle + 2 \end{array}\right\}, \\
h_\infty(v) &= \min\left\{\begin{array}{l} \left\langle\left(\begin{smallmatrix}-2\\0\end{smallmatrix}\right),v\right\rangle, \left\langle\left(\begin{smallmatrix}0\\-2\end{smallmatrix}\right),v\right\rangle, \left\langle\left(\begin{smallmatrix}2\\-2\end{smallmatrix}\right),v\right\rangle, \\ \left\langle\left(\begin{smallmatrix}2\\0\end{smallmatrix}\right),v\right\rangle + 2, \left\langle\left(\begin{smallmatrix}0\\2\end{smallmatrix}\right),v\right\rangle + 2, \left\langle\left(\begin{smallmatrix}-2\\2\end{smallmatrix}\right),v\right\rangle \end{array}\right\}.
\end{aligned}$$

We have $\underline{h}(n_{\rho_i}) = -2$ for $1 \leq i \leq 6$, providing us with the weight polytope $\Box_h$ which is pictured in Figure 2.4(b).

The following list displays the induced divisor $h^*(u)$ on $\mathbb{P}^1$ for every weight $u = (u_1, u_2) \in \Box_h$, where a triple (a,b,c) corresponds to the $\mathbb{Q}$-Cartier divisor $D(a,b,c) = a[0] + b[\infty] + c[1]$.

$(0,0)$	$\mapsto$	$(2,0,0)$	$(0,-1)$	$\mapsto$	$(1,0,0)$	$(1,1)$	$\mapsto$	$(2,-2,0)$
$(1,0)$	$\mapsto$	$(2,-1,0)$	$(0,-2)$	$\mapsto$	$(0,0,0)$	$(1,-1)$	$\mapsto$	$(1,0,0)$
$(2,0)$	$\mapsto$	$(2,-2,0)$	$(-1,1)$	$\mapsto$	$(2,0,-1)$	$(2,-1)$	$\mapsto$	$(1,-1,0)$
$(-1,0)$	$\mapsto$	$(2,0,-1)$	$(-2,1)$	$\mapsto$	$(2,0,-2)$	$(2,-2)$	$\mapsto$	$(0,0,0)$
$(-2,0)$	$\mapsto$	$(2,0,-2)$	$(-2,2)$	$\mapsto$	$(2,0,-2)$	$(1,-2)$	$\mapsto$	$(0,0,0)$
$(0,1)$	$\mapsto$	$(2,-1,0)$	$(-1,2)$	$\mapsto$	$(2,-1,-1)$			
$(0,2)$	$\mapsto$	$(2,-2,0)$	$(-1,-1)$	$\mapsto$	$(1,0,-1)$			

Summing up the dimensions of the vector spaces $\Gamma\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(D(a,b,c))\big)$ over all degrees yields $\dim \Gamma(X, -K_X) = 27$.

2.6 Outlook

Description of Cartier Divisors in Higher Complexity Hendrik Süß showed that the description of invariant Weil divisors (cf. Proposition 2.1) works equally well in higher complexity, see [PS, Proposition 3.13]. However, the description of invariant Cartier divisors in the general setting is still missing.

In complexity one, the marked fansy divisor $\Xi(\mathcal{S})$ yields a "standard covering" of $\mathrm{TV}(\mathcal{S})$. This provides us with easy criteria to decide whether a given support function is Cartier or not. In particular, open subsets of the base curve are either affine or complete and, more importantly, different prime divisors (i.e. points) on the base curve Y always have empty intersection.

This, of course, is no longer true in higher complexity. Although it is still possible and not hard to associate continuous piecewise affine linear functions over prime divisor slices $\mathcal{S}_D$ with a given invariant Cartier divisor on $\mathrm{TV}(\mathcal{S})$, the reverse procedure is much more difficult. Indeed, it is no longer sufficient to provide for continuous piecewise affine linear functions on the slices $\mathcal{S}_D$, since one also has to check/guarantee compatibility on intersections of different prime divisors to produce local equations on a covering of Y such that the latter glue together. So far, we do not have an effective method to encode these data.

Higher Cohomology Groups of Equivariant Line Bundles Let $\mathcal{O}_X(D_h)$ be a nef line bundle on the not necessarily toroidal complexity-one T-variety X. In vein of Proposition 2.35, it would be nice to have an analogous vanishing result even for this more general case.

One probably very unorthodox way to approach this problem would be to read the Leray spectral sequence which is associated to the map $r : \widetilde{X} \to X$ (see below) from right to left, instead of reading it from left to right. Namely, since we have a vanishing result for the nef line bundle $r^*\mathcal{O}_X(D_h)$ on $\widetilde{X}$ and $r_*r^*\mathcal{O}_X(D_h) \cong \mathcal{O}_X(D_h)$, we can try to use the convergence

$$H^p(X, R^q r_*(r^*\mathcal{O}_X(D_h))) \Longrightarrow H^{p+q}(\widetilde{X}, r^*\mathcal{O}_X(D_h))$$

to obtain more information about the $H^i(X, \mathcal{O}_X(D_h))$. Indeed, one could hope for the vanishing of the higher direct image sheaves $R^i r_*(r^*\mathcal{O}_X(D_h))$ for $i \geq 2$. Then, the associated initial diagram of the spectral sequence would reduce to two columns and might facilitate computations.

Conjecture 2.39. Let X be a complete complexity-one T-variety. For any nef T-invariant Cartier divisor D_h on X we have that $H^i(X, \mathcal{O}_X(D_h)) = 0$ for $i \geq 2$.

Another and maybe more conceptual approach for general cohomology group computations in this setting might consist in a translation of the Čech complex for a fixed degree $u \in M$ into some sort of "combinatorial structure" on the underlying curve Y, cf. Section 2.4.1 for the topological counterpart in toric geometry. Yet, we believe that it is only the rational case for which such an approach might be successful.

Chapter 3

Cox Rings

First, we briefly recall fundamental notions and some results of the theory of Cox rings which are essential for the following discussions. We then proceed to the construction of a polyhedral divisor on $\mathbb{P}^1$ which corresponds to the Cox ring of a complexity-one T-Mori dream space with free divisor class group. After studying some examples, we lastly provide a comparison with an Ansatz of Klaus Altmann and Jarek Wiśniewski.

3.1 General Setup

In his classical paper [Cox95], David Cox associated a so-called multigraded homogeneous coordinate ring, which is also known as the Cox ring, to a non-degenerate (e.g. complete) normal toric variety $\mathrm{TV}(\Sigma)$. In the following, we will shortly recall a natural extension of his construction to a wider class of varieties. For much more details on this subject the reader is referred to [Hau08, BH03].

Let X be a normal variety with only constant invertible global functions, i.e. $\Gamma(X, \mathcal{O}_X^*) = \mathbb{K}^*$, and a finitely generated divisor class group $\mathrm{Cl}(X)$. Then one can define a $\mathrm{Cl}(X)$-graded abelian group

$$\mathrm{Cox}(X) \ := \bigoplus_{D \in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).$$

Moreover, $\mathrm{Cox}(X)$ carries a ring structure which is canonical up to isomorphism. If $\mathrm{Cl}(X)$ is torsion free, it may be fixed by choosing a section $\mathrm{Cl}(X) \hookrightarrow \mathrm{Div}(X)$ of the natural surjection $\mathrm{Div}(X) \twoheadrightarrow \mathrm{Cl}(X)$. Instead, if $\mathrm{Cl}(X)$ has torsion, one has to make use of a finite presentation of $\mathrm{Cl}(X)$ by a finitely generated subgroup of $\mathrm{Div}(X)$ with relations. For a short but more detailed treatment of the latter case see [HS, Section 2].

The $\mathbb{K}$-algebra $\mathrm{Cox}(X)$ described above is referred to as the *total coordinate ring* or the *Cox ring* of X. From now on, we will mostly restrict to complete varieties with freely generated divisor class group and finitely generated Cox ring. Although we neither suppose X to be $\mathbb{Q}$-factorial nor projective, we still make use of the terminology of [HK00] and thus call X a Mori dream space (MDS). Two important consequences of this property are the following, cf. loc. cit.

- Nef divisors are automatically semi-ample.
- The data of the minimal model program are finite: Not only are the ample, movable, and effective cones

$$\mathrm{Nef}(X) \subseteq \mathrm{Mov}(X) \subseteq \mathrm{Eff}(X) \subseteq N^1_{\mathbb{R}}(X)$$

inside the Néron-Severi group of X polyhedral, but $\mathrm{Eff}(X)$ also carries a finite polyhedral subdivision such that the birational transformations X_i of X correspond to the cells of this subdivision. Actually, the X_i appear as GIT quotients of the total coordinate space $\mathrm{Spec}\,\mathrm{Cox}(X)$, and the polyhedral subdivision of $\mathrm{Eff}(X)$ corresponds to the GIT equivalence classes.

3.2 The Cox Ring of a Complexity-One T-Variety

Probably the most fundamental problem concerning the Cox ring of a variety is to present it in terms of generators and relations. Far reaching results in this direction were obtained by Jürgen Hausen and Hendrik Süß for the class of T-varieties (see [HS]). We will use this section to review some of those which are explicitly stated for complete rational complexity-one T-varieties $\mathrm{TV}(\mathcal{S})$.

The crucial idea is to relate the total coordinate ring of $\mathrm{TV}(\mathcal{S})$ to the Cox ring of the geometric quotient $X_0 \to X_0/T$, where $X_0 \subset \mathrm{TV}(\mathcal{S})$ denotes the non-empty T-invariant open subset of points $x \in \mathrm{TV}(\mathcal{S})$ with finite isotropy group T_x. Note that this quotient $\overline{Y} := X_0/T$ is irreducible and normal, but not necessarily separated. Nevertheless, one can define a Cox ring for $\overline{Y}$ and its separation $\overline{Y}_{\mathrm{sep}}$ which, in our case ($\mathrm{TV}(\mathcal{S})$ is rational and of complexity one), is equal to the so-called Chow quotient of $\mathrm{TV}(\mathcal{S})$, i.e. $X_0/T = \mathbb{P}^1$.

We have to introduce some notation. Consider a point $P \in \mathbb{P}^1$ together with the set $\mathcal{V}_P := \{D_{(P,v)} \mid v \in \mathcal{S}_P(0)\}$ of all vertical divisors lying over P, and define the tuple $\mu(P) := \big(\mu(v) \mid v \in \mathcal{S}_P(0)\big)$. The set of all *exceptional* points, i.e. points $P \in \mathbb{P}^1$ for which $\mu(P) \neq (1)$, is denoted by $\widetilde{\mathcal{P}}$. By definition, $\widetilde{\mathcal{P}}$ is a subset of $\mathcal{P}$ (cf. Definition 1.17). Moreover, let $1_P \in \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(P))$ denote the canonical section of the divisor P and recall that $\mathcal{R}$ denotes the set of extremal rays in $\mathrm{tail}\,\mathcal{S}$ (cf. see Definition 1.18). With this notation, Theorem 1.2 from [HS] says that

$$\mathrm{Cox}(\mathrm{TV}(\mathcal{S})) \cong \mathrm{Cox}(\mathbb{P}^1)\big[T_{D_{(P,v)}}, S_{D_\rho} \mid P \in \widetilde{\mathcal{P}}, \rho \in \mathcal{R}\big] \Big/ \Big\langle \prod_{\mathcal{V}_P} T^{\mu(v)}_{D_{(P,v)}} - 1_P \Big\rangle,$$

where the $\mathrm{Cl}(\mathrm{TV}(\mathcal{S}))$-grading on the right hand side is defined by

$$\deg T_{D_{(P,v)}} = [D_{(P,v)}], \quad \deg S_{D_\rho} = [D_\rho].$$

In addition, the ideal which gives rise to the quotient representation is homogeneous with respect to this grading. To get an even more explicit description of $\mathrm{Cox}(\mathrm{TV}(\mathcal{S}))$, we enumerate the elements of $\widetilde{\mathcal{P}} = \{P_0, \dots, P_r\}$ and choose a representation $\widetilde{a}_i \in \mathbb{K}^2$ for each of them. Denoting a basis of the relations among the homogeneous coordinates by $\mathrm{Rel}(\widetilde{a}_0, \dots, \widetilde{a}_r)$ finally yields

$$\mathrm{Cox}(X) \cong \mathbb{K}\big[T_{D_{(P,v)}}, S_{D_\rho} \mid P \in \widetilde{\mathcal{P}}, \rho \in \mathcal{R}\big] \Big/ \Big\langle \sum_{i=0}^{r} \beta_i \Big(\prod_{v \in \mathcal{V}_{P_i}} T^{\mu(v)}_{D_{(P_i,v)}} \Big) \Big\rangle$$

with $\beta = (\beta_0, \ldots, \beta_r) \in \operatorname{Rel}(\widetilde{a}_0, \ldots, \widetilde{a}_r)$.

Example 3.1. We return to our well-known threefold $\mathbb{P}(\Omega_{\mathbb{P}^2})$, see Example 1.28 or [HS, Example 4.4]. Recall that $\mathcal{R} = \emptyset$. There is exactly one relation among the points $\widetilde{a}_0 = (1, 0)$, $\widetilde{a}_1 = (1, 1)$, $\widetilde{a}_\infty = (0, 1) \in \mathbb{K}^2$, namely $\beta = (1, -1, 1)$. Since all vertices $v_1, v_2 \in \mathcal{S}_0(0)$, $v_3, v_4 \in \mathcal{S}_1(0)$, and $v_5, v_6 \in \mathcal{S}_\infty(0)$ are lattice points, we conclude that

$$\operatorname{Cox}(\mathbb{P}(\Omega_{\mathbb{P}^2})) \;=\; \mathbb{K}[T_1, \ldots, T_6]\big/(T_1T_2 - T_3T_4 + T_5T_6).$$

3.3 The Cox Ring as a Polyhedral Divisor

In contrast to the previous section we would like to approach the Cox ring of a complexity-one T-MDS X via the language of polyhedral divisors. The motivation for doing so is that this description allows for a detailed study of torus orbits and deformations of $\operatorname{Cox}(X)$.

The combinatorial ingredients which will show up in the construction of the polyhedral divisor in question can already be seen in the purely toric setting.

3.3.1 A Motivation from Toric Geometry

It was shown in [Cox95] that the total coordinate ring $\operatorname{Cox}(\mathrm{TV}(\Sigma))$ of a non-degenerate toric variety $\mathrm{TV}(\Sigma)$ is a polynomial ring whose variables correspond to the rays of Σ. Vice versa, a normal variety with only constant globally invertible regular functions, whose Cox ring is a polynomial ring, is toric.

While this describes $\operatorname{Cox}(\mathrm{TV}(\Sigma))$ completely in algebraic terms, we now come to a more polyhedral point of view. As before, we denote the first non-trivial lattice point on a ray $\rho \in \Sigma(1)$ by n_ρ. Then we consider the canonical map $\varphi : \mathbb{Z}^{\Sigma(1)} \to N$, $e_\rho \mapsto n_\rho$. It sends some faces (including the rays) of the positive orthant $\mathbb{Q}^{\Sigma(1)}_{\geq 0}$ to cones of the fan Σ. Applying the functor TV, we obtain a rational map $\operatorname{Spec} \mathbb{C}[z_\rho \,|\; \rho \in \Sigma(1)] \dashrightarrow \mathrm{TV}(\Sigma)$. In particular, we recover the affine spectrum of $\mathbb{C}[z_\rho \,|\; \rho \in \Sigma(1)] = \operatorname{Cox}(\mathrm{TV}(\Sigma))$ as the toric variety $\mathrm{TV}(\mathbb{Q}^{\Sigma(1)}_{\geq 0})$. Thus, the Cox ring of a toric variety gives rise to an affine toric variety itself, and the defining cone $\mathbb{Q}^{\Sigma(1)}_{\geq 0}$ can be seen as a polyhedral resolution of the given fan Σ, since all linear relations among the rays have been removed.

3.3.2 Combining Torus Actions

Since we suppose X to be an MDS with torsion free divisor class group $\operatorname{Cl}(X)$, we see that $\operatorname{Spec}\operatorname{Cox}(X)$ is a normal affine variety with an effective action of the so-called Picard torus $\operatorname{Hom}_{\mathbb{Z}}(\operatorname{Cl}(X), \mathbb{K}^*)$ where the latter is encoded by the $\operatorname{Cl}(X)$-grading. Thus, one could ask for a description of $\operatorname{Spec}\operatorname{Cox}(X)$ in terms of a polyhedral divisor on some Y, which was done by Klaus Altmann and Jarek Wiśniewski in [AW]. As X itself already comes with an effective torus action which then is inherited by $\operatorname{Cox}(X)$, we may combine it with the action of the Picard torus such that $\operatorname{Spec}\operatorname{Cox}(X)$ turns into a complexity-one T-variety, too.

Our goal now is to present $\operatorname{Cox}(X)$ as a polyhedral divisor $\mathcal{D}_{\mathrm{Cox}}$ on $\mathbb{P}^1$. And it turns out that the construction of $\mathcal{D}_{\mathrm{Cox}}$ is very much in the vein of the polyhedral resolution of a fan Σ as described in Section 3.3.1.

3.3.3 The Construction of $\mathcal{D}_{\text{Cox}}$

Let $\mathrm{TV}(\mathcal{S})$ be a complete complexity-one T-MDS with free abelian class group $\mathrm{Cl}(\mathrm{TV}(\mathcal{S}))$. Hence, $\mathcal{S} = \sum_{P\in\mathbb{P}^1} \mathcal{S}_P \otimes [P]$ is a complete divisorial fan on $Y = \mathbb{P}^1$. In particular, $\deg \mathcal{S} \subsetneq |\operatorname{tail}\mathcal{S}| = N_{\mathbb{Q}}$. We choose a finite non-empty set of points $\mathcal{P} \subseteq \mathbb{P}^1$, such that for all $P \in \mathbb{P}^1 \setminus \mathcal{P}$ the slice $\mathcal{S}_P$ is trivial, i.e. $\mathcal{S}_P = \operatorname{tail}\mathcal{S}$. Note that the only difference between $\mathcal{P}$ and the set we denoted by the very same letter in Definition 1.17 is the technical requirement to be non-empty.

For a vertex v of some slice of $\mathcal{S}$ we denote by $P(v) \in \mathbb{P}^1$ the point whose slice we have taken v from. The well known sets (cf. Definitions 1.17 and 1.18)

$$\mathcal{V} \;:=\; \{v \in \mathcal{S}_P(0) \mid P \in \mathcal{P}\} \quad \text{and} \quad \mathcal{R} \;:=\; \{\rho \in (\operatorname{tail}\mathcal{S})(1) \mid \rho \cap \deg\mathcal{S} = \emptyset\}$$

then lead to the definition of the following two natural maps

$$Q : \mathbb{Z}^{\mathcal{V}\cup\mathcal{R}} \to \mathbb{Z}^{\mathcal{P}}/\mathbb{Z} \quad \text{with} \quad e_v \mapsto \mu(v)\,\overline{e}_{P(v)} \quad \text{and} \quad e_\rho \mapsto 0\,,$$

and

$$\phi : \mathbb{Z}^{\mathcal{V}\cup\mathcal{R}} \to N \quad \text{with} \quad e_v \mapsto \mu(v)v \quad \text{and} \quad e_\rho \mapsto n_\rho.$$

Here, e_v and e_ρ denote elements of the natural basis of the lattice $\mathbb{Z}^{\mathcal{V}\cup\mathcal{R}}$. Furthermore, $\mathbb{Z}^{\mathcal{P}}/\mathbb{Z}$ is supposed to arise from the lattice $\mathbb{Z}^{\mathcal{P}}$ by imposing the relation $\sum_{P\in\mathcal{P}} e_P = 0$.

The description of the divisor class group $\mathrm{Cl}(\mathrm{TV}(\mathcal{S}))$ (cf. Corollary 2.4) now becomes even simpler since we can use the fact that $(\mathbb{Z}^{\mathcal{V}\cup\mathcal{R}})^\vee \subseteq \text{T-Div}\big(\mathrm{TV}(\mathcal{S})\big)$ (cf. Section 2.1). Indeed, we obtain an exact sequence which is analogous to the well known one from toric geometry.

Corollary 3.2. Let $\mathrm{TV}(\mathcal{S})$ be as above. Then one has a short exact sequence

$$0 \to (\mathbb{Z}^{\mathcal{P}}/\mathbb{Z})^\vee \oplus M \to (\mathbb{Z}^{\mathcal{V}\cup\mathcal{R}})^\vee \to \mathrm{Cl}(\mathrm{TV}(\mathcal{S})) \to 0$$

where the first map is induced from (Q,ϕ).

