Mean curvature flow of graphs with free boundaries

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Zusammenfassung

Die vorliegende Arbeit befasst sich in der Hauptsache mit der Evolution von Graphen unter dem mittleren Krümmungsfluss mit freien Rändern. Wir betrachten sowohl den mittleren Krümmungsfluss von Graphen mit einem vorgeschriebenen Winkel an einer festen glatten Hyperfläche Σ im Euklidischen Raum, als auch das gleiche Problem mit einem zweiten Rand mit Dirichlet-Randbedingungen. Die Neumann-Randbedingung erfordert, dass die Einheitsnormalenvektorfelder des Graphen und der Hyperflächen Σ senkrecht bleiben.

Unsere ersten Resultate betreffen radialsymmetrische Graphen. In diesem Zusammenhang beweisen wir, dass bis auf Bedigungen an den Anfangsgraphen und Σ entweder Langzeitexistenz und Konvergenz zu Minimlaflächen (in einigen Fällen Funktionen konstanter Höhe) oder Entwicklung einer Typ I Krümmungs- (und Gradienten-) Singularität erfolgt.

Die zweite Klasse von Ergebnissen, welche wir präsentieren, befasst sich mit einem bestimmten Beispiel. Wir betrachten den Fall, wenn die Kontaktfläche Σ die Einheitssphäre im $\mathbb{R}^3$ ist, und betrachten die Bewegung von Graphen senkrecht zu der Sphäre und mit einer festen Höhe bei einem festen Radius von der Sphäre. In dem Fall von spiegelsymmetrischen Graphen beweisen wir die Erhaltung der Grapheneigenschaft für alle Zeit der Existenz.

Die nächsten Klasse von Ergebnissen betrifft allgemeine Graphen, die sich unter dem mittleren Krümmungsfluss im $\mathbb{R}^3$ entwickeln, mit entweder einem freien Neumann-Rand oder einem freien Neumann-Rand und einem zusätzlichen Dirichlet-Rand. Wir präsentieren eine allgemeine Methode um Schranken an die Höhe zu erhalten und klassifizieren das Verhalten des Gradienten am Rand.

Unter Benutzung dieser erhalten wir ein Ergebnis zur Langzeitexistenz für den mittleren Krümmungsfluss mit einem freien Rand auerhalb eines Zylinders.

In höheren Dimensionen $n \geq 2$ beweisen wir auch Langzeitexistenz von anfangs konvexen (oder konkaven) Graphen über einem Halbraum senkrecht an einer Hyperfäche.

Abstract. In this thesis the chief object of study is the evolution of graphs under mean curvature flow with free boundaries. We study both the mean curvature flow of graphs with a prescribed angle condition on a fixed smooth hypersurface Σ in Euclidean space, and the same problem with a second boundary on which we prescribe a Dirichlet condition. The Neumann boundary condition requires that the unit normal vector field of the graph and that of the hypersurface Σ remain perpendicular.

Our first set of results are concerned with radially symmetric graphs and here we prove, up to conditions imposed on the initial graph and Σ, either long time existence and convergence to minimal surfaces (in some cases functions with constant height), or development of a Type I curvature (and gradient) singularity.

The second class of results we present treats one specific example. We consider the case where the contact surface Σ is the unit sphere in $\mathbb{R}^3$, and study the motion of graphs perpendicular to the sphere and with zero height at a fixed radius from the sphere. In the case of reflective symmetric graphs we prove that the graph property is preserved for all times of existence.

The next class of results is concerned with general graphs evolving by mean curvature flow in $\mathbb{R}^3$ with either one free Neumann boundary or both a free Neumann boundary and an additional Dirichlet boundary. We present here a general method for obtaining height bounds and classify the gradient behaviour on the boundary.

Using these, we obtain a long time existence result for the mean curvature flow with a free boundary outside a cylinder.

In dimensions $n \geq 2$, we also prove long time existence of initially convex (or concave) graphs over a half space supported perpendicularly on a hyperplane.

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CHAPTER 1

Introduction

1. Background and results

A hypersurface M_t in Euclidean space is said to be evolving by mean curvature flow if each point X of the surface moves, in time and space, in the direction of its unit normal ν with speed equal to the mean curvature H at that point. That is

$$\frac{dX}{dt} \;=\; -H(X)\;\nu.$$

Equivalently if one considers the mean curvature flow of a smooth family of immersions $F_t = F(\cdot, t) : M^n \to \mathbb{R}^{n+1}$ this is given by

$$\frac{\partial F}{\partial t}(p,t) \;=\; -H(F(p,t))\;\nu_{M_t}(F(p,t)),\;\forall (p,t) \in M^n \times [0,T).$$

Mean curvature flow has two major qualities. If we observe that $H(X) = \Delta^{M_t} X$, where we denoted by Δ^{M_t} the Laplace-Beltrami operator on the manifold M_t, then it is easy to consider mean curvature flow as a heat flow for manifolds. Mean curvature flow is also the steepest descent flow for the area functional, evolving hypersurfaces towards minimal surfaces. These properties have been extensively used throughout the literature. Mean curvature flow has been studied for some time, at least since 1956, when Mullins [**31**] considered a version of mean curvature flow in one dimension. In 1978 Brakke [**3**] studied the mean curvature flow of surfaces from the point of view of geometric measure theory.

There are two approaches to the study of mean curvature flow. One may work directly with the immersions or if the hypersurfaces obey a graph condition in some fixed direction in $\mathbb{R}^{n+1}$, one may study mean curvature flow with classical techniques by considering it as a quasilinear parabolic partial differential equation. A detailed passage from one to another can be found in [**8**] and [**10**].

There has been much work on the mean curvature flow problem for immersions and graphs with or without boundary conditions. In the compact setting, one result of great interest is that of Huisken [**21**]. There the author proves that under mean curvature flow, compact, initially convex surfaces retain their convexity and converge to a spherical point in finite time. The analogous result for the one dimensional case, the curve shortening flow, was obtained by Gage and Hamilton [**12, 13**], where it was proved that initially convex planar curves contract to points. This was later generalised by Grayson [**15**] for all closed embedded planar curves.

The study of entire graphs by Ecker and Huisken [**9, 10**] provides a detailed exposition including a long time existence theorem for locally Lipschitz initial data. The non-parametric mean curvature flow of graphs with either a ninety degree contact angle or Dirichlet boundary condition on cylindrical domains has been studied by Huisken

[**23**] and there proves a long time existence and convergence to minimal surfaces theorem. In the setting of non-parametric mean curvature flow, this was later generalised by Altschuler and Wu [**1**], where they allow arbitrary contact angles at the fixed boundary for two dimensional graphs. This in turn was also later generalised to arbitrary dimensions by Guan [**17**]. From the point of view of immersions mean curvature flow with Dirichlet boundary data has been studied by Stone [**37, 38**] in Euclidean space and Priwitzer in [**32**] in the setting of Riemannian manifolds.

Given the above a natural next step is to study the mean curvature flow of immersions satisfying a graph condition in time dependent domains with either Neumann or Dirichlet boundary conditions.

This begins with a series of works on the mean curvature flow of immersions with free boundary, where a restriction on the angle of contact with a fixed hypersurface in Euclidean space is imposed. The free boundary is given by a ninety degrees contact angle with the fixed hypersurface. Important works in this setting include Stahl [**36**] and Buckland [**4**]. In the first, Stahl proves a short time existence result by writing the evolving hypersurfaces as graphs over the initial hypersurface and then, following Ecker and Huisken [**10**], he is able to obtain local gradient estimates in the case of hypersurfaces with bounded curvature. This gives that either the solution exists for all time or develops a curvature singularity. In the case of umbilic surfaces of contact and initially convex data he proves that the flow will shrink the solution to a Type I hemispherical point singularity in finite time. In [**4**], Buckland focused on the case of finite time existence and the study of the nature of singularities which occur on the free boundary. As a start he obtains the analogue of the monotonicity formula of Huisken [**24**] for manifolds with a free boundary. The main tool used to obtain this result is inspired by the work of Grüter and Jost [**16**], and that is the concept of reflection of points across the free boundary. The main result is a classification of limit surfaces with non-negative mean curvature near singular points obtained after a parabolic rescaling of the flow. A Brakke regularity result for mean curvature flow with Neumann free boundaries has been obtained by Koeller in [**26**].

We are concerned in this work not only with the study of the same problem treated by Stahl and Buckland, but also with a variant of this. The problem is viewed as a separate approach to mean curvature flow with a free boundary as well as a natural continuation of works on the boundary value mean curvature flow of graphs. Here we consider the non-parametric mean curvature flow defined on a time dependent domain, which is a strict generalisation of the cylindrical setting previously considered.

There are two basic types of problems that we treat in this thesis. The first is the setting of Stahl. We consider the mean curvature flow of graphs with a ninety degree contact angle condition onto a fixed hypersurface. For the second problem we add a disjoint boundary to the flow on which we will impose a Dirichlet boundary condition independent of time. Both the problems are considered as a flow of immersions which have also the property of being graphs in a fixed direction in Euclidean space. This will allow us to transform the evolution equation for the immersion into that for a scalar function. This equation is parabolic and quasilinear but defined on a time dependent domain due to the free Neumann boundary. The time dependent domain will be a challenge in obtaining a unified result of short time existence for both our problems and also for the application of the maximum principle. More details about the difficulties with a time dependent domain will be presented in the following chapters. We are interested in the conditions one must impose initially and on the fixed hypersurface of contact such that long time existence is obtained.

The results of this thesis are divided into five chapters, in addition to the present one. Each of the chapters treats a particular case in the sense of the evolving graphs, the initial data or the contact hypersurface. We summarise the main contributions of this thesis as follows.

(Ch. 2) **Maximum principles.** In this chapter maximum principles for mean curvature flow in the scalar graph and immersion setting with free boundaries are proved. The theorems following from the maximum principle are powerful tools which are generally only enjoyed by solutions of second order differential equations.

The programme of obtaining the long time existence result consists of first obtaining bounds on the height and then obtaining bounds on the gradient of the solution of the associated scalar problem. Since these steps are highly dependent on the applicability of the tools which come from the maximum principle, we present here a collection of indispensable results following mostly the work of either Lieberman [**30**] for the scalar setting or Ecker [**8**] for that of the immersion.

Since our problems in the setting of scalar graphs are defined on a time dependent domain, we must consider the possibility of a space time domain which has a non-smooth boundary. That is to say, the space time boundary could contain corners: points around which the boundary is no longer C^1. These points are difficult to handle when one is looking to obtain a Hopf lemma argument, and we present here one way to overcome these difficulties.

(Ch. 3) **Short time existence.** We present arguments to prove that solutions of the two basic types of problems we study exist at least for a short time. There are two types of approaches which we use. The first, following for example Stahl [**35**], is to write the evolving hypersurfaces as graphs in the direction of the normal vector field to the initial hypersurface, suitably extended, and then appeal to classical theory. The second is to write the surfaces as graphs in a fixed arbitrary direction in Euclidean space, after which one appeals again to standard theory. Both the arguments are required for our work, the first because we can not include the second problem in the first setting and the second reason since the second setting for the short time existence result is used for the problem in Chapter 4 to prove preservation of the graph property.

(Ch. 4) **Mean curvature flow of graphs with free boundary outside the sphere.** Here we discuss in detail the mean curvature flow of graphs outside a unit sphere centred at the origin in $\mathbb{R}^3$ with a ninety degree contact angle on the sphere and an additional zero height Dirichlet boundary condition on a fixed circle outside the sphere.

We present this example in a chapter of its own since due to the perfect symmetry of the sphere we are able to prove that the reflectively symmetric mean curvature flow solution preserves the graph property for all times of existence, in a completely different way compared with chapters 5 and 6. This is done using the second setting of short time existence in which we write our graphs in a fixed direction in $\mathbb{R}^3$, a direction which is always tangent to the sphere and also perpendicular to one of Killing vector fields of rotation.

The way we prove gradient bounds is a new approach exploiting the linear and antisymmetric nature of the Killing vector fields. The result excludes the possibility that the graphs will obtain unbounded gradient on the free boundary.

(Ch. 5) **Mean curvature flow of radially symmetric graphs with free boundaries.** The results of this chapter are obtained regardless of dimension and with the most general choice of hypersurfaces of contact for the free Neumann boundary. We consider the evolution of a radially symmetric graph under mean curvature flow with a free boundary on a fixed hypersurface in Euclidean space and sometimes also with a fixed height on a separate disjoint Dirichlet boundary.

Up to conditions imposed on the initial graphs and on the fixed contact hypersurface we prove long time existence and convergence to minimal hypersurfaces. Here there are two kinds of long time existence, one where the height is bounded for all times by the initial values and the second where the height is bounded by a constant depending on the hypersurface of contact for the Neumann boundary. The main condition imposed on the hypersurface of contact is rotational symmetry.

The second major result of this chapter is convergence for time to infinity to constant functions. This complements the convergence to minimal surfaces given by the long time existence theorem in some cases of initial graphs. It is obtained by the use of an auxiliary function depending on the graphs for which one can prove long time existence as well. Again in this chapter we make extensive use of tools derived from the maximum principle.

The last major result we prove treats the problem which is defined on a set with two distinct boundaries, a free Neumann boundary and a fixed Dirichlet height on the second. For this type of problem we prove that for certain types of hypersurfaces of contact and initial data the graph solution exists only for a finite time and develops a Type I singularity on the axis of rotation. We use again the comparison principle with a self similar torus to obtain this theorem, and we give sufficient conditions on the initial graph for the existence of such a torus.

(Ch. 6) **Mean curvature flow of graphs with free boundaries in Euclidean space.** This chapter treats the most general problem, with no symmetries imposed on the evolving graphs and works exclusively with the immersion setting.

As in the case of the radially symmetric graphs the height bound can be obtained in two different ways. For some hypersurfaces of contact the height is bounded by initial values for all times. In the general case the height bound is obtained by the use of rotationally symmetric barriers.

The gradient bound on the Dirichlet boundary, in case we have a problem with two boundaries, is easily obtained by the usual barrier construction. The maximum principle thus implies that the first bad behaviour that the gradient of the graphs can have is on the Neumann boundary. We distinguish between two types of bad behaviour on the Neumann boundary, a 'tilt' and the movement onto horizontal parts of the contact surface. A tilt appears when the graphs are still smooth but the boundary curve turns in such a way that it loses the graph property on the contact surface. For the case of the tilt we provide a complete list of properties of the curvature and the derivatives of the curvature of the graph in such a point. The second bad gradient behaviour on the Neumann boundary is when the graphs evolve towards a point where the Σ surface is horizontal. We can exclude such behaviour for the class of graphs defined on sets in $\mathbb{R}^n$ topological equivalent to a disc.

For the mean curvature flow of graphs inside cylinders in $\mathbb{R}^3$ we present a long time existence result and convergence to minimal surfaces. This is the equivalent, in 3-dimensions of the result found in [**23**], but obtained without the integral estimates.

In this chapter we also include a section on n-dimensional graphs in a half spaces, with a ninety degrees angle condition on a hyperplane. We obtain uniform bounds on height, gradient and the mean curvature. For initially convex or concave hypersurfaces we uniformly bound the second fundamental form and we prove long time existence, independent of the initial height. This is also followed by the convergence to hyperplanes if the initial height is bounded. This result is the natural next step for the result on entire graphs found in [**9**].

This work also contains two appendices. The first appendix presents some additional results which characterise the tilt behaviour for the problem of Chapter 4, the mean curvature flow of graphs outside the unit sphere. This showcases a new approach in locally representing the surfaces generated by the graphs around a point of tilt on the free boundary.

The second appendix is provided only for the convenience of the reader. It contains a detailed exposition on how to obtain the well-known result of a gradient bound on Dirichlet boundaries by the construction of barriers. We have chosen to follow Trudinger [**39**] and add various comments on their applicability to our two problems.

There are a series of open questions related to the above results. Can one still obtain long time existence when one does not impose the fixed Dirichlet boundary, and instead considers graphs extending off to infinity? Can one exclude the tilt behaviour for general hypersurfaces of contact Σ? Is there a general way to describe a tilt in higher dimensions and what kind of conditions must one impose on the initial graphs such that the tilt behaviour will be prevented? What happens when we replace the fixed hypersurface on which there is the free boundary moving with a hypersurface generated by the evolution of another flow? Is it possible to construct graphical hypersurfaces which tilt and lose the graph property? The study of mean curvature flow with only free Neumann boundaries is perhaps especially interesting since it represents a more truthful representation of phenomenons found in nature, giving inspiration for other contact hypersurfaces for the Neumann boundary.

2. The problems

Although the most general problems are treated in detail only in the last chapter, since we study the special cases of radially and reflectively symmetric graphs, as well as state our short time existence of the problems before that, we include here for the ease of the reader the definition of the general setting of our problem in $n+1$ dimensional Euclidean space.

Let Σ denote an n-dimensional hypersurface smoothly embedded in $\mathbb{R}^{n+1}$. The first type of problem we study is where the solutions possess only a free Neumann boundary on the fixed hypersurface Σ. This is the problem studied earlier in [**35**], and we follow the conventions there. Let M^n to be a smooth, orientable n-dimensional manifold with smooth, compact boundary ∂M^n and set $M_0 := F_0(M^n)$, where $F_0 : M^n \to \mathbb{R}^{n+1}$ is a smooth embedding satisfying

$$\begin{aligned} \partial M_0 \equiv F_0(\partial M^n) &= M_0 \cap \Sigma, \\ \langle \nu_{M_0}, \nu_\Sigma \circ F_0 \rangle (p) &= 0 \ \forall p \ \in \partial M^n, \end{aligned}$$

where we have denoted by ν_{M_0} the unit normal to M_0.

Let $I \subset \mathbb{R}$ be an open interval and let $F_t = F(\cdot, t) : M^n \to \mathbb{R}^{n+1}$, $t \in I$ be a one-parameter family of smooth embeddings. The family of hypersurfaces $(M_t)_{t \in I}$, where $M_t = F_t(M^n)$, is said to be evolving by mean curvature flow with free Neumann boundary condition on Σ if

$$\begin{aligned} \frac{\partial F}{\partial t}(p,t) &= -\, H(p,t)\nu_{M_t}, \ \forall (p,t) \in M^n \times I, \\ F(\cdot, 0) &= F_0, \\ F(p,t) &\subset \Sigma, \ \forall (p,t) \in \partial M^n \times I, \\ \langle \nu_{M_t}, \nu_\Sigma \circ F \rangle (p,t) &= 0, \ \forall (p,t) \in \partial M^n \times I, \end{aligned} \tag{1}$$

where we have denoted by ν_{M_t} the unit normal to M_t and by H the mean curvature of M_t. Examples of this problem are the mean curvature flow inside the catenoid neck, studied in chapters 5 and 6, and the mean curvature flow inside a sphere, studied in [**35**].

The second type of problem is when the solutions possess a second boundary on which we prescribe a fixed height over the plane of definition as a graph. This problem includes the example we work on in Chapter 4 where the surface Σ is the unit sphere in $\mathbb{R}^3$. We define it as follows.

Let M^n to be a smooth, orientable n-dimensional manifold with two smooth, compact, disjoint boundaries which we denote by $\partial_N M^n$ for Neumann boundary and $\partial_D M^n$ for Dirichlet boundary. Set $M_0 := F_0(M^n)$, where $F_0 : M^n \to \mathbb{R}^{n+1}$ is a smooth embedding satisfying

$$\begin{aligned} \partial_N M_0 &\equiv F_0(\partial_N M^n) = M_0 \cap \Sigma, \\ \langle \nu_{M_0}, \nu_\Sigma \circ F_0 \rangle (p) &= 0 \ \forall p \ \in \partial_N M^n, \\ \partial_D M_0 &\equiv F_0(\partial_D M^n). \end{aligned}$$

Let $I \subset \mathbb{R}$ be an open interval and let $F_t = F(\cdot, t) : M^n \to \mathbb{R}^{n+1}$, $t \in I$ be a one-parameter family of smooth embeddings. The family of hypersurfaces $(M_t)_{t \in I}$, where $M_t = F_t(M^n)$, are said to be evolving by mean curvature flow with free Neumann boundary condition on Σ and an additional fixed Dirichlet boundary condition if it satisfies

$$
\begin{aligned}
\frac{\partial F}{\partial t}(p,t) &= - H(p,t)\nu_{M_t},\ \forall (p,t) \in M^n \times I,\\
F(\cdot,0) &= F_0,\\
F(p,t) &\subset \Sigma,\ \forall (p,t) \in \partial_N M^n \times I,\\
\langle \nu_{M_t}, \nu_\Sigma \circ F \rangle (p,t) &= 0,\ \forall (p,t) \in \partial_N M^n \times I,\\
F(p,t) &= F_0(p),\ \forall (p,t) \in \partial_D M^n \times I.
\end{aligned} \tag{2}
$$

Furthermore in the setup section of every chapter we will define the particular case on which we are working and impose additional constraints and compatibility conditions as applicable.

CHAPTER 2

Maximum and comparison principles

1. Introduction

An important tool in the study of second order parabolic problems, in particular here for the study of mean curvature flow, is the maximum principle. In general the maximum principle states that the maximum of a solution of a homogeneous linear or quasilinear parabolic equation in a domain must occur on the boundary of that domain. The part of the boundary where the maximum will occur is called the parabolic boundary. We will define and use the parabolic boundary also in the short time existence chapter. The parabolic boundary includes the domain at initial time. The strong maximum principle asserts that if the maximum occurs anywhere other than on the parabolic boundary, then the solution must be constant. This argument can be used to obtain uniqueness results and also has many other far-reaching consequences in boundary value problems such as ours.

In this chapter we provide a collection of comparison and maximum principles. We are particularly interested in two kinds of maximum principles. The first is applicable to scalar functions, either general functions or radially symmetric one dimensional equations. Although these are well-known from the literature we believe that a collection of them will be useful for the comprehension of this work. There is an extensive literature with various approaches in this direction, however here we will follow Lieberman [**30**].

The second case is that of mean curvature flow of immersions. In this setting the most well-known maximum and comparison principles are in the mean curvature flow of a compact manifold or for the mean curvature flow of an entire graph. In Section 3, dedicated to the mean curvature flow of immersions, we modify these to include the case of mean curvature flow of a manifold with boundary. Our main reference is Ecker [**10**], and for the comparison principle some additional work done by Huisken [**22**] is also relevant.

2. Mean curvature flow of graphs

Here we present a set of maximum principles for scalar functions which satisfy a parabolic evolution equation on a domain in $\mathbb{R}^n$. We follow Lieberman [**30**] and modify his results as required. The comparison and maximum principles will be used to obtain interior estimates, and since we have a boundary value problem, the estimates we give here will depend upon the boundary values.

We now define the parabolic boundary of a space-time domain as

$$\tilde{\Omega} = \bigcup_{t\in[0,T)} \Omega(t) \times \{t\},$$

where $\Omega(t) \subset \mathbb{R}^n$ depends on time. The parabolic boundary $\mathcal{P}\tilde{\Omega}$ is the set of all points $X = (x,t)$ in the topological boundary $\partial\tilde{\Omega}$ such that for every $\epsilon > 0$ the parabolic cylinder $\mathcal{Q}(X,\epsilon)$

$$\mathcal{Q}(X,\epsilon) = \{Y \in \mathbb{R}^{n+1} : \ |Y - X| < \epsilon, \ t < t_0\},$$

where $|X| = \max\{|x|_{\mathbb{R}^n}, \sqrt{|t|}\}$, contains points which are not in $\tilde{\Omega}$.

Let us define now the general quasilinear operator

$$Pu \ = \ -\frac{\partial u}{\partial t} + a^{ij}(X,u,\mathrm{D}u)\mathrm{D}^2_{ij}u \ + \ a(X,u,\mathrm{D}u),$$

for some u defined on $\tilde{\Omega} \subset \mathbb{R}^{n+1}$ as above such that $u \in C^{2,1}(\tilde{\Omega})$. The coefficients a^{ij} and a are assumed to be defined for all values of their arguments, that is $a^{ij}(X,z,p)$ and $a(X,z,p)$ are defined for all $(X,z,p) \in \tilde{\Omega} \times \mathbb{R} \times \mathbb{R}^n$. We say that P is parabolic in a subset $\mathcal{S}$ of $\tilde{\Omega} \times \mathbb{R} \times \mathbb{R}^n$ if the coefficient matrix $a^{ij}(X,z,p)$ is positive definite for all $(X,z,p) \in \mathcal{S}$, and we use λ and Λ to denote the smallest and largest eigenvalues of the matrix a^{ij}. Hence we have

$$\lambda(X,z,p)\ |\xi|^2_{\mathbb{R}^n} \ \le \ a^{ij}(X,z,p)\xi_i\xi_j \ \le \ \Lambda(X,z,p)\ |\xi|^2_{\mathbb{R}^n},$$

for all $\xi \in \mathbb{R}^n$, and P is parabolic in $\mathcal{S}$ if $\lambda > 0$ on $\mathcal{S}$. If the ratio $\frac{\Lambda}{\lambda}$ is uniformly bounded on $\mathcal{S}$ then we say that P is uniformly parabolic on $\mathcal{S}$. If $\mathcal{S} = \tilde{\Omega} \times \mathbb{R} \times \mathbb{R}^n$ we say that P is (uniformly) parabolic in $\tilde{\Omega}$ and if $\mathcal{S} = \{(X,z,p) : z = u(X), p = \mathrm{D}u(X)\}$ for some C^1 function u, we say that P is (uniformly) parabolic at u.

Before proving a maximum principle it is useful to prove first a comparison principle. Therefore we start with the following result.

Theorem 2.1 (Lieberman [**30**], 1996, Comparison principle). *Let P be the quasilinear operator as above. Suppose that $a^{ij}(X,z,p)$ is independent of z and that there is an increasing positive constant k such that $a(X,z,p) + k(M)z$ is a decreasing function of z on $\tilde{\Omega} \times [-M,M] \times \mathbb{R}^n$ for any $M > 0$. If u and v are functions in $C^{2,1}(\overline{\tilde{\Omega}} \sim \mathcal{P}\tilde{\Omega}) \cap C(\overline{\tilde{\Omega}})$ such that P is parabolic with respect to u or v, $Pu \ge Pv$ in $\overline{\tilde{\Omega}} \sim \mathcal{P}\tilde{\Omega}$, and $u \le v$ in $\mathcal{P}\tilde{\Omega}$, then $u \le v$ in $\overline{\tilde{\Omega}}$.*

Proof. Let $M = \max\{\sup|u|, \sup|v|\}$, and define the function $w = (u-v)e^{\lambda t}$, where we have λ a constant at our disposal to be chosen later. So we have $w \le 0$ on the parabolic boundary $\mathcal{P}\tilde{\Omega}$. Assume there exists a point $X_0 = (x_0, t_0)$ where w takes its first positive maximum. At this point we have

$$\begin{aligned}
\mathrm{D}w \ &= \ \mathrm{D}u \ - \ \mathrm{D}v \ = \ 0,\\
\mathrm{D}^2 w \ &= \ \left(\ \mathrm{D}^2 u - \mathrm{D}^2 v\ \right)\ e^{\lambda t} \ \le \ 0, \ \text{and}\\
\frac{\partial w}{\partial t} \ &= \ \left(\ \frac{\partial u}{\partial t} \ - \ \frac{\partial v}{\partial t}\ \right)\ e^{\lambda t} \ + \ \lambda(u-v)\ e^{\lambda t} \ > \ 0.
\end{aligned}$$

Thus there exists an $\epsilon > 0$ such that

$$\begin{aligned}
\mathrm{D}u \ &= \ \mathrm{D}v,\\
\mathrm{D}^2 u - \mathrm{D}^2 v \ &\le \ 0, \ \text{and}
\end{aligned}$$

$$\frac{\partial}{\partial t}(u-v) \ \geq \ -\lambda(u-v-\epsilon).$$

Let $R = \big(X_0, u(X_0), \mathrm{D}u(X_0)\big)$ and $S = \big(X_0, v(X_0), \mathrm{D}u(X_0)\big)$. We compute

$$Pu(X_0) - Pv(X_0) \ = \ a^{ij}(R)\mathrm{D}^2_{ij}(u-v) \ + \ \big(a(R)-a(S)\big) \ - \ \frac{\partial}{\partial t}(u-v).$$

Now using the hypothesis on the existence of the constant k and the above properties of w at the positive maximum X_0 we obtain

$$Pu(X_0) - Pv(X_0) \ \leq \ \big(k(M) \ + \ \lambda\big)(u-v).$$

Choosing $\lambda < -k(M)$ we have $u > v$ and thus

$$Pu(X_0) - Pv(X_0) \ < \ 0$$

which contradicts the hypothesis that $Pu \ \geq \ Pv$. So we cannot have an interior positive maximum of w, which gives us $u \ \leq \ v$ in $\tilde{\Omega}$. □

One result which comes directly from the comparison principle above is the uniqueness of a solution for a parabolic boundary value problem such as ours. We only state the result here and invite the reader to consult [**30**] for more details.

Corollary 2.2 (Lieberman [**30**], 1996, Uniqueness). *Suppose that P is as in Theorem 2.1 and that u and v belong to $C^{2,1}(\tilde{\Omega}) \cap C(\bar{\tilde{\Omega}})$. If $Pu = Pv$ in $\tilde{\Omega}$ and $u = v$ on $\mathcal{P}\tilde{\Omega}$, then $u = v$ in $\tilde{\Omega}$.*

One may wonder whether or not we may apply this result to our problem. The two cases of problems which we must check are the radially symmetric graphs moving by mean curvature flow and the general mean curvature flow of graphs. The evolution equations will be defined and explained in the next two chapters. In Chapter 4 we work with a particular case of graphs outside the unit sphere and in Chapter 5 we treat radially symmetric graphs with Neumann condition on general rotationally symmetric surfaces. Here we simply state the relevant evolution equations to determine the applicability of the comparison principle above. The evolution equation for the general mean curvature flow of graphs is

$$\frac{\partial u}{\partial t} \ = \ \sqrt{1+|\mathrm{D}u|^2}\ \mathrm{div}\Big(\frac{\mathrm{D}u}{\sqrt{1+|\mathrm{D}u|^2}}\Big) \tag{3}$$

and in case u is a radially symmetric graph function (3) reads

$$\frac{\partial \omega}{\partial t} \ = \ \frac{d^2\omega}{dy^2}\,\frac{1}{1+(\frac{d\omega}{dy})^2} \ + \ \frac{d\omega}{dy}\,\frac{n-1}{y}. \tag{4}$$

We now show that the comparison principle above applies to solutions of (3) and (4).

Proposition 2.3 (Comparison principle for the mean curvature flow of radially symmetric or general graphs). *Theorem 2.1 applies to solutions of* (3) *and* (4)*.*

Proof. Let us examine the hypothesis of Theorem 2.1. We start by rewriting (3) as

$$\frac{\partial u}{\partial t} \ = \ \delta_{ij}\mathrm{D}^2_{ij}u - \frac{1}{1+|\mathrm{D}u|^2}\ \mathrm{D}^2_{ij}u\ \mathrm{D}_iu\ \mathrm{D}_ju \ = \ \Big(\ \delta_{ij} - \frac{\mathrm{D}_iu\ \mathrm{D}_ju}{1+|\mathrm{D}u|^2}\ \Big)\ \mathrm{D}^2_{ij}u,$$

where δ_{ij} is the Kronecker symbol. The coefficients of our operator are $a^{ij}(X, u, \mathrm{D}u) = \delta_{ij} - \frac{\mathrm{D}_i u \mathrm{D}_j u}{1+|\mathrm{D}u|^2}$, which does not depend on u and $a = 0$. So the operator is parabolic in $\tilde{\Omega}$ with $\lambda = \frac{1}{1+|p|^2} > 0$. The result of Theorem 2.1 can be applied in this case.

For the radially symmetric case of mean curvature flow (4) the coefficients of the operator P are $a^{11}(y, \omega, \frac{d\omega}{dy}) = \frac{1}{1+(\frac{d\omega}{dy})^2}$ and $a = \frac{d\omega}{dy}\,\frac{n-1}{y}$, both not dependent on ω. Again the operator is parabolic with $\lambda = \frac{1}{1+p^2} > 0$. The coefficient a is not dependent on the function ω but only on the point, time and first derivative, which means that the comparison principle is also applicable for the radially symmetric evolution. This was to be expected since the case of a radially symmetric graph function is just a particular case of the general one. □

The next step is to develop a maximum principle using the comparison principle above.

THEOREM 2.4 (Maximum principle). *Let P be a parabolic operator whose coefficients $a^{ij}(X, z, p)$ and $a(X, z, p)$ do not depend on z. If $Pu \geq 0$ in $\tilde{\Omega}$ then*

$$\sup_{\tilde{\Omega}} u \;\leq\; \sup_{\mathcal{P}\tilde{\Omega}} u.$$

PROOF. We apply Theorem 2.1 to u and $v = \sup\limits_{P\tilde{\Omega}} u$. The function v is a constant and thus trivially satisfies $Pv = 0$ which gives us $Pu - Pv \geq 0$. This together with the fact that on the parabolic boundary $\mathcal{P}\tilde{\Omega}$ the hypothesis $u \leq v$ is satisfied lets us use the comparison principle. This completes the proof. □

REMARK. Note that (3) and (4) satisfy the hypothesis of Theorem 2.4.

Another powerful tool in the study of partial differential equations is the boundary point lemma of E. Hopf [**20**], which is normally called the Hopf Lemma. It is obvious that at a maximum point of a scalar function on a domain the directional derivative towards that point is non-negative. If this point is a boundary point and the scalar function satisfies a parabolic inequality, then the following result gives us a strict sign on the derivative in a direction away from the boundary. Here we prove a Hopf Lemma where the parabolic boundary is assumed to be at least C^1. This result can be found throughout the literature, for example in [**33**].

LEMMA 2.5 (Protter and Weinberger [**33**], 1984, Hopf Lemma). *Let $\tilde{\Omega}$ be a space-time domain with C^1-boundary in which u is a solution of the parabolic inequality*

$$Pu \geq 0$$

where P is a quasilinear parabolic operator with smooth coefficients. Suppose that $X_0 = (x_0, t_0)$ is a point on the boundary $\partial\tilde{\Omega}$ where the maximum value M of u occurs. Assume that there exists a sphere through X_0 whose interior lies entirely in $\tilde{\Omega}$ and in which $u < M$. Also suppose that the radial direction from the centre of the sphere to X is not parallel to the time axis. Then if $\frac{\partial}{\partial \nu}$ denotes any directional derivative away from the boundary, we have

$$\frac{\partial u}{\partial \nu} > 0 \text{ at } X_0.$$

Proof. If we assume that X_0 is a point where the solution u attains a maximum, then any directional derivative of u in a direction pointing towards the point X_0 will be non-negative. To obtain the strict sign we have to work a little bit more by considering a perturbation of the solution u to which we apply the maximum principle. So the proof of the Hopf lemma is essentially an application of the maximum principle.

We proceed with some notations. Denote the sphere appearing in the hypothesis as $S \subset \tilde{\Omega}$ with boundary ∂S and centre at $X_s = (x_s, t_s)$. Consider now another sphere K centred at X_0 and with boundary ∂K and with radius smaller than $|X_0 - X_s|_{\mathbb{R}^{n+1}} = \sqrt{|x_0 - x_s|^2_{\mathbb{R}^n} + |t_0 - t_s|^2}$. Here we have denoted by $|\cdot|_{\mathbb{R}^n}$ the usual Euclidean norm in $\mathbb{R}^n$, so for a point $x = (x_1, \dots, x_n)$ we have $|x|_{\mathbb{R}^n} = \sqrt{\sum_i^n x_i^2}$. We denote by C_1 and C_2 the portion of ∂K which is included in S, respectively the portion of the boundary ∂S included in K. We also add the end points of the arcs C_1, C_2 to obtain a closed lens-shaped domain which we denote by D. We thus have:

(*i*) $u < M$ on C_2 except at X_0.

(*ii*) $u = M$ at X_0.

(*iii*) There exists a sufficiently small constant $\mu > 0$ such that $u \leq M - \mu$ on C_1.

If S does not satisfy the first of the relations then a slightly smaller sphere osculating the boundary at X_0 will be contained in the interior of S, and so the condition $u < M$ will be satisfied everywhere on the arc C_2 except the point X_0. The second relation is simply a hypotheses, while third is satisfied since $u < M$ everywhere in the interior of S and the fact that C_1 is a closed subset of S.

Define the following auxiliary function

$$v(x,t) \;=\; e^{-\alpha|X-X_s|^2_{\mathbb{R}^{n+1}}} - e^{-\alpha|X_0-X_s|^2_{\mathbb{R}^{n+1}}}.$$

Pick α large enough such that

$$Pv(x,t) \; > 0 \;\text{ for all } (x,t) \text{ on } D \cup \partial D.$$

Consider now the function

$$w = u + \epsilon v$$

and observe that for every positive ϵ, $Pw = Pu + \epsilon Pv > 0$ everywhere in D. From relation (*iii*) there exists an ϵ so small that we have

$$w < M \text{ on } C_1. \tag{5}$$

By its definition $v = 0$ on ∂S, thus also on the arc C_2. This together with relation (*i*) gives

$$w < M \text{ on } C_2 \text{ except at } X_0, \tag{6}$$

and

$$w = M \text{ at } X_0. \tag{7}$$

Now apply the maximum principle for the function w and using (5), (6) and (7) we find that the maximum of the function w occurs only at the boundary point X_0. This gives

$$\frac{\partial w}{\partial \nu} = \frac{\partial u}{\partial \nu} + \epsilon \frac{\partial v}{\partial \nu} \geq 0 \tag{8}$$

for any outward pointing direction ν of the set D. Denote by n the outer pointing unit normal to the boundary $\partial\tilde{\Omega}$ at X_0. Since the given vector ν is also outward pointing then $\langle \nu, n \rangle > 0$. Choose a coordinate system such that X_s is the origin and let $r(X) = |X - X_s|_{\mathbb{R}^{n+1}}$. We may rewrite v as $v(x,t) = e^{-\alpha r^2} - e^{-\alpha |X_0 - X_s|^2_{\mathbb{R}^{n+1}}}$, and we compute

$$\frac{\partial v}{\partial x_i} = -2\alpha x_i e^{-\alpha r^2}.$$

Then

$$\frac{\partial v}{\partial \nu} = -2\alpha r e^{-\alpha r^2} \langle n, \nu \rangle < 0.$$

This together with (8) gives us the desired result

$$\frac{\partial u}{\partial \nu} > 0 \text{ at } X_0.$$

□

Unfortunately Lemma 2.5 will not be enough for us since it does not include space-time domains which have a non-smooth boundary. The fact that the boundary of the domain of definition for the evolving graphs is time dependent allows the possibility of space-time domains which have corners. A corner (see Figure 2.1) occurs when the graph function will move upwards in the time direction and then at a certain time change and move downwards or the other way around. The boundary of the space time domain will then form, at the time of changing the direction of motion, a corner. And so the hypothesis of the interior sphere condition of the above theorem is not satisfied for all types of domains. Also at a corner of the space-time domain the normal to the boundary of this domain is not well defined. But here we still have the outer normal to the space set being well defined. This normal points out horizontally of the space-time set.

This will be sufficient for us to prove that we have a similar result using now the directional derivative in this restricted normal direction. One may wonder why this is sufficient. The answer is simple: the general result stated above gives a strict sign on any directional derivative of the graph function as long as it is in the direction of an outward pointing vector for the space-time domain $\tilde{\Omega}$. When we will apply this result in later chapters, we will only need to use an outward pointing vector to the boundary of the space domain only. This boundary normal is just the projection onto the space domain of the full normal to the boundary of the space-time domain.

Remark (Inner pointing corner points). There are two cases of corner boundary points. If the corner develops from a movement of the graph decreasing the diameter of the domain on which it is defined and then changing to either increasing at a certain time or to becoming constant for some time, then the above interior sphere condition is satisfied. All we have to do is to decide with what we will replace the unit outer normal direction, which as discussed above is not well-defined at a corner point. It is

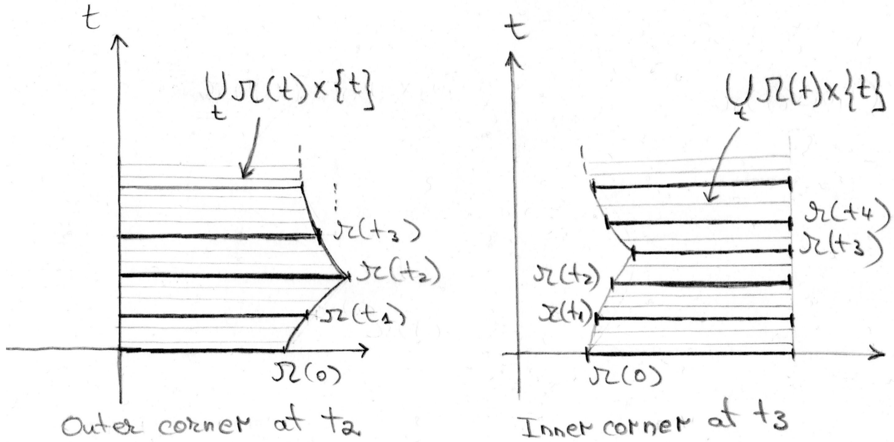

FIGURE 2.1. Outer corner and inner corner in the case of radially symmetric graphs with a 1-dimensional domain.

sufficient to consider here the radial direction of the interior sphere and use this as the outer normal. We call these points inner pointing corner points.

With the above observations, we are able to obtain Lemma 2.5 also for domains with inner pointing corners. The proof is similar to that of Lemma 2.5, with the only difference being that at an inner pointing corner we replace the unit normal with the vector ν_s defined in Proposition 2.6.

PROPOSITION 2.6 (Hopf Lemma in inner pointing corner points). *Let $\tilde{\Omega}$ be a space-time domain in which u is a solution of the parabolic inequality*

$$Pu \geq 0$$

where P is a quasilinear parabolic operator with smooth coefficients. Suppose that $X_0 = (x_0, t_0)$ is a point on the boundary $\partial\tilde{\Omega}$ where the maximum value M of u occurs. Assume that there exists a sphere through X_0 whose interior lies entirely in $\tilde{\Omega}$ and in which $u < M$. Also suppose that the radial direction ν_s from the centre of the sphere to X_0 is not parallel to the time axis. Then if $\frac{\partial}{\partial\nu}$ denotes any directional derivative satisfying $\langle \nu, \nu_s \rangle > 0$, we have

$$\frac{\partial u}{\partial \nu} > 0 \ \textit{at}\ X_0.$$

When the corner arises from the domain growing and then shrinking or remaining constant, we have an outer pointing corner. At these points the interior sphere condition is no longer satisfied. Fortunately, instead of imposing the interior sphere condition we may use a lower parabolic frustum around the point of boundary maxima. Following Lieberman [**30**], we define

$$PF(R, \mu, Y) \ = \ \{X = (x,t) \in \mathbb{R}^{n+1} \ : \ |x - y|^2_{\mathbb{R}^n} + \mu^2(s-t) < R^2, \ t < s\},$$

for some $Y = Y(y,s) \in \mathbb{R}^{n+1}$. We also define the unit outer normal we shall use. Suppose the centre of the frustum is at $Y = (y,s)$. Then at any other point $X = (x,s)$ at the same time s, define

$$\nu_{X,Y} = \frac{x-y}{|x-y|_{\mathbb{R}^n}}.$$

Note that this is a vector with n components.

The next result can be applied for any of the three cases of points on the boundary of the space-time domain, points where the boundary is C^1, inner pointing corners and outer pointing corners.

Proposition 2.7 (Lieberman [**30**], Hopf Lemma for domains with corners). *Let $\tilde{\Omega}$ be a space-time domain in which u is a solution of the parabolic inequality*

$$Pu \geq 0$$

where P is a quasilinear parabolic operator with smooth coefficients. Suppose that $X_0 = (x_0, t_0)$ is a point in the boundary $\partial\tilde{\Omega}$ where the maximum value M of u occurs in the following sense: there exists a lower parabolic frustum $PF(R,\mu,Y) \subset \tilde{\Omega}$ centred at some point $Y = Y(y,t_0) \in \tilde{\Omega}$ found in the same time slice as X_0 and two positive constants, $\mu > 0$ and the radius $R = |x_0 - y|_{\mathbb{R}^n}$ such that for all points except X_0 in $\overline{PF(R,\mu,Y)}$ we have $u < M$. Then for $\nu = (\nu_{X_0,Y}, 0)$ we have:

$$\frac{\partial u}{\partial \nu} \; > \; 0 \text{ at } X_0. \tag{9}$$

Proof. The proof follows similarly to that of Lemma 2.5 above. First let us define $r = r(X) = \sqrt{|x-y|^2_{\mathbb{R}^n} + \mu^2(t_0 - t)}$ everywhere in the parabolic frustum PF. We also denote the domain where we apply the maximum principle to the auxiliary function as $D = \{X \in PF(R,\mu,Y) |\, |x-y| > \frac{R}{2}\} \subset \overline{\tilde{\Omega}}$. The parabolic boundary of the domain D we denote by $\mathcal{P}D = \{X = X(x,t) \in \overline{D} : r(X) = R,\ t \leq t_0\} \cup \{X = X(x,t), |x-y| = \frac{R}{2},\ t \leq t_0\}$. Now notice that this boundary $\mathcal{P}D$ does not contain the "cap" of D, contained in the t_0 time slice. We are not concerned about this since Theorem 2.4 ensures that it is sufficient to look at the values on the parabolic boundary only, not the topological boundary.

We define the function v on D as

$$v(x,t) = e^{-\alpha r^2} - e^{-\alpha R^2},$$

and as before see that for sufficiently large α we have

$$Pv \geq 0$$

everywhere in D. Let us now define w as

$$w(x,t) = u(x,t) + \epsilon v(x,t),$$

which satisfies the following parabolic evolution in D

$$Pw = Pu + \epsilon Pv \geq 0.$$

Thus we may apply the maximum principle and obtain that for the function w any maximum in the space-time domain $\overline{D}$ appears only on the parabolic boundary $\mathcal{P}D$. We also know that the only maximum is attained at $X_0 \in \mathcal{P}D$. This is because, just as in the previous proof, on the first part of the parabolic boundary $\mathcal{P}D$ the set $\{X = X(x,t) \in D : r(X) = R,\ t \leq t_0\}$ of the domain D, we have $v = 0$. Thus w is equal to u there. On the second part of the parabolic boundary, since the values of u are strictly smaller than M and away from the point X_0 a choice of positive $\epsilon > 0$ will ensure that w is smaller than M. Thus w achieves its maximum at the point X_0 of the parabolic boundary of the frustum, just as the function u does.

As in the previous proofs we can now compute the directional derivative for w in the direction of ν and find that it is positive, since the function w increases towards the maxima X_0:

$$\frac{\partial w}{\partial \nu} = \frac{\partial u}{\partial \nu} + \epsilon \frac{\partial v}{\partial \nu} \geq 0 \text{ at } X_0.$$

Again we compute the directional derivative for v:

$$\begin{aligned} \frac{\partial v}{\partial \nu} &= \nu_{X_0,Y} \cdot \left(\frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_n} \right) = \frac{x_0 - y}{|x_0 - y|_{\mathbb{R}^n}} \cdot \mathrm{D} e^{-\alpha(|x-y|^2_{\mathbb{R}^n} + \mu^2(t-s))} \\ &= -\frac{2\alpha e^{-\alpha}}{|x_0 - y|_{\mathbb{R}^n}} < 0 \text{ at } X_0, \end{aligned}$$

which clearly gives the desired result

$$\frac{\partial u}{\partial \nu} > 0 \text{ at } X_0.$$

□

Remark (Use of lower parabolic frustum versus the interior sphere condition). As we can see in the above proposition, the use of the lower parabolic frustum has allowed the hypothesis on the subset where the maximum is attained to be relaxed, since now we are able to obtain (9) when we only have information for times immediately prior to t_0. This is the setting which we will normally use, treating maxima which appear at a certain time without knowing how the graph function will evolve past that time.

Note also that in all the variants of the Hopf Lemmas above the existence of a lower parabolic frustum can replace the interior sphere condition.

Remark (Relation between the parabolic and topological boundary). Another issue is the fact that each of the Hopf Lemmas use the topological boundary of the space-time domain $\tilde{\Omega}$ in which we have the parabolic evolution of the graphs, not the parabolic boundary $\mathcal{P}\tilde{\Omega}$. The major application of the Hopf lemma is to obtain some information on the derivative normal to the boundary at a boundary maxima. We know by the maximum principle that the boundary values are greater than the interior values. The parabolic boundary is in some cases a strict subset of the topological boundary. This means that in some cases Proposition 2.7 provides information on more than just the parabolic boundary.

REMARK. From now on when we refer to the 'Hopf Lemma', we invoke Proposition 2.7. Keep in mind however that things are simpler at some of the boundary points which we shall consider.

3. Mean curvature flow of immersions

Here we present results for the mean curvature flow of immersions, which can of course also be used in the particular case when the immersion satisfies a graph condition. These results are used in Chapter 6, where we treat the mean curvature flow of general graphs.

We start this section with a result used also in [**10**], a weak maximum principle for the mean curvature flow of immersions. The proof of this maximum principle follows the usual steps of a regular maximum principle but the required hypotheses are generalised for immersions. We present here the compact version and an easy extension to the case of compact hypersurfaces with boundary. We work first with functions defined intrinsically on the evolving hypersurfaces.

THEOREM 2.8 (Weak maximum principle, compact case). *Let $(M_t)_{t\in(t_0,t_1)}$ be a solution of mean curvature flow consisting of hypersurfaces $M_t = F_t(M^n)$ where $F(\cdot,t) = F_t : M^n \to \mathbb{R}^{n+1}$ and M^n is compact. Suppose $h : M^n \times [t_0,t_1) \to \mathbb{R}$ is sufficiently smooth for $t > t_0$, continuous on $M^n \times [t_0,t_1]$, and satisfies an inequality of the form*

$$\Big(\frac{\partial}{\partial t} - \Delta^{M_t}\Big)h \ \leq \ a \cdot \nabla^{M_t} h. \tag{10}$$

Then

$$\max_{M^n} h(\cdot,t) \leq \max_{M^n} h(\cdot,t_0) \tag{11}$$

for all $t \in [t_0,t_1]$. For the vector $a : M^n \times [t_0,t_1) \to \mathbb{R}^{n+1}$ we only require that it is well-defined and bounded in a neighbourhood of all maximum points of h.

PROOF. If we have h satisfying (10) then for any $\epsilon > 0$ the function $\tilde{h} = h - \epsilon t$, which agrees with h at time 0, satisfies the strict inequality

$$\Big(\frac{\partial}{\partial t} - \Delta^{M_t} - a \cdot \nabla^{M_t}\Big)\tilde{h} < 0. \tag{12}$$

At a point where for the first time $\max_{M_t} \tilde{h}$ reaches a value larger than $\max_{M_0} h$ the standard derivative criteria at a local maxima says that

$$\begin{aligned} \frac{\partial}{\partial t}\tilde{h} &\geq 0, \\ \partial_i \tilde{h} &= 0, \\ \partial_i\partial_j \tilde{h} &\leq 0, \end{aligned}$$

for any $1 \leq i,j \leq n$. Now we can compute the tangential derivative on M_t and the Laplacian on M_t by using Appendix A of [**10**] and obtain

$$\begin{aligned} \nabla^{M_t}\tilde{h} &= g^{ij}\partial_i\tilde{h}\partial_j F = 0, \\ \Delta^{M_t}\tilde{h} &= \mathrm{div}^{M_t}\nabla^{M_t}\tilde{h} = g^{ij}\Big(\partial_i\partial_j\tilde{h} - \Gamma_{ij}^k\partial_k\tilde{h}\Big) \leq 0, \end{aligned}$$

where g^{ij} is the inverse metric on M_t with $1 \leq i, j \leq n$, $\partial_i F$ are vectors tangent to M_t taken with respect to a choice of local coordinates on M^n, and Γ_{ij}^k are the corresponding Christoffel symbols on M_t. This, together with the above sign on the time derivative, gives

$$\Big(\frac{\partial}{\partial t} - \Delta^{M_t} - a \cdot \nabla^{M_t}\Big)\tilde{h} \geq 0,$$

which contradicts (12). Hence $\tilde{h}$ is bounded by the initial values of h at all positive times. Letting $\epsilon \to 0$ we see that the same is true for h. □

It is easy to see that in the case of compact solutions of mean curvature flow with boundary the same can be done and then (11) will depend upon the boundary values also. Below we state this extension of Theorem 2.8.

