What We Have Not Learnt Yet

Let us comment on some open questions and possible generalizations. As we have seen, implementable quasi–free endomorphisms of the CAR and CCR algebras are always accompanied by *Fock spaces of isometries.* This fact implies, among other things, that these endomorphisms restrict to reducible endomorphisms of the observable algebra. Gauge invariant endomorphisms behave in some respect like "master endomorphisms", because they carry, together with some "basic" representation of the gauge group, all higher (anti)symmetric tensor powers thereof. In typical cases, e.g. for the classical compact Lie groups, any irreducible representation of the group is equivalent to a subrepresentation of some tensor power of the defining representation. In such cases there will exist quasi–free endomorphisms which contain each superselection sector as a subrepresentation.

It is an interesting problem how to obtain irreducible subobjects of quasi–free endomorphisms. According to the general theory, one needs gauge invariant isometries fulfilling relation (0.7) on Fock space for this purpose. Such isometries would also permit to define direct sums of quasi–free endomorphisms, so that one would get the full Doplicher–Roberts category generated by quasi–free endomorphisms.

Another important question is whether one can exhibit classes of localized Bogoliubov operators with finite nonzero index, say on the single–particle space of the time–zero Dirac field, and such that the implementability condition holds. Recall that our construction of such Bogoliubov operators made essential use of the existence of local Fourier bases on the circle. It would be desirable to have a basis–independent characterization of such Bogoliubov operators in Minkowski space, but this is an unsolved functional analytic problem. Of particular interest would be the massive case in two dimensions, where one might hope to find localized quasi–free endomorphisms with non–Abelian ("plektonic") braid group statistics. But preliminary calculations based on our explicit formulas indicate that the implementers corresponding to irreducible subobjects of quasi–free endomorphisms can only have "anyonic" commutation relations (i.e. Abelian braid group statistics). A related question is what kind of quantum symmetries beyond compact groups can be realized by quasi–free endomorphisms.

Let us mention that our results might be relevant for the discussion of the superselection structure of the massless Thirring model. The sectors of this model have been described by Streater [Str74]. Later, Carey, Ruijsenaars and Wright constructed approximate cutoff fields which reproduce Klaiber's n–point functions of the Thirring model [CRW85]. The Thirring fields can be obtained as strong limits of the approximate fields. Whereas the latter are always (multiples of) unitary operators, it could well be that the Thirring fields are only (multiples of) isometries. They would then fall into the scope of our work. The Hilbert space chosen by Streater would then be too "big", in that the Thirring fields would not reach all sectors from the vacuum.

On the mathematical side, our investigations can be extended in several interesting directions. One could enlarge the class of representations under consideration, by studying the implementation of quasi–free endomorphisms in arbitrary quasi–free states, and not just in Fock states. Such an analysis should be possible because the main technical tools, the quasi–equivalence criteria, apply to arbitrary quasi–free states.

One could also enlarge the class of transformations and consider e.g. the implementation of completely positive quasi–free maps. Some work has been done in this context, notably by Evans [Eva77, Eva79, Eva80a]. The "Bogoliubov operators" corresponding to completely positive quasi–free maps are in general not isometric, but only contractive. As an analogue of implementation, Evans constructed "dissipators" for completely positive quasi–free maps, i.e. nonzero contractions on Fock space which are "dominated" by the completely positive map. But, as shown by Arveson in his generalization of Powers' index theory of E_0–semigroups to semigroups of completely positive maps [Arv96a, Arv96b], such maps also admit an implementation by "metric operator spaces", and this notion is very close to the implementation of endomorphisms by Hilbert spaces of isometries.

Let us finally mention that it would be interesting to study the representations of the Cuntz algebras that are generated by the implementers of quasi–free endomorphisms, and that one could examine whether the Fredholm index of CCR Bogoliubov operators, which is a finer invariant than the Jones–Kosaki index, has a counterpart on the algebraic level.