Proof. If we dealt with the whole projective line $\mathbb{P}^1$ instead of the finite subset $\mathcal{P}$, then $(\mathbb{Z}^{\mathbb{P}^1}/\mathbb{Z})^\vee$ would represent the principal divisors on $\mathbb{P}^1$, and the formula $\operatorname{div}\big(f\chi^u\big) = (Q,\phi)^\vee\big(\operatorname{div}(f), u\big)$ of Proposition 2.3 would provide the exactness of the sequence. However, for $P \in \mathbb{P}^1 \setminus \mathcal{P}$, we have $\mathcal{S}_P = \operatorname{tail}\mathcal{S}$, i.e. the corresponding $\mathbb{Z}$-summands of the first and second place cancel each other. $\square$

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & & \mathrm{Cl}(\mathrm{TV}(\mathcal{S})) \arrow[u] \arrow[r, equal] & \mathrm{Cl}(\mathrm{TV}(\mathcal{S})) \arrow[u] & \\
0 \arrow[r] & (\mathbb{Z}^{\mathcal{P}}/\mathbb{Z})^\vee \arrow[r, "Q^\vee"] & (\mathbb{Z}^{\mathcal{V}\cup\mathcal{R}})^\vee \arrow[u, "\Psi"] \arrow[r, "R^\vee"] & \widetilde{M} \arrow[u, "\alpha"'] \arrow[r] \arrow[l, bend left, "t^\vee"] & 0 \\
0 \arrow[r] & (\mathbb{Z}^{\mathcal{P}}/\mathbb{Z})^\vee \arrow[u, equal] \arrow[r] & (\mathbb{Z}^{\mathcal{P}}/\mathbb{Z})^\vee \oplus M \arrow[u, "(Q^\vee,\phi^\vee)"] \arrow[r] & M \arrow[u, "\beta"'] \arrow[r] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{3.1}$$

Note that the map Q is surjective since $\mathrm{Cl}(\mathrm{TV}(\mathcal{S}))$ is freely generated. We denote its kernel by $\widetilde{N}$ and fix a cosection $t : \mathbb{Z}^{\mathcal{V}\cup\mathcal{R}} \to \widetilde{N}$, i.e.

$$0 \longrightarrow \widetilde{N} \underset{t}{\overset{R}{\rightleftarrows}} \mathbb{Z}^{\mathcal{V}\cup\mathcal{R}} \xrightarrow{Q} \mathbb{Z}^{\mathcal{P}}/\mathbb{Z} \longrightarrow 0 \ .$$

Dualizing this exact sequence and combining it with the presentation of the lattice $\mathrm{Cl}(\mathrm{TV}(\mathcal{S}))$, we can visualize the whole picture in a single diagram, see (3.1). Note that the lower horizontal exact sequence splits trivially. The map β in the right hand column is defined as $\beta = R^\vee \circ \phi^\vee$. In addition, Ψ factorizes over $R^\vee$ and thus yields the map $\alpha : \widetilde{M} \to \mathrm{Cl}(\mathrm{TV}(\mathcal{S}))$. The $\widetilde{M}$-grading now displays the combined action of T and the Picard torus on the total coordinate space $\operatorname{Spec}\operatorname{Cox}(\mathrm{TV}(\mathcal{S}))$.

Lemma 3.3. The diagram from above is commutative. Its rows and columns are exact, and the maps α and β are well defined.

Proof. All we have to show is that the right hand column is exact and that the map α is well-defined. But the latter statement is clear, since preimages of $R^\vee$ only differ by images of $Q^\vee$ which itself are annihilated by Ψ. Clearly, $\alpha \circ \beta = 0$. Furthermore, α is surjective since for any element $\delta \in \mathrm{Cl}(\mathrm{TV}(\mathcal{S}))$ we conclude that $\alpha(R^\vee(\gamma)) = \delta$, where $\gamma \in (\mathbb{Z}^{\mathcal{V}\cup\mathcal{R}})^\vee$ is an arbitrary preimage of δ under the map Ψ. It remains to show that β is injective.

First, we note that $\phi^\vee$ is injective, because $\phi_{\mathbb{Q}}$ is surjective. Indeed, since $\deg(\mathcal{D}) = \sum_P \operatorname{conv} \mathcal{D}_P(0) + \operatorname{tail}(\mathcal{D}) \subsetneq \operatorname{tail}(\mathcal{D})$ for a single $\mathcal{D} \in \mathcal{S}$, every ray $\rho \in \operatorname{tail}(\mathcal{D})(1)$ either belongs to $\mathcal{R}$ (meaning that $\rho \cap \deg \mathcal{D} = \emptyset$), or ρ intersects $\sum_P \operatorname{conv} \mathcal{D}_P(0)$. Thus, every ray $\rho \in \operatorname{tail}(\mathcal{S})(1)$ either belongs to $\mathcal{R}$ or it intersects $\sum_P \operatorname{conv} \mathcal{S}_P(0)$ away from the origin. This means that non-zero elements of each ray of the tailfan occur in the image of the map $\phi_{\mathbb{Q}} : Q^{-1}(0) \to N_{\mathbb{Q}}$, i.e. it is surjective. Secondly, we have that $\operatorname{im}\phi^\vee \cap \operatorname{im} Q^\vee = \{0\}$. Hence, β is injective. ☐

We can now define the main polyhedral objects for the construction of $\mathcal{D}_{\mathrm{Cox}}$, namely the polytopes

$$\Delta_P^c \ := \ \operatorname{conv}\{e_v/\mu(v) \mid v \in \mathcal{S}_P(0)\} \subseteq Q^{-1}(\overline{e}_P) \subseteq \mathbb{Q}^{\mathcal{V}\cup\mathcal{R}},$$

and the polyhedral cone

$$\sigma \ := \ Q_{\mathbb{Q}}^{-1}(0) \cap \mathbb{Q}_{\geq 0}^{\mathcal{V}\cup\mathcal{R}} = \mathbb{Q}_{\geq 0} \cdot \prod_{P \in \mathcal{P}} \Delta_P^c \ + \ \mathbb{Q}_{\geq 0}^{\mathcal{R}}.$$

From these data we construct

$$\Delta_P \ \ := \ \ t\big(\Delta_P^c + \sigma\big) \ = \ t\big(Q_{\mathbb{Q}}^{-1}(\overline{e}_P) \cap \mathbb{Q}_{\geq 0}^{\mathcal{V}\cup\mathcal{R}}\big) \subseteq \widetilde{N}_{\mathbb{Q}}\,.$$

Theorem 3.4. Let $\mathrm{TV}(\mathcal{S})$ be a as above. The following polyhedral divisor

$$\mathcal{D}_{\mathrm{Cox}} \ := \ \sum_{P \in \mathbb{P}^1} \Delta_P \otimes [P]$$

on $\mathbb{P}^1$ with tailfan $t(\sigma)$ then corresponds to $\operatorname{Cox}(\mathrm{TV}(\mathcal{S}))$. In other words, we have that $A(\mathcal{D}_{\mathrm{Cox}}) \cong \operatorname{Cox}(\mathrm{TV}(\mathcal{S}))$ as $\mathrm{Cl}(\mathrm{TV}(\mathcal{S}))$-graded $\mathbb{K}$-algebras.

3.3.4 Proof of Theorem 3.4

Let $\delta \in \mathrm{Cl}(\mathrm{TV}(\mathcal{S}))$ be a divisor class, and denote by D a T-invariant representative. For the following, recall that

$$\operatorname{coeff}(D)^*(u) = \sum_P \min_{\mathrm{v}\in\mathcal{S}_\mathrm{P}} \left(\langle\, u, \cdot \,\rangle + \frac{\operatorname{coeff}_{D_{(P,\cdot)}} D}{\mu(\cdot)} \right) P\,,$$

see Definition 2.23. We claim that

$$\operatorname{coeff}(D)^*_P(w) = \min \langle Q_{\mathbb{Q}}^{-1}(\overline{e}_P) \cap \mathbb{Q}^{\mathcal{V}\cup\mathcal{R}}_{\geq 0}, D + \phi^\vee(w) \rangle\,.$$

To prove this, we first show that we have the following equality of sets

$$\left\{ \langle\, w, v \,\rangle + \tfrac{\operatorname{coeff}(D)_P(v)}{\mu(v)} \,\middle|\, v \in \mathcal{S}_P \right\}$$
$$\|$$
$$\left\{ \langle\, \widetilde{v}, D + \phi^\vee(w) \,\rangle \,\middle|\, \widetilde{v} \text{ vertex of } Q_{\mathbb{Q}}^{-1}(\overline{e}_P) \cap \mathbb{Q}^{\mathcal{V}\cup\mathcal{R}}_{\geq 0} \right\},$$

since there is a bijection between the vertices of $Q_{\mathbb{Q}}^{-1}(\overline{e}_P) \cap \mathbb{Q}^{\mathcal{V}\cup\mathcal{R}}_{\geq 0}$ and those of $\mathcal{S}_P$. Namely, $v \in \mathcal{S}_P$ corresponds to the vertex $\frac{e_v}{\mu(v)} \in Q_{\mathbb{Q}}^{-1}(\overline{e}_P) \cap \mathbb{Q}^{\mathcal{V}\cup\mathcal{R}}_{\geq 0}$. This correspondence establishes the equality via the identity

$$\left\langle \frac{e_v}{\mu(v)}\,, D + \phi^\vee(w) \right\rangle = \frac{\operatorname{coeff}(D)_P(v)}{\mu(v)} + \langle\, w, v \,\rangle\,.$$

Now, images of $Q^\vee$ correspond to principal divisors on $\mathrm{TV}(\mathcal{S})$ coming from $\mathbb{P}^1$. Hence, the vector space of their global sections is one-dimensional and concentrated in degree zero. Consider an arbitrary T-invariant Weil divisor D on $\mathrm{TV}(\mathcal{S})$. As

$$-D - \phi^\vee(w) + t^\vee(R^\vee(D + \phi^\vee(w))) =: Q^\vee(\Gamma^D_w)) \in \operatorname{im} Q^\vee$$

we can define the T-invariant Weil divisor $D + Q^\vee(\Gamma^D_w)$ which is linear equivalent to D.
Next, we want to establish an isomorphism of the following graded rings

$$\operatorname{Cox}(\mathrm{TV}(\mathcal{S})) = \bigoplus_{D\in\mathrm{Cl}(\mathrm{TV}(\mathcal{S}))} \Gamma(\mathrm{TV}(\mathcal{S}), \mathcal{O}(D)) \quad \text{and} \quad A(\mathcal{D}) = \bigoplus_{u\in\widetilde{M}} \Gamma(\mathbb{P}^1, \mathcal{D}(u))\,.$$

So let $\{E_1, \ldots, E_k\}$ denote an ordered basis of $\mathrm{Cl}(\mathrm{TV}(\mathcal{S}))$, and fix a non-zero section $\sigma^{E_i}_w \in \Gamma(\mathrm{TV}(\mathcal{S}), \mathcal{O}(Q^\vee(\Gamma^{E_i}_w)))$ for every element E_i of the basis and each weight $w \in M$. Recall that there is exactly one up to scalars. We can then extend this choice by linearity to every element D in $\mathrm{Cl}(\mathrm{TV}(\mathcal{S}))$.

Hence, we have an isomorphism of vector spaces

$$\Gamma(\mathrm{TV}(\mathcal{S}), \mathcal{O}(D))_w \xrightarrow{\cdot\sigma^D_w} \Gamma\big(\,\mathrm{TV}(\mathcal{S}), \mathcal{O}(D + Q^\vee(\Gamma^D_w))\big)_w\,.$$

But the right hand side is equal to $\Gamma(\mathbb{P}^1, \mathcal{D}(\beta(w) + R^\vee(D)))$, since

$$\begin{aligned}
& \operatorname{coeff}(D + Q^\vee(\Gamma^D_w))^*(w) \\
= \;& \textstyle\sum_{P\in Y} \min \langle\, Q_{\mathbb{Q}}^{-1}(\overline{e}_P) \cap \mathbb{Q}^{\mathcal{V}\cup\mathcal{R}}_{\geq 0}, t^\vee(R^\vee(D + \phi^\vee(w))) \,\rangle P \\
= \;& \textstyle\sum_{P\in Y} \min \langle\, t_{\mathbb{Q}}(Q_{\mathbb{Q}}^{-1}(\overline{e}_P) \cap \mathbb{Q}^{\mathcal{V}\cup\mathcal{R}}_{\geq 0}), R^\vee(D + \phi^\vee(w)) \,\rangle P \\
= \;& \textstyle\sum_{P\in Y} \min \langle\, \Delta_P, R^\vee(D) + \beta(w) \,\rangle P \\
= \;& \mathcal{D}(R^\vee(D) + \beta(w))\,.
\end{aligned}$$

The following commutative diagram then settles the compatibility of multiplicative structures and completes the proof.

$$\begin{tikzcd}
s \otimes t \arrow[r, mapsto] & s \cdot t \\
\Gamma(D_1)_{w_1} \otimes \Gamma(D_2)_{w_2} \arrow[r] \arrow[d, "\cdot \sigma_{w_1}^{D_1}"', "\cdot \sigma_{w_2}^{D_2}"] & \Gamma(D_1 + D_2)_{w_1+w_2} \arrow[d, "\cdot \sigma_{w_1+w_2}^{D_1+D_2}"] \\
\begin{matrix} \Gamma\big(\mathcal{D}(\beta(w_1) + R^\vee(D_1))\big) \\ \otimes \\ \Gamma\big(\mathcal{D}(\beta(w_2) + R^\vee(D_2))\big) \end{matrix} \arrow[r] & \Gamma\big(\mathcal{D}(\beta(w_1 + w_2) + R^\vee(D_1 + D_2))\big) \\
s\sigma_{w_1}^{D_1} \otimes t\sigma_{w_2}^{D_2} \arrow[r, mapsto] & st\sigma_{w_1}^{D_1}\sigma_{w_2}^{D_2} = st\sigma_{w_1+w_2}^{D_1+D_2}
\end{tikzcd}$$

Remark 3.5. There are complexity-one T-MDS such that the map Q is surjective although $\mathrm{Cl}(\mathrm{TV}(\mathcal{S}))$ is not torsion-free. As can be seen from the construction and the proof, the theorem remains true in this case. An example will be given in Section 3.4.

The general case in which $\mathrm{Cl}(\mathrm{TV}(\mathcal{S}))$ is not assumed to be freely generated is presented in [AP]. It involves the same polyhedral constructions, but the polyhedral divisor $\mathcal{D}_{\mathrm{Cox}}$ lives on a suitable finite covering of $\mathbb{P}^1$ which is induced by the finite cokernel of the map Q.

3.4 Examples

Projectivized Cotangent Bundles on Toric Surfaces A very prominent class of complexity-one T-MDS is given by projectivizations of rank two toric vector bundles over smooth projective toric varieties. For a detailed (algebraic) description of their Cox rings cf. [HS, Gonb].

From the description of the polyhedral fan associated to the projectivized cotangent bundle of a toric surface (see (1.2.3)) we deduce that Δ_P^c is an integer polytope, namely either a 1-simplex or a 2-simplex.

Example 3.6. We return to the projectivized cotangent bundle on $\mathbb{F}_1$, cf. Example 1.29. Its divisor class group $\mathrm{Cl}\left(\mathrm{TV}(\mathcal{S})\right)$ is free abelian of rank three and

$$Q = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 \end{pmatrix}.$$

The columns of the following matrix display the coordinates of the primitive generators of the twelve rays of the tailcone for a special choice of the cosection t:

$$\begin{pmatrix} 0 & -1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Combinatorially, it is the cone over the product of a quadrangle (product of two intervals) and a triangle. This realization is induced by the compact parts of the following polyhedral coefficients:

$$\begin{aligned}\Delta_0 &= \operatorname{conv}\{(0,0,0,0,0),(0,1,-1,0,0),(0,0,-1,1,0)\}+t(\sigma),\\ \Delta_1 &= \operatorname{conv}\{(0,0,0,0,0),(1,0,0,0,0)\}+t(\sigma),\\ \Delta_\infty &= \operatorname{conv}\{(-1,0,1,0,0),(0,0,0,0,1)\}+t(\sigma).\end{aligned}$$

Thus, we have that

$$\mathcal{D}_{\mathrm{Cox}} = \Delta_0 \otimes [0] + \Delta_1 \otimes [1] + \Delta_\infty \otimes [\infty] .$$

Example 3.7. We resume our discussion of the projectivized cotangent bundle on $\mathbb{P}^2$ from Example 1.28. Considering the non-trivial slices $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{S}_\infty$, we see that the three polytopes Δ_0^c, Δ_1^c and Δ_∞^c are compact edges. Furthermore, σ becomes a four-dimensional cone over a cube. Hence, the resulting polyhedral divisor is that of the affine cone over Grass$(2,4)$ from [AH08, p. 849].

Using the degeneration techniques developed in [Ilt10, IV], we construct a toric degeneration of $\mathrm{TV}(\mathcal{S}) = \mathbb{P}(\Omega_{\mathbb{P}^2})$ to the projective cone over the del Pezzo surface of degree six which we denote by $\mathrm{TV}(\mathcal{S}')$, see also [Süß, Example 5.1]. The divisorial fan $\mathcal{S}'$ is again defined over $\mathbb{P}^1$ and $\operatorname{tail}\mathcal{S}' = \operatorname{tail}\mathcal{S}$. Moreover, the marking is the same as for $\mathcal{S}$, namely $(\operatorname{tail}\mathcal{S})(1) \cup (\operatorname{tail}\mathcal{S})(2)$. The relevant slices are pictured in Figure 3.1. Note that the only non-trivial polyhedral subdivision $\mathcal{S}_0'$ is equal to the one which is induced by $\deg\mathcal{S}$. This means that for a given unbounded maximal polyhedron $\mathcal{D}_0' \subset \mathcal{S}_0'$ with $\operatorname{tail}\mathcal{D}_0' = \sigma$ we have that

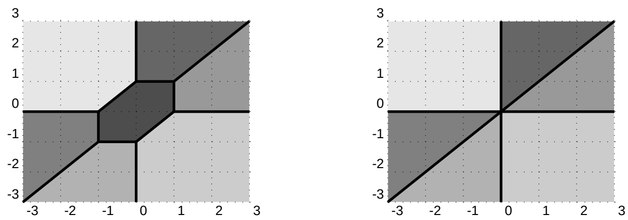
$$\mathcal{D}_0' = \mathcal{D}_0^\sigma + \mathcal{D}_\infty^\sigma + \mathcal{D}_1^\sigma .$$

Since $\mathrm{TV}(\mathcal{S}')$ is toric we know that its Cox ring is a polynomial ring. Applying our recipe, we can see from Figure 3.1 that the compact part $(\Delta_0')^c$ is a five-dimensional simplex.

Performing the analogous degeneration on the level of Cox rings, i.e. adding up all polyhedral coefficients of the polyhedral divisor described above, gives us a (toric) $\mathbb{K}$-algebra which is not a polynomial ring. Observe that the compact part of the only non-trivial polyhedral coefficient is the Minkowski sum of three edges, i.e. a three-dimensional cube.

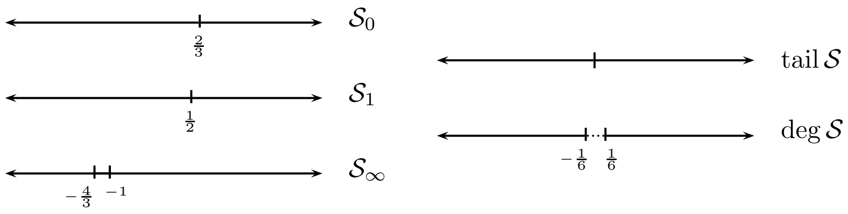
Further Examples

Example 3.8. Let $\mathrm{TV}(\mathcal{S})$ be the Gorenstein del Pezzo $\mathbb{C}^*$-surface of degree 3 with singularity type E_6. It has two elliptic fixed points, i.e. $\mathcal{R} = \emptyset$. The divisorial fan $\mathcal{S}$ is illustrated in Figure 3.2.



(a) $\mathcal{S}_0'$ (b) $\mathcal{S}_\infty'$

Figure 3.1: Divisorial fan of a toric degeneration of $\mathbb{P}(\Omega_{\mathbb{P}^2})$, cf. Example 3.7.



(a) Non-trivial slices of $\mathcal{S}$.　(b) Tailfan and degree.

Figure 3.2: Divisorial fan of a Gorenstein log del Pezzo $\mathbb{C}^*$-surface of singularity type E_6.