THEOREM 2.9 (Weak maximum principle, boundary case). *Let $(M_t)_{t\in(t_0,t_1)}$ be a solution of mean curvature flow consisting of hypersurfaces $M_t = F_t(M^n)$ where $F(\cdot,t) = F_t : M^n \to \mathbb{R}^{n+1}$ and M^n has a smooth boundary ∂M^n. Suppose $h : M^n \times [t_0, t_1) \to \mathbb{R}$ is sufficiently smooth for $t > t_0$, continuous on $M^n \times [t_0, t_1]$ and satisfies an inequality of the form* (10). *Then*

$$\sup_{M^n} h(\cdot,t) \leq \max\Big\{\sup_{M^n} h(\cdot,t_0),\ \sup_{\partial M^n} h(\cdot,t)\Big\}$$

for all $t \in [t_0, t_1]$. For the vector $a : M^n \times [t_0, t_1) \to \mathbb{R}^{n+1}$ we only require that it is well-defined and bounded in a neighbourhood of all maxima of h.

The same may be shown for functions defined extrinsically. That is, where $h(p,t) = f(X,t)$ and $X = F(p,t)$ is the position vector. The two types of arguments are equivalent and the condition (10) is modified to include the full time derivative

$$\Big(\frac{d}{dt} - \Delta^{M_t}\Big)f \leq a \cdot \nabla^{M_t} f. \tag{13}$$

Following the work done by Huisken [**22**] we give now a comparison principle for two solutions of mean curvature flow with free boundaries given by a fixed contact angle of ninety degrees on a fixed surface Σ. The only modifications required are to deal with possible contact of the evolving hypersurfaces on the boundary.

THEOREM 2.10 (Comparison principle for solutions of mean curvature flow with boundaries). *Let M_1 and M_2 be two smooth solutions of mean curvature flow* (1) *or* (2) *for time $0 \leq t \leq t_1$. If $\overline{M_1}$ and $\overline{M_2}$ are disjoint at time $t = 0$ then they remain disjoint for the whole interval $0 \leq t \leq t_1$.*

PROOF. Suppose there exists a time $t_0 > 0$ such that M_{1,t_0} and M_{2,t_0} touch for the first time. There are two cases. Either the intersection point X is in the interior of the hypersurfaces, or it is a boundary point.

The interior case may be dealt with exactly as in the proof for closed hypersurfaces presented in [**22**]. For the convenience of the reader we present it below.

Let S be some fixed reference hyperplane which is tangent to the surfaces M_{1,t_0} and M_{2,t_0} at the point X and take Gaussian coordinates in a neighbourhood of $X \in S$. Then locally around X we can write $M_{1,t}$ and $M_{2,t}$ for $t \in (t_0-\epsilon, t_0+\epsilon)$ as graphs of functions $u_1(t)$ and $u_2(t)$ on S. The unit normal to M_i is given by

$$\nu_i \;=\; (1+|\mathrm{D}u_i|^2)^{-\frac{1}{2}}(-\mathrm{D}u_i, 1),$$

and u_i satisfies the evolution

$$\frac{\partial}{\partial t}u_i \;=\; (1+|\mathrm{D}u_i|^2)^{\frac{1}{2}}H_i,$$

where H_i is the mean curvature of M_i. We also have that $\mathrm{D}u_1 \;=\; \mathrm{D}u_2 = 0$ at X since either one of them attains an interior minimum and the other an interior maximum or they both attain an interior maximum or minimum. Before time t_0 we had $u_1 - \delta \geq u_2$, for some $\delta > 0$ by assumption. Since $u_1 - \delta$ and u_2 satisfy the hypothesis of Theorem 2.1 in a neighbourhood of X, we obtain a contradiction to $\delta \leq 0$ at X.

The second case is when X is a boundary point, where the hypersurfaces meet on Σ perpendicularly. Choose again a reference hyperplane S which is tangential to M_{1,t_0} and M_{2,t_0} at the boundary point $X \in \partial S \subset \Sigma$. Once again we use local Gaussian coordinates around the point X and write our two surfaces as graphs u_1 and u_2 on S. On the boundary of the two surfaces we have that $\langle \nu_{M_i}, \nu_\Sigma\rangle = 0$. Let us choose our coordinate system around X such that we have

$$\nu_\Sigma|_{\partial S} \;=\; \big(\nu_{\partial S}, 0\big).$$

This transforms the boundary condition for the two graphs u_i into $\langle \mathrm{D}u_i, \nu_{\partial S}\rangle = 0$. So the two graphs satisfy the following evolution equation

$$\begin{aligned}\frac{\partial}{\partial t}u_i \;&=\; (1+|Du_i|^2)^{\frac{1}{2}}\; H_i \text{ on the interior } S\\ \langle \mathrm{D}u_i, \nu_{\partial S}\rangle \;&=\; 0 \text{ on the boundary } \partial S.\end{aligned}$$

This implies, using Proposition 2.7, that u_1 and u_2 have at the time of touching both negative and positive values, since otherwise the boundary point X where u_1 and u_2 are zero will be a maximum or minimum. This contradicts the previous argument which excludes interior points where the hypersurfaces touch. This completes the two cases of the proof. □

Finally we add a Hopf Lemma to our collection of maximum principles for hypersurfaces moving by mean curvature flow. The proof follows similarly to that of Proposition 2.7, where instead of using the maximum principle Theorem 2.4 we invoke Theorem 2.9 above. Here we shall use the extrinsic definition for a function.

LEMMA 2.11 (Hopf Lemma for mean curvature of hypersurfaces). *Let $(M_t)_{t\in(t_0,t_1)}$ be a smooth solution of mean curvature flow consisting of hypersurfaces $M_t = F_t(M^n)$ where $F(\cdot, t) = F_t : M^n \to \mathbb{R}^{n+1}$ and M^n has a smooth boundary ∂M^n. Suppose $f : \bigcup_{t\in[t_0,t_1)} M_t \times \{t\} \to \mathbb{R}$ is sufficiently smooth for $t > t_0$, continuous on M_t and satisfies an inequality of the form* (13). *Take $p^* \in \partial M^n$ and a time t^* such that at the point $X^* = F(p^*, t^*)$, f attains a first maximum. Then at this point we have*

$$\nabla^{M_{t^*}}_{\nu_{\partial M_{t^*}}} f|_{X^*} > 0,$$

where $\nu_{\partial M_t}$ *is the normal to the boundary* ∂M_t*. In case* X^* *is a point where* f *attains a first minimum then the sign of the inequality changes.*

Remark. The smoothness of M_t together with the assumption that we work only in a first maximum or minimum ensures that we have the existence of a parabolic frustum.

CHAPTER 3

Short time existence for the mean curvature flow of graphs with a free boundary

1. Introduction

The first step in solving the graph problems (1) and (2) is to show that the solution exists for a short time. Most importantly for us are the applications of short time existence. In particular, later we investigate long time existence, either proving height and gradient bounds which permit us to repeatedly and indefinitely apply the short time existence theorem indefinitely, or showing that our graph will cease to exist after some finite time.

The result of short time existence for hypersurfaces moving by mean curvature flow with a free Neumann boundary on a fixed hypersurface Σ is known, at least, from the work of Stahl [**35**]. The author writes the moving surfaces for a short time as graphs over the initial surface and obtains short time existence for the linearised problem using the results of Hamilton [**18**], and quoting regularity theory for parabolic problems (found in [**27**] for example) he obtains the desired short time existence result. Due to the fact that the problem (2) does not satisfy the conditions of the short time existence theorem found in [**35**] we present in this chapter another approach to proving short time existence for free boundary problems. We must choose a different reference for the 'standard parabolic theory', and use the book of Lieberman [**30**].

There are two approaches which we use. In the first we write our problem as a graph in a fixed direction over a fixed subset of $\mathbb{R}^n$. This approach is new and it is used to obtain preservation of the graph property in Chapter 4. The second is the well-known technique of writing the evolving surfaces as graphs in the direction of the normal to the initial hypersurface. Both of the settings can be used to obtain the short time existence result for any of our problems in the next chapters, up to conditions on the initial data. We have to mention here that the second setting, where we consider graphs over the initial hypersurface, is not enough to provide us with a long time existence result since the direction of the graph definition will change with the evolving surfaces; nevertheless it is an important tool. In the next section we start by presenting in detail the first approach. We then briefly summarise the setup of the second setting and invite the reader to follow the details in [**35**]. Finally, we explain the way one is able to use the standard parabolic theory found in [**30**] to obtain a short time existence result which fits both of the settings and applies to each of the problems.

Some of our work is restricted to surfaces, that is $n = 2$, such as in Chapters 4 and some sections in Chapter 6. But the existence results below are essentially independent of the dimension of the evolving hypersurface. In the next section we use the definition of the problems (1) and (2), but when working in the 2-dimensional setting we refer to the definitions in Chapter 6, that is (54) and (56).

2. Setup

Let us first note that writing our surfaces as graphs in the $e_3 = (0,0,1) \in \mathbb{R}^3$ direction gives rise to a problem defined on a domain with time dependent boundary. This makes our short time existence results more complicated, especially in the case of combined boundary conditions where the domain is an annulus with a changing width. Both of the approaches in the settings we present overcome this problem. The difference between them is the choice of direction for our graphs.

For the first setting we choose a vector field fixed in time which allows us to define our graphs over a fixed domain in $\mathbb{R}^2$. (Here and in what follows we identify $\mathbb{R}^2$ with the plane $\operatorname{span}\{e_1, e_2\} = \{ae_1 + be_2 : a, b \in \mathbb{R}\} \subset \mathbb{R}^3$.) The vector field is completely determined by the surface Σ. The Neumann boundary moves freely on the fixed surface Σ and we consider only surfaces Σ which are rotationally symmetric in the Killing vector field direction $K_3 : \mathbb{R}^3 \to \mathbb{R}^3$ given by $K_3(x_1, x_2, x_3) = (x_2, -x_1, 0)$. For the definition and properties of the Killing vector fields we refer the reader to Chapter 4. We can therefore find the perfect candidate for our graph direction as the vector field in $\mathbb{R}^3$ which is tangent to Σ and also perpendicular to K_3:

$$\xi|_\Sigma \;=\; K_3 \times \nu_\Sigma \in T\Sigma,$$

where we denote by $\times$ the vectorial product. For example when Σ is a sphere we have

$$\begin{aligned}
\xi(x_1,x_2,x_3) &= \big(-x_1x_3,\ -x_2x_3,\ x_1^2+x_2^2\big)\\
&= (-x_1)\ \big(\ x_3,\ 0,\ -x_1\big)\ +\ (-x_2)\ \big(0,\ x_3,\ -x_2\big)\\
&= (-x_1)\ K_1(x_1,x_2,x_3) + (-x_2)\ K_2(x_1,x_2,x_3),
\end{aligned}$$

where K_1 and K_2 are the other two Killing vector fields of rotation in $\mathbb{R}^3$. Note that ξ vanishes at some points on certain surfaces Σ but this will not cause a problem later. Define $\Omega_N = B_r\big((0,0)\big) \subset \mathbb{R}^2$ and $\Omega_{DN} = B_R\big((0,0)\big) \sim B_r\big((0,0)\big) \subset \mathbb{R}^2$. These sets serve as the domain of definition for the graph function: the first for problem (54) and the second for (56), in Chapter 6. We use R for the radius of the fixed boundary on which we have the Dirichlet boundary condition and use r to denote the radius of the ball obtained by the intersection of the rotational symmetric surface Σ with $\mathbb{R}^2$.

Examples include the mean curvature flow of graphs inside the catenoid neck where $r = 1$, and mean curvature flow outside the unit sphere with a fixed Dirichlet boundary condition on the circle of radius R. We denote the two choices of sets by a generic Ω and in the future we differentiate between them explicitly only when it is not clear by the context. The boundary of Ω consists of either one connected component or two disjoint connected components. We denote by $\partial\Omega_D$ the Dirichlet boundary and by $\partial\Omega_N$ the Neumann boundary.

If $M_0 = F_0(M^2)$ is the smooth initial immersion of M^2 into $\mathbb{R}^3$ satisfying the initial condition

$$\langle \xi, \nu_{M_0} \rangle \;>\; 0 \tag{14}$$

and (x_1, x_2) are coordinates on Ω, let Φ be the flow associated with this vector field ξ such that $\Phi(x_1, x_2, a)$ is the point in $\mathbb{R}^3$ which one obtains beginning at (x_1, x_2) and travelling a distance $|a|$ in the direction of the vector field ξ if $a > 0$ or in the direction of the vector field $-\xi$ if $a < 0$. That is

$$\Phi(x_1, x_2, a) \;=\; \int_0^a \frac{\partial \Phi}{\partial \mu}(x_1, x_2, \mu) d\mu \;=\; \int_0^a \xi\big(\Phi(x_1, x_2, \mu)\big) d\mu.$$

Using this vector field we can write our initial surface as

$$M_0 = \{X \in \mathbb{R}^3 : X = \Phi(x_1, x_2, w_0(x_1, x_2)),\ (x_1, x_2) \in \Omega\},$$

where $w_0 : \Omega \to \mathbb{R}$ is the scalar function given by $w_0(x_1, x_2) = a$, the unique a such that $\Phi(x_1, x_2, a)$ is contained in M_0. The uniqueness of a is guaranteed by the fact that ξ is not singular. In the case ξ is singular we consider the second setting for the short time existence argument.

Consider for $\epsilon > 0$ small

$$\mathcal{U}_\epsilon := \{\Phi(x_1, x_2, a) : (x_1, x_2) \in \Omega \text{ and } |\Phi(x_1, x_2, a) - \Phi(x_1, x_2, w_0(x_1, x_2))|_{\mathbb{R}^3} < \epsilon\},$$

a tubular neighbourhood of M_0. For $X = (x_1, x_2, x_3) = \Phi(x_1, x_2, a)$, the coordinates (x_1, x_2) represent the 'foot point' in Ω in which the flow starts and $x_3 = a$ is the signed distance one travels from this foot point along the vector field ξ. We say that a is the length of the flow line between the foot point in Ω and the point $X \in M_t$. At each point in $\mathcal{U}_\epsilon$, the set $\{\tilde{e}_1 = e_1, \tilde{e}_2 = e_2, \tilde{e}_3 = \xi\}$ defines a basis for $T\mathbb{R}^3$. This can be seen from the fact that we consider initial graphs, which implies that for some neighbourhood around the boundary of M_0 the contact surface Σ is not horizontal, i.e. ξ cannot be represented as a linear combination of e_1 and e_2. This basis induces a Riemannian metric on $\mathbb{R}^3$ with components

$$\gamma_{\alpha\beta} = \langle \tilde{e}_\alpha, \tilde{e}_\beta \rangle\,,$$

and corresponding Christoffel symbols

$$\tilde{\Gamma}^\rho_{\alpha\beta} = \frac{1}{2}\gamma^{\rho\sigma}\big(\partial_\alpha \gamma_{\sigma\beta} + \partial_\beta \gamma_{\sigma\alpha} - \partial_\sigma \gamma_{\alpha\beta}\big),$$

where $\alpha,\ \beta = 1, 2, 3$. In the above and in what follows we use the Einstein summation convention, unless otherwise noted. In this basis we write the normal to the fixed surface Σ as

$$\nu_\Sigma = \gamma^{\alpha\beta}\nu^\alpha_\Sigma \tilde{e}_\beta,$$

where $\nu^\alpha_\Sigma = \nu^\alpha_\Sigma(x_1, x_2, s)$, for each $\alpha = 1, 2, 3$ are the component functions of ν_Σ. For a short time $\delta > 0$ we can write in these coordinates the evolving surfaces M_t as the graphs of scalar function $w(\cdot, t)$

$$M_t \;=\; \{X \in \mathbb{R}^3 : X = \Phi(x_1, x_2, w(x_1, x_2, t)),\ (x_1, x_2) \in \Omega\}.$$

where $w : \Omega \times [0, \delta] \to \mathbb{R}$. In these coordinates our problem becomes

$$\begin{aligned} \frac{\partial w}{\partial t}(x_1, x_2, t) \;&=\; -\frac{1}{s}H,\ \forall (x_1, x_2, t) \in \Omega \times [0, \delta], \\ \sum_{i=1}^{2} \nu^i_\Sigma \mathrm{D}_i w \;&=\; \nu^3_\Sigma,\ \forall (x_1, x_2, t) \in \partial\Omega_N \times [0, \delta], \end{aligned} \tag{15}$$

$$\begin{aligned}\big(\ w(x_1,x_2,t) \ &= \ w_0(x_1,x_2),\ \forall (x_1,x_2,t)\in \partial\Omega_D\times[0,\delta]\ \big),\\ w(\cdot,0) \ &= \ w_0,\end{aligned}$$

where we define $s:\Omega\times[0,\delta]\to\mathbb{R}$ by

$$s(x_1,x_2,w(x_1,x_2,t)) = \langle \nu_w,\xi\rangle(x_1,x_2,w(x_1,x_2,t)),$$

and H is the mean curvature of the graph function w. The last of the boundary conditions only comes in for the combined boundary condition problem (56) and should be ignored for the pure Neumann problem (54).

Remark. This setting is mostly utilised in the study of the mean curvature flow of graphs outside a sphere, cf. Chapter 4. There it is used in the proofs of both short and preservation of the graph property. Despite this, we note that if the initial surface satisfies (14), then we can guarantee short time existence for any of the other problems we consider also. This is not contained in nor supersedes the classical notion of graphs.

The second type setting we consider here can be found in [**35**], and is perhaps by now a standard technique for obtaining short time existence for geometric flows of all flavours. It is when we write our evolving surfaces M_t as graphs over the initial surface M_0 in the direction normal to M_0. For us the situation is made slightly more complex by the presence of Σ. To this end, in the notation of the previous section, we set ξ to be the normal of the initial surface and further restrict it to be tangential to Σ. That is

$$\xi|_{M_0} \ = \ \nu_{M_0}\ ,\ \xi|_{\Sigma}\ \in\ T\Sigma,\ \text{and}\ |\xi|_{\mathbb{R}^3}\ =\ 1.$$

We omit the details since they are carefully explained in [**35**]. In this setting our problems become equivalent to the evolution of a scalar function $w: M^2\times[0,\delta]\to\mathbb{R}$ given by

$$\begin{aligned}\frac{\partial w}{\partial t}(x_1,x_2,t) \ &= \ -\frac{1}{s}H,\ \forall (x_1,x_2,t)\in M^2\times[0,\delta],\\ \sum_{i=1}^{2}\nu_\Sigma^i \mathrm{D}_i w \ &= \ \nu_\Sigma^3,\ \forall (x_1,x_2,t)\in \partial M^2{}_N\times[0,\delta],\\ \big(\ w(x_1,x_2,t) \ &= \ 0,\ \forall (x_1,x_2,t)\in \partial M^2{}_D\times[0,\delta]\ \big),\\ w(\cdot,0) \ &= \ 0,\end{aligned} \tag{16}$$

where we have used the same conventions as above.

The following proposition from [**35**] computes in detail some important quantities we use to define our graph evolution.

Proposition 3.1 (Stahl [**35**], 1994). *Let $F:\Omega\to\mathbb{R}^3$, $F(x_1,x_2):=(x_1,x_2,w(x_1,x_2,t))$. Then we have the following properties for the surfaces defined by this immersion:*

(1) *The standard tangent vectors of $F(\Omega)$ are*

$$X_k(x_1,x_2) \ = \ \Big(0,\ldots,1,0,\ldots,D_k w\Big)(x_1,x_2)\ \text{for}\ k=1,2$$

(2) *The unit normal vector field is given by*

$$\nu(x_1,x_2) \ = \ s(x_1,x_2,t)\ \Big(\gamma_{\alpha\beta}(x_1,x_2,w(x_1,x_2,t))\Big)^{-1}_{1\le\alpha,\beta\le 3}\cdot\ \Big(-D_1 w,-D_2 w,1\Big)^T,$$

where T denotes the vector transpose operator, and also we have

$$s(x_1,x_2,w(x_1,x_2,t))^{-1} = \sqrt{\gamma^{3,3} - 2\gamma^{k,3}D_k w + \gamma^{kl}D_k w\ D_l w}$$

(3) *The following equalities hold*

$$\langle \nu, e_k\rangle (x_1,x_2,w(x_1,x_2,t)) = -s(x_1,x_2,w(x_1,x_2,t))D_k w(x_1,x_2,t) \text{ for } k=1,2,$$
$$s(x_1,x_2,w(x_1,x_2,t)) = \langle \nu,\xi\rangle (x_1,x_2,w(x_1,x_2,t)),$$

(4) *The first and second fundamental forms of $F(\Omega)$ are given by*

$$g_{ij}(x_1,x_2) = (\gamma_{ij} + \gamma_{i,3}D_j w + \gamma_{3,j}D_i w + D_i w D_j w)(x_1,x_2) \text{ for } i,j=1,2,$$
$$h_{ij}(x_1,x_2) = s\ (-D_iD_j w + \tilde{\Gamma}^k_{\alpha\beta}X^\alpha_i X^\beta_j D_k w - \tilde{\Gamma}^3_{\alpha\beta}X^\alpha_i X^\beta_j)(x_1,x_2) \text{ for } i,j=1,2,$$

where in the above we have used the Einstein summation convention and the convention that the Latin indices α, β range from 1 to 3.

The next proposition gives the aforementioned equivalence between the solutions of (15) and (16) defined above and the mean curvature flow defined in (54) and (56).

Proposition 3.2 (Stahl [**35**], 1994). *For every solution of* (15) *or* (16) *there exists a solution of* (54) *or* (56)*, such that the two solutions are equivalent up to tangential diffeomorphisms.*

3. Short time existence for Neumann and combined boundary problems

In this section we present one theorem which guarantees the existence of a solution to (54) and (56) from Chapter 6, for a short time. In this section we again follow [**30**]. First we introduce some notation, then define the general problem for which we state the short time existence theorem. Finally we show how one may apply the theorem to the problems (15) and (16).

Let $\tilde{\Omega} \subset \mathbb{R}^{n+1}$ with $n \geq 1$ be a bounded, open, connected subset of Euclidean space. Points in $\tilde{\Omega}$ are denoted by $X=(x,t)\in\tilde{\Omega}$, where $x=(x_1,\cdots,x_n)\in\mathbb{R}^n$. The norms in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ for a point $X=(x_1,\cdots,x_n,t)=(x,t)$ are given by

$$|x|_{\mathbb{R}^n} = \sqrt{\sum_{i=1}^n x_i^2}, \qquad \text{and} \qquad |X| = \max\left\{|x|_{\mathbb{R}^n}, \sqrt{t}\right\}.$$

For a point $X_0=(x_1^0,\cdots,x_n^0,t_0)\in\tilde{\Omega}$ and a positive number $\epsilon>0$ we define the cylinder

$$\mathcal{Q}(X_0,R) = \{X\in\mathbb{R}^{n+1} : \ |X-X_0|<\epsilon,\ t<t_0\}.$$

We define the parabolic boundary $\mathcal{P}\tilde{\Omega}$ to be the set of all points in $X_0\in\partial\tilde{\Omega}$ such that for any $\epsilon>0$ the cylinder $\mathcal{Q}(X_0,\epsilon)$ contains at least one point not in $\tilde{\Omega}$. In the special case that $\tilde{\Omega}=\Omega\times[0,T)$ for some $\Omega\subset\mathbb{R}^n$ and $T>0$, $\mathcal{P}\tilde{\Omega}$ is the union of the sets $B\tilde{\Omega}=\Omega\times 0$ (which is the bottom of $\tilde{\Omega}$), $S\tilde{\Omega}=\partial\Omega\times(0,T)$ (which is the side of $\tilde{\Omega}$) and $C\tilde{\Omega}=\partial\Omega\times 0$ (which is the corner of $\tilde{\Omega}$).

We observe that our problems (15) and (16) are in fact defined on this type of cylindrical domain, since after redefining the graphical immersions as graphs in the ξ direction for (15), or in the direction of the initial normal for (16), the domain of definition is no longer time dependent. This contrasts directly with the case when we write the initial surface as a graph in the e_3 direction.

For completeness, we note that in the case where $\tilde{\Omega}$ is a set in $\mathbb{R}^{n+1}$ these special sets are defined as follows. $B\tilde{\Omega}$ is the set of all points $X_0 = X_0(x_0, t_0) \in \mathcal{P}\tilde{\Omega}$ such that there is a positive R with $\mathcal{Q}\big((x_0, t_0 + R^2)\big) \subset \tilde{\Omega}$, $C\tilde{\Omega} = \big(\overline{B\tilde{\Omega}} \cap \mathcal{P}\tilde{\Omega}\big) \sim B\tilde{\Omega}$ and $S\tilde{\Omega} = \mathcal{P}\tilde{\Omega} \sim \big(B\tilde{\Omega} \cup C\tilde{\Omega}\big)$.

The proof of short time existence for quasilinear partial differential equations follows in two steps. First we obtain the existence of a solution for the associated linear problem, and then extend the existence to the quasilinear case through a fixed point argument.

So let us first define a linear operator L acting on functions $u : \tilde{\Omega} \to \mathbb{R}$ by

$$Lu \;=\; -\frac{\partial}{\partial t}u \;+\; a^{ij}(X)\,\mathrm{D}^2_{ij}u \;+\; b^i(X)\,\mathrm{D}_i u \;+\; c(X)\,u, \tag{17}$$

where we have kept the sign convention used in [**30**]. We also assume that L is a weakly parabolic operator:

$$a^{ij}(X)\,\xi_i\,\xi_j \;\geq\; 0 \text{ for all } \xi \in \mathbb{R}^n \text{ and all } X \in \tilde{\Omega}.$$

Our problems may be viewed as possessing an oblique derivative condition on the Neumann boundary, and so we shall also define the boundary operator. For a point $X_0 \in \mathcal{P}\tilde{\Omega}$ we say that an $(n+1)-$vector β points into $\overline{\tilde{\Omega}}$ if $\beta^{n+1} \leq 0$ and there is a positive constant ϵ such that $X_0 + h\beta \in \overline{\tilde{\Omega}}$ if $0 < h < \epsilon$. Denote the full space time derivative of u to as $\partial u = \Big(\mathrm{D}u,\ \frac{\partial}{\partial t}u\Big)$. The boundary operator associated with L is defined as

$$Mu \;=\; \langle \beta, \partial u\rangle + \beta^0 u, \tag{18}$$

for some vector field $\beta \in R^{n+1}$ such that β points into $\overline{\tilde{\Omega}}$ for all $X_0 \in \mathcal{P}\tilde{\Omega}$ and some scalar function β^0. One can guarantee that β points into $\overline{\tilde{\Omega}}$ by assuming that $\mathcal{P}\tilde{\Omega} \in H_\delta$ for some $\delta \in (1, 2)$ and that

$$\langle \beta, \gamma\rangle \;\geq\; \chi, \tag{19}$$

for some positive constant χ, where $\gamma(X_0)$ is the unit inner normal to the boundary of $\tilde{\Omega}(t_0) = \{x \in \mathbb{R}^n : (x, t_0) \in \tilde{\Omega}\}$ at X_0.

We now define the above mentioned Hölder space H_δ, and other further spaces and norms which we shall require. For $\alpha \in (0, 1]$, we say that the function f defined on $\tilde{\Omega} \subset \mathbb{R}^{n+1}$ is Hölder continuous at X_0 with exponent α if the quantity

$$[f]_{\alpha;X_0} = \sup_{X \in \tilde{\Omega} \sim \{X_0\}} \frac{|f(X) - f(X_0)|}{|X - X_0|^\alpha}$$

is finite. If the quantity is finite for $\alpha = 1$, we say that f is Lipschitz continuous at X_0. Also it is easy to see that if f is Holder continuous at a point, then it is also continuous there. If the semi-norm

$$[f]_{\alpha;\tilde{\Omega}} = \sup_{X_0 \in \tilde{\Omega}} [f]_{\alpha;X_0}$$

is finite, we say that f is uniformly Hölder continuous in $\tilde{\Omega}$.

REMARK. Since functions are Lipschitz at any point of differentiability, we see that Hölder continuity can be viewed as a sort of fractional differentiability, as explained in [**30**]. In addition, the above definition is in agreement with the basic "rule" from second order parabolic partial differential equations, that "two space derivatives are equivalent to one time derivative", because the exponents with respect to x in the definition of $[f]$ are twice those with respect to t. To see this simply recall the norms defined on $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ from above.

Let us now define some higher order Hölder semi-norms. For $\beta \in (0, 2]$ let

$$< f >_{\beta;X_0} = \sup\left\{ \frac{|f(x_0,t) - f(X_0)|}{|t - t_0|^{\frac{\beta}{2}}} : (x_0, t) \in \tilde{\Omega} \sim \{X_0\} \right\}$$

and

$$< f >_{\beta;\tilde{\Omega}} = \sup_{X_0 \in \tilde{\Omega}} < f >_{\beta;X_0} .$$

Then for any $a > 0$, such that $a = k + \alpha$, where k is a non-negative integer and $\alpha \in (0, 1]$ we can define

$$\begin{aligned} < f >_{a;\tilde{\Omega}} &= \sum_{|\beta|+2j=k-1} < \mathrm{D}_x^\beta\, \mathrm{D}_t^j f >_{\alpha+1}, \\ [f]_{a;\tilde{\Omega}} &= \sum_{|\beta|+2j=k} \left[\mathrm{D}_x^\beta\, \mathrm{D}_t^j f\right]_\alpha, \\ |f|_{a;\tilde{\Omega}} &= \sum_{|\beta|+2j\le k} \sup |\mathrm{D}_x^\beta\, \mathrm{D}_t^j f| \;+\; [f]_{a;\tilde{\Omega}} + < f >_{a;\tilde{\Omega}} . \end{aligned}$$

We can verify that $|f|_a$ defines a norm on $H_a(\tilde{\Omega}) = \left\{f : \tilde{\Omega} \to \mathbb{R} \;:\; |f|_a < \infty\right\}$ which makes $H_a(\tilde{\Omega})$ a Banach space.

We also define some weighted Hölder norms; all of these definitions can be found in Chapter IV of [**30**]. We denote by $d : \tilde{\Omega} \to \mathbb{R}$ the distance from the parabolic boundary, defined as

$$d(X_0) = \inf\Big\{|X - X_0| : X \in \mathcal{P}\tilde{\Omega} \text{ and } t < t_0\Big\},$$

and we set $d(X, Y) = \min\{d(X), d(Y)\}$. To simplify the notation we also make the following definitions

$$[f]_0 = [f]_0^* = \operatorname*{osc}_{\tilde{\Omega}} f,$$

$$|f|_0 = |f|_0^* = \sup_{\tilde{\Omega}} |f|.$$

For $b \geq 0$, we define

$$|f|_0^{(b)} = \sup_{\tilde{\Omega}} d^b |f|,$$

and for $b < 0$ we define

$$|f|_0^{(b)} = (\mathrm{diam}\ \tilde{\Omega})^b \sup_{\tilde{\Omega}} |f|.$$

If $a > 0$ and $a+b \geq 0$ and we have as before $a = k+\alpha$, we define

$$
\begin{aligned}
[f]_a^{(b)} &= \sup_{X \neq Y \text{ in } \tilde{\Omega}} \sum_{|\beta|+2j=k} d(X,Y)^{a+b} \frac{|D_x^\beta D_t^j f(X) - D_x^\beta D_t^j f(Y)|}{|X-Y|^\alpha}, \\
< f >_a^{(b)} &= \sup_{X \neq Y \text{ in } \tilde{\Omega}, x=y} \sum_{|\beta|+2j=k-1} d(X,Y)^{a+b} \frac{|D_x^\beta D_t^j f(X) - D_x^\beta D_t^j f(Y)|}{|X-Y|^{1+\alpha}}, \\
|f|_a^{(b)} &= \sum_{|\beta|+2j \leq k} |D_x^\beta D_t^j f|_0^{(|\beta|+2j+b)} + [f]_a^{(b)} + < f >_a^{(b)}
\end{aligned}
$$

and if $a+b < 0$, we define $[f]_a^{(b)}$ and $< f >_a^{(b)}$ by replacing $d(X,Y)$ with diam $\tilde{\Omega}$. We can also define $|f|_a^* = |f|_a^{(0)}$, and we note that using the above defined norms we have H_a^* and $H_a^{(b)}$ (with the obvious definitions) to be also Banach spaces.

The next theorem is an interpolation inequality which is used in the proof of the short time existence theorem. The proof and a generalised version of it can be found in Lieberman [**30**], Chapter IV.

Theorem 3.3 (Lieberman [**30**], 1996). *Let $0 \leq a < b$ and let $\sigma \in (0,1)$. Then there is a constant C depending only by b such that*

$$|u|_{\sigma a+(1-\sigma)b} \;\leq\; C\,|u|_a^\sigma |u|_b^{1-\sigma}.$$

We also need to explain what it means that our parabolic boundary is included in one of the above Banach spaces, which is one of the hypotheses of the short time existence theorem. For $\delta \geq 1$, we say that $\mathcal{P}\tilde{\Omega} \in H_\delta$ if $B\tilde{\Omega} \subset \{t = t^*\}$ for some t^* and if there is an $\epsilon > 0$ such that for any $X_0 \in S\tilde{\Omega}$, there are a function $f \in H_\delta(\mathcal{Q}'(X_0,\epsilon))$ and a coordinate system centred at X_0 such that

$$\tilde{\Omega} \cap \mathcal{Q}(X_0,\epsilon) \;=\; \{Y \in \mathcal{Q}(X_0,\epsilon) : y^n > f(y', s)\}$$

in this coordinate system. Here we have denoted by $\mathcal{Q}'$ the cylinder in $\mathbb{R}^n$ (following the same definition of cylinders in $\mathbb{R}^{n+1}$) and by y' the first $n-1$ components of the point $Y \in \mathbb{R}^{n+1}$. The following result found in [**30**], Chapter V, gives us a short time solution for the linear problem.

Theorem 3.4 (Lieberman [**30**], 1996). *Let $\mathcal{P}\tilde{\Omega} \in H_\delta$ for some $\delta \in (1,2)$, suppose that there exist positive constants A, B, C, λ, Λ such that the coefficients of the operator L satisfy*

$$
\begin{aligned}
a^{ij}\xi_i\xi_j &\geq \lambda|\xi|_{\mathbb{R}^n} \text{ for all } \xi \in \mathbb{R}^n \text{ and all } X \in \tilde{\Omega}, \\
\sum_{i=1}^n a^{ii} &\leq \Lambda, \\
|a^{ij}|_\alpha &\leq A, \\
|b^i|_\alpha &\leq B, \\
|c|_\alpha &\leq C,
\end{aligned}
$$

for some $\alpha \in (0,1)$ *and that the coefficients of the boundary operator* M *satisfies* (19) *and the following conditions*

$$\begin{aligned} |\beta^j|_{\delta-1} \;&\leq\; B_1\ \chi \text{ for } j = 1,\dots,n+1, \\ |\beta| \;&\leq\; \mu\ \langle \beta, \gamma \rangle\,, \end{aligned}$$

for some positive constants μ *and* B_1 *and* χ *and* γ *as above. If* $f \in H_\alpha^{2-\delta}$, $\Psi \in H_{\delta-1}$ *and* ϕ *is a continuous function on* $B\tilde{\Omega} \cup C\tilde{\Omega}$*, then there is a unique solution of the problem*

$$\begin{aligned} Lu \;&=\; f \text{ in } \tilde{\Omega}, \\ Mu \;&=\; \Psi \text{ on } S\tilde{\Omega}, \\ u \;&=\; \phi \text{ on } B\tilde{\Omega}. \end{aligned}$$

If also $\phi \in H_\delta(B\tilde{\Omega})$ *and* $M\phi = \Psi$ *on* $C\tilde{\Omega}$*, then* $u \in H_{2+\alpha}^{-\delta}$ *and there is a constant* C *determined only by* A*,* B*,* B_1*,* C*,* n*,* t*,* α*,* δ *and* $\tilde{\Omega}$ *such that*

$$|u|_{2+\alpha}^{-\delta} \;\leq\; C\,\Big(|f|_\alpha^{2-\delta} \;+\; \frac{|\Psi|_{\delta-1}}{\chi} \;+\; |\phi|_\delta \Big).$$

After getting a solution for the linear boundary problem, we can go a bit further and obtain a solution for quasilinear operators by a fixed point argument. Details of this procedure can be found in Chapter VIII of [**30**].

For the convenience of the reader we also include here the definition of a general quasilinear operator

$$Pu \;=\; -\frac{\partial u}{\partial t} + a^{ij}(X,u,\mathrm{D}u)\mathrm{D}^2_{ij}u \;+\; a(X,u,\mathrm{D}u), \tag{20}$$

for some u defined on $\tilde{\Omega} \subset \mathbb{R}^{n+1}$ as above such that $u \in C^{2,1}(\tilde{\Omega})$. The coefficients a^{ij} and a are assumed to be defined for all values of their arguments, that is $a^{ij}(X,z,p)$ and $a(X,z,p)$ are defined for all $(X,z,p) \in \tilde{\Omega} \times \mathbb{R} \times \mathbb{R}^n$.

Before discussing the existence result we give the relevant definitions for the work in this chapter. We say that P is parabolic in a subset $\mathcal{S}$ of $\tilde{\Omega} \times \mathbb{R} \times \mathbb{R}^n$ if the coefficient matrix $a^{ij}(X,z,p)$ is positive definite for all $(X,z,p) \in \mathcal{S}$, and we use λ and Λ to denote the smallest and largest eigenvalue of the matrix a^{ij}. Hence we have

$$\lambda(X,z,p)\ |\xi|^2_{\mathbb{R}^n} \;\leq\; a^{ij}(X,z,p)\xi_i\xi_j \;\leq\; \Lambda(X,z,p)\ |\xi|^2_{\mathbb{R}^n},$$

for all $\xi \in \mathbb{R}^n$, and P is parabolic in $\mathcal{S}$ if $\lambda > 0$ on $\mathcal{S}$. If the ratio $\frac{\Lambda}{\lambda}$ is uniformly bounded on $\mathcal{S}$ then we say that P is uniformly parabolic on $\mathcal{S}$.

Also we require the above defined quasilinear operator P to have smooth coefficients and be parabolic in the sense that for any bounded subset $\mathcal{K} \subset \tilde{\Omega} \times \mathbb{R} \times \mathbb{R}^n$ we suppose the existence of a positive constant $\lambda_{\mathcal{K}}$ such that

$$\lambda_{\mathcal{K}}\ |\xi|^2_{\mathbb{R}^n} \;\leq\; a^{ij}(X,z,p)\xi_i\xi_j,$$

for any $(X,z,p) \in \mathcal{K}$ and any $\xi \in \mathbb{R}^n$. The maps $X \to a^{ij}(X,z,p)$ and $X \to a(X,z,p)$ must be Hölder continuous if u is smooth enough. This is satisfied if there exists $\alpha \in (0,1)$ such that we have a^{ij} and a are in $H_\alpha(\mathcal{K})$ for any bounded subset $\mathcal{K}$ of $\tilde{\Omega} \times \mathbb{R} \times \mathbb{R}^n$.

The following Brouwer fixed point theorem is used to pass from the linear results to the quasilinear ones. We omit the proof of this theorem and invite the reader to follow [**30**], Chapter VIII for details.

THEOREM 3.5 (Lieberman [**30**], 1996). *Let $\mathcal{S}$ be a compact, convex subset of a Banach space $\mathcal{B}$ and let J be a continuous map of $\mathcal{S}$ into itself. Then J has a fixed point.*

If the coefficients of the operator are independent of u and $\mathrm{D}u$, as in the linear case, then the result of Theorem 3.4 provides us with a $H_{2+\alpha}^{(-\delta)}$ solution of the boundary value problem for any $\delta \in (1,2)$ for boundary data which are smooth enough. If we have that the boundary data is in $H_{2+\alpha}$ then the solution is in $H_{2+\alpha}$ and the same results can be obtained for the quasilinear Dirichlet boundary value problem if $\tilde{\Omega}$ is small enough in the time direction using the fixed point argument from above.

The situation is more complicated for the oblique derivative boundary value problem, which is the case for our problems (15) and (16). The usual method for the Dirichlet boundary value problem of getting existence of a solution for the quasilinear problem from estimates on the linear problem solution by applying a simple fixed point argument does not work for the oblique derivative boundary problem. The idea of the proof is to construct a map defined on a convex set, which takes values in the same set and then apply the fixed point theorem and obtained a fixed point which gives us a solution of the quasilinear problem. Proving that the map takes values into its own domain is the key goal. For the Dirichlet problem Lieberman [**30**], uses the fact that Lipschitz functions with compact support are bounded by the diameter of the set of definition. The compact support is obtained by subtracting the boundary values from the solution. This argument does not work for an oblique derivative boundary value problem in a general space time domain.

In this case one can use a more complicated version of a fixed point argument, as is also found in Lieberman [**30**], called Caristi theorems. We will not follow this approach since we notice that there is a simpler way for our class of problems. The issues arising from the fact that we have an oblique boundary value problem can be overcame if the problem can be transformed to one defined on a cylindrical space time domain. We notice that in the setup section we have done just that. The problems (15) and (16) are defined on cylindrical space time domains and we are able to obtain the analogue of the linear result found in Lieberman for our quasilinear problems with oblique derivative boundary conditions on cylindrical domains.

We begin by defining a domain smaller in the time variable, $\tilde{\Omega}_\epsilon$ as

$$\tilde{\Omega}_\epsilon = \{X \in \tilde{\Omega} : t < t_0 + \epsilon\},$$

where we also assume that $B\tilde{\Omega} \subset \{t = t_0\}$.

Now we can state the result of short time existence for quasilinear problems with linear oblique derivative boundary conditions. The proof of the theorem is a modification of the result found in Lieberman [**30**] for Dirichlet boundary value problems in Chapter VIII.

THEOREM 3.6 (Short time existence for quasilinear problems). *Let $\tilde{\Omega}$ be a cylindrical space time domain, that is $\tilde{\Omega} = D \times [0,T]$ for some open set D in $\mathbb{R}^n$. Suppose that the coefficients of P satisfy the hypotheses of Theorem 3.4 and $f \in H_\alpha^{2-\delta}$, $\Psi \in H_{\delta-1}$ and ϕ*

is a continuous function on $B\tilde{\Omega} \cup C\tilde{\Omega}$. Then there exists a positive constant $\epsilon > 0$ such that the problem

$$\begin{aligned} Pu &= f \text{ in } \tilde{\Omega}_\epsilon, \\ Mu &= \Psi \text{ on } S\tilde{\Omega}_\epsilon, \\ u &= \phi \text{ on } B\tilde{\Omega}_\epsilon, \end{aligned}$$

has a solution $u \in H^{(-\delta)}_{2+\alpha}$.

PROOF. We first remark that one of the hypothesis of Theorem 3.4 was the regularity of the parabolic boundary, $\mathcal{P}\tilde{\Omega} \in H_\delta$ for some $\delta \in (1,2)$. This is satisfied by the assumption that the set $\tilde{\Omega}$ is a cylindrical set. The proof follows the steps of the Dirichlet quasilinear problems and it is based on the Schauder fixed point theorem. Let $\theta \in (1,\delta)$ and set $B_0 = 1 + |\phi|_\theta$ and for $\epsilon > 0$ to be chosen later and define the set

$$\mathcal{S} \;=\; \{v \in H_\theta(\tilde{\Omega}_\epsilon) : |v|_\theta \leq B_0\}.$$

Define the map $J : \mathcal{S} \to H_\theta$ by $Jv = u$ where u is a solution of the problem

$$\begin{aligned} -\frac{\partial u}{\partial t} \;+\; a^{ij}(X, v, \mathrm{D}v)\mathrm{D}^2_{ij}u \;+\; a(X, v, \mathrm{D}v) &= 0 \text{ in } \tilde{\Omega}_\epsilon, \\ Mu &= \Psi \text{ on } S\tilde{\Omega}_\epsilon, \\ u &= \phi \text{ on } B\tilde{\Omega}_\epsilon, \end{aligned}$$

which is the linear analogue of the quasilinear problem we are working with. The map is well defined since for each $v \in \mathcal{S}$ there exists a unique solution of the above linear problem by Theorem 3.4 and the solution is in $H^{(-\delta)}_{2+\alpha(\theta-1)}$. From the same linear existence result we have the estimate

$$|u|_1 \;\leq\; |u|_\delta \;\leq C\; |u|^{(-\delta)}_{2+\alpha(\theta-1)} \;\leq\; C_{lin},$$

where C comes from Theorem 3.4 and depends on all the initial data and boundary coefficients and also $C_{lin} = C_{lin}(A, B, B_1, C, n, \alpha, \delta, \tilde{\Omega}_\epsilon) < \infty$.

Now if we look at the definition of norm $|\cdot|_1$, we notice that in bounding this norm we also bound the weaker one $< \cdot >_1$ by at least the same constant, since the most powerful norm is a sum of all the others. Then we have by definition of $< \cdot >_1$,

$$\sup_{X_0 \in \tilde{\Omega}} \; \sup \left\{ \frac{|u(x_0,t) - u(x_0,t_0)|}{|t-t_0|^{\frac{1}{2}}} : \forall (x_0,t) \in \tilde{\Omega}_\epsilon, \; t \neq t_0 \right\} \;\leq\; C_{lin},$$

where we recall that space time points look like $X_0 = (x_0, t_0)$. Then for any $X_0 \in \tilde{\Omega}_\epsilon$ we have

$$\sup \left\{ \frac{|u(x_0,t) - u(x_0,t_0)|}{|t-t_0|^{\frac{1}{2}}} : \forall (x_0,t) \in \tilde{\Omega}, \; t \neq t_0 \right\} \;\leq\; C_{lin},$$

and then for all times $t \in [0, \epsilon]$ we have

$$\frac{|u(x_0,t) - u(x_0,t_0)|}{|t-t_0|^{\frac{1}{2}}} \;\leq\; C_{lin}.$$

Now since all our times are less than ϵ we obtain

$$|u(x_0,t)-u(x_0,t_0)| \;\leq\; C_{lin}|t-t_0|^{\frac{1}{2}} \;\leq\; C_{lin}\sqrt{\epsilon}.$$

Set now the point X_0 to be part of $B\tilde{\Omega}$, which means that $X_0=(x,0)$ where $x\in D$. So now we can rewrite the above as

$$|u(x,t)-\phi(x)| \;\leq\; C_{lin}\sqrt{\epsilon}$$

for all points $(x,t)\in D\times[0,\epsilon]$, which is the same as saying for all points in $\tilde{\Omega}$ since we are working only in cylindrical domains. Now we are ready to use the interpolation inequality Theorem 3.3 by choosing $\sigma=\frac{\delta-\theta}{\delta-1}$, $a=1$ and $b=\delta$:

$$|u-\phi|_\theta \;\leq\; C(\delta)\;|u-\phi|_1^{\frac{\delta-\theta}{\delta-1}}\;|u-\phi|_\delta^{1-\frac{\delta-\theta}{\delta-1}}.$$

We know that $|u|_\delta\leq C_{lin}$ and also $\phi\in H_\delta(\mathcal{P}\tilde{\Omega})$. Thus the above estimate containing ϵ for $|u|_1$ gives

$$\begin{aligned}|u-\phi|_\theta \;&\leq\; C(\delta)\;(C_{lin}\;\sqrt{\epsilon})^{\frac{\delta-\theta}{\delta-1}}\;(C_{lin})^{1-\frac{\delta-\theta}{\delta-1}}\\ &\leq\; C(\delta)\;\tilde{C}(C_{lin})\;C_{lin}^{\frac{\delta-\theta}{\delta-1}}\;(\sqrt{\epsilon})^{\frac{\delta-\theta}{\delta-1}},\end{aligned}$$

where $\tilde{C}=\tilde{C}(C_{lin})$ is the bound for the norm of $u-\phi$ in H_δ. Note that it is finite by the conditions on the initial data. Now we can choose ϵ small enough such that $C(\delta)\tilde{C}(C_{lin})C_{lin}^{\frac{\delta-\theta}{\delta-1}}(\sqrt{\epsilon})^{\frac{\delta-\theta}{\delta-1}}\leq 1$ which gives us

$$|u-\phi|_\theta\leq 1.$$

This implies

$$|u|_\theta\leq 1+|\phi|_\theta=B_0.$$

Hence J maps $\mathcal{S}$ into itself. The set $\mathcal{S}$ is a ball in the function space $H_\theta(\tilde{\Omega}_\epsilon)$, and so a convex set. Therefore we can apply Theorem 3.5 to conclude that the map J has a fixed point u, which is in $H^{(-\delta)}_{2+\alpha(\theta-1)}$ and which solves our quasilinear problem. This ends our proof. □

And finally we can use all of the above to guarantee the existence of a short time solution for the problems (15) and (16). The first thing we have to worry about is the parabolicity of our operator, then if it is quasilinear and if it does fit into the above developed program. For this, the next lemma is easily obtained from Stahl's Proposition 3.1 for (15) and (16).

Lemma 3.7 (Parabolic quasilinear oblique derivative boundary value problems). *The evolution problems* (15) *and* (16) *belong to the category of quasilinear parabolic problems with oblique derivative boundary conditions.*

Proof. We have to check two things for (15) and (16). One is that the interior evolution is parabolic and quasilinear and the second is that the boundary condition is oblique. The interior evolution of the graphs is given by

$$\frac{\partial w}{\partial t}(x_1,x_2,t) \;=\; -\frac{1}{s}H.$$

Using the results of Proposition 3.1 we can compute

$$\tilde{H}(x_1, x_2, t) \;=\; (g^{ij}h_{ij})(x_1, x_2, t) \;=\; -s\; g^{ij}\mathrm{D}^2_{ij}w(x_1, x_2, t) \;+\; f(x_1, x_2, t, w, \mathrm{D}w),$$

where we have seen that the inverse of the first fundamental form depends upon the point, time, the graph function and its first derivative: $g^{ij} = g^{ij}(x_1, x_2, t, w, \mathrm{D}w)$. Since it is a metric on the moving surfaces generated by the graphs evolution it is a positive definite tensor. This together with the positivity of the quantity s gives us the parabolicity of the evolution problem (15) and also for (16). The quasilinear property is easy to see also, and comes from the dependence of the coefficients of the second derivatives of the graph functions by only the time, point, the graph function and the first derivative as we have seen from the first fundamental form definition.

The second condition to verify is that we have a oblique derivative boundary condition. In our evolution we have either one or two boundary conditions depending if we are treating a purely Neumann problem or a combined Neumann and Dirichlet problem with two separate boundaries. The two cases correspond as we have earlier seen to two types of domains, either a topological disc or a topological annulus in $\mathbb{R}^2$.

