Appendix A

Derivation of Rytov's solution

A.1 First-order approximation

A solution of equation (2.38) is now constructed in the form of the expansion $\Psi = \Psi_1 + \Psi_2 + ..$, where Ψ_1 is of the order $\sqrt{\langle n^2 \rangle}$, Ψ_2 of the order $\langle n^2 \rangle$ etc. . This derivation considers the 2-D case. Note that the 3-D case may derived analogously. Equating terms of the same order in $\langle n^2 \rangle$ to zero, the following system of coupled, partial differential equations is obtained:

$$\begin{aligned} 2ik\frac{\partial \Psi_1}{\partial z} + \frac{\partial^2 \Psi_1}{\partial x^2} &= -2k^2 n \\ 2ik\frac{\partial \Psi_2}{\partial z} + \frac{\partial^2 \Psi_2}{\partial x^2} &= -\left(\frac{\partial \Psi_1}{\partial x}\right)^2 \\ \cdots \quad &\quad \cdots \quad . \end{aligned} \tag{A.1}$$

This set of equations describe the log-amplitude and phase fluctuations in the half-space $z > 0$ if the plane wave (2.35) enters from the half-space $z < 0$. The boundary conditions for equations (A.1) are then $\Psi_i(x, z = 0) = 0$. The second equation of the set (A.1) refers to the so-called second-order Rytov approximation which we are going to solve as soon as an expression for the first-order approximation is derived. Note that all these partial differential equations have the same structure. Applying the Green's function approach, any of these equations has the solution

$$\Psi_n(L, x) = \int_0^L dz' \int_{-\infty}^{\infty} dx' G(z - z', x - x') f_n(z', x') \,, \tag{A.2}$$

where $f_n(z, x)$ denotes the terms on the right side of the equations (A.1), respectively. G denotes the Green's function of the operator $2ik\frac{\partial}{\partial z} + \frac{\partial^2}{\partial x^2}$, i.e., it is determined by

$$2ik\frac{\partial G}{\partial z} + \frac{\partial^2 G}{\partial x^2} = f_n \tag{A.3}$$

with $f_n(z, x) = \delta(z)\delta(x)$. To determine G we assume that Ψ and f_n can be represented by a Fourier integral respective to the transverse coordinate:

$$\Psi_n(z, x) = \int_{-\infty}^{\infty} d\kappa \, e^{i\kappa x} \rho(z, \kappa) \qquad \rightleftharpoons \qquad \rho(z, \kappa) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' e^{-i\kappa x'} \Psi_n(z, x') \tag{A.4}$$

$$f_n(z, x) = \int_{-\infty}^{\infty} d\kappa \, e^{i\kappa x} \mu(z, \kappa) \qquad \rightleftharpoons \qquad \mu(z, \kappa) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' e^{-i\kappa x'} f_n(z, x') \,. \tag{A.5}$$

The Fourier transformation of equation (A.3) yields

$$2ik\frac{\partial\rho(z,\kappa)}{\partial z} - \kappa^2\rho(z,\kappa) = \mu(z,\kappa) \tag{A.6}$$

which has the solution

$$\rho(z,\kappa) = \frac{1}{2ik}\int_0^z dz' e^{-i\kappa^2\frac{z-z'}{2k}}\,\mu(z',\kappa)\,. \tag{A.7}$$

The inverse Fourier transform yields finally the Green's function

$$\begin{aligned} G(z-z',x-x') &= \frac{1}{4\pi ik}\int_{-\infty}^{\infty} d\kappa\; e^{i\left[\kappa(z-z')-\frac{\kappa^2(x-x')}{2k}\right]} \\ &= \frac{1}{4\pi ik}\sqrt{\frac{2k\pi}{i(z-z')}}\quad e^{\left[\frac{ik(x-x')^2}{2(z-z')}\right]}\,. \end{aligned} \tag{A.8}$$

The 3-D analogon can be found in Rytov et al. (1987) equation (2.37).

The calculation of the first Rytov solution is then straightforward:

$$\Psi_1(L,x) = \int_0^L dz' \int_{-\infty}^{\infty} dx' \frac{1}{2\pi ik}\sqrt{\frac{2k\pi}{i(z-z')}}\quad e^{\left[\frac{ik(x-x')^2}{2(z-z')}\right]}\left(-2k^2 n(z,x)\right). \tag{A.9}$$

Adopting the notation of Ishimaru (1978), we represent $n(z,x)$ by its spectral expansion

$$n(z,x) = \int_{-\infty}^{\infty} e^{i\kappa x} d\nu(z,\kappa)\,, \tag{A.10}$$

where $d\nu$ denotes the Fourier-Stieltjes differential. Its properties are thoroughly discussed in appendix A of Ishimaru's book. With equation (A.10) we express equation (A.9) after performing the integral over dz' (see Prudnikov et al., 1988, pp. 344, No. 13) as

$$\Psi_1(L,x) = ik\int_0^L dz' \int_{-\infty}^{\infty} d\nu(z',\kappa)\, e^{i\left[\kappa x - \frac{\kappa^2}{2k}(L-z')\right]}\,. \tag{A.11}$$

Note that this is exactly equation (B4) in Shapiro et al. (1996b), where it has been derived using the approach of Ishimaru (1978). The calculation of the variance of equation (A.11) is also shown in detail in this reference. The real part is given by (see also equation (B.1) for the 3-D case)

$$\sigma^2_{\chi\chi} = 2\pi k^2 L\int_0^{\infty} d\kappa\,\kappa\left(1-\frac{\sin(\kappa^2 L/k)}{\kappa^2 L/k}\right)\Phi^{2D}(\kappa)\,. \tag{A.12}$$

A.2 Second-order approximation

To show that equation (A.12) can be obtained from the second-order Rytov approximation, we solve now equation (A.1) with exactly the same Green's function approach. From equation (A.2) we obtain

$$\Psi_2(L,x) = \int_0^L dz' \int_{-\infty}^{\infty} dx' \frac{1}{4\pi ik}\sqrt{\frac{2k\pi}{i(z-z')}}\quad e^{\left[\frac{ik(x-x')^2}{2(z-z')}\right]}\left(-\left[\frac{\partial\Psi_1}{\partial x}\right]^2\right). \tag{A.13}$$