The divisor class group $\operatorname{Cl}\big(\operatorname{TV}(\mathcal{S})\big)$ is torsion free of rank one. Using the linear map

$$Q = \begin{pmatrix} 3 & 0 & -3 & -1 \\ 0 & 2 & -3 & -1 \end{pmatrix}$$

and choosing a suitable cosection t, we obtain a tailcone $t(\sigma)$ which is generated by the rays $(0,1)$ and $(-2,1)$. Furthermore,

$$\begin{array}{lcl} \Delta_0 & = & (0,-1/3) + t(\sigma), \\ \Delta_1 & = & (0,1/2) + t(\sigma), \\ \Delta_\infty & = & \operatorname{conv}\{(0,0),(-1/3,0)\} + t(\sigma), \end{array}$$

which leads to

$$\mathcal{D}_{\mathrm{Cox}} = \Delta_0 \otimes [0] + \Delta_1 \otimes [1] + \Delta_\infty \otimes [\infty].$$

Example 3.9. Finally, we consider a $\mathbb{K}^*$-surface which arises as a toric downgrade, namely from the complete toric surface $\operatorname{TV}(\Sigma)$ whose primitive generators of the rays are given in the following list:

$$\nu_1 = (1,2), \quad \nu_2 = (-1,3), \quad \nu_3 = (-1,-2), \quad \nu_4 = (1,-3), \quad \nu_5 = (3,-4).$$

The divisorial fan which is induced from the downgrade map $\mathbb{Z}^2 \xrightarrow{(1\ 0)} \mathbb{Z}^1 = \mathbb{Z}^{\mathcal{P}}/\mathbb{Z}$ is visualized in Figure 3.3. Observe that $\operatorname{Cl}\big(\operatorname{TV}(\Sigma)\big) \cong \mathbb{Z}^3 \oplus \mathbb{Z}/5\mathbb{Z}$, but the map

$$Q = \begin{pmatrix} 2 & 3 & 2 & 3 & 4 \end{pmatrix}$$

is obviously surjective. The columns of the following matrix display the coordinates of the primitive generators of the six rays of the tailcone for a special choice of the cosection t:

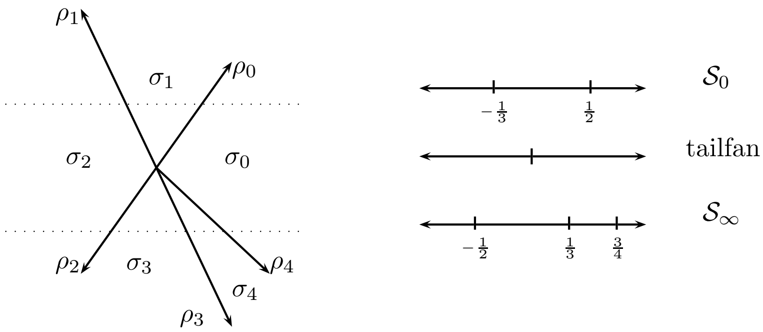
$$\begin{pmatrix} 0 & -1 & 0 & 1 & 4 & 2 \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & -1 \end{pmatrix}.$$

Combinatorially, it corresponds to the cone over the product of an interval and a triangle. Furthermore, we have that

$$\begin{array}{lcl} \Delta_0 & = & \operatorname{conv}\{(1/2,0,0,0),(1/3,0,0,0)\} + t(\sigma), \\ \Delta_\infty & = & \operatorname{conv}\{(-1/2,1/2,0,0),(-1/3,0,1/3,0),(0,0,0,-1/4)\} + t(\sigma), \end{array}$$

which gives us

$$\mathcal{D}_{\mathrm{Cox}} = \Delta_0 \otimes [0] + \Delta_\infty \otimes [\infty] \,.$$



(a) TV(Σ) (b) Its associated divisorial fan.

Figure 3.3: Toric downgrade of TV(Σ), cf. Example 3.9.

3.5 Comparing Polyhedral Divisors

There is another approach to the construction of a polyhedral divisor for the description of the Cox ring of an MDS X. In the case that $\mathrm{Cl}(X) =: M_X$ is a lattice, Klaus Altmann and Jarek Wiśniewski used stabilized multiplicities with respect to a set of exceptional divisors $\{D_i\}$ on the Chow quotient W of the total coordinate space $Z = \operatorname{Spec}\operatorname{Cox}(X)$ and the birational morphism $\psi : W \to X$ to define a polyhedral divisor $\mathcal{D}_{\mathtt{Cox}}$ (note the typewriter font of the index) such that $\operatorname{Cox}(X) = A(\mathcal{D}_{\mathtt{Cox}})$, cf. [AW]. Its support is contained in the finite set $\{D_i\}$. Recall that the so-called Chow quotient W is the normalization of the distinguished component of the inverse limit of the GIT quotients of Z, which is rather difficult to compute in general.

Let us now briefly oppose both approaches to each other. On the one hand, we have the triplet $(W, \mathrm{Cl}(X)^\vee, \mathcal{D}_{\mathtt{Cox}})$ which yields a finitely generated graded $\mathbb{K}$-algebra isomorphic to $\operatorname{Cox}(X)$, namely

$$\begin{aligned} A(\mathcal{D}_{\mathtt{Cox}}) &= \textstyle\bigoplus_{D \in \mathrm{Cl}(X) \cap \mathrm{tail}(\mathcal{D}_{\mathtt{Cox}})^\vee} \Gamma(W, \mathcal{D}_{\mathtt{Cox}}(D)) \\ &= \textstyle\bigoplus_{D \in \mathrm{Cl}(X)} \Gamma(W, \psi^* \mathcal{O}_X(D)) \\ &= \textstyle\bigoplus_{D \in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)) \\ &= \operatorname{Cox}(X) \,. \end{aligned}$$

On the other hand, we have our well known triplet $(\mathbb{P}^1, \widetilde{N}, \mathcal{D}_{\mathrm{Cox}})$ from Theorem 3.4 which gives us:

$$\begin{aligned} A(\mathcal{D}_{\mathrm{Cox}}) &= \textstyle\bigoplus_{u \in \widetilde{M} \cap \mathrm{tail}(\mathcal{D}_{\mathrm{Cox}})^\vee} \Gamma(\mathbb{P}^1, \mathcal{D}_{\mathrm{Cox}}(w)) \\ &= \textstyle\bigoplus_{D \in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)) \\ &= \operatorname{Cox}(X) \,. \end{aligned}$$

3.5.1 Downgrading $\mathcal{D}_{\mathrm{Cox}}$ to $\mathcal{D}_{\mathtt{Cox}}$

The following exact sequence

$$0 \longrightarrow \mathrm{Cl}(\mathrm{TV}(\mathcal{S}))^\vee \xrightarrow{\alpha^\vee} \widetilde{N} \xrightarrow{\beta^\vee} N \longrightarrow 0$$

will play the key role in this paragraph. Note that $\mathcal{D}_{\mathrm{Cox}}$ is associated to the torus $T_{\widetilde{N}}$ whereas $\mathcal{D}_{\mathtt{Cox}}$ comes with the action of $T_{\mathrm{Cl}(\mathrm{TV}(\mathcal{S}))^\vee}$. Hence, with the above sequence, we can realize $T_{\mathrm{Cl}(\mathrm{TV}(\mathcal{S}))^\vee}$ as a subtorus of $T_{\widetilde{N}}$ via the map $\alpha^\vee$.

Instead of describing $\mathrm{Spec}\,\mathrm{Cox}(\mathrm{TV}(\mathcal{S}))$ via the polyhedral divisor $\mathcal{D}_{\mathrm{Cox}}$ on $\mathbb{P}^1$, we would like to describe it by a natural downgrade construction coming from the above exact sequence and compare the outcome to $\mathcal{D}_{\mathtt{Cox}}$.

In essence, for a toroidal complexity-one T_N-MDS X with a free abelian class group the downgrade of $\mathcal{D}_{\mathrm{Cox}}$ should provide us with the following data:

1. a toroidal complexity-one T-variety $\mathrm{TV}(\mathcal{S})$ over $\mathbb{P}^1$ where T is associated to the lattice $\widetilde{N}/\,\mathrm{Cl}(X)^\vee = N$, and $\mathcal{S}$ is induced by the images of the polyhedral coefficients of $\mathcal{D}_{\mathrm{Cox}}$ under the map $\beta^\vee$;

2. a polyhedral divisor $\mathcal{E}$ on $\mathrm{TV}(\mathcal{S})$ such that $\mathrm{TV}(\mathcal{E}) \overset{!}{\cong} \mathrm{TV}(\mathcal{D}_{\mathrm{Cox}})$ comes with an action of $T_{\mathrm{Cl}(X)^\vee}$.

A construction might work as follows, cf. [IV10]. The slice $\mathcal{S}_P$ for $P \in \mathbb{P}^1$ is given as the subdivision which is induced by $\beta^\vee_{\mathbb{Q}}\big((\mathcal{D}_{\mathrm{Cox}})_P\big)$. Furthermore, with the choice of a section $s : N \to \widetilde{N}$ of the map $\beta^\vee$, we define

$$\begin{aligned} \mathcal{E}_{D_\rho} &= \big(\mathrm{tail}\,\mathcal{D}_{\mathrm{Cox}} - s(n_\rho)\big) \cap \mathrm{Cl}(X)^\vee_{\mathbb{Q}}, \\ \mathcal{E}_{D_{(P,v)}} &= \big((\mathcal{D}_{\mathrm{Cox}})_P - s(v)\big) \cap \mathrm{Cl}(X)^\vee_{\mathbb{Q}}. \end{aligned}$$

To keep our calculations simple we will from now on restrict to the class of $\mathbb{K}^*$-MDSurfaces, see also [AW, section 6]. Here, in the language of loc. cit., $W = X$, and $\mathcal{D}_{\mathtt{Cox}} : \mathrm{Eff}(X) \to \mathrm{Nef}(X)$ reflects the Zariski decomposition. Thus, we have a unique decomposition

$$D \equiv P + \sum_i a_i E_i\,,$$

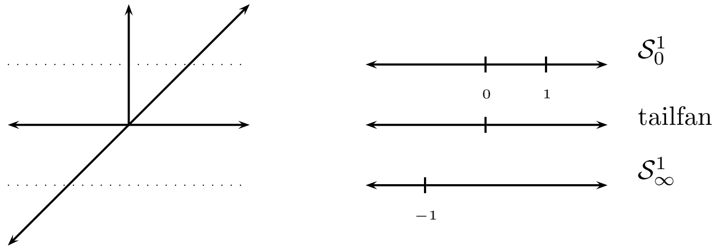
for any effective divisor D on X, where $\mathrm{Nef}(X) \ni P = \mathcal{D}_{\mathtt{Cox}}(D)$, and E_i are exceptional curves with $(P \cdot E_i) = 0$. Furthermore $a_i = \mathrm{mult}^{st}_{E_i} D$, and the intersection pairing allows us to identify $\mathrm{Cl}(Z)_{\mathbb{Q}}$ and $\mathrm{Cl}(Z)^\vee_{\mathbb{Q}}$.

3.5.2 Two Toric Examples

Example 3.10. We consider the toric variety $\mathrm{TV}(\Sigma^1)$ whose primitive generators of the numbered rays are given as the columns of the following matrix

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & -1 \end{pmatrix}.$$

Note that $\mathrm{TV}(\Sigma^1)$ is the blow up of $\mathbb{P}^2$ in two points, cf. [AW, Example 6.3(1)]. The subtorus action is fixed via the embedding $F : \mathbb{Z} \to \mathbb{Z}^2$ with $F = (1\ 0)^t$.



(a) $\mathrm{TV}(\Sigma^1)$ (b) Its associated divisorial fan.

Figure 3.4: Toric downgrade of $\mathrm{TV}(\Sigma^1)$, cf. Example 3.10.

This realizes $\mathrm{TV}(\Sigma^1)$ as a toroidal $\mathbb{K}^*$-surface with divisorial fan $\mathcal{S}^1$, see Figure 3.4. Let us proceed to the description of $\mathcal{D}_{\mathrm{Cox}}$. Choosing a suitable cosection t, we obtain the tailcone $t(\sigma)$ which is generated by four rays. Its primitive generators are given as the columns of the following matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Furthermore, we have that $\mathcal{D}_{\mathrm{Cox}} = \Delta_0 \otimes [0] + \Delta_\infty \otimes [\infty]$ with

$$\begin{aligned} \Delta_0 &= \operatorname{conv}\{(0,1,0,0),(1,1,0,0)\} + t(\sigma), \\ \Delta_\infty &= t(\sigma). \end{aligned}$$

Our recipe for downgrading $\mathcal{D}_{\mathrm{Cox}}$ to $\mathcal{D}_{\mathtt{Cox}}$ from the previous section gives us a divisorial fan $\mathcal{S}$ on $\mathbb{P}^1$ which is equal to the divisorial fan of $\mathrm{TV}(\Sigma^1)$ (associated to the subtorus action from above) up to a shift by the principal polyhedral divisor $\{1\} \otimes ([\infty] - [0])$. Thus, we see that $\mathrm{TV}(\mathcal{S}) = \mathrm{TV}(\Sigma^1)$.

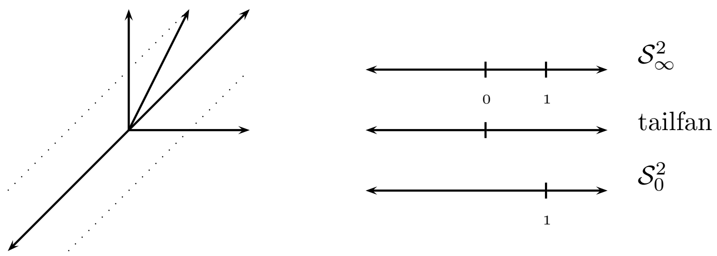
Now we move on to the construction of the polyhedral divisor $\mathcal{E}$ on $\mathrm{TV}(\mathcal{S})$. With a suitable choice of the section s we obtain that the tailcone of $\mathcal{E}$ is generated by three rays. Their primitive generators are given as the columns of the following matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

In addition, we have that

$$\begin{aligned} \mathcal{E}_{D_{([0],-1)}} &= \operatorname{conv}\{(1,0,0),(2,0,1)\} + \operatorname{tail}\mathcal{E}, \\ \mathcal{E}_{D_{([0],0)}} &= \operatorname{conv}\{(1,0,0),(1,1,0)\} + \operatorname{tail}\mathcal{E}, \\ \mathcal{E}_{D_{([\infty],0)}} &= \operatorname{tail}\mathcal{E}, \\ \mathcal{E}_{D_{[0,\infty)}} &= (0,1,0) + \operatorname{tail}\mathcal{E}, \\ \mathcal{E}_{D_{(-\infty,0]}} &= \operatorname{conv}\{(1,0,0),(1,0,1)\} + \operatorname{tail}\mathcal{E}. \end{aligned}$$

With a little effort, one can find a unimodular lattice transformation together with a shift by a principal polyhedral divisor which identifies $\mathcal{E}$ and the polyhedral divisor $\mathcal{D}_{\mathtt{Cox}}$ as presented in [AW, Example 6.3(1)].



(a) $\mathrm{TV}(\Sigma^2)$ (b) Its associated divisorial fan.

Figure 3.5: Toric downgrade of $\mathrm{TV}(\Sigma^2)$, cf. Example 3.11.

Example 3.11. We consider the toric variety $\mathrm{TV}(\Sigma^2)$ whose primitive generators of the numbered rays are given in the following matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 & -1 \\ 0 & 1 & 2 & 1 & -1 \end{pmatrix}.$$

Note that X is the blow up of $\mathbb{P}^2$ in two infinitesimally near points, cf. [AW, Example 6.3(2)]. The subtorus action is fixed via the embedding $F = \mathbb{Z} \to \mathbb{Z}^2$ with $F = (1\ 1)^t$. This realizes $\mathrm{TV}(\Sigma^2)$ as a toroidal $\mathbb{K}^*$-surface with divisorial fan $\mathcal{S}^2$, see Figure 3.5.

Choosing a suitable cosection t, we obtain the tailcone $t(\sigma)$ which is generated by four rays. Its primitive generators are given as the columns of the following matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Furthermore, we have that $\mathcal{D}_{\mathrm{Cox}} = \Delta_0 \otimes [0] + \Delta_\infty \otimes [\infty]$ with

$$\begin{aligned} \Delta_0 &= \mathrm{conv}\{(0,1,0,0),(1,1,0,0)\} + t(\sigma), \\ \Delta_\infty &= t(\sigma). \end{aligned}$$

Applying the recipe for the downgrade then gives us a divisorial fan $\mathcal{S}$ on $\mathbb{P}^1$ which is equal to $\mathcal{S}^2$ up to the shift by the principal polyhedral divisor $\{1\} \otimes ([0] - [\infty])$. Thus, we see that $\mathrm{TV}(\mathcal{S}) = \mathrm{TV}(\Sigma^2)$.

With a suitable choice of the section s we obtain a polyhedral divisor $\mathcal{E}$ on $\mathrm{TV}(\mathcal{S})$ whose tailcone is generated by three rays. Their primitive generators are given as the columns of the following matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

In addition, we have

$$\begin{aligned} \mathcal{E}_{D_{([0],1)}} &= \mathrm{conv}\{(0,1,0),(0,2,-1)\} + \mathrm{tail}\,\mathcal{E}, \\ \mathcal{E}_{D_{([0],2)}} &= \mathrm{conv}\{(0,4,-2),(0,2,-1),(1,2,-1)\} + \mathrm{tail}\,\mathcal{E}, \\ \mathcal{E}_{D_{([\infty],0)}} &= \mathrm{tail}\,\mathcal{E}, \\ \mathcal{E}_{D_{[0,\infty)}} &= \mathrm{conv}\{(1,0,0),(0,2,-1),(0,1,-1/2)\} + \mathrm{tail}\,\mathcal{E}, \\ \mathcal{E}_{D_{(-\infty,0]}} &= (0,-1,1) + \mathrm{tail}\,\mathcal{E}\,. \end{aligned}$$

As in the previous example, one can find a unimodular lattice transformation together with a shift by a principal polyhedral divisor which maps our polyhedral divisor $\mathcal{E}$ to the one given in [AW, Example 6.3(1)].

Remark 3.12. Unfortunately, we were not able to deduce a uniform method how to identify the downgrade of $\mathcal{D}_{\mathrm{Cox}}$ with $\mathcal{D}_{\mathtt{Cox}}$ since a lot of choices are involved in the construction. In particular, we lack a method to quickly identify the number of vertices of a polyhedral coefficient of the downgrade of $\mathcal{D}_{\mathrm{Cox}}$. Nonetheless, our calculations from above and further computations very much indicate that the following statement holds true.

Conjecture 3.13. Let $\mathrm{TV}(\mathcal{S})$ denote a complexity-one $\mathbb{K}^*$-MDSurface. Using the downgrade procedure "down" as described in Section 3.5.1, we have that

$$\mathcal{D}_{\mathtt{Cox}} \cong \mathrm{down}(\mathcal{D}_{\mathrm{Cox}}, \alpha)$$

over $\mathrm{TV}(\mathcal{S})$, where $\alpha : \widetilde{M} \twoheadrightarrow \mathrm{Cl}(\mathrm{TV}(\mathcal{S}))$ induces the toric downgrade.

3.6 Outlook

Refinement of the Theory of Cox Rings of Complexity-One T-Varieties
As usual, we would like to use toric geometry as a role model for further investigations of complexity-one T-varieties. Regarding Cox rings and their applications, the toric theory is very rich. Among others,

- it retrieves a canonical quotient construction for a non-degenerate toric variety $\mathrm{TV}(\Sigma)$,
- it relates homogeneous ideals of the total coordinate ring to closed subvarieties of $\mathrm{TV}(\Sigma)$,
- it relates graded modules over the total coordinate ring of $\mathrm{TV}(\Sigma)$ to sheaves on $\mathrm{TV}(\Sigma)$,
- it relates the cohomology of a sheaf on $\mathrm{TV}(\Sigma)$ to the local cohomology of a corresponding module over the total coordinate ring with respect to the so-called irrelevant ideal.

For details and exact statements, we refer the reader to [CLS, Sections 5 + 9.5]. The motivating question of this paragraph may now be stated like this:

Is it possible to extend some of the "toric" relations from above to a suitable class of complexity-one T-varieties?

Let us become more precise about which these relations we mean. Consider a (smooth) complexity-one T-Mori dream space $\mathrm{TV}(\mathcal{S})$ and denote its total coordinate space by $Z := \operatorname{Spec}\mathrm{Cox}(\mathrm{TV}(\mathcal{S}))$. Looking for an answer to the question above, one should first try to find the necessary ingredients for a good quotient representation of $\mathrm{TV}(\mathcal{S})$ (cf. [CLS, Theorem 5.1.10]), i.e.

$$\mathrm{TV}(\mathcal{S}) \cong \big(Z \setminus Z(\mathcal{S})\big)/\!\!/G\,,$$

where $G = \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Cl}(\mathrm{TV}(\mathcal{S})), \mathbb{K}^*\big)$. Taking a closer look at the description of $\mathcal{D}_{\mathrm{Cox}}$, one could try to determine $Z(\mathcal{S})$ by removing a minimal collection of

$T_{\widetilde{N}}$-orbits $\mathcal{F}$, such that $(\mathrm{id}_{\mathbb{P}^1}, \beta^\vee, \mathfrak{f})$ gives us an equivariant morphism from the $T_{\widetilde{N}}$-variety $Z \setminus \mathcal{F}$ to $\mathrm{TV}(\mathcal{S})$, where $\mathfrak{f} \in N \otimes \mathbb{K}(\mathbb{P}^1)^*$ is some principal polyhedral divisor.

The aim of the next step would be to give an explicit description of the irrelevant ideal $B(\mathcal{S}) \subset \mathrm{Cox}(\mathrm{TV}(\mathcal{S}))$ which is associated to $Z(\mathcal{S})$ in terms of a specific set of generators. Ultimately, one could hope for a correspondence between equivariant sheaves on $\mathrm{TV}(\mathcal{S})$ and graded $\mathrm{Cox}(\mathrm{TV}(\mathcal{S}))$-modules (cf. [CLS, Propositions 5.3.3 + 5.3.9]). Here, we restrict to the smaller grading which is induced by $\mathrm{Cl}(\mathrm{TV}(\mathcal{S}))$ instead of the one that comes from $\widetilde{M}$.

In particular, such a fundamental correspondence might yield another ansatz for the computation of the higher cohomology groups of line bundles, namely via local cohomology computations of the corresponding modules over $\mathrm{Cox}(\mathrm{TV}(\mathcal{S}))$, cf. [CLS, Theorem 9.5.10].

Chapter 4

Okounkov Bodies

In this chapter, we aim at computing Okounkov bodies of complexity-one T-varieties. After recalling Okounkov's construction and first results from toric geometry, we construct two types of admissible flags in the complexity-one setting and describe the associated Okounkov bodies. Furthermore, we compute them for various examples. Next, we come to degenerations and deformations and display their connection to Okounkov bodies. Finally, we proceed to the computation of the global Okounkov body and provide an outlook on topics which may be approached in the near future.

