The inverse mean curvature flow for hypersurfaces with boundary

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Eigenständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Hilfsmittel verwendet habe. Die Arbeit wird zum ersten Mal in einem Promotionsverfahren eingereicht.

Thomas Marquardt

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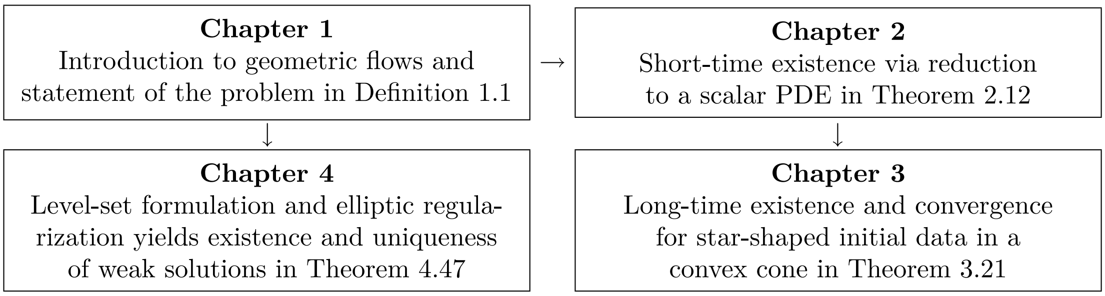
Contents

Preface

The evolution of hypersurfaces in the direction of the unit normal with speed equal to the reciprocal of the mean curvature is called inverse mean curvature flow (IMCF). In the case of closed hypersurfaces this flow is well studied. One of the classical results goes back to Gerhardt [16] (see also Urbas [65]). He proved long-time existence and convergence to a round sphere for star-shaped initial data with strictly positive mean curvature. A more recent result with a striking application to theoretical physics is due to Huisken and Ilmanen [29]. They defined weak solutions of IMCF and proved existence and uniqueness of such solutions. This was one of the main tools in their proof of the Riemannian Penrose inequality which gives an estimate for the mass in general relativity. In the current work we will investigate IMCF in the case where the hypersurfaces possess a boundary and move along, but stay perpendicular to, a fixed supporting hypersurface. The work is organized as follows:



Chapter 1
Introduction to geometric flows and statement of the problem in Definition 1.1

Chapter 2
Short-time existence via reduction to a scalar PDE in Theorem 2.12

Chapter 4
Level-set formulation and elliptic regularization yields existence and uniqueness of weak solutions in Theorem 4.47

Chapter 3
Long-time existence and convergence for star-shaped initial data in a convex cone in Theorem 3.21

We will use Chapter 1 to give a more detailed overview about geometric evolution equations in general and about IMCF for closed hypersurfaces in particular. Furthermore, we will specify our setup for hypersurfaces with boundary.

The first question which we have to answer is whether or not this flow has a solution for a small time. This short-time existence result is obtained in Chapter 2, Theorem 2.12 by writing the hypersurface as a graph over the initial hypersurface and reducing the equations to a scalar parabolic Neumann problem. This approach was also used by Stahl [59] for hypersurfaces with boundary evolving under mean curvature flow.

The counter example of a half-torus evolving on a plane shows that long-time existence cannot be expected in general. However, in the case where the supporting hypersurface is a convex cone and the initial hypersurface is star-shaped and has strictly positive mean curvature, we are able to prove long-time existence and convergence to a spherical cap. This work is carried out in Chapter 3. The main result is Theorem 3.21. This is the analogous statement to the one of Gerhardt [16] for closed hypersurfaces.

In order to deal with more general supporting hypersurfaces we follow the ideas of Huisken and Ilmanen [29] and define weak solutions in Chapter 4. First, we use a level-set approach together with a regularization procedure to obtain solutions for a family of regularized elliptic mixed boundary value problems in domains with corners. These solutions give rise to a converging sequence of weak solutions one dimension higher. Thanks to a compactness result we can finally prove that the limit is the unique minimizer of a certain functional related to the level-set problem. This program yields existence and uniqueness for weak solutions of IMCF in the case of hypersurfaces with boundary in Theorem 4.47.

1 Introduction

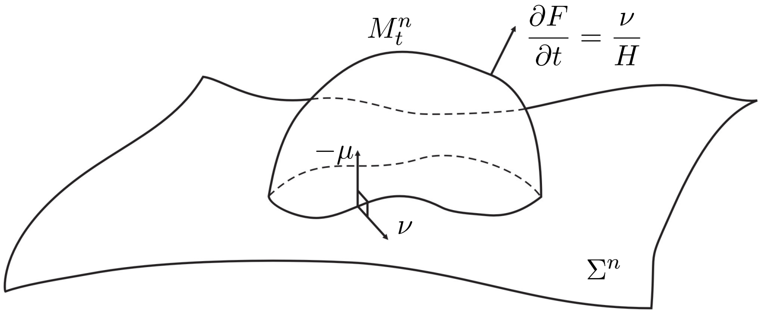


Figure 1.1: Inverse mean curvature flow for hypersurfaces with boundary.

An introduction to a thesis is definitely not the right place for a detailed summary of more than 50 years of research in the area of geometric evolution equations. Having said that, one cannot talk about the particular problem of inverse mean curvature flow (IMCF) without putting it in broader context and mentioning some of the cornerstones in the area of geometric evolution equations. So let us try to give an overview about important results related to geometric flows in Section 1 and then focus on IMCF for closed surfaces and surfaces with boundary in the Sections 2 and 3.

1.1 Geometric evolution equations

Geometric evolution equations, which are also called geometric flows, have been studied for more than fifty years now. As they describe the deformation of geometric quantities in terms of partial differential equations (PDEs), this topic is settled between differential geometry and the theory of PDEs. Often methods from the calculus of variations, geometric measure theory and functional analysis are also used to treat the problems. The motivation for looking at geometric flows arises from various areas such as topology, physics or even image processing. From the viewpoint of PDEs one can distinguish different flows by the type of equation used to describe them. Another way to distinguish them from the viewpoint of differential geometry is to divide them into extrinsic and intrinsic flows.

Intrinsic flows are defined by a PDE which changes an intrinsic geometric quantity. One family of examples is to change the metric g of the manifold (M, g) according to the law

$$\frac{\partial g}{\partial t} = f(g) \tag{1.1}$$

where f is a function depending on g and derivatives of g. A well known example of this type is the Ricci flow where f takes the form $f(g) := -2\,\mathrm{Ric}(g)$ and $\mathrm{Ric}(g)$ is the Ricci curvature of the manifold (M, g). Hamilton introduced it in 1981 as an approach to solve Thurston's geometrization conjecture, which is a topological classification for closed 3-manifolds. Based on Hamilton's work, Perelman [50, 51] achieved the outstanding task of proving this conjecture in 2003. As a corollary, the Poincaré conjecture – an open problem since 1904 – was also settled. It states that every simply connected, closed 3-manifold is homeomorphic to the 3-sphere. For more details see [12, 34, 64] and the references therein. A main tool used by Hamilton and Perelman is the so called surgery. It describes the process of cutting out certain regions of the evolving surface in order to prevent the formation of singularities.

Another intrinsic flow which was also introduced by Hamilton is the Yamabe flow. It can be written in the form (1.1) using $f(g) := (\overline{\mathrm{R}} - \mathrm{R})g$ where R is the scalar curvature of (M, g) and $\overline{\mathrm{R}}$ is its mean value over M. Hamilton introduced this flow as a tool to study the Yamabe problem. That is the problem of finding, for a given compact Riemannian manifold (M, g) of dimension $n \geq 3$, a positive scalar function φ such that φg has constant scalar curvature. After partial results of Trudinger, Aubin and others the Yamabe problem was finally solved by Schoen [53] in 1984. The proof involved the Riemannian positive mass theorem which he proved together with Yau [54] in 1979. It states that a 3-manifold of non negative scalar curvature has non negative ADM-mass. This concept of mass is due to Arnowitt, Deser and Misner [3]. For an asymptotically flat[1] 3-manifold the ADM-mass is obtained as the limit of a flux integral through the sphere at infinity

$$m_{\mathrm{ADM}} := \lim_{r \to \infty} \frac{1}{16\pi} \int_{\partial B_r(0)} \sum_{i,j} \left(\partial_j g_{ii} - \partial_i g_{ij}\right) \nu^j \, \mathrm{d}\mu,$$

where ν denotes the exterior unit normal to the sphere. A survey on the Yamabe problem and all the references to the results of Hamilton, Schoen and Yau can be found in the work of Lee and Parker [40].

A different family of interesting problems involves extrinsic flows. They are defined using extrinsic geometric quantities such as the mean curvature. Therefore, the manifold under consideration must be embedded (or more generally immersed) into an ambient manifold to make sense of the extrinsic quantities. The presence of an ambient manifold allows one to investigate the flow in different settings by changing the co-dimension or by choosing a Lorentzian ambient space instead of a Riemannian one.

Let us consider the case of Riemannian ambient spaces and one co-dimension. One way to describe the evolution of the embedded hypersurface is to do it in terms of the evolution of the embedding $F : M^n \to N^{n+1}$. We require that F satisfies

$$\frac{\partial F}{\partial t} = f\nu \tag{1.2}$$

so that every point on the embedded manifold moves in the direction of its unit normal ν with speed f. Here f is a function depending on some extrinsic quantities. An example of such a flow is the mean curvature flow (MCF) where $f := -H$ and H stands for the scalar mean curvature of M^n in N^{n+1}. The easiest setting one should have in mind for MCF is the case where the initial hypersurface is given by $S^n_{r_0}$, i.e. the n-sphere of radius r_0 embedded in $N^{n+1} := \mathbb{R}^{n+1}$ and ν is the outward pointing unit normal. Under MCF

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[1] Roughly speaking a manifold $M = C \cup D$ is asymptotically flat if C is compact and D is diffeomorphic to $\mathbb{R}^n \setminus K$ for some compact set K. See e.g. [29] for an exact definition.

the initial sphere stays a round sphere but shrinks to a point in finite time $T := r_0^2/2n$. The radius at time t is given by $r(t) = \sqrt{r_0^2 - 2nt}$.

MCF was first introduced by Mullins [49] in 1956 and independently by Brakke [6] in 1978 from the viewpoint of geometric measure theory. Since then the flow was widely studied. A detailed and chronological review of the developments in MCF can be found in the introductory part of Ecker [14] or Ilmanen [32]. One of the latest interesting developments is the classification result for 2-convex[2] surfaces by Huisken and Sinistrari [31] in 2009. The statement is that every smooth, closed, n-dimensional, 2-convex surface which is immersed in $\mathbb{R}^{n+1}$ is either diffeomorphic to S^n or to a finite connected sum of $S^{n-1} \times S^1$. A major tool in the proof was a surgery procedure for mean curvature flow similar to the surgery Hamilton used in Ricci flow. Furthermore, in 2011 Head [26] proved convergence of a sequence of surgery solutions to the weak solution of the level-set flow.

1.2 Inverse mean curvature flow (IMCF)

The flow we will be concerned with in this work is the inverse mean curvature flow (IMCF). Like MCF this is an extrinsic flow but here we define $f := 1/H$ in (1.2). In contrast to MCF the surfaces are expanding. If, as above, we consider the example of a sphere $S^n_{r_0}$ in $N := \mathbb{R}^{n+1}$ we observe that the initial sphere stays round under IMCF. The formula for the radius is $r(t) = r_0 e^{t/n}$. This behavior is a special case of a theorem of Gerhardt [16]. It states that under IMCF compact, star-shaped initial hypersurfaces with strictly positive mean curvature converge after suitable rescaling to a round sphere. In addition, examples of eternal solutions to IMCF are known. They are discussed by Huisken and Ilmanen in [27].

IMCF was put forward by Geroch [20] and Jang and Wald [33] in the seventies as an approach to the proof of the positive mass theorem. Geroch showed that as long as IMCF remains smooth it can be used to prove the Riemannian Penrose inequality and therefore the positive mass theorem. The Riemannian Penrose inequality states that an asymptotically flat, complete, connected 3-manifold with non negative scalar curvature and with one (to keep things simple here) compact minimal surface N_0 as its compact boundary satisfies the inequality

$$m_{\mathrm{ADM}} \geq \sqrt{\frac{|N_0|}{16\pi}}.$$

In a nut shell Geroch's argument was the following. He combined Hawking's observation that the so called Hawking quasi-local mass

$$m_{\mathrm{Haw}}(N_t) := \frac{|N_t|^{1/2}}{(16\pi)^{3/2}} \left(16\pi - \int_{N_t} H^2 \, \mathrm{d}\mu_t \right)$$

converges to m_{ADM} if the surfaces N_t converge to a sphere at infinity with his observation that $m_{\mathrm{Haw}}(N_t)$ is monotone increasing in t for smooth solutions of IMCF. Thus, if the initial hypersurface for IMCF is the minimal surface N_0, if $m_{\mathrm{Haw}}(N_t) \to m_{\mathrm{ADM}}$ and if the flow remains smooth one obtains

$$\sqrt{\frac{|N_0|}{16\pi}} \; = \; m_{\mathrm{Haw}}(N_0) \; \leq \; m_{\mathrm{Haw}}(N_t) \; \to \; m_{\mathrm{ADM}}$$

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[2] Two-convexity means that the sum of the two smallest principal curvatures is non-negative.

assuming the surfaces N_t become round in the limit. Unfortunately the flow does not remain smooth in general. This can be seen if one starts with a thin torus of positive mean curvature which is embedded in $\mathbb{R}^3$. Then one notes that it fattens up and therefore, after some time, the mean curvature reaches zero at some points. Thus, the classical flow has to break down.

In 2001 Huisken and Ilmanen [29] used a level-set approach and developed the notion of weak solutions for IMCF to overcome theses problems. They showed existence for weak solutions and proved that Geroch's monotonicity for the Hawking mass carries over to the weak setting. This enabled them to prove the Riemannian Penrose inequality which also gave an alternative proof for the Riemannian positive mass theorem. A summary about their work is given in [27] and [28]. In [30] Huisken and Ilmanen proved higher regularity for IMCF in $\mathbb{R}^n$ (see also Smoczyk [58] for $n = 2$). Their work also shows that weak solutions become star-shaped and smooth outside some compact region and thus (by the result of Gerhardt) round in the limit. A different proof of the most general form[3] fo the Riemannian Penrose inequality was given by Bray [7]. An overview about the different methods used by Huisken and Ilmanen and Bray can be found in [8]. An approach to solve the full Penrose inequality was brought up by Bray et. al. [9] defining a generalized IMCF. Despite that the full Penrose inequality is still an open problem.

Another remarkable result which was obtained using IMCF is the proof of the Poincaré conjecture for 3-manifolds with Yamabe invariant greater than that of $\mathbb{RP}^3$ by Bray and Neves [10] (see also [1]).

Schulze [55] used the level-set approach to study flows with speed equal to positive powers of the mean curvature. In [56] he used this formulation of the flow to give a new proof of the isoperimetric inequality. Furthermore, in a joint work with Metzger they proved the so-called no mass drop property for mean curvature flow [48].

1.3 IMCF for hypersurfaces with boundary

The project of this thesis is to consider IMCF in the case where the hypersurfaces possess a boundary and move along but stay perpendicular to a fixed supporting hypersurface (see Figure 1.1). The exact setting is contained in the following definition.

Definition 1.1. Let M^n be a compact, smooth, orientable, manifold with compact, smooth boundary ∂M^n. Let Σ^n be an orientable $C^{2,\alpha}$-hypersurface without boundary in the Riemannian ambient manifold $(N^{n+1}, \overline{\overline{\gamma}})$. Suppose that $F_0 : M^n \to N^{n+1}$ is a $C^{2,\alpha}$-immersion such that $M_0^n := F_0(M^n)$ has strictly positive mean curvature and satisfies

$$F_0\left(\partial M^n\right) = F_0\left(M^n\right) \cap \Sigma^n, \qquad \langle \nu_0, \mu \circ F_0 \rangle_{\overline{\overline{\gamma}}} \ = \ 0 \ \text{ on } \partial M^n,$$

where ν_0 and μ are the unit normal vector fields on M^n and Σ^n respectively.[4] We say that the one-parameter family of smooth immersions $F : M^n \times [0, T) \to N^{n+1}$ moves

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[3] Bray proved that $16\pi m_{\mathrm{ADM}}^2 \geq |\partial M|$ with no assumption on the connectedness of ∂M.

[4] Note that locally F_0 is an embedding so it makes sense to talk about a normal $\nu_0(x)$ but it makes no sense to write $\nu_0(F_0(x))$.

under inverse mean curvature flow if F satisfies $F\left(\partial M^n, t\right) = F\left(M^n, t\right) \cap \Sigma^n$ and

$$
\text{(IMCF)} \quad \begin{cases} \dfrac{\mathrm{d}F}{\mathrm{d}t} = \dfrac{\nu}{H} & \text{in } M^n \times (0,T) \\[2ex] \langle \nu, \mu \circ F \rangle_{\overline{\overline{\gamma}}} = 0 & \text{on } \partial M^n \times (0,T) \\[2ex] F(\,.\,,0) = F_0 & \text{on } M^n. \end{cases}
$$

Here ν is a choice of unit normal vector field on M^n and H is the scalar mean curvature of M^n in N^{n+1} which is supposed to be positive. Furthermore μ is chosen to point away from M_t i.e. for curves on M_t ending at $p \in \partial M_t$ with tangent vector $v(p)$ we have $\langle v, \mu \rangle_{\overline{\overline{\gamma}}}(p) \geq 0$.

Remark 1.2. The corresponding Neumann problem for mean curvature flow was first studied by Stahl [59–61]. It was followed by the work of Buckland [11] who analyzed the singularities and by the work of Koeller [35, 36] who proved further regularity results.

Currently, Alexander Volkmann [66] is using the level-set approach to study the Neumann problem for flows with speed equal to positive powers of the mean curvature.

Example 1.3. Let us assume that the supporting hypersurface Σ^n is the hyperplane $\{e_{n+1} = 0\}$ in $N^{n+1} = \mathbb{R}^{n+1}$ and the initial embedded hypersurface is a half-sphere of radius r_0 centered at the origin. Then the solution of (IMCF) exists for all time and is given at time t as the half sphere centered at the origin with radius $r(t) = r_0 e^{t/n}$. This example also shows that two half-spheres of radius r_0 which are centered at two points of distance $R > 2r_0$ would collide at time $T = n \ln(R/2r_0)$.

Notice, that as long as Σ^n is a hyperplane we can exploit the symmetry and obtain solutions using the results of IMCF for closed surfaces by reflecting the hypersurfaces with respect to Σ^n. Using this technique we see that a half-torus of positive mean curvature fattens up under (IMCF) and develops points of zero mean curvature in finite time. Thus, the evolution as it is described by (IMCF) breaks down after finite time.

Remark 1.4. Besides the description of the hypersurfaces M_t^n as images of an embedding F, i.e. $M_t^n = F(M^n, t)$ we will also consider the hypersurfaces as the t-level sets of a scalar function. To do this we need some notation. Let us denote by Ω all points on Σ^n and above Σ^n. For sets $A \subset \Omega$ we want to distinguish the boundary parts of A on Σ^n and inside Ω by writing

$$
\partial_\Omega A := \overline{\partial A \setminus \Sigma^n} \qquad \text{and} \qquad \partial_\Sigma A := \partial A \setminus \partial_\Omega A.
$$

The aim is to find a function $u : \Omega \to \mathbb{R}$ such that $M_t^n = \partial_\Omega \{u < t\}$. We will show that, as long as the mean curvature of M_t^n is strictly positive, the parabolic formulation (IMCF) is equivalent to

$$
(\star) \quad \begin{cases} \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = |Du| & \text{in } \Omega_0 := \Omega \setminus \overline{E_0} \\[2ex] D_\mu u = 0 & \text{on } \partial_\Sigma \Omega_0 \\[2ex] u = 0 & \text{on } \partial_\Omega E_0 \end{cases}
$$

where $E_0 = \{u < 0\}$ and μ is the normal to Σ^n. Note that $(\star)$ is a degenerate elliptic mixed boundary value problem in a non-smooth domain. As in the work of Huisken and Ilmanen [29] the formulation $(\star)$ is the starting point for the definition of weak solutions via $J_u^K(u) \leq J_u^K(v)$ for locally Lipschitz competitors v satisfying $\{u \neq v\} \subset\subset \Omega_0$. The functional is defined by

$$J_u^K : C^{0,1}_{loc}(\Omega_0) \to \mathbb{R} : v \mapsto J_u^K(v) := \int_K \left(|Dv| + v|Du|\right) \mathrm{d}\lambda \tag{1.3}$$

and the integration is performed over any compact set K containing $\{u \neq v\}$. It turns out that this formulation allows us to overcome the problems mentioned in Example 1.3.