The term in brackets requires that we differentiate equation (A.11) with respect to x, take the square and insert this into equation (A.13). Straightforward calculations yield

$$\Psi_2(L,x) = \frac{ik}{2}\int_0^L dz' \iint_0^{z'} dz''d\zeta'' \iint_{-\infty}^{\infty} d\nu(z'',\kappa)d\nu(\zeta'',\alpha)$$
$$\alpha\,\kappa\, e^{ix(\kappa+\alpha)}\, e^{i\frac{\kappa^2 z''}{2k}}\, e^{i\frac{\alpha^2\zeta''}{2k}}\, e^{-i\frac{(\kappa+\alpha)^2}{2k}L}\, e^{i\frac{\kappa\alpha}{k}z'}\,. \qquad \text{(A.14)}$$

It remains to determine the mean value of equation (A.14). Noting the identity $\langle d\nu(z'',\kappa)\, d\nu(\zeta'',\alpha)\rangle = F(z''-\zeta'',\kappa)\,\delta(\kappa+\alpha)\,d\kappa\, d\alpha$, where F denotes the 1-D Fourier transform of the autocorrelation function of $n(z,x)$, we obtain

$$\langle\Psi_2(L)\rangle = \frac{ik}{2}\int_0^L dz' \iint_0^{z'} dz''d\zeta'' \int_{-\infty}^{\infty} d\kappa\, F(z''-\zeta'',\kappa)\,(-\kappa^2)\; e^{i\frac{\kappa^2}{2k}(z''+\zeta'')}\, e^{-i\frac{\kappa^2}{k}z'}\,. \qquad \text{(A.15)}$$

Note that at this stage there is no more dependency on the transverse coordinate x involved. Introducing the difference and center of mass coordinates $\eta = z''-\zeta''$, $\theta = \frac{z''+\zeta''}{2}$ and adopting the approximation with respect to the area of integration as in chapter 17 of Ishimaru (1978), we can perform the integration of the space variable θ. Using the relationship

$$\Phi^{2D}(\kappa) = \frac{1}{\pi}\int_0^{\infty} d\eta\, F(\eta,\kappa) \qquad \text{(A.16)}$$

for the case of isotropic random functions, it is also possible to integrate with respect to η

$$\begin{aligned}\langle\Psi_2(L)\rangle &= i\pi k\int_0^L dz' \int_{-\infty}^{\infty} d\kappa\,\Phi^{2D}(\kappa)\,(-\kappa^2)\,\frac{k}{i\kappa^2}\left(1-e^{-i\frac{\kappa^2 z'}{k}}\right)\\ &= -\pi k^2\int_{-\infty}^{\infty} d\kappa\,\Phi^{2D}(\kappa)\left[L-\left(\frac{k}{-i\kappa^2}\left(e^{-i\kappa^2L/k}-1\right)\right)\right]. \end{aligned}\qquad \text{(A.17)}$$

Finally, separating real and imaginary part and noting that the integrand is an even function of κ yields

$$\Re\langle\Psi_2(L)\rangle = -2\pi k\int_0^{\infty} d\kappa\,\Phi^{2D}(\kappa)\left[L-\frac{k}{\kappa}\sin\left(\frac{\kappa^2 L}{k}\right)\right] \qquad \text{(A.18)}$$

$$\Im\langle\Psi_2(L)\rangle = -2\pi k\int_0^{\infty} d\kappa\,\Phi^{2D}(\kappa)\left[\frac{k}{\kappa^2}-\frac{k}{\kappa^2}\cos\left(\frac{\kappa^2 L}{k}\right)\right]. \qquad \text{(A.19)}$$

Comparing equations (A.18) and (A.12), we find the following relation between the mean value of $\Re\langle\Psi_2\rangle = \langle\chi\rangle$ obtained in the second-order Rytov approximation and its variance

$$\langle\chi\rangle = -\sigma^2_{\chi\chi}\,. \qquad \text{(A.20)}$$

Appendix B

Explicit expressions for the wave field attributes

B.1 Log-amplitude and phase variances

Simple expressions for the quantities $\sigma^2_{\chi\chi}, \sigma^2_{\chi\phi}$ and ϕ_c are known (Ishimaru, 1978 and Rytov et al., 1987) . The results corresponding to plane wave propagation in 3-D media are

$$\sigma^2_{\chi\chi} = 2\pi^2 k^2 L \int_0^\infty d\kappa\,\kappa \left(1 - \frac{\sin(\kappa^2 L/k)}{\kappa^2 L/k}\right) \Phi^{3D}(\kappa) \tag{B.1}$$

$$\sigma^2_{\chi\phi} = 4\pi^2 k^3 \int_0^\infty d\kappa\,\kappa \left(\frac{\sin^2(\kappa^2 L/2k)}{\kappa^2}\right) \Phi^{3D}(\kappa) \tag{B.2}$$

$$\sigma^2_{\phi\phi} = 2\pi^2 k^2 L \int_0^\infty d\kappa\,\kappa \left(1 + \frac{\sin(\kappa^2 L/k)}{\kappa^2 L/k}\right) \Phi^{3D}(\kappa) \quad . \tag{B.3}$$

In these equations $\Phi^{3D}(\kappa)$ denote the fluctuation spectra which are the 3-D Fourier transforms of media correlation functions. The terms in brackets are the so-called spectral filter functions or Fresnel filters (since they act on the fluctuation spectra like filters; their behavior for the different wave field ranges is thoroughly discussed in Ishimaru,1978). For the 2-D case the results can be obtained by skipping κ in the integral over $d\kappa$, dividing by π and using the 2-D fluctuation spectra.

For spherical waves the quantities $\sigma^2_{\chi\chi}, \sigma^2_{\chi\phi}$ are (Ishimaru, 1978, chapter 18, and Ishimaru, 1972):

$$\sigma^2_{\chi\chi} = 4\pi^2 k^2 \int_0^L d\eta \int_0^\infty d\kappa\,\kappa \sin^2\left(\frac{\eta}{L}\frac{(L-\eta)}{2k}\kappa^2\right) \Phi^{3D}(\kappa) \tag{B.4}$$

$$\sigma^2_{\chi\phi} = 4\pi^2 k^2 \int_0^L d\eta \int_0^\infty d\kappa\,\kappa \sin\left(\frac{\eta}{L}\frac{(L-\eta)}{k}\kappa^2\right) \Phi^{3D}(\kappa) \tag{B.5}$$

with the same notations as above. Changing the order of integration and integrate over η we find

$$\sigma^2_{\chi\chi} = 2\pi^2 k^2 L \int_0^\infty d\kappa\,\kappa\,\Phi^{3D}(\kappa) \left[1 - \frac{\cos(\pi/2A^2)\,C(A) + \sin(\pi/2A^2)\,S(A)}{A}\right] \tag{B.6}$$