The first case where Ω is topologically equivalent to a disc in $\mathbb{R}^2$ with only one boundary and the evolution problem has only a Neumann boundary condition fits easily into the oblique derivative problems defined by Lieberman. The boundary condition in (15) or (16) is given by

$$\sum_{i=1}^{2} \nu^i_\Sigma \mathrm{D}_i w \;=\; \nu^3_\Sigma,\ \forall (x_1, x_2, t) \in \partial\Omega_N \times [0, \delta].$$

We only have to pick β to be the vector formed by the first 2 components of the unit normal to Σ and have the last component equal to zero. We also pick the fixed function β^0 to be zero and the Ψ function to be the last component of the normal to Σ. That is

$$\begin{aligned} \beta &= \Big(\nu^1_\Sigma, \nu^2_\Sigma, 0\Big), \\ \beta^0 &\equiv 0, \\ \Psi &= \nu^3_\Sigma. \end{aligned}$$

For the second type of domain, the one which is topologically equivalent to an annulus with two boundaries, the inner circle on which we have a Neumann condition and the outer one on which we will have a Dirichlet condition, the boundary conditions from (15)

$$\begin{aligned} \sum_{i=1}^{2} \nu^i_\Sigma \mathrm{D}_i w &= \nu^3_\Sigma,\ \forall (x_1, x_2, t) \in \partial\Omega_N \times [0, \delta], \\ w(x_1, x_2, t) &= w_0(x_1, x_2),\ \forall (x_1, x_2, t) \in \partial\Omega_D \times [0, \delta], \end{aligned}$$

have to be put together in one boundary condition for the definition of the coefficients of the operator M. The same is valid for the problem (16). This is done by defining two cut-off functions such that one of them is 0 on one of the boundaries and 1 on the other one and the second with the same property but with switched boundaries. Define $\phi_1, \phi_2 : \Omega \to \mathbb{R}$ smooth functions such that $\phi_1|_{\partial\Omega_N} = 1$, $\phi_1|_{\partial\Omega_D} = 0$ and $\phi_2|_{\partial\Omega_D} = 1$,

$\phi_2|_{\partial\Omega_N} = 0$. Using these two functions the two boundary conditions can be expressed in the following way

$$\phi_1 \sum_{i=1}^{2} \nu_\Sigma^i \mathrm{D}_i w \ + \ \phi_2 \ w(x_1, x_2, t) \ = \ \phi_1 \ \nu_\Sigma^3 \ + \ \phi_2 \ w_0(x_1, x_2), \qquad \forall (x_1, x_2, t) \in (\partial\Omega_N \cup \partial\Omega_D) \times [0, \delta].$$

So now the coefficients in the oblique derivative boundary operator M are

$$\begin{aligned} \beta \ &= \ \phi_1 \left(\nu_\Sigma^1, \nu_\Sigma^2, 0\right), \\ \beta^0 \ &= \ \phi_2, \\ \Psi \ &= \ \phi_1 \ \nu_\Sigma^3 \ + \ \phi_2 \ w_0(x_1, x_2). \end{aligned}$$

Some of our functions, for example components of the unit normal to Σ, are only defined on one of the boundaries. This can be easily solved if we consider Σ as being defined by level sets. This ends our proof. □

THEOREM 3.8 (Short time existence). *For any $\alpha \in (0, 1)$ there exists a positive time $\epsilon > 0$ such that there exists a short time solution w satisfying the problem* (15), *respectively* (16) *and $w \in H_{2+\alpha}^{(-\delta)}(\Omega \times [0, \epsilon])$ respectively $w \in H_{2+\alpha}^{(-\delta)}(M^2 \times [0, \epsilon])$, for some $\delta \in (1, 2)$.*

PROOF. The proof is based on the results presented above and all we have to do is show that our problem satisfies the conditions and hypotheses of the two existence theorems, Theorem 3.4 and Theorem 3.6. This comes easily from Lemma 3.7 together with the cylindrical setting discussed earlier. □

CHAPTER 4

Mean curvature flow of graphs with a free boundary outside the sphere

1. Introduction

We consider in this chapter the mean curvature flow of graphs outside a sphere with one free Neumann boundary and a fixed Dirichlet boundary. On the Neumann boundary we require that the normal of the solution and the normal of the sphere always meet at ninety degrees. The Dirichlet boundary is assumed to be a fixed circle with centre equal to that of the sphere and radius strictly greater than that of the sphere. This problem, up to some initial conditions, is equivalent to the scalar mean curvature flow of graphs defined on an annular region of $\mathbb{R}^2$. The smaller radius of the annulus is given by the radius of the fixed sphere used as a free boundary and the larger radius is the radius of the circle used for the Dirichlet condition.

We begin with a short time existence result, explaining how the problem fits into the structure of the general local existence theorem from Chapter 3. In the same section we also give a list of conditions necessary for the preservation of the graph property.

After this we discuss the rotationally symmetric graphs setting. These results are a subset of those from Chapter 5 where we treat the most general setting of radially symmetric graphs moving outside or supported on a fixed radially symmetric surface in $\mathbb{R}^{n+1}$. The results highlight two of the behaviours of the radially symmetric graphs, depending on the initial conditions. If we have an initial graph with height bounded by the maximal height of the sphere then the solution exists for all time and converges as time goes to infinity to the flat annulus of the domain of definition. In case the initial graph satisfies an existence result for a self similar torus, then the mean curvature flow solution pinches off above the North Pole or below the South Pole of the sphere, developing a curvature and gradient singularity.

The final section treats general graphs in the setting of the chapter and uses the initial conditions imposed in the setup and short time existence sections to prove that for reflectively symmetric surfaces the graph property is preserved for all times of existence. We rewrite our problem on a fixed domain by using a linear combination of Killing vector fields in $\mathbb{R}^3$ which fits in the setting of short time existence chapter. The author wishes to thank Prof. Dr. Ben Andrews from ANU for suggesting the usage of a Killing vector field.

2. Setup

Let Σ be the two dimensional unit sphere centred at the origin $(0, 0, 0) \in \mathbb{R}^3$. The problem treated in this chapter belongs to the category of problems in (56), with a Neumann boundary moving freely on the sphere a fixed Dirichlet height on a circle of some radius outside the sphere. Let M^2 be a smooth, orientable 2-dimensional manifold with two smooth, compact, disjoint boundaries which we denote by $\partial_N M^2$ for the Neumann boundary and $\partial_D M^2$ for the Dirichlet boundary. Set $M_0 := F_0(M^2)$, where $F_0 : M^2 \to \mathbb{R}^3$ is a smooth embedding satisfying

$$\begin{aligned}
\partial_N M_0 \;&\equiv\; F_0(\partial_N M^2) \;=\; M_0 \cap \Sigma,\\
\langle \nu_0, \nu_\Sigma \circ F_0 \rangle (p) \;&=\; 0,\ \forall p\, \in \partial_N M^2,\\
\partial_D M_0 \;&\equiv\; F_0(\partial_D M^2) \;=\; \partial B_R\big((0,0)\big),
\end{aligned}$$

for some positive $R > 1$. Here we have denoted by $\partial B_R\big((0,0)\big)$ the boundary of the 2-dimensional disc $B_R\big((0,0)\big)$ of radius R centred at the origin in $\mathbb{R}^2$. Let $I \subset \mathbb{R}$ be an open interval and let $F_t = F(\cdot,t) : M^2 \to \mathbb{R}^3$ be one-parameter family of smooth embeddings for all $t \in I$. The family of surfaces $(M_t)_{t\in I}$, where $M_t = F_t(M^2)$, are said to be evolving by mean curvature flow with Neumann free boundary condition on Σ and a constant zero height on the Dirichlet boundary if it satisfies

$$\begin{aligned}
\frac{\partial F}{\partial t}(p,t) \;&=\; -\,H(p,t)\nu,\ \forall (p,t) \in M^2 \times I,\\
F(\cdot,0) \;&=\; F_0,\\
F(p,t) \;&\subset\; \Sigma,\ \forall (p,t) \in \partial_N M^2 \times I,\\
\langle \nu, \nu_\Sigma \circ F \rangle (p,t) \;&=\; 0,\ \forall (p,t) \in \partial_N M^2 \times I,\\
F(p,t) \;&=\; F_0(p),\ \forall (p,t) \in \partial_D M^2 \times I.
\end{aligned} \tag{21}$$

The convention is, throughout this work and when not stated otherwise, that the unit normal ν_Σ to Σ points outside the evolving surfaces. Here this means that is points into the sphere. Also we assume the initial data satisfies the graph condition

$$\langle \nu_{M_0}, e_3 \rangle \;>\; 0. \tag{22}$$

This problem is equivalent to the non-parametric mean curvature flow of graphs over a time dependent domain. We start by defining this domain. Let $\Omega(t) = B_R\big((0,0)\big) \sim B_{r(t)}\big((0,0)\big) \subset \mathbb{R}^2$. This set has two boundaries which we denote as done earlier by $\partial\Omega_D = \partial B_R\big((0,0)\big)$ for the Dirichlet boundary and $\partial\Omega_N(t) = \partial B_{r(t)}\big((0,0)\big)$ for the Neumann boundary. These sets depends on time or to be more precise on the way the solution slides up or down on the sphere. Using techniques as in Ecker [**8**] one can show that the immersion problem (21) is equivalent (up to tangential diffeomorphisms) to the evolution of a scalar function $u : \Omega(t) \times [0,T) \to \mathbb{R}$ satisfying the boundary value problem:

$$\begin{aligned}
\frac{\partial u}{\partial t} \;&=\; \sqrt{1+|\mathrm{D}u|^2}\,\mathrm{div}\Big(\frac{\mathrm{D}u}{\sqrt{1+|\mathrm{D}u|^2}}\Big) \ \text{ on } \ \Omega(t) \times [0,T),\\
\langle \nu_u, \nu_\Sigma \rangle \;&=\; 0 \ \text{ and } u(x,t)^2 \,+\, |x|^2 \;=\; 1 \ \text{ on } \ \partial\Omega_N(t) \times [0,T),\\
u(x,t) \;&\equiv\; 0 \ \text{ on } \ \partial\Omega_D \times [0,T),\\
u(x,0) \;&=\; u_0 \ \text{ on } \ \Omega(0).
\end{aligned} \tag{23}$$

Here we denoted by ν_u the unit normal to the graph. The Neumann boundary condition $u(x,t)^2 + |x|^2 = 1$ ensures us that our free boundary remains on the sphere of radius 1.

3. Short time existence

In the short time existence chapter we have seen that for combined boundary conditions, such as (21), where we have a Dirichlet boundary condition on one of the boundaries and a free Neumann boundary condition on the other, one may transform the problem into one over a time independent domain through the use of an appropriate coordinate transformation.

We wish to perform this transformation explicitly. We must find the vector field ξ tangent to Σ which helps us to write our problem in a domain which is fixed in time. In the case where Σ is a sphere, this vector field takes a special form involving Killing vector fields.

Let us start by giving the definition of a Killing vector field and some properties that are very useful in the rest of the chapter. For this we use as a reference Chapter 3 from [**7**]. A vector field X on $\mathbb{R}^{n+1}$ is a map $X : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$; we say that the field is linear if X is a linear map. A linear field on $\mathbb{R}^{n+1}$, defined by a matrix A, is a Killing vector field if and only if A is anti-symmetric. The first property we want to mention is equivalent to the definition.

Proposition 4.1 (Killing vector field equation). *X is a Killing vector field on $\mathbb{R}^{n+1}$ if and only if*

$$\langle D_Y X, Z\rangle + \langle D_Z X, Y\rangle = 0,$$

for all Y and Z in $\mathbb{R}^{n+1}$. Here we have denoted by D the covariant derivative on $\mathbb{R}^{n+1}$.

The next property comes directly from the linearity of a Killing vector field as a map between Euclidean spaces.

Proposition 4.2 (Linearity of a Killing vector field). *If X is a Killing vector field in $\mathbb{R}^{n+1}$ then*

$$D_Y D_Z X \equiv 0,$$

for any Y and Z vector fields in $\mathbb{R}^{n+1}$.

We omit the proof of these propositions since they are explained in detail and in greater generality in [**7**]. Let us now return to our sphere problem and see how can we construct a vector field ξ which may be used for the short time existence theorem. What makes the sphere such a special case is the fact that three of the six Killing vector fields in $\mathbb{R}^3$ are completely tangent to the sphere in every point, since they are the rotations of 3-dimensional Euclidean space. Using two of them we define

$$\begin{aligned} \xi(x_1,x_2,x_3) &= \big(-x_1x_3,\ -x_2x_3,\ {x_1}^2 + {x_2}^2\big) \\ &= (-x_1)\,\big(\, x_3,\ 0,\ -x_1\big) \ + \ (-x_2)\,\big(0,\ x_3,\ -x_2\big) \\ &= (-x_1)\ K_1(x_1,x_2,x_3) \ + \ (-x_2)\ K_2(x_1,x_2,x_3), \end{aligned} \tag{24}$$

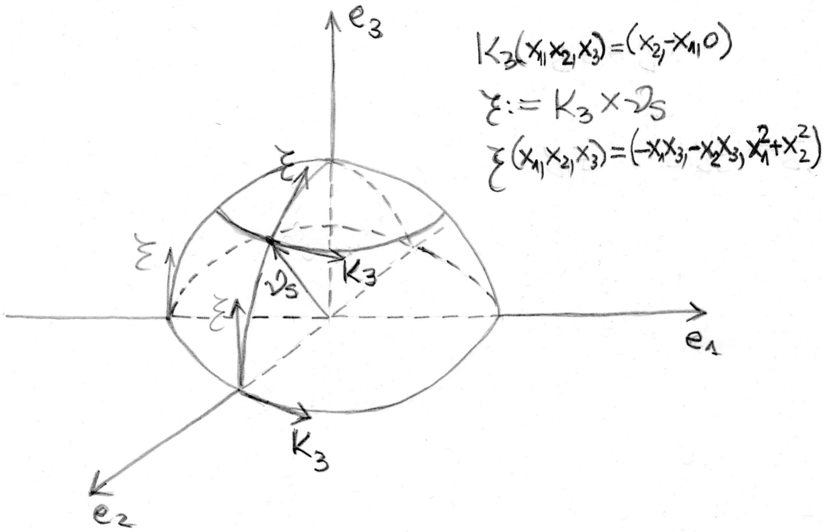

FIGURE 4.1. The ξ vector field.

where $K_1(x_1, x_2, x_3) = \big(x_3,\ 0,\ -x_1\big)$ and $K_2(x_1, x_2, x_3) = \big(0,\ x_3,\ -x_2\big)$ are two of the Killing vector fields of rotation tangent to the sphere. Note that ξ is singular along the x_3 axis. We denote the third Killing vector field of rotation by $K_3(x_1, x_2, x_3) = \big(x_2,\ -x_1,\ 0\big)$. This is the direction to which the normal to the evolving graphs becomes parallel in a tilt point, which is a point of infinite gradient for the graph function. For details about the two kinds of 'bad behaviour' for the gradient on the Neumann boundary, we invite the reader to follow our section on the gradient in Chapter 6. We define now three very useful quantities $s, s_1, s_2 : M_t \to R$ by

$$
\begin{aligned}
s_1(X) &= s_1\big(F(p,t)\big) := \langle \nu_{M_t}|_X, K_1(X)\rangle, \\
s_2(X) &= s_2\big(F(p,t)\big) := \langle \nu_{M_t}|_X, K_2(X)\rangle, \\
s(X) &= s\big(F(p,t)\big) := \langle \nu_{M_t}|_X, \xi(X)\rangle = (-x_1)s_1(X) + (-x_2)s_2(X),
\end{aligned} \tag{25}
$$

where we do not distinguish between the image $F(p, t)$ of a point $p \in M^2$ and its coordinate vector $X = X(p, t)$. For a point in $X \in M_t \subset \mathbb{R}^3$ we have $X = (x_1, x_2, x_3) = F(p, t)$ for some $p \in M^2$ and $t \in [0, T)$.

If we have initially strict positivity of the quantity s

$$
s\big(F_0(p)\big) = \big\langle \nu_{M_0}\big(F_0(p)\big), \xi\big(F_0(p)\big)\big\rangle > 0 \tag{26}
$$

everywhere on M_0, then we can apply Theorem 3.8 from Chapter 2 and obtain a solution of (21) for a short time. To apply the theorem first we write (21) as a graph in the ξ direction over the set $D := B_R\big((0,0)\big) \sim B_1\big((0,0)\big) \subset \mathbb{R}^2$ as $w : D \times [0, \delta) \to \mathbb{R}$, $\delta > 0$ satisfying

$$
\frac{\partial w}{\partial t}(x_1, x_2, t) = -\frac{1}{s}H,\ \forall (x_1, x_2, t) \in D \times [0, \delta], \tag{27}
$$

$$\sum_{i=1}^{2} \nu_\Sigma^i \mathrm{D}_i w = \nu_\Sigma^3, \ \forall (x_1,x_2,t) \in \partial B_1\big((0,0)\big) \times [0,\delta],$$
$$w(x_1,x_2,t) = w_0(x_1,x_2), \ \forall (x_1,x_2,t) \in \partial B_R\big((0,0)\big) \times [0,\delta]$$
$$w(\cdot,0) = w_0,$$

where H is the mean curvature of $M_t = \big((x_1,x_2), w(x_1,x_2,t)\big)$ and w_0 is such that $M_0 = \big((x_1,x_2), w_0(x_1,x_2)\big)$ with $w_0\big|_{\partial B_R((0,0))} \equiv 0$.

Theorem 4.3 (Short time existence for the motion outside the sphere). *For any $\alpha \in (0,1)$ there exists a positive time δ such that we have a short time solution w for the problem* (27) *and $w \in H^{(-\theta)}_{2+\alpha}(D \times [0,\delta])$ for some $\theta \in (1,2)$.*

We discuss now the extra conditions imposed on the initial graphs which allow us to prove preservation of the graph condition. Notice that (26) is implied by the following more restrictive set of sign conditions:

(28) $$\langle \nu_{M_0}, K_1\rangle < 0 \ \text{when} \ x_1 > 0,$$

(29) $$\langle \nu_{M_0}, K_1\rangle > 0 \ \text{when} \ x_1 < 0,$$

(30) $$\langle \nu_{M_0}, K_2\rangle < 0 \ \text{when} \ x_2 > 0,$$

(31) $$\langle \nu_{M_0}, K_2\rangle > 0 \ \text{when} \ x_2 < 0,$$

where $X_0 = F_0(p) \in M_0$ is $X_0 = (x_1,x_2,x_3)$. Of course, for reasons of smoothness, on the planes $x_1 = 0$ and $x_2 = 0$ the quantities $s_1 = \langle \nu_{M_0}, K_1\rangle$ and $s_2 = \langle \nu_{M_0}, K_2\rangle$ vanish:

$$s_1(X_0) = \langle \nu_{M_0}|_{X_0}, K_1(X_0)\rangle = 0 \ \text{when} \ X_0 = (0,x_2,x_3),$$

(32) $$s_2(X_0) = \langle \nu_{M_0}|_{X_0}, K_2(X_0)\rangle = 0 \ \text{when} \ X_0 = (x_1,0,x_3).$$

These conditions are more restrictive than imposing a sign on $s(X_0) = \langle \nu_{M_0}|X_0, \xi(X_0)\rangle = (-x_1)s_1(X_0) + (-x_2)s_2(X_0)$, but in the process of preserving this sign for later times the intermediate quantities s_1 and s_2 are of great help. These restrictions on our initial surface are necessary since we are not able to preserve (26) directly, and must instead preserve the more restrictive (28)–(31) and (32). For this the properties of Killing vector fields are of paramount importance. The change of sign from the x_1 positive coordinate half space to the negative half space comes from the orientation we have chosen for the Killing vector fields.

Here we must also discuss an additional condition we need to impose upon the initial graph for the solution to be locally well-posed in time. This comes from the fact that we have a fixed Dirichlet boundary, and it must be preserved. To do this we must start with an initial graph for which mean curvature vanishes on the Dirichlet boundary. Otherwise, the graph will dislodge itself and move away instantaneously from the fixed Dirichlet boundary.

$$H\big|_{\partial_D M_0} \equiv 0.$$

This condition and the reason why it is sufficient to state it only at initial time, is well explained in Chapter 6, Proposition 6.29. The following proposition shows that condition (22) is sufficient so that our problem is consistent with the above conditions (28) to (32) on the Dirichlet boundary.

Proposition 4.4 (Initial conditions on the Dirichlet boundary). *Let M_t satisfy* (21). *On the Dirichlet boundary $\partial_D M^2$ condition* (22) *implies*

$$s_1(X_0) \;=\; \langle \nu_{M_0}|_{X_0}, K_1(X_0)\rangle \;<\; 0$$

for $X_0 = (x_1, x_2, x_3) = F_0(p) \in \partial_D M_t \subset \mathbb{R}^3$ such that $x_1 \;>\; 0$; and

$$s_1(X_0) \;=\; \langle \nu_{M_0}|_{X_0}, K_1(X_0)\rangle \;>\; 0$$

for $X_0 = (x_1, x_2, x_3) = F_0(p) \in \partial_D M_t \subset \mathbb{R}^3$ such that $x_1 \;<\; 0$.

Proof. We compute the desired inner product by writing it in the canonical basis of $\mathbb{R}^3$:

$$\begin{aligned}\langle \nu_{M_0}|_{X_0}, K_1(X_0)\rangle \;=\; & \langle \nu_{M_0}|_{X_0}, e_1\rangle \langle e_1, K_1(X_0)\rangle + \langle \nu_{M_0}|_{X_0}, e_2\rangle \langle e_2, K_1(X_0)\rangle \\ & + \langle \nu_{M_0}|_{X_0}, e_3\rangle \langle e_3, K_1(X_0)\rangle\,.\end{aligned}$$

Notice that $\langle e_1, K_1(X_0)\rangle = x_3$, $\langle e_3, K_1(X_0)\rangle = -x_1$ and $\langle e_2, K_1(X_0)\rangle = 0$. Also since we are on the Dirichlet boundary that means we are on the plane $x_3 = 0$. This implies that

$$\langle \nu_{M_0}|_{X_0}, K_1(X_0)\rangle \;=\; -x_1\ \langle \nu_{M_0}|_{X_0}, e_3\rangle\,,$$

which together with (22) implies the result. □

Remark. The same result can be obtained similarly for the second quantity s_2, but depending instead on the x_2 coordinate.

4. Rotationally symmetric graphs moving outside the sphere

When we consider the rotationally symmetric problem (23) in n-dimensions, the scalar evolution is

$$\begin{aligned}\frac{\partial \omega}{\partial t} &= \frac{d^2\omega}{dy^2}\,\frac{1}{1+(\frac{d\omega}{dy})^2} + \frac{d\omega}{dy}\,\frac{n-1}{y} \quad \text{on} \quad \bigcup_{t\in[0,T)} (r(t), R)\times\{t\}, \\ \frac{d\omega}{dy}(r(t),t) &= \frac{\sqrt{1-r(t)^2}}{r(t)} \text{ and } \omega^2(r(t),t) + r^2(t) \;=\; 1 \;\text{ for all }\; t\in[0,T), \\ \omega(R,t) &= 0 \;\text{ on }\; [0,T), \\ \omega(y,0) &= \omega_0 \;\text{ on }\; (r(0),R),\end{aligned} \tag{33}$$

where $\omega : (r(t), R)\times[0,T) \to \mathbb{R}$. Here we have used the notation of Chapter 5 on rotationally symmetric graphs with ninety degree contact angle on general rotationally symmetric surfaces Σ where $y = |(x_1,\dots,x_n)|_{\mathbb{R}^n}$. We have also used the fact that Σ is a unit sphere centred at the origin of $\mathbb{R}^{n+1}$, specifically that $|\omega_\Sigma| \;=\; \sqrt{1-y^2}$ and $\nu_\Sigma = -\sqrt{1-y^2}(\frac{y}{\sqrt{1-y^2}},\ 1)$.

It is easy to see that this evolution problem satisfies the hypotheses of the theorems found in Chapter 5. Depending on the initial height of the graph, the solution could either exist for all time and converge to the flat annulus around the sphere or it could develop a Type I curvature singularity on the axis of rotation in finite time. Since the proof follows along similar lines to that found in Chapter 5 we only state the result and invite the reader to follow the details in that chapter.

THEOREM 4.5 (Rotationally symmetric graphs outside the sphere). *If ω satisfies* (33) *the following holds:*

(a) If $|\omega_0| \leq 1$, then $T = \infty$ and the solution converges as $t \to \infty$ to zero, that is the annulus around the sphere;

(b) If ω_0 satisfies the conditions of Lemma 5.10 then the solution exists for only finite time $T < \infty$ and the graphs $\omega(y,t)$ develop a Type I curvature singularity at $y = 0$ as $t \to T$.

REMARK. There is one thing here which needs to be explained in more detail. In the part of the proof of long time existence for rotationally symmetric graphs, Theorem 5.2, which deals with gradient bounds one uses the fact that the horizontal parts of Σ can not be reached by the flow. This is done by imposing a condition on the surface Σ such that in the region between the maximum and minimum of the initial graph there is no points where Σ is horizontal: that is condition (44).

Here one may notice that we do allow such 'bad' points, since we have taken the initial height of the graph ω_0 up to 1 or -1, so the North or South Pole of the sphere are included. The next theorem and proposition explains why taking the initial graph up to the maximal height 1 or minimal height -1 is still enough to prevent mean curvature flow from evolving the graphs ω to the North or South Pole of the sphere. These points are not desirable since there the gradient of the rotationally symmetric graphs will become infinite on the Neumann boundary as well as developing a curvature singularity.

Even more the following results prove that we can weaken the conditions imposed on the initial graph allowing a wider range of initial data. The improved condition requires that the initial graph is below a piece of a catenoid touching the sphere somewhere below the North Pole or above the South Pole, with a contact angle of ninety degrees or less.

THEOREM 4.6 (Catenoid comparison and long time existence). *Suppose ω satisfies* (33) *with initial data $\omega_0 : (r_0, R) \to \mathbb{R}$. If there exist constants d_i, C_i, $\epsilon_i \in [0,1)$ and $y_i \in (0, r_0)$ for $i = 1, 2$ such that*

$$-d_1 \textit{arccosh}\,(C_1 y) - \epsilon_1 \;<\; \omega_0(y) \;<\; d_2 \textit{arccosh}\,(C_2 y) + \epsilon_2, \;\; \forall y \in [r_0, R], \tag{34}$$

$$d_i \textit{arccosh}\,(C_i y_i) + \epsilon_i \;=\; \sqrt{1 - y_i^2}, \tag{35}$$

and

$$0 \;\leq\; (1 - d_i^2) c_i^2 y_i^2 - c_i^2 y_i^4 - 1 + y_i^2, \tag{36}$$

then there exists a solution $\omega : (r_0, R) \times [0, \infty) \to \mathbb{R}$ converging to the flat annulus around the sphere Σ.

PROOF. The proof follows the lines of Theorem 5.2, and here we only include the details which differ from the proof found in Chapter 5. Long time existence for (33) is obtained again from uniform height and gradient bounds. The latter follows from the radial symmetry and the fact that we exclude behaviour in which the evolving graphs reach points where the sphere has a horizontal point, so the North and South Pole. In the proof of Theorem 5.2 this behaviour is prevented by beginning the flow with a graph such that in the region between the maximal and minimal height value there is no point where Σ is horizontal. Then we preserve the height of the graphs for all times between the initial values. The preservation part of the argument is still valid but here we allow initial data which have a height above (or below) the critical value 1 (or -1).

To prove that the graphs do not move towards the North or South Pole of the sphere we apply the comparison principle with two pieces of a minimal surface. These pieces touch the sphere above and below the initial graph, as can be deduced from (34). These two pieces of minimal surfaces are given as graphs over a domain $[y_i, \infty)$ and they are both pieces of catenoids. Here the points y_i, $i = 1$ for the negative piece and $i = 2$ for the positive piece, represent the circles at which the catenoids meet the sphere, as is enforced by (35). The angle condition between the sphere as a graph over $\mathbb{R}$ and these two pieces is given by (36) and one can easily see from this relation that the angle is less than or equal to ninety degrees. The initial graph starts between these two catenoids and due to the choice of angle at the intersection of the sphere with the two catenoids, the comparison principle shows that for all times the graphs remain contained between the two catenoid pieces. One can prove this by either using the comparison principle Theorem 2.10 if we have an angle of ninety degrees or noticing that if the angle is strictly less than ninety degrees the surfaces will meet for the first time in the interior, a case excluded by the comparison principle found in [**22**]. □

This theorem can be used to allow the case where the initial graph attains heights greater than that of the sphere. We modestly apply this to allow the initial graph to reach 1 or -1.

PROPOSITION 4.7 (Existence of catenoid for $|\omega_0| \leq 1$ height bound). *Let ω satisfy (33) with $|\omega_0| \leq 1$. Then Theorem 4.6 is applicable.*

PROOF. For the existence of the catenoids used as a barriers in the above theorem we have to prove first that the Neumann boundary of the initial graph is not equal to the North or the South Pole of the sphere. This is the same as proving that there exists a strict positive constant ϵ_i for the choice of catenoid barriers. Once we have ϵ_i, it is easy to choose the other constants C_i, d_i and y_i.

Suppose that the initial graph satisfies $\omega_0(r_0) = 1$, which also implies that $r_0 = 0$. Thus we find ourselves at the North Pole of the Sphere where $\frac{d\omega_0}{dy}(r_0) = \frac{1}{r_0} = +\infty$. This implies that there exists a $y \in (r_0, R)$ such that $\omega_0(y) > 1$, which is a contradiction with the initial height bound. The same argument also contradicts the assumption that $\omega_0(r_0) = -1$.

Thus there exist positive constants $\epsilon_i \in [0, 1)$ for $i = 1, 2$. One is then easily able to choose the rest of the constants which characterise the two catenoidal pieces found in Theorem 4.6. □

5. Preservation of the graph property

Recall that we have considered the mean curvature flow of graphs with graph direction ξ, where ξ is determined by the initial values of the problem and by the surface Σ. In the following we prove that for all times of existence an initial graph in the ξ direction remains a graph. This is one of the steps required in the proof of long time existence.

This is similar to non-tilting arguments which can be found in Chapter 6. The main difference here is that the 'tilt' (which is when $s = 0$) is taken with respect to the

ξ direction and not to the usual e_{n+1} direction. The result below states that for the sphere, these two notions of tilt are equivalent. Note that this applies also for any other rotationally symmetric surface of contact Σ.

PROPOSITION 4.8 (Equivalence of boundary tilt points on the sphere). *Let ν be an arbitrary vector field in $\mathbb{R}^3$ tangential to the sphere, that is $\langle \nu, \nu_s \rangle = 0$. Then away from the x_3 coordinate axis*

$$\langle \nu, \xi \rangle = 0 \text{ if and only if } \langle \nu, e_3 \rangle = 0.$$

Here $\nu_s : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\nu_s(x_1, x_2, x_3) = (x_1, x_2, x_3)$ is the position vector corresponding to the identity in $\mathbb{R}^3$.

PROOF. The proof is very simple and comes down to the choice of two orthogonal bases of $\mathbb{R}^3$. We already have the canonical Euclidean basis $\mathcal{B}_1 = \{e_1, e_2, e_3\}$ and we define a new orthogonal (but not orthonormal) basis

$$\mathcal{B}_2 = \{K_3, \nu_s, \xi\}.$$

Now suppose that $\langle \nu, \xi \rangle = 0$. We write the ν vector in the $\mathcal{B}_2$ basis

$$\nu \;=\; \langle \nu, \xi \rangle \frac{\xi}{|\xi|^2} + \langle \nu, \nu_s \rangle \frac{\nu_s}{|\nu_s|^2} + \langle \nu, K_3 \rangle \frac{K_3}{|K_3|^2}.$$

Computing the other inner product $\langle \nu, e_3 \rangle$ and using the assumption $\langle \nu, \xi \rangle = 0$ with the fact that $\langle K_3, e_3 \rangle = 0$ we obtain

$$\langle \nu, e_3 \rangle \;=\; \frac{1}{|\nu_s|^2} \langle \nu, \nu_s \rangle \langle \nu_s, e_3 \rangle .$$

Using also $\langle \nu, \nu_s \rangle = 0$ we get the desired result.

Suppose now that we have $\langle \nu, e_3 \rangle = 0$. Then in the same orthogonal basis $\mathcal{B}_2$ we have

$$0 \;=\; \langle \nu, e_3 \rangle \;=\; \frac{1}{|\xi|^2} \langle \nu, \xi \rangle \langle \xi, e_3 \rangle + \frac{1}{|\nu_s|^2} \langle \nu, \nu_s \rangle \langle \nu_s, e_3 \rangle + \frac{1}{|K_3|^2} \langle \nu, K_3 \rangle \langle K_3, e_3 \rangle .$$

Use again $\langle K_3, e_3 \rangle = 0$ and also $\langle \nu, \nu_s \rangle = 0$ the above can be reduced to

$$0 \;=\; \langle \nu, \xi \rangle \langle \xi, e_3 \rangle .$$

Since $\xi = (-x_1 x_3, -x_2 x_3, {x_1}^2 + {x_2}^2)$, we see that $\langle \xi, e_3 \rangle = {x_1}^2 + {x_2}^2 > 0$ since we are away from the x_3 axis where ξ is not well defined. This implies that $\langle \nu, \xi \rangle = 0$ and thus completes our proof. □

REMARK. The condition 'away from the x_3 axis' is imposed to ensure that the ξ vector field is well-defined. The proposition works with a general vector field ν which is perpendicular to the to the position vector ν_s. This also corresponds to our case, when the normal to the graph is tangent to the unit sphere. The unit normal to a sphere is a scaled version of the position vector $\nu_\Sigma = -\nu_s/|\nu_s|$.

Before stating our theorem we need to include some additional results needed in the proof. One of them is the evolution of the quantities s_i.

PROPOSITION 4.9 (Evolution of s quantities). *Let F_t satisfy* (21) *(or any mean curvature flow evolution). The quantities $s_i = \langle \nu_{M_t}, K_i \rangle$, $i = 1, 2, 3$ satisfy the following evolution equations*

$$\left(\frac{d}{dt} - \Delta_{M_t}\right) s_i \;=\; |A^{M_t}|^2 s_i, \tag{37}$$

where we denoted by A^{M_t} the second fundamental form of M_t.

PROOF. We carry through the proof for s_1 and the other two work exactly the same. First we compute the time derivative using the time derivative of the normal, as in [**21**]

$$\frac{d}{dt} s_1 \;=\; \frac{d}{dt} \langle \nu_{M_t}, K_1 \rangle \;=\; \langle \nabla H, K_1 \rangle \;-\; H \left\langle \nu_{M_t}, \mathrm{D}_{\nu_{M_t}} K_1 \right\rangle,$$

where we have used the evolution of the immersion with the chain rule, since $K_1 = K_1(F_t)$. Let $\{\tau_i\}_{i=1,2}$ be an orthonormal basis of TM_t. This proof obviously works in arbitrary dimension, but we keep to the 2-dimensional case. Next we compute the Laplace-Beltrami operator applied to s_i:

$$\begin{aligned}
\nabla_{\tau_i} s_1 \;&= \nabla_{\tau_i} \langle \nu_{M_t}, K_1 \rangle = \sum_{p=1}^{2} h_{ip} \langle \tau_p, K_1 \rangle + \langle \nu_{M_t}, \mathrm{D}_{\tau_i} K_1 \rangle, \\
D_{\tau_j} \nabla_{\tau_i} s_1 \;&=\; \sum_{p=1}^{2} \nabla_{\tau_j} h_{ip} \langle \tau_p, K_1 \rangle + \sum_{p=1}^{2} h_{ip} \left\langle \mathrm{D}_{\tau_j} \tau_p, K_1 \right\rangle + \sum_{p=1}^{2} h_{ip} \left\langle \tau_p, \mathrm{D}_{\tau_j} K_1 \right\rangle \\
&\quad +\; \sum_{p=1}^{2} h_{jp} \langle \tau_p, \mathrm{D}_{\tau_i} K_1 \rangle + \left\langle \nu_{M_t}, \mathrm{D}^2_{\tau_i, \tau_j} K_1 \right\rangle + \left\langle \nu_{M_t}, \mathrm{D}_{\mathrm{D}_{\tau_j} \tau_i} K_1 \right\rangle,
\end{aligned}$$

where we used the Weingarten equation (or the definition of the second fundamental form) and denoted by h_{ij} the components of the second fundamental form A^{M_t}. We also compute

$$\mathrm{D}_{\tau_i} \tau_j = -h_{ij} \nu_{M_t} + \sum_{k=1}^{2} \Gamma_{ij}^k \tau_k,$$

using again the definition of the second fundamental form $A^{M_t} = \left(h_{ij}\right)_{1 \le i,j \le 2}$ and Christoffel symbols. For the ease of computation we choose an orthonormal basis of the tangent space such that the Christoffel symbols vanish at the point where the computation is evaluated, that is $\Gamma_{ij}^k = 0$ for all $i, j, k = 1, 2$. The local linearity of a Killing vector field, Proposition 4.2, causes the second derivative of the K_3 term also to vanish. These simplify the computation to:

$$\begin{aligned}
\Delta_{M_t} s_1 \;&=\; \sum_{i=1}^{2} \langle \tau_i, \mathrm{D}_{\tau_i} \nabla s_1 \rangle = \sum_{i=1}^{2} \sum_{p=1}^{2} \nabla_{\tau_i} h_{ip} \langle \tau_p, K_1 \rangle - \sum_{i=1}^{2} \sum_{p=1}^{2} h_{ip} h_{ip} \langle \nu_{M_t}, K_1 \rangle \\
&\quad +\; \sum_{i=1}^{2} \sum_{p=1}^{2} h_{ip} \langle \tau_p, \mathrm{D}_{\tau_i} K_1 \rangle + \sum_{i=1}^{2} \sum_{p=1}^{2} h_{ip} \langle \tau_p, \mathrm{D}_{\tau_i} K_1 \rangle - \sum_{i=1}^{2} h_{ii} \left\langle \nu_{M_t}, \mathrm{D}_{\nu_{M_t}} K_1 \right\rangle.
\end{aligned}$$

Using the Codazzi equation on the first term we obtain

$$\sum_{i=1}^{2}\sum_{p=1}^{2} \nabla_{\tau_i} h_{ip} \langle \tau_p, K_1 \rangle \;=\; \langle \nabla H, K_1 \rangle \,.$$

The antisymmetry of Killing vector fields (Proposition 4.1) implies that $\langle \mathrm{D}_V K_1, V \rangle = 0$ for every vector field V. This makes the last term in the computation of the Laplace-Beltrami vanish. To use this property on the rest of the terms we set pointwise a basis in which the second fundamental form takes a diagonal form. This eliminates all the first order terms containing $\mathrm{D}K_1$.

$$\Delta s_1 \;=\; \langle \nabla H, K_1 \rangle - \sum_{i=1}^{2} h_{ii}^2 \langle \nu_{M_t}, K_1 \rangle = \langle \nabla H, K_1 \rangle - |A_{M_t}|^2 \; s_1.$$

If we put this last result together with the time derivative computed above we finally obtain the desired evolution for s_1. □

Here we need to employ the following result from Stahl [**35**]. The problem treated in [**35**] is the mean curvature flow of immersions with a ninety degree contact angle on a fixed hypersurface in $\mathbb{R}^{n+1}$. The result of the proposition is obtained on the Neumann boundary, so it can be used for both our problems.

PROPOSITION 4.10 (Stahl [**35**], 1994). *Let F_t satisfy* (21)*. Let $X \in \Sigma \cap M_t$, $v \in T_X M_t$ and $w := v - \langle v, \nu_\Sigma \rangle \, \nu_\Sigma \in T_X(M_t \cap \Sigma)$ be the projection of v onto $T_X\Sigma$. Then:*

$$\begin{aligned} A^{M_t}(w, \nu_\Sigma) \;&=\; -A^{\Sigma}(w, \nu_{M_t}), \\ A^{M_t}(v, \nu_\Sigma) \;&=\; -A^{\Sigma}(w, \nu_{M_t}) \;+\; \langle v, \nu_\Sigma \rangle \, A^{M_t}(\nu_\Sigma, \nu_\Sigma). \end{aligned}$$

In a point of the Neumann boundary where we have $s_i = 0$ for some $i = 1, 2$, the components of the second fundamental form satisfy certain relations.

PROPOSITION 4.11 (Curvature property in a tilt point). *Let F_t be a solution of* (21)*, with the initial immersion F_0 satisfying conditions* (28)–(32)*,* (22) *and the initial Dirichlet boundary compatibility condition $H|_{\partial_D M_0} \equiv 0$. Consider a point on the Neumann boundary $X = F_t(p) \in \partial_N M_t \subset \Sigma$ for some $p \in \partial_N M^2$ where for the first time we have*

$$s_i(X) = 0$$

for some $i = 1, 2$. Then, for an orthogonal basis $\{\tau_1, \tau_2\}$ of $T_X M_t$ such that

$$\tau_1|_X \;=\; K_i(X) \quad \textit{and} \quad \tau_2|_X \;=\; \nu_\Sigma|_X \;=\; \nu_{\partial_N M_t}|_X,$$

we have

$$A^{M_t}\big|_X(\tau_1, \nu_\Sigma) \;>\; 0,$$

if s_i was previously negative, or

$$A^{M_t}\big|_X(\tau_1, \nu_\Sigma) \;<\; 0,$$

if s_i was previously positive.

Proof. From the conditions imposed on s_i at and around the point X, s_i has attained a boundary maximum (or minimum) at this point, after being negative (or positive) everywhere in the interior. We work with the first case when we have a maximum. The minimum argument works in the same way. Proposition 4.9 shows that s_i satisfies a nice parabolic evolution and allows us to apply the Hopf Lemma, Lemma 2.7, at the point X:

$$0 < \nabla_{\tau_2} s_i = \nabla_{\tau_2} \langle \nu_{M_t}, K_i \rangle = A^{M_t}(\tau_1, \tau_2) \langle \tau_1, K_i \rangle + A^{M_t}(\tau_2, \tau_2) \langle \tau_2, K_i \rangle + \langle \mathrm{D}_{\tau_2} K_i, \nu_{M_t} \rangle ,$$

where we have used the Gauss-Weingarten equations to express derivatives of the normal in tangential directions. Now we know that at X we have $\tau_1 = K_i$ and $\tau_2 = \nu_\Sigma = -\nu_s/|\nu_s|$. Here we wish to remind the reader that $\nu_s(x_1, x_2, x_3) = (x_1, x_2, x_3)$ is the position vector in $\mathbb{R}^3$ and it is always normal to the sphere Σ. Then

$$\langle \mathrm{D}_{\tau_2} K_i, \nu_{M_t} \rangle |_X = -\frac{1}{|\nu_s|} \langle \mathrm{D}_{\nu_s} K_i, \nu_{M_t} \rangle = -\frac{1}{|\nu_s|} \langle K_i, \nu_{M_t} \rangle = 0,$$

since $s_i = \langle \nu_{M_t}, K_i \rangle = 0$ at X. Also

$$A^{M_t}(\tau_1, \tau_2) \langle \tau_1, K_i \rangle |_X = A^{M_t}(\tau_1, \tau_2) \langle K_i, K_i \rangle = A^{M_t}(\tau_1, \tau_2) |K_i|^2$$

and

$$A^{M_t}(\tau_2, \tau_2) \langle \tau_2, K_i \rangle |_X = A^{M_t}(\tau_2, \tau_2) \langle \nu_\Sigma, K_i \rangle = -A^{M_t}(\tau_2, \tau_2) \langle \nu_s, K_i \rangle = 0.$$

We have thus shown that

$$0 < A^{M_t}|_X(\tau_1, \nu_\Sigma)|K_i|^2,$$

which gives us the desired result. □

The next result shows that we can preserve the sign of the s_i quantities with the use of the extra conditions (28)–(32).

Proposition 4.12 (Preservation of sign for Killing vector fields directions). *Let F_t satisfy* (21) *and be reflectively symmetric over the planes $\{x_1 = 0\}$ and $\{x_2 = 0\}$. If the initial immersion $M_0 = F_0(M^2)$ satisfies conditions* (28)–(32)*, the compatibility condition on the Dirichlet boundary $H|_{\partial_D M_0} \equiv 0$, the initial condition* (22) *and the additional initial height bound $| \langle F_0, e_3 \rangle | \leq 1$, then the flow preserves conditions* (28)–(32) *for all time.*

Proof. The proof is based on the application of the maximum principle for the two quantities s_1 and s_2 on a quadrant ${M_t}^+ := M_t \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0, x_2 > 0\}$. We carry through the case of s_1 in detail, and the proof for s_2 is identical. From the initial condition (28) we have that on ${M_0}^+$, $s_1(X_0) < 0$, with zero boundary values on ${M_0}^{x_1=0} = M_0 \cap \{(0, x_2, x_3) \in \mathbb{R}^3\}$. There are three more boundaries of the domain, the free boundary at the intersection with the sphere Σ which we denote by $\partial_N {M_0}^+$, the fixed Dirichlet boundary on the fixed radius outside the unit sphere, which we denote by $\partial_D {M_0}^+$, and ${M_0}^{x_2=0} = M_0 \cap \{(x_1, 0, x_3) \in \mathbb{R}^3\}$.

From Proposition 4.9 and Theorem 2.4 on ${M_t}^+$ we know that the sign of s_1 can be preserved for all times, if on the boundaries we do not get any "new" zero values (which also are maximal values of s_1 on $\overline{{M_t}^+}$). The two boundaries which come from the reflective symmetry can be ignored, since one can work ϵ-close to them, for any positive

ϵ. So we turn our attention to the two boundaries which can make a difference in the sign change of the s_1 quantity.

First we need to exclude the possibility that s_1 might take a zero value on the Neumann boundary. Suppose that there is a point $X = F_t(p)$ on $\partial_N M_t{}^+ \subset \Sigma$ for some $p \in \partial_N M^2$ where we have for the first time in the evolution of the graph that $s_1(X) = 0$. In this point of the boundary we consider an orthogonal basis $\{\tau_1, \tau_2\}$ of $T_X M_t$, chosen such that we have at X

$$\tau_1 \ = \ K_1 \ \text{ and } \ \tau_2 \ = \ \nu_\Sigma = \nu_{\partial_N M_t{}^+}.$$

Now using the result of Proposition 4.11 we see that at X

$$A^{M_t}(\tau_1, \nu_\Sigma) \ > \ 0, \tag{38}$$

where A^{M_t} is the second fundamental form of the graph immersions. Using a result of Stahl [**35**], which we quoted in Proposition 4.10, we know that

$$A^{M_t}(\tau_1, \nu_\Sigma) = -A^\Sigma(\tau_1, \nu_{M_t}).$$

This is helpful since at a boundary point the tangent space of Σ is spanned by $\{\tau_1, \nu_{M_t}\}$. Since Σ is a sphere and $\tau_1 = K_1$ at X and ν_{M_t} is orthogonal to τ_1, they are two principal directions at the point X. Thus the second fundamental form of Σ is diagonal at X. Using the relation between the off-diagonal elements of the second fundamental form of M_t and Σ we can see that

$$A^{M_t}(\tau_1, \nu_\Sigma) = -A^\Sigma(\tau_1, \nu_{M_t}) = 0$$

which contradicts (38). Therefore there does not exists a point on the Neumann boundary where s_1 changes sign.

Now the other problem is if the s_1 quantity changes sign on the Dirichlet boundary. This cannot be the case since we started with an initial graph in the e_3 condition by condition (22). The standard construction of barriers shows that this relation preserved for all times. Finally, using Proposition 4.4 we see that on the Dirichlet boundary relation (22) is equivalent to s_1 being negative.

This ends our proof of preserving condition (28) for the s_1 quantity. Relation (29) is implied by the reflective symmetry of the surfaces M_t. The same works for the s_2 quantity. Thus we have shown that we can preserve conditions (28)–(32) for all time. ☐

Remark. The condition imposed on the initial height, that $|\langle F_0, e_3\rangle| \leq 1$ is there to prevent the graphs from flowing to the North or South Pole of the sphere Σ, points in which the vector field ξ is not defined. The height bound can be preserved in at least two ways.

One of them is by constructing radially symmetric barriers which are above and below the maximal height of the initial graph. Since the radially symmetric solutions have a height bound from the results of the previous section, then our general reflective symmetric graph also enjoys a height bound.

The second method is to use the same arguments as can be found in Chapter 6 developed for general graphs, Theorem 6.1. The Neumann boundary condition and the convention that we take the unit normal to the sphere Σ to be pointing away from the evolving surfaces implies $\langle \nu_\Sigma, e_3\rangle \leq 0$ above the $\mathbb{R}^2$ plane and the opposite sign below. Using this one can prove that the height of the graphs remains bounded for all times by the initial bound. Using the result of Theorem 4.6 the initial height can be taken up to and including the maximal height of the sphere.

Perhaps a little surprisingly, one can show that while the gradient is bounded the mean curvature satisfies a *uniform* bound.

PROPOSITION 4.13 (Uniform bound for the mean curvature). *Let F_t satisfy* (21) *and be reflectively symmetric over the planes $\{x_1 = 0\}$, $\{x_2 = 0\}$. If the initial immersion $M_0 = F_0(M^2)$ satisfies conditions* (28)–(32)*, the compatibility condition on the Dirichlet boundary $H|_{\partial_D M_0} \equiv 0$, initial condition* (22) *and the additional initial height bound $|\langle F_0, e_3\rangle| \leq 1$, then there exists a global constant $C < \infty$ such that*

$$\sup_{M_t} |H| \leq C \sup_{M_0} |H|,$$

for all times $t < \infty$.

PROOF. The proof is based once again on the use of the maximum principle and the Hopf lemma. In the following we modify the idea of Ecker and Huisken [**9**] of bounding the curvatures after we have obtained a gradient bound. Proposition 4.12 gives us that the quantities s_1 and s_2 preserve the strict negative sign on the quadrant ${M_t}^+$, which is equivalent to a gradient bound. We use s_1 in the following, although the argument applies in all cases.

Consider the quantity $X \mapsto \frac{H^2}{s_1^2}(X) : M_t^+ \to \mathbb{R}$. Using the reflective symmetry we see that it is enough to work on the quarter space M_t^+. After the same computation as in [**9**] and using the evolution of the mean curvature found in [**22**] we find that $\frac{H^2}{s_1^2}$ satisfies a parabolic evolution in the interior:

$$\Big(\frac{d}{dt} - \Delta^{M_t}\Big)\ \frac{H^2}{s_1^2}\ \leq\ 2\frac{\nabla s_1}{s_1}\cdot\nabla\frac{H^2}{s_1^2}.$$

From the above evolution and the use of the maximum principle (Theorem 2.9) with the bounded vector field $a = \frac{\nabla s_1}{s_1}$, we see that as long as we exclude maximums of the above quantity on the boundaries we obtain the result.

Once again, as in the proof of Proposition 4.12, we can ignore the two boundaries which come from the reflective symmetry. The Dirichlet boundary $\partial_D M_t$ does not raise any problems, since Proposition 6.29 implies the compatibility condition $H|_{\partial_D M_0} \equiv 0$ is preserved for all times.