Outline. The work is organized as follows. The first question which we have to answer is whether or not (IMCF) has a solution for a small time. This short-time existence result is obtained in Chapter 2, Theorem 2.12 by writing the hypersurface as a graph over the initial hypersurface and reducing the equations to a scalar parabolic Neumann problem. This approach was also used by Stahl [59] for hypersurfaces with boundary evolving under mean curvature flow.

The counter example of a half-torus evolving on a plane shows that long-time existence cannot be expected in general. However, in the case where the supporting hypersurface is a convex cone and the initial hypersurface is star-shaped and has strictly positive mean curvature, we are able to prove long-time existence and convergence to a spherical cap. This work is carried out in Chapter 3. The main result of that chapter is Theorem 3.21. This is the analogous statement to the one of Gerhardt [16] for closed hypersurfaces.

In Chapter 4 we follow the ideas of Huisken and Ilmanen [29] and define weak solutions of $(\star)$ as the minimizers of the functional (1.3). To prove the existence of those weak solutions we regularize $(\star)$ to obtain solutions u^ε of a family of non-degenerate elliptic mixed boundary value problems $(\star)_\varepsilon$ in weighted Hölder spaces. These solutions give rise to a converging sequence $U^{\varepsilon_i}(x,z) := u^{\varepsilon_i}(x) - \varepsilon_i z$ of smooth solutions to (IMCF) one dimension higher. Thanks to a compactness result we can prove that in the limit as $\varepsilon_i \to 0$ there exists a sequence converging to $U(x,z) := u(x)$ which is the minimizer of the functional (1.3) one dimension higher. Finally, we use cut-off functions to prove that u is the unique weak solution of IMCF in the case of hypersurfaces with boundary. This is our main result which is stated in Theorem 4.47. The last section gives an outlook to a potential application of weak solutions indicated by the monotonicity of the Hawking mass for classical solutions of (IMCF).

2 Short-time existence

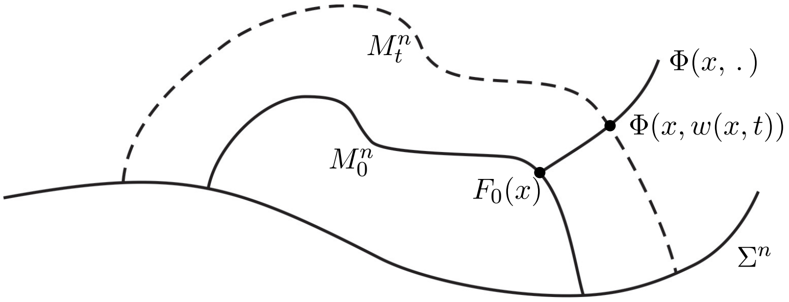

Figure 2.1: Generalized tubular neighborhood of M_0^n.

In order to prove short-time existence for (IMCF) we will write the evolving hypersurface at time t as a graph over the initial hypersurface. Therefore, we need some coordinates which are adapted to the geometry of the supporting hypersurface Σ^n. We will introduce these coordinates in Section 1. Then we will treat a scalar Neumann problem in Section 2 which will give rise to a solution of (IMCF) as we will encounter in Section 3. The main result of this chapter is the short-time existence result stated in Theorem 2.12. The same method was applied by Stahl [59] to prove short-time existence for hypersurfaces with boundary evolving under mean curvature flow.

2.1 Generalized tubular neighborhood

In general $M_0^n := F_0(M^n)$ is not embedded but only immersed in the ambient space N^{n+1}. Therefore, we will rather work on M^n than on $M_0^n \subset N^{n+1}$. We will need the following Lemma.

Lemma 2.1. *Let* $(N^{n+1}, \overline{\overline{\gamma}}), M^n, \Sigma^n$ *and* F_0 *be as in Definition 1.1. Then there is a generalized tubular neighborhood* $\mathscr{U}_\varepsilon \subset N^{n+1}$ *of* $M_0^n = F_0(M^n)$ *which is the immersed image of the product manifold* $M^n \times [-\varepsilon, \varepsilon]$ *and respects the geometry of* Σ^n*. More precisely there is an isometric immersion*

$$\Phi : \left(M^n \times [-\varepsilon, \varepsilon], \overline{\gamma}\right) \to \mathscr{U}_\varepsilon \subset \left(N^{n+1}, \overline{\overline{\gamma}}\right) : (x, s) \mapsto \Phi(x, s)$$

where $p = \Phi(x, s)$ *is the point on a curve* $\Phi(x,\,.\,)$ *which starts at* $F_0(x) \in M_0^n$ *in direction of the unit normal* $\nu(x)$ *such that the length from* p *to* $F_0(x)$ *is equal to* s[1]*.* Φ *respects the geometry of* Σ^n *in the sense that for* $x \in \partial M^n$ *we have* $\Phi(x, s) \in \Sigma^n$ *for all* $s \in [-\varepsilon, \varepsilon]$*.*

Proof. Let $x \in M^n$. There is a neighborhood $U_x \subset M^n$ of x and a neighborhood $V_x \subset M_0^n$ of $F_0(x)$ such that F_0 restricted to U_x is a smooth embedding. By $W_x \subset N^{n+1}$ we denote

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[1] The distance to points which are reached if one follows Φ starting in direction $-\nu$ gets a negative sign by definition.

a neighborhood in the ambient space such that $W_x \cap M_0^n = V_x$. Since M^n is compact we have

$$M^n \subset \bigcup_{x \in M^n} U_x \;\Rightarrow\; M^n \subset \bigcup_{k=1,\dots,N} U_{x_k}.$$

Furthermore, we can choose the cover in such a way that a small neighborhood W of M_0^n is contained in $W_{x_1} \cup \dots \cup W_{x_N}$.

If $V_{x_k} \cap \Sigma^n = \emptyset$ we define a vector field ξ_k in TW_{x_k} being the tangent field to the geodesic arcs $\Phi_k(x,\,.\,)$ in N^{n+1} starting at $F_0(x)$ in direction $\nu(x)$ for $x \in U_{x_k}$. If $V_{x_k} \cap \Sigma^n \neq \emptyset$ then $\Phi_k(x,\,.\,)$ is the integral curve with respect to a vector field $\xi_k \in TW_{x_k}$ which satisfies

$$\xi_k|_{V_{x_k}} \in NV_{x_k}, \qquad \xi_k|_{\Sigma^n} \in T\Sigma^n, \qquad \|\xi_k\|_{N^{n+1}} = 1.$$

Again $\Phi_k(x,\,.\,)$ is starting from $F_0(x)$ in direction ν.

Now we use a partition of unity, i.e. maps $\chi_i \in C_c^\infty(W_{x_i}, \mathbb{R})$ for $1 \leq i \leq N$ satisfying $\sum_{i=1}^N \chi_i \equiv 1$ in W. This allows us to construct the vector field

$$\xi : M^n \times [-\varepsilon, \varepsilon] \to TN^{n+1} : (x,s) \mapsto \xi(x,s) := \sum_{i=1}^N \chi_i(\Phi_i(x,s))\xi_i(\Phi_i(x,s))$$

from which we obtain a family of integral curves, i.e. a map $\Phi : M^n \times [-\varepsilon, \varepsilon] \to N^{n+1}$. Next we define

$$e_i(\Phi(x,s)) := \frac{\partial \Phi(x,s)}{\partial x^i}, \qquad e_{n+1}(\Phi(x,s)) := \frac{\partial \Phi(x,s)}{\partial s}$$

for $i = 1, \dots, n$ and notice that $\operatorname{rank}(\Phi)(x,0) = n+1$ since $\Phi(\,.\,,0) = F_0$ is an immersion and $e_{n+1} \in NM_0^n$. Thus, for $x \in M^n$ and small s we get $\operatorname{rank}(\Phi)(x,s) = n+1$ and Φ is an immersion. Therefore, if $\varepsilon > 0$ is sufficiently small and $M^n \times [-\varepsilon, \varepsilon]$ is equipped with the metric

$$\overline{\gamma}_{\alpha\beta}(x,s) := (\Phi^* \overline{\overline{\gamma}}_{\alpha\beta})(x,s) = \overline{\overline{\gamma}}(e_\alpha(\Phi(x,s)), e_\beta(\Phi(x,s))), \qquad 1 \leq \alpha, \beta \leq n+1$$

then Φ is an isometric immersion. □

Remark 2.2. If Σ is totally geodesic we can replace Φ by geodesics starting from $F_0(x)$ in the direction of $\nu(x)$. In this case we obtain a classical tubular neighborhood.

Remark 2.3. Since $\Phi(x,0) = F_0(x)$ the proof of Lemma 2.1 implies that the metric on $F_0(M^n)$ is given by $\gamma_{ij}(x) := \overline{\overline{\gamma}}(e_i(p), e_j(p))$ with $p = \Phi(x,0)$ and that for $t = 0$ we have

$$\overline{\gamma}(x,0) = \left(\begin{array}{ccc|c} & & & 0 \\ & \gamma_{ij}(x) & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 & 1 \end{array} \right)$$

where $1 \leq i, j \leq n$.

Remark 2.4. The idea is that we use Φ and a scalar function $w(\,.\,,t)$ to describe points p of the hypersurface M_t^n as $p = \Phi(x, w(x,t))$. This is shown in Figure 2.1. Since the immersion Φ is isometric we can locally identify $(M^n \times [-\varepsilon, \varepsilon], \overline{\gamma})$ and $(\mathscr{U}_\varepsilon, \overline{\overline{\gamma}})$. Furthermore, we can locally identify $(M^n, \overline{\gamma}|_{M^n})$ and $(M_0^n, \overline{\overline{\gamma}}|_{M_0^n})$. In that sense the hypersurface M_t^n can be described by

$$F_t : M^n \to M^n \times [-\varepsilon, \varepsilon] : x \mapsto (x, w(x,t)).$$

The next lemma is concerned with the geometry of those graphs.

Lemma 2.5. *Let $t \geq 0$ be fixed. Let $w(\,.\,,t) : M^n \to [-\varepsilon, \varepsilon]$ be in $C^2(M^n)$ and $M_t^n :=$ $\operatorname{graph}(w(\,.\,,t)) \subset (M^n \times [-\varepsilon, \varepsilon], \overline{\gamma})$. Let $p := (x, w(x,t))$ and e_α be the standard basis vectors of $T_p(M^n \times [-\varepsilon, \varepsilon])$. In the point p we have the following formulas.*

(i) The standard basis for $T_p M_t^n$ is given by:

$$\tau_k := e_k + D_k w e_{n+1}, \qquad 1 \leq k \leq n.$$

(ii) A unit normal to M_t^n in p is given by

$$\nu := v^{-1} \overline{\gamma}^{-1} \begin{pmatrix} -Dw \\ 1 \end{pmatrix}$$

with $v^2 := \overline{\gamma}^{n+1,n+1} - 2\overline{\gamma}^{k,n+1} D_k w + \overline{\gamma}^{kl} D_k w D_l w.$

(iii) Let ν be as in (ii) then we have the following relations

$$\langle \nu, e_k \rangle_{\overline{\gamma}} = -v^{-1} D_k w \quad 1 \leq k \leq n, \qquad \langle \nu, e_{n+1} \rangle_{\overline{\gamma}} = v^{-1}.$$

(iv) The metric and second fundamental form of $T_p M_t^n$ are given by

$$g_{ij} = \overline{\gamma}_{ij} + \overline{\gamma}_{i,n+1} D_j w + \overline{\gamma}_{n+1,j} D_i w + \overline{\gamma}_{n+1,n+1} D_i w D_j w,$$

$$h_{ij} = -v^{-1} \left(D_{ij} w - \overline{\Gamma}^k_{\alpha\beta} \tau_i^\alpha \tau_j^\beta D_k w + \overline{\Gamma}^{n+1}_{\alpha\beta} \tau_i^\alpha \tau_j^\beta \right)$$

where $\overline{\Gamma}$ denote the Christoffel-symbols with respect to the metric $\overline{\gamma}$.

Note that D_i and D_{ij} are not covariant but partial derivatives.

Proof. (i) This statement follows from the definition of $\tau_k = (F_t)_*(\partial/\partial x^k)$ with $F_t : M^n \to M^n \times [-\varepsilon, \varepsilon] : x \mapsto (x, w(x,t))$.

(ii) Using $\hat{\nu} := (-Dw, 1)$ we obtain

$$\langle \tau_k, \nu \rangle_{\overline{\gamma}} = \overline{\gamma}_{\alpha\beta} \tau_k^\alpha \nu^\beta = \overline{\gamma}_{\alpha\beta} \tau_k^\alpha \frac{1}{v} \overline{\gamma}^{\beta\rho} \hat{\nu}_\rho = \frac{1}{v} \tau_k^\alpha \hat{\nu}_\alpha = 0.$$

The vector $\nu = \frac{1}{v} \overline{\gamma}^{-1} \hat{\nu}$ has unit norm for $v := |\overline{\gamma}^{-1} \hat{\nu}|_{\overline{\gamma}}$ and

$$|\overline{\gamma}^{-1} \hat{\nu}|^2_{\overline{\gamma}} = \langle \overline{\gamma}^{-1} \hat{\nu}, \overline{\gamma}^{-1} \hat{\nu} \rangle_{\overline{\gamma}} = \overline{\gamma}^{n+1,n+1} - 2 D_k w \overline{\gamma}^{k,n+1} + D_k w D_l w \overline{\gamma}^{kl}.$$

(*iii*) This is clear from $\langle \nu, e_\delta\rangle_{\overline{\gamma}} = v^{-1}\overline{\gamma}_{\alpha\beta}\overline{\gamma}^{\alpha\rho}\hat{\nu}_\rho e^\beta_\delta = v^{-1}\hat{\nu}_\delta$ and the definition of $\hat{\nu}$.

(*iv*) For g the formula follows from

$$g_{ij} := \langle \tau_i, \tau_j\rangle_{\overline{\gamma}} = \overline{\gamma}_{kl}\tau_i^k\tau_j^l + \overline{\gamma}_{k,n+1}\tau_i^k\tau_j^{n+1} + \overline{\gamma}_{n+1,l}\tau_i^{n+1}\tau_j^l + \overline{\gamma}_{n+1,n+1}\tau_i^{n+1}\tau_j^{n+1}.$$

Using

$$-h_{ij}\nu := \overline{\nabla}_{\tau_i}\tau_j - \nabla_{\tau_i}\tau_j = D_{ij}F_t + D_iF_t^\alpha D_jF_t^\beta\overline{\Gamma}^\rho_{\alpha\beta}e_\rho - \Gamma^k_{ij}D_kF_t$$

we obtain for the second fundamental form

$$h_{ij} = \langle h_{ij}\nu, \nu\rangle_{\overline{\gamma}} = -\langle D_{ij}F_t, \nu\rangle_{\overline{\gamma}} - \overline{\Gamma}^\rho_{\alpha\beta}D_iF_t^\alpha D_jF_t^\beta \langle e_\rho, \nu\rangle_{\overline{\gamma}} + \Gamma^k_{ij}\langle D_kF_t, \nu\rangle_{\overline{\gamma}}.$$

The last inner product vanishes. Using the results from (ii), (iii) and the fact that $D_{ij}F_t = D_{ij}w\; e_{n+1}$ yields the result. □

2.2 Associated scalar Neumann problem

In this section we want to solve a parabolic Neumann problem for a scalar function w. This function occurs when we express the evolving hypersurface as a graph over the initial surface. The relation between w and the solution to (IMCF) will be discussed in the next section. The scalar Neumann problem is the following:

$$\text{(SP)}\begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{v}{H}(\,.\,, w, Dw, D^2w) = 0 & \text{in } M^n \times (0,T) \\ r^i(\,.\,, w)D_iw = s(\,.\,, w) & \text{on } \partial M^n \times (0,T) \\ w(\,.\,, 0) = 0 & \text{on } M^n \end{cases}$$

where $r(x,w) := r^i(x,w)e_i(x) \in T_xM^n$, $r(x,0) = \nu$ is the outward unit normal to ∂M^n and $s(x,0) = 0$. The idea is to obtain a solution to (SP) using the inverse function theorem. Before we can prove the existence of a solution we need two lemmas.

Lemma 2.6. *Suppose that M_0^n is a smooth immersed hypersurface with strictly positive mean curvature $H_0 \in C^{0,\alpha}(M^n)$. Let ν be the outward unit normal to ∂M^n. Then the auxiliary problem*

$$\text{(AP)}\begin{cases} \dfrac{\partial w}{\partial t} - \Delta w = \dfrac{1}{H_0} & \textit{in } M^n \times (0,T) \\ \nu^iD_iw = 0 & \textit{on } \partial M^n \times (0,T) \\ w(\,.\,, 0) = 0 & \textit{on } M^n \end{cases}$$

has a unique solution $w_0 \in C^{2+\alpha, 1+\frac{\alpha}{2}}(M^n \times [0,T])$.

Proof. Since $w(\,.\,,0) = 0$ the compatibility condition $\nu^i(x)D_iw(x,0) = 0$ is satisfied. As the directional derivative at ∂M^n is transversal the theory of linear parabolic equations (see Theorem A.4) yields a unique solution $w_0 \in C^{2+\alpha, 1+\frac{\alpha}{2}}(M^n \times [0,T])$. □

The role of (AP) will become clear in the existence proof for (SP). Before we come to that point we want to calculate the linearization of (SP) around w_0.

Lemma 2.7. *Let* $w_0 \in C^{2+\alpha,1+\frac{\alpha}{2}}(M^n \times [0,T])$ *be the solution of* (AP). *Let* $\zeta \in C^{\alpha,\frac{\alpha}{2}}(M^n \times [0,T])$, $\eta \in C^{1+\alpha,\frac{1+\alpha}{2}}(\partial M^n \times [0,T])$ *with* $\eta(\,.\,,0) = 0$. *Then there is some* $T > 0$ *such that the linearization of* (SP) *around* w_0 *given by*

$$\text{(LSP)}\quad \begin{cases} L_{w_0} w := \dfrac{\partial w}{\partial t} - a^{ij} D_{ij} w + b^k D_k w + cw = \zeta & \textit{in } M^n \times (0,T) \\ N_{w_0} w := r_0^i D_i w + s_0 w = \eta & \textit{on } \partial M^n \times (0,T) \\ w(\,.\,,0) = 0 & \textit{on } M^n \end{cases}$$

has a unique solution $w \in C^{2+\alpha,1+\frac{\alpha}{2}}(M^n \times [0,T])$.