If not stated otherwise all divisors in this chapter are supposed to be Cartier.

4.1 Okounkov's Construction

4.1.1 Preliminaries

In a series of papers [Oko96, Oko03] on log-concavity of multiplicities Andrei Okounkov gave a procedure to associate a convex set to a linear system on a projective variety. Although Okounkov essentially worked in the setting of ample line bundles, the construction works perfectly well for big divisor classes. Robert Lazarsfeld and Mircea Mustaţă thoroughly studied this setting and recovered many fundamental results from the asymptotic theory of linear series, cf. [LM].

Let us now briefly recall the construction of the so-called Okounkov body as presented in [LM, Section 1]. Denote by X a projective variety of dimension d and fix a flag

$$Y_\bullet \;:\; X = Y_0 \supset Y_1 \supset Y_2 \supset \cdots \supset Y_{d-1} \supset Y_d = \{\mathrm{pt}\}\,,$$

consisting of subvarieties Y_i of codimension i in X each of which is non-singular at the point Y_d. A flag $Y_\bullet$ as above will be called an *admissible flag*.

For any divisor D on X one can define a valuation-like function

$$\nu_{Y_\bullet,D} \;:(H^0(X,\mathcal{O}_X(D))\setminus\{0\})\to\mathbb{Z}^d,\; s\mapsto \nu_{Y_\bullet,D}(s)=(\nu_1(s),\ldots,\nu_d(s))$$

by an inductive procedure. Restricting to a suitable open neighborhood of the smooth point Y_d, we may assume that Y_{i+1} is a Cartier divisor on Y_i for $0\le i\le d-1$.

To begin with, we set $\nu_1(s) = \mathrm{ord}_{Y_1}(s)$ where $\mathrm{ord}_{Y_1}(s)$ denotes the vanishing order of s along Y_1. In other words, it is equal to $\mathrm{ord}_{Y_1}(\mathrm{div}(s) + D)$. By choosing a local equation for Y_1 in X, our section s determines in a natural way a section $\widetilde{s}_1 \in H^0(X, \mathcal{O}_X(D - \nu_1(s)Y_1))$ which does not vanish identically along Y_1. Restricting $\widetilde{s}_1$ to Y_1 gives us a non-zero section $s_1 \in H^0(Y_1, \mathcal{O}_{Y_1}(D - \nu_1(s)Y_1)$ and we set $\nu_2(s) = \mathrm{ord}_{Y_2}(s_1)$. We can define the remaining $\nu_i(s)$ analogously and thus obtain the valuation vector $(\nu_1(s), \ldots, \nu_d(s))$ associated to $s \in \Gamma(X, \mathcal{O}_X(D))$.

Observe that the valuation like function $\nu_{Y_\bullet, \cdot}$ has the following properties:

1. Ordering $\mathbb{Z}^d$ lexicographically, we have that
$$\nu_{Y_\bullet, D}(s_1 + s_2) \geq \min\{\nu_{Y_\bullet, D}(s_1), \nu_{Y_\bullet, D}(s_2)\}$$
for any $s_1, s_2 \in \Gamma(X, \mathcal{O}_X(D)) \setminus \{0\}$.

2. For $s \in \Gamma(X, \mathcal{O}_X(D)) \setminus \{0\}$ and $t \in \Gamma(X, \mathcal{O}_X(E)) \setminus \{0\}$ we have that
$$\nu_{Y_\bullet, D+E}(s \otimes t) = \nu_{Y_\bullet, D}(s) + \nu_{Y_\bullet, E}(t)\,.$$

Notation 4.1. Working with a fixed divisor D we will often simply write $\nu_{Y_\bullet}(s)$ instead of $\nu_{Y_\bullet, D}(s)$. Moreover, we denote by $\nu_{Y_\bullet}(D)$ the set of all $\nu_{Y_\bullet}(s)$ for $s \in \Gamma(X, \mathcal{O}_X(D)) \setminus \{0\}$.

Definition 4.2. Let X be a projective variety, D a divisor on X and $Y_\bullet$ a fixed admissible flag. The *graded semigroup* of D with respect to the flag $Y_\bullet$ is the subsemigroup
$$\Gamma_{Y_\bullet}(D) = \big\{(\nu_{Y_\bullet}(s), m) \,\big|\, s \in \Gamma(X, \mathcal{O}_X(mD)) \setminus \{0\},\, m \geq 0\big\} \subset \mathbb{N}^d \times \mathbb{N}\,.$$

Definition 4.3. Let X be a projective variety, D a divisor on X, and $Y_\bullet$ a fixed admissible flag. The *Okounkov body* of D with respect to the flag $Y_\bullet$ is defined as
$$\Delta_{Y_\bullet}(D) = \overline{\mathrm{conv}(\bigcup_{m \geq 1} 1/m \cdot \nu_{Y_\bullet}(mD))} \subset \mathbb{R}^d\,.$$

By construction, we have that $\Delta_{Y_\bullet}(D) \subset \mathbb{R}^d_{\geq 0}$. It is shown in [LM, Theorem 2.3] that $\mathrm{vol}_{\mathbb{R}^d}(\Delta_{Y_\bullet}(D)) = \mathrm{vol}_X(D)/d!$ for a big divisor D on a projective variety X of dimension d, where
$$\mathrm{vol}_X(D) = \limsup_{m \to \infty} \frac{\dim \Gamma(X, \mathcal{O}_X(mD))}{m^d/d!}.$$

If D is nef this quantity is equal to the top self-intersection number D^d, see [Laz04, p. 148]. In particular, $\Delta_{Y_\bullet}(D)$ has a non-empty interior. Furthermore, $\Delta_{Y_\bullet}(D)$ only depends on the numerical equivalence class of D (cf. [LM, Proposition 4.1]), and $\Delta_{Y_\bullet}(kD) = k \cdot \Delta_{Y_\bullet}(D)$. This equality moreover says that $\Delta_{Y_\bullet}(\xi)$ is well defined for big classes $\xi \in N^1(X)_{\mathbb{Q}}$. Indeed, we simply set
$$\Delta_{Y_\bullet}(\xi) := \frac{1}{k}\,\Delta_{Y_\bullet}(k \cdot \xi)$$
for some $k \in \mathbb{Z}_{\geq 1}$ such that $k \cdot \xi \in N^1(X)$.

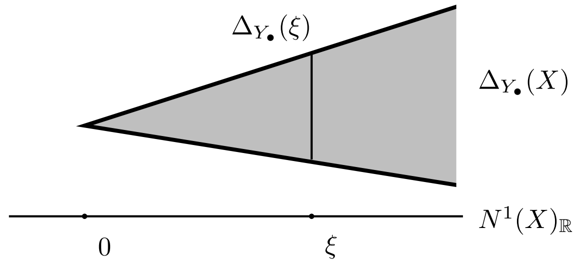


Figure 4.1: The global Okounkov body $\Delta_{Y_\bullet}(X)$ and its defining fibration.

Notation 4.4. Using the correspondence between Cartier divisors and line bundles on X, we will sometimes switch notation from $\Delta_{Y_\bullet}(D)$ to $\Delta_{Y_\bullet}(\mathcal{L})$ if $\mathcal{L} \cong \mathcal{O}_X(D)$.

It has to be pointed out that the computation of Okounkov bodies is very very far from being trivial. They may be non-polyhedral, and even when polyhedral they often are not rational, cf. [LM, 6.2-6.3]. Nonetheless, a very nice feature is that the set of Okounkov bodies $\Delta_{Y_\bullet}(\xi)$ for all big rational classes $\xi \in N^1(X)_{\mathbb{Q}}$ fit together to a global convex object.

Definition 4.5. Cf. [LM, Theorem 4.5]. Let X be a projective variety and $Y_\bullet$ a fixed admissible flag. The *global Okounkov body* $\Delta_{Y_\bullet}(X)$ of X with respect to the flag $Y_\bullet$ is defined as the closed convex cone $\Delta_{Y_\bullet}(X) \subset \mathbb{R}^d \times N^1(X)_{\mathbb{R}}$ such that the fiber of the projection $\mathbb{R}^d \times N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$ over any big class $\xi \in N^1(X)_{\mathbb{Q}}$ is equal to $\Delta_{Y_\bullet}(\xi)$.

Note that the global Okounkov body projects to the pseudoeffective cone

$$\overline{\mathrm{Eff}} = \overline{\mathrm{Big}(X)} \subset N^1(X)_{\mathbb{R}}$$

which is the closure of the big cone $\mathrm{Big}(X)$. The whole setting is illustrated in Figure 4.1, which is a copy of the neat picture given in [LM, Figure 2].

The complexity of the computation of these global bodies usually is quite frightening. Nevertheless, for a fixed torus invariant flag in a smooth projective toric variety $\mathrm{TV}(\Sigma)$, Okounkov's construction reappears as the well known correspondence between divisor classes and lattice polytopes. Furthermore, the global Okounkov body turns out to be the image of a positive orthant under a linear isomorphism [LM, Proposition 6.1]. We will give full details about these results in Section 4.1.2.

One can further generalize the construction and get rid of the choice of the flag $Y_\bullet$. For the sake of completeness we also briefly recall the construction of the convex bodies in this most general setting, cf. [LM, 5.2].

Definition 4.6. A property holds for a *very general* choice of data if it is satisfied away from a countable union of proper closed subvarieties of the relevant parameter space.

First, we fix a smooth point $x \in X$ together with a complete flag of subspaces

$$V_\bullet \ : \ T_xX = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_{d-1} \supset V_d = \{\mathrm{pt}\}\,,$$

in the tangent space to X at x. Blowing up X in x, we obtain an exceptional divisor $E = \mathbb{P}(T_x X)$ and an induced flag

$$F(x; V_\bullet) \; : \; \mathrm{Bl}_x X \supset E \supset \mathbb{P}(V_1) \supset \mathbb{P}(V_2) \supset \cdots \supset \mathbb{P}(V_{d-1}) = \{\mathrm{pt}\} ,$$

on the blow up $\mathrm{Bl}_x X \xrightarrow{\pi} X$. Now, for any divisor D on X and any $m \in \mathbb{Z}_{\geq 0}$ we have that

$$\Gamma(X, \mathcal{O}_X(mD)) = \Gamma\big(\mathrm{Bl}_x X, \mathcal{O}_{\mathrm{Bl}_x X}(m\pi^* D)\big) .$$

Hence, the flag $F(x; V_\bullet)$ in $\mathrm{Bl}_x X$ also defines a valuation-like function on the sections of D, which allows us to define

$$\Delta_{F(x;V_\bullet)}(D) := \Delta_{F(x;V_\bullet)}(\pi^* D).$$

It is shown in [LM, Proposition 5.3] that the corresponding Okounkov bodies $\Delta_{F(x,V_\bullet)}(D)$ for a big divisor D on X all coincide for a very general choice of $x \in X$ and $V_\bullet$. Moreover, the analogous statement holds for the global Okounkov bodies $\Delta_{F(x,V_\bullet)}(X)$. This generic invariance triggers the following definition, cf. [LM, Definition 5.4].

Definition 4.7. Let X be a projective variety and D a big divisor on X. The *infinitesimal Okounkov body* is defined as $\Delta(D) := \Delta_{F(x,V_\bullet)}(D)$ for a very general choice of $x \in X$ and flag $V_\bullet$ in $T_x X$. In analogy to Definition 4.5, we define the *infinitesimal global Okounkov body* $\Delta(X) := \Delta_{F(x,V_\bullet)}(X)$ for a very general choice of $x \in X$ and flag $V_\bullet$ in $T_x X$.

It seems to be almost hopeless to expect an explicit description of the infinitesimal Okounkov bodies even in rather simple examples, cf. [LM, Remark 5.5, Problem 7.5]. About some speculations in this directions see also Section 4.5.

4.1.2 Okounkov Bodies for Toric Varieties

Let $\mathrm{TV}(\Sigma)$ be a smooth projective toric variety of dimension d which is given by a fan Σ in $N_\mathbb{Q}$, and let m be the number of rays $\rho \in \Sigma(1)$ corresponding to the torus invariant prime divisors in X. Recall the exact sequence

$$0 \longrightarrow M \xrightarrow{\iota} \mathbb{Z}^m \xrightarrow{\mathrm{pr}} \mathrm{Pic}(\mathrm{TV}(\Sigma)) \longrightarrow 0 .$$

Supposing the admissible flag $Y_\bullet$ to be invariant, one can order the invariant prime divisors of $\mathrm{TV}(\Sigma)$ in such a way that $Y_i = D_1 \cap \cdots \cap D_i$. The set of the corresponding rays $\{\rho_1, \ldots \rho_d\}$ clearly spans a smooth d-dimensional cone which we denote by σ. It corresponds to the fixed point Y_d. Taking the primitive generators n_i of these rays as a basis for the lattice N, we obtain a splitting of the above exact sequence into

$$\psi : \; \mathbb{Z}^d \times \mathrm{Pic}(\mathrm{TV}(\Sigma)) \longrightarrow \mathbb{Z}^m$$

with $\psi^{-1}(D) = (q(D), \mathrm{pr}(D))$ and $q : \; \mathbb{Z}^m \to \mathbb{Z}^d$ being the projection onto the first d coordinates. We denote by $\phi : M \to \mathbb{Z}^d$ the map which is given by $\phi(u) = (\langle u, n_i \rangle)_{1 \leq i \leq d}$. Moreover, we denote by $P_D \subset M_\mathbb{R}$ the polytope whose lattice points correspond to the homogeneous global sections of $\mathcal{O}(D)$.

Proposition 4.8. Cf. [LM, Proposition 6.1]. Let $\mathrm{TV}(\Sigma)$ be a smooth projective toric variety, and let $Y_\bullet$ be an admissible flag of invariant subvarieties chosen as above.

1. Given any big equivariant line bundle $\mathcal{L}$ on $\mathrm{TV}(\Sigma)$, let D be the unique T-invariant divisor such that $\mathcal{L} \simeq \mathcal{O}(D)$ and its restriction to the affine chart U_σ is trivial. Then we have that
$$\Delta_{\mathrm{Y}_\bullet}(\mathcal{L}) = \phi_{\mathbb{R}}(P_D)\,.$$

2. The global Okounkov body $\Delta_{\mathrm{Y}_\bullet}(\mathrm{TV}(\Sigma))$ is the inverse image of the non-negative orthant $\mathbb{R}^m_{\geq 0} \subset \mathbb{R}^m$ under the isomorphism
$$\psi_{\mathbb{R}}:\ \mathbb{R}^d \times \mathrm{Pic}(\mathrm{TV}(\Sigma))_{\mathbb{R}} \xrightarrow{\cong} \mathbb{R}^m\,.$$

4.2 Divisorial Polytopes and Okounkov Bodies

We already saw that Okounkov's construction recovered a well known correspondence in toric geometry. Our next aim is to generalize this result for smooth projective complexity-one T-varieties. Hence, instead of lattice polytopes we will now deal with divisorial polytopes as introduced in Section 1.2.2.

4.2.1 Different Types of Admissible Flags

Let $\mathrm{TV}(\mathcal{S})$ be a projective T-variety of complexity one which contains at least one smooth point x_{fix} which is fixed under the torus action. The aim of this section is to construct T-invariant admissible flags $Y_\bullet$ in $\mathrm{TV}(\mathcal{S})$ with $Y_d = x_{\mathrm{fix}}$. These will then be used for the computation of Okounkov bodies.

As before, we denote by $\mathcal{P} \subset Y$ a non-empty finite set of points in Y such that the slice S_Q over a point $Q \in Y \setminus \mathcal{P}$ is trivial.

Definition 4.9. A point $Q \in Y \setminus \mathcal{P}$ is called *general*.

Recall that a slice $\mathcal{S}_Q$ for a general point $Q \in Y$ is equal to $\Sigma := \mathrm{tail}\,\mathcal{S}$, meaning that the fiber of the quotient map $\pi : \widetilde{\mathrm{TV}}(\mathcal{S}) \to Y$ is equal to the toric variety $\mathrm{TV}(\Sigma)$. In particular it is reduced and irreducible.

In the following, we will present the construction of several types of admissible T-invariant flags in $\mathrm{TV}(\mathcal{S})$ which will depend upon the choice of a smooth fixed point $x_{\mathrm{fix}} \in \mathrm{TV}(\mathcal{S})$. To begin with, we distinguish between the following two cases:

- **A** The maximal cone $\sigma_{\mathrm{fix}} \in \Sigma$ corresponding to x_{fix} is not marked, i.e. $\sigma_{\mathrm{fix}} \notin C(\mathcal{S})$ (see Definition 1.20).

- **B** The maximal cone $\sigma_{\mathrm{fix}} \in \Sigma$ corresponding to x_{fix} is marked, i.e. $\sigma_{\mathrm{fix}} \in C(\mathcal{S})$.

Note that we must have $Y = \mathbb{P}^1$ in case **B** since x_{fix} is smooth (see [Süß, Proposition 3.1]).

Construction in the First Case

$\mathbf{A}_1$ We assume that x_{fix} lies over a general point $Q \in Y$. We set $Y_1 := r(\pi^{-1}(Q)) \cong \mathrm{TV}(\Sigma)$ and proceed as in the toric case for the remaining elements of the flag:
We label the rays in the smooth cone σ_{fix} from 1 to $d-1$, i.e. $\sigma_{\text{fix}}(1) = \{\rho_1, \ldots, \rho_{d-1}\}$ and we define $\Delta_F(k) := \langle \rho_1, \ldots, \rho_k \rangle$. Thus, we see that $\Delta_F(k)$ corresponds to a T-orbit of codimension k in Y_1 which allows us to define $Y_{i+1} := r\big(\overline{\mathrm{orb}(\Delta_F(i))}\big) \subset Y_1$ for $1 \leq i \leq d-1$. It is not hard to see that such a flag is admissible.

$\mathbf{A}_2$ We assume that x_{fix} lies over a point $P \in \mathcal{P}$. Since x_{fix} is smooth we are in a formal-locally toric situation. Indeed, after a suitable refinement of the invariant covering, we can assume that x_{fix} is contained in an affine open subset $\mathrm{TV}(\mathcal{D}^{\sigma_{\text{fix}}}) \subset \mathrm{TV}(\mathcal{S})$ for a polyhedral divisor $\mathcal{D}^{\sigma_{\text{fix}}}$ with locus contained in $(Y \setminus \mathcal{P}) \cup \{P\}$ and tailfan σ_{fix}. According to [Süß, Theorem 3.3], we have that $\big(\mathrm{TV}(\mathcal{D}^{\sigma_{\text{fix}}}), x_{\text{fix}}\big)$ is formally isomorphic to the smooth affine toric variety $\big(\mathrm{TV}(\delta_{\text{fix}}), \mathrm{orb}(\delta_{\text{fix}})\big)$ with

$$\delta_{\text{fix}} = \overline{\mathbb{Q}_{\geq 0} \cdot \big(\{1\} \times \mathcal{D}_P^{\sigma_{\text{fix}}}\big)} \subset \mathbb{Q}_{\geq 0} \times N_{\mathbb{Q}} \,.$$

The rays of δ_{fix} are given through the vertices of $\mathcal{D}_P^{\sigma_{\text{fix}}}$ in height 1 and the rays of the tailcone σ_{fix} in height 0. The admissible flag then arises as in the toric setting by an enumeration of the rays of δ_{fix}, cf. Section 4.1.2.

Construction in the Second Case

$\mathbf{B_1}$ Assume that σ_{fix} is a smooth cone. We can now proceed as in the construction of an admissible flag of type $\mathbf{A_1}$ by picking a general point Q and identifying x_{fix} with the orbit that corresponds to the cone $\sigma_{\text{fix}} \in \operatorname{tail}\mathcal{S} = \mathcal{S}_Q$.
$\mathbf{B_2}$ Since σ_{fix} is marked and x_{fix} is smooth we are in a Zariski-locally toric situation. Indeed, according to [Süß, Proposition 3.1] we have an affine open T-invariant subset $\mathrm{TV}(\mathcal{D}^{\sigma_{\text{fix}}}) \subset \mathrm{TV}(\mathcal{S})$ such that

$$\mathcal{D}^{\sigma_{\text{fix}}} \cong \mathcal{D}_{P_1}^{\sigma_{\text{fix}}} \otimes [P_1] + \mathcal{D}_{P_2}^{\sigma_{\text{fix}}} \otimes [P_2] \,.$$

Thus, after adding a principal polyhedral divisor, we may assume that $\mathcal{D}^{\sigma_{\text{fix}}} \in \mathcal{S}$ is equal to the r.h.s. Hence, we see that $\mathrm{TV}(\mathcal{D}^{\sigma_{\text{fix}}})$ is isomorphic to the smooth toric variety $\mathrm{TV}(\delta_{\text{fix}})$ with

$$\delta_{\text{fix}} = \overline{\mathbb{Q}_{\geq 0} \cdot \big(\{1\} \times \mathcal{D}_{P_1}^{\sigma_{\text{fix}}} \,\cup\, \{-1\} \times \mathcal{D}_{P_2}^{\sigma_{\text{fix}}}\big)} \subset \mathbb{Q} \times N_{\mathbb{Q}} \,.$$

The rays of δ_{fix} are given through the vertices of $\mathcal{D}_{P_1}^{\sigma_{\text{fix}}}$ and $\mathcal{D}_{P_2}^{\sigma_{\text{fix}}}$. In particular, we have a natural upgrade of the torus action. We now construct an admissible flag as in the toric setting by numbering the rays of δ_{fix}.