On the Neumann boundary $\partial_N M_t$ we apply a Hopf Lemma argument. Assume that there is a point $X = F(p,t) \in \partial_N M_t$ such that $\frac{H^2}{s_1^2}$ attains a maximum at X. At this point choose an orthonormal basis $\{\tau_1, \tau_2\}$ of the tangent space TM_t such that $\tau_1 \in T\partial_N M_t$ and $\tau_2 = \nu_\Sigma$ at X. Then Lemma 2.11 implies

$$0\ <\ \nabla_{\nu_\Sigma}\frac{H^2}{s_1^2}\ =\ 2\frac{H}{s_1^2}\nabla_{\nu_\Sigma}H\ -\ 2\frac{1}{s_1^3}\nabla_{\nu_\Sigma}s_1. \tag{39}$$

Using Proposition 6.14 we replace in the first term

$$\nabla_{\nu_\Sigma}H\ =\ HA^\Sigma(\nu_{M_t},\nu_{M_t})\ =\ -H,$$

where we have used that Σ is a sphere and that the unit normal to Σ points away from the evolving surfaces. We now turn our attention to the second term in (39), and compute:

$$\begin{aligned}\nabla_{\nu_\Sigma} s_1 &= \nabla_{\nu_\Sigma} \langle \nu_{M_t}, K_1 \rangle \\ &= A^{M_t}(\tau_1, \nu_\Sigma)\ \langle \tau_1, K_1 \rangle \ +\ A^{M_t}(\nu_\Sigma, \nu_\Sigma)\ \langle \nu_\Sigma, K_1 \rangle \ +\ \langle \nu_{M_t}, \mathrm{D}_{\nu_\Sigma} K_1 \rangle \\ &= \langle \nu_{M_t}, \mathrm{D}_{\nu_\Sigma} K_1 \rangle ,\end{aligned}$$

where we have used, as in the proof of Proposition 4.12, the relation

$$A^{M_t}(\tau_1, \nu_\Sigma) \ =\ -\ A^\Sigma(\tau_1, \nu_{M_t}) \ =\ 0,$$

since $\tau_1 \in T\partial_N M_t \subset T\Sigma$, τ_1 is perpendicular to $\nu_{M_t} \in T\Sigma$, and Σ is a sphere. We have also used the fact that $\langle K_1, \nu_\Sigma \rangle = 0$. Noting that $\nu_\Sigma = -\nu_s$, where we remind the reader that ν_s is the position vector, the last term in the above computation becomes

$$\nabla_{\nu_\Sigma} s_1 \ =\ \langle \nu_{M_t}, \mathrm{D}_{\nu_\Sigma} K_1 \rangle \ =\ -\ \langle K_1, \nu_{M_t} \rangle \ =\ -s_1.$$

Returning to (39) we obtain a contradiction:

$$0 \ < \nabla_{\nu_\Sigma} \frac{H^2}{s_1^2} \ =\ -\ 2\frac{H^2}{s_1^2} \ +\ 2\frac{H^2}{s_1^2} \ =\ 0.$$

Therefore we do not have a Neumann boundary maximum for $\frac{H^2}{s_1^2}$ at any positive time. Thus

$$\sup_{M_t} |H| \ \le\ \frac{\sup_{M_t} |s_1|}{\inf_{M_0} |s_1|} \sup_{M_0} |H|.$$

Noting that $\sup\limits_{M_t} |s_1| \le \sup\limits_{M_t} |K_1| \le \sup\limits_{M_t} |\nu_s| \le \sup\limits_{M_0} |\nu_s|$, and using the fact that the height is bounded by the initial bound gives us the existence of the global constant $C < \infty$ as desired. □

Remark (Bounds on half spaces). The quantities s_1 and s_2 become zero on the two axes Ox_2 and Ox_1 respectively so the above argument has a problem on the axis Ox_2. This is easily overcome by noticing that we obtain the same result if we replace s_1 with s_2 in the above proposition. For the quantity $\frac{H^2}{s_2}$ the problem axis is Ox_1. The intersection of the two problem domains is only the origin, and this is never be part of the domain M^2 by the catenoid comparison result, or more restrictively by height estimates.

We obtain now as a corollary of Proposition 4.12 the fact that as long as the solution of (21) exists it remains a graph.

Theorem 4.14 (Preservation of the graph property). *Let F_t satisfy (21) for $t \in [0, T)$ and be reflectively symmetric over the planes $\{x_1 = 0\}$ and $\{x_2 = 0\}$. If the initial immersion $M_0 = F_0(M^2)$ satisfies conditions (28)–(32), the compatibility condition on the Dirichlet boundary $H|_{\partial_D M_0} \equiv 0$, initial condition (22) and the additional initial height bound $|\langle F_0, e_3 \rangle| \le 1$ then*

$$\langle \nu_{M_t}, \xi \rangle \ >\ 0$$

for all times $t \in [0, T)$. That is, (26) is preserved for all time and the solution is a graph.

Proof. As long as the immersion exists the non-tilting result from Proposition 4.12 can be applied for both the quantities s_1 and s_2. This together with the discussion in the short time existence section where we defined s_1 and s_2 gives that relation (26) is preserved for all times. That is for all times we have $s = \langle \nu_{M_t}, \xi \rangle > 0$. We can therefore write our immersions as graphs in the ξ direction for all times. □

Remark (Time dependent gradient bounds). The sign preservation of the relation (26) provides us with a bound for the gradient of the associated scalar function (27). By preserving for all times the positivity of the quantity s we know that for all times of existence the surfaces can be written as a graph in the ξ direction. The bound is not uniform in time, hence for a long time existence result one would also require bounds on the full second fundamental form of the evolving surfaces. The problem comes from the fact that the result of Proposition 4.12 is strongly dependent on the smoothness of the surface.

The usual proof of long time existence can take one of two paths. One either provides bounds for all derivatives of the immersion for all times as done in [**9**], or refers to standard parabolic theory applied to the associated scalar evolution. Bounding all the derivatives of the immersion requires information about these on the Neumann boundary, which at the moment we do not have.

In trying to apply the second approach we have encountered the following problem. The associated scalar graph evolution (27) for the problem (21) is quasilinear parabolic with an oblique derivative boundary condition on one of the boundaries and a Dirichlet condition on the other. The long time existence theorems for these types of problems use, as one can see from for example Corollary 8.10 and Theorem 8.3 in [**30**], require estimates on the $H_{1+\alpha}$ (for $\alpha \in (0,1)$) norm independent of time. Our gradient estimates are time dependent (in a non-obvious way), so obtaining the $H_{1+\alpha}$ estimates from bounds on the height and gradient provides us with a time dependent bound, without any control on how the bound grows in time. To our knowledge this can be overcome if we know that for all times we have a smooth surface. Then, even at some finite final time we are able to apply the non-tilting arguments and obtain bounds on the gradient and then restart the flow.

Remark. The initial height restriction of one is related to the surface Σ, which in our case is the radius of the sphere on which we have the free moving Neumann boundary. Again here, due to Theorem 4.6 from the radially symmetric section of this chapter, we are able to take heights up to the maximal value 1 and minimal value -1.

Remark (Graph in e_3 direction). One can easily see that the above theorem also states that an initial graph in e_3 direction with the extra conditions imposed by the theorem remains a graph in the e_3 direction for all times of existence. This comes from the parabolic evolution that the quantity $\langle \nu_{M_t}, e_3 \rangle$ satisfies on the interior (see Chapter 6 for details) and the fact that the bad behaviour on the two boundaries for this quantity is equivalent to bad behaviour for the quantity $\langle \nu_{M_t}, \xi \rangle$, which is prevented by the theorem.

6. Curvature singularity

In this section we state a result analogous to the results found in Chapter 5 on the development of a curvature singularity for radially symmetric graphs. For this we use the results of Angenent [**2**].

One can see from the previous section that one of the requirements of long time existence is a height bound below that of the sphere. If one does not have such a bound, and if also the graphs are 'high' and 'wide' enough, such that they contain a self-similar Angenent torus in the region bounded by the plane at height 1 and the initial graph (or -1 and the initial graph), then the evolving surfaces are forced towards the x_3-axis and become pinched. The immersions lose the graph property and will at best develop a Type I curvature singularity in finite time. We invite the reader to follow Chapter 5 for more details on what conditions can be imposed in a more general setting to ensure at most finite existence time, and even guarantee the development of a Type I singularity.

THEOREM 4.15 (Curvature singularity on the rotation axis). *Let F_t satisfy* (21) *with initial data F_0 such that there exists a self-similar torus in the region bounded by M_0 and the plane $\{(x_1, x_2, 1) \in \mathbb{R}^3\}$ or in the region bounded by M_0 and the plane $\{(x_1, x_2, -1) \in \mathbb{R}^3\}$. Then the solution exists only for finite time $T < \infty$.*

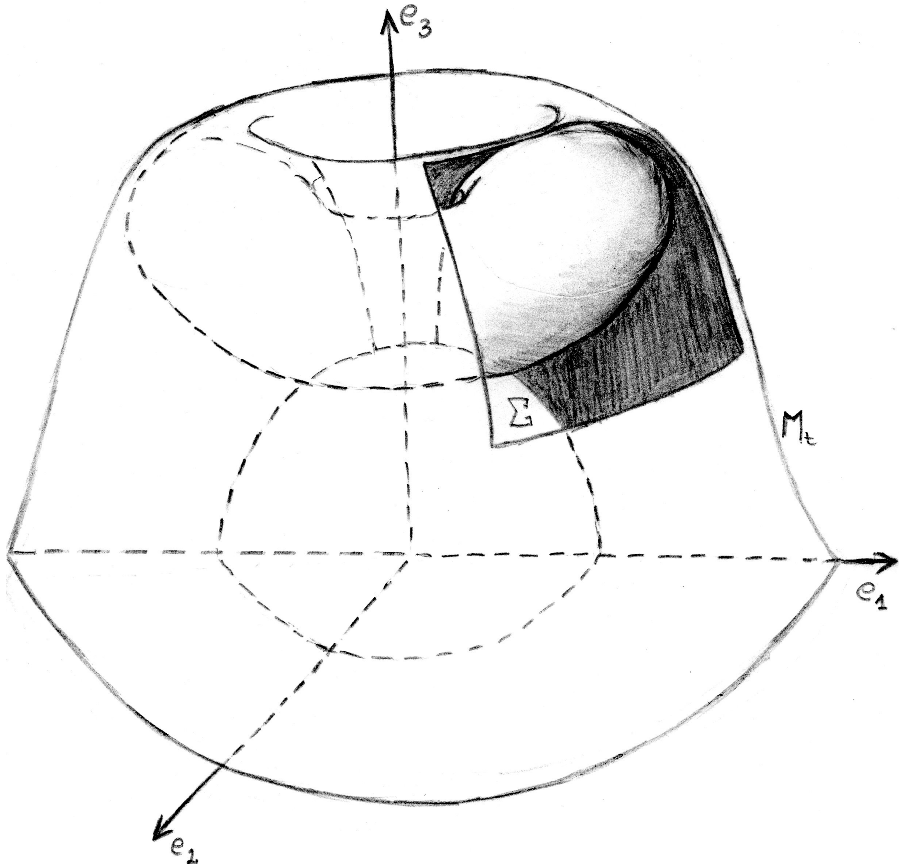

FIGURE 4.2. The self similar torus causing (at best) a Type I singularity for the motion of graphs outside the sphere.

CHAPTER 5

Mean curvature flow of radially symmetric graphs with a free boundary

1. Introduction

This chapter treats the case of n-dimensional initially radially symmetric graphs for the problems (1) and (2). The first is the mean curvature flow of radially symmetric graphs with a ninety degree contact angle on a fixed hypersurface in Euclidean space. This is a Neumann boundary value problem on a time dependent disc in $\mathbb{R}^n$. The second is defined on a time dependent annulus in $\mathbb{R}^n$ with a time dependent Neumann boundary and a fixed Dirichlet height. For these two problems we present here three main results. Specifically, we give sufficient conditions for long time existence, prove that the solutions converge to minimal surfaces, and show that under certain initial conditions a curvature singularity develops in finite time on the Neumann boundary.

2. Setup and short time existence

Let $x = (x_1, \ldots, x_n)$ be a point in $\mathbb{R}^n$, with $n \geq 2$ and denote by $y = \sqrt{\sum_{i=1}^n x_i^2}$ the length of the position vector corresponding to x. We denote by Σ an n-dimensional smooth hypersurface without boundary smoothly embedded in $\mathbb{R}^{n+1}$. The Neumann boundary of the immersions generated by our graphs is included in Σ. We only consider hypersurfaces Σ which are the union of two rotationally symmetric graphs ω_Σ^+ and ω_Σ^- where

$$\begin{aligned}
&\text{Dom } \omega_\Sigma^+ = \text{Dom } \omega_\Sigma^-,\\
&\omega_\Sigma^+(y) \geq 0,\\
&\omega_\Sigma^-(y) \leq 0,\\
&\omega_\Sigma^+(y) = 0 \Leftrightarrow \omega_\Sigma^-(y) = 0,
\end{aligned}$$

for all y. Each point $X \in \Sigma$ can be written as $X = (x_1, \ldots, x_n, \omega_\Sigma(y))$, where ω_Σ is either ω_Σ^+ or ω_Σ^-. We also impose that the graphs meet vertically, that is

$$\langle \nu_\Sigma(X), e_{n+1} \rangle = 0 \text{ when } \omega_\Sigma = 0,$$

where we denote by ν_Σ the normal to ω_Σ. A convention which we use in the following is that the normal to the fixed hypersurface Σ is pointing away from the moving graphs.

We first consider a free Neumann boundary problem defined on an interval $D(t) = (0, r(t)) \subset \mathbb{R}$ with the Neumann boundary given by the freely moving point $\partial D_N(t) = \{r(t)\}$. The other boundary point comes from the fact that the general graph over the disc of radius $r(t)$ centred at the origin associated with the radially symmetric graph is of genus zero. Thus the origin is treated as a boundary point for the radially symmetric problem. This causes no issue in the later arguments however since by symmetry and smoothness we have that at this point the radially symmetric graph is horizontal. The mean curvature flow of a radially symmetric graph $\omega : \bigcup_{t\in[0,T)} (0, r(t)) \times \{t\} \to \mathbb{R}$ attached to Σ at a ninety degree angle is then

$$(40)\qquad \begin{aligned} \frac{\partial \omega}{\partial t} &= \frac{d^2\omega}{dy^2}\,\frac{1}{1+(\frac{d\omega}{dy})^2} + \frac{d\omega}{dy}\,\frac{n-1}{y} \quad \text{on} \quad \bigcup_{t\in[0,T)} (0, r(t)) \times \{t\}, \\ \langle \nu_\omega, \nu_\Sigma\rangle\,(r(t)) &= 0 \text{ and } \omega(r(t),t) = \omega_\Sigma(r(t)) \quad \text{for all } t \in [0,T), \\ &\exists \lim_{y\to 0} \frac{1}{y}\frac{d\omega}{dy}(y) \\ \omega(y,0) &= \omega_0 \quad \text{on} \quad (0, r(0)). \end{aligned}$$

Examples for this problem include graphs evolving inside a catenoid neck, inside the hole of a torus or inside a sphere in $\mathbb{R}^{n+1}$.

The second case we consider is when besides the Neumann boundary condition we also have a fixed Dirichlet boundary condition. In this setting the domain of the general graph is homeomorphic to an annulus in $\mathbb{R}^n$. The domain of the radially symmetric graph is an interval away from the origin of the form $D(t) = (r(t), R)$. We denote by $\partial D_N(t) = \{r(t)\}$ and $\partial D_D = \{R\}$ the time dependent Neumann boundary and fixed Dirichlet boundary respectively. In this case the graphs are moving outside the fixed hypersurface Σ. The mean curvature flow of a radially symmetric graph $\omega : \bigcup_{t\in[0,T)} (r(t), R) \times \{t\} \to \mathbb{R}$ attached to Σ at a ninety degree angle and with height R_0 on some fixed circle is given by the evolution

$$(41)\qquad \begin{aligned} \frac{\partial \omega}{\partial t} &= \frac{d^2\omega}{dy^2}\,\frac{1}{1+(\frac{d\omega}{dy})^2} + \frac{d\omega}{dy}\,\frac{n-1}{y} \quad \text{on} \quad \bigcup_{t\in[0,T)} (r(t), R) \times \{t\}, \\ \langle \nu_\omega, \nu_\Sigma\rangle\,(r(t)) &= 0 \text{ and } \omega(r(t),t) = \omega_\Sigma(r(t)) \quad \text{for all } \; t \in [0,T), \\ \omega(R,t) &= R_0 \quad \text{for all } \; t \in [0,T), \\ \omega(y,0) &= \omega_0 \quad \text{on} \quad (r(0), R). \end{aligned}$$

Examples of this include graphs evolving outside a sphere, ellipsoid, cylinder, or the catenoid neck.

From now on we denote by $D(t)$ the interval used in the definition of the domain of the radially symmetric graphs and differentiate between the type of interval for the two problems only if necessary. Before discussing short time existence results we wish to simplify the form in which the Neumann condition is given. The ninety degree contact angle condition on the fixed hypersurface Σ which appears in the Neumann condition, $\langle \nu_\omega, \nu_\Sigma\rangle = 0$, can be written in a simpler way if we take into account that we are working with two graph functions. We have the upper unit normal to ω,

$$\nu_\omega = \frac{1}{\sqrt{1+(\frac{d\omega}{dy})}}\Big(-\frac{d\omega}{dy}, 1\Big),$$

and

$$\nu_\Sigma = \frac{1}{\sqrt{1+(\frac{d\omega_\Sigma}{dy})}}\big(-\frac{d\omega_\Sigma}{dy},1\big),$$

the upper unit normal to Σ. This transforms our Neumann boundary condition to

$$\frac{d\omega}{dy}(r(t)) = -\frac{1}{\frac{d\omega_\Sigma}{dy}(r(t))} \quad \text{for all } t \in [0,T). \tag{42}$$

The next step is to ensure that the two problems stated above at least exist for a short time. The two problems fit into the general frame of the short time existence theorems in Chapter 3, so we have the following result. Let $\tilde{\Omega} = \bigcup_{t\in[0,T)} D(t)\times\{t\}$, where $D(t) = (0, r(t))$ for (40) and $D(t) = (r(t), R)$ for (41) respectively.

THEOREM 5.1 (Short time existence). *For any $\alpha \in (0,1)$ there exists a $T > 0$ such that we have a solution ω for the problems* (40) *and* (41) *with $\omega \in H^{(-\delta)}_{2+\alpha}(\tilde{\Omega})$ for some $\delta \in (1,2)$.*

3. Long time existence

The main results of this section are the long time existence theorems stated below. The two theorems separate our problems into two major cases. Both of them obtain long time existence through uniform bounds on the height and the gradient of the radially symmetric graphs. The first theorem provides sufficient conditions for the height to be bounded by the initial values and the second theorem provides just a bound for the height function of the radially symmetric graphs. In the first theorem we also separate two cases, depending on the type of problem and also on the conditions imposed on the surface of contact Σ.

THEOREM 5.2 (Long time existence with height bound by initial values). *Let Σ and the graph function ω_0 be defined as above. Assume that there exists a constant C such that* $\sup|\omega_0| \leq C$ *and define the set $\mathcal{S} = \{y \in Dom(\omega_\Sigma) \ : \ |\omega_\Sigma(y)| \ \leq \ C\}$. The following hold:*
(a) If

$$\begin{aligned} \omega_\Sigma(y)\frac{d\omega_\Sigma}{dy}(y) \ &\geq \ 0 \ \forall y \in \mathcal{S} \text{ and} \\ \frac{d\omega_\Sigma}{dy}(y) \ &\neq \ 0 \ \ \forall y \in \mathcal{S} \end{aligned} \tag{43}$$

then there exists a solution to the problem (40) *for all times and it converges to a minimal surface as $t \to \infty$;*
(b) If the domain of the initial graph is $D(0) = (r(0), R)$, $R \notin Dom(\omega_\Sigma)$, $\omega_\Sigma(r) = 0$ for $r(0) \leq r < R$,

$$\begin{aligned} \omega_\Sigma(y)\frac{d\omega_\Sigma}{dy}(y) \ &\leq \ 0 \ \forall y \in \mathcal{S} \\ \frac{d\omega_\Sigma}{dy}(y) \ &\neq \ 0 \ \ \forall y \in \mathcal{S}, \end{aligned} \tag{44}$$

and finally also assuming that the initial graph satisfies the compatibility condition $H(\omega_0)|_{y=R} = 0$*, there exists a solution to the problem* (41) *for all times converging to a minimal surface as* $t \to \infty$.

Remark. The conditions imposed on the first derivative of the graphs of Σ are required to obtain a height bound for all times between the maximum and minimum of the initial height.

Remark. This first theorem applies for example to radially symmetric graphs moving inside the catenoid neck (case (a)) and motion outside the unit sphere (case (b)).

Proof of Theorem 5.2. The proof of the theorem is based on the usual strategy where one is tasked with obtaining uniform height and gradient bounds for the evolving graphs and we obtain these in Lemmas 5.3 and 5.4. Following Huisken [**23**] for example we see that uniform bounds on the height and gradient are sufficient for long time existence of graphs evolving by mean curvature flow. The basic reason for this is that with these estimates in hand we may recast our equation as a uniformly parabolic linear problem with Hölder continuous coefficients. For completeness we present here the most important steps in this well-known parabolic program of obtaining long time existence from uniform bounds for the height and gradient of the graph function.

We work with a one dimensional evolution so our problem fits into the framework found in [**30**], Theorem 12.2. The result states that from the uniform bounds of height and gradient we obtain H_α estimates in the interior for the gradient, for any $\alpha \in (0, 1)$. The estimate can be easily extended to the boundary by making use of Theorem 5.1 in Chapter 6 of [**27**]. For n-dimensional graphs this theorem requires the extra condition that there is a uniform bound on the time derivative of the graph, but in one dimension this extra condition can be waved due to the fact that our one dimensional evolution can be written in divergence form. Also uniform height and gradient bounds provide a bound for the time derivative of the graph function, as in [**27**], Chapter 6. Having H_α Hölder estimates on the gradient gives an $H_{1+\alpha}$ estimate on the graph function. The long time existence follows then from a similar argument as the one found in Theorem 8.3 of [**30**]. It is a completely standard application of the Arzelá-Ascoli Theorem and Theorem 3.8.

Convergence to minimal surfaces is also quite standard and follows from simple arguments whenever the surface area of the solution is uniformly bounded. We perform this argument in a more general setting in Proposition 6.28 and refer the reader to that proof. □

Next we provide the reader with a proof of how one may obtain the requisite height and gradient bounds. We start with the height bounds. The following result is valid for both case (a) and case (b).

Lemma 5.3 (Height bound). *If* ω *satisfies* (40) *or* (41) *in the domain* $\tilde{\Omega}$ *and the hypotheses of Theorem 5.2, then we have* $\sup_{D(t)} |\omega(y,t)| \leq \sup_{D(0)} |\omega_0|$ *for every* t.

PROOF. The maximum principle Theorem 2.4 applied to the quasilinear parabolic evolution of ω gives us:

$$\sup_{D(t)} |\omega(y,t)| \leq \max\Big\{ \sup_{D(0)} |\omega_0|, \sup_{\partial D_N(t)} |\omega|, \sup_{\partial D_D} |\omega| \Big\},$$

for all $t \in [0,T)$. If we consider the first type of boundary problem, problem (40) treated in case (a) of Theorem 5.2, the third term above does not appearing since there is no Dirichlet boundary. If we consider the second boundary problem, problem (41), treated in case (b) we know our Dirichlet boundary values are constant in time and equal to $R_0 < \infty$. Then the third term above can be estimated by the maximum of the initial values. In the end the only term we need to worry about is the one on the free Neumann boundary. Taking into account the way we have rewritten the Neumann boundary condition (42),

$$\frac{d\omega}{dy} = -\frac{1}{\frac{d\omega_\Sigma}{dy}},$$

we shall compute the derivative in the direction normal to the Neumann boundary by taking the outer normal to the boundary in the two different cases of domain. First, let us look at the problem (40) where $D(t) = (0, r(t))$. Here the choice of outer unit normal to the Neumann boundary is $\nu_{\partial D_N(t)} = \frac{y}{|y|} = 1$ and then the directional derivative in the direction of the outer unit normal to the Neumann boundary is

$$\frac{d\omega}{d\nu_{D_N(t)}} = -\frac{1}{\frac{d\omega_\Sigma}{dy}}.$$

If we find ourselves in the positive part of the Σ surface then we see that the condition (43) says that $\frac{d\omega_\Sigma}{dy} \geq 0$, so we can put a sign on our directional derivative from above

$$\frac{d\omega}{d\nu_{D_N(t)}} \leq 0. \tag{45}$$

If we assume there is a maximum on the Neumann boundary then the Hopf lemma tells us that this directional derivative should be strictly positive, which contradicts (45). So we have no maxima on the Neumann boundary where $\omega \geq 0$. The same argument can also be done where $\omega \leq 0$, but applying a minimum principle and then the Hopf lemma again. This tells us that the absolute value of ω can not attain a maximum on the Neumann boundary. On the $y = 0$ boundary point found on the axis of rotation, the same can be done since the 0 value of the directional derivative prevents both minima and maxima from appearing at this boundary point.

In case (b) we have the domain defined using $D(t) = (r(t), R)$, so the unit outer normal to the Neumann boundary of this domain is of the opposite sign to the one in the first problem. But here we have condition (44), where we also have an opposite sign to that of problem (a), so the same argument applies.

Therefore we have demonstrated the required height bound in each case and completes our proof. □

We now turn our attention to gradient bounds.

LEMMA 5.4 (Gradient bound). *If ω satisfies* (40) *or* (41) *on the domain $\tilde{\Omega}$ and the hypotheses of Theorem 5.2, then there exists a global constant $C = C(\omega_0, \Sigma) < \infty$ such that we have*

$$\sup_{D(t)} \left|\frac{d\omega}{dy}(y,t)\right| \leq C,$$

for every t.

PROOF. Following [**9**] we consider the quantity $v = \langle \nu_{M_t}, e_{n+1}\rangle^{-1}$ which is equal up to tangential diffeomorphisms to $\sqrt{1 + (\frac{d\omega}{dy})^2}$. The function v enjoys a parabolic evolution on the hypersurfaces M_t generated by the graphs:

$$\left(\frac{d}{dt} - \Delta_{M_t}\right)v \leq 0,$$

and this allows us to apply the maximum principle. Since we are in a non-compact setting we have that the maximum of the gradient is controlled by the maximum between the initial values and the boundary values

$$\sup_{D(t)} v \ \leq \ \max\Big\{\sup_{D(0)} v, \ \sup_{\partial D_N(t)} v, \ \sup_{\partial D_D} v\Big\},$$

for all $t \in [0, T)$. The two boundary maximums can be bounded as follows. Following the work of Huisken [**23**] which we quote in Theorem 6.3 concerning the Dirichlet boundary, a barrier construction provides us with the required bound on ∂D_D in a standard way.

On the Neumann boundary, the rotational symmetry of the solution (and the fact that this is preserved) prevents tilt behaviour. This occurs when the normal to the graph becomes parallel to the vector field of rotation for Σ. This behaviour is explained in much greater detail in Chapter 6.

Apart from this, we must argue why it is that our rotationally symmetric graph does not reach points on the Neumann boundary where the surface Σ is horizontal. In such points the boundary gradient becomes infinite simply by the Neumann condition (42). To avoid such behaviour we combine Lemma 5.3 with the initial conditions from the long time existence theorem: either condition (43) for the purely Neumann problem (40) or condition (44) for the combined Dirichlet and Neumann problem (41). These conditions say that on the Neumann boundary, in the area enclosed by the maximum and minimum of the initial graph $\mathcal{S}$, there is no point where Σ is horizontal. Now since Lemma 5.3 implies that the height at later times remains bounded by the initial height, this continues to hold and the graph is bounded away from these potentially troublesome areas of Σ. This completes our proof. □

REMARK. In general, one can not prevent a curvature singularity from occurring on the free boundary without a condition such as (43) or (44). An example of such behaviour is given in Theorem 5.7.

The next result gives long time existence for solutions of (41), but with a different set of initial conditions. These conditions do not imply that the height remains bounded by the initial height for all times.

THEOREM 5.5 (Long time existence without an optimal height bound). *Let* Σ *and the graph function* $\omega_0 : (r(0), R) \to \mathbb{R}$ *be as above. Assume that there exists a global constant* C *such that* $\sup|\omega_0| \leq C$ *and suppose that* ω_Σ *is taken such that* $R \in Dom(\omega_\Sigma)$ *and* $\omega_\Sigma(r) = 0$ *with* $r \leq r(0) < R$ *and* $C < |\omega_\Sigma(R)| < \infty$. *Define the set* $\mathcal{S} = \{y \in Dom(\omega_\Sigma) : |\omega_\Sigma(y)| \leq |\omega_\Sigma(R)|\}$. *If*

$$\begin{aligned} \omega_\Sigma(y)\frac{d\omega_\Sigma}{dy}(y) &\geq 0 \qquad &&\text{for all } y \in \mathcal{S}, \\ \frac{d\omega_\Sigma}{dy}(y) &\neq 0 \qquad &&\text{for all } y \in \mathcal{S}, \end{aligned} \tag{46}$$

and ω_0 *satisfies the compatibility condition* $H(\omega_0)|_{y=R} = 0$, *then there exists a solution to the problem* (41) *for all times and it converges as* $t \to \infty$ *to a minimal surface.*

REMARK. This theorem applies to the motion of radially symmetric graphs outside a catenoid neck with a fixed Dirichlet height on a circle of radius R. If we also suppose that $\omega_0(R) = 0$ one can show that the surface generated by the rotationally symmetric graphs converges as time goes to infinity to a piece of catenoid meeting the the fixed catenoid Σ at right angle.

PROOF OF THEOREM 5.5. We are again concerned with obtaining height and gradient bounds. This time we are not able to prove that the height remains bounded by initial values as before. But still we are able to obtain gradient bounds and then a height bound. One should keep in mind the picture of a solution evolving outside a catenoid neck with a fixed Dirichlet boundary at radius R. There it is intuitively obvious that a gradient bound implies a height bound: if the height of the solution grows without bound then it must 'cross itself,' thus losing the graph property on the interior.

As before we make use of the function v associated with our evolving graphs. The maximum principle implies

$$\sup_{D(t)} v \;\leq\; \max\Big\{\sup_{D(0)} v,\; \sup_{\partial D_N(t)} v,\; \sup_{\partial D_D} v\Big\}, \tag{47}$$

for all times $t \in [0, T)$. The term on the Dirichlet boundary is bounded again by the usual construction of barriers, cf. Theorem 6.3. The rotational symmetry and the Neumann boundary condition (42) implies also as before that the gradient does not become infinite on the Neumann boundary so long as the graph does not evolve towards a point where $\frac{d\omega_\Sigma}{dy} = 0$.

In the argument earlier it was easy to exclude such behaviour by assuming that Σ does not contain such points in the region between the maximal and minimal initial height, and then using the fact that the height remains bounded by initial values. Again here we prove that the solution only moves in a region where there are no points with $\frac{d\omega_\Sigma}{dy} = 0$. Condition (46) implies that such a region is $\mathcal{S}$. We know that our graph is initially defined on $D(0) = (r(0), R) \subset \mathcal{S}$ with $r \leq r(0) < R$, and so our strategy is to show that the graph is contained within this domain for all times and thus that the gradient on the Neumann boundary remains bounded.

From condition (46) one observes that $\mathcal{S} = [r, R]$ where $\omega_\Sigma(r) = 0$. This tells us that if there exists a time t^* such that the domain $D(t^*)$ is not included in $\mathcal{S}$ then there must also exist a point $y^* \in D(t^*)$ in the domain such that $y^* > R$ and very close to R. This implies that the evolving graphs have tipped over in the interior, and so there exists an interior point with infinite gradient. This contradicts the fact that the gradient remains bounded by the maximum of the boundary values and the initial values. These are in turn bounded by assumption (46) and the Dirichlet boundary barrier construction, Theorem 6.3. So there is no such point $y > R$ in $D(t)$ for any $t \in [0, T)$. This tells us that the domain of definition of the evolving graphs is always included in $\mathcal{S}$, that is $D(t) \subset \mathcal{S}$, and gives us gradient estimates for the evolving graphs.

This argument also gives a height bound. The maximum principle implies

$$\sup_{D(t)} |\omega(y,t)| \leq \max\{\sup_{D(0)} |\omega_0|, \sup_{\partial D_N(t)} |\omega|, \sup_{\partial D_D} |\omega|\},$$

for all times $t \in [0, T)$. The initial height and the Dirichlet boundary height are bounded by C. From the discussion above we already know that the graphs never evolve outside the height $\omega_\Sigma(R)$ on the Neumann boundary. This implies

$$\sup_{D(t)} |\omega(y,t)| \leq \max\{C, \ \omega_\Sigma(R)\} \ \leq \ \omega_\Sigma(R) < \infty$$

for all $t \in [0, T)$. Convergence to minimal surfaces follows from the long time existence and the fact that we have a uniform area bound, exactly as before. This argument is given in greater generality in Proposition 6.28, Chapter 6. □

4. Convergence

In this section we prove that in some cases long time existence implies that the graphs approach a constant. In the case of a mixed Dirichlet and Neumann problem this can obviously only hold if the height prescribed at the Dirichlet boundary is $R_0 = 0$.

THEOREM 5.6 (Convergence to a constant). *Under the assumptions of Theorem 5.2:*
(a) the solution of the problem (40) *converges to a constant function as* $t \to \infty$*;*
(b) if $R_0 = 0$ *the solution of the problem* (41) *converges to the annulus* $[r, R]$ *as* $t \to \infty$*, where* r *is such that* $\omega_\Sigma(r) = 0$.

PROOF. First we construct an auxiliary rotationally symmetric function and prove that it also exists for all time. The same function is used to obtain both (a) and (b), although the argument by necessity must differ slightly at some points. Let $g : \tilde{\Omega} \to \mathbb{R}$ be defined by

$$g(y,t) = t\ \omega^2(y,t) + \frac{1}{2} y^2 \sup_{D(0)} |\omega_0|^2.$$

We prove that this function exists for all times $t < \infty$ and from its height bound we obtain the desired convergence as time approaches infinity. At $t = 0$ we have

$$g(y,0) = \frac{1}{2} y^2 \sup_{D(0)} |\omega_0|^2,$$

and on the Dirichlet boundary

$$g(y,t) = \frac{1}{2} R^2 \sup_{D(0)} |\omega_0|^2,$$

since $\omega \equiv 0$ there by the condition $R_0 = 0$. Consider the following quasilinear parabolic operator

$$L = \frac{\partial}{\partial t} - \frac{d^2}{dy^2}\frac{1}{(\frac{d\omega}{dy})^2+1} - \frac{n-1}{y}\frac{d}{dy}.$$

By the gradient bound, Lemma 5.4, the coefficients of this operator remain bounded in the interior of our domains. Using Lemma 5.3 we can verify that g is a supersolution for this operator:

$$\begin{aligned}(Lg)(y,t) &= 2t\ \omega(y,t)(L\omega)(y,t) - \frac{\sup\limits_{D(0)}|\omega_0|^2}{(\frac{d\omega}{dy})^2+1} \\ &\quad - 2t\ \big(\frac{d\omega}{dy}\big)^2\frac{1}{(\frac{d\omega}{dy})^2+1} - (n-1)\ \sup_{D(0)}|\omega_0|^2 + \omega^2(y,t) \\ &\leq \omega^2(y,t) - \sup_{D(0)}|\omega_0|^2 \\ &\leq 0.\end{aligned}$$

Thus by the maximum principle

$$\begin{aligned}\sup_{D(t)} g(y,t) \leq \max\big\{ &\sup_{D(0)}\ \frac{1}{2}y^2, \sup_{D(0)}|\omega_0|^2,\ \sup_{\partial D_N(t)}\ g(y,t), \\ &\frac{1}{2}R^2\sup_{D(0)}\ |\omega_0|^2\big\},\end{aligned} \tag{48}$$

for all times $t \in [0,T)$. To exclude boundary maxima we apply a similar argument as in Lemma 5.3, by calculating the sign of the derivative of g in the direction normal to the Neumann boundary. Let us first compute the derivative:

$$\frac{dg}{dy} \ = \ 2t\ \omega\frac{d\omega}{dy} + y\ \sup_{D(0)}|\omega_0|^2. \tag{49}$$

Now using (42)

$$\begin{aligned}\frac{dg}{d\nu_{D_N(t)}} &= \ 2t\ \omega|_{y=r(t)}\frac{d\omega}{dy}\Big|_{y=r(t)}\ \nu_{D_N(t)} + r(t)\sup_{D(0)}|\omega_0|^2\ \nu_{D_N(t)} \\ &= -2t\ \omega_\Sigma\frac{1}{\frac{d\omega_\Sigma}{dy}}\ \nu_{D_N(t)}\ +\ r(t)\sup_{D(0)}|\omega_0|^2\ \nu_{D_N(t)} \\ &= -2t\ \omega_\Sigma^2\frac{1}{\omega_\Sigma\frac{d\omega_\Sigma}{dy}}\ \nu_{D_N(t)}\ +\ r(t)\sup_{D(0)}|\omega_0|^2\ \nu_{D_N(t)}.\end{aligned} \tag{50}$$

Our argument must now differ for each of the cases (a) and (b). We begin with (a).

In the case of the Neumann problem (40), with $D(t) = (0, r(t))$ and boundary $\partial D_N(t) = \{r(t)\}$. The 0 point on the rotation axis is here regarded as a boundary point also. For a radially symmetric graph we have $\frac{d\omega}{dy}\big|_{y=0} = 0$ by smoothness and symmetry as explained before. The normal to the Neumann boundary $\partial D_N(t)$ is $\nu_{D_N(t)} = \frac{y}{|y|} = 1$.

In this case we cannot exclude the appearance of a maximum for g on the Neumann boundary by contradicting condition (43) with the Hopf Lemma. Instead we proceed in a manner somewhat analogous to the proof of Theorem 5.5 above.

First we shall dismiss the appearance of maxima at the point $y = 0$ using a Hopf Lemma argument. Since we have

$$\frac{dg}{dy}\Big|_{y=0} \; = \; 2t\,\omega\frac{d\omega}{dy}\Big|_{y=0} + y\,\sup_{D(0)}|\omega_0|^2\Big|_{y=0} = 0,$$

the height of the g function satisfies

$$\sup_{D(t)}\; g(y,t) \le \max\big\{\sup_{D(0)}\frac{1}{2}y^2\sup_{D(0)}\;|\omega_0|^2,\; \sup_{\partial D_N(t)}\; g(y,t)\big\} \tag{51}$$

for all times $t \in [0, T)$. Recall that we do not have a Dirichlet boundary term in the case of the problem (40). Thus from Lemma 5.3 we obtain that the height of the function g is finite at all times $t < \infty$.

We also need a gradient bound for g. The same arguments as for the solution ω imply the existence of a constant $C = C(t) < \infty$ such that $\frac{dg}{dy} \le C$. Further, the growth of C in time is at worst linear. This together with the height bound above and standard parabolic theory gives that the function g exists for all times $t < \infty$.

Returning to the height estimate (51), we compute

$$\begin{aligned} t\,\omega^2(y,t) + \frac{1}{2}y^2\sup_{D(0)}|\omega_0|^2 \le \max\,\big\{&\sup_{D(0)}\frac{1}{2}y^2\sup_{D(0)}|\omega_0|^2,\\ &\sup_{\partial D_N(t)}\big\{\,t\,\omega^2(y,t) + \frac{1}{2}y^2\sup_{D(0)}|\omega_0|^2\big\}\big\}\\ \le \max\,\big\{&\frac{1}{2}r(0)^2\sup_{D(0)}|\omega_0|^2,\\ &\sup_{y=r(t)}\;\big\{t\omega^2(r(t),t) + \frac{1}{2}r(t)^2\sup_{D(0)}|\omega_0|^2\big\}\big\}\\ \le \max\,\big\{&\frac{1}{2}r(0)^2\sup_{D(0)}|\omega_0|^2,\\ &t\omega^2(r(t),t) + \frac{1}{2}r(t)^2\sup_{D(0)}|\omega_0|^2\big\}. \end{aligned}$$

Thus

$$\begin{aligned} t\omega^2(y,t) + \frac{1}{2}y^2\sup_{D(0)}|\omega_0|^2 \;\le\; &\frac{1}{2}r(0)^2\sup_{D(0)}|\omega_0|^2 \;+\; t\omega^2(r(t),t)\\ &+\frac{1}{2}r(t)^2\sup_{D(0)}\;|\omega_0|^2, \end{aligned}$$

where for the last inequality we used the fact that both the quantities compared are positive. Therefore

$$t\omega^2(y,t) \le \frac{1}{2}r(0)^2\sup_{D(0)}\;|\omega_0|^2 \;+\; t\omega^2(r(t),t)$$

$$+\frac{1}{2}r(t)^2 \sup_{D(0)} |\omega_0|^2 - \frac{1}{2}y^2 \sup_{D(0)} |\omega_0|^2,$$

and so

$$\begin{aligned}\omega^2(y,t) \leq & \frac{1}{2t}r(0)^2 \sup_{D(0)} |\omega_0|^2 \ + \ \omega^2(r(t),t) + \frac{1}{2t}r(t)^2 \sup_{D(0)} |\omega_0|^2 \\ & - \frac{1}{2t}y^2 \sup_{D(0)} |\omega_0|^2.\end{aligned}$$

Now let $t \to \infty$ and using the fact that for any $t < \infty$, $r(t)$ and $y \in [0, r(t)]$ are bounded and Lemma 5.3 we obtain

$$\omega^2(y,\infty) \leq \omega^2(r(\infty),\infty) \text{ for all } y \in [0, r(\infty)].$$

One can see that $r(t) < \infty$ for all $t \in [0,T)$ by using the result of Theorem 5.2 that is satisfied by the graphs and obtaining a uniform height bound. Due to the uniform bound we also have $r(\infty) < \infty$. Another approach for proving this is to use the property of the mean curvature flow of being an area minimising flow and notice that we have started with a bounded area graph. If for some time t, $r(t) = \infty$ for $D(t) = (0, r(t))$ this implies that the area of the hypersurface generated by the graph is infinite at this time and contradicts the initial bound.

The last estimate gives us a maximum of the height of the solution on the Neumann boundary at time $T = \infty$. By the same argument as in the height bound Lemma 5.3, this can not occur. This together with the above relation at $T = \infty$ tells us that the graphs must have reached a constant height at $T = \infty$.

We now turn our attention to case (b). For the problem (41) we have two boundary conditions, a Neumann and a Dirichlet condition. Here we have $D(t) = (r(t), R)$ with the boundaries being $\partial D_D = \{R\}$ and $\partial D_N(t) = \{r(t)\}$. The situation is easier than in the previous case since on the Neumann boundary we can prove directly that we do not have any maxima of the function g. This comes from (50) together with the fact that the outer normal to the Neumann boundary is $\nu_{\partial D_N(t)} = -\frac{y}{|y|} = -1$:

$$\frac{dg}{d\nu_{D_N(t)}} \ = \ 2t\omega_\Sigma^2 \frac{1}{\omega_\Sigma \frac{d\omega_\Sigma}{dy}} - r(t) \sup_{D(0)} |\omega_0|^2 \leq 0,$$

where the last inequality is implied by condition (44). This implies that, similar to case (a) treated above, we have that g exists for all times $t < \infty$.

The Hopf Lemma tells us then that there do not exist any maxima of g on the Neumann boundary. The previous application of the maximum principle in (48) gives us that the height of g is bounded by the maximum of the initial values and of the values on the Dirichlet boundary. This implies

$$\begin{aligned}t\omega^2(y,t) + \frac{1}{2}y^2 \sup_{D(0)} |\omega_0|^2 \leq & \max\Big\{\frac{1}{2}R^2 \sup_{D(0)} |\omega_0|^2, \ \sup_{(r(0),R)} \frac{1}{2}y^2 \sup_{D(0)} |\omega_0|^2\Big\}, \\ \leq & \frac{1}{2}R^2 \sup_{D(0)} |\omega_0|^2\end{aligned}$$

which leads to

$$\omega^2(y,t) \leq \frac{1}{2t}(R^2 - y^2) \sup_{D(0)} |\omega_0|^2, \quad \text{for every } t < \infty.$$

Taking $t \to \infty$ in the last line gives us that $\omega^2(y,t) \leq 0$ as $t \to \infty$. This implies ω converges to the zero function defined over an annulus. □

Remark (C^∞ convergence). The convergence in the above theorem is only in the C^0 topology, since we have only explicitly shown that the height converges to a constant as $t \to \infty$. To obtain convergence in the C^∞ topology, where all derivatives must also converge, one must apply interior estimates (such as can be found in [**6, 10**]) after one has already established long time existence.

Remark (Examples). Theorem 5.6 is applicable, for example, to the motion of radially symmetric graphs inside the catenoid neck (case (a)) and to the motion of radially symmetric graphs outside the sphere (case (b)). The first converges to the flat disc inside the catenoid neck with zero height. The second example, flow by mean curvature of radially symmetric graphs outside the sphere supported at ninety degrees on the sphere and with a fixed zero height at some fixed radius outside the sphere, can be applied to the graphs considered in the previous chapter as a barrier for the reflectively symmetric graphs. By the result of the theorem the radially symmetric graphs converge to the annulus around the sphere.

5. Curvature singularity on the free boundary in finite time

The results of this section only apply to (41). We use a self-similar solution of mean curvature flow to show that for some specific initial data parts of the graphs evolve towards the rotation axis $y = 0$ where the surface Σ is horizontal, developing a curvature singularity by being pinched. We first state the most general form for this result and then give sufficient conditions for the initial graph to observe such behaviour.

In Theorem 5.2 we gave sufficient conditions for which the height of the graphs for both of our problems (40) and (41) remains bounded by the initial values. The initial conditions also include a relation which states that between the maximum and minimum of the initial height there is no point on which the surface Σ is horizontal. These two conditions are enough to prevent the graphs ω from developing an infinite gradient on the Neumann boundary.

When the surface Σ has a point in which it is horizontal in the above mentioned region, that is there exists a point between the maximum and minimum of the height of the initial graph such that $|\langle \nu_\Sigma, e_3 \rangle| = 1$, then there is essentially no obstruction to the evolving graphs moving towards those points. When one of these points lies on the axis of rotation for Σ, a curvature singularity can also develop. This is the case which interests us in this section.

In the following consider rotationally symmetric surfaces Σ which are diffeomorphic to spheres with rotation axis $y = 0$.

THEOREM 5.7 (Curvature singularity on the boundary). *Suppose* Σ *is such that* $Dom(\omega_\Sigma) = [0, r]$ *with* $\omega_\Sigma(r) = 0$ *and satisfying in addition:*

$$\begin{aligned} \omega_\Sigma(y)\frac{d\omega_\Sigma}{dy}(y) &\leq 0 && \text{for all } y \in Dom(\omega_\Sigma), \\ \frac{d\omega_\Sigma}{dy}(y) &\neq 0 && \text{for all } y \in Dom(\omega_\Sigma) \sim \{0\}, \\ \frac{d\omega_\Sigma}{dy}(0) &= 0. \end{aligned}$$

Let ω *satisfy* (41) *for* $D(t) = (r(t), R)$ *such that* $D(0) = (r(0), R)$, $R \notin Dom(\omega_\Sigma)$ *and* $r(0) \leq r < R$. *If for the initial graph function* ω_0 *there exists a self-similar torus in the region bounded by the initial graph* ω_0 *and the line* $z = \omega_\Sigma(0)$ *if the graph* ω_0 *has positive values, or there exists a self-similar torus in the region bounded by the initial graph* ω_0 *and the line* $z = -\omega_\Sigma(0)$ *if the graph* ω_0 *has negative values, then the solution for the problem* (41) *exists for only a finite time* $T < \infty$ *and the graphs* $\omega(\cdot, t)$ *develop a Type I curvature singularity at* $y = 0$ *as* $t \to T$.

PROOF. We apply the comparison principle to obtain that the moving graphs and the enclosed evolving torus never touch. If we are careful in the choice of torus, then this will force the desired pinching behaviour.

Despite the presence of boundaries in our problem, the compact case of the comparison principle, found for example in [**22**], is sufficient here. This tells us that the solutions of our moving graphs and the evolving tori remain disjoint for all time. Note also that our previous results imply that the solution ω continues to exist until the appearance of the first gradient singularity on the boundary. Due to the rotational symmetry, this is only possible on the Neumann boundary. This ensures that the solution exists at least for as long as the torus beneath (or above) it exists. We are able to apply the compact version of the comparison principle since, from the hypothesis of the theorem, the torus is above the maximum of the height of Σ, so in particular above the Neumann boundary also. By the choice of our initial torus we have that it exists for a finite quantum of time T until it becomes a point, forcing the solution to pinch at the point $y = 0$ as $t \to T$.

To prove that it is a type I curvature singularity we apply again the comparison principle. This time we place a sphere centred on the rotation axis above the maximum height of Σ, or below the minimum height if we find ourselves on the negative side of the graphs ω_Σ. Taking the initial sphere to be disjoint from the evolving graphs one obtains that they remain disjoint for all times of existence. Therefore the evolving graph can only become singular on the axis of rotation as fast as the sphere. Now since a sphere evolving by mean curvature flow develops a Type I curvature singularity when it finally contracts at the centre point on the rotation axis $y = 0$ (in fact we could place any strictly convex surface there, as one can infer from [**21**]), the radially symmetric graphs must also develop a Type I curvature singularity there. □

REMARK. The plane $z = \omega_\Sigma(0)$ (or $z = -\omega_\Sigma(0)$) which bounds the torus comes from the maximal (or minimal) height of the surface Σ. This condition can be weakened by replacing the plane with a backwards cone which still gives enough freedom for the comparison principle to work for the self-similar torus of Angenent. We have decided to use the plane condition instead since requiring the self-similar torus in the hypothesis of the theorem is very strong condition. With the plane condition, one has the freedom to choose any torus which will simply pinch the graphs at the point $y = 0$, not necessarily shrinking self-similarly. That is, any torus which does not collapse to a circle will do.

REMARK. One may carry through a similar idea for graphs without any symmetry, however the statement obtained is much weaker. Of course the comparison principle continues to apply and the torus continues to force the graph to cease existing after a finite amount of time. The problem is that as of this moment, we have not been able to prevent the occurrence of a singularity for the gradient on the free boundary. For our techniques to work, it seems one requires at least some symmetry assumption. This implies that the graph may not even exist as long as the torus: it may develop a singularity for the gradient (or the curvature) on the free boundary before the torus pinches off. The upshot of this is that without the rotational symmetry assumption we only obtain a maximal time estimate from above, and do not have any information about the type of singularity which develops in finite time.

Our theorem requires the following result due to Angenent [**2**].

THEOREM 5.8 (Angenent [**2**], 1989). *For $n \geq 2$ there exist embeddings $X_n : S^1 \times S^{n-1} \to \mathbb{R}^{n+1}$ for which $X_n(p,t) = \sqrt{2(1-t)} \cdot X_n(p)$ is a solution of the flow by mean curvature equation.*

REMARK. This theorem states the existence of a self-similar torus. The solution for the compact mean curvature flow given by the embedding in the theorem above states that the torus shrinks to the origin by dilatations, and it will become singular at time $T = 1$.

There are three quantities which characterise a torus, and Angenent finds conditions on these for which the torus is self similar. We consider the torus instead as a surface of revolution. The width of the torus is the maximal distance from the rotation axis taken pointwise and the radius of the hole is the minimum of all distances of points from the curve to the axis of rotation. The third quantity is the 'fatness' of the torus or the maximum height that the plane curve takes as a graph. Let us denote the three quantities by r_1 for the radius of the hole, r_2 for the width of the torus, and h for the maximum height. In case we are looking at a perfect torus, obtained by the rotation of a circle, the three quantities will be: r_1 is the difference between the radius of the rotation and the radius of the rotated circle, r_2 is be the sum of r_1 and the diameter of the rotated circle and h is just the radius of the circle which is being rotated. Following the work of Angenent [**2**] one finds, quite surprisingly, that the self-similar torus is not a 'perfect torus', obtained by the rotation of a circle. It is in fact somewhat egg-shaped, so working directly with the self-similar torus is not easy. This is why we make use of a little trick.

Since we are working with a general graph we want to give the most general condition on the three quantities for which the existence of a self-similar torus is assured. The most direct method is to fit a big "box" in the region where we wish to place the curve generating the self-similar torus, which then is contained in the "box". From discussions

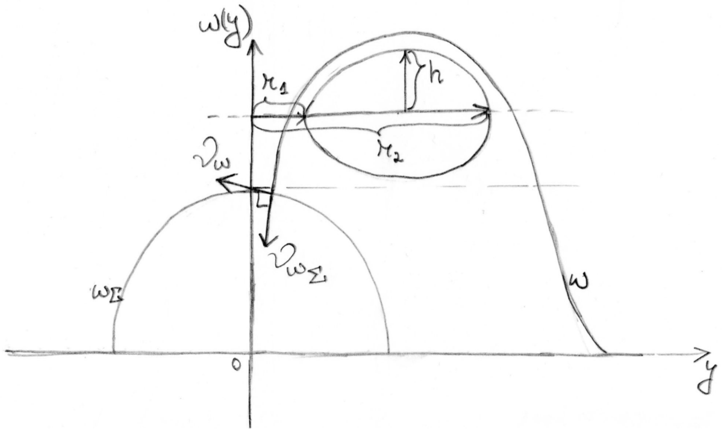

FIGURE 5.1. The three quantities of the torus.

found in [**2**] we obtain relations between the three quantities, for which the torus is a self-similar shrinking solution.