$$\sigma^2_{\chi\phi} = 2\pi^2 k^2 L \int_0^\infty d\kappa\,\kappa\,\Phi^{3D}(\kappa) \left[\frac{\sin(\pi/2A^2)\,C(A) - \cos(\pi/2A^2)\,S(A)}{A}\right] , \tag{B.7}$$

where $A = \sqrt{\frac{\kappa^2 L}{2\pi k}}$ and C, S denote the Fresnel integrals

$$C(x) = \int_0^x dt \cos\left(\frac{\pi}{2}t^2\right) \qquad S(x) = \int_0^x dt \sin\left(\frac{\pi}{2}t^2\right), \tag{B.8}$$

respectively. Again, the corresponding expressions for 2-D media are obtained by skipping κ in the integral over $d\kappa$, dividing by π and using the 2-D fluctuation spectra.

Explicit expressions for the coherent phase are obtained with help of the Bourret approximation. The result in 3-D media is

$$\phi_c - \phi_0 = \pi k^2 L \int_0^\infty d\kappa\, \kappa \ln\left(\frac{2k+\kappa}{2k-\kappa}\right)^2 \Phi^{3D}(\kappa) \quad , \tag{B.9}$$

whereas in 2-D media one obtains

$$\phi_c - \phi_0 = 4\pi k^3 L \int_{2k}^\infty d\kappa \frac{\Phi^{2D}(\kappa)}{\sqrt{\kappa^2 - 4k^2}} \quad . \tag{B.10}$$

Like in equations (B.1)-(B.2), $\Phi(\kappa)$ are the fluctuation spectra.

B.2 Exponential and Gaussian media

Here the explicit results for exponential and Gaussian random media are presented. Equations (3.17) and (3.18) can be further simplified if we choose a special type of of correlation function. Namely, for exponentially correlated fluctuations, i.e. the correlation function and its 2-D Fourier transform are given by

$$B_n(r) = \sigma_n^2 e^{-r/a} \qquad \rightleftharpoons \qquad \Phi^{2D}(\kappa) = \frac{\sigma_n^2 a^2}{2\pi(1+\kappa^2 a^2)^{3/2}} \quad , \tag{B.11}$$

where a is called the correlation length and σ_n^2 is the variance of the velocity fluctuations, we obtain

$$\begin{aligned} \langle\chi\rangle &= -\frac{1}{2}\sigma_n^2 (ak)^3 D \left[1 + \frac{\pi}{2}\left(J_1(D/4)\cos(D/4) + Y_1(D/4)\sin(D/4)\right)\right] && \text{(B.12)} \\ \langle\phi\rangle &= \sigma_n^2 (ak)^4 D \left[g\left(F(\frac{\pi}{2}, g) - E(\frac{\pi}{2}, g)\right) - \frac{2}{Dka^2}\int_0^\infty d\kappa \frac{\sin^2(\kappa^2 D a^2/4)}{\kappa^2(1+\kappa^2 a^2)^{\frac{3}{2}}}\right] &&, \text{(B.13)} \end{aligned}$$

J_1, Y_1 are the Bessel functions of first and second kind, their argument D is the wave parameter. F, E are the elliptic integrals of the first and second kind with the second argument $g = \frac{1}{\sqrt{1+4a^2k^2}}$.

For Gaussian correlated fluctuations and its corresponding 2-D Fourier transform

$$B(r) = \sigma_n^2 e^{-r^2/a^2} \qquad \rightleftharpoons \qquad \Phi^{2D}(\kappa) = \frac{\sigma_n^2 a^2}{4\pi} e^{-\kappa^2 a^2/4} \quad , \tag{B.14}$$

we obtain:

$$\begin{aligned} \langle\chi\rangle &= -\frac{\sqrt{\pi}}{4}\sigma_n^2 (ak)^3 D \left(1 - \frac{1}{D}(1+4D^2)^{\frac{1}{4}} \sin\left[\frac{1}{2}\arctan(2D)\right]\right) && \text{(B.15)} \\ \langle\phi\rangle &= -\frac{1}{4}\sigma_n^2 (ak)^4 D \left[\exp(-\frac{k^2a^2}{4})K_0(\frac{k^2a^2}{2}) - \frac{1}{ka}\sqrt{\frac{2\pi}{D}}\right. \\ &\quad \left.\left(\left(1+\frac{1}{4D^2}\right)^{\frac{1}{4}} \cos\left(\frac{1}{2}\arcsin\left[1+\frac{1}{4D^2}\right]^{-\frac{1}{2}}\right) - \sqrt{\frac{1}{2D}}\right)\right] &&, \text{(B.16)} \end{aligned}$$

where K_0 is the modified Bessel function of the second kind (MacDonald function).

For an exponential correlation function with its 3-D Fourier transform

$$\Phi^{3D}(\kappa) = \frac{\sigma_n^2 a^3}{\pi^2(1+\kappa^2 a^2)^2} \quad , \tag{B.17}$$

we can find explicit expressions for the mean log amplitude and phase fluctuations:

$$\begin{aligned} \langle\chi\rangle &= -\frac{1}{8}(ak)^3\sigma_n^2 D^2 \left[D A_{\frac{1}{2}} + \frac{\pi}{2}\, {}_1F_2\left(1, [2, \frac{1}{2}], \frac{-D^2}{16}\right)\right] \\ \langle\phi\rangle &= \sigma_n^2 (ak)^3 D \left[akg^2 - \frac{\pi}{2D}\sin(\frac{D}{2}) + \frac{\pi}{4}\cos(\frac{D}{2}) + \frac{D}{4}A_0\right] \quad . \end{aligned} \tag{B.18}$$

In equations (B.18) we use

$$A_l = \sum_{i=0}^{\infty}\left[-\Psi(\frac{1}{2}+l+i) - \Psi(2+l+i) + \ln\left(\frac{D^2}{16}\right)\right]\left(\frac{-D^2}{16-24l}\right)^i \frac{l+1/2+i}{\Gamma(3+2l+2i)} \tag{B.19}$$

with the Gamma function Γ and the Psi or di-Gamma function $\Psi(x) = \frac{d}{dx}\ln\Gamma(x)$ (Gradstheyn (1983), chapter 8.36); ${}_1F_2$ is the generalized hypergeometric function, which can be simplified in the region $D \ll 1$:

$${}_1F_2 \approx \frac{1}{\sqrt{\pi}}\left(1 - \frac{D^2}{16}\right) \quad . \tag{B.20}$$