Remark 4.10. Considering a toric variety $\mathrm{TV}(\Sigma)$ as a complexity-one T-variety via the downgrade method we presented in Section 1.1.1, one can easily see that the admissible invariant flags constructed above comprise those which are invariant under the original (big) torus action.

Definition 4.11. An admissible flag $Y_\bullet$ of T-invariant subvarieties $Y_i \subset \mathrm{TV}(\mathcal{S})$ constructed as described in $\mathbf{A_1}$, or $\mathbf{B_1}$ is called *general*. An admissible flag $Y_\bullet$ as constructed in $\mathbf{A_2}$ or $\mathbf{B_2}$ is called *toric*.

Remark 4.12. Note that the existence of one admissible general flag already implies the existence of a one parameter family of these since Q may be chosen from $\mathbb{P}^1 \setminus \mathcal{P}$.

The subsequent lemmata show that the computation of $\nu_{Y_\bullet}(s)$ for a section $s \in \Gamma(X, \mathcal{O}_X(D))$ can be reduced to the calculation of the $\nu_{Y_\bullet}(s_{u_i})$, where $s = \sum_i s_{u_i}$ is the decomposition into M-homogeneous components. The main ingredient is [Gona, Lemma 4.5] which we recall here for the convenience of the reader.

Lemma 4.13. Let X be an affine T-variety together with an admissible flag $Y_\bullet : X = Y_0 \supset \cdots \supset Y_d$ of normal T-invariant subvarieties such that $Y_{i+1} = \operatorname{div} h_{u_i} \subset Y_i$ for a rational semi-invariant function h_{u_i}, $(1 \leq i \leq d-1)$, $u_i \in M$. For a rational function $g \in \mathbb{K}(X)$ that decomposes as $g = \sum g_{u_j}$ into homogeneous components with respect to elements $u_j \in M$, we have that $\nu_{Y_\bullet}(g) \in \{\nu_{Y_\bullet}(g_{u_j})\}$.

Lemma 4.14. Let $Y_\bullet$ be an admissible flag (of type $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{B_1}$ or $\mathbf{B_2}$) on a projective complexity-one T-variety $\mathrm{TV}(\mathcal{S})$. Let furthermore $s = \sum_i s_{u_i}$ be a section of the T-invariant big line bundle $\mathcal{O}(D)$ given by the T-invariant divisor D. Then we have that $\nu_{Y_\bullet}(s) \in \{\nu_{Y_\bullet}(s_{u_i})\}$.

Proof. Since this is a local computation, we may restrict to the affine case. Hence, consider an element $\mathcal{D} \in \mathcal{S}$ with $\operatorname{tail} \mathcal{D} = \sigma_{\mathrm{fix}}$ and $x_{\mathrm{fix}} \in \mathrm{TV}(\mathcal{D})$. After possibly shrinking the support of $\mathcal{D}$ (only for types $\mathbf{A_1}$ and $\mathbf{A_2}$), the T-invariant open affine subset $X' := \operatorname{Spec} A(\mathcal{D})$ becomes smooth. Furthermore, it contains the fixed point x_{fix} and comes with the induced admissible flag $Y'_\bullet := Y_\bullet \cap X'$. Since the pair $(X', Y'_\bullet)$ fulfills the preconditions of the previous lemma we have completed the proof. ☐

Remark 4.15. In [Gona, Section 4.1], José González gave a construction for a T-invariant flag on projectivized rank two toric vector bundles over smooth projective toric varieties. Using the description of these projectivized bundles in terms of polyhedral divisors as given in [AHS08, Proposition 8.4], one sees that thosee flags are exclusively of type $\mathbf{B_2}$.

4.2.2 Okounkov Bodies for General Flags

Let us recall some notions which were introduced in Chapter 2. Given a T-invariant Cartier divisor D_h on $\mathrm{TV}(\mathcal{S})$ we denote by $D_{\underline{h}}$ the Cartier divisor which is defined on $\mathrm{TV}(\operatorname{tail} \mathcal{S})$ via the linear part of h. Furthermore, we have the map

$$h_P^* : \Box_h \to \mathbb{Q}\,, \quad u \mapsto \min_{\mathrm{v} \in \mathcal{S}_\mathrm{P}} (u - h_P)\,,$$

for every point $P \in Y$. For the ease of later computations, we introduce the following notion.

Definition 4.16. A T-invariant divisor D_h on $\mathrm{TV}(\mathcal{S})$ is called *normalized* with respect to the general flag $Y_\bullet$ if $h_Q|_{\mathcal{D}_Q^{\sigma_{\mathrm{fix}}}} \equiv 0$. In particular, this implies that $h_Q^* \equiv 0$.

Theorem 4.17. Let $\mathrm{TV}(\mathcal{S})$ be a rational projective T-variety of complexity one together with a general flag $Y_\bullet$. Consider a T-invariant big divisor D_h on $\mathrm{TV}(\mathcal{S})$ which is normalized with respect to $Y_\bullet$. Denote by $D_{\underline{h}}$ the associated invariant divisor on the toric variety $Y_1 = \mathrm{TV}(\Sigma)$, where $\Sigma = \operatorname{tail}\mathcal{S}$, and consider the induced flag $Y_{\geq 1}$ on Y_1. Then we have that

$$\Delta_{Y_\bullet}(D_h) = \Big\{(x,w) \in \mathbb{R} \times \mathbb{R}^{d-1} \;\Big|\; w \in \Delta_{Y_{\geq 1}}(D_{\underline{h}}),\ 0 \leq x \leq \deg\, h^*(\phi_{\mathbb{R}}^{-1}(w))\Big\},$$

where $\phi_{\mathbb{R}}$ is equal to the map which was introduced in the toric setting of Section 4.1.2. Moreover, $\Delta_{Y_{\geq 1}}(D_{\underline{h}}) = \phi_{\mathbb{R}}(\Box_{\underline{h}})$ denotes the Okounkov body of $D_{\underline{h}}$ on Y_1 with respect to the flag $Y_{\geq 1}$. In particular, $\Delta_{Y_\bullet}(D_h)$ is a rational polytope.

Proof. Note that

$$m\, h_P^*\big(\frac{1}{m}u\big) = (mh)_P^*(u) \ \text{ for } m \in \mathbb{Z}_{\geq 1}\,,\ u \in \Box_h\,.$$

Let us first prove the inclusion "$\subset$". It is enough to show that $\frac{1}{m}\nu(s)$ is an element of the r.h.s. for any homogeneous non-zero section $s \in \Gamma(\mathrm{TV}(\mathcal{S}), \mathcal{O}(D_{mh}))$ and any $m \geq 1$. We write $s = f\chi^u$ where $u = \phi_{\mathbb{R}}^{-1}(w)$ denotes the weight of s and $f \in \mathbb{K}(\mathbb{P}^1)$. Due to convexity and the results in the toric setting for $Y_{i\geq 1}$, it is enough to show that

$$\deg\, h^*\big(\frac{1}{m}u\big) \geq \frac{1}{m}\nu_1(s) \geq 0\,.$$

Since D_h is normalized we have that $h_Q^* \equiv 0$ and $\nu_1(s) = \mathrm{ord}_Q(f) \geq 0$. Furthermore, $\nu_1(s)$ is bounded above by $\sum_{P\in\mathbb{P}^1} \lfloor (mh)_P^*(u) \rfloor$. Thus, we arrive at

$$m \deg\, h^*\big(\frac{1}{m}u\big) = \deg\, (mh)^*(u) \geq \sum_{P\in\mathbb{P}^1} \lfloor (mh)_P^*(u) \rfloor \geq \nu_1(s) \geq 0\,.$$

For the other inclusion, consider a point $(x,w) \in \mathbb{Q} \times \mathbb{Q}^{d-1}$ of the right hand side, i.e.

$$\Delta_{Y_{\geq 1}}(D_{\underline{h}}) \ni w = \phi_{\mathbb{Q}}(u)$$

for some $u \in \Box_h$. Due to convexity and the fact that $x \leq \deg\, h^*(u)$ with $u = \phi_{\mathbb{Q}}^{-1}(w)$ it is enough to show that $(\deg\, h^*(u), w) \in$ l.h.s.

Since we only have finitely many non-trivial slices, each of which is a finite subdivision of $N_{\mathbb{Q}}$, there exists a natural number N such that $\Box_{Nh} \ni Nu \in M$, and $(Nh)_P^*(Nu)$ is an integer for every $P \in \mathcal{P}$. So the rounddown is no longer necessary and we have

$$N \deg\, h^*(u) = N \sum_{P\in\mathbb{P}^1} h_P^*(u) = N \sum_{P\in\mathbb{P}^1} (Nh)_P^*\big(Nu\big) = \sum_{P\in\mathbb{P}^1} \lfloor (Nh)_P^*\big(Nu\big) \rfloor \geq 0.$$

But then there is a homogeneous section $s \in \Gamma(\mathrm{TV}(\mathcal{S}), \mathcal{O}(D_{Nh}))$ of weight Nu such that $\nu_1(s) = N \deg h^*(u)$. Scaling by $1/N$ completes the proof. ∎

Thus, Theorem 4.17 also gives us a link between rational divisorial polytopes $(\Psi, \Box, \mathbb{P}^1)$ and their corresponding Okounkov bodies $\Delta_{Y_\bullet}(D_{\Psi^*})$.

Corollary 4.18. Given a divisorial polytope $(\Psi, \Box, \mathbb{P}^1)$ the associated Okounkov body $\Delta_{Y_\bullet}(D_{\Psi^*})$ arises, up to translation, from the convex hull of the graph of the function $\sum_{P \in \mathbb{P}^1} \Psi_P$ over the polytope $\Box$.

Remark 4.19. In contrast to the Okounkov body $\Delta_{Y_\bullet}(D_h)$, the concave graph $\sum_{P \in \mathcal{P}} h_P^* \circ \phi_{\mathbb{R}}^{-1}$ over the induced toric Okounkov body does not have to lie in the positive orthant. Examples can already be found in the realm of toric surfaces when considered as $\mathbb{K}^*$-surfaces via a downgrade. That is why we have to intersect the graph with the half space $\mathbb{R}_{\geq 0} \times \mathbb{R}^{d-1}$ which is already implicit in the equation given in the proposition. Nonetheless, if D is also semiample this intersection is no longer necessary.

Remark 4.20. Fixing a big T-invariant divisor D_h together with an enumeration of the rays of σ_{fix}, Theorem 4.17 also shows that all resulting Okounkov bodies are identical sincethere is no dependence on $Q \in Y \setminus \mathcal{P}$.

4.2.3 Okounkov Bodies for Toric Flags

Before proceeding to the computation of Okounkov bodies with respect to toric flags, let us introduce some further useful notation.

Definition 4.21. Let $\mathrm{TV}(\mathcal{S})$ be a rational projective T-variety of complexity one together with a fixed toric flag $Y_\bullet$ of type $\mathbf{A_2}$ or $\mathbf{B_2}$. A T-invariant divisor D_h is called *normalized* with respect to $Y_\bullet$ if $h|_{\mathcal{D}^{\sigma_{\text{fix}}}} \equiv 0$.

Moreover, we introduce the map

$$c_{Y_\bullet} : \text{T-CaDiv}\left(\mathrm{TV}(\mathcal{S})\right)_{\mathbb{Q}} \to \mathbb{Q}^d, \quad c_{Y_\bullet}(D_h)_i = \operatorname{coeff}_{D_i} D_h \,,$$

where D_i is the Weil divisor in $\mathrm{TV}(\mathcal{S})$ which is associated to the i'th ray of the cone δ_{fix} arising from $Y_\bullet$.

Using a Toric Flag of Type $\mathbf{A_2}$ Recall that x_{fix} lies over a point $P \in \mathcal{P}$ and the associated cone σ_{fix} is not marked. Embedding $\mathcal{D}_P^{\sigma_{\text{fix}}}$ into $\{1\} \times N_{\mathbb{Q}}$, we obtain a smooth cone $\delta_{\text{fix}} \subset \mathbb{Q} \times N_{\mathbb{Q}}$. Since D_h is normalized, a global section $f\chi^u \in \Gamma(\mathrm{TV}(\mathcal{S}), \mathcal{O}(D_h))_u$ turns into the rational function of weight $[\operatorname{ord}_P f, u] \in \mathbb{Z} \times M$ where $\mathbb{Z} \times M$ is the character lattice of the big torus acting upon $\mathrm{TV}(\delta_{\text{fix}})$. In addition, we define

$$\phi : \mathbb{Z} \times M \to \mathbb{Z}^d, \quad \phi\big([\operatorname{ord}_P f, u]\big) = \Big(\big\langle [\operatorname{ord}_P f, u], n_i \big\rangle\Big)_{1 \leq i \leq d} \,,$$

where n_i is the primitive generator of the i-th ray of δ_{fix} fixed by the flag $Y_\bullet$. Note the similarity to the map used in Section 4.1.2. The next statement now follows easily from the toric discussion in Section 4.1.2 and the definition of $\nu_{Y_\bullet, D_h}$.

Lemma 4.22. Let $\mathrm{TV}(\mathcal{S})$ and $Y_\bullet$ be as above. For a T-invariant divisor D_h on $\mathrm{TV}(\mathcal{S})$ we have

$$\nu_{Y_\bullet, D_h}(f\chi^u) = \phi\big([\operatorname{ord}_P f, u]\big) + c_{Y_\bullet}(D_h) \,,$$

where the last summand vanishes if D_h is normalized.

Proposition 4.23. Let $\mathrm{TV}(\mathcal{S})$ be a rational projective T-variety of complexity one together with a toric flag $Y_\bullet$ of type $\mathbf{A_2}$ and a normalized big T-invariant divisor D_h. The Okounkov body $\Delta_{Y_\bullet}(D_h)$ then results from the image of the rational polytope

$$W(h) \;:=\; \overline{\Big\{(x,u) \in \mathbb{Q} \times \Box_{\underline{h}} \;\big|\; 0 \leq x + h_P^*(u) \leq \deg h^*(u)\Big\}} \subset \mathbb{R} \times \mathbb{R}^{d-1}.$$

under a lattice isomorphism which is induced by the ordered set $\{n_1, \ldots, n_d\}$ of the primitive generators of the rays of δ_{fix}. Namely, an element $w \in W(h)$ gives us

$$\big(\langle w, n_1\rangle, \ldots, \langle w, n_d\rangle\big) \in \Delta_{Y_\bullet}(D_h) \subset \mathbb{R}^d.$$

In particular, $\Delta_{Y_\bullet}(D_h)$ is a rational polytope.

Proof. The proof is essentially analogous to the proof of Theorem 4.17. We start with the inclusion "$\subset$". So we have to show that $\frac{1}{m}\nu(f\chi^u)$ lies inside the set described above for any non-zero section $f\chi^u \in \Gamma(\mathrm{TV}(\mathcal{S}), \mathcal{O}(D_{mh}))$ and any $m \geq 1$. Thus, we only have to prove that $\big([\frac{1}{m} \operatorname{ord}_P f, \frac{1}{m}u]\big) \in W(h)$. But this claim follows from the two subsequent inequalities which can easily be derived from the description of $\operatorname{div} f\chi^u$ (cf. Proposition 2.3) and the description of $\Gamma(\mathrm{TV}(\mathcal{S}), \mathcal{O}(D_{mh}))$ (cf. Proposition 2.22):

$$0 \leq \operatorname{ord}_P f + (mh)_P^*(u)\,, \qquad \operatorname{ord}_P f + (mh)_P^*(u) \leq \deg\,(mh)^*(u)\,.$$

For the other inclusion "$\supset$", we consider a point $(x,u) \in W(h) \cap \mathbb{Q} \times \mathbb{Q}^{d-1}$. Our aim now is to find a section $f\chi^{Nu} \in \Gamma(\mathrm{TV}(\mathcal{S}), \mathcal{O}(D_{Nh}))$ for some $N > 0$ such that $[\frac{1}{N} \operatorname{ord}_P f, u] = (x,u)$. Since we only have finitely many non-trivial slices, each of which is a finite subdivision of $N_\mathbb{Q}$, there exists a natural number N such that $(Nx, Nu) \in \mathbb{Z} \times (\Box_{Nh} \cap M)$, and $(Nh)_{P'}^*(Nu)$ is an integer for every point $P' \in \mathcal{P}$. So the rounddown is no longer necessary and we have that

$$\begin{aligned} N \sum_{P \in \mathbb{P}^1} h_P^*(u) \;&=\; \sum_{P \in \mathbb{P}^1} (Nh)_P^*\big(Nu\big) = \sum_{P \in \mathbb{P}^1} \lfloor (Nh)_P^*\big(Nu\big) \rfloor \\ &=\; \deg\big((Nh)^*(Nu)\big) \geq 0\,. \end{aligned}$$

But then there is also a homogeneous section $f\chi^{Nu} \in \Gamma(\mathrm{TV}(\mathcal{S}), \mathcal{O}(ND_h))$ of weight Nu such that $\operatorname{ord}_P f = Nx$. Scaling by $1/N$ completes the proof. ☐

Using a Toric Flag of Type $\mathbf{B_2}$ For the computation of the Okounkov body with respect to an admissible flag of type $\mathbf{B_2}$ we assume that

$$\mathcal{D}^{\sigma_{\mathrm{fix}}} = \mathcal{D}_{P_1}^{\sigma_{\mathrm{fix}}} \otimes [P_1] + \mathcal{D}_{P_2}^{\sigma_{\mathrm{fix}}} \otimes [P_2]\,.$$

Embedding $\mathcal{D}_{P_1}^{\sigma_{\mathrm{fix}}}$ in $\{1\} \times N_\mathbb{Q}$ and $\mathcal{D}_{P_2}^{\sigma_{\mathrm{fix}}}$ in $\{-1\} \times N_\mathbb{Q}$, we obtain a smooth cone $\delta_{\mathrm{fix}} \subset \mathbb{Q} \times N_\mathbb{Q}$. As D_h is normalized, a global section $f\chi^u \in \Gamma(\mathrm{TV}(\mathcal{S}), \mathcal{O}(D_h))_u$ turns into the rational function of weight $[\operatorname{ord}_{P_1} f, u] = [-\operatorname{ord}_{P_2} f, u] \in \mathbb{Z} \times M$. Again, the latter lattice is the character lattice of the big torus that acts upon $\mathrm{TV}(\delta_{\mathrm{fix}})$. With the very same notation as in the previous section, we define

$$\phi : \mathbb{Z} \times M \to \mathbb{Z}^d, \quad \phi\big([\operatorname{ord}_{P_1} f, u]\big) = \Big(\big\langle [\operatorname{ord}_{P_1} f, u], n_i \big\rangle\Big)_{1 \leq i \leq d}.$$

This gives us

Lemma 4.24. Let $\mathrm{TV}(\mathcal{S})$ and $Y_\bullet$ be as above. For a T-invariant divisor D_h on $\mathrm{TV}(\mathcal{S})$ we have that

$$\nu_{Y_\bullet, D_h}(f\chi^u) = \phi\big([\mathrm{ord}_{P_1} f, u]\big) + c_{Y_\bullet}(D_h)\,,$$

where the last summand vanishes if D_h is normalized.

Proposition 4.25. Let $\mathrm{TV}(\mathcal{S})$ be a rational projective T-variety of complexity one together with a toric flag $Y_\bullet$ of type $\mathbf{B_2}$ and a normalized big T-invariant divisor D_h. The Okounkov body $\Delta_{Y_\bullet}(D_h)$ then results from the pairing of the rational polytope

$$W(h) \;:=\; \overline{\Big\{(x,u) \in \mathbb{Q} \times \Box_{\underline{h}} \;\Big|\; 0 \leq x + h^*_{P_1}(u) \leq \deg h^*(u)\Big\}} \subset \mathbb{R} \times \mathbb{R}^{d-1}.$$

with the ordered set of primitive generators of the rays of δ_{fix}, i.e. an element $w \in W(h)$ gives us

$$\big(\langle w, n_1\rangle, \ldots, \langle w, n_d\rangle\big) \in \mathbb{R}^d.$$

Hence, $\Delta_{Y_\bullet}(D_h)$ is also a rational polytope.

Proof. Replacing P by P_1 the proof is identical to the proof of Proposition 4.23. □

Remark 4.26. In particular, this result gives us a uniform framework to compute Okounkov bodies of equivariant line bundles on projectivized rank two toric vector bundles over smooth projective varieties, cf. [Gona]. Computations in this setting were already done for special cases (e.g. pull back bundles) in loc. cit., but a general answer to this problem was still missing.

4.2.4 Examples

Revisiting Toric Geometry

Considering a toric variety $\mathrm{TV}(\Sigma)$ with big torus T, we may downgrade to a torus action of complexity one where we denote the smaller torus by T'. It turns out that the set of T'-invariant admissible flags we have described in the previous section is much bigger than the set of admissible flags which are invariant under the action of the big torus T. Indeed, for type $\mathbf{A_1}$ or $\mathbf{B_1}$ we essentially have a one-parameter family of choices for a general flag (depending on the choice of the point $Q \in \mathbb{P}^1 \setminus \mathcal{P}$) whereas, in the toric setting, we are restricted to the flags which are associated to the finite (and possibly empty) set of T-fixed points, cf. Section 4.1.2. This means, for example, that we will now be able to compute Okounkov bodies even if there is no smooth T-invariant fixed point at all (see Example 4.29). Moreover, computations of Okounkov bodies for a line bundle $\mathcal{O}(D)$ with respect to some general T'-invariant flags can yield convex bodies that differ considerably from P_D.