LEMMA 5.9 (Angenent [**2**], 1989). *There exists a smallest $r_2^* \geq \sqrt{2n}$ for which we have a self-similar torus with $r_1^* \geq \frac{1}{r_2^*}$ and $h \leq \frac{C}{r_2^*}$, where C does not depend on r_2^*.*

For $n = 2$ one can also estimate numerically [**2, 5**] the approximate values of these three quantities for the self-similar torus.

REMARK (Angenent [**2**], 1989, Chopp [**5**], 1994, Approximate values for the self--similar torus for $n = 2$). For the above existence result in the case $n = 2$ we have the following approximate values

$$\begin{aligned} r_2^* &= 3.4, \\ r_1^* &= 0.45, \\ h^* &= 0.87, \\ C &< 3 \end{aligned}$$

up to scaling.

We do not wish to use only the values given by the existence result of the above lemma since one may always find a "smaller" torus which also shrinks self-similarly. Thus we must be very careful in the scaling behaviour of these three quantities. In the following we look at the the conditions that a scaled version of the self-similar torus must satisfy. Assuming we start with another value for the width of the torus, which we denote as before with r_2, we need to introduce a scaling factor:

$$\lambda = \frac{r_2}{r_2^*} \leq \frac{r_2}{\sqrt{2n}}.$$

The conditions imposed on the three quantities are scaled appropriately as:
(a) $r_1 \geq \lambda^2 \frac{1}{r_2}$,
(b) $h \leq \lambda^2 \frac{C}{r_2}$.

As announced in the lemma above, note that the constant C does not depend on the scaling.

Next we wish to give sufficient conditions for the self-similar torus of Theorem 5.7 to exist. We work assuming that the initial graph is positive. The negative or mixed cases are treated similarly. Let us set

$$\mathcal{Q} = \{z \in \mathbb{R} : \exists y \in [r(0), R] \text{ such that } z = \omega_0(y) \text{ and } z > \omega_\Sigma(0)\}$$

to be the set of all initial values above the fixed height line $z = \omega_\Sigma(0)$ from Theorem 5.7. Let

$$M = \sup_{z \in \mathcal{Q}} \omega_0^{-1}(z)$$

be the farthest point away from the rotation axis $y = 0$ for which the initial graph is above the maximal value of the Σ graphs. Then we have the following lemma.

Lemma 5.10 (Conditions on the initial graph). *Suppose* ω_0 *is an initial graph for the problem* (41). *If there exists* $0 < R_2 \leq M$ *and there exist* $z_1, z_2 \in \mathcal{Q}$ *with* $z_1 \leq z_2$ *and* $z_2 - z_1 \geq C\, \frac{2R_2}{r_2^{*2}}$ *such that* $|\max \omega_0^{-1}(z) - \min \omega_0^{-1}(z)| \;\geq\; R_2 - \frac{R_2}{r_2^{*2}}$ *for all* $z \in [z_1, z_2]$, *then there exists the self-similar torus required for Theorem 5.7.*

Proof. The two conditions of the lemma are sufficient to enable us to construct a "box" high enough and wide enough such that we are able to fit the curve which generates the self-similar torus in the region bounded by the constant height $z = \omega_\Sigma(0)$ and the initial graph ω_0. The box permits our curve to satisfy the above scaled conditions (a) and (b). ☐

6. Examples

We want to mention here two examples for the two types of problems (40) and (41). One of them is the motion by mean curvature flow of radially symmetric graphs inside the catenoid neck and the second one is the motion by mean curvature flow of radially symmetric graphs outside the unit sphere combined with a fixed zero Dirichlet boundary height on a fixed radius outside the sphere. In the first problem our results imply that the mean curvature flow solution exists for all times and converges to the flat disc inside the catenoid neck. The second shows that the problem (41) has a long time solution if we start with an initial graph below the height of the sphere and converge as $t \to \infty$ to the annulus around the sphere. In the case where we do not have such an initial bound on the height and instead satisfy the conditions of the Theorem 5.7, the graphs move towards the North Pole of the sphere or the South Pole in case we find ourselves with a negative graph. The graph develops a Type I curvature singularity at either of these poles in finite time.

CHAPTER 6

Mean curvature flow of graphs with a free boundary on general hypersurfaces in Euclidean space

1. Introduction

This chapter treats the general setting for our two types of problems. The majority of the results in this chapter are explicitly proved in $\mathbb{R}^3$, even though most of the results carry over without difficulty to higher dimensions.

First we define in detail the two problems, the evolution of graphs by mean curvature flow with Neumann boundary condition on a fixed contact surface in $\mathbb{R}^3$ and the evolution of graphs with both Neumann and Dirichlet boundary conditions.

In section 3 we obtain height bounds in two different ways. The first is strongly related to the information enclosed in the contact surface Σ and it is more restrictive. As examples here we have mean curvature flow of graphs outside the sphere or inside the catenoid neck. The second type of height bound follows simply from the radially symmetric chapter results with no extra condition imposed on the surface Σ and it applies to any of the situations considered thus far: motion outside the sphere, inside or outside the catenoid neck, and more.

Section 4 takes the reader through a well-known method for obtaining bounds on the Dirichlet boundary.

It is followed by the section concerned with gradient bounds on the Neumann boundary. We classify the 'bad' behaviour that a graph might exhibit on the Neumann boundary. There are two types of bad gradient behaviour. The first one we call a tilt and it occurs when the surfaces will become non-graphical on the Neumann boundary by the unit normal becoming parallel to the vector field of rotation for the surface Σ. We present here an extensive list of conditions on curvature and derivatives of curvature in a first boundary point of tilt. The second type of bad gradient behaviour on the Neumann boundary is when the evolving graphs move towards a point where the surface of contact Σ is horizontal. We can exclude such behaviour in certain situations. The result is obtained for the evolution of graphs which are defined on two dimensional topological disks.

Section 6 treats the particular case when the contact surface is a cylinder in $\mathbb{R}^3$ for the purely Neumann problem (1). We obtain uniform bounds for height, gradient, and mean curvature. These estimates imply long time existence. This result has been previously obtained by Huisken [**23**] through the use of integral estimates, a completely different method.

In section 7 we treat the mean curvature flow of graphs in a half space with free boundary on a hyperplane. This is the natural next step in developing the results obtained for entire graphs by Ecker and Huisken [**9**]. Here we work directly with the general n-dimensional flow and we obtain uniform bounds for the height, gradient and the mean curvature. Initially convex or concave (in the sense of eigenvalues of the second fundamental form) hypersurfaces remain so, and thus we also obtain uniform bounds on the full second fundamental form. These estimates imply long time existence for the solution. Further, we prove that the solution converges to a hyperplane if the initial height is bounded. This can be contrasted with the following well-known result from minimal surface theory: any complete minimal surface bounded between two planes must be itself a plane. This is obvious from the maximum principle; here we face the difficulty of having to prove that the solution converges to a minimal surface with bounded height. The standard technique relies heavily upon the surface area remaining bounded, and in this situation that is obviously not the case. We instead directly prove that the norm of the second fundamental form decays to zero.

The last section contains a collection of useful results for general graphs evolving by mean curvature flow with free boundary. Here we discuss the compatibility condition on the Dirichlet boundary and also show that solutions with bounded area which exist for all time must converge to a minimal surface. Here we also apply a technique developed in this chapter, section 5, to obtain long time existence for some of the radially symmetric flows with weaker conditions on Σ.

2. Setup

As in Chapter 5 we only consider contact surfaces Σ which are the union of two rotationally symmetric graphs ω_Σ^+, ω_Σ^- where

$$\begin{aligned}&\text{Dom } \omega_\Sigma^+ = \text{Dom } \omega_\Sigma^-,\\ &\omega_\Sigma^+(y) \geq 0,\\ &\omega_\Sigma^-(y) \leq 0,\\ &\omega_\Sigma^+(y) = 0 \Leftrightarrow \omega_\Sigma^-(y) = 0,\end{aligned}$$

for all $y = |(x_1, x_2)|$. Each point $X \in \Sigma$ can be written as $X = \big((x_1, x_2), \omega_\Sigma(y)\big)$, where ω_Σ is either ω_Σ^+ or ω_Σ^-. We also impose that the graphs meet vertically, that is

$$\langle \nu_\Sigma(y), e_3 \rangle = 0 \text{ when } \omega_\Sigma = 0,$$

where we denote by ν_Σ the normal to ω_Σ. A convention which we continue to use is that the normal to the fixed surface Σ is pointing away from the moving graphs.

For example we have $\omega_\Sigma^\pm(y) = \pm\text{arccosh}(y)$ and $\omega_\Sigma^\pm(y) = \pm\sqrt{1 - y^2}$ for the catenoid and the sphere respectively. This rotational symmetry is consistent with our previous work. Note also that for rotationally symmetric surfaces of contact Σ the Killing vector field $K_3 : \mathbb{R}^3 \to \mathbb{R}^3$ defined in Chapter 4 as $K_3(\ x_1,\ x_2,\ x_3)\ =\ (\ x_2,\ -\ x_1,\ 0)$ is in every point tangent to Σ:

$$\langle K_3, \nu_\Sigma \rangle \ = \ 0. \tag{52}$$

We also impose the following condition on the contact surface Σ

$$\omega_\Sigma \, \langle \nu_\Sigma, e_3 \rangle \ \leq \ 0 \text{ everywhere on } \Sigma, \tag{53}$$

where ν_Σ is a choice of unit normal to the fixed surface Σ pointing for all times outside the evolving mean curvature flow surfaces defined below. This condition ensures that we are able to obtain height estimates for our problems using a maximum principle approach. Condition (53) is satisfied for motion outside the sphere or inside the catenoid neck. For motion inside the sphere and outside the catenoid neck the sign is reversed.

REMARK. We see in the next section on height bounds that we do not need to impose condition (53) to obtain height bounds. We nevertheless state it here, since it is a way to obtain height bounds using the maximum principle.

We are now ready to redefine our problems. Since most of the results we present here are for the case $n = 2$, we shall for the moment restrict ourselves to this. There are two types of problem that we consider. The first is the mean curvature flow of graphs with a Neumann boundary condition on the fixed surface Σ. The boundary condition is given by enforcing a ninety degree contact angle with the surface Σ.

Let M^2 to be a smooth, orientable 2-dimensional manifold with smooth, compact boundary ∂M^2 and set $M_0 := F_0(M^2)$, where $F_0 : M^2 \to \mathbb{R}^3$ is a smooth embedding satisfying

$$\begin{aligned} \partial M_0 \equiv F_0(\partial M^2) &= M_0 \cap \Sigma, \\ \langle \nu_{M_0}, \nu_\Sigma \circ F_0 \rangle (p) &= 0 \ \forall p \ \in \partial M^2, \end{aligned}$$

where we have denoted by ν_{M_0} the unit normal to M_0. Let $I \subset \mathbb{R}$ be an open interval and let $F_t = F(\cdot, t) : M^2 \to \mathbb{R}^3$ be a one-parameter family of smooth embeddings for all $t \in I$. The family of surfaces $(M_t)_{t\in I}$, where $M_t = F_t(M^2)$, are said to be evolving by mean curvature flow a with free Neumann boundary condition on Σ if

$$\begin{aligned} \frac{\partial F}{\partial t}(p,t) &= -\, H(p,t)\nu_{M_t}, \ \forall (p,t) \in M^2 \times I, \\ F(\cdot, 0) &= F_0, \\ F(p,t) &\subset \Sigma, \ \forall (p,t) \in \partial M^2 \times I, \\ \langle \nu_{M_t}, \nu_\Sigma \circ F \rangle (p,t) &= 0, \ \forall (p,t) \in \partial M^2 \times I, \end{aligned} \tag{54}$$

where we have denoted by ν_{M_t} the unit normal to M_t and by H the mean curvature of M_t.

The second type of problem is when the evolving surfaces also have a second boundary on which we prescribe a fixed height over the plane of definition as a graph. This is the general setting which includes the example we have worked on in Chapter 4 where the surface Σ was the unit sphere in $\mathbb{R}^3$. We define this problem as follows.

Let M^2 to be a smooth, orientable 2-dimensional manifold with two smooth, compact, disjoint boundaries which we denote by $\partial_N M^2$ for Neumann boundary and $\partial_D M^2$ for Dirichlet boundary. Set $M_0 := F_0(M^2)$, where $F_0 : M^2 \to \mathbb{R}^3$ is a smooth embedding satisfying

$$\begin{aligned} \partial_N M_0 &\equiv F_0(\partial_N M^2) = M_0 \cap \Sigma, \\ \langle \nu_{M_0}, \nu_\Sigma \circ F_0 \rangle (p) &= 0 \ \forall p \ \in \partial_N M^2, \\ \partial_D M_0 &\equiv F_0(\partial_D M^2). \end{aligned}$$

We will impose also another condition on the Dirichlet boundary which is a necessary condition in obtaining the Dirichlet boundary estimates in Section 3 of this chapter. If we consider the Dirichlet boundary of the associated graph in the e_3 direction of M_0, that is the projection onto the $\mathbb{R}^2$ plane $pr_{R^2}\partial_D M_0$, then the following must hold:

$$H\big|_{pr_{R^2}\partial_D M_0} \;\geq\; 0. \tag{55}$$

Let $I \subset \mathbb{R}$ be an open interval and let $F_t = F(\cdot, t) : M^2 \to \mathbb{R}^3$ be a one-parameter family of smooth embeddings for all $t \in I$. The family of surfaces $(M_t)_{t \in I}$, where $M_t = F_t(M^2)$, is said to be evolving by mean curvature flow with free Neumann boundary condition on Σ and an additional fixed Dirichlet boundary condition if it satisfies

$$\begin{aligned} \frac{\partial F}{\partial t}(p,t) \;&=\; -\,H(p,t)\nu_{M_t},\ \forall (p,t) \in M^2 \times I, \\ F(\cdot,0) \;&=\; F_0, \\ F(p,t) \;&\subset\; \Sigma,\ \forall (p,t) \in \partial_N M^2 \times I, \\ \langle \nu_{M_t}, \nu_\Sigma \circ F\rangle (p,t) \;&=\; 0,\ \forall (p,t) \in \partial_N M^2 \times I, \\ F(p,t) \;&=\; F_0(p),\ \forall (p,t) \in \partial_D M^2 \times I. \end{aligned} \tag{56}$$

We also assume that, in both problems, M_0 can be written as a graph in the direction of e_3. So initially we have

$$\langle \nu_{M_0}, e_3 \rangle \;>\; 0. \tag{57}$$

In the following we show that (54) and (56), for special cases of the surface Σ and conditions on the initial surface M_0, preserve the initial graph condition, which leads us to gradient bounds for the associated scalar graph problem.

3. Height bounds

As announced in the introduction of this chapter we use two methods for obtaining height bounds for the problems under study. The first one is when we make use of condition (53) and prove that the height of the graphs remain for all times bounded by the initial values. The second comes from using the radially symmetric graphs as barriers. Let us start with the first method.

Following [**9**] we define the height function $u : M_t \to \mathbb{R}$

$$u(X) \;:=\; \langle X, e_3 \rangle \;=\; \langle F(p,t), e_3 \rangle$$

where we freely alternate between using $F(p,t)$ for a point $p \in M^2$ and the position vector $X = F(p,t)$. This function measures the distance of a fixed point on the moving surface to the hyperplane (orthogonal to e_3) over which the surface is initially defined as a graph. Having a bound on this height is equivalent to having a bound on the height of the associated graph function.

In a local orthonormal frame $\{\tau_i\}_{1 \leq i \leq 2}$ on M_t we also have the formula

$$\nabla^{M_t} u \;=\; \sum_{i=1}^{2} \langle \tau_i, e_3 \rangle\, \tau_i. \tag{58}$$

From the work on entire graphs by Ecker and Huisken [**9**], one observes that this height function satisfies the heat equation on M_t:

$$\Big(\frac{d}{dt} - \Delta^{M_t}\Big)\, u \;=\; 0. \tag{59}$$

This evolution allows us to bound the height at all times, barring poor boundary behaviour. In both (54) and (56) one sees that because u is constant on the Dirichlet boundary, the only bad behaviour can occur at the Neumann boundary. Imposing the extra condition (53) on the surface Σ one can exclude such behaviour as follows.

On the Neumann boundary the unit normal of Σ is in every point a member of the tangent space of M_t, so we can always choose $\tau_2 = \nu_\Sigma = \nu_{\partial M_t}$ on the boundary. Then we get the following result.

THEOREM 6.1 (Height bounds using an additional condition and the maximum principle). *Let Σ be a rotationally symmetric surface in $\mathbb{R}^3$ satisfying* (53). *Let M_t be a graph satisfying* (54) *or* (56). *Then the height function u is bounded by the supremum of its initial values for all time*

$$\sup_{M_t} |u| \;\le\; \sup_{M_0} |u|.$$

PROOF. From the parabolic evolution of the height function (59) we obtain that the square of the height function also satisfies

$$\Big(\frac{d}{dt} - \Delta^{M_t}\Big) u^2 \;=\; -|\nabla^{M_t} u|^2 \;\le\; 0.$$

By Theorem 2.9 this implies

$$\sup_{M_t} u^2 \;\le\; \max\Big\{ \sup_{M_0} u^2, \max_{\partial_D M_t} u^2, \max_{\partial_N M_t} u^2 \Big\}.$$

Since $\max\limits_{\partial_D M_t} u^2 \le \sup\limits_{M_0} u^2$ this simplifies to

$$\sup_{M_t} u^2 \;\le\; \max\Big\{ \sup_{M_0} u^2, \max_{\partial_N M_t} u^2 \Big\}. \tag{60}$$

To exclude a maximum of the square of the height function on the Neumann boundary we will make use of (53) and the Neumann angle condition. Suppose there exists a point $X = X(p,t) \in \partial_N M_t$ such that the function u^2 has a first boundary maximum at X. Then by the Hopf Lemma 2.7, using the parabolic evolution of u^2, we get a sign on the derivative of the height function in the normal direction to the boundary, that is

$$\big\langle \nabla u^2, \nu_{\partial M_t} \big\rangle \big|_X \;=\; 2u \,\langle \nabla u, \nu_\Sigma \rangle \big|_X \;>\; 0 \tag{61}$$

where we have chosen $\nu_{\partial M_t} = \nu_\Sigma$. Here and henceforth we abbreviate $\nabla^{M_t} u$ by ∇u. Now by (58) we also have

$$\begin{aligned} \langle \nabla u, \nu_\Sigma \rangle \;&=\; \sum_{i=1}^{2} \langle \tau_i, e_3 \rangle \;\langle \tau_i, \nu_\Sigma \rangle \\ &=\; \langle \tau_2, e_3 \rangle \\ &=\; \langle \nu_\Sigma, e_3 \rangle \end{aligned}$$

where we have used the choice of our orthonormal frame. This implies

$$\left\langle \nabla u^2, \nu_{\partial M_t} \right\rangle \big|_X \;=\; 2u \left\langle \nu_\Sigma, e_3 \right\rangle \big|_X \;=\; 2\omega_\Sigma \left\langle \nu_\Sigma, e_3 \right\rangle \big|_X,$$

where in the last equality we have used that $u = \omega_\Sigma$ on the Neumann boundary, since the evolving graphs move on the surface Σ. From the assumption (53) on Σ we have

$$\left\langle \nabla u^2, \nu_{\partial M_t} \right\rangle \big|_X \;\leq\; 0,$$

and this contradicts the sign given by the Hopf Lemma (61) at a maximum on the boundary. This finally implies that there is no maximum of the height squared on the Neumann boundary at any time. Returning to (60) we obtain the desired result. □

REMARK (Examples). This type of height bound is applicable for the motion of graphs outside the sphere or inside the catenoid neck, or to a graph evolving outside any ellipsoid in $\mathbb{R}^3$.

The second way of obtaining height bounds is less restrictive and uses additional conditions on Σ similar to those imposed in the previous chapter on radially symmetric graphs.

THEOREM 6.2 (Height bounds using comparison principle and barriers). *Let Σ be a rotationally symmetric surface in $\mathbb{R}^3$ as defined in Section 2. Let M_t satisfy* (54) *or* (56). *If Σ satisfies condition* (43) *or* (44), *for problem* (54) *or* (56) *respectively, then we have for all time*

$$\sup_{M_t} |u| \;\leq\; \sup_{M_0} |u|.$$

If Σ satisfies instead condition (46) *for the problem* (56) *then there exists a global constant $C < \infty$ such that for all times*

$$\sup_{M_t} |u| \;\leq\; C.$$

PROOF. The proof is simple and is based on the results on radially symmetric graphs from Chapter 5. It works in the same way for all our various problems. We can always find radially symmetric graphs ω^+ and ω^- such that the surfaces generated by them are barriers for the initial manifold M_0. By the comparison principle, Theorem 2.10, they remain barriers for all surfaces M_t so long as the immersions F_t exist. From Lemma 5.3 and the proof of Theorem 5.5 we see that the radially symmetric graphs satisfy height bounds for all times. These bounds are given by the initial values in the case of conditions (43) and (44).

In case we only have condition (46) for problem (56), we can not conclude that the height of the solution remains bounded by the supremum of the initial values. For this case we apply Theorem 5.5 to the radially symmetric barrier. Thus the constant C is the equivalent of $\omega_\Sigma(R)$ in that proof. There R denotes the radius of the rotation on the Dirichlet boundary. So the constant C can be computed after the choice of the radially symmetric barrier as $C = \omega_\Sigma(R)$, where R is the Dirichlet boundary point for the radially symmetric graphs. □

REMARK (Examples). This type of height bound applies to all of the examples we have considered thus far: the evolution of graphs outside the sphere, inside or outside the catenoid neck, outside ellipsoids, and so on.

REMARK. The computations in this section and the comparison with rotationally symmetric barriers apply equally in all dimensions, and so the results of this section apply also to the mean curvature flow of hypersurfaces in $\mathbb{R}^{n+1}$ for each of the problems under consideration, with the same hypotheses.

4. Dirichlet boundary estimates

In this section we briefly review the well-known method used to obtain estimates on the Dirichlet boundary through the construction of barriers. For more details on general results in this field and references we invite the reader to follow Appendix B. Here we only state the result that is most applicable to our work. For mean curvature flow with Dirichlet boundary conditions the use of barriers was observed for the first time by Huisken in [**23**]. There one makes use of the well-known work of Serrin [**34**] to construct barriers for the elliptic analogue of the parabolic operator. The two barriers (an 'upper' and a 'lower') bound the initial data at the Dirichlet boundary, and (following Huisken) one can apply the strong maximum principle to bound the gradient at the Dirichlet boundary for all times.

THEOREM 6.3 (Huisken [**23**], 1989, Estimates on the Dirichlet boundary). *Let M_t satisfy* (56) *with smooth initial data M_0 which satisfies conditions* (55) *and* (57)*. Then there exists a global constant $C > 0$ such that for all time*

$$\langle \nu_{M_t}, e_3\rangle \; > C \text{ on the Dirichlet boundary } \partial_D M_t. \tag{62}$$

PROOF. To follow the proof of Huisken [**23**] we need to first write our problem as a non-parametric mean curvature flow, as in [**8**] and as in Chapter 4 for the sphere problem. The immersion flow (56) is equivalent up to tangential diffeomorphisms to the following scalar evolution of a graph $h : \tilde{\Omega} \to \mathbb{R}$: in the e_3 direction, defined on a time dependent domain $\tilde{\Omega} = \bigcup_{t\in[0,T)} \Omega(t) \times \{t\}$:

$$\begin{aligned} \frac{\partial h}{\partial t} \;&=\; \sqrt{1+|\mathrm{D}h|^2}\,\mathrm{div}\Big(\frac{\mathrm{D}h}{\sqrt{1+|\mathrm{D}h|^2}}\Big) \text{ on } \tilde{\Omega}, \\ \langle \nu_h, \nu_\Sigma\rangle \;&=\; 0 \text{ and } h(x,t) \;=\; \omega_\Sigma(|x|) \text{ on } \partial\Omega_N(t) \text{ for all } t \in [0,T), \\ h(x,t) \;&=\; h_0 \text{ on } \partial\Omega_D \text{ for all } t\in[0,T), \\ h(x,0) \;&=\; h_0 \text{ on } \Omega(0), \end{aligned} \tag{63}$$

where $\Omega(t) \subset \mathbb{R}^2$ has two distinct smooth boundaries: the Neumann boundary $\partial\Omega_N(t)$ and the Dirichlet boundary $\partial\Omega_D$. Here we have also denoted by ν_h the unit normal to the graph function h. The Neumann boundary condition is given by the restriction that the graphs must meet the surface Σ perpendicularly and also be at the same height as the graph ω_Σ which generates the rotationally symmetric surface Σ. The function ω_Σ is as in (53).

The result of this theorem is that on the Dirichlet boundary we enjoy a uniform (in time) gradient bound along the flow. The link between the gradient and the inner product $\langle \nu_{M_t}, e_3\rangle$ is well explained in [**8, 9**], so we omit it here.

Following the work of Serrin [**34**], we are able to find barriers δ^- and δ^+ defined locally in a neighbourhood $\mathcal{U}$ of the Dirichlet boundary $\partial\Omega_D$ such that

$$\begin{aligned}
\mathrm{div}\Big(\frac{\mathrm{D}\delta^-}{\sqrt{1+|\mathrm{D}\delta^-|^2}}\Big) \;&\leq\; 0 \text{ on } \Omega(0)\cap\mathcal{U},\\
\mathrm{div}\Big(\frac{\mathrm{D}\delta^+}{\sqrt{1+|\mathrm{D}\delta^+|^2}}\Big) \;&\geq\; 0 \text{ on } \Omega(0)\cap\mathcal{U},\\
\delta^- \;=\; \delta^+ \;&=\; h_0 \text{ on } \partial\Omega_D,\\
\delta^- \;\leq\; h_0 \;&\leq\; \delta^+ \text{ on } \Omega(0)\cap\mathcal{U}.
\end{aligned}$$

The choice of barriers is made locally around the Dirichlet boundary independent of time. The existence of the barriers δ^+ and δ^- is by construction and can be found in Appendix B, starting with relation (136). In the notation here one should keep in mind that $M = \sup\limits_{\Omega(0)} h_0$.

Using the above and the strong maximum principle we obtain that the barrier functions bound the graph function from above and below for all times:

$$\delta^- \;\leq\; h(\cdot,t) \;\leq\; \delta^+ \text{ on } \big(\Omega(t)\cap\mathcal{U}\big)\ \forall t\in[0,T).$$

Thus there exists a constant $C<\infty$ depending on $\partial\Omega_D$ and h_0 such that we get the gradient bound for all times of existence

$$|Dh| \;\leq\; C \text{ uniformly on } \partial\Omega_D\times[0,T),$$

which implies (62). □

5. Neumann boundary gradient behaviour and bounds

Following [**9**] we consider two quantities $s, v : M_t \to \mathbb{R}$:

$$\begin{aligned}
s(X) \;&:=\; \langle \nu_{M_t}(X), e_3\rangle\\
v(X) \;&:=\; \langle \nu_{M_t}(X), e_3\rangle^{-1} \;=\; s^{-1}(X).
\end{aligned}$$

Up to tangential diffeomorphisms, one can express the second quantity as $v=\sqrt{1+|Dh|^2}$. This makes it clear that an upper bound on v gives the desired gradient bound.

As discussed in the introduction we start with an initial graph, and so at $t=0$ we have $s(\cdot,0) > 0$. Preserving the sign of this function gives us some control on the gradient of the graph function, since when the quantity s stays bounded away from zero, v is bounded from above. Preserving the sign of $s(\cdot,t)>0$ for all times ensures that M_t remains a graph.

Following Ecker and Huisken [**9**] we see that these quantities satisfy parabolic evolutions.

PROPOSITION 6.4 (Ecker, Huisken [**9**], 1989, Evolution of v). *The quantity* $v : M_t \to \mathbb{R}$ *satisfies the evolution:*

$$\Big(\frac{d}{dt} - \Delta^{M_t}\Big)v \;=\; -\,|A^{M_t}|^2 v \;-\; 2v^{-1}|\nabla^{M_t} v|^2 \;\leq\; 0.$$

PROOF. From [**21**], we have that the time derivative of the normal satisfies $\frac{d}{dt}\nu_{M_t} = \nabla^{M_t} H$, thus

$$\frac{d}{dt}v \;=\; -v^2 \left\langle \nabla^{M_t} H, e_3 \right\rangle .$$

In terms of a local orthonormal frame $\{\tau_i\}_{1\leq i\leq 2}$ on M_t we compute:

$$\begin{aligned}\Delta_{M_t} v \;&=\; \sum_{i=1}^{2} \mathrm{D}_{\tau_i}\big(-v^2 \left\langle \nabla_{\tau_i}, e_3\right\rangle \big) \;=\; \sum_{i=1}^{2}\sum_{l=1}^{2} \mathrm{D}_{\tau_i}\big(-v^2 \left\langle h_{il}\tau_l, e_3\right\rangle \big)\\ &=\; -v^2 \left\langle \nabla^{M_t} H, e_3 \right\rangle \;+\; v|A^{M_t}|^2 \;+\; 2v^{-1}|\nabla^{M_t} v|^2,\end{aligned}$$

where we have used the Weingarten and Codazzi equations and the fact that in the interior we can choose coordinates such that the Christoffel symbols vanish at a point. Bringing the last two equalities together we obtain the desired result. □

This proof also implies the following evolution of s.

PROPOSITION 6.5 (Ecker, Huisken [**9**], 1989, Evolution of s). *The quantity* $s : M_t \to \mathbb{R}$ *satisfies the evolution:*

$$\Big(\frac{d}{dt} - \Delta^{M_t}\Big)s \;=\; |A^{M_t}|^2 s.$$

On the Neumann boundary contained in the fixed surface Σ there are two important cases when v quantity becomes unbounded (and s becomes zero). In the following we explain each of them in detail and present results which help to avoid such bad behaviour.

First however we wish to quote a result which prevents the boundary curve ∂M_t from losing regularity by developing corners. This result can be found in Stahl [**35**], Theorem 7.19, as a local in time and space gradient bound on the Neumann boundary contained in Σ.

THEOREM 6.6 (Stahl [**35**], 1994, Local gradient estimate at the boundary). *Let* $X_0 = F(p_0, t_0) \in \partial_N M_{t_0}$. *Then* $\forall \epsilon > 0\ \exists r_1 > 0,\ \delta > 0$ *such that*

$$\left\langle \nu_{M_t}(F(p,t)), \nu_{M_{t_0}}(X_0)\right\rangle^{-1} \;<\; 1 \;+\; \epsilon,\ \forall (p,t) \text{ with } F(p,t) \in \mathcal{Z}_{r_1}(X_0),\ 0 \leq t - t_0 \leq \delta$$

where $r_1 = r_1(\epsilon,\ \Sigma,\ n)$ *and* $\mathcal{Z}_{r_1}(X_0)$ *is the infinite space time cylinder centre at* X_0 *with* r_1 *radius of the circle.*

Using this result we know that the first bad behaviour that can happen on the Neumann boundary is loss of the graph property. First let us recall that we require our surface $\Sigma \subset \mathbb{R}^3$ to be rotationally symmetric. The first bad behaviour, when the gradient can become infinite on the Neumann boundary is, when the evolving surface tilts. That is, $\langle \nu_{M_t}, e_3\rangle = 0$ because ν_{M_t} becomes parallel to the rotation vector field of

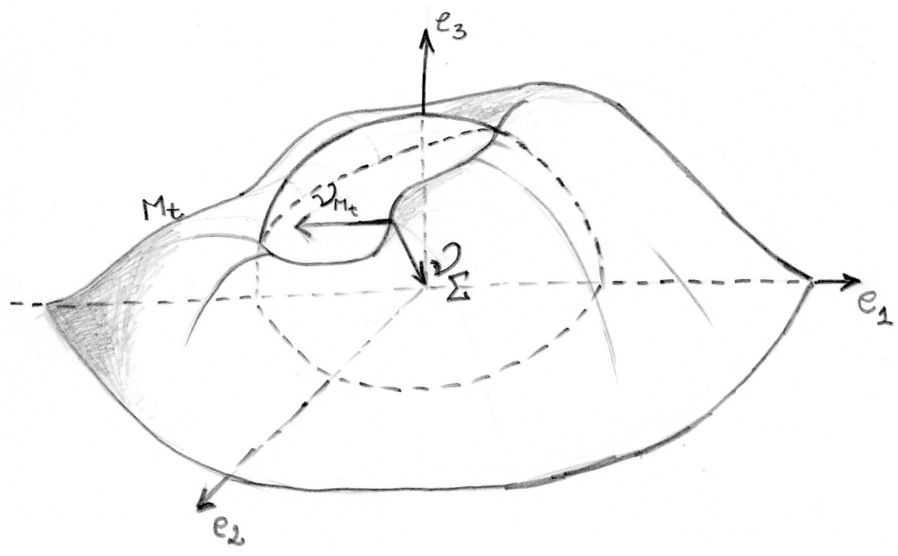

FIGURE 6.1. Tilt in the motion of graphs outside the sphere.

Σ, $\nu_{M_t} = \frac{K_3}{|K_3|}$ up to a sign. We call this bad behaviour *tilting*. The second case is when the moving surfaces reach a point on the Neumann boundary where Σ is horizontal: $|\langle \nu_\Sigma, e_3 \rangle = 1|$. This is also not desirable since the Neumann boundary condition then states that ν_{M_t} is then tangent to a plane parallel to the plane of definition of the graph, thus $\langle \nu_{M_t}, e_3 \rangle = 0$. We shall exclude this behaviour in certain cases.

REMARK. For some of the initial graphs the second type of bad behaviour on the Neumann boundary, that is evolution towards horizontal parts of the contact surface Σ, can not be prevented. Indeed, we were able to prove (cf. Theorem 4.15) that the graphs will actually move toward these points and develop a curvature (and gradient) singularity there.

Let us start with the first case, the tilting, and see what properties the surface M_t possesses in the first point of tilt. We state here results which can be applied to both (54) and (56). Below we work with an orthonormal frame $\{\tau_i\}_{1 \leq i \leq 2}$ spanning TM_t chosen such that for all $X \in \partial M_t$ we have

$$\begin{aligned} \tau_1 &\in T_X \partial_N M_t, \\ \tau_2 &= \nu_\Sigma = \nu_{\partial_N M_t}. \end{aligned}$$

The following proposition collects the properties that the surface M_t exhibits in the first point of tilt on the Neumann boundary.

PROPOSITION 6.7 (Properties in a first boundary tilt). *Let F_t satisfy* (54) *or* (56). *Suppose that $X \in \partial_N M_t \subset \Sigma$ is the first point of boundary tilt. Then at X the following properties hold*

(64) $\langle \nu_{M_t}, e_3 \rangle = 0$ *at X and* $\langle \nu_{M_t}, e_3 \rangle > 0$ *everywhere else on M_s with $s \in [0, t]$*,

(65) $h_{11} = h_{12} = 0$,

(66) $h_{22} \langle \nu_\Sigma, e_3 \rangle < 0$,

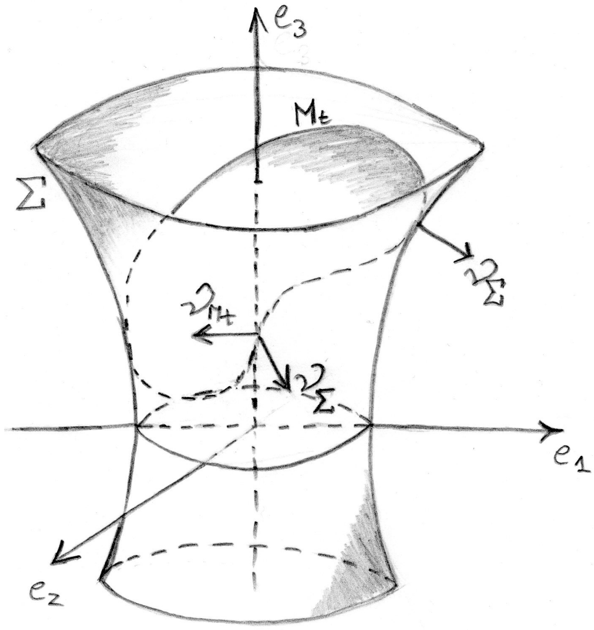

FIGURE 6.2. Tilt in the motion of graphs inside the catenoid neck.

$$\nabla_1 h_{11} \langle \tau_1, e_3 \rangle \;\geq\; 0, \tag{67}$$

$$\nabla_1 h_{12} \;=\; h_{22}\, h^\Sigma_{11}, \tag{68}$$

$$\nabla_1 h_{22}\, \langle \tau_1, e_3 \rangle \;\leq\; -\, h_{22}\, h^\Sigma_{22}\, \langle \nu_\Sigma, e_3 \rangle\,, \tag{69}$$

$$\nabla_2 h_{22} \;=\; h_{22}\, \big(\, h^\Sigma_{22} \;-\; h^\Sigma_{11}\,\big), \tag{70}$$

where we have denoted by h_{ij} and h^Σ_{ij} the components of the second fundamental forms A^{M_t} and A^Σ of M_t and Σ respectively.

The proof of the proposition is contained in the results below. We first prove the following curvature properties, which we keep separate from the other properties since they are sufficient on their own to prevent tilt in the cases where the contact surface is a cylinder or a hyperplane.

PROPOSITION 6.8 (The second fundamental form in a first boundary tilt point). *Let F_t satisfy* (54) *or* (56). *Let $X \in \partial_N M_t \subset \Sigma$ be a first point of boundary tilt, that is $\langle \nu_{M_t}, e_3 \rangle = 0$ by $\nu_{M_t} = \frac{K_3}{|K_3|}$ up to a sign. Then the second fundamental form of M_t at X has the following properties:*

$$\begin{aligned} A^{M_t}(\tau_1, \tau_1) \;&=\; 0, \\ A^{M_t}(\tau_1, \tau_2) \;&=\; 0, \\ \langle \nu_\Sigma, e_3 \rangle \;\; A^{M_t}(\tau_2, \tau_2) \;&<\; 0. \end{aligned}$$

PROOF. This proof is similar to that of Lemma 4.11. The quantity s satisfies a nice evolution, as we have seen in Proposition 6.5. Since we are in a boundary point where for the first time the quantity s takes a zero value after being positive for all earlier times, we can apply the Hopf Lemma, Lemma 2.11. This gives a sign on the derivative normal to the boundary ∂M_t and also the value of the derivative in the direction of the tangent to the boundary curve:

$$\begin{aligned} \nabla_{\nu_{\partial_N M_t}} s|_X &< 0, \\ \nabla_{\tau_1} s|_X &= 0. \end{aligned}$$

We have used again the convention that the unit normal to Σ points away from the evolving graphs. Let us now compute the general form of the gradient of s in the 2-dimensional setting:

$$\nabla_{\tau_i} s \;=\; \nabla_{\tau_i} \langle \nu_{M_t}, e_3 \rangle \;=\; \langle \nabla_{\tau_i} \nu_{M_t}, e_3 \rangle \;=\; \sum_{l=1}^{2} h_{il} \langle \tau_l, e_3 \rangle .$$

Note that here we used the Weingarten equations. We also have denoted by h_{il} the component $A^{M_t}(\tau_i, \tau_l)$ of the second fundamental form of M_t in the τ_i and τ_l direction. Using our choice of orthonormal frame we compute

$$\begin{aligned} \nabla_{\nu_{\partial_N M_t}} s|_X &= \nabla_{\nu_\Sigma} s|_X = \nabla_{\tau_2} s|_X \\ &= A^{M_t}(\tau_1, \tau_2)|_X \langle \tau_1, e_3 \rangle |_X + A^{M_t}(\tau_2, \tau_2)|_X \langle \tau_2, e_3 \rangle |_X \\ &< 0, \\ \nabla_{\tau_1} s|_X &= A^{M_t}(\tau_1, \tau_1)|_X \langle \tau_1, e_3 \rangle |_X + A^{M_t}(\tau_1, \tau_2)|_X \langle \tau_2, e_3 \rangle |_X \\ &= 0. \end{aligned}$$

Note that $\langle \tau_1, e_3 \rangle |_X$ is not zero, since X is a tilt point. Now observe that, as in the sphere case of Lemma 4.11, that in a boundary tilt point we have $\nu_{M_t} = \frac{K_3}{|K_3|} \in T_X\Sigma$ (up to a sign). Also by our choice of basis the other tangent vector to Σ is τ_1. So $\{\frac{K_3}{|K_3|}, \tau_1\}$ is an orthonormal basis of the tangent space $T_X\Sigma$. Further, since these vectors are orthogonal and Σ is a surface of rotation with rotation vector K_3 we know that these two tangent vectors are in fact principal directions for Σ at X. Then, at X the second fundamental form of Σ is diagonal. Thus

$$A^\Sigma(\tau_1, \nu_{M_t})|_X = 0.$$

Using now Proposition 4.10 from Stahl [**35**], we have

$$A^{M_t}(\tau_1, \nu_\Sigma) \;=\; -\, A^\Sigma(\tau_1, \nu_{M_t}) \text{ everywhere on } \partial M_t,$$

and thus obtain that the second fundamental form of the moving graphs is also in diagonal form:

$$A^{M_t}(\tau_1, \tau_2)|_X = 0.$$

This is the third relation, and combining this with the above computation of the directional derivatives of s gives the other two desired relations. □

Using this result one can prove that tilting is not possible when Σ is a cylinder.

Proposition 6.9 (Non-tilting in the case of cylinders). *Let Σ be a cylinder in $\mathbb{R}^3$, and suppose F_t satisfies* (54) *or* (56). *Then there is no tilting behaviour on the Neumann boundary for the evolving surfaces M_t.*

Proof. Suppose that M_t tilts at $X \in \partial_N M_t$. Then Proposition 6.8 implies

$$\langle \nu_\Sigma, e_3 \rangle \ A^{M_t}(\tau_2, \tau_2)|_X \ < \ 0.$$

This is in direct contradiction with the fact that when the surface Σ is a cylinder we have

$$\langle \nu_\Sigma, e_3 \rangle \ = \ 0.$$

This ends the proof. □

Remark. This result is consistent with previous work of Huisken [**23**]. We investigate cylinders in detail in the next section where, using uniform bounds on the height and gradient, we obtain long time existence. The estimates are obtained in a different way than the integral estimates in the above mentioned work. The argument of non-tilting presented above is independent of this type of problem. The tilting behaviour is excluded for evolutions inside (as considered in [**23**]) or outside the cylinder independent of the dimension of the set. Although excluding tilting does not provide a uniform bound, it states that for all times of existence the surfaces remain graphs. This holds even if they are defined outside the cylinder on an infinite domain.

We continue with the results required for Proposition 6.7. We are working with a surface of contact Σ which is rotationally symmetric, and the vector field of rotation is given by the Killing vector field $K_3 : \mathbb{R}^3 \to \mathbb{R}^3$. This together with the tilt point information provides us with an extra property on the first derivatives of components of the second fundamental form of M_t.

Lemma 6.10 (Property of the rotation vector of Σ in a tilt point). *Let F_t satisfy* (54) *or* (56). *Let $X \in \partial_N M_t \subset \Sigma$ be a first point of boundary tilt. Then at X we have*

$$D_{\tau_1} K_3 \in (T_X M_t)^\perp.$$

Proof. Since we know by the Neumann boundary condition that $\partial_N M_t$ is included in Σ we can differentiate relation (52) in directions tangent to the boundary of the moving graphs locally around X. Since the boundary is only one dimensional, this is only in the τ_1 direction. We compute:

$$d(\langle K_3, \nu_\Sigma \rangle)(\tau_1) \ = \ 0,$$
$$\langle \mathrm{D}_{\tau_1} K_3, \nu_\Sigma \rangle \ + \ \langle K_3, \mathrm{D}_{\tau_1} \nu_\Sigma \rangle \ = \ 0.$$

Now using the fact that $\tau_1 \in T_X\Sigma$ and that K_3 is also a tangent vector to Σ we have

$$\langle \mathrm{D}_{\tau_1} K_3, \nu_\Sigma \rangle \ = \ - \ \langle K_3, \mathrm{D}_{\tau_1} \nu_\Sigma \rangle \ = \ - \ A^\Sigma(K_3, \tau_1). \tag{71}$$

Let us recall that X is a tilt point and Proposition 6.8 implies that in such points we have $A^\Sigma(K_3, \tau_1) \ = \ |K_3| \ A^\Sigma(\nu_{M_t}, \tau_1) \ = \ 0$, where we the first equality holds up to a sign. This gives us

$$\langle \mathrm{D}_{\tau_1} K_3, \nu_\Sigma \rangle \ = \ 0. \tag{72}$$

Now from the properties of Killing vector fields in $\mathbb{R}^3$, Proposition 4.1, we also have that

$$\langle \mathrm{D}_{\tau_1} K_3, \tau_1 \rangle \;=\; 0.$$

This together with (72) and the fact that at X we can take as an orthonormal basis of $\mathbb{R}^3$ the basis $\{\tau_1, \nu_\Sigma, \nu_{M_t}\}$ give the desired result. □

Using the above result one is able to compute some of the components of the 3-tensor of the first derivative of the second fundamental form A^Σ of Σ at the tilt point X.

Proposition 6.11 (Property of second fundamental form of Σ in a tilt point). *Let F_t satisfy* (54) *or* (56)*. Let $X \in \partial_N M_t \subset \Sigma$ be a first point of boundary tilt. Then at X we have*

$$(\nabla^\Sigma_{\tau_1} A^\Sigma)(\tau_1, \nu_{M_t}) \;=\; 0,$$

where we have denoted by ∇^Σ the covariant derivative on Σ.

Proof. This proposition uses the results and similar computations as in the proof of Lemma 6.10. We begin by looking at (71). This has been obtained by differentiation of relation (52) on the boundary curve in direction τ_1 tangent to the boundary and tangent to Σ. We can once more differentiate in the same direction of the tangent to the boundary τ_1 to obtain

$$\begin{aligned} &\left\langle \mathrm{D}^2_{\tau_1,\tau_1} K_3, \nu_\Sigma \right\rangle \;+\; \left\langle \mathrm{D}_{\mathrm{D}_{\tau_1}\tau_1} K_3, \nu_\Sigma \right\rangle \;+\; \langle \mathrm{D}_{\tau_1} K_3, \mathrm{D}_{\tau_1}\nu_\Sigma \rangle \;= \\ &-\;(\nabla^\Sigma_{\tau_1} A^\Sigma)(\tau_1, K_3) \;-\; A^\Sigma(\nabla^\Sigma_{\tau_1}\tau_1, K_3) \;-\; A^\Sigma(\nabla^\Sigma_{\tau_1} K_3, \tau_1). \end{aligned} \tag{73}$$

We look at the terms in the above relation and estimate them at our tilt point X. The easiest is the second derivative which completely vanishes by Proposition 4.2, that is

$$\left\langle \mathrm{D}^2_{\tau_1,\tau_1} K_3, \nu_\Sigma \right\rangle = 0.$$

Another term also vanishes:

$$A^\Sigma({\nabla^\Sigma}_{\tau_1} K_3, \tau_1) \;=\; 0,$$

using the result of Lemma 6.10 and the fact that in a tilt point the second fundamental form of the surface Σ is of diagonal form, an argument which can be found in Proposition 6.8.

Let us expand the next term:

$$A^\Sigma(\nabla^\Sigma_{\tau_1}\tau_1, K_3) \;=\; \langle \mathrm{D}_{\tau_1}\tau_1, \tau_1 \rangle \; A^\Sigma(\tau_1, K_3) \;+\; \langle \mathrm{D}_{\tau_1}\tau_1, \nu_{M_t} \rangle \; A^\Sigma(\nu_{M_t}, K_3)$$

where we have used that $\langle \mathrm{D}_{\tau_1}\tau_1, \tau_1 \rangle = \left\langle \nabla^\Sigma_{\tau_1}\tau_1, \tau_1 \right\rangle$ and also $\langle \mathrm{D}_{\tau_1}\tau_1, \nu_{M_t} \rangle = \left\langle \nabla^\Sigma_{\tau_1}\tau_1, \nu_{M_t} \right\rangle$ at X on Σ since ν_{M_t} and τ_1 are tangent to the surface Σ. Also recall that we are working at a point X on the boundary so the third component of an orthonormal basis of $\mathbb{R}^3$ is the normal of the moving graphs ν_{M_t}.

The first inner product is zero at X, using the results found in the proof of Proposition 6.8 where we have up to a sign

$$A^\Sigma(\tau_1, K_3)|_X \;=\; |K_3| \; A^\Sigma(\tau_1, \nu_{M_t})|_X \;=\; 0.$$

The second term is a curvature of M_t at X:

$$\langle \mathrm{D}_{\tau_1}\tau_1, \nu_{M_t} \rangle \;=\; -\; A^{M_t}(\tau_1, \tau_1).$$

Now using the results in Proposition 6.8 we see that this curvature in particular is 0 at X. Combining the arguments above we have that the full term vanishes at X:

$$A^\Sigma(\nabla^\Sigma_{\tau_1}\tau_1, K_3) \;=\; 0.$$

The next term is

$$\langle \mathrm{D}_{\tau_1}K_3, \mathrm{D}_{\tau_1}\nu_\Sigma\rangle \;=\; \langle \mathrm{D}_{\tau_1}K_3, \nu_{M_t}\rangle \; A^\Sigma(\nu_{M_t}, \tau_1) \;=\; 0,$$

where we used the result of Lemma 6.10 and the proof of Proposition 6.8, in particular the fact that the second fundamental form of Σ at a tilt point is diagonal.

Now we look at the last term:

$$\begin{aligned}\left\langle \mathrm{D}_{\mathrm{D}_{\tau_1}\tau_1}K_3, \nu_\Sigma\right\rangle \;=\;& \langle \mathrm{D}_{\tau_1}\tau_1, \tau_1\rangle \; \langle \mathrm{D}_{\tau_1}K_3, \nu_\Sigma\rangle \;+\; \langle \mathrm{D}_{\tau_1}\tau_1, \nu_\Sigma\rangle \; \langle \mathrm{D}_{\nu_\Sigma}K_3, \nu_\Sigma\rangle \\ &+\; \langle \mathrm{D}_{\tau_1}\tau_1, \nu_{M_t}\rangle \; \left\langle \mathrm{D}_{\nu_{M_t}}K_3, \nu_\Sigma\right\rangle .\end{aligned}$$

From the properties of Killing vector fields in $\mathbb{R}^3$ we know that $\langle \mathrm{D}_{\nu_\Sigma}K_3, \nu_\Sigma\rangle \;=\; 0$ so the middle term vanishes. Now looking at the last one it also disappears from Proposition 6.8, as

$$\langle \mathrm{D}_{\tau_1}\tau_1, \nu_{M_t}\rangle\,|_X \;=\; -\; A^{M_t}(\tau_1,\tau_1)|_X \;=\; 0.$$

Finally, the first term is also zero by Lemma 6.10

$$\langle \mathrm{D}_{\tau_1}K_3, \nu_\Sigma\rangle \;=\; \langle \mathrm{D}_{\tau_1}K_3, \nu_{M_t}\rangle \; \langle \nu_{M_t}, \nu_\Sigma\rangle\,.$$

Therefore the only remaining term in (73) at a tilt point X is the one we were interested in, and it must also be zero. ☐

We need to use another result of Stahl [**35**] which we quote below.