Furthermore, we present the explicit results of $\langle\chi\rangle$ and $\langle\phi\rangle$ for a Gaussian correlation function with its 3-D Fourier transform

$$\Phi^{3D}(\kappa) = \frac{\sigma_n^2 a^3}{8\pi^{\frac{3}{2}}} e^{-\kappa^2 a^2/4} \quad . \tag{B.21}$$

We find

$$\langle\chi\rangle = -\frac{\sqrt{\pi}}{8}(ak)^3\sigma_n^2\left[2D - \arctan(2D)\right] \tag{B.22}$$

$$\langle\phi\rangle = \frac{\sqrt{\pi}}{16}(ak)^3\sigma_n^2\left[8\sqrt{\pi}D\,\mathrm{daw}(ka) - \ln(4D^2+1)\right] \quad . \tag{B.23}$$

In equation (B.23) we use the Dawson-Integral $\mathrm{daw}(x) = e^{-x^2}\int_0^x dt\, e^{t^2}$. Note that equation (B.22) coincides with equation (IV,2.80) of Rytov et al. (1989).

Appendix C

Dispersion relations with n subtractions

The derivation of the Kramers-Kronig relations for a transfer function of a passive and linear medium is presented in this section. More comprehensive derivations can be found in Weaver and Pao (1981), Mobley et al. (2000) and Nussenzveig (1972). Aki and Richards (1980) and Ben-Menhahem and Singh (1981) give also a brief outline of the causality principle. Discussions of the use of the causality principle in the framework of multiple scattering theories can be found in Weaver (1986) and Beltzer (1989).

Let $H(\omega) = A(\omega) + iB(\omega)$ denote the Fourier transform of the impulse response $h(t)$ of a linear system. Then by means of Titchmarsh's theorem the real and imaginary part of $H(\omega)$ are related by a pair of Hilbert transforms:

$$\begin{aligned} A(\omega) &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{B(\omega')}{\omega' - \omega} d\omega' \\ B(\omega) &= -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{A(\omega')}{\omega' - \omega} d\omega' , \end{aligned} \tag{C.1}$$

where P denotes the principle value of the integral. The latter equations express the causality principle, i.e., the impulse response is zero for negative times ($h(t < 0) = 0$) for finite energy signals ($\int_{-\infty}^{\infty} |h(t)|^2 dt < \infty$). Note that equations (C.1) only hold provided that the function $H(\omega + iy)$ is analytic for $y > 0$, i.e., in the upper half plane and has no poles on the real axis. Equations (C.1) are known as Kramers-Kronig dispersion relations. In general, these relations involve the constants $A(\infty)$ and $B(\infty)$. Here, these constants disappear because of the finite energy condition.

The transfer function H in our notation is given by

$$H(\omega, L) = \exp(iK(\omega, L)L) \tag{C.2}$$

where K is the complex wavenumber as introduced in (3.2): $K = \varphi + i\alpha$. Therefore, we can write

$$\frac{\ln H(\omega, L)}{L} = iK(\omega, L) \equiv \gamma(\omega, L) . \tag{C.3}$$

If H is analytic in the upper half plane, then $\gamma = \ln(H)/L$ is also analytic except for the zeroes of H. For finite travel-distances L there are no zeroes of H in the upper half plane. This can be verified by inspection of equations (B.9)-(B.10). However, noting Paresvals theorem, the finite energy condition $\int_{-\infty}^{\infty}|h(t)|^2dt = \int_{-\infty}^{\infty}|H(\omega)|^2d\omega < \infty$ means that $H(\omega + iy) = 0$ for $|\omega + iy| \to \infty$. This applies also for our wave field attributes as shown in the Figures (3.12) and (3.13). Then γ is divergent and not square integrable. To circumvent this fact, Nussenzveig introduced the method of subtractions. First, a new complex function Λ is formed. If γ diverges as ω^n, then the function Λ_n can be obtained in terms of γ

$$\begin{aligned}\Lambda_n(\omega,\omega_0) &= \frac{\gamma(\omega)-\gamma(\omega_0)-(d/d\omega)\gamma(\omega)|_{\omega=\omega_0}-..-(d^{n-1}/d\omega^{n-1})\gamma(\omega)|_{\omega=\omega_0}(\omega-\omega_0)^{n-1}/(n-1)!}{(\omega-\omega_0)^n}\\ &= \frac{1}{(\omega-\omega_0)^n}\left[\gamma(\omega)-\sum_{i=0}^{n}\frac{d^i}{d\omega^i}\gamma(\omega)|_{\omega=\omega_0}\frac{(\omega-\omega_0)^i}{i!}\right]. \qquad (C.4)\end{aligned}$$

Real and imaginary parts of Λ form a Hilbert transform pair and constitute the dispersion relations with n subtractions, where ω_0 denotes the subtraction frequency. Λ is analytic everywhere,where γ is, but is also convergent as $|z| \to \infty$ and thus square integrable.

Noting that

$$P\int_{-\infty}^{\infty}\frac{d\omega'}{\omega'-\omega} = 0 \qquad (C.5)$$

we can remove the principle value notation and find

$$\begin{aligned}\Im\Lambda_n(\omega,\omega_0) &= -\frac{1}{\pi}\int_0^{\infty}\frac{\Re\Lambda_n(\omega',\omega_0)-\Re\Lambda_n(\omega,\omega_0)}{\omega'-\omega}d\omega'\\ &\quad+\frac{1}{\pi}\int_0^{\infty}\frac{\Re\Lambda_n(-\omega',\omega_0)-\Re\Lambda_n(\omega,\omega_0)}{\omega'+\omega}d\omega' \qquad (C.6)\end{aligned}$$

and

$$\begin{aligned}\Re\Lambda_n(\omega,\omega_0) &= \frac{1}{\pi}\int_0^{\infty}\frac{\Im\Lambda_n(\omega',\omega_0)-\Im\Lambda_n(\omega,\omega_0)}{\omega'-\omega}d\omega'\\ &\quad-\frac{1}{\pi}\int_0^{\infty}\frac{\Im\Lambda_n(-\omega',\omega_0)-\Im\Lambda_n(\omega,\omega_0)}{\omega'+\omega}d\omega'. \qquad (C.7)\end{aligned}$$