In the following, we will illustrate a few new features of Okounkov bodies associated to ample line bundles on some genuinely toric $\mathbb{K}^*$-surfaces.

Example 4.27. We consider the n'th Hirzebruch surface $\mathbb{F}_n$ (see Example 1.16) together with the ample line bundle $\mathcal{L} = \mathcal{O}(D_{\rho_2} + D_{\rho_3})$. Our aim is to perform computations with respect to all flags discussed so far.

The Toric Setting. The toric Okounkov body $\Delta_{Z_\bullet}(\mathcal{L})$ with respect to the flag

$$Z_\bullet : \quad \mathbb{F}_n \supset D_{\rho_0} \supset (D_{\rho_0} \cap D_{\rho_1})$$

then is given by the polytope which is pictured in Figure 4.2(a).

A General Flag of Type $\mathbf{A_1}$*.* To give such a flag, we choose a parabolic fixed point x_{fix} represented by the interval $[0\ \infty)$ in the slice $\mathcal{S}_Q$ for $Q \in \mathbb{P}^1 \setminus \{0, \infty\}$. Then we define

$$Z^1_\bullet : \quad \mathbb{F}_n \supset r(\pi^{-1}(Q)) \supset (Q, [0\ \infty))\,.$$

Note that $D_h = D_{\rho_2} + D_{\rho_3}$ already is normalized with repsect to $Z^1_\bullet$. Using Theorem 4.17, an easy calculation then shows that

$$\Delta_{Z^1_\bullet}(\mathcal{L}) = \operatorname{conv}\{(0,0), (1,0), (1,1), (0, n+1)\}.$$

A General Flag of Type $\mathbf{B_1}$*.* We consider a general point $Q \in \mathbb{P}^1 \setminus \{0, \infty\}$ together with the flag

$$Z^2_\bullet : \quad \mathbb{F}_n \supset r(\pi^{-1}(Q)) \supset (Q, (-\infty\ 0])\,.$$

Note that $(Q, (-\infty\ 0])$ corresponds to the elliptic fixed point. We take $D_h = (n+1)D_{\rho_0} + D_{\rho_1}$ which is linear equivalent to $D_{\rho_2} + D_{\rho_3}$ and normalized with respect to $Z^2_\bullet$. An easy computation then shows that

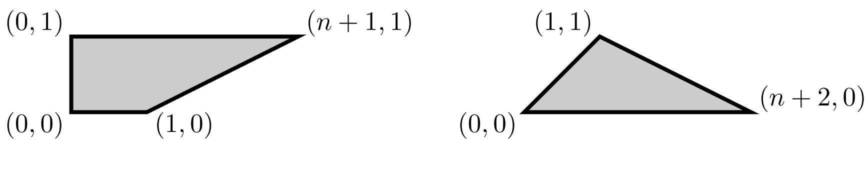
$$\Delta_{Z^2_\bullet}(D_h) = \operatorname{conv}\{(0,0), (0, n+1), (1, n), (1, n+1)\}.$$

A Toric Flag of Type $\mathbf{A_2}$*.* Let us consider an admissible flag of type $\mathbf{A_2}$ which is associated to the hyperbolic fixed point x_{fix} represented by the interval $[-1/n\ \ 0]$ in the slice $\mathcal{S}_0$. Numbering the rays of the cone δ_{fix} by $(r_1, r_2) := \big(\mathbb{Q}_{\geq 0}(1,0), \mathbb{Q}_{\geq 0}(n,-1)\big)$, we set

$$Z^3_\bullet : \quad \mathbb{F}_n \supset D_{r_1} \supset (D_{r_1} \cap D_{r_2})\,.$$

Using linear equivalence, we pass from $D_{\rho_2} + D_{\rho_3}$ to $D_h := D_{\rho_0} + D_{\rho_3}$ to obtain a divisor which is normalized with respect to $Z^3_\bullet$. The graphs of the functions h^*_0 and h^*_∞ are given in Figure 4.3. Furthermore, we compute that

$$W(h) = \operatorname{conv}\{(0,-1), (0,0), (1,-1), (1,0), (1,n)\} \subset \mathbb{R}^2\,.$$



(a) $\Delta_{Z_\bullet}(\mathcal{L})$.

(b) $\Delta_{Y_\bullet}(\mathcal{L})$, where the first and second coordinate have been interchanged.

Figure 4.2: Okounkov bodies associated to different flags for an ample line bundle $\mathcal{L}$ on $\mathbb{F}_n$, cf. Examples 4.27 and 4.28.

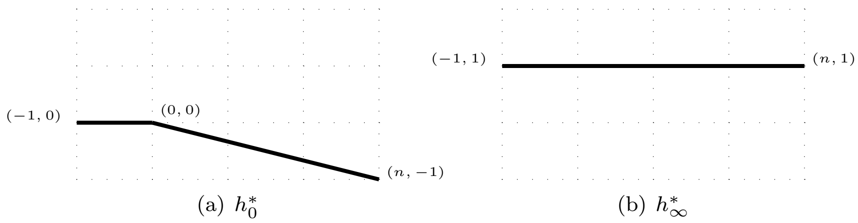


Figure 4.3: Graphs of h_0^* and h_∞^*, cf. Example 4.27.

Pairing $W(h)$ with (n_{r_1}, n_{r_2}) gives us

$$\Delta_{Z^3_\bullet}(D_h) = \operatorname{conv}\{(0,1),(0,0),(1,n+1),(1,0)\}.$$

Moreover, it is not hard to check that flags of type $\mathbf{A_2}$ with respect to the parabolic fixed points over 0 or ∞ yield identical Okounkov bodies.

A Toric Flag of Type $\mathbf{B_2}$. Finally, we compute the Okounkov body for a flag of type $\mathbf{B_2}$ with respect to the elliptic fixed point x_{fix}. First, we order the rays of the induced cone δ_{fix} by $(r_1, r_2) := \big(\mathbb{Q}_{\geq 0}(1,-1), \mathbb{Q}_{\geq 0}(-1,0)\big)$ and set

$$Z^4_\bullet : \quad \mathbb{F}_n \supset D_{r_1} \supset (D_{r_1} \cap D_{r_2})\,.$$

Moreover, we take $D_h = (n+1)D_{\rho_0} + D_{\rho_1}$ which is linear equivalent to $D_{\rho_2} + D_{\rho_3}$ and normalized with respect to $Z^4_\bullet$. With these data we compute

$$W(h) = \operatorname{conv}\{(0,-n-1),(0,0),(-1,-n-1),(-1,-n)\}$$

and obtain

$$\Delta_{Z^4_\bullet}(D_h) = \operatorname{conv}\{(n+1,0),(0,0),(n,1),(n-1,1)\}.$$

Concluding Remark. This particular downgrade did not give us any "new" polytopes. Indeed, one easily checks that all of them can be transformed into $\Delta_{Z_\bullet}(\mathcal{L})$ by an affine lattice isomorphism.

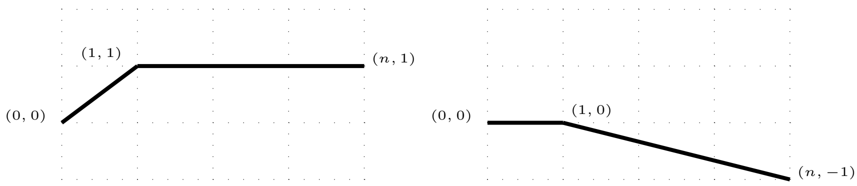
Example 4.28. In contrast to Example 1.16, we now choose the subtorus action which arises from the following data:

$$F = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & -1 \end{pmatrix}, \qquad s = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

The associated divisorial fan is given in Figure 4.5. Furthermore, let $Q \in \mathbb{P}^1 \setminus \{0, \infty\}$ be a general point and fix the following admissible flag of type $\mathbf{B_1}$:

$$Y_\bullet : \quad \mathbb{F}_n \supset r(\pi^{-1}(Q)) \supset (Q, [0\ \ \infty))\,.$$

See Figure 4.4 for the graphs of h_P^* and the right hand polytope in Figure 4.2 for a picture of the resulting Okounkov body $\Delta_{Y_\bullet}(\mathcal{L})$. Observe that this polytope corresponds to the toric variety $\mathbb{P}(1,1,n+2)$.



(a) h_0^* (b) h_∞^*

Figure 4.4: Graphs of h_0^* and h_∞^*, cf. Example 4.28.

Example 4.29. We return to the toric surface TV(Σ) from Example 3.9. Since all of its fixed points are singular we cannot find an admissible flag as chosen in Section 4.1.2. Nevertheless, we may perform a downgrade and choose the subtorus action that comes from

$$F = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad P = \begin{pmatrix} 2 & -1 \end{pmatrix}, \qquad s = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

The associated divisorial fan is given in Figure 4.6. Next, we consider the ample line bundle $\mathcal{L} = \mathcal{O}(5D_{\rho_1} + 15D_{\rho_2} + 5D_{\rho_3})$ which is given by the following generators of the global sections over the affine charts U_{σ_i}:

$$u_{\sigma_0} = [0\ \ 0], \quad u_{\sigma_1} = [2\ -1], \quad u_{\sigma_2} = [11\ \ 2], \quad u_{\sigma_3} = [7\ \ 4], \quad u_{\sigma_4} = [4\ \ 3].$$

Let $Q \in \mathbb{P}^1 \setminus \{0, \infty\}$ and fix the following general flag of type $\mathbf{A_1}$:

$$Y_\bullet: \quad \mathrm{TV}(\Sigma) \supset r(\pi^{-1}(Q)) \supset (Q, [0\ \ \infty)).$$

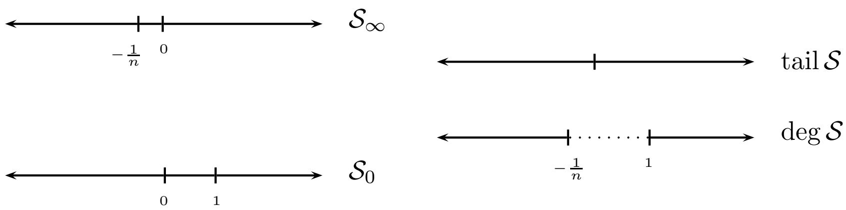
Invoking Theorem 4.17, we see that $\Delta_{Y_\bullet}(\mathcal{L})$ is equal to

$$\mathrm{conv}\{(0,0), (1,0), (2,10), (2,15), (0,15)\}.$$

For an illustration of this convex body (in $\mathbb{R}^2$ with interchanged coordinate axes), see Figure 4.7.

The Anti-Canonical Bundle on $\mathbb{P}(\Omega_{\mathbb{P}^2})$

A Toric Flag For the construction of an admissible flag of type $\mathbf{B_2}$, we consider the polyhedral divisor $\mathcal{D}_{\sigma_{\mathrm{fix}}}$ whose tailcone σ_{fix} is generated by the rays



(a) Relevant slices of $\mathcal{S}$. (b) Tailfan and degree of $\mathcal{S}$.

Figure 4.5: Divisorial fan associated to $\mathbb{F}_n$, cf. Example 4.28.

$(1,0)$ and $(1,1)$. Note that the polyhedral coefficient $\mathcal{D}_\infty$ is trivial. Hence, we have that $\mathrm{TV}(\mathcal{D}_{\sigma_{\text{fix}}}) = \mathrm{TV}(\delta_{\text{fix}}) = \mathbb{A}^3$ with δ_{fix} being spanned by the rays ρ_1, ρ_2 and ρ_3 whose primitive generators are $(1,0,0)$, $(1,0,1)$ and $(-1,1,0)$, respectively. We also take this enumeration for the definition of our flag, i.e.

$$Y_\bullet : \quad \mathrm{TV}(\delta_{\text{fix}}) \supset D_{\rho_1} \supset (D_{\rho_1} \cap D_{\rho_2}) \supset (D_{\rho_1} \cap D_{\rho_2} \cap D_{\rho_3}) = x_{\text{fix}} .$$

An easy calculation according to Proposition 4.25 now shows that $\Delta_{Y_\bullet}(-K_X) \subset \mathbb{R}^3$ is the convex polytope whose vertices are represented as the columns of the following matrix

$$\begin{pmatrix} 0 & 2 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 4 \end{pmatrix}.$$

Interchanging the y and z-axes and scaling with the factor $1/2$ gives us the same polytope which was computed in [Gona, Example 6.1]. Moreover, one checks that

$$\nu_{Y_\bullet}(\Gamma(X, -K_X)) = \Delta_{Y_\bullet}(-K_X) \cap \mathbb{Z}^3.$$

A General Flag Let $Q \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. We consider the maximal cone spanned by the rays $\delta_1 = \mathbb{Q}_{\geq 0} \cdot (1,0)$ and $\delta_2 = \mathbb{Q}_{\geq 0} \cdot (1,1)$ together with the induced admissible flag of type $\mathbf{B_1}$:

$$Y_\bullet : \quad \mathbb{P}(\Omega_{\mathbb{P}^2}) \supset r\big(\pi^{-1}(Q) = \mathrm{TV}(\Sigma)\big) \supset r(D_{\delta_1}) \supset r(D_{\delta_1} \cap D_{\delta_2}) = x_{\text{fix}} .$$

Applying Theorem 4.17, we obtain that $\Delta_{Y_\bullet}(-K_X) \subset \mathbb{R}^3$ is the convex polytope with vertices represented as the columns of the following matrix

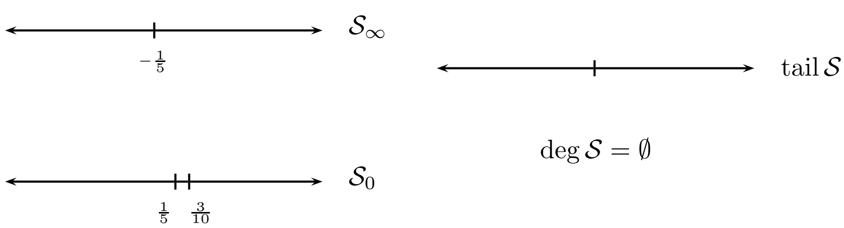
$$\begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 4 & 2 & 4 \\ 0 & 0 & 2 & 2 & 2 & 4 & 4 \end{pmatrix}.$$

Again, one easily checks that

$$\nu_{Y_\bullet}(\Gamma(X, -K_X)) = \Delta_{Y_\bullet}(-K_X) \cap \mathbb{Z}^3.$$

The Anti-Canonical Bundle on the Smooth Quadric

Recall from Example 1.27 that every maximal cone in the tailfan $\mathcal{S}(Q)$ is marked and singular. Hence, we may only construct a toric flag of type $\mathbf{B_2}$.



(a) Relevant slices of $\mathcal{S}$. (b) Tailfan and degree of $\mathcal{S}$.

Figure 4.6: Divisorial fan associated to $\mathrm{TV}(\Sigma)$, cf. Example 4.29.

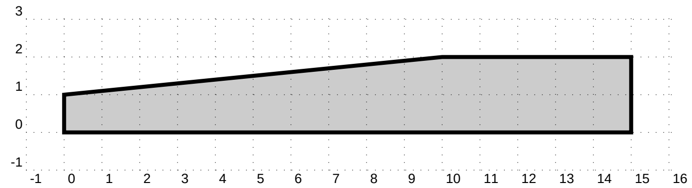


Figure 4.7: Okounkov body of the ample line bundle $\mathcal{L}$ on $\mathrm{TV}(\Sigma)$ with respect to flag of type $\mathbf{A_1}$, cf. Example 4.29.

So let us consider the polyhedral divisor $\mathcal{D}_{\sigma_{\mathrm{fix}}}$ whose tailcone σ_{fix} is generated by the rays $(1,1)$ and $(1,-1)$. Note that the polyhedral coefficient $\mathcal{D}_1$ is trivial. Hence, we have that $X(\mathcal{D}_{\sigma_{\mathrm{fix}}}) = \mathrm{TV}(\delta_{\mathrm{fix}}) = \mathbb{A}^3$ with δ_{fix} being spanned by the rays ρ_1, ρ_2 and ρ_3 whose primitive generators are $(1,0,0)$, $(1,0,-1)$ and $(-2,1,1)$, respectively. We also take this enumeration for the definition of our flag, i.e.

$$Y_\bullet : \quad \mathrm{TV}(\delta_{\mathrm{fix}}) \supset D_{\rho_1} \supset (D_{\rho_1} \cap D_{\rho_2}) \supset (D_{\rho_1} \cap D_{\rho_2} \cap D_{\rho_3}) = x_{\mathrm{fix}}\,.$$

An easy calculation now shows that

$$\Delta_{\mathrm{Y}_\bullet}(-K_X) = \mathrm{conv}\{(0,0,0),(3,0,0),(0,3,0),(0,0,6)\} \subset \mathbb{R}^3\,,$$

whose normal fan corresponds to $\mathbb{P}(1,1,2,2)$. As before, one checks that

$$\nu_{\mathrm{Y}_\bullet}(\Gamma(X,-K_X)) = \Delta_{\mathrm{Y}_\bullet}(-K_X) \cap \mathbb{Z}^3.$$

4.3 Degenerations and Deformations

4.3.1 Anderson's Approach

We will use our results from the previous section to investigate toric degenerations with a focus upon Dave Anderson's article [And]. This paper will be our guideline for the presentation and outline of the main notions which we will need in this section.

Let K denote a field (which we always think of as a function field $\mathbb{K}(X)$ over $\mathbb{K}$) and equip $\mathbb{Z}^d$ with the lexicographic order. We fix a $\mathbb{Z}^d$-valuation ν on K. For a finite-dimensional $\mathbb{K}$-subspace $V \subset K$ we denote by $V^m \subset K$ the subspace which is spanned by elements of the form $f_1 \cdots f_m$ with $f_i \in V$. Furthermore, we set

$$\Gamma(V) \;:=\; \Gamma_\nu(V) \;:=\; \big\{(m,\nu(f)) \in \mathbb{N}\times\mathbb{Z}^d \mid f \in V^m \setminus 0\big\} \subset \mathbb{N}\times\mathbb{Z}^d\,,$$

which is a graded semigroup. By $\operatorname{cone}\Gamma(V) \subset \mathbb{R}\times\mathbb{R}^d$ we mean the closure of the convex hull of $\Gamma(V)$. Following [KK], one may define the *Newton-Okounkov body* of V as

$$\Delta(V) \;:=\; \Delta_\nu(V) \;:=\; \operatorname{cone}\Gamma(V) \cap \big(\{1\}\times\mathbb{R}^d\big).$$

Proposition 4.30. Let $(X,\mathcal{L})$ be a rational smooth projective T-variety of complexity one together with a fixed general or toric flag $Y_\bullet$ and an equivariant ample line bundle $\mathcal{L}$ on X. Furthermore, let $V := H^0(X,\mathcal{L})$, $\nu := \nu_{Y_\bullet}$ and assume that $\Delta_{\nu_{Y_\bullet}}(V) = \Delta_{\mathrm{Y}_\bullet}(\mathcal{L})$. Then the following assertions hold:

1. The associated semigroup $\Gamma := \Gamma_\nu(V)$ is finitely generated.

2. The image $X(V)$ of X in $\mathbb{P}(V)$ via $|\mathcal{L}|$ (which is basepoint-free since X is an MDS) with $V := H^0(X, \mathcal{L})$ admits a flat degeneration to the not necessarily normal toric variety

$$X(\Gamma) = \operatorname{Proj} k[\Gamma]$$

whose normalization is the toric variety associated to $\Delta_{Y_\bullet}(V)$.

Proof. The ampleness of $\mathcal{L} = \mathcal{O}_X(D_h)$ gives us that the Okounkov body $\Delta_{Y_\bullet}(V)$ is a lattice polytope. Indeed, this follows from Theorem 4.17, Propositions 4.23 and 4.25, and the description of divisorial polytopes, cf. Definition 1.24 (3.).

1. To prove that Γ is finitely generated it suffices to show that cone(Γ) is generated by $\Gamma \cap (\{1\} \times \mathbb{Z}^n)$, see e.g. [And, Lemma 2.2]. But since all vertices of $\Delta_{Y_\bullet}(V)$ are already contained in $\Gamma \cap (\{1\} \times \mathbb{Z}^n)$ we conclude that the latter is a generating set for cone(Γ).

2. This follows from [And, Theorem 5.4].

$\square$

To make this degeneration more explicit, we have to introduce further notions and notation. For more details, the reader may consult [KK], and [And]. Let $V \subset K$ and ν be as above. We define the following graded $\mathbb{K}$-algebra

$$R \;:=\; R(V) \;:=\; \bigoplus_{m \geq 0} V^m \,.$$

One may equip the latter with a valuation $\widehat{\nu} : R(V) \setminus \{0\} \to \mathbb{N} \times \mathbb{Z}^d$ defined by $\widehat{\nu}(f) = (m, \nu(f_m))$, where f_m denotes the homogeneous component of lowest degree in f. Furthermore, we consider $\mathbb{N} \times \mathbb{Z}^d$ with the total ordering given by

$$(m_1, u_1) \leq (m_2, u_{,2}\,) \Longleftrightarrow m_1 < m_2 \;\text{ or }\; \big(m_1 = m_2 \text{ and } u_1 \geq u_2\big).$$

Moreover, for any $(m, u) \in \Gamma(V)$ we set $R_{\leq(m,u)} = \{f \in R \mid \widehat{\nu}(f) \leq (m, u)\}$. This is a finite dimensional $\mathbb{K}$-vector subspace of R, cf. [And, Lemma 2.1]. Furthermore, one has that $R_{\geq(m,u)} \cdot R_{\geq(m',u')} \subset R_{\geq(m+m',u+u')}$. Altogether, these data give rise to the $\Gamma(V)$-graded ring

$$\operatorname{gr} R \;:=\; \bigoplus_{(m,u) \in \Gamma} R_{\leq(m,u)}/R_{<(m,u)} \,.$$

For the sake completeness, we also recall Proposition 5.1 from loc. cit.