Proof. The PDE in (SP) can be written as $\partial w/\partial t - Q(x, w, Dw, D^2 w) = 0$ with

$$Q : M^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R} : (x,z,p,A) \mapsto \frac{v(x,z,p)}{g^{ij}(x,z,p) h_{ij}(x,z,p,A)}.$$

Let $w_\varepsilon := w_0 + \varepsilon w$. We obtain the linearized operator L_{w_o} of $\partial/\partial t - Q$ around w_0 as

$$\begin{aligned} L_{w_0} w &:= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \left(\frac{\partial w_\varepsilon}{\partial t} - Q(x, w_\varepsilon, Dw_\varepsilon, D^2 w_\varepsilon) \right) \\ &= \frac{\partial w}{\partial t} - Q_{A_{ij}} D_{ij} w - Q_{p_k} D_k w - Q_z w \end{aligned}$$

where the indices on Q denote the differentiation with respect to the index variable and the derivative is taken at $(x, w_0, Dw_0, D^2 w_0)$. Due to the regularity of w_0 the coefficients of L_{w_0} are in $C^{\alpha,\frac{\alpha}{2}}(M^n \times [0,T])$. Furthermore, from the definition of g^{ij}, h_{ij} and H we see that

$$a^{ij} := Q_{A_{ij}}(\,.\,, w_0, Dw_0, D^2 w_0) = \frac{g^{ij}(\,.\,, w_0, Dw_0)}{H^2(\,.\,, w_0, Dw_0, D^2 w_0)}.$$

At $t = 0$ we have $a^{ij} = \gamma^{ij}/H_0^2$ where H_0 is the mean curvature of the initial hypersurface which is strictly positive. Thus, L_{w_0} is uniformly parabolic in some small time interval $[0,T]$. The Neumann condition can be expressed as $N(x, w, Dw) = 0$ where

$$N : \partial M^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} : (x,z,p) \mapsto r^i(x,z) p_i - s(x,z).$$

The linearized operator N_{w_0} of N around w_0 is given by

$$\begin{aligned} N_{w_0} w &:= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} N(\,.\,, w_\varepsilon, Dw_\varepsilon) \\ &= r^i(\,.\,, w_0) D_i w + \big(r_z^i(\,.\,, w_0) D_i w_0 - s_z(\,.\,, w_0)\big) w. \end{aligned}$$

The compatibility condition is satisfied since $N_{w_0} w(\,.\,,0) = 0$ and $\eta(\,.\,,0) = 0$ on ∂M^n. The transversality condition is satisfied in a small time interval $[0,T]$ since for $t = 0$

$$r_0^i(x, w_0(x,0)) e_i(x) = r_0^i(x,0) e_i(x) = \nu$$

is the unit normal to ∂M^n in x. Therefore, the theory of linear parabolic equations (see Theorem A.4) yields the result. ☐

Now we can prove the existence of a unique solution to (SP).

Proposition 2.8. *Let M^n be a compact, smooth manifold with compact, smooth boundary ∂M^n. Suppose that the mean curvature H_0 of M_0^n is strictly positive. Then there exists some $T > 0$ and a unique solution $w \in C^{2+\alpha,1+\frac{\alpha}{2}}(M^n \times [0,T])$ to* (SP).

Proof. We want to translate the solvability of (SP) to the question of invertibility of some operator A between suitable Banach spaces. We define $Q_T := M^n \times (0,T)$, $S_T := \partial M^n \times (0,T)$ and the spaces

$$X := \{w \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q_T}) \mid w(x,0) = 0 \quad \forall\, x \in M^n\},$$

$$Y := \{(\zeta,\eta) \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q_T}) \times C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{S_T}) \mid \eta(\,.\,,0) = 0\}.$$

X is a closed subspace of the Banach space $C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)$ equipped with the usual norm and Y is a Banach space with respect to the norm $\|(\zeta,\eta)\|_Y := \|\zeta\|_{\alpha,\frac{\alpha}{2},\overline{Q_T}} + \|\eta\|_{1+\alpha,\frac{1+\alpha}{2},\overline{S_T}}$. Let Q and N be defined as in the proof of the last Lemma. The solvability of (SP) now follows from the invertibility of

$$A : X \to Y : w \mapsto Aw := \left(\frac{\partial w}{\partial t} - Q(\,.\,,w,Dw,D^2w), N(\,.\,,w,Dw)\right)$$

in some neighborhood which contains $(0,0)$. The inverse function theorem (see e.g. [13], 10.2.5.) states that if A is continuously (Fréchet-) differentiable in a neighborhood V_{w_0} of some $w_0 \in X$ and if $DA(w_0)$ is a linear homeomorphism from X to Y then there exists a neighborhood $U_{w_0} \subset V_{w_0}$ such that $A : U_{w_0} \to A(U_{w_0})$ is a homeomorphism.

Let w_0 be the solution of the auxiliary problem (AP). Then $DA(w_0)$ is given by

$$DA(w_0) : X \to Y : w \mapsto DA(w_0)(w) := (L_{w_0}w, N_{w_0}w)$$

with L_{w_0} and N_{w_0} as in the last lemma. From the Lemma 2.7 we know that for any $(\zeta,\eta) \in Y$ there is a unique solution $w \in X$ to (LSP). This shows that $DA(w_0)$ is invertible. Since the norm of w in X is bounded by the norm of $(\zeta,\eta) \in Y$ we see that $DA(w_0)$ is a linear homeomorphism from X to Y. Therefore A is invertible in a neighborhood U_{w_0} of w_0. This means that for all $\overline{T} > 0$ there exists a $\delta(\overline{T}) > 0$ such that for all $(\zeta,\eta) \in Y$ satisfying

$$\|A(w_0) - (\zeta,\eta)\|_Y < \delta(\overline{T})$$

there exists a unique $w \in X$ satisfying: $A(w) = (\zeta,\eta)$. Thus there is a unique solution to (SP) if $(\zeta,\eta) = (0,0)$ is close to $A(w_0)$. Due to the choice of w_0 we have

$$\begin{aligned}
&\|A(w_0)\|_Y \\
&= \left\|\frac{\partial w_0}{\partial t} - Q(\,.\,,w_0,Dw_0,D^2w_0)\right\|_{\alpha,\frac{\alpha}{2},\overline{Q_T}} + \left\|N(\,.\,,w_0,Dw_0)\right\|_{1+\alpha,\frac{1+\alpha}{2},\overline{S_T}} \\
&= \left\|\left(\Delta w_0 + \frac{1}{H_0(x)}\right) - \frac{v}{H}\right\|_{\alpha,\frac{\alpha}{2},\overline{Q_T}} + \left\|r^i(\,.\,,w_0)D_iw_0 - s(\,.\,,w_0)\right\|_{1+\alpha,\frac{1+\alpha}{2},\overline{S_T}}.
\end{aligned}$$

Since $w_0(x,0) = 0$ we see that this expression vanishes for $t = 0$. Therefore, arguing as in [62], Lemma 2.1.0 there exists some $T \le \overline{T}$ such that $\|A(w_0)\|_Y < \delta(\overline{T})$. □

The reason why we can not expect higher regularity up to $t = 0$ is that this would require higher order compatibility conditions and therefore more conditions on the initial data. Despite that fact we get smooth solutions away from zero.

Lemma 2.9. *Let* $w \in C^{2+\alpha,1+\frac{\alpha}{2}}(M^n \times [0,T])$ *be a solution to* (SP). *Then for every* $\varepsilon > 0$ *and every* $k \in \mathbb{N}$ *we get*

$$w \in C^{2+\alpha,1+\frac{\alpha}{2}}(M^n \times [0,T]) \cap C^{2k+\alpha,k+\frac{\alpha}{2}}(M^n \times [\varepsilon,T]).$$

Proof. For $1 \leq i \leq n$ we consider the difference quotients

$$v_h^0(x,t) := \frac{u(x,t+h) - u(x,t)}{h} \qquad \text{and} \qquad v_h^i(x,t) := \frac{u(x+he_i,t) - u(x,t)}{h}$$

in space and time and use the fact that these functions are solutions to linear parabolic equations. Note that one has to distinguish the cases of interior points where the v_h^i satisfy a Dirichlet problem and boundary points where they are solutions to a Neumann problem. One uses cut-off functions to localize the estimates. This yields the result for $k = 1$. The higher order estimates are proved by induction over k. For more details see e.g. [18], Theorem 2.5.10. and [19]. ☐

2.3 Short-time existence

Following the ideas of the previous sections one could think that a map

$$\tilde{F} : M^n \times [0,T] \to M^n \times [-\varepsilon,\varepsilon] : (x,t) \mapsto F(x,t) := \big(x, w(x,t)\big) \tag{2.1}$$

with a suitable scalar function w is a good candidate for a solution to (IMCF). But if we look at F more carefully we see that points starting at the initial surface always evolve in e_{n+1} direction in $M^n \times [-\varepsilon,\varepsilon]$, i.e. along the integral curves of $\Phi(x,\,.\,)$ in N^{n+1}. Since we want to create an evolution in normal direction we have to adjust our definition. Therefore, we make the ansatz

$$F : M^n \times [0,T] \to M^n \times [-\varepsilon,\varepsilon] : (x,t) \mapsto \tilde{F}(\varphi(x,t),t) \tag{2.2}$$

for some map $\varphi : M^n \times [0,T] \to M^n$ which should be bijective for fixed t and map boundary points into boundary points since we do not want the surface to lift off from Σ^n. Before we prove short-time existence of (IMCF) we will prove the existence of such a map φ.

Lemma 2.10. *Let* $w \in C^{2+\alpha,1+\frac{\alpha}{2}}(M^n \times [0,T]) \cap C^\infty(M^n \times (0,T])$ *be a solution to* (SP) *and* $\tilde{F}$ *defined as above. Let* $(\,.\,)^\top$ *denote the projection onto the tangent space of* $\tilde{M}_t^n := \tilde{F}(M^n,t)$. *Then there is a unique map*

$$\varphi \in C^{2+\alpha,1+\frac{\alpha}{2}}(M^n \times [0,T], M^n) \cap C^\infty(M^n \times (0,T], M^n)$$

solving

$$\text{(ODE)} \begin{cases} \dfrac{d\varphi}{dt} = \left(-\left(D_x\tilde{F}\right)^{-1} \left(\dfrac{\partial \tilde{F}}{\partial t}\right)^\top \right)(\varphi,t) & \textit{in } M^n \times (0,T) \\[2ex] \varphi(.,0) = \mathrm{id} & \textit{on } M^n. \end{cases}$$

Furthermore, φ keeps ∂M^n invariant, i.e. for $x \in \partial M^n$ it follows that $\varphi(x,t) \in \partial M^n$ and for fixed t, $\varphi(\,.\,,t)$ is a diffeomorphism[2].

Proof. The vector field on the right hand side is smooth away from $t = 0$, smooth in x even for $t = 0$ and $C^{\frac{\alpha}{2}}$ in the t-variable up to $t = 0$. The existence and regularity theory for ODEs implies the desired existence and regularity and shows that the map $\varphi(\,.\,,t)$ is a diffeomorphism for fixed t (see e.g. [17], chapter 9). To see that φ keeps ∂M^n invariant we will show that for $\varphi(x,t) \in \partial M^n$ we have $\frac{\mathrm{d}\varphi}{\mathrm{d}t} \in T_x\partial M^n$. The result then follows from the uniqueness of ODEs. We calculate

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = -\left(D_x\tilde{F}\right)^{-1}\left(\frac{\partial \tilde{F}}{\partial t}\right)^{\top} = -\frac{\partial w}{\partial t}\left(D_x\tilde{F}\right)^{-1} e_{n+1}^{\top}.$$

Next we observe that due to the Neumann condition for w the surface $\tilde{M}_t^n$ touches $\partial M^n \times [-\varepsilon,\varepsilon]$ orthogonally since

$$\langle\tilde{\nu},\mu\rangle_{\overline{\gamma}} = \overline{\gamma}_{\alpha\beta}\tilde{\nu}^{\alpha}\mu^{\beta} = v^{-1}\left(\overline{\gamma}^{\alpha k}(-D_k w) + \overline{\gamma}^{\alpha\, n+1}\right)\mu_{\alpha} = v^{-1}\left(\mu^{n+1} - \mu^k D_k w\right) = 0$$

at $p = \tilde{F}(\varphi(x,t),t)$. Therefore the fact that $e_{n+1} \in T_p(\partial M^n \times [-\varepsilon,\varepsilon])$ implies that $e_{n+1}^{\top} \in T_p\partial\tilde{M}_t^n$ and since $\tilde{F}$ maps ∂M^n into $\partial\tilde{M}_t^n$ we see that $\left(D_x\tilde{F}\right)^{-1} e_{n+1}^{\top} \in T_x\partial M^n$. □

Now we can relate the existence of solutions to (SP) and (IMCF) as we promised in the last section.

Proposition 2.11. *Given a solution $w \in C^{2+\alpha,1+\frac{\alpha}{2}}(M^n \times [0,T]) \cap C^{\infty}(M^n \times (0,T])$ of* (SP) *there is a unique map $\varphi \in C^{2+\alpha,1+\frac{\alpha}{2}}(M^n \times [0,T], M^n) \cap C^{\infty}(M^n \times (0,T], M^n)$ such that*

$$F : M^n \times [0,T] \to M^n \times [-\varepsilon,\varepsilon] : (x,t) \mapsto F(x,t) := (\varphi(x,t), w(\varphi(x,t),t)) \tag{2.3}$$

is a solution to (IMCF). *On the other hand, given a solution $F \in C^{2+\alpha,1+\frac{\alpha}{2}}(M^n \times [0,T], M^n \times [-\varepsilon,\varepsilon]) \cap C^{\infty}(M^n \times (0,T], M^n \times [-\varepsilon,\varepsilon])$ of* (IMCF) *there is a unique map φ such that $w \in C^{2+\alpha,1+\frac{\alpha}{2}}(M^n \times [0,T]) \cap C^{\infty}(M^n \times (0,T])$ defined by* (2.3) *solves* (SP).

Proof. We first show that F is a solution to (IMCF). Let w be a solution to (SP). By Lemma 2.10 this yields a unique solution φ to (ODE). Let $\tilde{F}$ and F be defined by (2.1) and (2.2). Notice that this definition implies

$$\tilde{\nu}(\varphi(x,t),t) = \nu(x,t), \qquad \tilde{H}(\varphi(x,t),t) = H(x,t).$$

The initial condition is satisfied since

$$F(x,0) = (\varphi(x,0), w(\varphi(x,0),0)) = (x, w(x,0)) = (x,0) = F_0(x).$$

From the fact that φ maps ∂M^n into ∂M^n we see[3] that $F(\partial M^n) = F(M^n) \cap \Sigma^n$ and for the Neumann condition we calculate

$$\langle\nu,\mu\rangle_{\overline{\gamma}} = \overline{\gamma}_{\alpha\beta}\nu^{\alpha}\mu^{\beta} = v^{-1}\mu_{\alpha}(-\overline{\gamma}^{\alpha k}D_k w + \overline{\gamma}^{\alpha\, n+1}) = v^{-1}(-\mu^k D_k w + \mu^{n+1}) = 0.$$

---

[2] Note that φ is smooth in x for fixed t but since it is only $C^{1+\frac{\alpha}{2}}$ in the t variable we chose the natural Hölder space $C^{2+\alpha,1+\frac{\alpha}{2}}$ in the regularity statement. Furthermore, this is the regularity we will finally obtain for the solution to (IMCF).

[3] Recall that we identified Σ^n with $\partial M^n \times [-\varepsilon,\varepsilon]$

By construction of φ and w the evolution equation holds too. Remember that Lemma 2.5 implies $\langle e_{n+1}, \nu\rangle = v^{-1}$. We obtain:

$$\begin{aligned}
\frac{d}{dt}F(x,t) &= \frac{d}{dt}\tilde{F}(\varphi(x,t),t) = D_x\tilde{F}(\varphi(x,t),t)\frac{d}{dt}\varphi(x,t) + \frac{\partial}{\partial t}\tilde{F}(\varphi(x,t),t)\\
&= -\left(\frac{\partial}{\partial t}\tilde{F}\right)^{\top}(\varphi(x,t),t) + \frac{\partial}{\partial t}\tilde{F}(\varphi(x,t),t) = \left\langle \frac{\partial}{\partial t}\tilde{F}, \tilde{\nu}\right\rangle_{\overline{\gamma}} \tilde{\nu}(\varphi(x,t),t)\\
&= \frac{\partial w}{\partial t}\left\langle e_{n+1}, \tilde{\nu}\right\rangle_{\overline{\gamma}}\tilde{\nu}(\varphi(x,t),t) = \frac{1}{\tilde{H}}\tilde{\nu}(\varphi(x,t),t) = \frac{1}{H}\nu(x,t).
\end{aligned}$$

This shows that the above defined map F solves (IMCF). The regularity of F is clear from the regularity of w and φ. Now let $F \in C^{2+\alpha,1+\frac{\alpha}{2}}(M^n \times [0,T], M^n \times [-\varepsilon,\varepsilon]) \cap C^\infty(M^n \times (0,T], M^n \times [-\varepsilon,\varepsilon])$ be a solution of (IMCF) we can implicitly define a function w and a map φ by

$$(\varphi(x,t), w(\varphi(x,t),t)) := F(x,t).$$

We see that $\varphi \in C^{2+\alpha,1+\frac{\alpha}{2}}(M^n \times [0,T], M^n) \cap C^\infty(M^n \times (0,T], M^n)$ and therefore we also have $w \in C^{2+\alpha,1+\frac{\alpha}{2}}(M^n \times [0,T]) \cap C^\infty(M^n \times (0,T])$. Since

$$(x,0) = F_0(x) = F(x,0) = (\varphi(x,0), w(\varphi(x,t),0))$$

we see that $\varphi(x,0) = x$ and $w(x,0) = 0$. So they satisfy the right initial conditions. With the same calculation as above we obtain for the Neumann condition

$$\langle \nu, \mu\rangle_{\overline{\gamma}} = v^{-1}(-\mu^k D_k w + \mu^{n+1}) = 0.$$

Finally, we calculate the evolution equation for w.

$$\begin{aligned}
\frac{\partial}{\partial t}w(\varphi(x,t),t) &= \frac{d}{dt}w(\varphi(x,t),t) - D_i w(\varphi(x,t),t)\left(\frac{d}{dt}\varphi(x,t)\right)^i\\
&= \left(\frac{d}{dt}F(x,t)\right)^{n+1} - D_i w(\varphi(x,t),t)\left(\frac{d}{dt}F(x,t)\right)^i\\
&= \frac{1}{H}\left(\nu^{n+1} - D_i w \nu^i\right)(\varphi(x,t),t)\\
&= \frac{v}{H}\left\langle \frac{1}{v}\overline{\gamma}^{-1}\begin{pmatrix} -Dw \\ 1 \end{pmatrix}, \nu\right\rangle_{\overline{\gamma}}(\varphi(x,t),t)\\
&= \frac{v}{H}(\varphi(x,t),t).
\end{aligned}$$

Thus w satisfies (SP). □

Now we can conclude the desired short-time existence result for (IMCF).

Theorem 2.12 (**Short-time existence**). *Let N^{n+1}, M^n, Σ^n and F_0 be as in Definition 1.1. Then there exists some $T > 0$ and a unique solution $F \in C^{2+\alpha,1+\frac{\alpha}{2}}(M^n \times [0,T], N^{n+1}) \cap C^\infty(M^n \times (0,T], N^{n+1})$ satisfying* (IMCF).

Proof. By Remark 2.4 we can use Φ to identify a tubular neighborhood of $M_0^n \subset N^{n+1}$ with the product $M^n \times [-\varepsilon, \varepsilon]$. So we can regard F as a map from $M^n \times [0, T]$ to $M^n \times [-\varepsilon, \varepsilon]$. By Proposition 2.8 there exists a solution

$$w \in C^{2+\alpha, 1+\frac{\alpha}{2}}(M^n \times [0, T]) \cap C^\infty(M^n \times (0, T])$$

to (SP). Then by Proposition 2.11 there is a tangential diffeomorphism

$$\varphi \in C^{2+\alpha, 1+\frac{\alpha}{2}}(M^n \times [0, T], M^n) \cap C^\infty(M^n \times (0, T], M^n)$$

such that the map F defined by

$$F : M^n \times [0, T] \to M^n \times [-\varepsilon, \varepsilon] : (x, t) \mapsto F(x, t) := (\varphi(x, t), w(\varphi(x, t), t))$$

is in $C^{2+\alpha, 1+\frac{\alpha}{2}}(M^n \times [0, T], M^n \times [-\varepsilon, \varepsilon]) \cap C^\infty(M^n \times (0, T], M^n \times [-\varepsilon, \varepsilon])$ and solves (IMCF). Now suppose there are two solutions F_1 and F_2 to (IMCF). By Theorem 2.11 there are unique tangential diffeomorphisms φ_1, φ_2 and solutions w_1, w_2 of (SP) such that

$$F_1(x, t) = (\varphi_1(x, t), w_1(\varphi_1(x, t), t)), \qquad F_2(x, t) = (\varphi_2(x, t), w_2(\varphi_2(x, t), t)).$$

By Theorem 2.8 (SP) has a unique solution. Therefore $w_1 = w_2$ and from Lemma 2.10 we see that then also $\varphi_1 = \varphi_2$. This shows that $F_1 = F_2$. $\square$

3 Expansion in a cone

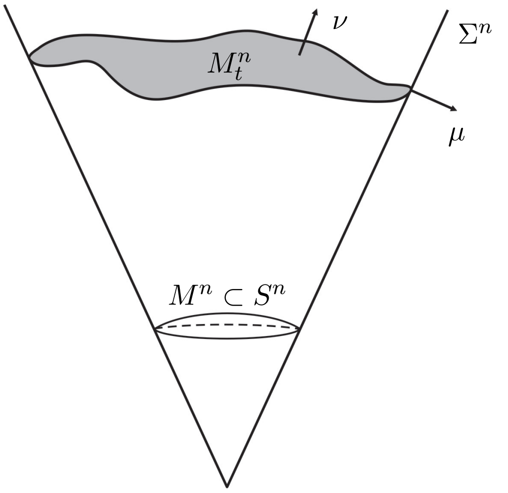

Figure 3.1: Evolution of a star-shaped hypersurface M_t^n in the cone Σ^n.