Proposition 6.12 (Stahl [**35**], 1994). *Let F_t satisfy* (54) *or* (56)*. On the boundary $\partial_N M_t \subset \Sigma$, the derivatives of components of the second fundamental forms of M_t and Σ are related by the equation:*

$$\begin{aligned}(\nabla_{\tau_1}A^{M_t})(\nu_\Sigma, \tau_1) \;=\;& -\; (\nabla^\Sigma_{\tau_1})A^\Sigma(\nu_{M_t}, \tau_1) \;-\; 2\; A^{M_t}(\tau_1,\tau_1)A^\Sigma(\tau_1,\tau_1) \\ &+\; A^{M_t}(\nu_\Sigma,\nu_\Sigma)\; A^\Sigma(\tau_1,\tau_1) \;+\; A^{M_t}(\tau_1,\tau_1)\; A^\Sigma(\nu_{M_t},\nu_{M_t}).\end{aligned}$$

The next Proposition relates the above result of Stahl and our previous Proposition 6.11.

Proposition 6.13 (Property of derivative of curvature in a first boundary tilt point). *Let F_t satisfy* (54) *or* (56)*. Let $X \in \partial_N M_t \subset \Sigma$ be a first point of boundary tilt. Then at X we have*

$$(\nabla_{\tau_1}A^{M_t})(\tau_1, \nu_\Sigma) \;=\; \nabla_1 h_{12} \;=\; A^{M_t}(\nu_\Sigma,\nu_\Sigma)\; A^\Sigma(\tau_1,\tau_1) \;=\; h_{22}h^\Sigma_{11}$$

where we have denoted by ∇ the covariant derivative on M_t.

Proof. This follows directly by inserting the result of Proposition 6.11 into Proposition 6.12 and observing that in a first boundary tilt point we have that $A^{M_t}(\tau_1,\tau_1) = 0$ by Proposition 6.8. ☐

One result of great help to us is the following proposition which is also due to Stahl [**35**]. It follows simply from differentiating the Neumann boundary condition in time.

PROPOSITION 6.14 (Stahl [**35**], 1994). *Let $X \in \partial_N M_t$. Then*

$$\langle \nabla H, \nu_\Sigma \rangle |_X \;=\; H A^\Sigma(\nu_{M_t}, \nu_{M_t})|_X.$$

This result can be used, as Stahl has done, to preserve the initial sign of the mean curvature in the case of the problem (54). We are ready now to prove Proposition 6.7.

PROOF OF PROPOSITION 6.7. The first property is just the definition of a first boundary tilt point. The next two properties have been proved in Proposition 6.8. Relation (68) follows from the result of Proposition 6.13. We focus now on the rest of the relations. We make use of the above propositions and lemmas.

To prove (67) one has to notice that in the first boundary tilt point the quantity s takes a first minimum. That implies

$$\mathrm{D}^2_{\tau_1,\tau_1} s \geq 0.$$

Using the Weingarten equations we can expand this to

$$\mathrm{D}_{\tau_1}\big(h_{11}\langle \tau_1, e_3\rangle \;+\; h_{12}\langle \nu_\Sigma, e_3\rangle\big) \;\geq\; 0$$

and finally

$$\begin{aligned}
&\nabla_1 h_{11}\langle \tau_1, e_3\rangle \;+\; 2h_{11}\left\langle \nabla^{M_t}_{\tau_1}\tau_1, \tau_1\right\rangle\langle \tau_1, e_3\rangle \;+\; 2h_{12}\left\langle \nabla^{M_t}_{\tau_1}\tau_1, \nu_\Sigma\right\rangle\langle \tau_1, e_3\rangle \\
&+\; h_{11}\langle \mathrm{D}_{\tau_1}\tau_1, e_3\rangle \;+\; \nabla_1 h_{12}\langle \nu_\Sigma, e_3\rangle \;+\; h_{12}\left\langle \nabla^{M_t}_{\tau_1}\tau_1, \tau_1\right\rangle\langle \nu_\Sigma, e_3\rangle \;+\; h_{22}\left\langle \nabla^{M_t}_{\tau_1}\tau_1, \nu_\Sigma\right\rangle\langle \nu_\Sigma, e_3\rangle \\
&+\; h_{11}\left\langle \nabla^{M_t}_{\tau_1}\nu_\Sigma, \tau_1\right\rangle\langle \nu_\Sigma, e_3\rangle \;+\; h_{12}\left\langle \nabla^{M_t}_{\tau_1}\nu_\Sigma, \nu_\Sigma\right\rangle\langle \nu_\Sigma, e_3\rangle \;+\; h_{12}\langle \mathrm{D}_{\tau_1}\nu_\Sigma, e_3\rangle \;\geq\; 0.
\end{aligned}$$

Now using the fact that $h_{11} = h_{12} = 0$ we obtain

$$\nabla_1 h_{11}\langle \tau_1, e_3\rangle \;+\; \nabla_1 h_{12}\langle \nu_\Sigma, e_3\rangle - h_{22}h^\Sigma_{11}\langle \nu_\Sigma, e_3\rangle \;\geq\; 0$$

where we have also used $h^\Sigma_{11} \;=\; -\langle \mathrm{D}_{\tau_1}\tau_1, \nu_\Sigma\rangle \;=\; -\left\langle \nabla^{M_t}_{\tau_1}\tau_1, \nu_\Sigma\right\rangle$. From (68) and the above we can conclude relation (67).

Next we shall prove (70). Using Proposition 6.14, keeping in mind our choice of orthonormal basis where $\tau_2 = \nu_\Sigma$ on the Neumann boundary $\partial_N M_t$, we have

$$\nabla_2 H \;=\; H h^\Sigma_{22} \;=\; (h_{11} + h_{22})h^\Sigma_{22}.$$

The first component of the second fundamental form of M_t vanishes at a point of tilt and simplifies the above to

$$\begin{aligned}
\nabla_2 H \;&=\; h_{22}h^\Sigma_{22}, \text{ and} \\
\nabla_2 h_{11} \;+\; \nabla_2 h_{22} \;&=\; h_{22}h^\Sigma_{22}.
\end{aligned}$$

Applying the Codazzi equation to the last identity gives

$$\nabla_2 h_{22} \;=\; h_{22}h^\Sigma_{22} \;-\; \nabla_1 h_{12},$$

and together with (68) ends the proof of (70).

The last relation to be proven is (69). It is an expression of the fact that at a first boundary tilt point $s = 0$, and so the time derivative of s is non-positive.

$$\frac{d}{dt}s \;\leq\; 0, \text{ so}$$

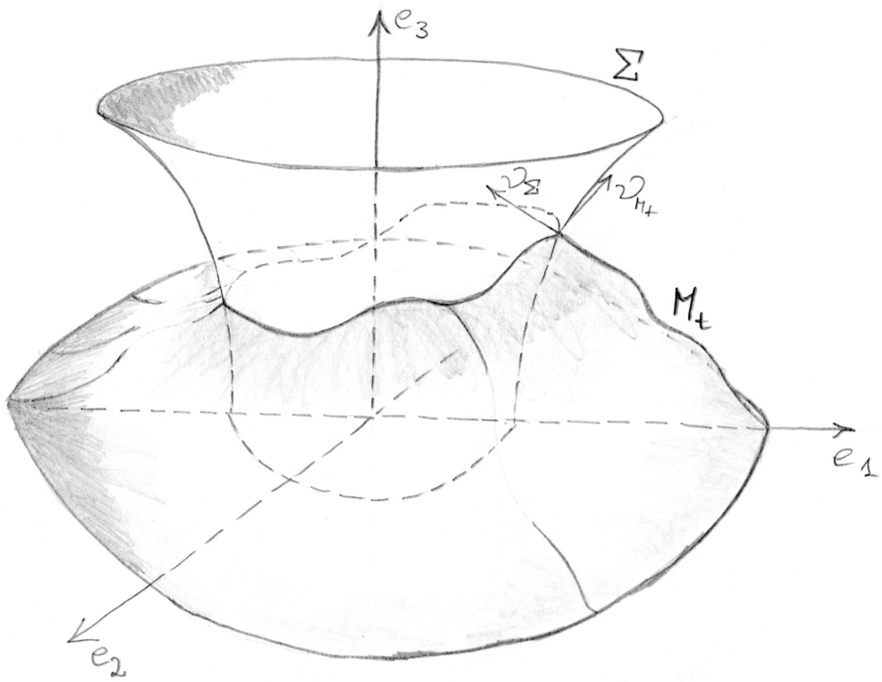

FIGURE 6.3. Motion outside the catenoid neck.

$$\frac{d}{dt}\langle \nu_{M_t}, e_3\rangle \;\leq\; 0.$$

Using the evolution of the unit normal (as can be found in [**21**] for example) we compute

$$\begin{aligned} \langle \nabla H, e_3\rangle \;&\leq\; 0, \\ \nabla_1 H \;\langle \tau_1, e_3\rangle \;+\; \nabla_2 H \;\langle \nu_\Sigma, e_3\rangle \;&\leq\; 0. \end{aligned}$$

From Proposition 6.14 and the fact that $h_{11} = 0$ at X we replace the second term as follows:

$$\begin{aligned} \nabla_1 H \;\langle \tau_1, e_3\rangle \;&\leq\; -h_{22}h^{\Sigma}_{22}\;\langle \nu_\Sigma, e_3\rangle\,, \\ \big(\nabla_1 h_{11} \;+\; \nabla_1 h_{22}\big)\,\langle \tau_1, e_3\rangle \;&\leq\; -h_{22}h^{\Sigma}_{22}\,\langle \nu_\Sigma, e_3\rangle\,. \end{aligned}$$

Replacing the first term on the left side using relation (67) we obtain (69). This ends the proof of the proposition. $\square$

The second case of bad gradient behaviour on the Neumann boundary is when the evolving graphs move towards a point where Σ is horizontal, that is a point where $|\langle \nu_\Sigma, e_3\rangle| = 1$. If the surface Σ contains no horizontal points in the horizontal slab defined by the minimum and maximum height of the initial graph and the surface Σ satisfies the requirements of Proposition 6.1, then the height bound in terms of the supremum of the initial values gives that the graphs do not reach any points where Σ is horizontal.

For example if we have motion of graphs outside the unit sphere in $\mathbb{R}^3$ with an initial height bound for the maximum and minimum of the initial graph below the maximum and minimum height of the sphere respectively, then the graphs do not reach the North or South Pole. If we do not have the initial height bound, then as we have seen in Proposition 4.15 some graphs do move towards the North or South Pole and develop a curvature singularity as well as unbounded gradient.

If Σ does not satisfy (53), then we are still able to exclude such bad behaviour for a certain class of initial graphs even without initial height bounds below the height of points where Σ is horizontal. This result is contained in Theorem 6.17. The class of initial graphs is quite general and it is defined by the inclusion of the origin point of $\mathbb{R}^2$ in the domain of definition of the associated scalar graph. One should think of this as requiring that the scalar graph is defined on a topological two dimensional disc. Also there are some conditions on the contact surface Σ as to where the horizontal points appear. An example for which these conditions are satisfied is the mean curvature flow of graphs inside a torus hole. For this example the following result states that the graph does not reach points where $\nu_\Sigma = e_3$ or $\nu_\Sigma = -e_3$, even if the height is initially much larger than the maximal height of the torus (or smaller than the minimal height).

This result is somewhat surprising, since one can imagine a situation where the initial values are far above the maximal height of the surface Σ, and it might actually be optimal for the moving graphs, in taking minimal area, to slide up to the points where Σ surface becomes horizontal. The theorem does not apply to motion outside the sphere since the domain of definition of the associated scalar graphs is an annulus in $\mathbb{R}^2$.

Let us first define $w : M_t \to \mathbb{R}$ as the length of the projection of the position vector onto the plane of definition of the graphs

$$w(X) \ := \ |F(p,t)|^2 \ - \ u^2(X), \tag{74}$$

where here we used u to denote the height of the graphs and $F(p,t)$ is the position vector. As earlier, below we work in an orthonormal frame $\{\tau_i\}_{1\leq i\leq 2}$ on M_t such that

$$\begin{aligned} \tau_1 \ &\in \ T_X \partial_N M_t, \\ \tau_2 \ &= \ \nu_\Sigma = \nu_{\partial_N M_t}. \end{aligned}$$

on the Neumann boundary $\partial_N M_t \subset \Sigma$. First we compute the evolution of w.

Proposition 6.15 (Evolution of horizontal length). *Let F_t satisfy* (54) *or* (56) *(or any mean curvature flow evolution). Then the horizontal length $w : M_t \to \mathbb{R}$ has the following parabolic evolution*

$$(\frac{d}{dt} \ - \ \Delta^{M_t})w \ \leq \ 0.$$

Proof. Using relation (58) and the evolution of the height function u from (59) we compute

$$\begin{aligned} (\frac{d}{dt} - \Delta^{M_t})w =& 2\left\langle (\frac{d}{dt} - \Delta^{M_t})F, F \right\rangle - 2u(\frac{d}{dt} - \Delta^{M_t})u \\ & - \ 2\sum_{i=1}^{2} \langle \tau_i, \tau_i \rangle \ + \ 2\sum_{i=1}^{2} \left\langle \nabla^{M_t} u, \tau_i \right\rangle^2 \end{aligned}$$

$$= -\ 2\sum_{i=1}^{2}|\tau_i|^2 \ +\ 2\sum_{i=1}^{2}\langle e_3,\tau_i\rangle^2 .$$

Since

$$\langle e_3,\tau_i\rangle^2 \ \leq\ |\tau_i|^2\ |e_3|^2$$

and $|e_3| = 1$ this gives the desired result

$$\big(\frac{d}{dt} - \Delta^{M_t}\big)w \ \leq\ 0.$$

□

REMARK. Note that the previous evolution is strict unless the tangent vector along the boundary is parallel to e_3.

PROPOSITION 6.16 (Derivative normal to the boundary in a horizontal part of Σ). *Let F_t satisfy* (54) *or* (56) *and $w : M_t \to \mathbb{R}$ be the horizontal length defined as above. Let $X \in \partial_N M_t \subset \Sigma$ be a point on the Neumann boundary where Σ has a horizontal point. Then the derivative normal to the boundary $\partial_N M_t$ vanishes at point X:*

$$\langle\nabla w, \nu_{\partial M_t}\rangle \ =\ 0.$$

PROOF. With the above choice of the orthonormal frame as above we compute

$$\langle\nabla w, \nu_{\partial M_t}\rangle \ =\ \langle\nabla w, \tau_2\rangle =\ 2\,\langle\tau_2, F\rangle \ -\ 2u\,\langle\tau_2, e_3\rangle\,,$$

where in the last equality we have used relation (58). At a horizontal point of Σ we have $\nu_\Sigma = -e_3$ or $\nu_\Sigma = e_3$. Using for example the first case and we simplify the above to

$$\langle\nabla w, \nu_{\partial M_t}\rangle \ =\ -\ 2\,\langle e_3, F\rangle \ +\ 2\,\langle F, e_3\rangle\,\langle e_3, e_3\rangle \ =\ 0,$$

where we have also used the definition of the height $u = \langle F, e_3\rangle$. The same applies for the other case. This ends our proof. □

THEOREM 6.17 (Excluding evolution to horizontal points of Σ). *Let Σ be a rotationally symmetric surface in $\mathbb{R}^3$ as in Section 2 and let F_t satisfy* (54) *where M^2 is topologically equivalent to a disk. Suppose $\tilde{X} \in \Sigma$, $\tilde{X} = \big((\tilde{x}_1,\tilde{x}_2), \omega_\Sigma(|(\tilde{x}_1,\tilde{x}_2)|)\big)$ is a point such that $|\,\langle\nu_\Sigma, e_3\rangle\,| = 1$ at $\tilde{X}$ and $|(\tilde{x}_1,\tilde{x}_2)|^2 > \max_{M_0}(F_0^2 - u^2)$. Then the graphs M_t do not reach $\tilde{X}$.*

REMARK. The condition imposed on the initial graph simply states that the Neumann boundary of the graph is away from the horizontal points of Σ. One should keep in mind the picture of a mean curvature flow solution inside a catenoidal neck which flattens out before ∞. The theorem can be applied if the initial graph boundary is away from those horizontal points. The maximum of the initial graph's height can be above those points though and that is what makes this result useful, since only the usual procedure of bounding the height by the initial values does not give us enough information to exclude evolution towards the horizontal parts of Σ.

Proof of Theorem 6.17. The proof is by contradiction. Let us assume that for some time $\tilde{t}$ the surface $M_{\tilde{t}}$ has reached for the first time the point $\tilde{X}$. That implies $\tilde{X} = F(\tilde{p}, \tilde{t}) \in \partial M_{\tilde{t}} \subset \Sigma$ for some $\tilde{p} \in M^2$. From Proposition 6.15 we have a parabolic evolution for the horizontal length w and knowing that this is the first time when our moving surfaces evolve towards a horizontal point of surface Σ, where the length w attains a boundary maximum by the conditions imposed on the point $\tilde{X}$, then the Hopf Lemma, Lemma 2.7, gives

$$\langle \nabla w, \nu_{\partial M_t} \rangle \ > \ 0.$$

This relation is contradicted by the result of Proposition 6.16 and this ends the proof of the theorem. □

Remark. The proof of the above theorem is independent of the graph condition (57) so the result still stands for an immersion evolution. This result is also independent of dimension.

There are two special cases of contact surfaces Σ for which we are able to prove uniform bounds for the gradient. These are treated in the two following sections.

6. Mean curvature flow of graphs with a free boundary on a cylinder

This section treats the problem (54) when the contact surface Σ is a cylinder in $\mathbb{R}^3$, that is mean curvature flow of graphs inside any cylinder in 3-dimensional Euclidean space.

In Proposition 6.9 we proved that titling does not occur in the case where Σ is a cylinder. In this setting we are also able to prove uniform bounds of the height, gradient and mean curvature. This result is the equivalent of the one found in [**23**], but obtained without the use of integral estimates. We want to remind the reader that the unit normal to the cylinder is taken such that it points away from the evolving surfaces.

The uniform estimates are contained in the theorem below.

Theorem 6.18 (Uniform bounds for mean curvature flow with free boundary on a cylinder). *Let Σ be a cylinder in $\mathbb{R}^3$ such that $A^{\Sigma}(K_3, K_3) > 0$ and $\langle \nu_\Sigma, e_3 \rangle = 0$. Let F_t satisfy* (54) *with F_0 satisfying the initial graph condition* (57). *Then the following uniform bounds hold*

$$\sup_{M_t} |u| \ \leq \ \sup_{M_0} |u|, \tag{75}$$

$$\inf_{M_t} s \ \geq \ \inf_{M_0} s, \tag{76}$$

$$\sup_{M_t} \left| \frac{H}{s} \right| \ \leq \ \sup_{M_0} \left| \frac{H}{s} \right|, \tag{77}$$

for all times $t \in [0, T)$.

Proof. The proof is based on repeated applications of the maximum principle. For the height bound (75) we only need to apply Proposition 6.1 after noticing that the cylinder Σ satisfies (53).

The gradient bound (76), which is equivalent to a lower bound on the $s = \langle \nu_{M_t}, e_3\rangle$ quantity comes from the following. Using the parabolic evolution of the s (from Proposition 6.5) we have

$$\inf_{M_t} s \;\geq\; \min\big\{\inf_{M_0} s,\; \inf_{\partial_N M_t} s\big\}.$$

Suppose that there exists a point $X^* = F(p^*, t^*) \in \partial_N M_{t^*}$ such that s attains a new minimum value at X^*. Then by applying the Hopf lemma 2.11 with a choice of orthonormal frame $\{\tau_1,\ \tau_2\}$ of TM_t such that $\tau_1|_{\partial_N M_t} \in T\partial_N M_t$ and $\tau_2|_{\partial_N M_t} = \nu_\Sigma$ we have at X^*

$$\begin{aligned} 0 \;>\; \nabla_{\nu_\Sigma} s \;&=\; \nabla_{\nu_\Sigma} \langle \nu_{M_t}, e_3\rangle \;=\; h_{12}\,\langle \tau_1, e_3\rangle \;+\; h_{22}\,\langle \nu_\Sigma, e_3\rangle \\ &=\; h_{12}\,\langle \tau_1, e_3\rangle\,, \end{aligned}$$

where we have used the definition of the second fundamental form and, in the last equality, the fact that Σ is a cylinder with e_3 being a tangent direction. Using Proposition 4.10 we relate the curvature h_{12} to the h^Σ_{12} and compute this curvature in the basis formed by the principle directions of the cylinder, $\frac{K_3}{|K_3|}$ and e_3:

$$\begin{aligned} h_{12} \;&=\; -h^\Sigma_{12} = -A^\Sigma\big(\tau_1, \nu_{M_t}\big) \\ &= -A^\Sigma\Big(\frac{K_3}{|K_3|}, \frac{K_3}{|K_3|}\Big)\left\langle \tau_1, \frac{K_3}{|K_3|}\right\rangle \left\langle \nu_{M_t}, \frac{K_3}{|K_3|}\right\rangle - A^\Sigma\Big(\frac{K_3}{|K_3|}, e_3\Big)\left\langle \tau_1, \frac{K_3}{|K_3|}\right\rangle \langle \nu_{M_t}, e_3\rangle \\ &\quad - A^\Sigma\Big(e_3, \frac{K_3}{|K_3|}\Big)\langle \tau_1, e_3\rangle \left\langle \nu_{M_t}, \frac{K_3}{|K_3|}\right\rangle - A^\Sigma(e_3, e_3)\,\langle \tau_1, e_3\rangle\,\langle \nu_{M_t}, e_3\rangle \\ &= -A^\Sigma\Big(\frac{K_3}{|K_3|}, \frac{K_3}{|K_3|}\Big)\left\langle \tau_1, \frac{K_3}{|K_3|}\right\rangle \left\langle \nu_{M_t}, \frac{K_3}{|K_3|}\right\rangle. \end{aligned}$$

The Hopf lemma result translates to

$$0 \;>\; -\,A^\Sigma\Big(\frac{K_3}{|K_3|}, \frac{K_3}{|K_3|}\Big)\left\langle \tau_1, \frac{K_3}{|K_3|}\right\rangle \left\langle \nu_{M_t}, \frac{K_3}{|K_3|}\right\rangle \langle \tau_1, e_3\rangle\,. \tag{78}$$

We also express $\frac{K_3}{|K_3|}$ in the orthonormal basis of $\mathbb{R}^3$ given by $\{\tau_1,\ \nu_{M_t},\ \nu_\Sigma\}$

$$\begin{aligned} \frac{K_3}{|K_3|} \;&=\; \left\langle \tau_1, \frac{K_3}{|K_3|}\right\rangle \tau_1 \;+\; \left\langle \nu_{M_t}, \frac{K_3}{|K_3|}\right\rangle \nu_{M_t} \;+\; \left\langle \nu_\Sigma, \frac{K_3}{|K_3|}\right\rangle \nu_\Sigma \\ &=\; \left\langle \tau_1, \frac{K_3}{|K_3|}\right\rangle \tau_1 \;+\; \left\langle \nu_{M_t}, \frac{K_3}{|K_3|}\right\rangle \nu_{M_t}, \end{aligned}$$

where we have used that K_3 is always tangent to the cylinder. Combining $\langle K_3, e_3\rangle = 0$ with the above implies

$$-\left\langle \tau_1, \frac{K_3}{|K_3|}\right\rangle \langle \tau_1, e_3\rangle \left\langle \nu_{M_t}, \frac{K_3}{|K_3|}\right\rangle \;=\; \left\langle \nu_{M_t}, \frac{K_3}{|K_3|}\right\rangle^2 \langle \nu_{M_t}, e_3\rangle \;\geq\; 0.$$

returning to (78) this implies that the curvature coming from the rotation of the cylinder satisfies

$$A^\Sigma(K_3, K_3) \;<\; 0,$$

and this contradicts our hypothesis imposed on the cylinder Σ. Thus there does not exist y minimum of the quantity s on the Neumann boundary, and so we have for all times the gradient bound (76):

$$\inf_{M_t} s \;\geq\; \inf_{M_0} s.$$

The estimate (77) follows again from an application of the maximum principle. After the same computation as in [**9**] and using the evolution of the mean curvature found in [**22**] we can see that $\frac{H^2}{s^2}$ satisfies a parabolic evolution in the interior

$$\Big(\frac{d}{dt} - \Delta^{M_t}\Big)\frac{H^2}{s^2} \;\leq\; 2\frac{\nabla s}{s}\cdot\nabla\frac{H^2}{s^2}.$$

From the above evolution and the use of the maximum principle Theorem 2.9 with the bounded vector field $a = \frac{\nabla s}{s}$, we see that so long as we exclude maximums of the above quantity on the boundary we obtain the theorem.

Suppose that there exists a point $X^* = F(p^*, t^*) \in \partial_N M_{t^*}$ such that $\frac{H^2}{s^2}$ attains a maximum value at X^*. From the Hopf lemma we have at X^*

$$0 \;<\; \nabla_{\nu_\Sigma}\frac{H^2}{s^2} \;=\; 2\frac{H}{s^2}\nabla_{\nu_\Sigma}H \;-\; 2\frac{H^2}{s^3}\nabla_{\nu_\Sigma}s. \tag{79}$$

From Proposition 6.14 the first term becomes

$$\begin{aligned} 2\frac{H}{s^2}\nabla_{\nu_\Sigma}H &= 2\frac{H^2}{s^2}A^\Sigma(\nu_{M_t},\nu_{M_t}) \\ &= 2\frac{H^2}{s^2}A^\Sigma\Big(\frac{K_3}{|K_3|},\frac{K_3}{|K_3|}\Big)\left\langle \nu_{M_t}, \frac{K_3}{|K_3|}\right\rangle^2, \end{aligned}$$

where for the last equality we have used, just as before, the decomposition of the unit normal ν_{M_t} in the basis formed by the principle directions of the cylinder.

The second term in (79) can be computed using the same discussion as in the bound (76) as follows

$$\begin{aligned} -\,2\frac{H^2}{s^3}\nabla_{\nu_\Sigma}s &= -\,2\frac{H^2}{s^3}A^\Sigma\Big(\frac{K_3}{|K_3|},\frac{K_3}{|K_3|}\Big)\left\langle \nu_{M_t}, \frac{K_3}{|K_3|}\right\rangle^2\langle\nu_{M_t},e_3\rangle \\ &= -\,2\frac{H^2}{s^2}A^\Sigma\Big(\frac{K_3}{|K_3|},\frac{K_3}{|K_3|}\Big)\left\langle \nu_{M_t}, \frac{K_3}{|K_3|}\right\rangle^2, \end{aligned}$$

where in the last equality we have also used the definition of $s = \langle\nu_{M_t}, e_3\rangle$. Replacing the two terms in (79) we obtain

$$0 \;<\; \nabla_{\nu_\Sigma}\frac{H^2}{s^2} \;=\; 0.$$

This contradicts the assumption of a maximum of $\frac{H^2}{s^2}$ on the Neumann boundary and thus we have the claimed uniform bound (77):

$$\sup_{M_t}\Big|\frac{H}{s}\Big| \;\leq\; \sup_{M_0}\Big|\frac{H}{s}\Big|.$$

□

Remark (Mean curvature bound). From the bound (77) and $s = \langle \nu_{M_t}, e_3 \rangle \leq 1$ one can easily obtain an uniform bound on the mean curvature of the evolving surfaces

$$\sup_{M_t} |H| \;\leq\; \sup_{M_0} \left| \frac{H}{s} \right|. \tag{80}$$

Theorem 6.19 (Long time existence for mean curvature flow with free boundary on a cylinder). *Let Σ be a cylinder in $\mathbb{R}^3$ such that $A^{\Sigma}(K_3, K_3) > 0$ and $\langle \nu_\Sigma, e_3 \rangle = 0$. Let F_t satisfy* (54) *with F_0 satisfying the initial graph condition* (57). *Then there exists a solution for all times and it converges as $t \to \infty$ to a disk.*

Proof. Once one has obtained uniform height and gradient estimates, the proof of the theorem is completely standard. For example, in Huisken [**23**] one can see that uniform bounds on the height and gradient are sufficient for the long time existence of graphs evolving by mean curvature flow with a ninety degree angle condition on a cylinder.

For completeness we present here the most important steps in this well-known parabolic program for obtaining long time existence from uniform bounds for height and gradient. The scalar graph problem associated with (54) fits into the frame of work found in Chapter 6 of [**27**]. The uniform estimates on height and gradient give us a uniform $H^{1+\alpha}$ estimate for the solution by applying Theorem 1.1 of Chapter 6 in [**27**] for any time $t \geq 0$ and in any interior point. The estimate can be easily extended to the boundary by making use Theorem 2.1 of Chapter 6 in [**27**] and our uniform bound on $\frac{H^2}{s^2}$. This bound is in fact a uniform bound on the time derivative of the scalar graph function, and can also be obtained from the uniform bounds on the height and gradient. This implies that for any time $t > 0$ we have $H^{1+\alpha}$ estimates of the solution up to and including the boundary. Thus the coefficients of our problem are at least Hölder continuous. The long time existence now follows from Corollary 8.10 in [**30**]. The proof is based on considering the problem as uniformly parabolic, linear with Hölder continuous coefficients, obtaining good estimates, and applying the Arzelá-Ascoli theorem in combination with Theorem 3.8.

Now since the area is uniformly bounded, Proposition 6.28 implies that the solution is a minimal surface. Comparison with moving planes or an estimate of $\int_{M_t} v d\mu$ as in (12) of [**23**] implies that the gradient of the solution approaches zero, and so it must be a disk. ☐

7. Mean curvature flow of graphs with free boundary on a hyperplane

Suppose Σ is a hyperplane in $\mathbb{R}^{n+1}$ with $n \geq 2$ defined by

$$\langle \nu_\Sigma, e_{n+1} \rangle \;=\; 0,$$

everywhere on Σ. Define the two half spaces generated by Σ in $\mathbb{R}^{n+1}$ as $\mathbb{R}^{n+1}_+$ and $\mathbb{R}^{n+1}_-$. Suppose M^n is a smooth, orientable n-dimensional manifold with smooth boundary $\partial_N M^n$ and set $M_0 := F_0(M^n) \subset \mathbb{R}^{n+1}_+$ where $F_0 : M^n \to \mathbb{R}^{n+1}$ is a smooth embedding satisfying

$$\begin{aligned} \partial_N M_0 \equiv F_0(\partial_N M^n) \;&=\; M_0 \cap \Sigma, \\ \langle \nu_{M_0}, \nu_\Sigma \circ F_0 \rangle (p) \;&=\; 0 \;\forall p \;\in \partial_N M^n, \end{aligned}$$

where we have denoted by ν_{M_0} the unit normal vector field on M_0.

Let $I \subset \mathbb{R}$ be an open interval containing zero and let $F_t = F(\cdot, t) : M^n \to \mathbb{R}^{n+1}$, $t \in I$ be a one-parameter family of smooth embeddings. The family of hypersurfaces $(M_t)_{t\in I}$, where $M_t = F_t(M^n)$, is said to be evolving by mean curvature flow with free Neumann boundary condition on Σ if

$$
\begin{aligned}
\frac{\partial F}{\partial t}(p,t) &= -\, H(p,t)\nu_{M_t},\ \forall (p,t) \in M^n \times I,\\
F(\cdot,0) &= F_0,\\
F(p,t) &\subset \Sigma,\ \forall (p,t) \in \partial M^n \times I,\\
\left\langle \nu_{M_t}, \nu_\Sigma \circ F \right\rangle (p,t) &= 0,\ \forall (p,t) \in \partial_N M^n \times I,
\end{aligned}
\tag{81}
$$

where we have denoted by ν_{M_t} the unit normal to M_t and by H the mean curvature of M_t.

For this problem we are able to prove uniform bounds for the gradient of the associated scalar graph problem as well as uniform bounds on the mean curvature which together with an initial convexity or concavity condition brings us to long time existence. This result is the most natural next step from the result on entire graphs of Ecker and Huisken [**9**]. The restriction of convexity or concavity is not used to obtain uniform bounds, rather it enters when one desires long time existence for the graphs without an initial height bound. The idea is that this allows one to bound the full second fundamental form in terms only of the mean curvature.

Note that the higher dimensional analogues of the s and u quantities are

$$s = \left\langle \nu_{M_t}, e_{n+1} \right\rangle, \text{ and } u = \left\langle F, e_{n+1} \right\rangle,$$

respectively.

THEOREM 6.20 (Uniform bounds for mean curvature flow of graphs in a half space). *Let Σ be a hyperplane in $\mathbb{R}^{n+1}$ with $n \geq 2$ and $\langle \nu_\Sigma, e_{n+1}\rangle = 0$. Let F_t satisfy* (81) *with F_0 satisfying the initial graph condition*

$$\left\langle \nu_{M_0}, e_{n+1} \right\rangle \ \geq\ C\ >\ 0,$$

everywhere on M_0. Then the following uniform bounds hold

$$\sup_{M_t} |u| \ \leq\ \sup_{M_0} |u|, \tag{82}$$

$$\inf_{M_t} s \ \geq\ \inf_{M_0} s, \tag{83}$$

$$\sup_{M_t} \left|\frac{H}{s}\right| \ \leq\ \sup_{M_0} \left|\frac{H}{s}\right| \tag{84}$$

for all times $t \in [0, T)$.

PROOF. The proof is based again on repeated applications of the maximum principle. For the height bound (82) we only need to apply Proposition 6.1 in the n-dimensional setting by noticing that the hyperplane Σ satisfies the n-dimensional version of (53).

The gradient bound (83), which is equivalent to a lower bound on s,comes as follows. The parabolic evolution of s implies

$$\inf_{M_t} s \ \geq\ \min\left\{ \inf_{M_0} s,\ \inf_{\partial_N M_t} s \right\}.$$

Suppose that there exists a point $X^* = F(p^*, t^*) \in \partial_N M_{t^*}$ such that s attains a new minimum value at X^*. Then by applying the Hopf Lemma 2.11 with a choice of orthonormal frame $\{\tau_i\}_{i=1,n}$ of TM_t such that $\tau_i|_{\partial_N M_t} \in T\partial_N M_t$ for all $i = 1, \ldots, n-1$ and $\tau_n|_{\partial_N M_t} = \nu_\Sigma$ we have at X^*

$$\begin{aligned} 0 \;>\; \nabla_{\nu_\Sigma} s \;&=\; \nabla_{\nu_\Sigma} \langle \nu_{M_t}, e_{n+1} \rangle \;=\; \sum_{i=1}^{n-1} h_{in} \langle \tau_i, e_{n+1} \rangle \;+\; h_{nn} \langle \nu_\Sigma, e_{n+1} \rangle \\ &=\; \sum_{i=1}^{n-1} h_{in} \langle \tau_i, e_{n+1} \rangle \,, \end{aligned}$$

where we have used the definition of the second fundamental form and, in the last equality, the fact that Σ is a hyperplane with $e_{n+1} \in T\Sigma$. Using Proposition 4.10 (this holds without any difficulty in higher dimensions) we shall relate the curvatures h_{in} to the curvatures h^Σ_{in} of Σ, for each $i = 1, \ldots, n-1$. Since Σ is a hyperplane and $\{\tau_i, \nu_{M_t}\}$ for $i = 1, \ldots, n-1$ forms an orthonormal basis of $T\Sigma$ we have that

$$h_{in} \;=\; -\, h^\Sigma_{in} \;=\; 0,$$

for all $i = 1, \ldots, n-1$. This transforms the result of the Hopf lemma to

$$0 \;>\; \nabla_{\nu_\Sigma} s \;=\; 0,$$

and contradicts the existence of a minimum of s on the Neumann boundary. Thus we have the uniform gradient bound (83)

$$\inf_{M_t} s \;\geq\; \inf_{M_0} s.$$

The estimate (84) follows again from an application of the maximum principle. After the same computation as in [**9**] and using the evolution of the mean curvature found in [**22**] we can see that $\frac{H^2}{s^2}$ satisfies

$$\Big(\frac{d}{dt} - \Delta^{M_t}\Big)\frac{H^2}{s^2} \;\leq\; 2\frac{\nabla s}{s} \cdot \nabla \frac{H^2}{s^2}.$$

From the above evolution and the use of the maximum principle with the bounded vector field $a = \frac{\nabla s}{s}$, we see that so long as we exclude maximums of the above quantity on the boundary we obtain the desired uniform estimate.

Consider the same orthonormal frame $\{\tau_i\}_{i=1,n}$ of the tangent space TM_t as before. Suppose that there exists a point $X^* = F(p^*, t^*) \in \partial_N M_{t^*}$ such that $\frac{H^2}{s^2}$ attains a new maximum value at X^*. From the Hopf Lemma we have at X^*

$$0 \;<\; \nabla_{\nu_\Sigma} \frac{H^2}{s^2} \;=\; 2\frac{H}{s^2} \nabla_{\nu_\Sigma} H \;-\; 2\frac{H^2}{s^3} \nabla_{\nu_\Sigma} s. \tag{85}$$

Now note that Proposition 6.14 also obviously holds in higher dimensions. Thus the first term becomes

$$2\frac{H}{s^2} \nabla_{\nu_\Sigma} H \;=\; 2\frac{H^2}{s^2} A^\Sigma(\nu_{M_t}, \nu_{M_t}) \;=\; 0$$

where for the last equality we have used the fact that Σ is a hyperplane.

The second term in (85) also vanishes from the same discussion as in the proof of the bound (83)

$$-\, 2\frac{H^2}{s^3}\nabla_{\nu_\Sigma} s \;=\; 0.$$

Replacing the two terms in (85) we obtain

$$0 \;<\; \nabla_{\nu_\Sigma}\frac{H^2}{s^2} \;=\; 0.$$

Thus there does not exist any new maxima of $\frac{H^2}{s^2}$ on the Neumann boundary and we obtain (84):

$$\sup_{M_t}\Big|\frac{H}{s}\Big| \;\le\; \sup_{M_0}\Big|\frac{H}{s}\Big|.$$

□

REMARK (Mean curvature bound for mean curvature flow of graphs in a half space). From the bound (84) and $s = \langle \nu_{M_t}, e_{n+1}\rangle \le 1$ we obviously also have

$$\sup_{M_t}|H| \;\le\; \sup_{M_0}\Big|\frac{H}{s}\Big|. \tag{86}$$

In the case of Σ being a hyperplane we can prove also that the mean curvature decays in time.

PROPOSITION 6.21 (Decay of mean curvature). *Let Σ be a hyperplane in $\mathbb{R}^{n+1}$ with $n \ge 2$ and $\langle \nu_\Sigma, e_{n+1}\rangle = 0$. Let F_t satisfy* (81) *with F_0 satisfying the initial graph condition*

$$\langle \nu_{M_0}, e_{n+1}\rangle \;\ge\; C \;>\; 0,$$

everywhere on M_0. Then the mean curvature satisfies

$$\frac{2t}{n}H^2 \;\le\; \sup_{M_0}\frac{1}{s^2} - 1,$$

for all times $t \ge 0$.

PROOF. Following the same idea as Ecker and Huisken [**9**] we can compute the evolution on M_t of $\frac{2t}{n}\frac{H^2}{s^2}+\frac{1}{s^2}$ using the evolution of $\frac{H^2}{s^2}$ computed in the previous theorem:

$$\Big(\frac{d}{dt} - \Delta^{M_t}\Big)\Big(\frac{2t}{n}\frac{H^2}{s^2} + \frac{1}{s^2}\Big) \;\le\; 2\frac{\nabla s}{s}\cdot\nabla\frac{2t}{n}\frac{H^2}{s^2} \;+\; \frac{2}{n}\frac{H^2}{s^2} \;-\; \frac{2}{s^2}|A|^2 \;-\; 6\frac{|\nabla s|^2}{s^4}.$$

Using the fact that $\frac{1}{n}H^2 \le |A|^2$ we simplify the above to

$$\Big(\frac{d}{dt} - \Delta^{M_t}\Big)\Big(\frac{2t}{n}\frac{H^2}{s^2} + \frac{1}{s^2}\Big) \;\le\; 2\frac{\nabla s}{s}\cdot\nabla\Big(\frac{2t}{n}\frac{H^2}{s^2} + \frac{1}{s^2}\Big).$$

This implies that we can apply the maximum principle, Theorem 2.9, and bound $\frac{2t}{n}\frac{H^2}{s^2}+\frac{1}{s^2}$ by the maximum between the supremum of the initial values and the supremum on the boundary.

We shall contradict the appearance of new maxima on the Neumann boundary. Suppose that there exists a point $X^* = F(p^*, t^*) \in \partial_N M_{t^*}$ such that $\frac{2t}{n}\frac{H^2}{s^2} + \frac{1}{s^2}$ attains a new maximum value at X^*. From the Hopf Lemma we have at X^*

$$0 \;<\; \nabla_{\nu_\Sigma}\left(\frac{2t}{n}\frac{H^2}{s^2} + \frac{1}{s^2}\right) \;=\; \frac{2t}{n}\nabla_{\nu_\Sigma}\frac{H^2}{s^2} \;+\; \nabla_{\nu_\Sigma}\frac{1}{s^2} \;=\; 0. \tag{87}$$

Where in the last equality we have used (cf. the proof of Proposition 6.20) Proposition 6.14 and the fact that Σ is a hyperplane. This contradicts the existence of a new maximum on the boundary for our quantity and thus we conclude

$$\sup_{M_t}\left(\frac{2t}{n}\frac{H^2}{s^2} + \frac{1}{s^2}\right) \;\leq\; \sup_{M_0}\frac{1}{s^2}.$$

Using $s \leq 1$ we can simplify the above to

$$\frac{2t}{n}H^2 \;\leq\; s^2\,\sup_{M_0}\frac{1}{s^2} \;-\; 1 \;\leq\; \sup_{M_0}\frac{1}{s^2} \;-\; 1.$$

This completes the proof. $\square$

We know from Theorem 10.4 of Stahl [**35**] that when Σ is umbilic, an initially convex (or concave) hypersurface, in the sense that all the eigenvalues of the second fundamental form are positive, remains so under the flow for all times of existence.

THEOREM 6.22 (Stahl [**35**], 1994, Convexity). *Let Σ be umbilic and let F_t satisfy* (54) *with M_0 convex in the sense of eigenvalues. Then M_t is convex for all $t \in [0, T]$.*

This theorem together with the uniform bound on the mean curvature from (86) provides us with a uniform bound on the second fundamental form.

PROPOSITION 6.23 (Second fundamental form bound for mean curvature flow of graphs in a half space). *Let Σ be a hyperplane in $\mathbb{R}^{n+1}$ with $n \geq 2$ and $\langle \nu_\Sigma, e_{n+1}\rangle = 0$. Let F_t satisfy* (81) *with M_0 convex or concave in the sense of eigenvalues. Then*

$$\sup_{M_t}|A|^2 \;\leq \sup_{M_0}\left|\frac{H}{s}\right|^2,$$

for all times $t \in [0, T)$.

To prove long time existence for this problem we make use of the following result of Hamilton.

LEMMA 6.24 (Hamilton [**19**], 1982, Equivalent metrics). *Let g_{ij} be a time dependent metric on M for $0 \leq t < T \leq \infty$. Suppose*

$$\int_{t=0}^{T}\max_M\left|\frac{\partial}{\partial t}g_{ij}\right|_M d\tau \leq C < \infty.$$

Then the metrics $g_{ij}(t)$ are at each time equivalent, and they converge as $t \to T$ uniformly to a positive-definite metric tensor $g_{ij}(T)$ which is continuous and also equivalent.

REMARK. In the above lemma the norm of the time derivative of the metric is taken with respect with the metric as

$$|\frac{\partial}{\partial t}g_{ij}|_M = \sqrt{g^{ij}g^{kl}\frac{\partial}{\partial t}g_{ik}\frac{\partial}{\partial t}g_{jl}},$$

where the indices are $i, j, k, l = 1, \ldots, n$ and we have used the Einstein summation convention. We can also relate this norm to the second fundamental form using the time derivative of the metric from [**21**],

$$\frac{\partial}{\partial t}g_{ij} = -2Hh_{ij},$$

so

$$|\frac{\partial}{\partial t}g_{ij}|_M = 2\sqrt{H^2 g^{ij}g^{kl}h_{ik}h_{jl}} = 2|H||A|.$$

This result is useful since in some cases one does not have an initial bound on the height. We present here two different results of long time existence.

If we have an initial height bound the result of long time existence follows in a similar way as in the case of a bounded domain of definition for the associated scalar graph evolution. Here the initial height is viewed as the property that initial hypersurface lies between two hyperplanes.

THEOREM 6.25 (Long time existence of graphs in half spaces with free boundary on a hyperplane and initial height bounds). *Let Σ be a hyperplane in $\mathbb{R}^{n+1}$ with $n \geq 2$ and $\langle \nu_\Sigma, e_{n+1}\rangle = 0$. Let F_t satisfy* (81) *with F_0 satisfying an initial height bound*

$$\sup_{M_0}|u| = \sup_{M_0}|\langle F_0, e_{n+1}\rangle| \leq C_0 < \infty.$$

Then there exits a solution for (81) *for all times $t \geq 0$ and it converges to a minimal surface as $t \to \infty$.*

PROOF. The problem (81) is equivalent to a scalar graph evolution on a half space with oblique derivative boundary condition. Since the initial graph has a height bound we can apply Theorem 6.1 in Chapter 4 of [**27**] and obtain a unique solution for the associated linearised problem. The same fixed point argument as in Theorem 3.6 can be applied to obtain a short time solution of the quasilinear scalar graph evolution in a half space with oblique derivative boundary condition. For any time $t \geq 0$ we have H_1 uniform estimates on the solution, which together with the uniform estimate of the time derivative for the scalar graph function, given by relation (84), imply uniform $H_{1+\alpha}$ (where $\alpha \in (0, 1)$) as in Theorem 2.1, Chapter 6 of [**27**]. The $H_{1+\alpha}$ uniform estimates for all times $t \geq 0$ imply long time existence by a standard application of the Arzelá - Ascoli Theorem as in Corollary 8.10, [**30**].

The convergence part of the argument is follows using Proposition 6.21 and taking $t \to \infty$ in

$$H^2 \leq \frac{n}{2t}\big(\sup_{M_0}\frac{1}{s} - 1\big),$$

which implies that $H \to 0$. □

REMARK. Standard comparison with moving planes, as referred to in the introduction, shows that the solution is asymptotic to a plane as $t \to \infty$.

If there is no initial height bound then we are still able to obtain long time existence if we use the result of Hamilton, Lemma 6.24, and impose initial convexity or concavity of the hypersurface.

THEOREM 6.26 (Long time existence of graphs in half spaces with a free boundary on a hyperplane and initial convexity or concavity). *Let Σ be a hyperplane in $\mathbb{R}^{n+1}$ with $n \geq 2$ and $\langle \nu_\Sigma, e_{n+1} \rangle = 0$. Let F_t satisfy* (81) *with M_0 convex (concave) in the sense of eigenvalues then there exits a solution for* (81) *for all times $t \geq 0$.*

PROOF. Even without the initial height bound, (81) is equivalent to a scalar graph evolution. Writing the problem as the evolution of a graph over the initial hypersurface and using the results found in [**27**] we obtain the existence of a short time solution. Suppose that there exists a finite maximal time $T < \infty$ such that the solution for (81) exists only on $[0, T)$. Then for all times in $[0, T)$ we have uniform bounds on s, $\frac{H}{s}$ and H by Theorem 6.20. Also from the result of Stahl, Theorem 6.22 we can preserve the convexity (concavity) of M_t for all times in $[0, T]$ which implies that we get a uniform bound on the second fundamental form $|A|^2$ for all times $[0, T)$ by Proposition 6.23. We can therefore apply Lemma 6.24 and obtain that the hypersurfaces M_t converge uniformly to a limit hypersurface M_T which is H_1, since the limit metric is continuous. This implies that the unit normal on M_T is defined and continuous. Due to the uniform estimates for s we get that on M_T the s quantity is also uniformly bounded which implies as in the proof of Theorem 6.20 that the mean curvature H is bounded. Using the preservation of convexity (concavity) up to the time T we also get the bound on $|A|^2$ and obtain that the limit hypersurface M_T is H_2. We can therefore reapply short time existence and this contradicts the maximality of T. □

8. Additional tools and results

One is able to apply the techniques of the previous section on Neumann gradient bounds to improve the long time existence for some cases of radially symmetric graphs.

The theorem below treats the case of the radially symmetric problem (40). In Chapter 5, when obtaining long time existence for this problem we were forced to impose the condition (43) on Σ. This condition states that the surface Σ has no horizontal points between the maximum and minimum height of the initial graph. This property together with the height bound of the evolving graphs between the initial values prevents the graphs of moving on the Neumann boundary to a point where Σ is horizontal.

However, one can also obtain this result using Theorem 6.17. This theorem tells us that we are able to relax the condition of having no horizontal points for the surface Σ in the range spanned by the initial height of the graph. For problem (40) the initial height of the graph is allowed to be greater than the height of points where Σ is horizontal. The condition has been relaxed to no horizontal points of Σ in the horizontal slab bounded by the maximum and minimum of the Neumann boundary values. As one can see in the remark following Theorem 6.17 this slab can be much smaller than the one which is bounded by the maximum and minimum of the initial height. This allows us to consider contact surfaces Σ which are worse behaved, in the sense of existence of horizontal points.

Theorem 6.27 (Long time existence of radially symmetric graphs - improved conditions). *Let Σ and the graph function ω_0 be defined as in Chapter 5. Define the set*

$$\mathcal{S} = \{y \in Dom(\omega_\Sigma) \ : \ |\omega_\Sigma(y)| \ \leq \ |\omega_0(r_0)|\}$$

where r_0 is as in Theorem 5.2. Then if

$$\omega_\Sigma(y)\frac{d\omega_\Sigma}{dy}(y) \ \geq \ 0 \ \forall y \in \mathcal{S} \tag{88}$$

$$\frac{d\omega_\Sigma}{dy}(y) \ \neq \ 0 \ \forall y \in \mathcal{S} \tag{89}$$

there exists a solution to the problem (40) *for all times and it converges to a minimal surface as $t \to \infty$.*

Proof. The proof follows the exact lines of Theorem 5.2 with the addition that excluding evolution towards the horizontal points of Σ is given by Theorem 6.17 applied to the radially symmetric graph. □

In case we have long time existence for solutions of (1) or (2) it is easy to give sufficient conditions such that the solution converges to a minimal surface. In Chapter 5, we have worked with the most general surfaces Σ which still permit us to obtain long time existence for the evolving radially symmetric graphs. In some cases of radially symmetric graphs, as in Theorem 5.6, we can prove that the graphs converge to constant functions, so minimal surfaces of constant height above the plane of definition as a graph. But there are other cases where the graphs exist for all times but they do not converge to constant functions, as in Theorem 5.5. If we have long time existence and uniformly bounded area then following standard convergence argument can be applied.

Proposition 6.28 (Convergence to minimal surfaces). *Let M_t be a solution of* (1) *or* (2) *which exists for all time. Assume that there exists a global constant $C \ < \ \infty$ with $|M_0| \ \leq \ C$. Then the surfaces M_t converge to a smooth minimal surface M_∞ as $t \to \infty$.*

Proof. The proof of this result is the equivalent of the proof found in Huisken [**23**], but in the setting of immersions evolving by mean curvature flow instead of non-parametric mean curvature flow. We begin by observing that

$$\frac{d}{dt}\int_{M_t} d\mu_t \ = \ - \int_{M_t} H^2 d\mu_t,$$

where we have used the time evolution of the area element, computed as in [**23**]. We integrate in time and obtain

$$\int_0^t \int_{M_t} H^2 d\mu_t \ = \ |M_0| \ - \ |M_t|,$$

where we have denoted by $|M_t| \ = \ \int_{M_t} d\mu_t$ the area of the surface M_t. Using the fact that the initial area is bounded we have that there exists a constant $\tilde{C} \ < \ \infty$ such that

$$\int_0^t \int_{M_t} H^2 d\mu_t \ < \ \tilde{C} \ < \ \infty.$$

Now since we do have long time existence, this means $T = \infty$ we can take time to infinity in the above and conclude

$$\int_0^\infty \int_{M_t} H^2 d\mu_t < \tilde{C} < \infty.$$

Due to the uniform estimates of the long time existence (either uniform bounds on height and gradient or gradient and second fundamental form) one can obtain by smoothness estimates as in [**9**] that the time derivative of H^2 is also bounded globally in time and space. Hence, using the bound on area, the derivative of the map

$$t \rightarrow \int_{M_t} H^2 d\mu_t$$

is uniformly bounded for $t \in [0, \infty)$. This together with the above bound on $\int_0^\infty \int_{M_t} H^2 d\mu_t$ and the non-negativity of the above time map gives us that $\int_{M_t} H^2 d\mu_t \rightarrow 0$ as $t \rightarrow \infty$. And this implies that $H \rightarrow 0$ as time $t \rightarrow \infty$ and ends the proof. □

The next result treats the additional compatibility condition that one must impose on the initial graph at the Dirichlet boundary. To prevent the graphs from moving away from the fixed height at which we want to prescribe them on the Dirichlet boundary of the problem (2) one needs to impose

$$H\big|_{\partial_D M_0} \equiv 0. \tag{90}$$

It is sufficient to impose this condition at the initial time since we can preserve it for all times as shown below.