Proposition 4.31. Let $R = R(V)$, and assume that $\operatorname{gr} R$ is finitely generated. Then there is a finitely generated, $\mathbb{N}$-graded, flat $\mathbb{K}[t]$-subalgebra $\mathcal{R} \subset R[t]$, such that $\mathcal{R}/t\mathcal{R} \cong \operatorname{gr} R$ and $\mathcal{R}[t^{-1}] \cong R[t, t^{-1}]$ as $\mathbb{K}[t, t^{-1}]$-algebras.

More specifically, there is a linear projection $\pi : \mathbb{Z} \times \mathbb{Z}^d \to \mathbb{Z}$ such that the $\mathbb{N}$-filtration $R_{\leq k} = \{f \in R \mid \pi \circ \widehat{\nu}(f) \leq k\}$ has $\operatorname{gr} R$ as the associated graded algebra. The Rees algebra $\mathcal{R} = \bigoplus_{k \geq 0} (R_{\leq k}) t^k$ for this filtration then has the desired properties.

Finally, the $\mathbb{N}$-grading on R via powers of V is compatible with the one on $\mathcal{R}$ via powers of the variable t. Hence, $\mathcal{R}$ carries a natural $\mathbb{N} \times \mathbb{N}$-grading.

Let us use these objects to give a concrete description of the degeneration mentioned above. Considering a very ample line bundle $\mathcal{L}$ on a smooth projective variety X, one can construct a flat family $\operatorname{Proj}\mathcal{R} = \mathcal{X} \to \mathbb{A}^1$ such that the general fiber $\mathcal{X}_t$ for $t \neq 0$ is isomorphic to $\operatorname{Proj} R(V) = X(V) \cong X$, and the normalization of the special fiber $\mathcal{X}_0 = \operatorname{Proj}(\operatorname{gr} R)$ is isomorphic to the toric variety associated to $\Delta_{Y_\bullet}(V)$:

$$\begin{tikzcd} \mathcal{X}_0 \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d] \\ 0 \arrow[r, hook] & \mathbb{A}^1 . \end{tikzcd}$$

Finally, a sufficient criterion for the limit $\mathcal{X}_0$ to be normal is that

$$\nu(V) = \Delta_{Y_\bullet}(V) \cap \mathbb{Z}^d ,$$

cf. [And, Theorem 1.5].

Given a smooth projective polarized complexity-one T-variety X in terms of a divisorial polytope $(\Psi, \square, \mathbb{P}^1)$, i.e. $X = \mathrm{TV}(\Xi(\Psi))$, we translate this criterion into a criterion on the map Ψ.

Proposition 4.32. Let $(\Psi, \square, \mathbb{P}^1)$ and X be as above, and set $V := \Gamma(X, \mathcal{L}(\Psi))$. Then, we have that $\nu(V) = \Delta_{Y_\bullet}(V) \cap \mathbb{Z}^d$ for a general flag $Y_\bullet$ in X if and only if

$$\sum_{P \in \mathbb{P}^1} \Psi_P(u) - \sum_{P \in \mathbb{P}^1} \lfloor \Psi_P(u) \rfloor < 1 \quad \text{for all} \quad u \in \square \cap M.$$

Proof. First, we assume that $\nu_{Y_\bullet}(V) = \Delta_{Y_\bullet}(V) \cap \mathbb{Z}^d$ and the above difference is greater than or equal to 1 for a specific weight $u \in \square \cap M$. Then we have at least one lattice point $(k, \phi_{\mathbb{R}}(u))$, $0 \leq k \leq \deg \Psi(u)$, inside $\Delta_{Y_\bullet}(V)$ which cannot be the image of a section $f\chi^u$ under the valuation $\nu_{Y_\bullet}$ since the degree of $\sum_P \lfloor \Psi_P(u) \rfloor P$ is too small. There simply are not enough sections which is a contradiction to our assumption.

On the other hand, assume that the condition from above holds. Note that it is enough to prove the inclusion $\nu_{Y_\bullet}(V) \supset \Delta_{Y_\bullet}(V) \cap \mathbb{Z}^d$ because the other one holds trivially. Consider a lattice point $(k, \phi_{\mathbb{R}}(u)) \in \Delta_{Y_\bullet}(V)$. Since k is an integer and $k \leq \sum_{P \in \mathbb{P}^1} \Psi_P(u)$, we must also have that $k \leq \sum_{P \in \mathbb{P}^1} \lfloor \Psi_P(u) \rfloor$. Hence, there is a section $f\chi^u$ whose evaluation is actually equal to $(k, \phi_{\mathbb{R}}(u))$. ∎

Before concluding this section with a result on degenerations of rational smooth projective $\mathbb{K}^*$-surfaces, we recall the notion of an extremal point of a convex set.

Definition 4.33. Let K be a convex set in a real vector space V. A point $v \in K$ is called *extremal* if it does not lie in the interior of a compact line segment contained in K.

Proposition 4.34. Allowing for normalization after each degeneration step, every rational smooth projective $\mathbb{K}^*$-surface X degenerates to a weighted projective space.

Proof. We may assume that X is toric with Picard rank ≥ 2. Note, however, that X need not be smooth. Choosing an invariant ample divisor D and considering ∂P_D as a circular graph whose vertices are the extremal points of P_D, we can find two extremal points $u_1, u_2 \in P_D$ of distance 2, i.e. there is exactly one extremal point in between. In the next step, we choose a primitive generator v of $(u_1 - u_2)^\perp \subset N$ to define a downgrade via the exact sequence

$$0 \longrightarrow (u_1 - u_2)^\perp \xrightarrow{v} N \longrightarrow N/(u_1 - u_2)^\perp \longrightarrow 0\,.$$

It follows that there is at least one elliptic fixed point x_{fix} (corresponding to the unique extremal point in between u_1 and u_2). We continue by constructing a general flag $Y_\bullet$ of type $\mathbf{B_1}$ with respect to x_{fix}. The resulting Okounkov body $\Delta_{Y_\bullet}(D)$ then has one extremal point less than P_D since u_1 and u_2 lie in the same fiber with respect to $v^\vee : M \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big((u_1 - u_2)^\perp, \mathbb{Z}\big)$. Hence, the Picard rank drops and we may proceed by induction. $\square$

Remark 4.35. We would like to point out that Proposition 4.34 can also be shown by using so called degeneration diagrams, as presented in [Ilt10, Section 6.2].

4.3.2 Ilten's Approach

The previous section focused on degenerations. Now, we reverse our point of view and investigate the link between "decompositions" of Okounkov bodies and T-deformations. For the development and detailed treatment of the latter we refer the reader to [Ilt10]. In the following, we briefly recall the fundamental notion of Section 7.3 in loc. cit.

We begin by recalling the notion of a decomposition of a divisorial polytope, cf. [Ilt10, Definition 7.3.1].

Definition 4.36. Let $\Psi : \square \to \mathrm{Div}_{\mathbb{Q}} \mathbb{P}^1$ be a divisorial polytope. An α-*admissible one-parameter decomposition* of Ψ consists of two piecewise affine functions $\Psi_0^0, \Psi_0^1 : \square \to \mathbb{Q}$ such that:

1. The graph of the map Ψ_0^i has lattice vertices for $i = 0, 1$.
2. $\Psi_0(u) = \Psi_0^0(u) + \alpha\Psi_0^1(u)$ for all $u \in \square$.
3. For any full-dimensional polyhedron in $\square$ on which Ψ_0 is affine, Ψ_0^i has non-integral slope on this polyhedron for at most one $i \in \{0, 1\}$.
4. If $\alpha \neq 1$, then Ψ_0^1 always has integral slope.

It is explained in loc. cit. how to construct a one-parameter T-deformation $\pi : \mathcal{X} \to B$ of $\mathrm{TV}(\Xi(\Psi))$ over the affine base $0 \in B \subset \mathbb{A}^1 = \mathrm{Spec}\,\mathbb{K}[t]$ from such a decomposition of Ψ. In order to be more precise, let $\mathcal{X}_s$ denote the fiber $\pi^{-1}(s)$ and let $y_P \in \mathbb{K}(\mathbb{P}^1)$ be a rational function with its sole zero at $P \in \mathbb{P}^1$. Then, for any $s \in B$, one can show that the map $\Psi^{(s)} : \square \to \mathrm{Div}_{\mathbb{Q}} \mathbb{P}^1$ with

$$\Psi^{(s)}(u) = \sum_{P \neq 0} \Psi_P(u) \otimes V(y_P) + \Psi_0^0(u) \otimes V(y_0) + \Psi_0^1(u) \otimes V(y_0^\alpha - s)$$

defines a divisorial polytope such that $\mathcal{X}_s = \mathrm{TV}(\Xi(\Psi^{(s)}))$, cf. [Ilt10, Theorem 7.3.2]. The following diagram once more illustrates the situation:

$$\begin{tikzcd} \mathrm{TV}(\Xi(\Psi)) \arrow[r, equal] & \mathcal{X}_0 \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d] & \mathcal{X}_s \arrow[l, hook'] \arrow[r, equal] \arrow[d] & \mathrm{TV}(\Xi(\Psi^{(s)})) \\ & 0 \arrow[r, hook] & B & s \arrow[l, hook'] & \end{tikzcd}$$

Moreover, the author shows that if D_{Ψ^*} is very and ample and gives us a projectively normal embedding, then the deformation can be realized as an embedded deformation with respect to the embedding induced by the linear system $|\mathcal{O}(D_{\Psi^*})|$, cf. [Ilt10, Theorem 7.3.2].

Remark 4.37. Definition 4.36 is very much related to Proposition 4.32. Indeed, given a divisorial polytope $(\Psi, \Box, \mathbb{P}^1)$, let us assume that the following conditions hold.

1. The graph of the map Ψ_P has lattice vertices for all $P \in \mathbb{P}^1$.
2. For any full-dimensional polyhedron in $\Box$ on which Ψ_P is affine for all $P \in \mathbb{P}^1$ at most one Ψ_P has non-integral slope on this polyhedron.

It is not hard to see that the preconditions of Proposition 4.32 are now fulfilled, i.e. the degeneration yields a normal toric variety $\mathrm{TV}(\Xi(\Psi'))$ which arises from the divisorial polytope given by $\Psi'_0 = \sum_P \Psi_P$ and $\Psi'_\infty \equiv 0$.

4.3.3 Examples

Example 4.38. We return to Example 4.28. Checking our condition from above yields that $\mathbb{F}_n$ degenerates to $\mathbb{P}(1, 1, n + 2)$ which recovers a classical result.

Note that we may also describe this setting as a T-deformation with special fiber $\mathbb{P}(1, 1, n+2)$ and general fiber $\mathbb{F}_n$ in terms of an α-admissible one-parameter decomposition of $\Delta_{Y_\bullet}(D_h)$, cf. [Ilt10, Section 7.3]. To do so, we identify the Okounkov body with the divisorial polytope

$$\Psi : [0, n] \to \mathrm{Div}_{\mathbb{Q}}\, \mathbb{P}^1\,, \text{ with } \begin{cases} \Psi_P(u) &= h_0^*(u) + h_\infty^*(u), \quad P \text{ a fixed point,} \\ \Psi_Q &\equiv 0\,, \qquad\qquad\qquad\quad Q \in \mathbb{P}^1 \setminus P\,. \end{cases}$$

It is not hard to see that Ψ yields $\mathbb{P}(1, 1, n + 2)$ together with the ample line bundle $\mathcal{O}(D_{\Psi^*})$.

Example 4.39. Revisiting the Okounkov body of the (ample) anti-canonical bundle on $X = \mathbb{P}(\Omega_{\mathbb{P}^2})$ with respect to a flag $Y_\bullet^1$ of type $\mathbf{B_1}$ or a flag $Y_\bullet^2$ of type $\mathbf{B_2}$, we see that X degenerates to the toric variety X_*^i which is associated to the normal fan of $\Delta_{Y_\bullet^i}(-K_X)$. Moreover, it is not difficult to find two decompositions such that X_*^i T-deforms into X when the latter decompositions are applied one after another to the divisorial polytope associated to $\Delta_{Y_\bullet^i}(-K_X)$.

Example 4.40. The final remark in the discussion of the Okounkov body of the (ample) anti-canonical bundle on the smooth quadric in $\mathbb{P}^4$ with respect to a flag of type $\mathbf{B_2}$ shows that Q degenerates to the weighted projective space

$X_* := \mathbb{P}(1,1,2,2)$. As above, considering $\Delta_{Y_\bullet}(-K_Q)$ as a divisorial polytope, it is not hard to construct two decompositions of $\Delta_{Y_\bullet}(-K_Q)$ such that there is a T-deformation with special fiber X_* and general fiber Q which arises from the concatenation of these two decompositions.

4.4 The Global Okounkov Body

4.4.1 Lemmata on Polyhedra

Let $v \in \mathbb{R}^k$ be a point, $b = (b_1, \dots, b_k)$ an orthonormal basis of $\mathbb{R}^k$, and $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k_{>0}$. Using these data, we construct a piecewise linear object $T(v, b, \lambda) \subset \mathbb{R}^k$ in the following way:

$$\begin{aligned} T_1 &= \{v + \kappa b_1 \mid 0 \le \kappa \le \lambda_1\} \\ T_i &= T_{i-1} \cup \{v + \textstyle\sum_{j=1}^{i-1} \frac{\lambda_j}{2} b_j + \kappa b_i \mid 0 \le \kappa \le \lambda_i\}\,. \end{aligned}$$

Finally, we arrive at $T_k =: T(v, b, \lambda)$. See Figure 4.8 for an illustration in dimension 2.

Lemma 4.41. Let $f : \mathbb{R}^k \to \mathbb{R}$ be a concave function, and $T := T(v, b, \lambda) \subset \mathbb{R}^k$ as above. If f is affine linear on T then it is also affine linear on its convex hull $\operatorname{conv} T$.

Proof. We give a proof for dimension $k = 2$ and leave the general case to the reader.

Let f^{aff} denote the affine linear extension of $f|_T$ to $\operatorname{conv} T$. Since f is concave we have that $f \ge f^{\text{aff}}$ on $\operatorname{conv} T$. So let us assume that there is a point $v' = v + \kappa\lambda_1 b_1 + \mu\lambda_2 b_2 \in \operatorname{conv} T$ which is not in T (wlog $\kappa > 1/2$) such that $f(v') > f^{\text{aff}}(v')$. Define $v'' := v + (1-\kappa)\lambda_1 b_1 + \mu\lambda_2 b_2 \in \operatorname{conv} T$ ("reflection along the middle axis", see Figure 4.8). Since $f(v'') \ge f^{\text{aff}}(v'')$ and $f(v') > f^{\text{aff}}(v')$ we must have that

$$f(v + 1/2\lambda_1 b_1 + \mu\lambda_2 b_2) > f^{\text{aff}}(v + 1/2\lambda_1 b_1 + \mu\lambda_2 b_2)$$

which is absurd since the argument $v + 1/2\lambda_1 b_1 + \mu\lambda_2 b_2$ is an element of T where both functions agree. ☐

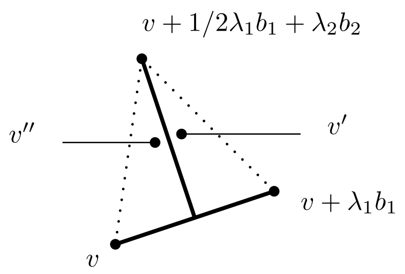


Figure 4.8: An illustration of $T(v, b, \lambda)$ in dimension 2.

Definition 4.42. Consider a set $T := T(v, b, \lambda) \subset \mathbb{R}^k$ as constructed above, and define
$$\mathbb{S}_T^{k-1} := \frac{1}{\epsilon}\big(\operatorname{conv} T \cap \mathbb{S}^{k-1}(v, \epsilon)\big)\,,$$
where $\mathbb{S}^{k-1}(v, \epsilon)$ denotes the sphere of radius $0 < \epsilon \ll 1$ around $v \in \mathbb{R}^k$. The sphere is supposed to be small enough such that no other extremal point of $\operatorname{conv} T$ apart from v is contained in the ball $B(v, \epsilon)$.

Lemma 4.43. Let $P \subset \mathbb{R}^k$ be a k-dimensional polytope and $f : P \to \mathbb{R}_{\geq 0}$ a non-negative concave function. Assume that the set
$$Q_S := \{(x, y) \mid x \in S,\ 0 \leq y \leq f(x)\} \subset \mathbb{R}^{k+1}$$
over any line segment $S \subset P$ is a polytope. Then
$$Q := \{(x, y) \mid x \in P,\ 0 \leq y \leq f(x)\} \subset \mathbb{R}^{k+1}$$
is also a polytope.

Crucial input in the following proof was provided by Christian Haase.

Proof. An element v of P is called a *vertex* if and only if $(v, f(v))$ is an extremal point of Q_S for all line segments $S \subset P$ containing v. Then Q will be the convex hull of
$$V := \{(v, f(v)) \mid v \text{ vertex}\} \cup (P \times \{0\})\,,$$
since Q is the convex hull of its extremal points which are, by definition, all contained in V. To see this, recall that a point of a convex set is called extremal if it does not lie in the middle of a compact line segment contained within this set.

We are left to show that the set of vertices is isolated in Q. So let $v \in P$ be a vertex, and denote by $K_v \subset \mathbb{S}^{k-1}$ the compact subset of directions from v which see other points of P, and define $S_l := P \cap \{v + \kappa l \mid 0 \leq \kappa < \infty\}$ for $l \in K_v$. Then set
$$r_v(l) := \min\{\alpha > 0 \mid (v{+}\alpha l, f(v{+}\alpha l)) \neq (v, f(v)) \text{ is an extremal point of } Q_{S_l}\}\,.$$
Showing that there is an $\epsilon > 0$ such that $r_v(l) \geq \epsilon$ for all $l \in K_v$ will complete the proof, since vertices must appear as extremal points on some Q_{S_l}. So let us consider a direction $l_1 \in K_v$, and set $\lambda_1 = r_v(l)$. From $v_1 := v + \frac{\lambda_1}{2} l$ we can proceed along a direction $l_2 \in (l_1)^\perp$ to set $\lambda_2 := r_{v_1}(l_2)$, and $v_2 = v_1 + \frac{\lambda_2}{2} l_2$. Again, we can walk along a direction $l_3 \in (\langle l_1 \rangle + \langle l_2 \rangle)^\perp$ with $\lambda_3 = r_{v_2}(l_3)$. Iterating this procedure, we finally obtain an object $T(l) = T(v, \lambda(l))$ on which f will be affine linear. So by Lemma 4.41 it will be affine linear on the whole k-dimensional polytope $\operatorname{conv} T(l)$.

Doing this for all elements $l \in K_v$ gives us an infinite covering of K_v by $\mathbb{S}^{k-1}_{T(l)}$. Since K_v is compact we can choose a finite number of them to cover it. Hence, we find an $\epsilon > 0$ such that there is no vertex $v' \in P$ with $d\big((v, f(v)), (v', f(v'))\big) < \epsilon$. $\square$

4.4.2 The Main Result

Before stating the main theorem of this section, we give a short description of the pseudo-effective cone $\overline{\mathrm{Eff}}(\mathrm{TV}(\mathcal{S}))$ of a smooth projective complexity-one T-variety $\mathrm{TV}(\mathcal{S})$.

As in toric geometry, there is an exact sequence describing the divisor class group $\mathrm{Cl}(\mathrm{TV}(\mathcal{S}))$ of a complete rational complexity-one T-variety $\mathrm{TV}(\mathcal{S})$. We denote by $\mathcal{P} \subset \mathbb{P}^1$ the set of points with non-trivial slices $\mathcal{S}_P$. Then we have

$$0 \longrightarrow (\mathbb{Z}^{\mathcal{P}}/\mathbb{Z})^* \oplus M \xrightarrow{\ \iota\ } \text{T-Div}\big(\mathrm{TV}(\mathcal{S})\big) \xrightarrow{\ \mathrm{pr}\ } \mathrm{Cl}(\mathrm{TV}(\mathcal{S})) \longrightarrow 0\,,$$

where $\text{T-Div}\big(\mathrm{TV}(\mathcal{S})\big) \cong \mathbb{Z}^{\mathcal{V}\cup\mathcal{R}}$, cf. Corollary 3.2. So we obtain $\overline{\mathrm{Eff}}(\mathrm{TV}(\mathcal{S}))$ as the image of the positive orthant $\mathbb{R}^{\mathcal{V}\cup\mathcal{R}}_{\geq 0} \subset \mathbb{R}^{\mathcal{V}\cup\mathcal{R}}$ under the map pr, i.e.

$$\overline{\mathrm{Eff}}(\mathrm{TV}(\mathcal{S})) = \mathrm{pr}(\mathbb{R}^{\mathcal{V}\cup\mathcal{R}}_{\geq 0}) \subset \mathrm{Cl}(\mathrm{TV}(\mathcal{S}))_{\mathbb{R}}\,.$$

By definition it is rational polyhedral. Indeed, after choosing a basis in every of these lattices, the maps ι and pr become integer matrices.