C.1 Twice-subtracted dispersion relations

For $n = 2$ which in our case is used for 3-D random media equation (C.7) leads to an expression for the attenuation coefficient

$$\begin{aligned}\alpha(\omega) = &-\frac{\omega^2}{\pi}\int_0^{\infty}\left[\frac{\varphi(\omega')-\omega' d/d\omega'\varphi(\omega')|_{\omega'=0}}{\omega'^2(\omega'-\omega)}-\frac{\varphi(\omega)-\omega d/d\omega\varphi(\omega)|_{\omega=0}}{\omega^2(\omega'-\omega)}\right]d\omega'\\ &-\frac{\omega^2}{\pi}\int_0^{\infty}\left[\frac{\varphi(\omega')-\omega' d/d\omega'\varphi(\omega')|_{\omega'=0}}{\omega'^2(\omega'+\omega)}-\frac{\varphi(\omega)-\omega d/d\omega\varphi(\omega)|_{\omega=0}}{\omega^2(\omega'+\omega)}\right]d\omega', \qquad (C.8)\end{aligned}$$

where we set $\alpha(\omega_0) = 0$ for $\omega_0 = 0$. Some algebraic manipulations yield

$$\alpha(\omega) = -\frac{2\omega^2}{\pi}\int_0^{\infty}\frac{d\omega'}{\omega'^2-\omega^2}\left[\frac{\varphi(\omega')}{\omega'}-\frac{\varphi(\omega)}{\omega}\right]. \qquad (C.9)$$

This corresponds to equation (3.20).

C.2 Triple-subtracted dispersion relations

For 2-D random media it turns out that the twice-subtracted dispersion relation results in a non-convergent integral. This is because the ω dependency of the Bourret part of $\varphi(\omega)$ is of the order $\mathcal{O}(\omega^3)$ (see equation (B.10)). Therefore, a triple-subtracted dispersion relation has to be applied. To do so, we use equation (C.7) with $n = 3$ and note that equation (B.10) corresponds to

$$\Im\Lambda_3 = \frac{4\pi}{c^3}\int_0^\infty \frac{H(\kappa - 2\omega/c)\Phi(\kappa)d\kappa}{\sqrt{\kappa^2 - 4\omega^2/c^2}}\,. \tag{C.10}$$

Inserting this into equation (C.7) with $n = 3$ we come up with the following equation

$$\begin{aligned}\Re\Lambda_3(\omega) \quad &= \quad \frac{4}{c^3}\int_0^\infty d\kappa\,\Phi(\kappa) \\ &\left[\frac{H(\kappa - 2\omega/c)}{\sqrt{\kappa^2 - 4\omega^2/c^2}}\int_0^\infty d\omega'\left(\frac{1}{\omega' + \omega} - \frac{1}{\omega' - \omega}\right) + \right. \\ &\left.\int_0^\infty d\omega'\frac{H(\kappa - 2\omega'/c)\Phi(\kappa)d\kappa}{\sqrt{\kappa^2 - 4\omega'^2/c^2}}\frac{1}{\omega' - \omega} - \int_0^\infty d\omega'\frac{H(\kappa - 2\omega'/c)\Phi(\kappa)d\kappa}{\sqrt{\kappa^2 - 4\omega'^2/c^2}}\frac{1}{\omega' + \omega}\right]\,.\end{aligned} \tag{C.11}$$

The integration of the terms in the second line yields

$$-i\pi\frac{H(\kappa - 2\omega/c)}{\sqrt{\kappa^2 - 4\omega^2/c^2}}\,. \tag{C.12}$$

The first integral in the third line can be evaluated by integration by parts. We obtain

$$-\frac{c\pi}{4(\kappa c/2 - \omega)} + \frac{\operatorname{arctanh}(\kappa c/\sqrt{\kappa^2c^2 - 4\omega^2})}{\sqrt{\kappa^2c^2 - 4\omega^2}} + \frac{i\pi/2H(\kappa - 2\omega/c) + i\pi/2}{\sqrt{\kappa^2c^2 - 4\omega^2}}\,. \tag{C.13}$$

Note that the function $\operatorname{arctanh}(z)$ is defined by

$$\operatorname{arctanh}(z) = \ln\left(\sqrt{\frac{1+z}{1-z}}\right) \tag{C.14}$$

and can be used for complex arguments provided that the complex square root and logarithm functions are defined properly. Analogously, we can evaluate the second integral in the third line of equation (C.12). A chain of manipulations yields finally the real function

$$\begin{aligned}\Re\Lambda_3(\omega) \quad &= \quad \frac{4}{c^3}\int_0^\infty d\kappa\,\Phi(\kappa)\left[\frac{\pi\kappa c^2/4}{(\kappa c/2)^2 - \omega^2} - H(2\omega/c - \kappa)\frac{\arctan(\kappa/\sqrt{|\kappa^2 - 4\omega^2/c^2|})}{\sqrt{|\kappa^2 - 4\omega^2/c^2|}}\right. \\ &\left. + H(\kappa - 2\omega/c)\frac{\ln\left(\sqrt{\frac{\kappa c/\sqrt{\kappa^2c^2 - 4\omega^2} + 1}{|\kappa c/\sqrt{\kappa^2c^2 - 4\omega^2} - 1|}}\right)}{\sqrt{|\kappa^2 - 4\omega^2/c^2|}}\right]\,.\end{aligned} \tag{C.15}$$

Equation (C.15) can be expressed in terms of the attenuation coefficient

$$
\begin{aligned}
\alpha(k) &= -8k^3 \int\limits_0^\infty d\kappa\, \Phi(\kappa) \left[\frac{\pi\kappa}{2(\kappa^2 - 4k^2)} - H(2k-\kappa) \frac{\arctan(\kappa/\sqrt{|\kappa^2 - 4k^2|})}{\sqrt{|\kappa^2 - 4k^2|}} + \right. \\
&\left. H(\kappa - 2k) \frac{\ln\left(\frac{\kappa/\sqrt{\kappa^2-4k^2}+1}{|\kappa/\sqrt{\kappa^2-4k^2}-1|} \right)^{\frac{1}{2}}}{\sqrt{\kappa^2 - 4k^2}} \right] .
\end{aligned} \tag{C.16}
$$

Considering only the leading terms in equation (C.16) we can write

$$\alpha(k) \approx -2\pi k^2 \int_{2k}^\infty d\kappa\, \Phi(\kappa)\,. \tag{C.17}$$