Proposition 6.29 (Preservation of the compatibility condition at the Dirichlet boundary). *Let M_t satisfy* (2) *and the initial condition* (90)*. Then for all time*

$$H\big|_{\partial_D M_t} \equiv 0.$$

Proof. The proof is just a basic computation, and in principle is contained in the study of the local existence of (2).

On the Dirichlet boundary we have the immersions at any time being equal to the values of the initial immersion at those points, so time independent:

$$F(p,t) = F_0(p), \ \forall (p,t) \in \partial_D M^n \times [0,T).$$

Differentiating the above we obtain

$$\frac{\partial}{\partial t} F \equiv 0 \text{ on } \partial_D M^n \times [0,T).$$

This implies

$$\frac{\partial}{\partial t} F|_{\partial_D M_t} = - H\big|_{\partial_D M_t} \nu_{M_t}|_{\partial_D M_t}.$$

The last two relations give us the conclusion of the proposition. □

APPENDIX A

The sphere problem - more properties of a tilt on the Neumann boundary

1. Introduction

In the previous chapters our goal has been to find the most general conditions for which the problems (54) or (56), starting with an initial graph preserve the graph property and exist for all times.

We have proved in the case of radially symmetric graphs that long time existence can be shown with the expected additional conditions on the contact surfaces Σ.

The second type of condition is assuming that the initial graphs have a reflective symmetry. This type of condition allows us to prove that the graph property is preserved for all times of existence in the case when the contact surface is a sphere. As explained in Chapter 6, for general graphs there are two ways by which the gradient bound on the Neumann boundary is not satisfied. One of them is when the surface Σ contains a horizontal section and the second and most important is the so called tilt behaviour. This is when the moving graph immersions lose the graph property in the sense of the quantity $s = \langle \nu, e_3 \rangle$ becoming 0 by $\nu_{M_t} = \frac{K_3}{|K_3|}$ up to a sign. In this appendix we want to present one additional result which characterises the first tilt point on the Neumann boundary for the motion of graphs outside the unit sphere in $\mathbb{R}^3$. The setting in which we work is applicable to more than one example but the computations become far too difficult if we do not have the extra properties given by the surface of contact Σ being a sphere. If one tries to construct a surface which has a first tilt point on the Neumann boundary for the problem (21), such that this surface is used as an initial surface later, then we find out that in this point the time derivative of the quantity characterising the graph property $s = \langle \nu_{M_t}, e_3 \rangle$ is zero. The bigger picture tells us then that we are not able to construct a counterexample for long time existence of (21) by means of continuous dependence on initial data.

The sections below are organised as follows. We start with the construction of a general local parametrisation of a surface having a ninety degrees contact angle with the surface Σ as in our two problems (54) and (56) and compute in this setting all the necessary quantities: the metric, the components of the second fundamental form and the components of the third tensor which is the first covariant derivative of the second fundamental form.

The next section treats only the case when the surface Σ is a unit sphere in $\mathbb{R}^3$, that is the problem (21) and shows how one can use the first tilt properties from Proposition 6.7 to prove that the time derivative of the s quantity is zero in a first tilt point.

Here I would like to thank Prof. Dr. Ben Andrews for his suggestions, interest and many exciting discussions on this topic during my visits to The Australian National University in Canberra.

2. Construction of the general parametrisation

In this section we construct a general parametrisation of an immersion which has a ninety degrees contact angle with a general surface Σ. The construction is valid for any initial surface which is an embedding and which satisfies the Neumann condition of the two studied problems, (54) and (56). In the setting of this parametrisation we compute all the quantities which are used in Proposition 6.7 to characterise a first point of tilt.

We start by defining the parametrisation $X : I_1 \times I_2 \to \mathbb{R}^3$. We denote by M^* the surface generated by X. Let I_1 and I_2 be two real intervals such that $0 \in I_1$ and $0 \in \overline{I_2} \sim I_2$. We want to construct our parametrisation on a small ball around the tilting point so the construction is defined on a Cartesian product of small length real intervals such that the tilting point occurs at point $(0, 0)$.

Let us define the parametrisation X

$$X(u, v) \;:=\; \sigma(u, v) \;+\; v\; \nu_\Sigma(\sigma(u, v)) \tag{91}$$

where $\sigma : I_1 \times I_2 \to \Sigma$ is a family of curves in Σ. This type of parametrisation constructs the surface M^* starting from a family of curves out of Σ and that is for each v, $\sigma(u, v)$ is a curve in Σ. Every point from M^* found at distance v from Σ is projected onto Σ by the unit normal of the contact surface Σ. At $v = 0$ we find the curve of the boundary of the surface, that is $\sigma(\cdot, 0) = \partial M^* \subset \Sigma$.

For this type of parametrisation we compute all the quantities from the above list of properties found in Proposition 6.7: the metric, curvatures and first derivatives of curvature.

The first condition that the M^* has to satisfy is the Neumann contact angle condition $\langle \nu_{M^*}, \nu_\Sigma \rangle \;=\; 0$. This condition is satisfied if we impose the following restriction on the curve $\sigma(\cdot, 0)$ since as we have seen above the contact with the surface Σ and thus the boundary curve is the curve obtained for $v = 0$

$$\frac{\partial \sigma}{\partial v}|_{v=0} \;\equiv\; 0. \tag{92}$$

Next we compute the metric on M^* at the boundary.

Proposition A.1 (The metric of M^*). *With the above choice of parametrisation the metric on the boundary ∂M^* is given by*

$$\begin{aligned} g_{11} &= \left|\frac{\partial \sigma}{\partial u}\right|^2, \\ g_{12} &= 0, \\ g_{22} &= 1. \end{aligned}$$

PROOF. The first step is to compute the tangent basis for TM^* from the parametrisation definition (91)

$$\begin{aligned} \tau_1 &= \frac{\partial X}{\partial u} = \frac{\partial \sigma}{\partial u} + v\, \mathrm{D}_{\frac{\partial \sigma}{\partial u}} \nu_\Sigma, \\ \tau_2 &= \frac{\partial X}{\partial v} = \frac{\partial \sigma}{\partial v} + v\, \mathrm{D}_{\frac{\partial \sigma}{\partial v}} \nu_\Sigma + \nu_\Sigma(\sigma(u,v)). \end{aligned} \tag{93}$$

These simplify at $v = 0$, which is true on the boundary ∂M^*, and using (92) to

$$\begin{aligned} \tau_1 &= \frac{\partial \sigma}{\partial u}\Big|_{(u,v)=(u,0)}, \\ \tau_2 &= \nu_\Sigma(\sigma(u,0)). \end{aligned} \tag{94}$$

By definition the metric components at the point $(u,v) = (u,0)$ are

$$\begin{aligned} g_{11} &= \langle \tau_1, \tau_1 \rangle = \left|\frac{\partial \sigma}{\partial u}\right|^2_{(u,v)=(u,0)}, \\ g_{12} &= \langle \tau_1, \tau_2 \rangle = \left\langle \frac{\partial \sigma}{\partial u}, \nu_\Sigma \right\rangle = 0, \\ g_{22} &= \langle \tau_2, \tau_2 \rangle = |\nu_\Sigma|^2 = 1. \end{aligned}$$

where we have also used relation (92) and the fact that $\sigma(\cdot, 0)$ is a curve in Σ, so its tangent vectors are perpendicular to the normal of Σ. This ends our proof. □

As one can see the basis of our tangent vectors to the M^* surface generated by the parametrisation is an orthogonal one but not an orthonormal one at the chosen point of tilting. That means that in all our future computations we need to be careful about scaling things in the proper way.

The next step is to choose a unit normal to the M^* surface. This choice has to be in agreement with the extra condition that the patch of surface that we define using the parametrisation is a graph in the e_3 direction, except at the point of tilting $(u,v) = (0,0)$. This means that we need to satisfy relation (64).

Let us make the choice of unit normal to be the normed cross product of the two tangent vectors defined above.

$$\nu_{M^*} = \frac{\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}}{\left|\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}\right|} = \frac{\tau_1 \times \tau_2}{|\tau_1 \times \tau_2|}. \tag{95}$$

The next proposition computes the components of the second fundamental form of the surface M^* on the boundary.

PROPOSITION A.2 (Second fundamental form of M^*). *With the above choice of parametrisation the second fundamental form of the surface M^* on the boundary ∂M^* is given by*

$$\begin{aligned} h_{11} &= -\frac{1}{\left|\frac{\partial \sigma}{\partial u} \times \nu_\Sigma\right|} \left\langle \frac{\partial \sigma}{\partial u} \times \nu_\Sigma, \frac{\partial^2 \sigma}{\partial u^2} \right\rangle, \\ h_{12} &= -\frac{1}{\left|\frac{\partial \sigma}{\partial u} \times \nu_\Sigma\right|} \left\langle \frac{\partial \sigma}{\partial u} \times \nu_\Sigma, D_{\frac{\partial \sigma}{\partial u}} \nu_\Sigma \right\rangle, \end{aligned}$$

$$h_{22} \;=\; -\,\frac{1}{|\frac{\partial\sigma}{\partial u}\times\nu_\Sigma|}\;\left\langle \frac{\partial\sigma}{\partial u}\times\nu_\Sigma, \frac{\partial^2\sigma}{\partial v^2}\right\rangle.$$

PROOF. By definition the components of the second fundamental form are

$$\begin{aligned} h_{11} \;&=\; -\left\langle \nu_{M^*}, \frac{\partial^2 X}{\partial u^2}\right\rangle,\\ h_{12} \;&=\; -\left\langle \nu_{M^*}, \frac{\partial^2 X}{\partial u\partial v}\right\rangle,\\ h_{22} \;&=\; -\left\langle \nu_{M^*}, \frac{\partial^2 X}{\partial v^2}\right\rangle. \end{aligned}$$

So all we have to do is use the choice of unit normal from (95) and compute the second derivatives of the parametrisation.

$$\begin{aligned} \frac{\partial^2 X}{\partial u^2} \;&=\; \frac{\partial}{\partial u}\big(\frac{\partial X}{\partial u}\big) \;=\; \frac{\partial^2\sigma}{\partial u^2} \;+\; v\,\mathrm{D}^2_{\frac{\partial\sigma}{\partial u},\frac{\partial\sigma}{\partial u}}\nu_\Sigma \;+\; v\,\mathrm{D}_{\frac{\partial^2\sigma}{\partial u^2}}\nu_\Sigma,\\ \frac{\partial^2 X}{\partial u\partial v} \;&=\; \frac{\partial}{\partial u}\big(\frac{\partial X}{\partial v}\big) \;=\; \frac{\partial^2\sigma}{\partial u\partial v} \;+\; v\,\mathrm{D}^2_{\frac{\partial\sigma}{\partial u},\frac{\partial\sigma}{\partial v}}\nu_\Sigma \;+\; v\,\mathrm{D}_{\frac{\partial^2\sigma}{\partial u\partial v}}\nu_\Sigma \;+\; \mathrm{D}_{\frac{\partial\sigma}{\partial u}}\nu_\Sigma,\\ \frac{\partial^2 X}{\partial v^2} \;&=\; \frac{\partial}{\partial v}\big(\frac{\partial X}{\partial v}\big) \;=\; \frac{\partial^2\sigma}{\partial v^2} \;+\; \mathrm{D}_{\frac{\partial\sigma}{\partial v}}\nu_\Sigma \;+\; v\,\mathrm{D}^2_{\frac{\partial\sigma}{\partial v},\frac{\partial\sigma}{\partial v}}\nu_\Sigma \;+\; v\,\mathrm{D}_{\frac{\partial^2\sigma}{\partial v^2}}\nu_\Sigma \;+\; \mathrm{D}_{\frac{\partial\sigma}{\partial v}}\nu_\Sigma. \end{aligned} \tag{96}$$

And these become on the boundary ∂M^* where $v=0$ and where we have relation (92)

$$\begin{aligned} \frac{\partial^2 X}{\partial u^2} \;&=\; \frac{\partial^2\sigma}{\partial u^2},\\ \frac{\partial^2 X}{\partial u\partial v} \;&=\; \mathrm{D}_{\frac{\partial\sigma}{\partial u}}\nu_\Sigma,\\ \frac{\partial^2 X}{\partial v^2} \;&=\; \frac{\partial^2\sigma}{\partial v^2} \end{aligned} \tag{97}$$

where we have used the fact that (92) implies

$$\frac{\partial^2\sigma}{\partial u\partial v} \;\equiv\; 0. \tag{98}$$

Returning to the definition of the second fundamental components ends our proof. □

Let us now compute the first derivatives of components of the second fundamental form. By Codazzi we see that we only need to compute four of them, and that is $\nabla_1 h_{11}$, $\nabla_1 h_{12}$, $\nabla_1 h_{22}$ and $\nabla_2 h_{22}$. Also we do not compute the second one since it is only used in the computation of others and it brings no further restrictions on the future construction. We put the remaining three in the following proposition.

PROPOSITION A.3 (First derivative of second fundamental form of M^*). *With the above choice of parametrisation the first covariant derivatives of the second fundamental form of M^* on the boundary ∂M^* are given by*

$$\nabla_1 h_{11} \;=\; -\,\frac{2h_{11}}{|\frac{\partial\sigma}{\partial u}|^2}\;\left\langle\frac{\partial^2\sigma}{\partial u^2},\frac{\partial\sigma}{\partial u}\right\rangle \;+\; 2h^\Sigma_{11}h_{12} \;+\; \frac{\partial}{\partial u}\Big(\frac{1}{|\frac{\partial\sigma}{\partial u}\times\nu_\Sigma|}\Big)\Big|\frac{\partial\sigma}{\partial u}\times\nu_\Sigma\Big|h_{11}$$

$$- \frac{1}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|} \left\langle \frac{\partial\sigma}{\partial u} \times \nu_\Sigma, \frac{\partial^3\sigma}{\partial u^3} \right\rangle - \frac{1}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|^2} h^\Sigma_{12} \left\langle \frac{\partial\sigma}{\partial u} \times \left[\frac{\partial\sigma}{\partial u} \times \nu_\Sigma \right], \frac{\partial^2\sigma}{\partial u^2} \right\rangle, \tag{99}$$

$$\begin{aligned} \nabla_1 h_{22} \;=\; & - \frac{1}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|} \left\langle \frac{\partial^2\sigma}{\partial u^2} \times \nu_\Sigma, \frac{\partial^2\sigma}{\partial v^2} \right\rangle - \frac{h^\Sigma_{12}}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|^2} \left\langle \frac{\partial\sigma}{\partial u} \times \left[\frac{\partial\sigma}{\partial u} \times \nu_\Sigma \right], \frac{\partial^2\sigma}{\partial v^2} \right\rangle \\ & - \frac{1}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|} \left\langle \frac{\partial\sigma}{\partial u} \times \nu_\Sigma, \frac{\partial^3\sigma}{\partial u \partial v^2} \right\rangle - \frac{h_{22}}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|^2} \left\langle \frac{\partial^2\sigma}{\partial u^2} \times \nu_\Sigma, \frac{\partial\sigma}{\partial u} \times \nu_\Sigma \right\rangle \\ & - \frac{2h^\Sigma_{11} h_{12}}{|\frac{\partial\sigma}{\partial u}|^2}, \end{aligned} \tag{100}$$

$$\begin{aligned} \nabla_2 h_{22} \;=\; & - \frac{1}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|} \left\langle D_{\frac{\partial\sigma}{\partial u}} \nu_\Sigma \times \nu_\Sigma, \frac{\partial^2\sigma}{\partial v^2} \right\rangle - \frac{1}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|} \left\langle \frac{\partial\sigma}{\partial u} \times \nu_\Sigma, \frac{\partial^3\sigma}{\partial v^3} \right\rangle \\ & - 3 \frac{1}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|} \left\langle \frac{\partial\sigma}{\partial u} \times \nu_\Sigma, D_{\frac{\partial^2\sigma}{\partial v^2}} \nu_\Sigma \right\rangle - 2h_{22} \left\langle \frac{\partial^2\sigma}{\partial v^2}, \nu_\Sigma \right\rangle \\ & - 2 \frac{h_{12}}{|\frac{\partial\sigma}{\partial u}|^2} \left\langle \frac{\partial^2\sigma}{\partial v^2}, \frac{\partial\sigma}{\partial u} \right\rangle \\ & - \frac{h_{22}}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|^2} \left[\left\langle D_{\frac{\partial\sigma}{\partial u}} \nu_\Sigma \times \nu_\Sigma, \frac{\partial\sigma}{\partial u} \times \nu_\Sigma \right\rangle + \left\langle \frac{\partial\sigma}{\partial u} \times \frac{\partial^2\sigma}{\partial v^2}, \frac{\partial\sigma}{\partial u} \times \nu_\Sigma \right\rangle \right] \end{aligned} \tag{101}$$

where we have neglected to expand some of the terms since they are not useful in the below construction argument.

Proof. To obtain the above relations we shall use the following formula

$$\nabla_k h_{ij} = \frac{\partial}{\partial x_k} (h_{ij}) - A^{M^*} \Big(\Big(\frac{\partial^2 X}{\partial x_k \partial x_i} \Big)^{TM^*}, \frac{\partial X}{\partial x_j} \Big) - A^{M^*} \Big(\frac{\partial X}{\partial x_i}, \Big(\frac{\partial^2 X}{\partial x_k \partial x_j} \Big)^{TM^*} \Big),$$

where $x_i,\ x_k,\ x_j$ can be either u or v and the indices $i,\ j,\ k$ go from 1 to 2.

Let us start with the first relation

$$\nabla_1 h_{11} = \frac{\partial}{\partial u}(h_{11}) - 2\, A^{M^*} \Big(\Big(\frac{\partial^2 X}{\partial u^2} \Big)^{TM^*}, \frac{\partial X}{\partial u} \Big).$$

On the boundary ∂M^*, where $v = 0$ and (92) is true we have $\frac{\partial^2 X}{\partial u^2} = \frac{\partial^2\sigma}{\partial u^2}$, so

$$\Big(\frac{\partial^2 X}{\partial u^2}\Big)^{TM^*} = \left\langle \frac{\partial^2\sigma}{\partial u^2}, \frac{\partial\sigma}{\partial u} \right\rangle \frac{\frac{\partial\sigma}{\partial u}}{|\frac{\partial\sigma}{\partial u}|^2} + \left\langle \frac{\partial^2\sigma}{\partial u^2}, \nu_\Sigma \right\rangle \nu_\Sigma.$$

Then the second term of the relation becomes

$$2\, A^{M^*} \Big(\Big(\frac{\partial^2 X}{\partial u^2} \Big)^{TM^*}, \frac{\partial X}{\partial u} \Big) = 2 \left\langle \frac{\partial^2\sigma}{\partial u^2}, \frac{\partial\sigma}{\partial u} \right\rangle \frac{h_{11}}{|\frac{\partial\sigma}{\partial u}|^2} + 2 \left\langle \frac{\partial^2\sigma}{\partial u^2}, \nu_\Sigma \right\rangle h_{12}.$$

Also here we can use that $\frac{\partial X}{\partial u} = \frac{\partial\sigma}{\partial u}$ on $\partial M^* \subset \Sigma$ is also tangent to Σ to see that

$$\left\langle \frac{\partial^2\sigma}{\partial u^2}, \nu_\Sigma \right\rangle = - h^\Sigma_{11}$$

and expand the second term as

$$2\ A^{M^*}\Big(\ \Big(\frac{\partial^2 X}{\partial u^2}\Big)^{TM^*},\frac{\partial X}{\partial u}\ \Big)\ =\ 2\ \Big\langle\frac{\partial^2\sigma}{\partial u^2},\frac{\partial\sigma}{\partial u}\Big\rangle\ \frac{h_{11}}{|\frac{\partial\sigma}{\partial u}|^2}\ -\ 2\ h^{\Sigma}_{11}\ h_{12}.$$

Now let us return to the first term of the relation and expand it using Proposition A.2

$$\begin{aligned}\frac{\partial}{\partial u}(h_{11})\ &=\ -\frac{\partial}{\partial u}\Big[\ \frac{\Big\langle\frac{\partial\sigma}{\partial u}\times\nu_\Sigma,\frac{\partial^2\sigma}{\partial u^2}\Big\rangle}{|\frac{\partial\sigma}{\partial u}\times\nu_\Sigma|}\Big]\\ &=\ -\frac{\partial}{\partial u}\Big(\ \frac{1}{|\frac{\partial\sigma}{\partial u}\times\nu_\Sigma|}\Big)\ \Big\langle\frac{\partial\sigma}{\partial u}\times\nu_\Sigma,\frac{\partial^2\sigma}{\partial u^2}\Big\rangle\ -\ \frac{1}{|\frac{\partial\sigma}{\partial u}\times\nu_\Sigma|}\frac{\partial}{\partial u}\Big(\ \Big\langle\frac{\partial\sigma}{\partial u}\times\nu_\Sigma,\frac{\partial^2\sigma}{\partial u^2}\Big\rangle\Big).\end{aligned}$$

We leave the first term in the same form and only notice that the inner product is actually a curvature of M^*

$$\Big\langle\frac{\partial\sigma}{\partial u}\times\nu_\Sigma,\frac{\partial^2\sigma}{\partial u^2}\Big\rangle\ =\ -h_{11}\ |\frac{\partial\sigma}{\partial u}\times\nu_\Sigma|.$$

The second term expands as follows

$$\frac{\partial}{\partial u}\Big(\ \Big\langle\frac{\partial\sigma}{\partial u}\times\nu_\Sigma,\frac{\partial^2\sigma}{\partial u^2}\Big\rangle\Big)\ =\ \Big\langle\frac{\partial^2\sigma}{\partial u^2}\times\nu_\Sigma,\frac{\partial^2\sigma}{\partial u^2}\Big\rangle\ +\ \Big\langle\frac{\partial\sigma}{\partial u}\times\mathrm{D}_{\frac{\partial\sigma}{\partial u}}\nu_\Sigma,\frac{\partial^2\sigma}{\partial u^2}\Big\rangle\ +\ \Big\langle\frac{\partial\sigma}{\partial u}\times\nu_\Sigma,\frac{\partial^3\sigma}{\partial u^3}\Big\rangle.$$

First we notice that $\Big\langle\frac{\partial^2\sigma}{\partial u^2}\times\nu_\Sigma,\frac{\partial^2\sigma}{\partial u^2}\Big\rangle\ =\ 0$ and then we expand the derivative of the unit normal to Σ on the boundary ∂M^* in the following way

$$\mathrm{D}_{\frac{\partial\sigma}{\partial u}}\nu_\Sigma\ =\ \Big\langle\mathrm{D}_{\frac{\partial\sigma}{\partial u}}\nu_\Sigma,\nu_{M^*}\Big\rangle\,\nu_{M^*}\ +\ \Big\langle\mathrm{D}_{\frac{\partial\sigma}{\partial u}}\nu_\Sigma,\frac{\partial\sigma}{\partial u}\Big\rangle\ \frac{\frac{\partial\sigma}{\partial u}}{|\frac{\partial\sigma}{\partial u}|^2}\ +\ \Big\langle\mathrm{D}_{\frac{\partial\sigma}{\partial u}}\nu_\Sigma,\nu_\Sigma\Big\rangle\,\nu_\Sigma.$$

The last term vanishes using the fact that ν_Σ is a choice of unit normal to the surface Σ and the rest of the inner products are as follows

$$\begin{aligned}\Big\langle\mathrm{D}_{\frac{\partial\sigma}{\partial u}}\nu_\Sigma,\nu_\Sigma\Big\rangle\ &=\ 0,\\ \Big\langle\mathrm{D}_{\frac{\partial\sigma}{\partial u}}\nu_\Sigma,\nu_{M^*}\Big\rangle\ &=\ h^{\Sigma}_{12},\\ \Big\langle\mathrm{D}_{\frac{\partial\sigma}{\partial u}}\nu_\Sigma,\frac{\partial\sigma}{\partial u}\Big\rangle\ &=\ h^{\Sigma}_{11}.\end{aligned}$$

Returning one step in our computation we get

$$\begin{aligned}\Big\langle\frac{\partial\sigma}{\partial u}\times\mathrm{D}_{\frac{\partial\sigma}{\partial u}}\nu_\Sigma,\frac{\partial^2\sigma}{\partial u^2}\Big\rangle\ &=\ h^{\Sigma}_{12}\ \Big\langle\frac{\partial\sigma}{\partial u}\times\nu_{M^*},\frac{\partial^2\sigma}{\partial u^2}\Big\rangle\ +\ \frac{h^{\Sigma}_{11}}{|\frac{\partial\sigma}{\partial u}|^2}\ \Big\langle\frac{\partial\sigma}{\partial u}\times\frac{\partial\sigma}{\partial u},\frac{\partial^2\sigma}{\partial u^2}\Big\rangle\\ &=\ h^{\Sigma}_{12}\ \Big\langle\frac{\partial\sigma}{\partial u}\times\nu_{M^*},\frac{\partial^2\sigma}{\partial u^2}\Big\rangle.\end{aligned}$$

Now returning the whole way in our computation of $\nabla_1 h_{11}$ and replacing the choice of unit normal to M^* by (95) we obtain the expected result

$$\nabla_1 h_{11}\ =\ -\ \frac{2h_{11}}{|\frac{\partial\sigma}{\partial u}|^2}\ \Big\langle\frac{\partial^2\sigma}{\partial u^2},\frac{\partial\sigma}{\partial u}\Big\rangle\ +\ 2h^{\Sigma}_{11}h_{12}\ +\ \frac{\partial}{\partial u}\Big(\frac{1}{|\frac{\partial\sigma}{\partial u}\times\nu_\Sigma|}\Big)\Big|\frac{\partial\sigma}{\partial u}\times\nu_\Sigma\Big|h_{11}$$

$$- \frac{1}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|} \left\langle \frac{\partial\sigma}{\partial u} \times \nu_\Sigma, \frac{\partial^3\sigma}{\partial u^3} \right\rangle - \frac{1}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|^2} h_{12}^\Sigma \left\langle \frac{\partial\sigma}{\partial u} \times \left[\frac{\partial\sigma}{\partial u} \times \nu_\Sigma\right], \frac{\partial^2\sigma}{\partial u^2} \right\rangle.$$

Again we mention that the terms we did not expand vanish in a tilting point in the following construction so we did not bother with them.

We now do the same for the second derivative of curvature

$$\nabla_1 h_{22} = \frac{\partial}{\partial u}\left(h_{22} \right) - 2\, A^{M^*}\left(\frac{\partial X}{\partial v}, \left(\frac{\partial^2 X}{\partial u \partial v}\right)^{TM^*} \right). \tag{102}$$

Using (96) we see that on the boundary ∂M^* we have

$$\left.\frac{\partial^2 X}{\partial u \partial v}\right|_{M^*} = \mathrm{D}_{\frac{\partial\sigma}{\partial u}} \nu_\Sigma.$$

Now if we decompose this in directions tangent to M^* we have

$$\begin{aligned}\left(\frac{\partial^2 X}{\partial u \partial v}\right)^{TM^*} &= \left\langle \mathrm{D}_{\frac{\partial\sigma}{\partial u}} \nu_\Sigma, \nu_\Sigma \right\rangle \nu_\Sigma + \left\langle \mathrm{D}_{\frac{\partial\sigma}{\partial u}} \nu_\Sigma, \frac{\partial\sigma}{\partial u} \right\rangle \frac{\frac{\partial\sigma}{\partial u}}{|\frac{\partial\sigma}{\partial u}|^2} \\ &= \frac{h_{11}^\Sigma}{|\frac{\partial\sigma}{\partial u}|^2} \frac{\partial\sigma}{\partial u},\end{aligned}$$

where we have used that

$$\left\langle \mathrm{D}_{\frac{\partial\sigma}{\partial u}} \nu_\Sigma, \nu_\Sigma \right\rangle = 0 \text{ and } \left\langle \mathrm{D}_{\frac{\partial\sigma}{\partial u}} \nu_\Sigma, \frac{\partial\sigma}{\partial u} \right\rangle = h_{11}^\Sigma.$$

That implies that the last term in (102) is given by

$$2A^{M^*}\left(\frac{\partial X}{\partial v}, \left(\frac{\partial^2 X}{\partial u \partial v}\right)^{TM^*} \right) = 2\, \frac{h_{11}^\Sigma h_{12}}{|\frac{\partial\sigma}{\partial u}|^2}. \tag{103}$$

We return to the first term in (102) and expand it using Proposition A.2

$$\begin{aligned}\frac{\partial}{\partial u} h_{22} &= \frac{\partial}{\partial u}\left[- \frac{\left\langle \frac{\partial\sigma}{\partial u} \times \nu_\Sigma, \frac{\partial^2\sigma}{\partial v^2} \right\rangle}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|} \right] \\ &= - \frac{\partial\sigma}{\partial u}\left(\frac{1}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|} \right) \left\langle \frac{\partial\sigma}{\partial u} \times \nu_\Sigma, \frac{\partial^2\sigma}{\partial v^2} \right\rangle - \frac{1}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|} \frac{\partial}{\partial u}\left(\left\langle \frac{\partial\sigma}{\partial u} \times \nu_\Sigma, \frac{\partial^2\sigma}{\partial v^2} \right\rangle \right).\end{aligned} \tag{104}$$

Let us compute the two terms separately and we start by looking at the above norm from the first term as a squared root of the inner product of the two vectors

$$\frac{\partial}{\partial u}\left|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma\right| = \frac{\left\langle \frac{\partial^2\sigma}{\partial u^2} \times \nu_\Sigma, \frac{\partial\sigma}{\partial u} \times \nu_\Sigma \right\rangle + \left\langle \frac{\partial\sigma}{\partial u} \times \mathrm{D}_{\frac{\partial\sigma}{\partial u}} \nu_\Sigma, \frac{\partial\sigma}{\partial u} \times \nu_\Sigma \right\rangle}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|}.$$

The second term in the above can be made to vanish by the following decomposition of the derivative $\mathrm{D}_{\frac{\partial\sigma}{\partial u}} \nu_\Sigma$

$$\left\langle \frac{\partial\sigma}{\partial u} \times \mathrm{D}_{\frac{\partial\sigma}{\partial u}} \nu_\Sigma, \frac{\partial\sigma}{\partial u} \times \nu_\Sigma \right\rangle = \left\langle \frac{\partial\sigma}{\partial u} \times \frac{\partial\sigma}{\partial u}, \frac{\partial\sigma}{\partial u} \times \nu_\Sigma \right\rangle \frac{h_{11}^\Sigma}{|\frac{\partial\sigma}{\partial u}|^2}$$

$$+ \left\langle \frac{\partial\sigma}{\partial u} \times \left[\frac{\partial\sigma}{\partial u} \times \nu_\Sigma \right], \frac{\partial\sigma}{\partial u} \times \nu_\Sigma \right\rangle \frac{h^\Sigma_{12}}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|}$$
$$+ \left\langle \frac{\partial\sigma}{\partial u} \times \nu_\Sigma, \frac{\partial\sigma}{\partial u} \times \nu_\Sigma \right\rangle \left\langle \mathrm{D}_{\frac{\partial\sigma}{\partial u}} \nu_\Sigma, \nu_\Sigma \right\rangle$$
$$= 0.$$

where we have used properties of the cross product and the fact that ν_Σ is a choice of unit normal to the contact surface Σ. This give us

$$\frac{\partial}{\partial u}\left|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma\right| = \frac{\left\langle \frac{\partial^2\sigma}{\partial u^2} \times \nu_\Sigma, \frac{\partial\sigma}{\partial u} \times \nu_\Sigma \right\rangle}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|}$$

and also

$$\frac{\partial}{\partial u}\frac{1}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|} = -\frac{\left\langle \frac{\partial^2\sigma}{\partial u^2} \times \nu_\Sigma, \frac{\partial\sigma}{\partial u} \times \nu_\Sigma \right\rangle}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|^3}.$$

We return at (104) to compute the second part

$$\frac{\partial}{\partial u} \left\langle \frac{\partial\sigma}{\partial u} \times \nu_\Sigma, \frac{\partial^2\sigma}{\partial v^2} \right\rangle = \left\langle \frac{\partial^2\sigma}{\partial u^2} \times \nu_\Sigma, \frac{\partial^2\sigma}{\partial v^2} \right\rangle + \left\langle \frac{\partial\sigma}{\partial u} \times \mathrm{D}_{\frac{\partial\sigma}{\partial u}} \nu_\Sigma, \frac{\partial^2\sigma}{\partial v^2} \right\rangle$$
$$+ \left\langle \frac{\partial\sigma}{\partial u} \times \nu_\Sigma, \frac{\partial^2\sigma}{\partial u \partial v^2} \right\rangle.$$

We can expand the middle term by expressing the derivative of the unit normal Σ in terms of the chosen basis of $\mathbb{R}^3$ at points on the boundary ∂M^* where $(u, v) = (u, 0)$, as we have done before.

$$\frac{\partial}{\partial u} \left\langle \frac{\partial\sigma}{\partial u} \times \nu_\Sigma, \frac{\partial^2\sigma}{\partial v^2} \right\rangle = \left\langle \frac{\partial^2\sigma}{\partial u^2} \times \nu_\Sigma, \frac{\partial^2\sigma}{\partial v^2} \right\rangle + \frac{h^\Sigma_{12}}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|} \left\langle \frac{\partial\sigma}{\partial u} \times \left[\frac{\partial\sigma}{\partial u} \times \nu_\Sigma \right], \frac{\partial^2\sigma}{\partial v^2} \right\rangle$$
$$+ \left\langle \frac{\partial\sigma}{\partial u} \times \nu_\Sigma, \frac{\partial^2\sigma}{\partial u \partial v^2} \right\rangle.$$

This gives us that (104) transforms to

$$\frac{\partial}{\partial u} h_{22} = \frac{\left\langle \frac{\partial^2\sigma}{\partial u^2} \times \nu_\Sigma, \frac{\partial\sigma}{\partial u} \times \nu_\Sigma \right\rangle}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|^3} \left\langle \frac{\partial\sigma}{\partial u} \times \nu_\Sigma, \frac{\partial^2\sigma}{\partial v^2} \right\rangle - \frac{1}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|} \Bigg(\left\langle \frac{\partial^2\sigma}{\partial u^2} \times \nu_\Sigma, \frac{\partial^2\sigma}{\partial v^2} \right\rangle$$
$$+ \frac{h^\Sigma_{12}}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|} \left\langle \frac{\partial\sigma}{\partial u} \times \left[\frac{\partial\sigma}{\partial u} \times \nu_\Sigma \right], \frac{\partial^2\sigma}{\partial v^2} \right\rangle + \left\langle \frac{\partial\sigma}{\partial u} \times \nu_\Sigma, \frac{\partial^2\sigma}{\partial u \partial v^2} \right\rangle \Bigg).$$

Replace the above relation and (103) into (102) and we get the second relation stated in the results

$$\nabla_1 h_{22} = -\frac{1}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|} \left\langle \frac{\partial^2\sigma}{\partial u^2} \times \nu_\Sigma, \frac{\partial^2\sigma}{\partial v^2} \right\rangle - \frac{h^\Sigma_{12}}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|^2} \left\langle \frac{\partial\sigma}{\partial u} \times \left[\frac{\partial\sigma}{\partial u} \times \nu_\Sigma \right], \frac{\partial^2\sigma}{\partial v^2} \right\rangle$$
$$- \frac{1}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|} \left\langle \frac{\partial\sigma}{\partial u} \times \nu_\Sigma, \frac{\partial^3\sigma}{\partial u \partial v^2} \right\rangle - \frac{h_{22}}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|^2} \left\langle \frac{\partial^2\sigma}{\partial u^2} \times \nu_\Sigma, \frac{\partial\sigma}{\partial u} \times \nu_\Sigma \right\rangle$$

$$- \frac{2h^{\Sigma}_{11}h_{12}}{|\frac{\partial\sigma}{\partial u}|^2}.$$

We turn our attention now to the third result of the proposition.

$$\nabla_2 h_{22} = \frac{\partial}{\partial v}\big(h_{22} \big) - 2\, A^{M^*}\Big(\Big(\frac{\partial^2 X}{\partial v^2} \Big)^{TM^*}, \frac{\partial X}{\partial v} \Big). \tag{105}$$

The second term is easier to handle in the same way as before. First notice that on the boundary ∂M^*, from (97) we have

$$\frac{\partial^2 X}{\partial v^2}\Big|_{\partial M^*} = \frac{\partial^2 \sigma}{\partial v^2}$$

and this allows us to express the tangential part on the boundary ∂M^* as follows

$$\Big(\frac{\partial^2 X}{\partial v^2} \Big)^{TM^*} = \Big(\frac{\partial^2 \sigma}{\partial v^2} \Big)^{TM^*} = \Big\langle \frac{\partial^2 \sigma}{\partial v^2}, \nu_\Sigma \Big\rangle \nu_\Sigma + \Big\langle \frac{\partial^2 \sigma}{\partial v^2}, \frac{\partial \sigma}{\partial u} \Big\rangle \frac{\frac{\partial\sigma}{\partial u}}{|\frac{\partial\sigma}{\partial u}|^2}.$$

That gives us that the second term in (105) is

$$\begin{aligned} 2\, A^{M^*}\Big(\Big(\frac{\partial^2 X}{\partial v^2} \Big)^{TM^*}, \frac{\partial X}{\partial v} \Big) &= 2\, A^{M^*}\Big(\Big(\frac{\partial^2 X}{\partial v^2} \Big)^{TM^*}, \nu_\Sigma \Big) \\ &= 2\, h_{22} \Big\langle \frac{\partial^2 \sigma}{\partial v^2}, \nu_\Sigma \Big\rangle + 2\, \frac{h_{12} \Big\langle \frac{\partial^2 \sigma}{\partial v^2}, \frac{\partial \sigma}{\partial u} \Big\rangle}{|\frac{\partial\sigma}{\partial u}|^2} \end{aligned} \tag{106}$$

where we have used that $\frac{\partial\sigma}{\partial v}\Big|_{\partial M^*} = \nu_\Sigma$. We return now to expand the first term of (105) and we have to notice that here we can not use the definition of the components of the second fundamental form from Proposition (A.2) since we differentiate in direction v which is normal to the boundary, but the full definition using the tangent vectors of the parametrisation. That goes as follows. First we use the definition of the second fundamental form component

$$h_{22} = - \frac{\Big\langle \frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}, \frac{\partial^2 X}{\partial v^2} \Big\rangle}{|\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}|}$$

and then we differentiate in the v variable

$$\begin{aligned} \frac{\partial}{\partial v} h_{22} = &- \frac{1}{|\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}|} \Big\langle \frac{\partial^2 X}{\partial u \partial v} \times \frac{\partial X}{\partial v}, \frac{\partial^2 X}{\partial v^2} \Big\rangle - \frac{1}{|\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}|} \Big\langle \frac{\partial X}{\partial u} \times \frac{\partial^2 X}{\partial v^2}, \frac{\partial^2 X}{\partial v^2} \Big\rangle \\ &- \frac{1}{|\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}|} \Big\langle \frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}, \frac{\partial^3 X}{\partial v^3} \Big\rangle - \frac{\partial}{\partial v} \frac{1}{|\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}|} \Big\langle \frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}, \frac{\partial^2 X}{\partial v^2} \Big\rangle. \end{aligned} \tag{107}$$

There are four terms that we have to deal with. We start here with the first term by using the form of tangent vectors of M^* and the second derivative of the parametrisation at the boundary ∂M^* from (94) and (97)

$$- \frac{1}{|\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}|} \Big\langle \frac{\partial^2 X}{\partial u \partial v} \times \frac{\partial X}{\partial v}, \frac{\partial^2 X}{\partial v^2} \Big\rangle = - \frac{\Big\langle \mathrm{D}_{\frac{\partial\sigma}{\partial u}} \nu_\Sigma \times \nu_\Sigma, \frac{\partial^2 \sigma}{\partial v^2} \Big\rangle}{|\frac{\partial\sigma}{\partial u} \times \nu_\Sigma|}. \tag{108}$$

The second term vanishes using properties of the cross product

$$-\frac{1}{|\frac{\partial X}{\partial u}\times\frac{\partial X}{\partial v}|}\left\langle\frac{\partial X}{\partial u}\times\frac{\partial^2 X}{\partial v^2},\frac{\partial^2 X}{\partial v^2}\right\rangle = 0. \tag{109}$$

For the third term we need to compute the third derivative of the parametrisation in the v direction starting with the general second derivative from (96)

$$\begin{aligned}\frac{\partial^3 X}{\partial v^3} &= \frac{\partial}{\partial v}\Big(\frac{\partial^2\sigma}{\partial v^2} + \mathrm{D}_{\frac{\partial\sigma}{\partial v}}\nu_\Sigma + v\,\mathrm{D}^2_{\frac{\partial\sigma}{\partial v},\frac{\partial\sigma}{\partial v}}\nu_\Sigma + v\,\mathrm{D}_{\frac{\partial^2\sigma}{\partial v^2}}\nu_\Sigma + \mathrm{D}_{\frac{\partial\sigma}{\partial v}}\nu_\Sigma\Big)\\ &= \frac{\partial^3\sigma}{\partial v^3} + 3\,\mathrm{D}^2_{\frac{\partial\sigma}{\partial v},\frac{\partial\sigma}{\partial v}}\nu_\Sigma + v\,\mathrm{D}^3_{\frac{\partial\sigma}{\partial v},\frac{\partial\sigma}{\partial v},\frac{\partial\sigma}{\partial v}}\nu_\Sigma\\ &\quad + 3\,v\,\mathrm{D}^2_{\frac{\partial^2\sigma}{\partial v^2},\frac{\partial\sigma}{\partial v}}\nu_\Sigma + 3\,\mathrm{D}_{\frac{\partial^2\sigma}{\partial v^2}}\nu_\Sigma + v\,\mathrm{D}_{\frac{\partial^3\sigma}{\partial v^3}}\nu_\Sigma.\end{aligned}$$

On the boundary where (92) is true and also we have $v = 0$ this becomes

$$\frac{\partial^3 X}{\partial v^3} = \frac{\partial^3\sigma}{\partial v^3} + 3\,\mathrm{D}_{\frac{\partial^2\sigma}{\partial v^2}}\nu_\Sigma,$$

and then we can compute the third term as follows

$$\begin{aligned}-\frac{1}{|\frac{\partial X}{\partial u}\times\frac{\partial X}{\partial v}|}\left\langle\frac{\partial X}{\partial u}\times\frac{\partial X}{\partial v},\frac{\partial^3 X}{\partial v^3}\right\rangle &= -\frac{1}{|\frac{\partial\sigma}{\partial u}\times\nu_\Sigma|}\left\langle\frac{\partial\sigma}{\partial u}\times\nu_\Sigma,\frac{\partial^3\sigma}{\partial v^3}\right\rangle\\ &\quad - 3\,\frac{1}{|\frac{\partial\sigma}{\partial u}\times\nu_\Sigma|}\left\langle\frac{\partial\sigma}{\partial u}\times\nu_\Sigma,\mathrm{D}_{\frac{\partial^2\sigma}{\partial v^2}}\nu_\Sigma\right\rangle.\end{aligned} \tag{110}$$

The fourth and last term can be computed as below by first computing the derivative

$$\frac{\partial}{\partial v}\frac{1}{|\frac{\partial X}{\partial u}\times\frac{\partial X}{\partial v}|} = -\frac{\left\langle\frac{\partial^2 X}{\partial v\partial u}\times\frac{\partial X}{\partial v},\frac{\partial X}{\partial u}\times\frac{\partial X}{\partial v}\right\rangle + \left\langle\frac{\partial X}{\partial u}\times\frac{\partial^2 X}{\partial v^2},\frac{\partial X}{\partial u}\times\frac{\partial X}{\partial v}\right\rangle}{|\frac{\partial X}{\partial u}\times\frac{\partial X}{\partial v}|^3}.$$

Using the fact that we are on the boundary ∂M^* where $v = 0$ and (92) is true the fourth term becomes

$$\begin{aligned}-\frac{\partial}{\partial v}\frac{1}{|\frac{\partial X}{\partial u}\times\frac{\partial X}{\partial v}|}\left\langle\frac{\partial X}{\partial u}\times\frac{\partial X}{\partial v},\frac{\partial^2 X}{\partial v^2}\right\rangle &= -\frac{h_{22}}{|\frac{\partial\sigma}{\partial u}\times\nu_\Sigma|^2}\Big[\left\langle\mathrm{D}_{\frac{\partial\sigma}{\partial u}}\nu_\Sigma\times\nu_\Sigma,\frac{\partial\sigma}{\partial u}\times\nu_\Sigma\right\rangle\\ &\quad + \left\langle\frac{\partial\sigma}{\partial u}\times\frac{\partial^2\sigma}{\partial v^2},\frac{\partial\sigma}{\partial u}\times\nu_\Sigma\right\rangle\Big].\end{aligned} \tag{111}$$

Now replacing (108), (109), (110) and (111) into (107), and then replacing (107) and (106) into (105) we get the last result of this proposition

$$\begin{aligned}\nabla_2 h_{22} &= -\frac{1}{|\frac{\partial\sigma}{\partial u}\times\nu_\Sigma|}\left\langle\mathrm{D}_{\frac{\partial\sigma}{\partial u}}\nu_\Sigma\times\nu_\Sigma,\frac{\partial^2\sigma}{\partial v^2}\right\rangle - \frac{1}{|\frac{\partial\sigma}{\partial u}\times\nu_\Sigma|}\left\langle\frac{\partial\sigma}{\partial u}\times\nu_\Sigma,\frac{\partial^3\sigma}{\partial v^3}\right\rangle\\ &\quad - 3\,\frac{1}{|\frac{\partial\sigma}{\partial u}\times\nu_\Sigma|}\left\langle\frac{\partial\sigma}{\partial u}\times\nu_\Sigma,\mathrm{D}_{\frac{\partial^2\sigma}{\partial v^2}}\nu_\Sigma\right\rangle - 2h_{22}\left\langle\frac{\partial^2\sigma}{\partial v^2},\nu_\Sigma\right\rangle - 2\,\frac{h_{12}}{|\frac{\partial\sigma}{\partial u}|^2}\left\langle\frac{\partial^2\sigma}{\partial v^2},\frac{\partial\sigma}{\partial u}\right\rangle\\ &\quad - \frac{h_{22}}{|\frac{\partial\sigma}{\partial u}\times\nu_\Sigma|^2}\Big[\left\langle\mathrm{D}_{\frac{\partial\sigma}{\partial u}}\nu_\Sigma\times\nu_\Sigma,\frac{\partial\sigma}{\partial u}\times\nu_\Sigma\right\rangle + \left\langle\frac{\partial\sigma}{\partial u}\times\frac{\partial^2\sigma}{\partial v^2},\frac{\partial\sigma}{\partial u}\times\nu_\Sigma\right\rangle\Big].\end{aligned}$$

And this completes our proof. □

We are now ready to start working on the particular case when Σ surface is a unit sphere.

3. Mean curvature flow of graphs outside the sphere

This section is reserved for the problem (56) when the contact surface Σ is a unit sphere in $\mathbb{R}^3$. More details on this problem can be found in Chapter 4. We construct here the parametrisation of the surface on which there is a first point of tilt in time and space. The conditions required by such a surface tell us more about the behaviour of a mean curvature flow solution which tilts.

If the contact surface Σ is the unit sphere in $\mathbb{R}^3$ centred at the origin then we have

$$\begin{aligned} h_{11}^{\Sigma} &= h_{22}^{\Sigma} = 1 \text{ and,} \\ h_{12}^{\Sigma} &= 0, \end{aligned} \tag{112}$$

for any orthonormal basis of the tangent space to Σ. Also here we have made the choice of unit normal to Σ such that

$$\langle \nu_{\Sigma}, e_3 \rangle \geq 0, \tag{113}$$

which is the opposite of the convention of Proposition 6.7 so we are careful to modify the results accordingly. That is why the inequality of (66) changes as in (116). This change of convention is easier to use in the setting of the previous section. In case we do not work with an orthonormal basis of the tangent space of Σ, but only an orthogonal one as is the case of the local parametrisation of the previous section then the curvatures of the sphere need to be scaled by the length of the tangent vectors. This is the reason why property (119) in the below proposition is slightly changed as to the (69) in Proposition 6.7. For Σ being a unit sphere in $\mathbb{R}^3$ the results of Proposition 6.7 change as below.

Proposition A.4 (Tilt properties for a unit sphere). *Let M_t be a family of immersions satisfying (21) where Σ be a unit sphere centred at the origin in $\mathbb{R}^3$. Suppose that $X(p,t) \in \partial_N M_t \subset \Sigma$ is the first point where $s = 0$ by $\nu_{M_t} = \frac{K_3}{|K_3|}$ up to a sign. At this point take an orthogonal basis of the tangent space of M_t to be $\{\tau_1, \tau_2\}$ where τ_1 is tangent to the boundary curve contained in Σ and τ_2 is the choice of unit normal to Σ such that it points into the surface of the graph M_t. Then at X the following properties hold*

$$\langle \nu_{M_t}, e_3 \rangle = 0 \text{ at } X \text{ and } \langle \nu_{M_t}, e_3 \rangle > 0 \text{ everywhere else} \tag{114}$$

$$h_{11} = h_{12} = 0 \tag{115}$$

$$h_{22} \langle \nu_{\Sigma}, e_3 \rangle > 0 \tag{116}$$

$$\nabla_1 h_{11} \langle \tau_1, e_3 \rangle \geq 0 \tag{117}$$

$$\nabla_1 h_{12} = h_{22} \tag{118}$$

$$\nabla_1 h_{22} \langle \tau_1, e_3 \rangle \leq - h_{22} \langle \nu_{\Sigma}, e_3 \rangle |\tau_1|^2 \tag{119}$$

$$\nabla_2 h_{22} = 0 \tag{120}$$

where we have denoted by h_{ij} and h_{ij}^{Σ} the components of the second fundamental forms A^{M_t} and A^{Σ} of M_t and Σ respectively.

REMARK. One can see that we have taken the opposite convention as in earlier chapters for the unit normal to the surface Σ. We are moving on the outside of the sphere and the normal of the sphere is chosen such that the convention is that the sphere is convex. In case we work with the (56) problem and we move on the interior of the sphere then the curvatures of the sphere are negative and the above result changes appropriately.