In Chapter 2 we proved short-time existence for general supporting hypersurfaces Σ^n. It turns out that long-time existence cannot be expected in general unless one uses a weaker notion of solutions or one imposes stricter conditions on Σ^n and the initial hypersurface M_0^n. In this chapter we deal with the latter case by considering hypersurfaces in the ambient space $N^{n+1} = \mathbb{R}^{n+1}$. Furthermore we are restricting ourselves to supporting hypersurfaces Σ^n which are convex cones. The initial hypersurface is required to be star-shaped with respect to the vertex of the cone and has to have strictly positive mean curvature (see figure 3.1).

In Section 1 we will derive the associated scalar Neumann problem by writing the surface as a graph over some piece of the sphere. In Section 2 we will use the maximum principle to derive a priori estimates. The central geometric estimate will be a bound on the slope of the height function. We will see that the convexity of the cone allows us to control this quantity. In Section 3 we will prove Hölder estimates which then yields long-time existence and convergence to a spherical cap in Section 4, Theorem 3.21. This is the main result of this chapter. In the case of closed hypersurfaces the corresponding result has been obtained by Gerhardt [16] (see also Urbas [65]).

3.1 Graphs over a spherical cap

Due to the assumption of star-shapedness and the choice of a cone as supporting hypersurface we can write the evolving hypersurface as a graph over some part of the sphere. This yields more explicit coordinates and simplifies most of the formulas. We start by defining the cone Σ^n.

Definition 3.1. Let $S^n \subset \mathbb{R}^{n+1}$ be the sphere of radius one. Let $M^n \subset S^n$ be some domain in S^n with smooth boundary. Then Σ^n defined by

$$\Sigma^n := \left\{ rx \in \mathbb{R}^{n+1} \;\middle|\; r > 0,\ x \in \partial M^n \right\} \tag{3.1}$$

is called a smooth cone. We say that Σ^n is convex if the second fundamental form of ∂M^n is positive definite with respect to the outward unit co-normal $n \in T_x M^n \cap N_x \partial M^n$.

To find a solution to (IMCF) we make the ansatz

$$\tilde{F} : M^n \times [0,T) \to \mathbb{R}^{n+1} : (x,t) \mapsto u(x,t)x$$

for some function $u : M^n \times [0,T) \to \mathbb{R}_+$. If the initial hypersurface M_0^n is a star-shaped $C^{2,\alpha}$-hypersurface there exists a scalar function $u_0 \in C^{2,\alpha}(M^n)$ such that F_0 can be expressed as $F_0 : M^n \to \mathbb{R}^{n+1} : x \mapsto u_0(x)x$. Analogous to Lemma 2.5 we have the following lemma for graphs over a spherical cap.

Lemma 3.2. *Let $t \geq 0$ be fixed. Let $\tilde{M}_t^n := \tilde{F}(M^n, t)$ and let $\{\sigma_{ij}\}_{i,j=1,\dots,n}$ denote the metric on M^n. We define $p := \tilde{F}(x,t)$ and assume that a point on M^n is described by local coordinates that is $x = x(\xi^i)$. The following formulas hold:*

(i) Let $v := \sqrt{1 + u^{-2}|\nabla u|^2}$ and $1 \leq i \leq n$. Then the tangent vectors $\tau_i \in T_p \tilde{M}_t^n$ and the unit normal $\nu \in N_p \tilde{M}_t^n$ are given by

$$\tau_i = x\,\nabla_i u + u\,\nabla_i x, \qquad \nu = \frac{1}{v}\Big(x - u^{-1}\nabla^i u \nabla_i x\Big)$$

where we used the same symbol for the position vector and the point x.

(ii) The metric $\{g_{ij}\}_{i,j=1,\dots,n}$ and inverse metric $\{g^{ij}\}_{i,j=1,\dots,n}$ on $T_p\tilde{M}_t^n$ are given by

$$g_{ij} = u^2\sigma_{ij} + \nabla_i u \nabla_j u, \qquad g^{ij} = \frac{1}{u^2}\left(\sigma^{ij} - \frac{\nabla^i u \nabla^j u}{u^2 + |\nabla u|^2}\right).$$

(iii) The second fundamental form $\{h_{ij}\}_{i,j=1,\dots,n}$ of $T_p\tilde{M}_t^n$ is given by

$$h_{ij} = \frac{u}{v}\Big(\sigma_{ij} + 2u^{-2}\nabla_i u \nabla_i u - u^{-1}\nabla^2_{ij} u\Big).$$

(iv) Let $p \in \Sigma^n$ and $\hat{\mu}(p)$ be the normal to Σ^n in p. Let $\mu = \mu^k(x)e_k(x)$ be the normal to Σ^n in x and e_k the basis vectors of $T_x S^n$. Then

$$\big\langle \hat{\mu}(p), \nu(p) \big\rangle = 0 \;\Leftrightarrow\; \mu^k(x)\nabla_k u(x,t) = 0.$$

The scalar mean curvature of $\tilde{M}_t^n$ is given by $H = g^{ij}h_{ij}$. In contrary to Lemma 2.5 all derivatives are covariant derivatives with respect to the metric $\{\sigma_{ij}\}_{i,j=1,\dots,n}$ on $M^n \subset S^n$.

Proof. (i) The formula for the τ_i is clear and one easily checks that $\langle \tau_i, \nu \rangle = 0$ and $\langle \nu, \nu \rangle = 1$.

(ii) The metric is obtained directly from the definition $g_{ij} := \langle \tau_i, \tau_j \rangle$ and since $g^{-1}g = gg^{-1} = \mathrm{id}$ we see that g^{-1} is the correct inverse metric.

(iii) The second fundamental form is obtained as in Lemma 2.5. In addition we replaced the partial derivatives by covariant derivatives with respect to σ using $\nabla^2_{ij}u = D_{ij}u - {}^{\sigma}\Gamma^k_{ij} D_k u$.

(iv) Let $p \in \Sigma^n$. Let $\hat{\mu}(p)$ be the normal to Σ^n in p and $\mu(x) = \mu^k(x)e_k(x)$ be the normal to Σ^n in x. Using the definition of ν and the fact that $\langle x, \mu \rangle = 0$ we see that

$$\langle \hat{\mu}(p), \nu(p) \rangle_{\mathbb{R}^{n+1}} = 0 \;\Leftrightarrow\; \hat{\mu}^k(p)\nabla_k u(x,t) = 0.$$

Since Σ^n is a cone in $\mathbb{R}^{n+1}$ we know that the normal at p and x coincide. Furthermore, we see that the tangent vectors to $S^n_{u(x,t)}$ at p and the tangent vectors to S^n at x only differ by the factor $u(x,t)$ i.e. $e_k(p) = u(x,t)e_k(x)$. Therefore, $\hat{\mu}^k(p)u(x,t) = \mu^k(x)$ which implies the result. □

So far $\tilde{F}$ only allows the evolution of points in radial direction. Since we want the surface to move in normal direction we modify the ansatz by defining

$$F : M^n \times [0,T) \to \mathbb{R}^{n+1} : (x,t) \mapsto \tilde{F}(\varphi(x,t),t)$$

for some map $\varphi : M^n \times [0,T) \to M^n$ which has to be bijective for fixed t and has to satisfy $\varphi(\partial M^n, t) = \partial M^n$. As in Chapter 2 the problem of solving (IMCF) reduces to solving

$$\text{(SP)} \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{v}{H} & \text{in } \; M^n \times (0,T) \\[2mm] \nabla_\mu u = 0 & \text{on } \; \partial M^n \times (0,T) \\[2mm] u(\,.\,,0) = u_0 & \text{on } \; M^n \end{cases}$$

as is stated in the next Lemma.

Lemma 3.3. *Let Σ^n be a smooth cone. Let the initial hypersurface be given by $F_0 : M^n \to \mathbb{R}^{n+1} : x \mapsto u_0(x)x$ with $u_0 \in C^{2,\alpha}(M^n)$ positive. Assume that $M^n_0 := F_0(M^n)$ has strictly positive mean curvature and meets Σ^n orthogonally, i.e. $F_0(\partial M^n) \subset \Sigma^n$ and $\nabla_\mu u_0 = 0$ on ∂M^n. Then there exists some $T > 0$, a unique function*

$$u \in C^{2+\alpha, 1+\frac{\alpha}{2}}(M^n \times [0,T]) \cap C^\infty(M^n \times (0,T])$$

and a unique diffeomorphism $\varphi : M^n \times [0,T] \to M^n$ which is $C^{1+\alpha}$ in time up to $t = 0$, such that the above defined map F solves (IMCF).

Proof. Besides the fact that we express the hypersurface as a graph over some piece of the sphere we are in the same situation as in the previous chapter. The Neumann problem for the height function u is now posed on $M^n \subset S^n \subset \mathbb{R}^{n+1}$ and the ODE for φ is more explicit than before:

$$\text{(ODE)} \begin{cases} \dfrac{d\varphi}{dt} = -\left(D\tilde{F}\right)^{-1} \left(\dfrac{\partial \tilde{F}(x,t)}{\partial t}\right)^{\top} = \dfrac{-1}{u^2 v H}\nabla u & \text{in } \; M^n \times (0,T), \\[2mm] \varphi(\,.\,,0) = \mathrm{id} & \text{on } \; M^n. \end{cases}$$

As in Chapter 2 for a short time there is a solution u of (SP) and a solution φ of (ODE) both with the desired regularity. Using (SP) and (ODE) we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}F = \left(\nabla^i u \frac{\mathrm{d}\varphi^i}{\mathrm{d}t} + \frac{\partial u}{\partial t}\right)\varphi + u\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{1}{vH}\varphi - \frac{\nabla^i u}{uvH}\nabla_i\varphi = \frac{1}{H}\nu.$$

The initial conditions for u and φ follow from the condition $F(x,0) = F_0(x)$. The Neumann condition for u follows from Lemma 3.2, (iv). $\square$

Remark 3.4. Note that (SP) from Chapter 2 looks slightly different since in Chapter 2 we considered graphs over the initial hypersurface whereas here we consider graphs over some part $M^n \subset S^n \subset \mathbb{R}^{n+1}$. Whenever we write (SP) in this chapter we refer to the problem on $M^n \subset S^n$.

Definition 3.5. The maximal existence time for (SP) is the largest value T^* such that there is a solution $u \in C^{2,1}(M^n \times [0,T^*)) \cap C^\infty(M^n \times (0,T^*))$ which solves (SP). The function u is called an admissible solution. Given an admissible solution u there is a diffeomorphism φ solving (ODE). The map F defined above is then called an admissible solution to (IMCF).

In order to prove long-time existence we will argue by contradiction. That means we will prove a priori estimates for an admissible solution u which tells us that u can be extended to be a solution on the closed time interval $[0,T^*]$. Using the short-time existence result we can therefore extend u beyond T^* which causes a contradiction. We start with a priori estimates which can be obtained using the maximum principle.

3.2 Maximum principle estimates

It turns out that the transformation $w := \ln u$ is useful. In terms of w the problem (SP) is the following.

Lemma 3.6. *The function u is a solution to* (SP) *if and only if $w := \ln u$ is a solution to*

$$(\mathrm{SP})' \begin{cases} \dfrac{\partial w}{\partial t} = Q(\nabla w, \nabla^2 w) & \text{in} \quad M^n \times (0,T) \\ \nabla_\mu w = 0 & \text{in} \quad \partial M^n \times (0,T) \\ w(.,0) = \ln u_0 & \text{in} \quad M^n \end{cases}$$

with

$$Q : \mathbb{R}^n \times \mathbb{R}^{n\times n} : (p,A) \mapsto Q(p,A) := \frac{1+|p|^2}{n - \left(\sigma^{ij} - \dfrac{p^i p^j}{1+|p|^2}\right)A_{ij}}.$$

Proof. This follows from the fact that the metric, second fundamental form and the mean curvature transform in the following way

$$g_{ij} = e^{2w}\Big(\sigma_{ij} + \nabla_i w \nabla_j w\Big), \qquad g^{ij} = e^{-2w}\left(\sigma^{ij} - \frac{\nabla^i w \nabla^j w}{1+|\nabla w|^2}\right)$$

$$h_{ij} = \frac{e^w}{\sqrt{1+|\nabla w|^2}}\Big(\sigma_{ij} + \nabla_i w \nabla_j w - \nabla^2_{ij} w\Big)$$

and

$$H = g^{ij} h_{ij} = \frac{1}{uv}\left(n - \left(\sigma^{ij} - \frac{\nabla^i w \nabla^j w}{1+|\nabla w|^2}\right)\nabla^2_{ij} w\right).$$

□

Remark 3.7. Note that Q is a nonlinear second order operator but in contrast to the equation for u there is no dependence on the function itself. We will use the following notation

$$Q^{ij}(\xi, B) := \frac{\partial Q(z,A)}{\partial A_{ij}}\bigg|_{(z,A)=(\xi,B)}, \qquad Q^k(\xi,B) := \frac{\partial Q(z,A)}{\partial z_k}\bigg|_{(z,A)=(\xi,B)}$$

and see that

$$Q^{ij}(\nabla w, \nabla^2 w) = \frac{v^2}{\left[n - \left(\sigma^{ij} - \frac{\nabla^i w \nabla^j w}{1+|\nabla w|^2}\right)\nabla^2_{ij} w\right]^2}\left(\sigma^{ij} - \frac{\nabla^i w \nabla^j w}{1+|\nabla w|^2}\right) = \frac{1}{H^2} g^{ij}$$

is positive definite once we have estimates for H.

In the following we will use (SP)$'$ to derive estimates for $|u|, |\partial u/\partial t|, |\nabla u|$ and $|H|$. We start with an estimate for $|u|$.

Lemma 3.8. *Let u be an admissible solution of* (SP)*. Let Σ^n be a smooth cone. Then u satisfies*

$$R_1 := \min_{M^n} u_0 \ \le\ u(x,t)e^{-t/n} \ \le\ \max_{M^n} u_0 =: R_2$$

for all $(x,t) \in M^n \times [0,T]$.

Proof. Let $w(x,t) := \ln u(x,t)$ and $w^+(x,t) := \ln(\max_{M^n} u_0) + t/n$. Both satisfy (SP)$'$. Using

$$R^{ij} := \int_0^1 Q^{ij}(\nabla w_\theta, \nabla^2 w_\theta)d\theta, \qquad S^k := \int_0^1 Q^k(\nabla w_\theta, \nabla^2 w_\theta)d\theta$$

with $w_\theta := \theta w^+ + (1-\theta)w$, we see that $\psi := w^+ - w$ satisfies

$$\begin{cases} \dfrac{\partial \psi}{\partial t} &= R^{ij}\nabla^2_{ij}\psi + S^k \nabla_k \psi & \text{in} \quad M^n \times (0,T) \\ \nabla_\mu \psi &= 0 & \text{on} \quad \partial M^n \times (0,T) \\ \psi(\,.\,,0) &\ge 0 & \text{on} \quad M^n. \end{cases}$$

The maximum principle (see Theorem A.6 and Corollary A.7) implies $\psi \ge 0$ in $M^n \times [0,T]$ and thus the upper bound. The lower bound is obtained in the same way using $w^-(x,t) := \ln(\min_{M^n} u_0) + t/n$. □

Remark 3.9. From a geometric point of view this estimate says that the rescaled surfaces $F(M^n,t)e^{-t/n}$ always stay between the two spherical caps which enclose the initial surface.

Next we want to estimate $\dot{u} := \partial u/\partial t$.

Lemma 3.10. *Let u be an admissible solution of* (SP)*. Let Σ^n be a smooth cone. Then $\dot{u} := \partial u/\partial t$ satisfies*

$$\left(\frac{R_1}{R_2}\right)\min_{M^n}\frac{v_0}{H_0} \le \dot{u}(x,t)e^{-t/n} \le \left(\frac{R_2}{R_1}\right)\max_{M^n}\frac{v_0}{H_0}$$

for all $(x,t) \in M^n \times [0,T]$, where $H_0 = H(\,.\,,0)$, $v_0 = v(\,.\,,0)$ and R_1, R_2 are defined as in Lemma 3.8.

Proof. Let u satisfy (SP) and $w := \ln u$. Then $\dot{w} := \partial w/\partial t$ satisfies

$$\begin{cases} \dfrac{\partial \dot{w}}{\partial t} &= Q^{ij}\nabla^2_{ij}\dot{w} + Q^k\nabla_k\dot{w} & \text{in} & M^n\times(0,T) \\ \nabla_\mu \dot{w} &= 0 & \text{on} & \partial M^n\times(0,T) \\ \dot{w}(\,.\,,0) &= Q(\nabla w_0, \nabla^2 w_0) & \text{on} & M^n \end{cases}$$

with $Q(\nabla w_0, \nabla^2 w_0) \ge 0$. The evolution equation follows directly by differentiating the evolution equation for w with respect to t. The initial value $\dot{w}(\,.\,,0)$ is also obtained from the evolution equation of w at time zero. For the Neumann condition we note that $\nabla_\mu w$ is differentiable in t for $t>0$ and equal to zero for all $t>0$. Thus,

$$0 = \frac{\partial}{\partial t}\left(\nabla_\mu w\right) = \nabla_{\dot{\mu}} w + \nabla_\mu \dot{w} = \nabla_\mu \dot{w}$$

since Σ^n is a cone and thus μ does not depend on t. Therefore, the maximum principle (see Theorem A.6 and Corollary A.7) implies

$$\min_{M^n}\frac{v_0}{u_0H_0} = \min_{M^n}\dot{w}(\,.\,,0) \le \dot{w}(x,t) \le \max_{M^n}\dot{w}(\,.\,,0) = \max_{M^n}\frac{v_0}{u_0H_0}.$$

Using the estimate for u and the fact that $\dot{w} = u^{-1}\dot{u}$ we obtain the desired result. □

For the estimate of $|\nabla u|$ we have to make use of the convexity of Σ^n.

Lemma 3.11. *Let u be an admissible solution of* (SP)*. Let Σ^n be a smooth, convex cone. Then*

$$|\nabla u(x,t)|e^{-t/n} \le \left(\frac{R_2}{R_1}\right)\max_{M^n}|\nabla u_0|$$

for all $(x,t) \in M^n \times [0,T]$.