Appendix D

Two-frequency mutual coherence function

D.1 The Markovian approach

The calculation of the two-frequency mutual coherence function in the 3-D case can be found in Sato and Fehler (1998). Here we present briefly the 2-D analogon. The differential equation for the two-frequency mutual coherence function $\Gamma_2(k', k'', x', x'', z)$ in 2-D reads (see for the 3-D equation e.g., Sato and Fehler (1998) and Ishimaru (1978), equation (20-92)):

$$2i\frac{\partial \Gamma_2}{\partial z} + \left[\frac{\partial^2}{\partial x'^2}\frac{1}{k'} - \frac{\partial^2}{\partial x''^2}\frac{1}{k''}\right]\Gamma_2 + i\left[(k'^2 + k''^2)A(0) - 2k'k''A(x' - x'')\right]\Gamma_2 = 0 \quad . \tag{D.1}$$

This so-called master equation for Γ_2 is obtained from the parabolic wave equation in the Markov approximation. The function $A(x)$ is the integral along the mean propagation direction z of the correlation function of the heterogeneities $B(x, z)$ and is also related to the fluctuation spectrum:

$$A(x) = \int_{-\infty}^{\infty} dz\, B(x, z) = (2\pi)^2 \int_{0}^{\infty} d\kappa\, J_0(\kappa x)\Phi(\kappa)\kappa \quad , \tag{D.2}$$

where J_0 is the Bessel function. For quasi-monochromatic waves with frequencies around ω_c, it is expedient to introduce center-of-mass and difference coordinates for the wave numbers ($k_c = (k' + k'')/2, \quad k_d = k' - k''$) which transform equation (D.1) into

$$\frac{\partial \Gamma_2}{\partial z} + i\frac{k_d}{2k_c^2}\frac{\partial^2 \Gamma_2}{\partial x^2} + k_c^2(A(0) - A(x_d))\Gamma_2 + \frac{k_d^2}{2}A(0)\Gamma_2 = 0 \quad , \tag{D.3}$$

where $x_d = x' - x''$. The dependence on the spatial difference coordinate x_d follows from the assumption that the random medium under consideration is statistically homogeneous. According to Sato and Fehler the wave package experiences additional to the broadening due to scattering the so-called wandering, which is caused by statistical averaging of the phase fluctuations. A factorization of Γ_2 into

$$\Gamma_2 = {}_0\Gamma_2 e^{-k_d^2 A(0) z/2} \tag{D.4}$$

separates these effects. Replacing Γ_2 by ${}_0\Gamma_2$ in equation (D.3) is therefore equivalent to a description of the envelope broadening due to multiple scattering in a single realization.

Note that $(-k_d^2 A(0)z/2$ equals the mean field attenuation coefficient obtained in the Markov approximation. The master equation for ${}_0\Gamma_2$ is then

$$\frac{\partial\, {}_0\Gamma_2}{\partial z} + i\frac{k_d}{2k_c^2}\frac{\partial^2\, {}_0\Gamma_2}{\partial x^2} + k_c^2(A(0) - A(x_d)){}_0\Gamma_2 = 0 \tag{D.5}$$

with the initial condition

$${}_0\Gamma_2(\omega_c, \omega_d, x = 0, z = 0) = 1 \tag{D.6}$$

which corresponds to $\hat{I}(t, z = 0) = \delta(t)$. For Gaussian correlated fluctuations the function $A(x)$ is

$$A(x_d) \approx \sqrt{\pi}\sigma_n^2 a \left(1 - \frac{x_d^2}{a^2}\right) \quad . \tag{D.7}$$

Substituting this into equation (D.5) and introducing the following non-dimensional variables

$$\tau = z/L \qquad\qquad \gamma = x_d/a_\perp \quad , \tag{D.8}$$

where $a_\perp$ means the correlation length in the transverse direction, we obtain

$$\frac{\partial\, {}_0\Gamma_2}{\partial \tau} + i\frac{k_d}{k_m}\frac{\partial^2\, {}_0\Gamma_2}{\partial \gamma^2} + \gamma^2\, {}_0\Gamma_2 = 0 \quad , \tag{D.9}$$

where $k_m = \frac{2k_c^2 a_\perp^2}{L}$. Equation (D.9) can be solved by choosing the ansatz

$${}_0\Gamma_2 = \frac{\exp(\nu(\tau)\gamma^2)}{w(\tau)} \quad . \tag{D.10}$$

Substituting (D.10) into equation (D.9) we obtain

$$\gamma^2 \left[\frac{d\nu}{d\tau} + 4i\frac{k_d}{k_m}\nu + 1\right] + \left[2i\nu\frac{k_d}{k_m} - \frac{1}{w}\frac{dw}{d\tau}\right] = 0 \quad ; \tag{D.11}$$

this equation can only be satisfied if each term in brackets is identical to zero. With the initial conditions $\nu(0) = 0$ and $w(0) = 1$ we find

$$\begin{aligned} \nu(\tau) &= -\nu_0 \exp(-3\pi i/4)\tanh(\frac{\tau}{\nu_0}) \\ w(\tau) &= \sqrt{\cosh\frac{\tau}{\nu_0}} \end{aligned} \tag{D.12}$$

with $\nu_0 = \frac{\exp(-3\pi i/4)}{2\sqrt{k_d/k_m}}$. Substituting (D.12) into (D.10) and considering ${}_0\Gamma_2(\tau = 1, \gamma = 0)$ we obtain finally the solution

$${}_0\Gamma_2 = \sqrt{\operatorname{sech}\left[2\exp(i3\pi/4)\sqrt{k_d/k_m}\right]} \tag{D.13}$$

noting that $\operatorname{sech}(x) = 1/\cosh(x)$.