Our parametrisation is defined using the family of curves $\sigma : I_1 \times I_2 \to \mathbb{R}^3$. Let us denote the components of σ, $\sigma_1, \sigma_2 : I_1 \times I_2 \to \mathbb{R}$ as below

$$\sigma(u,v) \;=\; \Big(\sigma_1(u,v), \sigma_2(u,v), \sqrt{1-\sigma_1(u,v)^2-\sigma_2(u,v)^2}\Big),$$

where the last component is defined such that we have that for every u and v the curve $\sigma(u,v)$ is contained in the unit sphere Σ. Using the properties of a first tilt point we impose different conditions on the two components of σ. Before we start we need to mention that here since we are working with the surface Σ being an unit sphere in $\mathbb{R}^3$ and using the definition of our parametrisation as in (91) the unit normal to the surface Σ is just the position vector of the curve σ

$$\nu_\Sigma \;=\; \sigma. \tag{121}$$

The first condition that we want to satisfy is (114). Use the above remark (121) and the definition of the unit normal to the parametrisation (95) and see that (114) transforms to the following

$$\begin{aligned}\frac{\partial \sigma_1}{\partial u}(u,v)\sigma_2(u,v)-\frac{\partial \sigma_2}{\partial u}(u,v)\sigma_1(u,v) \;&>\; 0 \;\;\forall (u,v)\neq(0,0) \;\text{ and}\\ \frac{\partial \sigma_1}{\partial u}(0,0)\sigma_2(0,0) \;&=\; \frac{\partial \sigma_2}{\partial u}(0,0)\sigma_1(0,0).\end{aligned}$$

One can also write the above in the following form

$$\begin{aligned}\frac{\partial}{\partial u}\frac{\sigma_1}{\sigma_2}(u,v) \;&>\; 0 \quad \forall (u,v)\neq(0,0) \;\text{ and}\\ \frac{\partial}{\partial u}\frac{\sigma_1}{\sigma_2}(0,0) \;&=\; 0.\end{aligned}$$

This leads us to the following constraint on the parametrisation (91) of a surface with a first point of tilt at $(u,v)=(0,0)$

$$\begin{aligned}\sigma_1(u,v) \;&=\; \sigma_2(u,v)\; g(u,v) \text{ where } g : I_1\times I_2 \to \mathbb{R} \text{ and}\\ \frac{\partial g}{\partial u}(u,v) \;&>\; 0 \;\;\forall (u,v)\neq(0,0) \text{ and } \frac{\partial g}{\partial u}(0,0) \;=\; 0.\end{aligned} \tag{122}$$

From now on we look to impose restrictions on σ_2 and the new introduced g function.

The next condition we look at is the Neumann condition imposed on the parametrisation (92). This gives the following relations on σ_2 and g functions

$$\frac{\partial \sigma_2}{\partial v}(u,0) \;=\; 0 \text{ and } \frac{\partial g}{\partial v}(u,0) \;=\; 0 \;\;\forall u. \tag{123}$$

We turn our attention now to the two curvature properties (115) and (116). From the first one we only use the information given by the component h_{11}, the other one being trivial since we are working with a orthogonal basis on the unit sphere Σ. Again we make use of (95) and compute the unit normal to the moving surface which contains a first point of tilt at $(0, 0)$ to be

$$\nu_{M^*} = \frac{1}{|\frac{\partial \sigma_2}{\partial u}(0,0)|\sqrt{1+g^2(0,0)}}\Big(\frac{\partial \sigma_2}{\partial u}(0,0),\ -\frac{\partial \sigma_2}{\partial u}(0,0)g(0,0),\ 0 \Big). \tag{124}$$

We also need to compute the second derivative in u and v of $\sigma(u, v)$ at $(0, 0)$ as

$$\frac{\partial^2 \sigma}{\partial u^2}(0,0) = \Big(\frac{\partial^2 \sigma_2}{\partial u^2}(0,0)g(0,0) + \frac{\partial^2 g}{\partial u^2}(0,0)\sigma_2(0,0),\ \frac{\partial^2 \sigma_2}{\partial u^2}(0,0),\ \dots \Big), \tag{125}$$

and

$$\frac{\partial^2 \sigma}{\partial v^2}(0,0) = \Big(\frac{\partial^2 \sigma_2}{\partial v^2}(0,0)g(0,0) + \frac{\partial^2 g}{\partial v^2}(0,0)\sigma_2(0,0),\ \frac{\partial^2 \sigma_2}{\partial v^2}(0,0),\ \dots \Big), \tag{126}$$

where we have used the conditions (122) and (123) and also omitted the last component since it does not come into the following computation. The relations (115) and (116) give us the following

$$\frac{\partial \sigma_2}{\partial u}(0,0)\sigma_2(0,0)\frac{\partial^2 g}{\partial u^2}(0,0) = 0, \tag{127}$$

$$\frac{\partial \sigma_2}{\partial u}(0,0)\sigma_2(0,0)\frac{\partial^2 g}{\partial v^2}(0,0) < 0. \tag{128}$$

One can see that these together simplify the (127) to be

$$\frac{\partial^2 g}{\partial u^2}(0,0) = 0. \tag{129}$$

The next relation we want to look at is (117). Using relations (99), (121) and the properties (115) and (116) this becomes

$$-\frac{1}{|\frac{\partial \sigma}{\partial u}(0,0)\times\sigma(0,0)|}\Big\langle \frac{\partial \sigma}{\partial u}(0,0)\times\sigma(0,0), \frac{\partial^3 \sigma}{\partial u^3}(0,0)\Big\rangle \Big\langle \frac{\partial \sigma}{\partial u}(0,0), e_3\Big\rangle \geq 0.$$

To compute the last scalar product we once again use relation (122)

$$\Big\langle \frac{\partial \sigma}{\partial u}(0,0), e_3\Big\rangle = -\frac{\frac{\partial \sigma_2}{\partial u}(0,0)\sigma_2(0,0)(1+g(0,0)^2)}{\sqrt{1-\sigma_2^2(0,0)(1+g^2(0,0))}}.$$

We also need to compute the third partial derivative in u for σ

$$\begin{aligned}\frac{\partial^3 \sigma}{\partial u^3}(0,0) = \Big(&\frac{\partial^3 \sigma_2}{\partial u^3}(0,0)g(0,0) + \frac{\partial^3 g}{\partial u^3}(0,0)\sigma_2(0,0) + 3\frac{\partial^2 g}{\partial u^2}(0,0)\frac{\partial \sigma_2}{\partial u}(0,0),\\ &\frac{\partial^3 \sigma_2}{\partial u^3}(0,0),\ \dots \Big),\end{aligned}$$

where we have also used (122) and (123) and omitted the last term since it does not play a role in the computation. Returning to our (117) and applying (127) we get another property of the two components of σ

$$\frac{\partial^3 g}{\partial u^3}(0,0) \;\geq\; 0. \tag{130}$$

The last relation we are going to work on needs a little modification before we are able to obtain our desired property. This is relation (119). Starting with relation (100) in which we compute the component of the three tensor of the derivative of curvatures of the surface described by our parametrisation, we apply first the two properties (115) and (116) and also use relation (121) to get

$$\begin{aligned}\nabla_1 h_{22} \;=\; & -\; \frac{1}{|\frac{\partial\sigma}{\partial u}\times\sigma|}\left\langle \frac{\partial^2\sigma}{\partial u^2}\times\sigma, \frac{\partial^2\sigma}{\partial v^2}\right\rangle \;-\; \frac{1}{|\frac{\partial\sigma}{\partial u}\times\sigma|}\left\langle \frac{\partial\sigma}{\partial u}\times\sigma, \frac{\partial^3\sigma}{\partial u\partial v^2}\right\rangle \\ & -\; \frac{h_{22}}{|\frac{\partial\sigma}{\partial u}\times\sigma|^2}\left\langle \frac{\partial^2\sigma}{\partial u^2}\times\sigma, \frac{\partial\sigma}{\partial u}\times\sigma\right\rangle.\end{aligned}$$

Using the same techniques as the ones found in the proof of Proposition A.3 we can expand the scalar product from the first term from above into the following

$$\begin{aligned}\left\langle \frac{\partial^2\sigma}{\partial u^2}\times\sigma, \frac{\partial^2\sigma}{\partial v^2}\right\rangle \;=\; & \frac{1}{|\frac{\partial\sigma}{\partial u}|^2}\left\langle \frac{\partial^2\sigma}{\partial u^2}\times\sigma, \frac{\partial\sigma}{\partial u}\right\rangle\left\langle \frac{\partial\sigma}{\partial u}, \frac{\partial^2\sigma}{\partial v^2}\right\rangle \;+\; \left\langle \frac{\partial^2\sigma}{\partial u^2}\times\sigma, \sigma\right\rangle\left\langle \sigma, \frac{\partial^2\sigma}{\partial v^2}\right\rangle \\ & +\; \frac{1}{|\frac{\partial\sigma}{\partial u}\times\sigma|^2}\left\langle \frac{\partial^2\sigma}{\partial u^2}\times\sigma, \frac{\partial\sigma}{\partial u}\times\sigma\right\rangle\left\langle \frac{\partial\sigma}{\partial u}\times\sigma, \frac{\partial^2\sigma}{\partial v^2}\right\rangle \\ =\; & \frac{1}{|\frac{\partial\sigma}{\partial u}|^2}\left\langle \frac{\partial^2\sigma}{\partial u^2}\times\sigma, \frac{\partial\sigma}{\partial u}\right\rangle\left\langle \frac{\partial\sigma}{\partial u}, \frac{\partial^2\sigma}{\partial v^2}\right\rangle \;-\; \frac{h_{22}}{|\frac{\partial\sigma}{\partial u}\times\sigma|}\left\langle \frac{\partial^2\sigma}{\partial u^2}\times\sigma, \frac{\partial\sigma}{\partial u}\times\sigma\right\rangle,\end{aligned}$$

where in the last equality we have used the definition of the second fundamental form components as seen in Proposition A.2 and properties of the cross product. One can also simplify this more by noticing that

$$\left\langle \frac{\partial^2\sigma}{\partial u^2}\times\sigma, \frac{\partial\sigma}{\partial u}\right\rangle \;=\; 0.$$

This comes from the following short computation. First by the properties of the cross product we always have

$$\left\langle \frac{\partial\sigma}{\partial u}\times\sigma, \frac{\partial\sigma}{\partial u}\right\rangle \;=\; 0 \;\; \forall (u,v).$$

This allows us to differentiate in u and obtain

$$\left\langle \frac{\partial^2\sigma}{\partial u^2}\times\sigma, \frac{\partial\sigma}{\partial u}\right\rangle \;=\; -\left\langle \frac{\partial\sigma}{\partial u}\times\frac{\partial\sigma}{\partial u}, \frac{\partial\sigma}{\partial u}\right\rangle \;-\; \left\langle \frac{\partial\sigma}{\partial u}\times\sigma, \frac{\partial^2\sigma}{\partial u^2}\right\rangle \;=\; -\;\left|\frac{\partial\sigma}{\partial u}\times\sigma\right| h_{11} \;=\; 0,$$

where we have used in the last equality the property (115) in a first point of tilt and the properties of the cross product.

Now returning to our first term of (100) we have

$$\left\langle \frac{\partial^2\sigma}{\partial u^2}\times\sigma, \frac{\partial^2\sigma}{\partial v^2}\right\rangle \;=\; -\;\frac{h_{22}}{|\frac{\partial\sigma}{\partial u}\times\sigma|}\left\langle \frac{\partial^2\sigma}{\partial u^2}\times\sigma, \frac{\partial\sigma}{\partial u}\times\sigma\right\rangle.$$

That gives us that the first and last term found in (100), which do not cancel from the other properties are in sum vanishing. In the end (100) becomes

$$\nabla_1 h_{22} \;=\; -\,\frac{1}{|\frac{\partial\sigma}{\partial u}\times\sigma|}\left\langle \frac{\partial\sigma}{\partial u}\times\sigma, \frac{\partial^3\sigma}{\partial u\partial v^2}\right\rangle. \tag{131}$$

We now use this relation to compute the relation given by (119) and obtain

$$\begin{aligned}&\left\langle \frac{\partial\sigma}{\partial u}\times\sigma, \frac{\partial^3\sigma}{\partial u\partial v^2}\right\rangle \frac{\frac{\partial\sigma_2}{\partial u}\sigma_2(g^2+1)}{\sqrt{1-\sigma_2^2(g^2+1)}}(0,0)\\ &\qquad\le \left\langle \frac{\partial\sigma}{\partial u}\times\sigma, \frac{\partial^2\sigma}{\partial v^2}\right\rangle \sqrt{1-\sigma_2^2(g^2+1)}\frac{\left(\frac{\partial\sigma_2}{\partial u}\right)^2(g^2+1)}{1-\sigma_2^2(g^2+1)}(0,0).\end{aligned}$$

First we have to compute the third mixed derivative of σ in u and v as follows

$$\begin{aligned}\frac{\partial^3\sigma}{\partial u\partial v^2}(0,0) \;=\; \Big(&\frac{\partial^3\sigma_2}{\partial u\partial v^2}(0,0)g(0,0)+\frac{\partial^2 g}{\partial v^2}(0,0)\frac{\partial\sigma_2}{\partial u}(0,0)+\frac{\partial^3 g}{\partial u\partial v^2}(0,0)\sigma_2(0,0),\\ &\frac{\partial^3\sigma_2}{\partial u\partial v^2}(0,0),\ \ldots\Big)\end{aligned}$$

where we have used (122) and (123) to eliminate some terms and again omitted the last term of the vector for it does not come into the computation. Returning to our relation from one computation above we get after simplifications

$$\frac{\partial^3 g}{\partial u\partial v^2}(0,0) \;\le\; 0. \tag{132}$$

All the above constraints on the parametrisation of a surface which develops a first point of tilt can be found in the following proposition.

Proposition A.5 (Properties of a parametrisation in a first tilt point). *Let M_t satisfying the evolution (21). Let M_{t^*} be the first surface such that it has on the Neumann boundary $\partial_N M_{t^*}\subset\Sigma$ the first point of tilt in time and space. Let M_{t^*} be represented using the parametrisation (91) such that the point of tilt is at $(0,0)\in I_1\times I_2$, then the following restrictions are imposed on the components of the parametrisation*

$$\begin{aligned}&\sigma_1(u,v) \;=\; \sigma_2(u,v)\;g(u,v)\ \textit{where}\ g: I_1\times I_2\to\mathbb{R}\ \textit{and}\\ &\frac{\partial g}{\partial u}(u,v) \;>\; 0\ \ \forall (u,v)\neq(0,0)\ \textit{and}\ \frac{\partial g}{\partial u}(0,0) \;=\; 0,\\ &\frac{\partial\sigma_2}{\partial v}(u,0) \;=\; 0\ \textit{and}\ \frac{\partial g}{\partial v}(u,0) \;=\; 0\ \forall u,\\ &\frac{\partial^2 g}{\partial u^2}(0,0) \;=\; 0,\\ &\frac{\partial\sigma_2}{\partial u}(0,0)\sigma_2(0,0)\frac{\partial^2 g}{\partial v^2}(0,0) \;<\; 0,\\ &\frac{\partial^3 g}{\partial u^3}(0,0) \;\ge\; 0,\\ &\frac{\partial^3 g}{\partial u\partial v^2}(0,0) \;\le\; 0.\end{aligned}$$

Using all the relations stated above we are able to prove that in a first point of tilt the time derivative of the quantity s which characterises the graph property of an immersion vanishes.

Proposition A.6 (Time derivative of s at a first tilt point). *Let M_t satisfying the evolution* (21). *Let M_{t^*} be the first surface such that it has on the Neumann boundary $\partial_N M_{t^*} \subset \Sigma$ the first point of tilt X^* in time and space. Then*

$$\frac{d}{dt}s(X^*) \;=\; \frac{d}{dt}\left\langle \nu_{M_{t^*}}, e_3\right\rangle\Big|_{X^*} \;=\; 0.$$

Proof. First we have to note here that the way we represented the surfaces using locally the parametrisation defined with the help of the family of curves σ can be done for any smooth immersed surface M_t. So having the proof in a setting where we work locally around the tilt point is sufficient for the result of the proposition to work for any surface evolving by (21).

Let us now start with the proof. We denote by $f : I_2 \to \mathbb{R}$ the partial derivative of the function g in the u variable at the point of tilt, that is

$$f(v) \;:=\; \frac{\partial g}{\partial u}(0, v).$$

Using relation (122) one can see that

$$f(0) \;=\; 0 \text{ and } f(v) \;>\; 0 \;\; \forall v \in I_2,\;\; v \neq 0$$

and this tells us that the function f has a minimum at $v = 0$ which is a boundary point. Also at this point we have

$$\frac{df}{dv}(0) = 0,$$

using relation (123). Then the minimum at $v = 0$ is obtained in such a way that

$$\frac{df}{dv^2}(0) \;=\; \frac{\partial^3 g}{\partial u \partial v^2}(0, 0) \;\geq\; 0.$$

The last inequality together with property (132) tells us that we can only have

$$\frac{\partial^3 g}{\partial u \partial v^2}(0, 0) \;=\; 0.$$

This also implies that the inequality (119) can only be satisfied with equality, so

$$\nabla_1 h_{22}\;\langle \tau_1, e_3\rangle \;=\; -\,h_{22}\;\;\langle \nu_\Sigma, e_3\rangle\;\;|\tau_1|^2.$$

If one looks back on the proof of how this inequality was obtained in the previous chapter, Proposition 6.7 relation (69), then we get that at the point of first tilt X^* we have

$$\begin{aligned} 0 \;\geq\; \frac{d}{dt}s\Big|_{X^*} \;&=\; \langle \nabla H, e_3\rangle\Big|_{X^*} \;=\; \nabla_1 H\,\langle \tau_1, e_3\rangle\,\frac{1}{|\tau_1|^2}\Big|_{X^*} \;+\; \nabla_2 H\,\langle \nu_\Sigma, e_3\rangle\Big|_{X^*} \\ &=\; \nabla_1 h_{11}\,\langle \tau_1, e_3\rangle\,\frac{1}{|\tau_1|^2}\Big|_{X^*} \;+\; \nabla_1 h_{22}\,\langle \tau_1, e_3\rangle\,\frac{1}{|\tau_1|^2}\Big|_{X^*} \;+\; \nabla_2 h_{11}\,\langle \nu_\Sigma, e_3\rangle\Big|_{X^*} \\ &\quad+\; \nabla_2 h_{22}\,\langle \nu_\Sigma, e_3\rangle\Big|_{X^*} \end{aligned}$$

$$= \nabla_1 h_{11} \langle \tau_1, e_3 \rangle \frac{1}{|\tau_1|^2}\Big|_{X^*} - h_{22} \ \langle \nu_\Sigma, e_3 \rangle \Big|_{X^*} + h_{22} \ \langle \nu_\Sigma, e_3 \rangle \Big|_{X^*}$$
$$= \nabla_1 h_{11} \langle \tau_1, e_3 \rangle \frac{1}{|\tau_1|^2}\Big|_{X^*}.$$

where the first inequality comes from the fact that we have the first minimum of the s quantity in time and space and in the rest we have used the Codazzi equations, the above equality in the relation (119) and properties (118) and (120) from Proposition A.4. One can see that this implies

$$\nabla_1 h_{11} \langle \tau_1, e_3 \rangle \Big|_{X^*} \le 0,$$

which together with property (117) tells us that it can only be true with equality. So we obtain

$$\nabla_1 h_{11} \langle \tau_1, e_3 \rangle \Big|_{X^*} = 0$$

and also by this we conclude our result

$$\frac{d}{dt} s \Big|_{X^*} = 0.$$

□

REMARK (Second time derivative behaviour). One might continue the above discussion by treating the second derivative in time of the s quantity. This tells us if the surface with the first tilt becomes better after the time of the tilt or maintains the tilt for some time. One can easily see that the second time derivative can not be strictly negative at the tilt point X^*, since that will imply that it has a maximum in time and that is not possible since before the time of the tilt all values are bigger than at the time of tilt. So the two cases remaining are if the second derivative is strictly positive or vanishes. If the second time derivative will be strictly positive then the tilt will immediately become better and the surfaces will regain their graph property. If we have a second derivative in time also vanishing at the time of tilt then we are in a true point of inflection in time for the s quantity and one needs to bring in discussion the third derivative of time to be able to characterise the tilt point.

APPENDIX B

Dirichlet boundary estimates

1. Introduction

In this appendix we discuss in detail the problem of obtaining Dirichlet boundary estimates and how these fit into the big picture of long time existence for our graph solutions. As we recall, we are working with graph problems which sometimes have two boundaries. On one of them the boundary condition is a free Neumann given by a ninety degrees angle on a fixed $2-$dimensional surface, Σ in $\mathbb{R}^3$ and on the other one we have a fixed Dirichlet boundary condition which keeps our graph functions at some height, which for example in the sphere problem treated in Chapter 4 is constant and 0.

The problem of obtaining bounds for the gradient on the Dirichlet boundary is well understood and there are many results as one can see in the below list of problems treated by different authors.

For mean curvature flow with Dirichlet boundary conditions the barriers have been used for the first time by Huisken in [**23**]. There one makes use of the well known work of Serrin [**34**] to construct barriers for the elliptic problem of the parabolic operator. The two barriers bound the initial values at the Dirichlet boundary. Following the steps of Huisken, one can apply the strong maximum principle and bound the gradient on the Dirichlet boundary for all times.

The above use of results is enough for an experienced reader, but for the completion of the work we include here a choice of exposition which contains the construction of barriers and the way they are used. Due to the huge amount of work done on the subject, we had to make a choice of material to follow and we apologise for omitting most of the references available. We have chosen the work of Trudinger, [**39**] since it is self contained treating the cases elliptic to parabolic, and also since it treats a general type of boundaries, weaker than convex and time dependent height functions on the Dirichlet boundary. This generality is later explained in this appendix.

The Cauchy-Dirichlet problems have been treated extensively in literature, for elliptic as well as parabolic evolutions. We want to give here a short history on how much work has been done on the general types of problem and explain our choices on which results to follow and why. We apologise in advance for omitting some of the names and results but we are restricting ourself only to the ones which we further use.

The existence of solutions for uniformly elliptic equations has been shown by Ladyzhenskaya and Ural'tseva in [**28**]. This was followed by the fundamental work of Serrin, [**34**], where were obtained very general conditions under which the Dirichlet problem for quasilinear elliptic equations with arbitrary smooth boundary values is solvable in a given domain. He also showed that these conditions are sharp, in the sense that the problem is not solvable for some (infinitely differentiable) boundary values when the conditions are violated.

The next step was to try to do the same for the parabolic case of the problems and here we find the work of Ladyzhenskaya, Ural'tseva and Solonnikov, [**27**], which appeared in the same time as their elliptic work but treated the uniformly parabolic equations. There are also more general and extensive works which treat the parabolic problems and we mention here couple of them which caught our attention. The book of Lieberman [**30**], which we have used in the earlier chapter of short time existence, contains a quite general and complete work on parabolic boundary value problems. Also here we have to mention the work of Edmunds and Peletier [**11**], which is the analogue of Serrin's elliptic result but in the parabolic setting.

So as we see, there are many works which treat the elliptic and parabolic Dirichlet boundary value problems in great details. We restrict ourself to a more specialised result, the one of Trudinger [**39**] in which he obtains the analogue of the elliptic general case of Serrin [**34**] for the boundary gradient.

From the fundamental work of Ladyzhenskaya, Ural'tseva and Solonnikov, [**27**], we see that the solvability of the Dirichlet boundary value parabolic problems depends upon the establishment of C^1 estimates for the graph functions. The derivation of these estimates can be naturally split into three steps. The first is a height estimate of the graph function on the boundary, followed by the entire domain height estimate. The second and the most difficult is the boundary gradient estimate. And the third, which ends the argument is an estimate of the first derivative of the graph function on the whole domain.

If we look at our problems, we notice that the height boundary estimate is given by the fixed height on which the Dirichlet boundary value keeps our graphs. The Neumann boundary estimates can be obtained for some types of problems by imposing different conditions on the contact surface Σ. The interior estimates come as we have seen in earlier chapters from maximum principle. The third step is also an application of the maximum principle since we can use a beautiful equivalent definition of the first derivative of a graph which generates a surface moving by mean curvature flow, found in [**9**]. So what we are left to do and what is treated in the following sections is to obtain bounds for the first derivative of the graphs on the Dirichlet boundary. And this is the reason we decided to follow the work done by Trudinger combined with the standard construction of barrier functions which can be found in many places in the literature starting with Serrin.

2. Setup

The following work can be found in [**39**]. We keep the sign convention related to the mean curvature and orientation of the boundary normal. This convention is that the mean curvature of convex surfaces, like for example a sphere is positive with respect to the inner normal to the sphere. We have decided to keep the convention since its more natural when working with distance functions and also has been used extensively by all the big works we are quoting from, like Serrin, Lieberman, Trudinger, Giusti and so on.

Let Ω be a bounded domain in the n-dimensional Euclidean space, with $n \geq 2$ and boundary $\partial\Omega$. We consider the space time cylinder $\tilde{\Omega} = \Omega \times [0, T)$. Before the parabolic quasilinear problem is treated it is necessary to look at the elliptic case of the problem. This is because the barrier construction of the elliptic case carries through the parabolic one also. For a parabolic operator P, as the one defined in (20) in the short time existence chapter, we consider the elliptic operator associated

$$Qu \;=\; a^{ij}(x,u,\mathrm{D}u)\,\mathrm{D}^2_{ij}u \;+\; a(x,u,\mathrm{D}u), \tag{133}$$

where $1 \leq i, j \leq n$. The coefficients a^{ij} and a are continuous in all their arguments and the ellipticity condition that the matrix a^{ij} is positive definite is satisfied.

We follow the construction of barriers for the elliptic operator. First we explain what the existence of above and bellow barriers for the graph functions implies. Consider the elliptic problem

$$Qu \;=\; 0 \text{ in } \Omega.$$

Suppose that in a neighbourhood $\mathcal{U}_{x_0}$ of a boundary point $x_0 \in \partial\Omega$ there exist two $C^2(\mathcal{U}_{x_0} \cap \overline{\Omega})$ functions, w^- and w^+ satisfying

$$\begin{aligned} &(i)\;\; Qw^+ \;\leq\; 0 \text{ and } Qw^- \geq 0 \text{ in } \mathcal{U}_{x_0} \cap \Omega,\\ &(ii)\;\; w^+(x_0) \;=\; w^+(x_0) \;=\; u(x_0),\\ &(iii)\;\; w^-(x) \;\leq\; u(x) \;\leq\; w^+(x), \quad x \in \partial(\mathcal{U}_{x_0} \cap \Omega). \end{aligned}$$

It follows from (i) that the difference functions $u - w^+$ and $w^- - u$ satisfy maximum principles in $\mathcal{U}_{x_0} \cap \Omega$ and then by the boundary conditions (iii) we have that

$$w^-(x) \;\leq\; u(x) \leq w^+(x), \quad x \in \mathcal{U}_{x_0} \cap \Omega.$$

Using (ii) we get

$$w^-(x) - w^-(x_0) \;\leq\; u(x) - u(x_0) \;\leq\; w^+(x) - w^+(x_0).$$

Consequently the normal derivatives of u, w^+ and w^- satisfy the following

$$-\Big|\frac{\partial w^-}{\partial \nu}(x_0)\Big| \;\leq\; \Big|\frac{\partial u}{\partial \nu}(x_0)\Big| \;\leq\; \Big|\frac{\partial w^+}{\partial \nu}(x_0)\Big|,$$

for any direction ν tangent to the graphs. This estimate gives us a bound for the first derivative of the graph function on the Dirichlet boundary, provided we have bounded first derivative for the two barriers.

We call the functions w^+ and w^-, upper and lower barriers for the operator Q and function u at x_0. The above argument is the proof of the following proposition.

Proposition B.1 (Trudinger [**39**], 1972). *Let u be a function satisfying $Qu = 0$. If we have the existence of a lower and upper barrier for any point of the boundary $\partial\Omega$ then we have a bound for the first derivative of the function u on the boundary.*

Denote by $\lambda(x,u,p)$ and $\Lambda(x,u,p)$ the minimum, respectively the maximum eigenvalue of the matrix operator $a^{ij}(x,u,p)$ such that we have

$$0 \;<\; \lambda|\xi|^2_{\mathbb{R}^n} \;\leq\; a^{ij}\xi_i\xi_j \;\leq\; \Lambda|\xi|_{\mathbb{R}^n}, \quad \forall \xi \in \mathbb{R}^n\ \xi \neq 0\ .$$

Following the work found in Serrin or Bernstein we can define the quantity $\mathcal{E} = \mathcal{E}(x,u,p) = a^{ij}p_ip_j$. By the above we have

$$\lambda|p|^2 \;\leq\; \mathcal{E} \;\leq\; \Lambda|p|^2.$$

Let us compute these expressions for the minimal surface equation, which is the elliptic version of the mean curvature flow, that is

$$\mathcal{M}u \;=\; (1+|\mathrm{D}u|^2)\;\Delta u - \mathrm{D}^2_{ij}u\;\mathrm{D}_i u\;\mathrm{D}_j u.$$

For this operator we have

$$\begin{aligned}\mathcal{E} \;&=\; |p|^2,\\ \lambda \;&=\; 1,\\ \Lambda \;&=\; (1+|p|^2).\end{aligned} \tag{134}$$

For us it is interesting to see what are these quantities for the elliptic part of the parabolic operator for graphs moving by mean curvature flow. As one can see this operator is better behaved than the minimal surface operator since its smallest and biggest eigenvalue are smaller than the ones above. They are a scaled version of those. If we define the scaled operator to be

$$\tilde{\mathcal{M}}u \;=\; \Delta u \;-\; \frac{1}{1+|\mathrm{D}u|^2}\;\mathrm{D}^2_{ij}\;u\;\mathrm{D}_i u\;\mathrm{D}_j u,$$

then it is easy to see that its smallest and largest eigenvalues are

$$\begin{aligned}\mathcal{E} \;&=\; \frac{|p|^2}{1+|p|^2},\\ \lambda \;&=\; \frac{1}{1+|p|^2},\\ \Lambda \;&=\; 1.\end{aligned} \tag{135}$$

From now on, when following the notes of Trudinger on the minimal surface operator and then on the parabolic minimal surface problem we always add comments which relate us to the mean curvature flow, which is just a scaled version of the parabolic problem treated by Trudinger.

There are two steps in the construction of barriers in the paper of Trudinger and those are: constructing barriers when our graph problem has zero Dirichlet boundary conditions and then from this step the second one which is the general boundary value problems is obtained by defining a change of variables.

3. Construction of barriers

The work of Trudinger [**39**] generalises the construction of barriers from the usual convex condition of the boundary set to a more general convexity condition. The construction starts with the case when the Dirichlet boundary is convex in the easier sense that there exists a hyperplane which contains a piece of the boundary. That makes things easier since it assumes that the boundary around the point where we construct the barrier is flat. After obtaining barriers for this case, Trudinger modifies the estimates to include the case when the boundary satisfies an outside sphere condition. The next step is to generalise those estimates and get a general $Q-$convexity condition for the boundaries for which we can construct local boundary barriers. This condition is less restrictive than the one found in Serrin and includes as a particular case the positive mean curvature boundaries. Here the parabolic version of the minimal surface operator, with Dirichlet boundary values is stated as a Corollary, by showing that sets with boundaries with positive mean curvature are $Q-$convex with respect to the minimal surface operator. Also the results are carried through from Dirichlet boundary conditions which are not dependent of time to the time dependent version. Our problems are the simplest version of the result presented below. We have a fixed in time Dirichlet boundary condition on a boundary with positive mean curvature.

We follow the construction of barriers in the first two steps from Trudinger, going from convex boundaries in the sense of flatness to the one of outside sphere condition and $Q-$convex boundaries and we omit some of the proofs inviting the reader to [**39**] for more details. Along the way we state our observation on how the mean curvature flow fits into the program.

Suppose that $\partial\Omega$ is convex at a point x_0, in the sense that there exists a hyperplane $\mathcal{P}$ satisfying $x_0 \in \mathcal{P} \cap \overline{\Omega} \subset \mathcal{P} \cap \partial\Omega$. Define the linear function $d(x) = dist(x, \mathcal{P})$ as the distance from a point $x \in \Omega$ to the hyperplane $\mathcal{P}$. This function satisfies:

$$|\mathrm{D}d| \;=\; 1 \;\text{ and }\; a^{ij}\,\mathrm{D}^2_{ij}d \;=\; 0.$$

Define the function $w = \phi(d)$ where we take $\phi \in C^2[0,\infty)$ such that $\phi' \neq 0$ and check if the conditions from the definition of the barrier functions are satisfied.

$$\overline{Q}w \;=\; a^{ij}(x,u,\mathrm{D}w)\,\mathrm{D}^2_{ij}w \;+\; a(x,u,\mathrm{D}w) \;=\; \frac{\phi''}{(\phi')^2}\,\mathcal{E} \;+\; a.$$

If we assume the existence of a non-decreasing function ν_0 such that we have

$$|a| \;\leq\; \nu_0(|u|)\,\mathcal{E}, \quad \forall x \in \Omega, \quad |p| \;>\; 1, \tag{136}$$

it give us

$$\Big(\frac{\phi''}{(\phi')^2} - \nu\Big)\,\mathcal{E} \;\leq\; \overline{Q}w \;\leq\; \Big(\frac{\phi''}{(\phi')^2} \;+\; \nu\Big)\,\mathcal{E},$$

provided that $|\phi'| \geq 1$ where $\nu = \nu_0(M)$, $M = \sup\limits_{\Omega} u$. Define now

$$\phi(d) \;=\; \frac{1}{\nu}\;\log(1+kd),$$

for $k > 0$ and also consider the neighbourhood $\mathcal{U} = \mathcal{U}_{x_0} = \{x : d < a\}$, $a > 0$. We have that $\phi'' = -\nu(\phi')^2$ in $\mathcal{U}$. If we take $ka = e^{\nu M} - 1$ then

$$\phi(a) \;=\; \frac{1}{\nu}\;\log(1+ka) \;=\; M$$

and

$$\begin{aligned}\phi'(d) \;&=\; \frac{k}{\nu(1+kd)} \;\geq\; \frac{k}{\nu(1+ka)} \quad \text{in } \mathcal{U}\cap\Omega\\ &=\; \frac{k}{\nu e^{\nu M}}\\ &=\; 1 \;\text{ if } k \;=\; \nu e^{\nu M}.\end{aligned}$$

Then with the above choices of k and a the functions $w^+ = +\phi(d)$ and $w^- = -\phi(d)$ are respectively upper and lower barriers at x_0 for the operator $\overline{Q}$ and for the function u

provided $u = 0$ on $\mathcal{U} \cap \Omega$. This is the first most simplest construction. From now on we modify the estimate (136) for more general type of domains.

Let us consider a new geometric configuration of the boundary $\partial\Omega$ at x_0. Assume that $\partial\Omega$ satisfies an exterior sphere condition at x_0 and that is, there exists a sphere $S = S(b)$ of radius $b > 0$, such that $x_0 \in \overline{S} \cap \overline{\Omega} \subset \overline{S} \cap \partial\Omega$. The above hyperplane construction can be useful if we change coordinates such that we can convert $\partial\Omega$ from satisfying an exterior sphere condition to the convexity in the sense of a hyperplane existence as above. After doing the change of coordinates the estimate (136) transforms into

$$(137) \qquad |a| \ + \ |p| \ \Lambda \ \leq \ \nu_0(|u|) \ \mathcal{E}, \quad x \in \Omega, \quad |p| \ > \ 1.$$

This estimate can be also obtained by direct construction of barriers now using the distance to the boundary of the sphere, $\partial\overline{S}$, $d(x) = dist(x, \partial\overline{S})$. Define $w = \phi(d)$ in the same way and apply the same operator

$$\overline{Q}w \ = \ \phi' a^{ij} \ \mathrm{D}^2_{ij} \ d \ + \ \frac{\phi''}{(\phi')^2} \ \mathcal{E} \ + \ a.$$

These two barrier constructions allow us to use Proposition B.1 and obtain gradient bounds for zero Dirichlet boundary data problems for boundaries which are either flat or satisfy an exterior sphere condition.

Remark. In Trudinger's [**39**] setting the sign of the mean curvature is computed with respect to the inner normal, opposite to the convention in Huisken [**23**].

We want now to pass to arbitrary boundary values. This is done by considering the evolution of the difference between the graph solution and the boundary values. For this we define the following functional which comes into the operator definition

$$\mathcal{F}(x, u, p, q) \ = \ a^{ij}(x, u, p)(p_i - q_i)(p_j - q_j).$$

Let now $u = f$ on $\partial\Omega$ with $f \in C^2(\overline{\Omega})$. Then the first generalisation of (136) is

$$(138) \qquad |a| \ + \ \Lambda \ \sup_{\Omega} |\mathrm{D}^2 f| \ \leq \ \nu_1(|u|) \ \mathcal{F}(x, u, p, Df), \quad x \in \Omega, \ , \ |p_k - D_k f| \ \geq \ 1,$$

where we denote by ν_1 a non decreasing function depending on ν_0 and $|f|_{C^2(\Omega)}$.

If we are in the case of domains satisfying the exterior sphere condition the estimate (137) changes for arbitrary Dirichlet data into

$$(139) \qquad |a| + \Lambda|p - Df| + \Lambda \sup_{\Omega} |D^2 f| \leq \nu_1(|u|)\mathcal{F}(x, u, p, Df), \ x \in \Omega, \ |p - Df| \geq 1,$$

where again we denote by ν_1 a non decreasing function depending on ν_0 and $|f|_{C^2(\Omega)}$.

The next step is to generalise these conditions needed for the existence of the barrier functions to domains which have convexity in terms of the elliptic operator. Note that although the exterior sphere condition is satisfied by C^2 domains, it is easier to regard it as a geometric condition rather than a regularity one. For a quasilinear operator Q of the form considered as above we say that $\partial\Omega$ is $Q-$convex at $x_0 \in \partial\Omega$ if there exists a C^3 domain $S = S_{x_0}$ satisfying

(i) $x_0 \in \partial S \cap \overline{\Omega} \subset \partial S \cap \partial\Omega$,

$$(ii)\ k \ = \ k_{x_0} \ = \ \sup_{\Omega} \frac{dist(x, \partial S \cap \partial\Omega)}{d(x)} \ < \ \infty \ \text{ or } \mathrm{D}^2 d \ = \ 0,$$
$$(iii)\ Q(d) \ \leq \ 0 \text{ and } \ -Q(-d) \ \leq \ 0 \ \text{ for } x \in \partial\Omega \cap \partial S$$

where d denotes the distance function $d(x) = dist(x, \partial S)$. By C^3 we mean that d is a C^3 function in some neighbourhood of ∂S, let us say in the set where $d \leq d_0$.

We call $\partial\Omega$ to be $Q-$convex if it is $Q-$convex at each $x_0 \in \partial\Omega$ and uniformly convex if k_{x_0} and $|d|_{C^3(\overline{\Omega}_{x_0})}$ where $\overline{\Omega}_{x_0} = \overline{\Omega} \cap \{d \leq d_0\}$ are bounded independent of x_0.

Thus if $\partial\Omega \in C^3$ and condition (iii) is satisfied with $S = \Omega$, then $\partial\Omega$ is uniformly $Q-$convex. Also it is easy to see that the $Q-$convexity contains also the two convexity cases we have discussed above, in the following way. Any convex domain in the sense of flatness is $Q-$convex with respect to any operator Q with $a = 0$. Also if $\partial\Omega$ satisfies an exterior sphere condition then it is convex with respect to the $Q = 0$ operator.

Let us look now at two operators which are of interest for us, the minimal surface operator and the elliptic part of the non-parametric mean curvature flow, which is a scaled version of the first one. We start with the following proposition.

Proposition B.2 (Trudinger [**39**], 1972, Sets with boundary with positive mean curvature). *Suppose that $\partial\Omega \in C^3$ and the mean curvature $H = H_{x_0}$ of $\partial\Omega$ at $x_0$satisfies $H \geq 0$, $\forall x_0 \in \partial\Omega$. Then $\partial\Omega$ is $Q-$convex with respect to the operators $\mathcal{M}^\infty$ and $\tilde{\mathcal{M}}^\infty$ given by*

$$\mathcal{M}^\infty u \ = \ \Delta u \ - \ \frac{D_i u \ D_j u \ D^2_{ij} u}{|Du|^2},$$
$$\tilde{\mathcal{M}}^\infty u \ = \ \Delta u.$$

Proof. From the smoothness property of the boundary $\partial\Omega \in C^3$ the first two conditions are satisfied for the operator M^∞. What we need to check is the third one. From the properties of the distance function, following discussions found [**34**], we can see that $\Delta d = -(n-1)H$ and the second term in $\mathcal{M}^\infty d$ vanishes since

$$\mathrm{D}^2 d(x) \ = \ -diag\left\{\frac{k_1}{1 - k_1 d(x)}, \ \ldots, \ \frac{k_{n-1}}{1 - k_{n-1} d(x)}, \ 0\right\}$$

in a set of coordinates where we also have

$$\mathrm{D}d(x) \ = \ (0, \ldots, 0, 1).$$

We want to remind the reader that we have kept the sign conventions from [**39**]. Also let us notice that in the work of Trudinger, the mean curvature is scaled by the dimension of the boundary set. The boundary set for us is a $n-1$-dimensional set so we have

$$H \ = \ \frac{1}{n-1} \sum_{i=1}^{n-1} k_i.$$

This constant is not changing the sign in the inequality on the mean curvature of the boundary set. To prove that $\mathcal{M}^\infty d \leq 0$ we need to have $-(n-1)H \leq 0$ which is equivalent to our hypothesis that $H \geq 0$. □

To generalise the necessary estimates for the existence of barriers from the two simpler cases of convexity to the more general one of $Q-$convexity one must distinguish a part of the operator Q, for which one solves the Dirichlet boundary value problem, like $\mathcal{M}^\infty$ or $\tilde{\mathcal{M}}^\infty$This part of the operator is used to show that the boundary set is convex with respect to the $Q-$convexity definition. Therefore Trudinger and us following his work consider the following splitting of an elliptic operator

$$\begin{aligned} Qu \;&=\; \Lambda Q^\infty u \;+\; Q^0 u \;\text{ where} \\ Q^\infty u \;&=\; a^{ij}_\infty(x,u,\sigma)\mathrm{D}^2_{ij}u \;+\; |\mathrm{D}u|\; C_\infty(x,u,\sigma), \quad \sigma = \frac{\mathrm{D}u}{|\mathrm{D}u|}, \\ Q^0 u \;&=\; a^{ij}_0(x,u,\mathrm{D}u)\mathrm{D}^2_{ij} \;+\; a_0(x,u,\mathrm{D}u) \end{aligned} \tag{140}$$

where we have $a^{ij}_\infty, C_\infty, a^{ij}_0, a_0 \in C^1(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n)$.

Corollary B.3 (Non-parametric mean curvature flow operator). *For the minimal surface operator, $\mathcal{M}$ and the elliptic operator of the non-parametric mean curvature flow, $\tilde{\mathcal{M}}$ we define $\mathcal{M}^\infty$ and $\tilde{\mathcal{M}}^\infty$ as in Proposition B.2.*

Note that here we have used relations (134) and (135), for our biggest and smallest eigenvalues for the two operators.

From [**39**], we see that for an elliptic problem with zero Dirichlet boundary values and a boundary which satisfies the $Q-$convexity, the dual inequality to (136) and (138), is

$$\Lambda \;+\; |p|\;\Lambda_0 \;+\; |a_0| \;\le\; \nu_0\;(|u|)\;\mathcal{E}, \quad |p| \;\ge\; 1,\; x\in\Omega. \tag{141}$$

This is obtained by the same construction of barriers using the distance function and computing everything using the decomposition of the operator. The last step is to obtain the dual of the inequalities (137) and (139) for some arbitrary boundary data. This follows the exact computations as in the simpler cases. We can now state the general theorem of estimates on the boundary for the first derivative of a solution to an elliptic problem with general Dirichlet data.

Theorem B.4 (Trudinger [**39**], 1972). *Let the operator Q satisfy the decomposition (140) and the estimate*

$$\Lambda \;+\; |p|\Lambda_0 \;+\; |a_0| \;\le\; \nu_1(|u|)\;\mathcal{F}(x,u,p,Df), \quad x\in\Omega,\; |p_k - D_k f| \;\ge\; 1$$

and suppose that $\partial\Omega$ is uniformly $Q^\infty-$convex. Then if u is a $C^2(\overline{\Omega})$ solution of the problem

$$\begin{aligned} Qu \;&=\; 0 \text{ in } \Omega \\ u \;&=\; f \text{ on } \partial\Omega, \end{aligned}$$

we have

$$|Du| \;\le\; C \text{ on } \partial\Omega$$

where C depends only on ν_0, S, k, $\sup_\Omega |u|$ and $|f|_{C^2(\overline{\Omega})}$.

This is the result we base our parabolic version of the first derivative bound on the Dirichlet boundary.

4. Estimates on the Dirichlet boundary for mean curvature flow

Consider now the general parabolic operator, defined also in the short time existence chapter

$$Pu \;=\; \frac{\partial u}{\partial t} \;-\; Q_t \;=\; \frac{\partial u}{\partial t} \;-\; a^{ij}(t,x,u,\mathrm{D}u)\;\mathrm{D}^2_{ij}u \;-\; a(t,x,u,\mathrm{D}u)$$

again with $1 \le i,j \le n$ and continuous coefficients now also in time, $a^{ij}, a \in C^0(\tilde{\Omega})$, where we define $\tilde{\Omega} = \Omega \times [0,T)$, $T > 0$. We use the quantities defined in the construction of barriers made in the elliptic case and think of them as also time dependent by adding a subscript of time, for example Q the elliptic operator becomes Q_t, for a time dependent coefficient operator.

We also redefine the parabolic boundary for our case, a cylinder domain to be $\mathcal{P}\tilde{\Omega} = \partial\Omega \times (0,T) \cup \Omega \times \{0\}$.

Since our discussion is set locally around the Dirichlet boundary then the following theorem is also applicable for our case when the set has two boundaries, and here we are talking about our Chapter 4, 5 and 6 or any other cases when we have combined boundary conditions.

Using the above section of construction of barriers for the elliptic case and noticing that the barriers in the elliptic case can also be used for the parabolic problems we can state the theorem which gives us the Dirichlet boundary bounds for the first derivative of the graphs. First define the two operators

$$\begin{aligned} Q^\infty_t u \;&=\; a^{ij}_\infty(t,x,u,\sigma)\;\mathrm{D}^2_{ij}u \;-\; |\mathrm{D}u|\;C_\infty(t,x,u,\sigma), \quad \text{for } \sigma = \frac{Du}{|Du|}, \\ Q^0_t u \;&=\; a^{ij}_0(t,x,u,\mathrm{D}u)\;\mathrm{D}^2_{ij}u \;+\; a_0(t,x,u,\mathrm{D}u), \end{aligned}$$

where $a^{ij}_\infty, C_\infty \in C^1(\overline{\tilde{\Omega}} \times \mathbb{R} \times \mathbb{R}^n)$. These operators are used to separate the coefficients containing the first derivative in the elliptic operator. We denote by Λ_0 the largest eigenvalue of the operator's Q^0_t second order coefficient. The operator Q^∞ is the one which we use to get the generalised convexity of the boundary set $\partial\Omega$.

Before stating the estimate we want to remind the reader about one functional definition, used in the elliptic construction of barriers which can be extended to the parabolic version by including the time variable

$$\mathcal{F}(t,x,u,p,q) \;=\; a^{ij}(t,x,u,p)(p_i - q_i)(p_j - q_j) \text{ for vectors } p \text{ and } q.$$

Theorem B.5 (Trudinger [**39**], 1972, Dirichlet gradient bound). *Let the operator P satisfy the following two conditions*

$$Q_t u \;=\; \Lambda Q^\infty_t u \;+\; Q^0_t u$$

and

(142)

$$\Lambda \;+\; |\frac{\partial f}{\partial t}| \;+\; |p|\;\Lambda_0 \;+\; |a_0| \;\le\; \nu_1(|u|)\;\mathcal{F}(t,x,u,p,Df), \quad (x,t) \in \tilde{\Omega},\; |p_k - D_k f| \ge 1$$

and suppose that $\partial\Omega$ is uniformly Q_t^∞ convex, $\forall t \in [0,T]$. Then if u is a $C^{2,1}(\overline{\tilde{\Omega}})$ solution of

$$\begin{aligned} Pu &= 0 \ in\ \tilde{\Omega}, \\ u &= f \ on\ \mathcal{P}\tilde{\Omega}, \end{aligned}$$

we have the following boundary first derivative bound

$$|Du| \ \leq \ C \ on\ \partial\Omega \times (0,T),$$

where C depends on ν_1, Ω, $|f|_{C^{2,1}(\overline{\tilde{\Omega}})}$ and $\sup_{\tilde{\Omega}} |u|$.

This theorem is followed by a Corollary which explains why the parabolic minimal surface equation satisfies the requirements of the theorem.

Corollary B.6 (Trudinger [**39**], 1972, Minimal surface equation). *The first initial boundary value problem*

$$\begin{aligned} n\frac{\partial u}{\partial t} &= (1+|Du|)^{-\frac{3}{2}}\Big\{(1+|Du|^2)\ \Delta u \ - \ D^2_{ij}u\ D_iu\ D_ju\Big\} \ in\ \tilde{\Omega}, \\ u &= f \ on\ \mathcal{P}\tilde{\Omega}, \end{aligned}$$

is solvable uniquely in $\tilde{\Omega}$ provided the mean curvature H of the boundary $\partial\Omega$ satisfies $H \geq \frac{n}{n-1} \sup_{(0,T)} |\frac{\partial f}{\partial t}|$ at each point of the boundary $\partial\Omega$.

Proof. The smoothness $C^{2,1}(\tilde{\Omega})$ is obtained by the estimates from the short time existence chapter in the same way for the minimal surface operator as for the mean curvature flow of graphs. However we cannot say that the estimate (142) is satisfied for arbitrary functions f_t. The theorem still applies since $\lim_{|p|\to\infty} \frac{\frac{\partial f}{\partial t}}{|p|\Lambda} = \frac{\partial f}{\partial t}$ and we can reduce the problem to the zero boundary value problem. There f_t can be incorporated in Q_t^∞. Working now with a zero boundary value problem, when we use $u - f$ we can redo the same computation as in obtaining Prop B.2 and see that we have an extra free term $n|\frac{\partial f}{\partial t}|$. The set $\partial\Omega$ is $\mathcal{M}^\infty-$convex if

$$\mathcal{M}^\infty d \ = \ -(n-1)H \ + \ n|\frac{\partial f}{\partial t}| \ \leq \ 0$$

which is true from the assumptions of the theorem. We are thus able to construct barriers which permit us to bound the height and gradient of the solution. □

The same can be done now for the non-parametric mean curvature flow which is just a scaled version of the parabolic minimal surface equation and a better behaved one.

Corollary B.7 (Non-parametric mean curvature flow). *If u is a $C^{2,1}(\overline{\tilde{\Omega}})$ solution of*

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u \ - \ \frac{1}{1+|Du|^2}\ D^2_{ij}u\ D_iu\ D_ju \ in\ \tilde{\Omega}, \\ u &= f \ on\ \mathcal{P}\tilde{\Omega}, \end{aligned}$$

and suppose that we have the mean curvature of the boundary $\partial\Omega$ *satisfying* $H \geq \frac{1}{n-1} \sup_{(0,T)} |\frac{\partial f}{\partial t}|$ *at each point of the boundary* $\partial\Omega$ *then there exists a constant* C *depending only on* ν_1, Ω, $|f|_{C^{2,1}(\bar{\tilde{\Omega}})}$ *and* $\sup_{\tilde{\Omega}} |u|$ *such that*

$$|Du| \leq C \text{ on } \partial\Omega \times (0,T).$$

Proof. The proof follows the exact lines of the above proof of corollary. We use the result of Corollary B.3, which also looks at the operator coming from the elliptic part of the mean curvature flow of graphs. □

Remark. It is easy to see that the required estimate for the mean curvature of the boundary is easily satisfied when the Dirichlet boundary function is not time dependent and the mean curvature of the boundary is positive. This is the situation in two type of problems we consider throughout the Chapters 4, 5 and 6. We can therefore state that the construction of barriers for our case is achieved. Also in our case we use the mean curvature as being the sum of principal curvatures, unscaled by the dimension of the set $\partial\Omega$ so our hypothesis for the mean curvature can be reduced to $H \geq \sup_{(0,T)} |\frac{\partial f}{\partial t}| \geq 0$ at each point of the boundary $\partial\Omega$.

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