Proof. By assumption $w = \ln u$ satisfies (SP)$'$. As in [16] we want to find a boundary value problem for $\psi := |\nabla w|^2/2$. Therefore, we first calculate

$$\nabla_k\psi = \nabla^2_{mk}w\nabla^m w, \qquad \nabla^2_{ij}\psi = \nabla^3_{mij}w\nabla^m w + \nabla^2_{mi}w\nabla^{2m}_{j}w.$$

Using the rule for interchanging covariant derivatives on S^n we get

$$\nabla^3_{mij} w \;=\; \nabla^3_{imj} w \;=\; \nabla^3_{ijm} w + R^l_{imj} \nabla_l w \;=\; \nabla^3_{ijm} w + \sigma_{ij} \nabla_m w - \sigma_{im} \nabla_j w$$

which implies

$$\nabla^2_{ij} \psi = \nabla^3_{ijm} w \nabla^m w + \sigma_{ij} |\nabla w|^2 - \sigma_{im} \nabla_j w \nabla^m w + \nabla^2_{mi} w \nabla^{2m}_j w.$$

This leads to

$$\begin{aligned}
\dot{\psi} &= \nabla_m \dot{w} \nabla^m w \\
&= \nabla_m Q(\nabla w, \nabla^2 w) \nabla^m w \\
&= Q^{ij} \nabla^3_{ijm} w \nabla^m w + Q^k \nabla^2_{km} w \nabla^m w \\
&= Q^{ij} \nabla^2_{ij} \psi - Q^{ij} \sigma_{ij} |\nabla w|^2 + Q^{ij} \sigma_{im} \nabla_j w \nabla^m w - Q^{ij} \nabla^2_{mi} w \nabla^{2\,m}_{j} w + Q^k \nabla_k \psi.
\end{aligned}$$

Using the special form of Q^{ij} we see that

$$\begin{aligned}
&- Q^{ij} \sigma_{ij} |\nabla w|^2 + Q^{ij} \sigma_{im} \nabla_j w \nabla^m w \\
&\qquad = \frac{1}{u^2 H^2} \left(\sigma^{ij} - \frac{\nabla^i w \nabla^j w}{1 + |\nabla w|^2} \right) \left(\nabla_i w \nabla_j w - \sigma_{ij} |\nabla w|^2 \right) = \frac{(1-n) |\nabla w|^2}{u^2 H^2}
\end{aligned}$$

and

$$\begin{aligned}
&Q^{ij} \nabla^2_{mi} w \nabla^{2\,m}_{j} w \\
&\qquad = \frac{1}{u^2 H^2} \left(\sigma^{ij} - \frac{\nabla^i w \nabla^j w}{1 + |\nabla w|^2} \right) \nabla^2_{mi} w \nabla^{2\,m}_{j} w = \frac{|\nabla^2 w|^2}{u^2 H^2} - \frac{|\nabla \psi|^2}{u^2 v^2 H^2}.
\end{aligned}$$

Thus the evolution equation for ψ can be written as

$$\frac{\partial \psi}{\partial t} = Q^{ij} \nabla^2_{ij} \psi + \left(Q^k + \frac{\nabla^k \psi}{u^2 v^2 H^2} \right) \nabla_k \psi - \frac{2(n-1)}{u^2 H^2} \psi - \frac{|\nabla^2 w|^2}{u^2 H^2}. \tag{3.2}$$

For the Neumann condition we use the fact that for $t > 0$ the function $\nabla_\mu w$ is differentiable and $\nabla_\mu w \equiv 0$. Since $\nabla_\mu \psi$ is a coordinate invariant expression (a (0,0)-tensor) we use an orthonormal frame for the calculation. Let $e_1, ..., e_{n-1} \in T_x \partial M^n$ and $e_n = \mu$. Then we have

$$\begin{aligned}
\nabla_\mu \psi &= \sum_{i=1}^{n-1} \nabla^2 w(e_i, e_n) \nabla_{e_i} w = \sum_{i=1}^{n-1} \left(\nabla_{e_i} \nabla_{e_n} w - (\nabla_{e_i} e_n)(w) \right) \nabla_{e_i} w \\
&= - \sum_{i=1}^{n-1} \left((\nabla_{e_i} e_n)(w) \right)^\top \nabla_{e_i} w = - \sum_{i,j=1}^{n-1} \langle \nabla_{e_i} e_n, e_j \rangle \nabla_{e_i} w \nabla_{e_j} w \\
&= - \sum_{i,j=1}^{n-1} {}^{\partial M^n} h_{ij} \nabla_{e_i} w \nabla_{e_j} w
\end{aligned}$$

with ${}^{\partial M^n}h_{ij}$ being the second fundamental form of the boundary ∂M^n. As initial value we can choose $\psi(\,.\,,0) = |\nabla w_0|^2/2$. Since Σ^n is convex we see that ψ satisfies the inequalities

$$\begin{cases} \dfrac{\partial \psi}{\partial t} & \leq Q^{ij}\nabla^2_{ij}\psi + \left(Q^k + \dfrac{\nabla^k\psi}{u^2v^2H^2}\right)\nabla_k\psi & \text{in} \quad M^n \times (0,T) \\ \nabla_\mu \psi & \leq \; 0 & \text{on} \quad \partial M^n \times (0,T) \\ \psi(.\,,0) & = \; |\nabla w_0|^2/2 & \text{on} \quad M^n. \end{cases}$$

Using the maximum principle (see Theorem A.6 and Corollary A.7) we obtain

$$\psi = \frac{|\nabla w|^2}{2} = \frac{|\nabla u|^2}{2u^2} \leq \max_{M^n} \frac{|\nabla w_0|^2}{2} = \max_{M^n} \frac{|\nabla u_0|^2}{2u_0^2}.$$

Together with the estimate for u we obtain the desired result. □

A more geometric way to derive the gradient estimate is to estimate the quantity $f := \langle F, \nu\rangle$. Even though the preservation of star-shapedness already follows from an estimate for ∇u we want to include this estimate due to its nice geometric nature.

Lemma 3.12. *Let F be an admissible solution to* (IMCF)*. Let Σ^n be a smooth, convex cone. If the initial hypersurface is star-shaped with respect to the center of the cone, i.e. $0 < R_1 \leq \langle F_0, \nu_0\rangle \leq R_2$. Then the hypersurfaces remain star-shaped and satisfy*

$$R_1 \leq \; \langle F, \nu\rangle e^{-t/n} \leq R_2$$

for all $(x,t) \in M^n \times [0,T]$.

Proof. Let F be an admissible solution to (IMCF). We first prove the upper bound using the same argument as Huisken and Ilmanen in [30]. We first calculate

$$\frac{\partial |F|^2}{\partial t} = \frac{2}{H}\langle F, \nu\rangle \leq \frac{2|F|}{H} \leq \frac{2|F|^2}{n}.$$

The last inequality follows from the observation that at the point most distant form the origin $H \geq n|F|^{-1}$. From the growth of solutions to this ODE we obtain

$$\langle F, \nu\rangle \leq |F| \leq \max |F(\,.\,,0)| e^{t/n} = \max\langle F_0, \nu_0\rangle e^{t/n} \leq R_2 e^{t/n}.$$

The equality comes from the fact that at the maximum of $|F_0|$ we have $|F_0| = \langle F_0, \nu_0\rangle$. For the lower bound we try to find a Neumann problem to be able to apply the maximum principle. Notice that the calculations are carried out on the surface M_t^n, i.e. with respect to the induced metric g and not with respect to σ. First we calculate $\partial\nu/\partial t$. For this calculation we use the fact that $\partial\nu/\partial t \in T_p M_t^n$ and that ν is orthogonal to the tangent vectors $\partial F/\partial x^i$. We see that

$$\begin{aligned} \frac{\partial \nu}{\partial t} &= g^{ij}\left\langle \frac{\partial \nu}{\partial t}, \frac{\partial F}{\partial x^i}\right\rangle \frac{\partial F}{\partial x^j} = -g^{ij}\left\langle \nu, \frac{\partial}{\partial x^i}\left(\frac{\nu}{H}\right)\right\rangle \frac{\partial F}{\partial x^j} \\ &= -g^{ij}\left[\frac{\partial}{\partial x^i}\left\langle \nu, \frac{\nu}{H}\right\rangle - \left\langle \frac{\partial \nu}{\partial x^i}, \frac{\nu}{H}\right\rangle\right]\frac{\partial F}{\partial x^j} = \frac{1}{H^2} g^{ij}\frac{\partial H}{\partial x^i}\frac{\partial F}{\partial x^j} = \frac{1}{H^2}\, {}^{g}\nabla H. \end{aligned}$$

Therefore, we obtain the following expression for the time derivative of $f = \langle F, \nu \rangle$:

$$\frac{\partial}{\partial t} \langle F, \nu \rangle = \left\langle \frac{\partial F}{\partial t}, \nu \right\rangle + \left\langle F, \frac{\partial \nu}{\partial t} \right\rangle = \frac{1}{H} + \frac{1}{H^2} \langle F, {}^g\nabla H \rangle .$$

Using the fact that $\overline{\Delta}\nu = -|A|^2\nu + {}^g\nabla H$ (see e.g. [14], (A.9)) we get

$$\begin{aligned} \Delta_g \langle F, \nu \rangle &= \left\langle \overline{\Delta} F, \nu \right\rangle + 2g^{ij} \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial \nu}{\partial x^j} \right\rangle + \left\langle F, \overline{\Delta}\nu \right\rangle \\ &= H - |A|^2 \langle F, \nu \rangle + \langle F, {}^g\nabla H \rangle . \end{aligned}$$

Altogether we see that f satisfies the evolution equation which was already used in [30]:

$$\frac{\partial f}{\partial t} = \frac{1}{H^2}\Delta_g f + \frac{|A|^2}{H^2} f.$$

In order to compute the normal derivative ${}^g\nabla_\mu f$ we want to use an orthonormal frame as in Lemma 3.11. This time we choose a frame such that $e_1, ..., e_{n-1} \in T_p\Sigma^n \cap T_pM_t^n$, $e_n = \mu$ and $e_{n+1} = \nu$. We first recall two relations which were derived by Stahl in [59]. We see that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle F, \mu \rangle &= \left\langle \frac{\nu}{H}, \mu \right\rangle + \left\langle F, \frac{\mathrm{d}\mu}{\mathrm{d}t} \right\rangle = \left\langle F, \mathrm{d}\mu \left(\frac{\mathrm{d}F}{\mathrm{d}t} \right) \right\rangle \\ &= \frac{1}{H} \langle F, \mathrm{d}\mu(\nu) \rangle = \frac{1}{H} \left\langle F, \overline{\nabla}_\nu \mu \right\rangle = \frac{1}{H} \sum_{k \neq n} \langle F, e_k \rangle \, {}^{\Sigma^n} h_{\nu k}. \end{aligned}$$

and for $1 \leq i \leq n-1$ we have

$$\begin{aligned} 0 &= \overline{\nabla}_i \langle \nu, \mu \rangle = \left\langle \overline{\nabla}_i \nu, \mu \right\rangle + \left\langle \nu, \overline{\nabla}_i \mu \right\rangle \\ &= \sum_{k \neq n+1} \langle e_k, \mu \rangle \, {}^{M_t^n} h_{ik} + \sum_{l \neq n} \langle \nu, e_l \rangle \, {}^{\Sigma^n} h_{il} = {}^{M_t^n} h_{i\mu} + {}^{\Sigma^n} h_{i\nu}. \end{aligned}$$

This allows us to calculate

$$\begin{aligned} &{}^g\nabla_\mu \langle F, \nu \rangle \\ &= \left\langle \overline{\nabla}_\mu F, \nu \right\rangle + \left\langle F, \overline{\nabla}_\mu \nu \right\rangle = \langle \mu, \nu \rangle + \sum_{k \neq n+1} \langle F, e_k \rangle \, {}^{M_t^n} h_{\mu k} \\ &= \sum_{k=1}^{n-1} \langle F, e_k \rangle \, {}^{M_t^n} h_{\mu k} + \langle F, \mu \rangle \, {}^{M_t^n} h_{\mu\mu} = -\sum_{k=1}^{n-1} \langle F, e_k \rangle \, {}^{\Sigma^n} h_{k\nu} + \langle F, \mu \rangle \, {}^{M_t^n} h_{\mu\mu} \\ &= -H \frac{\mathrm{d}}{\mathrm{d}t} \langle F, \mu \rangle + \langle F, \nu \rangle \, {}^{\Sigma^n} h_{\nu\nu} + \langle F, \mu \rangle \, {}^{M_t^n} h_{\mu\mu} = \langle F, \nu \rangle \, {}^{\Sigma^n} h_{\nu\nu}. \end{aligned}$$

The last equality holds since Σ^n is a cone. So we see that f satisfies the following Neumann problem

$$\begin{cases} \dfrac{\partial f}{\partial t} &= \dfrac{1}{H^2}\Delta_g f + \dfrac{|A|^2}{H^2} f \quad \text{in} \quad M^n \times (0, T) \\ {}^g\nabla_\mu f &= {}^{\Sigma^n} h_{\nu\nu} f \quad \text{on} \quad \partial M^n \times (0, T) \\ f(\,.\,, 0) &= f_0 \quad \text{on} \quad M^n. \end{cases}$$

Using the fact that $|A|^2/H^2 \geq 1/n$ and the fact that ${}^{\Sigma^n}h_{\nu\nu}$ is positive definite we see that $R_1 e^{t/n}$ is a subsolution to this problem. Therefore, the maximum principle (see Theorem A.6 and Corollary A.8) implies the lower bound. □

Next, we present the geometric version of the estimate for $\rho := \partial w/\partial t$. It will be useful for proving a Hölder estimate in the following section and also yields an estimate for the mean curvature H.

Lemma 3.13. *Let F be an admissible solution to* (IMCF)*. Let Σ^n be a smooth, convex cone and R_1, R_2 be defined as in Lemma 3.12. Then H satisfies*

$$\left(\frac{R_1}{R_2}\right)\min_{M^n} H_0 \;\leq\; H(x,t)e^{t/n} \;\leq\; \left(\frac{R_2}{R_1}\right)\max_{M^n} H_0$$

for all $(x,t) \in M^n \times [0,T]$.

Proof. We will investigate the evolution of $\rho := 1/(Hf)$ with $f = \langle F, \nu\rangle$ as above. Note that $\rho = \dot{w}$. So the only difference to Lemma 3.10 is that we do the calculations with respect to the induced metric g. The evolution equation for ρ was derived by Huisken and Ilmanen in [30]. We want to mention the ingredients for the sake of completeness. Using the evolution equation of the metric, inverse metric and second fundamental

$$\frac{\partial g_{ij}}{\partial t} = \frac{2}{H}h_{ij}, \qquad \frac{\partial g^{ij}}{\partial t} = -\frac{2}{H}h^{ij}$$

$$\frac{\partial h_{ij}}{\partial t} = \frac{1}{H^2}\Delta_g h_{ij} + \frac{|A|^2}{H^2}h_{ij} - \frac{2}{H^3}\,{}^g\nabla_i H\,{}^g\nabla_j H$$

one obtains the evolution equations for f and H

$$\frac{\partial H}{\partial t} = \frac{1}{H^2}\Delta_g H - \frac{|A|^2}{H^2}H - \frac{2|{}^g\nabla H|^2}{H^3} \tag{3.3}$$

$$\frac{\partial f}{\partial t} = \frac{1}{H^2}\Delta_g f + \frac{|A|^2}{H^2}f \tag{3.4}$$

and thus the evolution equation for ρ

$$\frac{\partial \rho}{\partial t} = \frac{1}{H^2}\Delta_g\rho - 2\frac{|{}^g\nabla\rho|^2}{\rho H^2} - \frac{2}{H^3}\,{}^g\nabla^i H\,{}^g\nabla_i\rho = \operatorname{div}_g\left(\frac{{}^g\nabla\rho}{H^2}\right) - 2\frac{|{}^g\nabla\rho|^2}{\rho H^2}. \tag{3.5}$$

In order to calculate the normal derivative we first calculate $\nabla_\mu H$. Similar to Stahl in [59] we differentiate $\langle \nu, \mu\rangle = 0$ in time and use the time derivative of ν from the last proof to obtain

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\langle \nu, \mu\rangle = \left\langle \frac{\mathrm{d}\nu}{\mathrm{d}t}, \mu\right\rangle + \left\langle \nu, \frac{\mathrm{d}\mu}{\mathrm{d}t}\right\rangle$$

$$= \frac{1}{H^2}\langle\, {}^g\nabla H, \mu\rangle + \frac{1}{H}\langle \nu, \mathrm{d}\mu(\nu)\rangle = \frac{{}^g\nabla_\mu H}{H^2} + \frac{{}^{\Sigma^n}h_{\nu\nu}}{H}. \tag{3.6}$$

Together with the Neumann condition for f this implies

$${}^g\nabla_\mu\rho = -\frac{1}{H^2 f}\,{}^g\nabla_\mu H - \frac{1}{Hf^2}\,{}^g\nabla_\mu f = \frac{{}^{\Sigma^n}h_{\nu\nu}}{Hf} - \frac{{}^{\Sigma^n}h_{\nu\nu}}{Hf} = 0.$$

Therefore, we see that ρ satisfies the following Neumann problem.

$$\begin{cases} \dfrac{\partial \rho}{\partial t} &= \operatorname{div}_g\left(\dfrac{{}^g\nabla\rho}{H^2}\right) - 2\dfrac{|{}^g\nabla\rho|^2}{\rho H^2} \qquad \text{in} \quad M^n \times (0,T) \\ {}^g\nabla_\mu \rho &= 0 \qquad \text{on} \quad \partial M^n \times (0,T) \\ \rho(\,.\,,0) &= \rho_0 \qquad \text{on} \quad M^n. \end{cases}$$

Thus, the maximum principle (see Theorem A.6 and Corollary A.7) implies

$$\frac{1}{R_2 \max\limits_{M^n} H_0} \le \min_{M^n} \rho_0 \le \rho = \frac{1}{Hf} \le \max_{M^n} \rho_0 \le \frac{1}{R_1 \min\limits_{M^n} H_0}.$$

Finally, the estimates for f yield the desired estimates for H. ☐

Remark 3.14. Note that the surfaces M_t^n tend to infinity as time tends to infinity. From the estimate for u we see that rescaling by the factor $e^{-t/n}$ implies a bound on u. Therefore, we can only expect good estimates for the rescaled solution $\hat{u} = ue^{-t/n}$ or in terms of $w = \ln u$ for $\hat{w} := w - t/n$.

We want to summarize the scaling of the important quantities in the next Lemma.

Lemma 3.15. *Let F be a solution to* (IMCF)*. We obtain the rescaled solution by defining $\hat{F} := Fe^{-t/n}$. This implies the following rescalings*

$$\hat{u} = ue^{-t/n}, \qquad \nabla\hat{u} = \nabla u e^{-t/n}, \qquad \frac{\partial \hat{u}}{\partial t} = \left(\frac{\partial u}{\partial t} - \frac{u}{n}\right) e^{-t/n},$$

$$\hat{w} = w - \frac{t}{n}, \qquad \nabla\hat{w} = \nabla w, \qquad \frac{\partial \hat{w}}{\partial t} = \frac{\partial w}{\partial t} - \frac{1}{n},$$

$$\hat{g}_{ij} = g_{ij}e^{-2t/n}, \qquad \hat{g}^{ij} = g^{ij}e^{2t/n}, \qquad \hat{h}_{ij} = h_{ij}e^{-t/n}, \qquad \hat{H} = He^{t/n}.$$

Proof. From the definition of F we see that the rescaling of F implies the rescaling for u. The other formulas follow by direct calculation. ☐

3.3 Higher order Hölder estimates

We will first prove estimates for the Hölder coefficients of $\nabla\hat{u}$ and $\partial\hat{u}/\partial t$. They imply a Hölder estimate for the mean curvature $\hat{H}$ which will finally yield the full $C^{2+\alpha,1+\frac{\alpha}{2}}$-estimate for $\hat{u}$. We start with the estimate for the gradient.

Lemma 3.16. *Let u be an admissible solution to* (SP)*. Let Σ^n be a smooth, convex cone. Then there exists some $\beta > 0$ such that the rescaled function $\hat{u}(x,t) := u(x,t)e^{-t/n}$ satisfies*

$$[\nabla\hat{u}]_{x,\beta} + [\nabla\hat{u}]_{t,\frac{\beta}{2}} \;\le\; C.$$

Here $[f]_{z,\gamma}$ denotes the γ-Hölder semi-norm of f in $M^n \times [0,T]$ with respect to the z-variable and $C = C\left(\|u_0\|_{2+\alpha,M^n}, n, \beta, M^n\right)$.