D.2 Γ_2 in Rytov and Bourret approximation

From the two-frequency correlation functions at lag $\tau = 0$ (Ishimaru, 1978, equations (19.49)) we recover the two-frequency variances in 3-D for the plane wave case:

$$\sigma^2_{\chi_1\chi_2}(k_1, k_2, z = L) = 2\pi^2 k_1 k_2 \int_0^\infty d\kappa\, \kappa\, \Phi(\kappa) \int_0^L d\eta\, g_\chi(\kappa, \eta) \tag{D.14}$$

with

$$\begin{aligned} g_\chi(\kappa, \eta) &= \Re\{h_1 h_2^\star - h_1 h_2\} \\ h_1 &= e^{-i\frac{L-\eta}{2k_1}\kappa^2} \\ h_2 &= e^{-i\frac{L-\eta}{2k_2}\kappa^2} . \end{aligned}$$

Integration with respect to η yields

$$\sigma^2_{\chi_1\chi_2} = 2\pi^2 k_1 k_2 \int\limits_0^\infty d\kappa\, \kappa\, \Phi(\kappa) \frac{2}{\kappa^2} \left[\frac{\sin\left(L\kappa^2/2\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\right)}{\frac{1}{k_1} - \frac{1}{k_2}} - \frac{\sin\left(L\kappa^2/2\left(\frac{1}{k_1} + \frac{1}{k_2}\right)\right)}{\frac{1}{k_1} + \frac{1}{k_2}} \right] . \tag{D.15}$$

In the same way we compute

$$\sigma^2_{\phi_1\phi_2} = 2\pi^2 k_1 k_2 \int\limits_0^\infty d\kappa\, \kappa\, \Phi(\kappa) \frac{2}{\kappa^2} \left[\frac{\sin\left(L\kappa^2/2\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\right)}{\frac{1}{k_1} - \frac{1}{k_2}} + \frac{\sin\left(L\kappa^2/2\left(\frac{1}{k_1} + \frac{1}{k_2}\right)\right)}{\frac{1}{k_1} + \frac{1}{k_2}} \right] \tag{D.16}$$

$$\sigma^2_{\chi_2\phi_1} - \sigma^2_{\chi_1\phi_2} = -16\pi^2 \int\limits_0^\infty d\kappa\, \frac{\Phi(\kappa)}{\kappa} \frac{k_1 k_2}{\frac{1}{k_1} - \frac{1}{k_2}} \sin^2\left(L\kappa^2/4\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\right). \tag{D.17}$$

With the above equations and the approximation for the logarithmic wave field attributes given by equations (B1)-(B9) we can express the real and imaginary part of the exponent of Γ_2 as

$$\begin{aligned} \bar{\chi} &= -\frac{1}{2}\left[\sigma^2_{\chi_1\chi_1} + \sigma^2_{\chi_2\chi_2} + \sigma^2_{\phi_1\phi_1} + \sigma^2_{\phi_2\phi_2}\right] + \sigma^2_{\chi_1\chi_2} + \sigma^2_{\phi_1\phi_2} \\ &= -2\pi^2 \int\limits_0^\infty d\kappa\, \frac{\Phi(\kappa)}{\kappa} \left[\kappa^2 L(k_1^2 + k_2^2) + 4\frac{k_1^2 k_2^2}{k_1 - k_2} \sin\left(L\kappa^2/2\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\right)\right] \\ &= -2\pi^2 (k_1^2 + k_2^2) L \int\limits_0^\infty d\kappa\, \kappa\, \Phi(\kappa) \left[1 - \frac{2k_1 k_2}{k_1^2 + k_2^2} \frac{\sin\left(L\kappa^2/2\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\right)}{L\kappa^2/2\left(\frac{1}{k_1} - \frac{1}{k_2}\right)}\right] \end{aligned} \tag{D.18}$$

$$\begin{aligned} \bar{\phi} &= \phi_{c_1} - \phi_{c_2} + \sigma^2_{\chi_2\phi_1} - \sigma^2_{\chi_1\phi_2} \\ &= \pi L \int\limits_0^\infty d\kappa\, \kappa\, \Phi(\kappa) \left[k_1^2 \ln\left(\frac{2k_1 + \kappa}{2k_1 - \kappa}\right)^2 - k_2^2 \ln\left(\frac{2k_2 + \kappa}{2k_2 - \kappa}\right)^2 - \right. \\ &\qquad \left. \frac{16\pi}{L\kappa^2} \frac{k_1 k_2}{\left(\frac{1}{k_1} - \frac{1}{k_2}\right)} \sin^2\left(L\kappa^2/4\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\right)\right] . \end{aligned} \tag{D.19}$$

Introducing center-of-mass and difference coordinates $k_c = (k_1 + k_2)/2$ and $k_d = k_1 - k_2$ one obtains

$$
\begin{aligned}
\bar{\chi}(k_c, k_d, L) &= -2\pi^2 (2k_c^2 + k_d^2/2) L \int\limits_0^\infty d\kappa \, \kappa \, \Phi(\kappa) \\
&\qquad \left[1 - \frac{2k_c^2 - k_d^2/2}{2k_c^2 + k_d^2/2} \frac{\sin\left(L\kappa^2/2 \frac{k_d}{k_c^2 - k_d^2/4} \right)}{L\kappa^2/2 \frac{k_d}{k_c^2 - k_d^2/4}} \right] \qquad \text{(D.20)} \\
\bar{\phi}(k_c, k_d, L) &= \pi L \int\limits_0^\infty d\kappa \, \kappa \, \Phi(\kappa) \left[k_2^2 \ln \left(\frac{2\,k_1^2 + \kappa}{2\,k_1^2 - \kappa} \right)^2 - k_2^2 \ln \left(\frac{2\,k_2^2 + \kappa}{2\,k_2^2 - \kappa} \right)^2 - \right. \\
&\qquad \left. \frac{16\pi}{L\kappa^2} \frac{\sin^2\left(-L\kappa^2/4 \frac{k_d}{k_c^2 - k_d^2/4} \right)}{\frac{k_d}{k_c^2 - k_d^2/4}} \right], \qquad \text{(D.21)}
\end{aligned}
$$

where we used $k_1 = k_c + k_d/2$ and $k_2 = k_c - k_d/2$. Inserting for example the fluctuation spectrum of the Gaussian correlation function, explicit results for $\bar{\chi}$ and $\bar{\phi}$ are obtained.