Remark 4.44. For the proof of the following theorem we would like to make some identifications.

Elements of the rational pseudo-effective cone $\overline{\mathrm{Eff}}(\mathrm{TV}(\mathcal{S}))_{\mathbb{Q}} := \overline{\mathrm{Eff}}(\mathrm{TV}(\mathcal{S})) \cap N^1(\mathrm{TV}(\mathcal{S}))_{\mathbb{Q}}$ will be denoted by ξ. Having fixed a general or toric flag $Y_\bullet$ in $\mathrm{TV}(\mathcal{S})$, the rational polytopes $\Box_{\underline{\xi}}$ and $W(\xi)$ are defined as $\Box_{\underline{h}}$ and $W(h)$, respectively, for the unique normalized $\mathbb{Q}$-Cartier divisor D_h with $[D_h] = \xi$. In the same vein, for every $P \in \mathbb{P}^1$ we define the map $\xi_P^* : \Box_{\underline{\xi}} \to \mathbb{Q}$ as the map $h_P^* : \Box_{\underline{h}} \to \mathbb{Q}$. Apart from these identifications we will also make use of the following linear map

$$\gamma : N^1(\mathrm{TV}(\mathcal{S}))_{\mathbb{Q}} \to N^1(\mathrm{TV}(\operatorname{tail}\mathcal{S}))_{\mathbb{Q}}, \quad \xi \mapsto \gamma(\xi) = \underline{\xi}\,,$$

whose image of $\overline{\mathrm{Eff}}(\mathrm{TV}(\mathcal{S}))_{\mathbb{Q}}$ lies inside $\overline{\mathrm{Eff}}(\mathrm{TV}(\operatorname{tail}\mathcal{S}))_{\mathbb{Q}}$.

Theorem 4.45. The global Okounkov body $\Delta_{Y_\bullet}(\mathrm{TV}(\mathcal{S}))$ for a rational projective complexity-one T-variety $\mathrm{TV}(\mathcal{S})$ with respect to a general or toric flag $Y_\bullet$ is rational polyhedral.

Proof. Let $Y_\bullet$ be a general flag and denote by $C \subset \mathbb{Q}^{d-1} \oplus N^1(\mathrm{TV}(\mathcal{S}))_{\mathbb{Q}}$ the cone over $\overline{\mathrm{Eff}}(\mathrm{TV}(\mathcal{S}))_{\mathbb{Q}}$ with fiber $\phi_{\mathbb{Q}}(\Box_{\underline{\xi}})$. Its closure $\overline{C}$ in $\mathbb{R}^{d-1} \oplus N^1(\mathrm{TV}(\mathcal{S}))$ is rational polyhedral, since $\overline{\mathrm{Eff}}(\mathrm{TV}(\mathcal{S}))$ is rational polyhedral ($\mathrm{TV}(\mathcal{S})$ is an MDS) and $\overline{C}$ arises as the pull-back of the global Okounkov body $\Delta_{Y_{\geq 1}}\big(\mathrm{TV}(\operatorname{tail}\mathcal{S}))\big)$ along $\mathrm{id}_{\mathbb{R}^{d-1}} \oplus \gamma_{\mathbb{R}}$. By forgetting the first coordinate, we get a projection

$$p : \Delta_{Y_\bullet}(\mathrm{TV}(\mathcal{S})) \longrightarrow \mathbb{R}^{d-1} \oplus N^1(\mathrm{TV}(\mathcal{S}))_{\mathbb{R}}$$

with image exactly equal to $\overline{C}$. Moreover, one can reconstruct $\Delta_{\mathrm{Y}_\bullet}(\mathrm{TV}(\mathcal{S}))$ from C by considering the graph of h^*. Namely, we have that $\Delta_{\mathrm{Y}_\bullet}(\mathrm{TV}(\mathcal{S}))$ is equal to the closure of

$$\left\{ (x, u, \xi) \in \mathbb{Q} \oplus \mathbb{Q}^{d-1} \oplus N^1(\mathrm{TV}(\mathcal{S}))_{\mathbb{Q}} \;\middle|\; \begin{array}{l} (u,\xi) \in C \subset \mathbb{Q}^{d-1} \oplus N^1(\mathrm{TV}(\mathcal{S}))_{\mathbb{Q}}\,, \\ 0 \leq x \leq \sum_{P \in \mathcal{P}} \xi_P^*(u) \end{array} \right\}$$

inside $\mathbb{R} \oplus \mathbb{R}^{d-1} \oplus N^1(\mathrm{TV}(\mathcal{S}))_{\mathbb{R}}$. Observe that the map

$$\mathbf{h}_{\overline{\mathrm{Eff}}} : \mathbb{Q}^{d-1} \oplus N^1(\mathrm{TV}(\mathcal{S}))_{\mathbb{Q}} \supset C \longrightarrow \mathbb{Q}\,,\ (u, \xi) \mapsto \sum_{P \in \mathbb{P}^1} \xi_P^*(u) = \deg \xi^*(u)$$

is concave and linear on rays, i.e.

$$\mathbf{h}_{\overline{\mathrm{Eff}}}(\lambda \cdot (u, \xi)) = \lambda \cdot \mathbf{h}_{\overline{\mathrm{Eff}}}(u, \xi)\,, \quad \lambda \geq 0.$$

We claim that this map varies piecewise affine linearly along any compact line segment

$$S(c_1, c_2) = \{\lambda c_1 + (1 - \lambda) c_2 \mid 0 \leq \lambda \leq 1\} \subset C$$

between two distinct points $c_1 = (u_1, \xi_1),\ c_2 = (u_2, \xi_2) \in C \subset \mathbb{Q}^{d-1} \oplus N^1(\mathrm{TV}(\mathcal{S}))_{\mathbb{Q}}$ with only a finite number of breaks in the linear structure. Note that it is enough to check this for a single summand

$$(\lambda \xi_1 + (1 - \lambda) \xi_2)_P^*(\lambda u_1 + (1 - \lambda) u_2)$$

for an arbitrary but fixed point $P \in \mathbb{P}^1$. Recall that

$$h_P^*(u) = \min\{u(v) - h_P(v) \mid v \in \mathcal{S}_P(0)\}.$$

where $u - h_P$ is a piecewise affine linear function on $N_{\mathbb{Q}}$ and $\mathcal{S}_P(0)$ is a finite set. Note that there exists a real number $\epsilon > 0$ such that, for $0 \leq \lambda, \mu \leq 1$, the functions $\lambda u_1 + (1-\lambda) u_2 - (\lambda (\xi_1)_P + (1-\lambda)(\xi_2)_P)$ and $\mu u_1 + (1-\mu) u_2 - (\mu (\xi_1)_P + (1-\mu)(\xi_2)_P)$ attain their minimum at the same vertex $v \in \mathcal{S}_P(0)$ whenever $|\lambda - \mu| < \epsilon$. Hence, we can partition the line segment $S(c_1, c_2)$ into a finite number of segments along which $(\lambda \xi_1 + (1 - \lambda) \xi_2)_P^*(\lambda u_1 + (1 - \lambda) u_2)$ is in fact affine linear. Taking a rational polytopal cross section of the cone $\overline{\mathrm{Eff}}(\mathrm{TV}(\mathcal{S}))$ and applying Lemma 4.43 then shows that a cross section of $\Delta_{Y_\bullet}(\mathrm{TV}(\mathcal{S}))$ is a rational polytope. Since $\Delta_{Y_\bullet}(\mathrm{TV}(\mathcal{S}))$ arises as the cone over this rational polytopal cross section it has to be rational polyhedral, too.

Finally, let $Y_\bullet$ be a toric flag and denote by $C \subset \mathbb{Q}^{d-1} \oplus N^1(\mathrm{TV}(\mathcal{S}))_{\mathbb{Q}}$ the cone over $\overline{\mathrm{Eff}}(\mathrm{TV}(\mathcal{S}))_{\mathbb{Q}}$ with fiber $W(\xi)$ for $\xi \in \overline{\mathrm{Eff}}(\mathrm{TV}(\mathcal{S}))_{\mathbb{Q}}$. We only have to show that the closure $\overline{C} \subset \mathbb{R}^{d-1} \oplus N^1(\mathrm{TV}(\mathcal{S}))_{\mathbb{R}}$ is rational polyhedral since $\Delta_{\mathrm{Y}_\bullet}(\mathrm{TV}(\mathcal{S}))$ arises as the pairing of $\overline{C}$ with primitive generators of the rays of δ_{fix}, cf. Propositions 4.23 and 4.25. But this claim follows easily from the explicit description of the rational polytope $W(h)$ and the general arguments concerning the piecewise affine structure of " $\cdot_P^*$ " as a function in ξ we have given above in the first part of the proof. □

Remark 4.46. Theorem 4.45 shows that the global Okounkov body of a rational projective complexity-one T-variety $\mathrm{TV}(\mathcal{S})$ is determined by the global Okounkov body of the general fiber $\mathrm{TV}(\operatorname{tail} \mathcal{S})$ and the function " * " which maps an element $\xi \in N^1(\mathrm{TV}(\mathcal{S}))_{\mathbb{Q}}$ to ξ^*.

Our result generalizes Theorem 5.2 from [Gona] which states that the global Okounkov body of a rank two toric vector bundle on a smooth projective toric surface with respect to a flag of type $\mathbf{B_2}$ (cf. Remark 4.15) is rational polyhedral. However, it does not give us any explicit equations for $\Delta_{\mathrm{Y}_\bullet}(\mathrm{TV}(\mathcal{S}))$ as they were obtained in loc. cit.

4.5 Outlook

Okounkov Bodies The downgrade approach provided us with new insights into the theory of Okounkov bodies of toric varieties. Nevertheless, there still are many open questions one might address. Three of these are:

Can one explicitly describe the global Okounkov body of a smooth projective toric variety $\mathrm{TV}(\Sigma)$ *when considering it as a complexity-one* T*-variety together with a general flag* $Y_\bullet$*?*

Using the downgrade method, how "close" are we to the computation of the infinitesimal Okounkov body for a smooth projective toric variety?

Can one partially extend the proof of Theorem 4.17 to flags of type $\mathbf{A_1}$ *on non-rational complexity-one* T *varieties?*

Relations to Cox Rings By explicitly describing the Cox rings of complete rational complexity-one T-varieties, Jürgen Hausen and Hendrik Süß showed in [HS] that the latter are Mori dream spaces. Shortly after, a different approach towards the computation of the Cox ring for the subclass of projectivized rank two toric vector bundles was presented by José González in [Gona] and [Gonb]. His detailed description of the global Okounkov body of a projectivized rank two toric vector bundle $\mathbb{P}(\mathcal{E})$ also provided him with a method to show that the total coordinate ring of these objects is finitely generated.

Is it possible to extend this proof to show that the Cox ring of a rational projective complexity-one T*-variety is finitely generated?*

More generally, is it possible to establish a link between some/any global Okounkov body of a variety and the Cox ring, i.e. does the "combinatorial complexity" of some $\Delta_{\mathrm{Y}_\bullet}(X)$ *relate to the finite generation of* $\mathrm{Cox}(X)$*?*

In this sense, one might be tempted to define an Okounkov dream space (ODS) as a projective variety X which admits a flag $Y_\bullet$ such that $\Delta_{\mathrm{Y}_\bullet}(X)$ is rational polyhedral. This leads us to the following question:

Are the notions of MDS and ODS equivalent?

Bibliography

[AH06] Klaus Altmann and Jürgen Hausen. Polyhedral divisors and algebraic torus actions. *Math. Ann.*, 334(3):557–607, 2006.

[AH08] Klaus Altmann and Georg Hein. A fansy divisor on $\overline{M}_{0,n}$. *J. Pure Appl. Algebra*, 212(4):840–850, 2008.

[AHS08] Klaus Altmann, Jürgen Hausen, and Hendrik Süß. Gluing affine torus actions via divisorial fans. *Transformation Groups*, 13(2):215–242, 2008.

[And] Dave Anderson. Okounkov bodies and toric degenerations. Eprint. arXiv:1001.4566.

[AP] Klaus Altmann and Lars Petersen. Cox rings of rational complexity-one T-varieties. Eprint. arXiv:1009.0478.

[AW] Klaus Altmann and Jarosław Wiśniewski. P-Divisors of Cox rings. Eprint. arXiv:0911.5167.

[BH03] Florian Berchtold and Jürgen Hausen. Homogeneous coordinates for algebraic varieties. *J. Algebra*, 266:636–670, 2003.

[CLS] David Cox, John Little, and Hal Schenck. Toric Varieties (preliminary version). Temporarily available at http://www.cs.amherst.edu/∼dac/toric.html.

[Cox95] David Cox. The homogeneous coordinate ring of a toric variety. *J. Algebraic Geometry*, 4(1):17–50, 1995.

[Dan78] V. I. Danilov. The geometry of toric varieties. *Russ. Math. Surv.*, 33(2):97–154, 1978.

[FMSS95] W. Fulton, R. MacPherson, F. Sottile, and B. Sturmfels. Intersection theory on spherical varieties. *J. Algebraic Geometry*, 4:181–193, 1995.

[Ful93] William Fulton. *Introduction to Toric Varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[Gona] José Luis González. Okounkov bodies on projectivizations of rank two toric vector bundles. Eprint. arXiv:0911.2287.

[Gonb] José Luis González. Projectivized rank two toric vector bundles are Mori dream spaces. Eprint. arXiv:1001.0838.

[Hau08] Jürgen Hausen. Cox rings and combinatorics II. *Mosc. Math. J.*, 8:711–757, 2008.

[HK00] Y. Hu and S. Keel. Mori Dream Spaces and GIT. *Michigan Math. J.*, 48:331–348, 2000.

[HS] Jürgen Hausen and Hendrik Süß. The Cox ring of an algebraic variety with torus action. Eprint. arXiv:0903.4789.

[Ilt10] Nathan Owen Ilten. *Deformations of Rational Varieties with Codimension-One Torus Action.* PhD thesis, Freie Universität Berlin, 2010.

[ISa] Nathan Owen Ilten and Hendrik Süß. AG Codes from polyhedral divisors. Eprint. arXiv:0811.2696.

[ISb] Nathan Owen Ilten and Hendrik Süß. Polarized complexity-one T-varieties. Eprint. arXiv:0910.5919.

[IV] Nathan Owen Ilten and Robert Vollmert. Deformations of rational T-varieties. Eprint. arXiv:0903.1393.

[IV10] Nathan Owen Ilten and Robert Vollmert. Upgrading and downgrading torus actions. Work in progress, 2010.

[KK] K. Kaveh and A. G. Khovanskii. Newton convex bodies, semigroups of integral points, graded algebras and intersection theory. Eprint. arXiv:0904.3350.

[KKMSD73] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal Embeddings I.* Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1973.

[Kly90] A. A. Klyachko. Equivariant bundles on toral varieties. *Math. USSR Izvestiya*, 35:337–375, 1990.

[Laz04] Robert Lazarsfeld. *Positivity in Algebraic Geometry I.* Springer, Berlin, 2004.

[LM] Robert Lazarsfeld and Mircea Mustaţă. Convex bodies associated to linear series. Eprint. arXiv:0805.4559.

[Mav] Anvar R. Mavlyutov. Deformations of toric varieties via Minkowski sum decompositions of polyhedral complexes. Eprint. arXiv:0902.0967.

[Mum74] David Mumford. *Abelian varieties. With appendices by C. P. Ramanujam and Yuri Manin. 2nd ed.* Tata Institute of Fundamental Research Studies in Mathematics. Oxford University Press, London, 1974.

[Oda88] Tadao Oda. *Convex Bodies and Algebraic Geometry*, volume 15 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer, Berlin, 1988.

[Oko96] Andrei Okounkov. Brunn-Minkowski inequality for multiplicities. *Invent. Math.*, 125:405–411, 1996.

[Oko03] Andrei Okounkov. Why would multiplicities be log-concave?, in *The orbit method in geometry and physics. Progr. Math.*, 213:329–347, 2003.

[PS] Lars Petersen and Hendrik Süß. Torus invariant divisors. Eprint. arXiv:0811.0517.

[Süß] Hendrik Süß. Canonical divisors on T-varieties. Eprint. arXiv:0811.0626.

[Süß10] Hendrik Süß. *Drei Klassifikationsprobleme für Varietäten mit Toruswirkung der Komplexität eins.* PhD thesis, Brandenburgische Technische Universität Cottbus, 2010.

[Tim97] D. A. Timashev. Classification of G-varieties of complexity 1. *Math. USSR-Izv.*, 61(2):363–397, 1997.

[Tim00] D. A. Timashev. Cartier Divisors and Geometry of Normal G-varieties. *Transformation Groups*, 5(2):181–204, 2000.

[Vol11] Robert Vollmert. *Deformations of Affine T-Varieties.* PhD thesis, Freie Universität Berlin, 2011.

Zusammenfassung

Der Schwerpunkt dieser Arbeit liegt auf der Beschreibung äquivarianter Geradenbündel auf T-Varietäten der Komplexität eins sowie auf zwei Anwendungen, die sich aus jener Beschreibung ergeben.

Grundlegend für diese Dissertation ist dabei die Sprache der polyedrischen Divisoren und divisoriellen Fächer, die von Klaus Altmann, Jürgen Hausen und Hendrik Süß entwickelt wurde und in Analogie zur Korrespondenz zwischen torischen Varietäten und polyedrischen Fächern eine solche für T-Varietäten und divisorielle Fächer liefert. Die für die vorliegende Schrift wesentlichen Aspekte dieser Theorie werden im ersten Kapitel präsentiert.

Um die Notation im folgenden zu erleichtern, sei eine T-Varietät der Komplexität eins zum divisoriellen Fächer $\mathcal{S}$ über der Kurve Y mit $\mathrm{TV}(\mathcal{S})$ und eine durch $\mathcal{S}$ gegebene polyedrische Unterteilung über dem Punkt $P \in Y$ mit $\mathcal{S}_P$ bezeichnet.

Im zweiten Kapitel werden äquivariante Geradenbündel auf $\mathrm{TV}(\mathcal{S})$ mit sogenannten Trägerfunktionen in Verbindung gebracht, wobei letztere stetige, affin lineare Funktionen darstellen, die auf den polyedrischen Unterteilungen $\mathcal{S}_P$ definiert sind. Hierbei wird gezeigt, dass die Gruppe der Cartier-Trägerfunktionen auf $\mathcal{S}$ isomorph zur Gruppe der T-invarianten Cartier-Divisoren auf $\mathrm{TV}(\mathcal{S})$ ist. Für ein gegebenes äquivariantes Geradenbündel ermöglicht die sich daraus ergebende teils kombinatorische, teils geometrische Beschreibung eine explizite Berechnung des zugehörigen graduierten Moduls der globalen Schnitte. Ferner wird ein Kohomologie-Verschwindungssatz für numerisch effektive Divisoren auf toroidalen T-Varietäten der Komplexität eins bewiesen.

Im anschließenden dritten Kapitel werden obige Erkenntnisse zur Theorie der äquivarianten Geradenbündel benutzt, um in einer ersten Anwendung eine Beschreibung des Cox-Rings kompletter rationaler T-Varietäten $\mathrm{TV}(\mathcal{S})$ mit freier Divisorenklassengruppe durch einen polyedrischen Divisor $\mathcal{D}_{\mathrm{Cox}}$ zu geben. Motiviert durch ein entsprechendes Resultat aus der torischen Geometrie wird gezeigt, dass sich $\mathcal{D}_{\mathrm{Cox}}$ insofern auf natürliche Weise konstruieren lässt, als er die affin linearen Abhängigkeiten zwischen den Ecken der polyedrischen Unterteilungen $\mathcal{S}_P$ auflöst. Abschließend wird auf einen interessanten Zusammenhang mit einer bereits bekannten, aber wesentlich verschiedenen Konstruktion von Klaus Altmann und Jarek Wiśniewski hingewiesen.

Als zweite Anwendung wird im vierten und letzten Kapitel die Berechnung von Okounkov-Körpern rationaler, projektiver T-Varietäten der Komplexität eins zum Gegenstand der Betrachtungen. Dazu werden zwei Typen von T-invarianten Flaggen konstruiert – allgemeine sowie torische. Es wird gezeigt, dass die jeweils resultierenden Okounkov-Körper rational polyedrisch sind. Im weiteren Verlauf werden spezielle torische Degenerationen, wie sie von Dave Anderson beschrieben wurden, mit Okounkov-Körpern in Verbindung gebracht und anhand diverser Beispiele diskutiert. Umgekehrt hat Nathan Ilten Zerlegungen von divisoriellen Polytopen mit T-Deformationen in Beziehung gesetzt, was hier an einzelnen Beispielen in vollkommener Analogie auch für Zerlegungen von Okounkov-Körpern getan wird. Schlussendlich wird gezeigt, dass auch der globale Okounkov-Körper einer rationalen, projektiven T-Varietät der Komplexität eins bzgl. beider Flaggentypen rational polyedrisch ist. Dies verallgemeinert ein Ergebnis von José González, der obige Eigenschaften für globale Okounkov-Körper projektivierter torischer Vektorbündel vom Rang 2 nachgewiesen hat.