Proof. First note that the a priori estimates for $|\nabla u|$ and $|\partial\hat{u}/\partial t|$ imply a bound for $[\hat{u}]_{x,\beta}$ and $[\hat{u}]_{t,\frac{\beta}{2}}$. The bound for $[\nabla\hat{u}]_{t,\frac{\beta}{2}}$ follows from a bound for $[\hat{u}]_{t,\frac{\beta}{2}}$ and [38], Chapter 2, Lemma 3.1 once we have a bound for $[\nabla\hat{u}]_{x,\beta}$. As $\nabla\hat{u} = \hat{u}\nabla w$ it is enough to bound $[\nabla w]_{x,\beta}$. To get this bound we fix t and rewrite $(\mathrm{SP})'$ as an elliptic Neumann problem with PDE

$$\operatorname{div}_\sigma\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right) + \left(\frac{\sqrt{1+|\nabla w(x,t)|^2}}{\dot{w}(x,t)} - \frac{n}{\sqrt{1+|\nabla w(x,t)|^2}}\right) = 0. \tag{3.7}$$

The equation is of the form $\nabla_i\big(a^i(p)\big) + a(x,t) = 0$. Since $\dot{w}$ and ∇w are bounded we see that a is a bounded function in x and t. Let us define $a^{ij}(p) := \partial a^i/\partial p^j$. Integrating the equation against the test function η and integration by parts yields

$$\int_{M^n}\Big(a^i(\nabla w)\nabla_i\eta - a(x,t)\eta\Big)\,\mathrm{d}\mu = 0. \tag{3.8}$$

The particular choice $\eta := \nabla_l\xi$ with $\xi \in W^{1,2}_{loc}(M^n)$ and another integration by parts shows that

$$\int_{M^n}\Big(a^{ij}(\nabla w)\nabla_{jl}w\nabla_i\xi - a(x,t)\nabla_l\xi\Big)\,\mathrm{d}\mu = 0.$$

Therefore $f := \nabla_l w$ satisfies (in a weak sense) a linear uniformly elliptic equation

$$\int_{M^n}\Big(a^{ij}(\nabla w)\nabla_j f - \sigma^i_l(x)a(x,t)\Big)\nabla_i\xi\,\mathrm{d}\mu = 0$$

with bounded and measurable coefficients. Thus [37], Chapter 3, Theorem 14.1 yields[1] an interior estimate of the form

$$[\nabla_l w]_{\beta,\overset{\circ}{M^n}} \le C\left(\operatorname{dist}(\overset{\circ}{M^n},\partial M^n), |\nabla w|, |\dot{w}|\right)$$

for some $\beta > 0$. To obtain the estimate near the boundary we proceed as in [37], Chapter 10, Section 2. We choose some boundary point x_0 and use a chart which locally flattens the boundary. Once more we use the weak formulation (3.8) but this time we choose $\eta := \nabla_r\xi$ and $\xi := \zeta^2\max\{\nabla_r w - k, 0\}$ where ζ is an arbitrary smooth function with values in $[0,1]$ defined in some neighborhood of an arbitrary boundary point. First let $r \ne n$, where e_n is supposed to be the direction normal to the boundary. This yields

$$\int_{M^n}\Big[\zeta^2 a^{ij}\nabla_i f\nabla_j f + 2\zeta a^{ij}\nabla_i\zeta\nabla_j f(f-k) - 2a\zeta\nabla_r\zeta(f-k) - a\zeta^2\nabla_r f\Big]\,\mathrm{d}\mu = 0$$

with $f := \nabla_r w$. Since a is bounded we denote its maximum by $\bar{a}$. Furthermore, the smallest and largest Eigenvalues λ_{min} and λ_{max} of a^{ij} are controlled due to the estimate for $|\nabla w|$.

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[1] Note that this result is stated for a domain in Eucledian space. But since only the known metric σ is involved we can translate this local result using a coordinate chart.

Using Young's inequality with ε on the second and last term and the same inequality with $\varepsilon = 1$ for the third term we obtain

$$\begin{aligned}\lambda_{min} \int_{A_{k,r}} |\nabla f|^2 \zeta^2 \,\mathrm{d}\mu \le \int_{A_{k,r}} \Big[&\lambda_{max} \varepsilon^2 \zeta^2 |\nabla f|^2 + \frac{\lambda_{max}}{\varepsilon^2} |\nabla \zeta|^2 |f-k|^2 \\ &+ \overline{a}^2 + |\nabla \zeta|^2 |f-k|^2 + \frac{\overline{a}^2 \zeta^2}{2\varepsilon^2} + \frac{\varepsilon^2}{2} |\nabla f|^2 \zeta^2 \Big] \,\mathrm{d}\mu\end{aligned}$$

where $A_{k,r} := B_r(x_0) \cap \Omega \cap \operatorname{spt} \xi$. Choosing ε small enough this yields

$$\int_{A_{k,r}} |\nabla f|^2 \zeta^2 \,\mathrm{d}\mu \le C(|\nabla w|, |a|) \int_{A_{k,r}} \Big(|\nabla \zeta|^2 |f-k|^2 + 1 \Big) \,\mathrm{d}\mu$$

This inequality for $\nabla_r w$ and the corresponding inequality for $-\nabla_r w$ imply (see [37], Chapter 2, Theorem 7.2) the Hölder continuity for $\nabla_r u$ in the case $r \neq n$. This result can be stated in the form of a Morrey estimate (compare [37], Chapter 2, Lemma 4.1), i.e.

$$\int_{B_r(x_0) \cap \Omega} |\nabla_r u|^2 \,\mathrm{d}\mu \le C r^{n-2+2\beta}.$$

To see that the same estimate also holds for $r = n$ one solves (3.7) for $\nabla_{nn} w$ to obtain $\nabla_{nn} w = b^{ir} \nabla_{ir} w + b$ where b^{ir} and b are bounded and the summation in r stops at $r = n-1$. Combining this with the Morrey estimate for $r < n$ we see that the Morrey estimate and therefore the Hölder continuity in the neighborhood of the boundary holds for $\nabla_r u$ up to $r = n$. The global result follows from a covering argument since M^n is compact. □

In the next step we estimate the Hölder coefficient for $\partial \hat{u} / \partial t$.

Lemma 3.17. *Let u be an admissible solution to* (SP). *Let Σ^n be a smooth, convex cone. Then there exists some $\beta > 0$ such that the rescaled function $\hat{u}(x,t) := u(x,t) e^{-t/n}$ satisfies*

$$\left[\frac{\partial \hat{u}}{\partial t} \right]_{x,\beta} + \left[\frac{\partial \hat{u}}{\partial t} \right]_{t,\frac{\beta}{2}} \;\le\; C.$$

Here $[f]_{z,\gamma}$ denotes the γ-Hölder norm of f in $M^n \times [0,T]$ with respect to the z-variable and $C = C\left(\|u_0\|_{2+\alpha, M^n}, n, \beta, M^n \right)$.

Proof. Similar to the last proof we want to use the weak formulation. This time we exploit the parabolic equation for ρ. We want to follow the argument in [38], Chapter 5, §7 pages 478 ff. Therefore we first note that $\rho = v/(uH) = \partial w / \partial t$ and therefore

$$\frac{\partial \hat{u}}{\partial t} = \left(\frac{\partial e^w}{\partial t} - \frac{u}{n} \right) e^{-t/n} = \frac{\partial w}{\partial t} e^w e^{-t/n} - \frac{\hat{u}}{n} = \hat{u} \left(\rho - \frac{1}{n} \right).$$

So the estimate for ρ will imply the estimate for $\partial \hat{u} / \partial t$. Next we remember from (3.5) that ρ satisfies the evolution equation

$$\frac{\partial \rho}{\partial t} = \operatorname{div}_{\hat{g}} \left(\frac{\nabla \rho}{\hat{H}^2} \right) - \frac{2 |\nabla \rho|^2_{\hat{g}}}{\rho \hat{H}^2}.$$

The weak formulation of this equation is

$$\int_{t_0}^{t_1}\int_{M_t^n}\left[\frac{\partial\rho}{\partial t}\eta+\frac{\nabla_i\rho\nabla^i\eta}{\hat{H}^2}+\frac{2|\nabla\rho|^2}{\rho\hat{H}^2}\eta\right]\mathrm{d}\mu_t\,\mathrm{d}t=0. \tag{3.9}$$

This time the argument for regions close to the boundary and those lying in the interior is the same. This is a special case since the right hand side of the Neumann condition is zero and the boundary integrals all vanish. We choose $\eta:=\xi^2\rho$ where ξ is an arbitrary smooth function with values in $[0,1]$. The first term can be written as

$$\int_{t_0}^{t_1}\int_{M_t^n}\frac{\partial\rho}{\partial t}\eta\,\mathrm{d}\mu_t\,\mathrm{d}t=\frac{1}{2}\int_{M_t^n}(\rho\xi)^2\,\mathrm{d}\mu_t\bigg|_{t_0}^{t_1}-\int_{t_0}^{t_1}\int_{M_t^n}(\rho)^2\,\xi\frac{\partial\xi}{\partial t}\,\mathrm{d}\mu_t\,\mathrm{d}t$$

and the second term in (3.9) equals

$$\int_{t_0}^{t_1}\int_{M_t^n}\frac{\nabla_i\rho\nabla^i\eta}{\hat{H}^2}\,\mathrm{d}\mu_t\,\mathrm{d}t=\int_{t_0}^{t_1}\int_{M_t^n}\left[\frac{2\xi\rho\nabla_i\rho\nabla^i\xi}{\hat{H}^2}+\frac{\xi^2\nabla_i\rho\nabla^i\rho}{\hat{H}^2}\right]\mathrm{d}\mu_t\,\mathrm{d}t.$$

Together this yields

$$\begin{aligned}\frac{1}{2}\|\rho\xi\|^2_{2,M_t^n}\bigg|_{t_0}^{t_1}+\int_{t_0}^{t_1}\int_{M_t^n}\left[\frac{\xi^2|\nabla\rho|^2}{\hat{H}^2}+\frac{2|\nabla\rho|^2\xi^2\rho}{\hat{H}^2\rho}\right]\mathrm{d}\mu_t\,\mathrm{d}t&\\ =\int_{t_0}^{t_1}\int_{M_t^n}\left[\rho^2\xi\frac{\partial\xi}{\partial t}-\frac{2\xi\rho\nabla_i\rho\nabla^i\xi}{\hat{H}^2}\right]\mathrm{d}\mu_t\,\mathrm{d}t.&\end{aligned}$$

Using the estimate

$$\frac{\xi^2|\nabla\rho|^2}{\hat{H}^2}\left(1+\frac{2\rho}{\rho}\right)\geq\frac{\xi^2|\nabla\rho|^2}{\max\hat{H}^2}$$

and Young's inequality we obtain

$$\begin{aligned}&\frac{1}{2}\|\rho\xi\|^2_{2,M_t^n}\Big|_{t_0}^{t_1}+\frac{1}{\max\hat{H}^2}\int_{t_0}^{t_1}\int_{M_t^n}\xi^2|\nabla\rho|^2\,\mathrm{d}\mu_t\,\mathrm{d}t\\ &\leq\int_{t_0}^{t_1}\int_{M_t^n}\left[\rho^2\xi|\frac{\partial\xi}{\partial t}|+\frac{2\xi\rho|\nabla\rho||\nabla\xi|}{\hat{H}^2}\right]\mathrm{d}\mu_t\,\mathrm{d}t\\ &\leq\int_{t_0}^{t_1}\int_{M_t^n}\left[\rho^2\xi|\frac{\partial\xi}{\partial t}|+\frac{\varepsilon\xi^2|\nabla\rho|^2}{\min\hat{H}^2}+\frac{\rho^2|\nabla\xi|^2}{\varepsilon\min\hat{H}^2}\right]\mathrm{d}\mu_t\,\mathrm{d}t.\end{aligned}$$

Choosing $\varepsilon:=\min\hat{H}^2/(2\max\hat{H}^2)$ this finally yields

$$\begin{aligned}&\frac{1}{2}\|\rho\xi\|^2_{2,M_t^n}\Big|_{t_0}^{t_1}+\frac{1}{2\max\hat{H}^2}\int_{t_0}^{t_1}\int_{M_t^n}\xi^2|\nabla\rho|^2\,\mathrm{d}\mu_t\,\mathrm{d}t\\ &\leq\left(1+\frac{2\max\hat{H}^2}{\min\hat{H}^4}\right)\int_{t_0}^{t_1}\int_{M_t^n}\rho^2\left[\xi|\frac{\partial\xi}{\partial t}|+|\nabla\xi|^2\right]\mathrm{d}\mu_t\,\mathrm{d}t.\end{aligned}$$

This inequality is of the same kind as the one in [38], Chapter 2, Remark 7.2. Therefore, Theorem 8.1 and Remark 8.2 in the same chapter imply[2] that ρ is Hölder continuous in the x and t variable. The global result follows from the local results and a covering argument. ☐

These two estimates directly imply an estimate for the mean curvature.

Lemma 3.18. *Let u be an admissible solution to* (SP)*. Let Σ^n be a smooth, convex cone. Then there exists some $\beta > 0$ such that the rescaled mean curvature $\hat{H} = He^{t/n}$ satisfies*

$$\left[\hat{H}\right]_{x,\beta} + \left[\hat{H}\right]_{t,\frac{\beta}{2}} \leq C.$$

Here $[f]_{z,\gamma}$ denotes the γ-Hölder norm of f in $M^n \times [0,T]$ with respect to the z-variable and $C = C\left(\|u_0\|_{2+\alpha,M^n}, n, \beta, M^n\right)$.

Proof. This follows from the fact that

$$\hat{H} = He^{t/n} = \frac{\sqrt{1+|\nabla w|^2}}{e^w \dot{w}} e^{t/n} = \frac{\sqrt{1+|\nabla w|^2}}{\hat{u}\dot{w}}$$

together with the Hölder estimates for $|\nabla w|$, $\dot{w}$ and $\hat{u}$. Note that the Hölder estimate for $\hat{u}$ follows trivially from the estimates on $|\nabla \hat{u}|$ and $|\partial \hat{u}/\partial t|$. ☐

Finally we obtain the full second order a priori estimates.

Lemma 3.19. *Let u be an admissible solution to* (SP)*. Let Σ^n be a smooth, convex cone. Then there exists some $\beta > 0$ such that*

$$\|u\|_{2+\beta,1+\frac{\beta}{2},M^n\times[0,T]} \leq C$$

with $C = C\left(\|u_0\|_{2+\alpha,M^n}, n, \beta, M^n\right)$.

Proof. We define $v := \sqrt{1+|\nabla w|^2}$ and use the formula for the mean curvature to write

$$uvH = n - \left(\sigma^{ij} - \frac{\nabla^i w \nabla^j w}{1+|\nabla w|^2}\right)\nabla^2_{ij} w = n - u^2 \Delta_g w.$$

Thus we obtain

$$\frac{\partial w}{\partial t} = \frac{v}{uH} = -\frac{uv}{u^2H^2}H + \frac{2v}{uH} = \frac{1}{\hat{H}^2}\Delta_{\hat{g}} w + \left(\frac{2v}{\hat{u}\hat{H}} - \frac{n}{\hat{u}^2\hat{H}^2}\right)$$

which is a linear, uniformly parabolic equation with Hölder continuous coefficients. Therefore the linear theory (e.g. [38], Chapter 4, Theorem 5.3) yields the result. ☐

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[2] Again the arguments in [38] work in Euclidean space but since the arguments are local and the chart only involves the metric $\hat{g}$ which is controlled (due to the estimates for $\hat{u}$ and $\nabla \hat{u}$) this does not cause any problems.

3.4 Long-time existence and convergence

From the definition of the maximal existence time we see that we have to show that all derivatives stay bounded up to T^* in order to be able to obtain a contradiction to the maximality of T^*. Therefore, we first prove a statement on higher regularity.

Lemma 3.20. *Let u be an admissible solution to* (SP). *Let Σ^n be a smooth, convex cone. Then there exists some $\beta > 0$ and some $t_0 > 0$ such that for all $k \in \mathbb{N}$*

$$\|u\|_{2k+\beta,k+\frac{\beta}{2},M^n\times[t_0,T]} \leq C$$

where C only depends on $\|u(\,.\,,t_0)\|_{2k+\alpha,M^n}, n, \beta$ and M^n.

Proof. Using the $C^{2+\beta,1+\frac{\beta}{2}}$-estimate from Lemma 3.19 we can consider the equations for $\dot{w}$ and $\nabla_i w$ as linear uniformly parabolic equations on the time interval $[t_0, T]$. At the initial time t_0 all compatibility conditions are satisfied and the initial function $u(\,.\,,t_0)$ is smooth. This implies (in two steps) a $C^{3+\beta,\frac{3+\beta}{2}}$-estimate for $\nabla_i w$ and (in one step) a $C^{2+\beta,1+\frac{\beta}{2}}$-estimate for $\dot{w}$. Together this yields the result for $k = 2$. From [45], chapter 4, Theorem 4.3, Exercise 4.5 and the preceding arguments one can see that the constants are independent of T. Higher regularity is proved by induction over k. □

Recall that $M^n \subset S^n \subset \mathbb{R}^{n+1}$, that the cone Σ^n is defined in (3.1) and that we consider the problem[3]

$$\text{(IMCF)}\begin{cases} \dfrac{\partial F}{\partial t} = \dfrac{\nu}{H} \circ F & \text{in } M^n \times (0,\infty) \\ \langle \mu \circ F, \nu \circ F\rangle = 0 & \text{on } \partial M^n \times (0,\infty) \\ F(\,.\,,0) = F_0 & \text{on } M^n \end{cases}$$

where ν is the unit normal to $M^n_t := F(M^n, t)$ pointing away from the center of the cone. Collecting all the a priori estimates we can prove the main result of this chapter.

Theorem 3.21 (Expansion in a cone). *Let $n \geq 2$. Let Σ^n be a smooth, convex cone with outward unit normal μ. Let $F_0 : M^n \to \mathbb{R}^{n+1}$ be such that $M^n_0 := F_0(M^n)$ is a compact $C^{2,\alpha}$-hypersurface which is star-shaped with respect to the center of the cone and has strictly positive mean curvature. Furthermore, assume that M^n_0 meets Σ^n orthogonally, i.e. $F_0(\partial M^n) \subset \Sigma^n$ and $\langle \mu \circ F_0, \nu_0 \circ F_0\rangle|_{\partial M^n} = 0$ where ν_0 is the unit normal to M^n_0. Then there exists a unique embedding*

$$F \in C^{2+\alpha,1+\frac{\alpha}{2}}(M^n \times [0,\infty), \mathbb{R}^{n+1}) \cap C^\infty(M^n \times (0,\infty), \mathbb{R}^{n+1})$$

with $F(\partial M^n, t) \subset \Sigma^n$ for $t \geq 0$, satisfying (IMCF). *Furthermore, the rescaled embedding $F(\,.\,,t)e^{-t/n}$ converges smoothly to an embedding F_∞, mapping M^n into a piece of a round sphere of radius $r_\infty = (|M^n_0|/|M^n|)^{(1/n)}$.*

Proof. From Lemma 3.3 we know that a solution with the desired regularity exists at least for a short time and using Lemma 3.20 we see that the Hölder norm of $u = \hat{u}e^{t/n}$ can not blow up as T tends to $T^* < \infty$. Therefore, u can be extended to be a solution

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[3] The only difference to (IMCF) in Definition 1.1 is that here M^n is a submanifold of $N^{n+1} = \mathbb{R}^{n+1}$.

to (SP) in $[0, T^*]$. The short-time existence result of Lemma 3.3 together with Lemma 3.20 imply the existence of a solution beyond T^* which is smooth away from $t = 0$. This is a contradiction to the choice of T^* and therefore $T^* = \infty$. To investigate the rescaled embedding as t tends to infinity we have to examine the behavior of $\hat{u} = ue^{-t/n}$. The a priori estimates allow us to read (3.2) of Lemma 3.11 as

$$\frac{\partial \psi}{\partial t} \ \leq \ Q^{ij}\nabla_{ij}\psi + B^k\nabla_k\psi - \gamma\psi.$$

with some $\gamma > 0$ which implies an exponential decay of ψ. The maximum principle (see Theorem A.6 and Corollary A.8) implies that

$$|\nabla\hat{u}| \leq \left(\frac{R_2}{R_1}\right)\max_{M^n}|\nabla u_0|e^{-\gamma t}.$$

Therefore, the gradient of $\hat{u}$ is decaying to zero. Using the formula for the first variation of area (see e.g. [57]) and the fact that $\operatorname{div}_{M_t^n}\nu = H$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}|M_t^n| = \int_{M_t^n}\operatorname{div}_{M_t^n}\left(\frac{1}{H}\nu\right)\,\mathrm{d}\mu_t = \int_{M_t^n}\sum_{i=1}^{n}\left\langle\nabla_{e_i}\left(\frac{1}{H}\nu\right), e_i\right\rangle\,\mathrm{d}\mu_t = |M_t^n|$$

where $\{e_i\}_{1\leq i\leq n}$ is some orthonormal frame of TM_t^n. Thus the surface area grows exponentially and the rescaled hypersurfaces have constant surface area. Using the Arzelà-Ascoli theorem and the decay of the gradient we see that every subsequence must converge to a constant function. The constant surface area implies $|M_0^n| = |\hat{M}_\infty^n| = r_\infty^n|M^n|$ and shows that $\hat{u}(\,.\,,t)$ is converging in $C^1(M^n)$ to the constant function $\hat{u}_\infty = r_\infty$.