Appendix E

Elastic Rytov and Bourret approximation

E.1 Elastic Rytov approximation

We follow Gold (1997) in order to derive the elastic Rytov approximation. The following calculations are done for the case of an incident P-wave in 3-D. The calculations for an incident S-wave and the 2-D case are analogous. We start with the elastodynamic wave equation (2.44), where the perturbations are introduced in the following way:

$$\begin{aligned} \rho(\mathbf{r}) &= \rho_0 + \delta\rho(\mathbf{r}) \\ \lambda(\mathbf{r}) &= \lambda_0 + \delta\lambda(\mathbf{r}) \\ \mu(\mathbf{r}) &= \mu_0 + \delta\mu(\mathbf{r}) \quad . \end{aligned} \tag{E.1}$$

That means the Lamé parameters and density fluctuate around their mean values (ρ_0, λ_0, μ_0). Further we assume small fluctuations: $\delta\rho(\mathbf{r}) \ll \rho_0$, $\delta\lambda(\mathbf{r}) \ll \lambda_0$, $\delta\mu(\mathbf{r}) \ll \mu_0$. The elastodynamic differential operator corresponding to a homogeneous isotropic medium is

$$X_{ik} = \omega^2 \rho_0 \delta_{ik} + (\lambda_0 + \mu_0)\partial_i \partial_k + \mu_0 \partial_m \partial_m \delta_{kj} \tag{E.2}$$

such that

$$X_{ik} G_{kj} = \delta(\mathbf{r} - \mathbf{r}')\delta_{ij} \quad , \tag{E.3}$$

where G_{kj} is the Green's function of an infinite elastic volume:

$$G_{kj}(R) = \frac{1}{4\pi\rho_0\omega^2} \left[\delta_{kj}\beta^2 \frac{e^{i\beta R}}{R} - \partial_k \partial_j \left(\frac{e^{i\alpha R}}{R} - \frac{e^{i\beta R}}{R} \right) \right] \quad . \tag{E.4}$$

Here α and β are the wave numbers of P- and S-waves, respectively; R is defined as $|\mathbf{r} - \mathbf{r}'|$.

In the following we consider the propagation in z-direction of an initially plane P-wave. In analogy to the acoustic case, we choose the following ansatz for the displacement vector:

$$\mathbf{u} = (u_x(\mathbf{r}), u_y(\mathbf{r}), 1)^T \exp(ikz) \exp(\Psi) \tag{E.5}$$

which corresponds to a time-harmonic plane wave and Ψ defined in section 2.1 (the time dependence $\exp(-i\omega t)$ is omitted). Assuming that Ψ, u_x and u_y are small quantities compared with unity, we insert equations (2.45), (E.1) and the ansatz (E.5) into the wave equation (2.44) and obtain:

$$\begin{aligned}(\lambda_0 + 2\mu_0)(2ik\Psi_{,z} + \Psi_{,zz}) + \mu_0(\Psi_{,xx} + \Psi_{,yy}) + \omega^2\delta\rho + ik(\lambda_0 + \mu_0)\\(u_{x,x} + u_{y,y}) + (\lambda_0 + \mu_0)(u_{x,xz} + u_{y,yz}) - k^2(\delta\lambda + 2\delta\mu) + ik(\delta\lambda + 2\delta\mu)_{,z} &= 0,\end{aligned}$$

where terms of second and higher order of the small quantities are neglected. Like Ishimaru (1978), we want to obtain an equation for Ψ. To do so, we define

$$\mathbf{L} = (u_x(\mathbf{r}), u_y(\mathbf{r}), \Psi(\mathbf{r}))^T \exp(ikz) \tag{E.6}$$

and apply the operator X_{jk} to this quantity what yields the result for $j = z$:

$$\begin{aligned}e^{-ikz}X_{zk}L_k = (\lambda_0 + \mu_0)(u_{x,xz} + u_{y,yz} + ik(u_{x,x} + u_{y,y})) +\\(\lambda_0 + 2\mu_0)(2ik\Psi_{,z} + \Psi_{,zz}) + \mu_0(\Psi_{,xx} + \Psi_{,yy}) \quad .\end{aligned} \tag{E.7}$$

Now we subtract (E.6) from this equation and obtain for $L_z = \Psi$

$$e^{-ikz}X_{zz}\Psi = \omega^2\delta\rho - k^2(\delta\lambda + 2\delta\mu) + ik(\delta\lambda + 2\delta\mu)_{,z} \quad . \tag{E.8}$$

This corresponds to an inhomogeneous wave equation for the complex exponent Ψ, which can be now determined by a Green's function approach. We obtain

$$\Psi = e^{-ikz}\int d^3r'[\omega^2\delta\rho - k^2(\delta\lambda + 2\delta\mu)]e^{ikz'}G_{33} \quad . \tag{E.9}$$

If we assume forward scattering and neglect the near field terms of G_{33}, the following approximation can be made:

$$G_{33}(R) \approx \frac{k^2}{\rho_o\omega^2}\frac{e^{ikR}}{4\pi R} = \frac{k^2}{\rho_o\omega^2}G^{3D}(R) \quad , \tag{E.10}$$

with the acoustic Green's function G^{3D}. Inserting this into equation (E.9) and noting that in the elastic case the velocity fluctuations n_v are in a first order approximation

$$2n_v = \delta\rho/\rho - (\delta\lambda + 2\delta\mu)/(\lambda + 2\mu)\,, \tag{E.11}$$

we finally obtain the result (3.63).

E.2 Elastic Bourret approximation

The detailed derivation of the coherent phase for elastic random media can be found in Gold et al. (2000). See their equations (A-19) and (A-20) for the effective wave numbers in 3-D random media. Its real part multiplied by the travel-distance defines the searched for coherent phase. In 2-D random media the results are (Gold, 1997);

$$\begin{aligned}\phi_c^P &= \alpha L\,\Re\Bigg\{\Bigg(1-\frac{1}{\rho_0\omega^2}\int\limits_{-\infty}^{\infty} d^2r\, e^{-i\alpha z}\,[\,\omega^4 B_{\rho\rho}G_{33}-\alpha^2 B_{\lambda\lambda}G_{jl,jl}-4\alpha^2 B_{\mu\mu}G_{33,33}\\ &\quad +4i\omega^2\alpha B_{\rho\mu}G_{33,3}+2i\rho_0\omega^2\alpha B_{\lambda\rho}G_{m3,m}\; -4\alpha^2 B_{\lambda\mu}G_{3m,3m}\Big]\Bigg)^{-1/2}\Bigg\} \qquad \text{(E.12)}\\ \phi_c^S &= \beta L\,\Re\Bigg\{\Bigg(1-\frac{1}{\rho_0\omega^2}\int\limits_{-\infty}^{\infty} d^2r\, e^{-i\beta z}\,[\,\omega^4 B_{\rho\rho}G_{11}-\beta^2 B_{\mu\mu}\,(G_{33,11}+2G_{13,13}\\ &\quad +G_{11,33}-4G_{33,33})+2i\omega^2\beta B_{\rho\mu}\,(G_{11,3}+G_{13,1}-2G_{33,3})\;])^{-1/2}\Bigg\}\quad . \qquad \text{(E.13)}\end{aligned}$$

G_{ij} means the 2-D Green's function (Hudson, 1980) and B_{xy} denotes the (cross-) correlation function of the quantities x, y.