Now assume that $\hat{u}(\,.\,,t)$ converges in $C^k(M^n)$ to r_∞. Since $\hat{u}(\,.\,,t)$ is uniformly bounded in $C^{k+1+\beta}(M^n)$ by Arzelà-Ascoli there exists a subsequence which converges to r_∞ in $C^{k+1}(M^n)$. Finally every subsequence must converge and the limit has to be r_∞. Thus $\hat{u}(\,.\,,t)$ converges in $C^{k+1}(M^n)$. This finishes the induction and shows that the convergence is smooth. $\square$

4 Existence of weak solutions

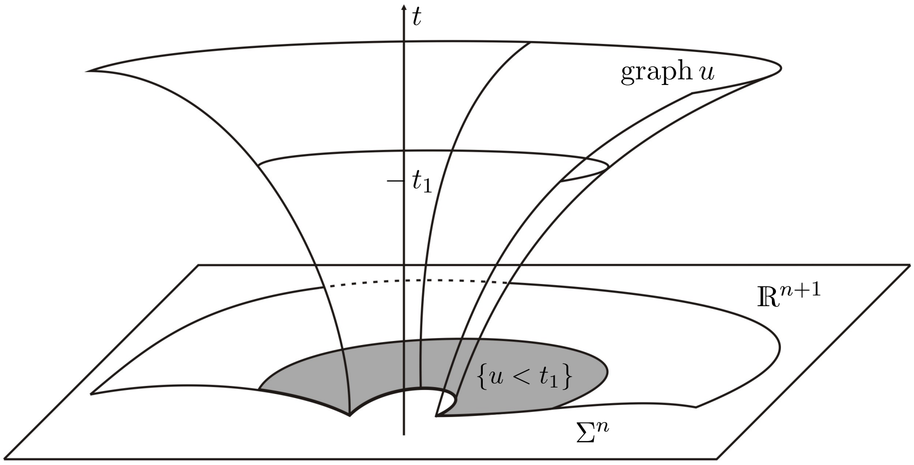

Figure 4.1: Level set description: $M^n_{t_1} = \partial\{u < t_1\}$.

So far we have considered the surfaces M^n_t as the image of the surface M^n under the embedding $F(\,.\,, t)$. Now we change our point of view. We introduce a scalar, time-independent function u such that the hypersurface M^n_t is given as the t-level set of the function u (see figure 4.1). In this setting the problem (IMCF) can be reformulated as a degenerate elliptic mixed boundary value problem for this level-set function in a domain with corners.

In Section 4.1 we will derive the level-set formulation and define a family of approximating problems which will have more regular solutions. We will use Section 4.2 to derive a priori estimates for the solutions of these approximating problems. This yields an existence and uniqueness result for the approximating problems in Section 4.3, Theorem 4.21. Guided by the ideas of Huisken and Ilmanen [29] we define a notion of weak solutions in Section 4.4. Furthermore, we show that the sequence of approximating solutions gives rise to a sequence of weak solutions one dimension higher. Using a compactness result we can finally prove that the limit of this sequence is the unique minimizer of a certain functional related to the level-set problem. This program yields existence and uniqueness for weak solutions of IMCF in the case of hypersurfaces with boundary in Theorem 4.47.

The last section gives an outlook to a potential application of weak solutions indicated by the monotonicity of the Hawking mass for classical solutions to (IMCF).

4.1 Level-set description and approximation

In the sequel we will be interested in sets which lie on one side of the oriented hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$. Therefore, we need the following definition.

Definition 4.1. Let Σ^n be an oriented hypersurface in $\mathbb{R}^{n+1}$ with a unit normal μ. We define the set of points lying on and above Σ^n using curves $\gamma : [0,1] \to \mathbb{R}^{n+1}$.

$$\Omega := \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \exists\, \gamma \text{ s.t. } \gamma([0,1]) \cap \Sigma^n = \gamma(0),\ \gamma(1) = x,\ \gamma'(0) = -\mu \right\} \cup \Sigma^n.$$

Furthermore, for a set $A \subset \Omega$ we define the boundary parts

$$\partial_\Omega A := \overline{\partial A \setminus \Sigma^n} \qquad \text{and} \qquad \partial_\Sigma A := \partial A \setminus \partial_\Omega A.$$

With the help of this definition we can describe the evolutionary problem in the level-set formalism.

Lemma 4.2. *Let F satisfy* (IMCF) *such that $M_t^n = F(M^n, t)$. Let $u : \Omega \to \mathbb{R}$ be the level-set function such that $M_t^n = \partial_\Omega \{u < t\}$ holds. As long as the mean curvature of the hypersurfaces M_t^n is strictly positive problem* (IMCF) *is equivalent to*

$$(\star) \quad \begin{cases} \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = |Du| & \text{in } \Omega_0 := \Omega \setminus \overline{E_0} \\ D_\mu u = 0 & \text{on } \Sigma_0 := \partial_\Sigma \Omega_0 \\ u = 0 & \text{on } \partial_\Omega E_0 \end{cases}$$

where $E_0 = \{u < 0\}$ and μ is the unit normal to Σ^n.

Proof. First we note that given a solution u to $(\star)$ in Ω_0 we can extend u to Ω such that $u \leq 0$ in E_0. In terms of u the outward unit normal to M_t^n is $\nu = Du/|Du|$. Since the mean curvature is the divergence of the normal we have

$$H = \operatorname{div}(\nu) = \operatorname{div}\left(\frac{Du}{|Du|}\right).$$

Let $\delta > 0$. We choose a curve $\gamma : [t-\delta, t+\delta] \to \mathbb{R}^{n+1}$ such that $\gamma(t) \in M_t^n$ and $\dot\gamma || \nu$. Then the point $\gamma(t)$ moves in time with the speed $|\dot\gamma(t)| = 1/H$ and $t = u(\gamma(t))$. Differentiating this expression in t yields

$$1 = \left\langle Du, \dot\gamma(t) \right\rangle_{\mathbb{R}^{n+1}} = \left\langle Du, \frac{\nu}{H} \right\rangle_{\mathbb{R}^{n+1}} = \frac{|Du|}{H}.$$

Therefore, $H = |Du|$ which justifies the PDE. The boundary condition on Σ_0 is equivalent to the orthonormality condition since

$$0 = \langle \mu, \nu \rangle_{\mathbb{R}^{n+1}} = \left\langle \mu, \frac{Du}{|Du|} \right\rangle_{\mathbb{R}^{n+1}}.$$

The initial condition $F(M^n, 0) = M_0^n$ is equivalent to $u = 0$ on $\partial_\Omega E_0$ since $\partial_\Omega E_0 = M_0^n = \{u = 0\}$. □

Remark 4.3. In the preceding lemma we used the fact that for $H > 0$ we have $M_t^n = \{u = t\}$. This does not coincide with $\partial_\Omega\{u < t\}$ if u is allowed to have plateaus. Furthermore, even for $|Du| > 0$

$$\operatorname{div}_{\mathbb{R}^{n+1}}\left(\frac{Du}{|Du|}\right) = \frac{1}{|Du|}\left(\delta^{ij} - \frac{D^i u D^j u}{|Du|^2}\right) D_{ij} u =: a^{ij}(Du) D_{ij} u$$

and a^{ij} is degenerate since the Eigenvalue in direction of Du is zero.

Example 4.4 (**Expanding half spheres**). In Example 1.3 we already saw that starting with an upper half sphere of radius r_0 as initial hypersurface and choosing $\Sigma^n := \{x_{n+1} = 0\}$ and $\mu := -e_{n+1}$ the half spheres expand exponentially such that $M_t^n = S_{r(t)}^{n,+}$ with $r(t) = r_0 e^{t/n}$. In this case the sets described above are

$$\begin{aligned}\Omega &:= \Big\{x \in \mathbb{R}^{n+1} \;\Big|\; x_{n+1} \geq 0\Big\}\\ E_0 &:= \Big\{x \in \mathbb{R}^{n+1} \;\Big|\; x_{n+1} \geq 0 \quad \text{and} \quad |x| < r_0\Big\}\\ \partial_\Omega E_0 &:= \Big\{x \in \mathbb{R}^{n+1} \;\Big|\; x_{n+1} \geq 0 \quad \text{and} \quad |x| = r_0\Big\}\\ \partial_\Sigma \Omega_0 &:= \Big\{x \in \mathbb{R}^{n+1} \;\Big|\; x_{n+1} = 0 \quad \text{and} \quad |x| > r_0\Big\}\end{aligned}$$

and the solution to $(\star)$ is given by $u(x) = n\ln(|x|/r_0)$.

In order to solve $(\star)$ we want to consider a family of non degenerate problems in a bounded domain. It turns out that we also have to deform the given set E_0 in order to be able to solve the non degenerate problem in the right weighted Hölder spaces.

Definition 4.5. Let $E_0 \subset \Omega$ be open and bounded. Assume that $\partial_\Omega E_0$ is a $C^{2,\alpha}$-hypersurface which meets Σ^n orthogonally. We define the set

$$E_{0,\varepsilon} := E_0 \setminus \Big\{x \in E_0 \;\Big|\; \operatorname{dist}(x, \Sigma^n) < \varepsilon \quad \text{and} \quad \operatorname{dist}(x, \partial E_0) < \xi_\varepsilon(x)\;\Big\} \tag{4.1}$$

where

$$\xi_\varepsilon(x) := \varepsilon^3 \exp\left(1 - \left(\frac{\varepsilon}{\varepsilon - \operatorname{dist}(x, \Sigma^n)}\right)^2\right).$$

So $E_{0,\varepsilon}$ is a subset of E_0 which coincides with E_0 for points far from Σ^n. The function ξ_ε is arranged in such a way that for the exterior normal to $E_{0,\varepsilon}$ given by $\nu_{\partial E_{0,\varepsilon}}$ we have $\theta_1(\varepsilon) := \measuredangle\left(\nu_{\partial E_{0,\varepsilon}}, \mu\right) \in (0, \frac{\pi}{2})$ or in other words

$$D_\mu \operatorname{dist}(\,.\,, \partial_\Omega E_{0,\varepsilon}) > 0 \quad \text{on} \quad \Sigma^n \cap \partial_\Omega E_{0,\varepsilon}. \tag{4.2}$$

As we will see this property ensures the existence of more regular solutions. To define a family of approximating problems in bounded domains we also have to introduce an artificial outer Dirichlet boundary.

Definition 4.6. Let $F_{L_\varepsilon} \subset \Omega$ be open in Ω. Assume that $\partial_\Omega F_{L_\varepsilon}$ is a $C^{2,\alpha}$-hypersurface and that $F_{L_\varepsilon} \supset E_{0,\varepsilon}$. Furthermore, assume that $\theta_2(\varepsilon) := \measuredangle\left(-\nu_{\partial F_{L_\varepsilon}}, \mu\right) \in (0, \frac{\pi}{2})$ where $\nu_{\partial F_{L_\varepsilon}}$ is the exterior unit normal to F_{L_ε}. We define

$$\Omega_\varepsilon := \overbrace{F_{L_\varepsilon} \setminus E_{0,\varepsilon}}^{\circ}, \qquad \Sigma_\varepsilon := \partial_\Sigma \Omega_\varepsilon \tag{4.3}$$

and consider the following family of ε-regularized level-set problems in bounded domains

$$(\star)_{\varepsilon,\tau}\begin{cases} Q^{\varepsilon}u^{\varepsilon,\tau} := \operatorname{div}\left(\dfrac{Du^{\varepsilon,\tau}}{\sqrt{\varepsilon^2+|Du^{\varepsilon,\tau}|^2}}\right) - \sqrt{\varepsilon^2+|Du^{\varepsilon,\tau}|^2} = 0 & \text{in } \Omega_\varepsilon \\ D_\mu u^{\varepsilon,\tau} = 0 & \text{on } \Sigma_\varepsilon \\ u^{\varepsilon,\tau} = 0 & \text{on } \partial_\Omega E_{0,\varepsilon} \\ u^{\varepsilon,\tau} = \tau & \text{on } \partial_\Omega F_{L_\varepsilon} \end{cases}$$

for $\varepsilon > 0$ and $\tau \in [0, L_\varepsilon]$ (see Figure 4.2).

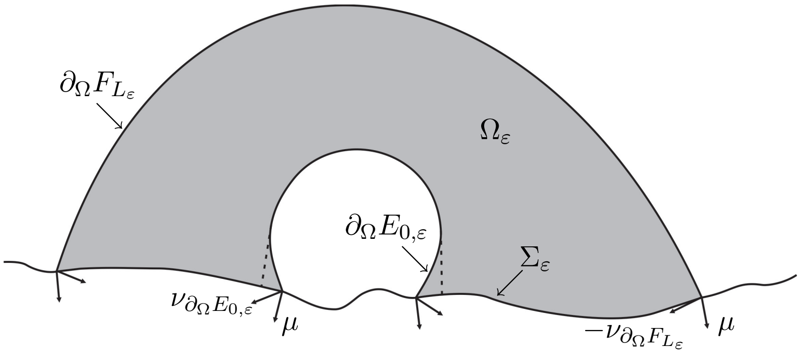

Figure 4.2: Domain and boundaries for $(\star)_{\varepsilon,\tau}$. The dotted line denotes $\partial_\Omega E_0$.

The idea is that for $\varepsilon \to 0$ the sets F_{L_ε} become larger, $\partial_\Omega E_{0,\varepsilon}$ deforms back to $\partial_\Omega E_0$ and $L_\varepsilon \to \infty$. Thus, we recover the problem $(\star)$ in the limit. The choice of F_{L_ε} and the largest possible value L_ε will depend on the availability of a subsolution as we will see in the next section.

4.2 Estimates for the approximating problems

Similar to the procedure in Chapter 3 we will now prove a priori estimates for $|u^{\varepsilon,\tau}|$ and $|Du^{\varepsilon,\tau}|$. To obtain estimates for $|u^{\varepsilon,\tau}|$ we will construct super- and subsolutions. To estimate $|Du^{\varepsilon,\tau}|$ on the Neumann boundary we use the maximum principle. The estimate of $|Du^{\varepsilon,\tau}|$ on the Dirichlet boundary will be obtained by constructing suitable barriers.

We will see that we can prove the existence of solutions to $(\star)_{\varepsilon,\tau}$ in weighted Hölder spaces which guarantees that the solutions are in particular in $C^{2,\alpha}(\Omega_\varepsilon) \cap C^{1,\beta}(\overline{\Omega_\varepsilon})$ for some $\alpha, \beta \in (0,1)$. To shorten the notation we make the following definition.

Definition 4.7. Let $\alpha, \beta \in (0,1)$. A function $u \in C^{2,\alpha}(\Omega_\varepsilon) \cap C^{1,\beta}(\overline{\Omega_\varepsilon})$ is called admissible.

We start with the estimate for u from above.

Lemma 4.8 (**Existence of a supersolution**). *Let $u^{\varepsilon,\tau}$ be an admissible solution of $(\star)_{\varepsilon,\tau}$. Then $v^+ :\equiv \tau$ is a supersolution and $u^{\varepsilon,\tau} \leq \tau$.*

Proof. The constant function $v^+(x) :\equiv \tau$ lies above $u^{\varepsilon,\tau}$ on both Dirichlet boundaries $\partial_\Omega E_{0,\varepsilon}$ and $\partial_\Omega F_{L_\varepsilon}$ and satisfies the Neumann condition $D_\mu v^+ = 0$ on Σ_ε. Furthermore, $Q^\varepsilon v^+ = -\varepsilon \leq 0$ in Ω_ε. Therefore, the maximum principle in Proposition A.12 implies the result. ☐

Unfortunately, the function $v \equiv 0$ is not a subsolution. The reason is that for every non-constant function the sign of the quantity $D_\mu v^-$ has to be controlled everywhere on Σ_ε. To achieve this we assume that Σ^n is globally given as the graph of a C^1-function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ such that all tangent lines to graph f in radial directions hit the x^{n+1}-axis above the point $x_0 := (0, ..., 0, -c_0)$, i.e.

$$\min_{x\in\mathbb{R}^{n+1}} \{f(x) - \langle Df(x), x\rangle_{\mathbb{R}^{n+1}}\} > -c_0 \tag{4.4}$$

for some positive $c_0 >$ sufficiently large (see Figure 4.3).

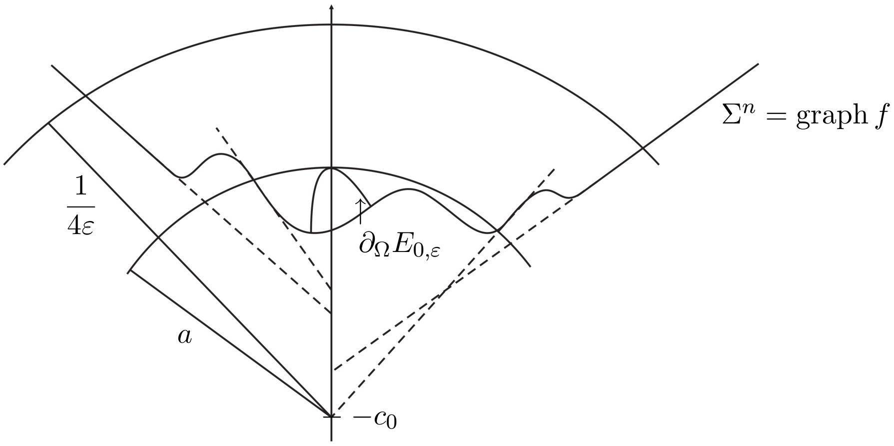

Figure 4.3: Asymptotically cone-like graphs allow for rotationally symmetric subsolutions

Lemma 4.9 (**Existence of a subsolution**). *Let* $n \geq 2$. *Let* Σ^n *be globally given as the graph of a* C^1*-function* $f : \mathbb{R}^{n+1} \to \mathbb{R}$ *such that* (4.4) *holds. Let* F_{L_ε} *be defined by*

$$F_{L_\varepsilon} := \left\{ x \in \Omega \;\middle|\; \operatorname{dist}(x, x_0) < \frac{1}{4\varepsilon} \right\}$$

and $\Omega_\varepsilon, \Sigma_\varepsilon$ *be defined by* (4.3). *Then an admissible solution* $u^{\varepsilon,\tau}$ *of* $(\star)_{\varepsilon,\tau}$ *with*

$$0 \leq \tau \leq L_\varepsilon := \frac{|\ln(4\varepsilon a)|}{2}, \qquad a := \max_{\partial_\Omega E_{0,\varepsilon}} \operatorname{dist}(\,.\,, x_0), \qquad \varepsilon < \frac{1}{4a}$$

satisfies the estimate

$$u^{\varepsilon,\tau}(x) \;\geq\; v^-_\tau(x) \;:=\; v^-_{L_\varepsilon}(x) - L_\varepsilon + \tau \;:=\; \frac{1}{2}\ln\left(\frac{\operatorname{dist}(x, x_0)}{a}\right) - L_\varepsilon + \tau. \tag{4.5}$$

In the limit as $\varepsilon \to 0$ *we see that* $\Omega_\varepsilon \to \Omega_0$ *and* $L_\varepsilon \to \infty$.

Proof. To obtain the lower bound we construct a subsolution of $(\star)_{\varepsilon,L_\varepsilon}$ of the form

$$v^-(x) := \lambda \ln\left(\frac{r(x)}{a}\right), \qquad L_\varepsilon := \lambda \ln\left(\frac{R}{a}\right), \qquad \lambda > 0, \qquad R > a$$

where $r(x) := \operatorname{dist}(x, x_0)$, $a := \max_{\partial_\Omega E_{0,\varepsilon}} r$ and λ and R will be specified later. By definition $v^- \leq 0$ on $\partial_\Omega E_{0,\varepsilon}$ and $v^- = L_\varepsilon$ on $\partial_\Omega F_{L_\varepsilon}$. We define $r_i := (x - x_0)_i$ and compute that

$$D_i v^- = \lambda \frac{r_i}{r^2}, \qquad \sqrt{\varepsilon^2 + |Dv^-|^2} = \frac{1}{r}\sqrt{\varepsilon^2 r^2 + \lambda^2}, \qquad D_{ij} v^- = \frac{\lambda}{r^2}\left(\delta_{ij} - \frac{2 r_i r_j}{r^2}\right).$$

For the Neumann condition we obtain

$$D_{\mu(x)}v^-(x) = \langle \mu(x), Dv^-(x)\rangle = \frac{\lambda}{r^2(x)}\,\langle \mu(x), x - x_0\rangle \qquad \text{for } x \in \Sigma_\varepsilon$$

so we see that $D_\mu v^- \leq 0$ as long as $x - x_0$ is pointing inside the domain or is at most tangential to the boundary. This is true by the choice of x_0. It is left to prove the inequality for the operator Q^ε. We obtain

$$\begin{aligned}
Q^\varepsilon(v^-) &= \operatorname{div}\left(\frac{Dv^-}{\sqrt{\varepsilon^2 + |Dv^-|^2}}\right) - \sqrt{\varepsilon^2 + |Dv^-|^2}\\
&= \frac{1}{\sqrt{\varepsilon^2 + |Dv^-|^2}}\left(\delta^{ij} - \frac{D^i v^- D^j v^-}{\varepsilon^2 + |Dv^-|^2}\right) D_{ij}v^- - \sqrt{\varepsilon^2 + |Dv^-|^2}\\
&= \frac{r}{\sqrt{\varepsilon^2 r^2 + \lambda^2}}\left(\delta^{ij} - \frac{\lambda^2 r^{-4} r^i r^j}{r^{-2}(\varepsilon^2 r^2 + \lambda^2)}\right)\frac{\lambda}{r^2}\left(\delta_{ij} - \frac{2 r_i r_j}{r^2}\right) - \frac{1}{r}\sqrt{\varepsilon^2 r^2 + \lambda^2}\\
&= \frac{\lambda}{r\sqrt{\varepsilon^2 r^2 + \lambda^2}}\left(n - 2 - \frac{\lambda^2}{\varepsilon^2 r^2 + \lambda^2} + \frac{2\lambda^2}{\varepsilon^2 r^2 + \lambda^2}\right) - \frac{1}{r}\sqrt{\varepsilon^2 r^2 + \lambda^2}\\
&= \frac{1}{r(\varepsilon^2 r^2 + \lambda^2)^{3/2}}\Big(\lambda(n-2)(\varepsilon^2 r^2 + \lambda^2) + \lambda^3 - (\varepsilon^2 r^2 + \lambda^2)^2\Big)\\
&= \frac{1}{r(\varepsilon^2 r^2 + \lambda^2)^{3/2}}\Big(- \lambda^4 + (n-1)\lambda^3 - 2\varepsilon^2 r^2 \lambda^2 + (n-2)\varepsilon^2 r^2 \lambda - \varepsilon^4 r^4\Big).
\end{aligned}$$

This yields

$$Q^\varepsilon(v^-) \;\geq \frac{1}{r(\varepsilon^2 r^2 + \lambda^2)^{3/2}}\Big(- \lambda^4 + \lambda^3 - 2\varepsilon^2 r^2 \lambda^2 - \varepsilon^4 r^4\Big) \tag{4.6}$$

provided $n \geq 2$. Thus, if we choose $\lambda = 1/2$, $r \leq R := \dfrac{1}{4\varepsilon}$ and $\varepsilon < \frac{1}{4a}$ then $Q^\varepsilon(v^-) > 0$ and the maximum principle in Proposition A.12 implies $v^- \leq u^{\varepsilon,\tau}$ in $\overline{\Omega_\varepsilon}$. Furthermore,

$$L_\varepsilon := \lambda \ln\left(\frac{R}{a}\right) = \frac{1}{2}\ln\left(\frac{1}{4\varepsilon a}\right) = \frac{|\ln(4\varepsilon a)|}{2} \to \infty$$

and $\Omega_\varepsilon \to \Omega_0$ since $F_{L_\varepsilon} \to \Omega$ as $\varepsilon \to 0$. So far we obtained a subsolution for $(\star)_{\varepsilon, L_\varepsilon}$ so we rename v^- to $v^-_{L_\varepsilon}$ and we see that the function $v^-_\tau := v^-_{L_\varepsilon} - L_\varepsilon + \tau$ is a subsolution for the problem $(\star)_{\varepsilon,\tau}$. ☐

We saw that a subsolution can be used to define F_{L_ε} and L_ε. Unfortunately, the estimate $u^{\varepsilon,\tau} \geq v^-$ is not very accurate near $\partial_\Omega E_{0,\varepsilon}$ since we only get $u^{\varepsilon,\tau} \geq -c(\varepsilon)$ but the estimate does not tell us that $u^{\varepsilon,\tau}$ becomes non-negative as ε tends to zero. Using subsolutions which are less steep (see Figure 4.4) we can fix this problem.

Lemma 4.10 (Improved lower bound). *Suppose the assumptions of Lemma 4.9 hold and $\varepsilon \leq 1 \cdot 10^{-20}$. If we restrict $(\star)_{\varepsilon,\tau}$ to smaller domains and boundary values, i.e.*

$$F_{L\varepsilon} := \left\{x \in \Omega \;\middle|\; \operatorname{dist}(x, x_0) < \frac{1}{10\varepsilon^{1/32}}\right\}, \qquad L_\varepsilon := \frac{|\ln(10a\varepsilon^{1/64})|}{2} \tag{4.7}$$

with $a := \max_{\partial_\Omega E_{0,\varepsilon}} \operatorname{dist}(\,.\,, x_0)$ *and* $\varepsilon < (10a)^{-64}$*. Then in addition to* (4.5) *an admissible solution* $u^{\varepsilon,\tau}$ *of* $(\star)_{\varepsilon,\tau}$ *satisfies*

$$u^{\varepsilon,\tau} \geq u^-(x) := \varepsilon^{21/16}\left(\frac{1}{2}\ln\left(\frac{\operatorname{dist}(x,x_0)}{a}\right) - L_\varepsilon + \tau\right). \tag{4.8}$$

In particular $u^{\varepsilon,\tau} \geq -\varepsilon^{5/4}$ *for*

$$\varepsilon \leq \min\left\{1\cdot 10^{-20}, C^{-16}, (10a)^{-64}\right\}, \qquad C := \left|\ln\left(\frac{b}{a}\right)\right| + \left|\ln(10a)\right| \tag{4.9}$$

where $b := \min_{\Omega_\varepsilon} \operatorname{dist}(\,.\,, x_0)$*.*

Remark 4.11. For the gradient estimate of $u^{\varepsilon,\tau}$ on $\partial_\Omega E_{0,\varepsilon}$ from below it will be important to have an estimate of the form $u^{\varepsilon,\tau} \geq -\varepsilon^{1+\gamma}$ for some $\gamma \in (0,1)$.

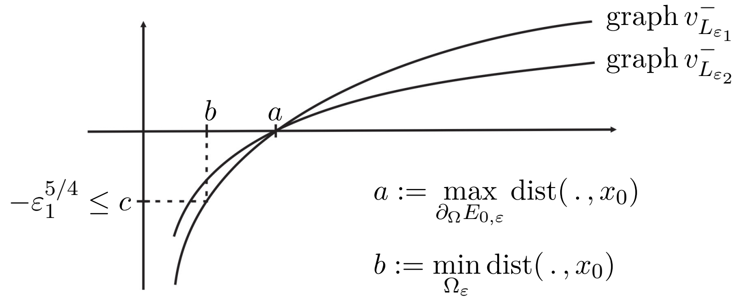

Figure 4.4: Improving lower bound for ε tending to zero ($\varepsilon_2 < \varepsilon_1$).

Proof of Lemma 4.10. We define a new subsolution of $(\star)_{\varepsilon,\tau}$ by

$$w^-(x) := \eta v^-(x) = \eta\left(\frac{1}{2}\ln\left(\frac{r(x)}{a}\right) - L_\varepsilon + \tau\right), \qquad L_\varepsilon := \frac{1}{2}\ln\left(\frac{R}{a}\right), \qquad R > a$$

with $r(x) := \operatorname{dist}(x, x_0)$ and $a := \max_{\partial_\Omega E_{0,\varepsilon}} r$. We see that for $\eta \in [0,1]$ the function $w^- = \eta v^-$ satisfies the right inequalities at the boundary

$$w^-\big|_{\partial_\Omega E_{0,\varepsilon}} \leq 0, \qquad w^-\big|_{\partial_\Omega F_{L_\varepsilon}} \leq \tau, \qquad D_\mu w^-\big|_{\Sigma_\varepsilon} \leq 0.$$

Now we use (4.6) to calculate $Q^\varepsilon(w^-)$. For $n \geq 2$ we obtain

$$\begin{aligned}
Q^\varepsilon(w^-) &= Q^\varepsilon(\eta v^-) = Q^\varepsilon\left(\frac{\eta}{2}\ln\left(\frac{r}{a}\right)\right) \\
&\geq \frac{1}{r\left(\varepsilon^2 r^2 + \left(\frac{\eta}{2}\right)^2\right)^{3/2}}\left(-\left(\frac{\eta}{2}\right)^4 + \left(\frac{\eta}{2}\right)^3 - 2\varepsilon^2 r^2\left(\frac{\eta}{2}\right)^2 - \varepsilon^4 r^4\right) \\
&\geq \frac{1}{r\left(\varepsilon^2 r^2 + \left(\frac{\eta}{2}\right)^2\right)^{3/2}}\left(\frac{1}{8}\eta^3 - \eta^4 - 2\varepsilon^2 r^2 \eta^2 - \varepsilon^4 r^4\right).
\end{aligned}$$

If we choose $\varepsilon \leq 1$ and $\eta := \varepsilon^{21/16}$ we derive

$$\begin{aligned} Q^\varepsilon(v^-) &\geq \frac{1}{r\left(\varepsilon^2 r^2 + \dfrac{\varepsilon^{21/8}}{4}\right)^{3/2}} \left(\frac{1}{8}\varepsilon^{63/16} - \varepsilon^{84/16} - 2r^2\varepsilon^{74/16} - r^4\varepsilon^{64/16}\right) \\ &\geq \frac{\varepsilon^{63/16}}{r\left(\varepsilon^2 r^2 + \dfrac{\varepsilon^{21/8}}{4}\right)^{3/2}} \left(\frac{1}{8} - \varepsilon^{1/16}(1+r^2)^2\right). \end{aligned}$$

The last expression is positive if

$$r \;\leq\; \sqrt{\frac{1}{\sqrt{8\varepsilon^{1/16}}} - 1} \;=\; \sqrt{\frac{1-\sqrt{8\varepsilon^{1/16}}}{\sqrt{8}}\frac{1}{\varepsilon^{1/64}}}.$$

The choice $\varepsilon \leq 1 \cdot 10^{-20}$ implies $\sqrt{8\varepsilon^{1/16}} \leq 3/4$ and allows us to choose $r \leq 1/(10\varepsilon^{1/64}) =: R$. Furthermore, by definition we have

$$L_\varepsilon := \frac{1}{2}\ln\left(\frac{R}{a}\right) = \frac{|\ln(10a\varepsilon^{1/64})|}{2}.$$

The maximum principle in Proposition A.12 implies that

$$\begin{aligned} u^{\varepsilon,\tau}(x) \geq w^-(x) &\geq \varepsilon^{21/16}\left(\frac{1}{2}\min_{\overline{\Omega_\varepsilon}} \ln\left(\frac{\operatorname{dist}(\,.\,,x_0)}{a}\right) - L_\varepsilon\right) \\ &\geq -\varepsilon^{5/4}\varepsilon^{1/16}\Big(C + |\ln(\varepsilon^{1/16})|\Big) \geq -\varepsilon^{5/4} \end{aligned}$$

for $\varepsilon \leq C^{-16}$. The value C is given by

$$C := \left|\ln\left(\frac{b}{a}\right)\right| + \Big|\ln(10a)\Big|, \qquad b := \min\nolimits_{\Omega_\varepsilon} \operatorname{dist}(\,.\,,x_0).$$

Note that we used the estimate $y|\ln(y)| \leq 1/e$ on $[0,1]$ in the last inequality. □

In the next steps we estimate the gradient. We start with the gradient estimate on the Dirichlet boundary parts $\partial_\Omega E_{0,\varepsilon}$ and $\partial_\Omega F_{L_\varepsilon}$. On $\partial_\Omega F_{L_\varepsilon}$ we can directly use the super- and subsolutions v^+ and v^- as barriers.

Lemma 4.12 (**Gradient estimate on $\partial_{\Omega}\mathbf{F}_{\mathbf{L}_\varepsilon}$**). *Assume that there exists an admissible subsolution $v^-_{L_\varepsilon}$ of $(\star)_{\varepsilon,L_\varepsilon}$ with $F_{L_\varepsilon} := \{v^-_{L_\varepsilon} < L_\varepsilon\}$. Let $u^{\varepsilon,\tau}$ be an admissible solution of $(\star)_{\varepsilon,\tau}$. Then the gradient of $u^{\varepsilon,\tau}$ satisfies the estimate*

$$0 \leq D_\nu u^{\varepsilon,\tau} \leq D_\nu v^-_{L_\varepsilon} \qquad \text{on } \partial_\Omega F_{L_\varepsilon}$$

where ν is the exterior unit normal to $\partial_\Omega F_{L_\varepsilon}$ with respect to the set F_{L_ε}. Under the assumptions of Lemma 4.9 we obtain the more explicit estimate $D_\nu u^{\varepsilon,\tau} \leq 2\varepsilon$.

Proof. Since $v^+ := \tau$ is a supersolution of $(\star)_{\varepsilon,\tau}$ in Ω_ε and coincides with $u^{\varepsilon,\tau}$ on $\partial_\Omega F_{L_\varepsilon}$ we see that v^+ is an upper barrier for the solution and thus

$$D_\nu u^{\varepsilon,\tau} \geq D_\nu v^+ = 0 \quad \text{on } \partial_\Omega F_{L_\varepsilon}$$

where ν is the exterior unit normal to Ω_ε on $\partial_\Omega F_{L_\varepsilon}$. In the same way $v_\tau^- := v_{L_\varepsilon}^- - L_\varepsilon + \tau$ is a subsolution for $(\star)_{\varepsilon,\tau}$ in Ω_ε which coincides with $u^{\varepsilon,\tau}$ on $\partial_\Omega F_{L_\varepsilon}$. Therefore, v_τ^- can be used as a barrier from below. This yields

$$D_\nu u^{\varepsilon,\tau} \leq D_\nu v_\tau^- \leq D_\nu v_{L_\varepsilon}^- \quad \text{on } \partial_\Omega F_{L_\varepsilon}.$$

Under the assumptions of Lemma 4.9 we obtain an explicit subsolution in (4.5). This yields the estimate

$$D_\nu u^{\varepsilon,\tau} \leq \frac{1}{2} D_\nu \ln\left(\frac{\operatorname{dist}(\,.\,, x_0)}{a}\right) \leq \frac{1}{2R}\left\langle \nu, \frac{x-x_0}{R}\right\rangle \leq \frac{1}{2R} = 2\varepsilon \quad \text{on } \partial_\Omega F_{L_\varepsilon}.$$

The last inequality holds since $\partial_\Omega F_{L_\varepsilon} = \{v_{L_\varepsilon}^- = L_\varepsilon\} = \{\operatorname{dist}(\,.\,, x_0) = R\}$ and $R = (4\varepsilon)^{-1}$. ☐

Now we estimate the gradient on $\partial_\Omega E_{0,\varepsilon}$. This will be done by constructing barriers of the form $\rho(x) := f\big(\operatorname{dist}(\,.\,, \partial_\Omega E_{0,\varepsilon})\big) \cdot g\big(\operatorname{dist}(\,.\,, \Sigma_\varepsilon)\big)$. In a first step we calculate $Q^\varepsilon(\rho)$ and $D_\mu \rho$ for this type of barriers.

Lemma 4.13 (**Formulas for barriers having a product structure**). *Let* $d := \operatorname{dist}_{\partial_\Omega E_{0,\varepsilon}}$, $s := \operatorname{dist}_{\Sigma_\varepsilon}$ *and assume that the distance functions are evaluated in a region where they are* C^2. *Let* $f, g \in C^2(\mathbb{R})$. *Then a barrier of the form* $\rho(x) := f(d(x)) \cdot g(s(x))$ *satisfies*

$$\big(|f'g| - |fg'|\big)^2 \;\leq\; |D\rho|^2 \;\leq\; \big(|f'g| + |fg'|\big)^2. \tag{4.10}$$

The Neumann condition reads

$$D_{\mu(x)}\rho(x)\big|_{\Sigma_\varepsilon} = f'(d(x))\big|_{\Sigma_\varepsilon} g(0) D_{\mu(x)} d(x)\big|_{\Sigma_\varepsilon} - f(d(x))\big|_{\Sigma_\varepsilon} g'(0) \tag{4.11}$$

and for the differential operator Q^ε *we obtain*

$$\begin{aligned}
&\sqrt{\varepsilon^2 + |D\rho|^2}\, Q^\varepsilon \rho \\
&= f'g\left(\delta^{ij} - \frac{f^2 g'^{\,2} D^i s D^j s}{\varepsilon^2 + |D\rho|^2}\right) D_{ij} d + fg'\left(\delta^{ij} - \frac{f'^{\,2} g^2 D^i d D^j d}{\varepsilon^2 + |D\rho|^2}\right) D_{ij} s \\
&\quad - \varepsilon^2 - |D\rho|^2 + \frac{f'' g}{\varepsilon^2 + |D\rho|^2}\Big(\varepsilon^2 + f^2 g'^{\,2}(1 - \langle Dd, Ds\rangle^2)\Big) \\
&\quad + \frac{f g''}{\varepsilon^2 + |D\rho|^2}\Big(\varepsilon^2 + f'^{\,2} g^2 (1 - \langle Dd, Ds\rangle^2)\Big) \\
&\quad + \frac{2f'g'}{\varepsilon^2 + |D\rho|^2}\Big(\varepsilon^2 \langle Dd, Ds\rangle + ff'gg'(\langle Dd, Ds\rangle^2 - 1)\Big).
\end{aligned} \tag{4.12}$$

Proof. The i-th derivative of ρ is $D_i\rho = f'gD_id + fg'D_is$ and

$$|D\rho|^2 \;=\; f'^{\,2}g^2 + 2fgf'g'\langle Dd, Ds\rangle + f^2g'^{\,2}.$$

The fact that $|Dd| = 1$ and $|Ds| = 1$ implies the formula for $|D\rho|^2$. Using $-\mu = Ds$ yields the formula for the directional derivative $D_\mu\rho$. To calculate $Q^\varepsilon\rho$ we first note that

$$D_{ij}\rho = f''gD_idD_jd + f'g'(D_idD_js + D_isD_jd) + fg''D_isD_js + f'gD_{ij}d + fg'D_{ij}s$$

and

$$D^i\rho D^j\rho = f'^{\,2}g^2D^idD^jd + ff'gg'(D^idD^js + D^isD^jd) + f^2g'^{\,2}D^isD^js.$$

Using once more $|Dd| = 1$ and $|Ds| = 1$ we see that $D^idD_{ij}d = 0$ and $D^isD_{ij}s = 0$. This yields

$$\begin{aligned}
&\sqrt{\varepsilon^2 + |D\rho|^2}\,Q^\varepsilon\rho\\
&= \left(\delta^{ij} - \frac{D^i\rho D^j\rho}{\varepsilon^2 + |D\rho|^2}\right) D_{ij}\rho - \varepsilon^2 - |D\rho|^2\\
&= \left(\delta^{ij} - \frac{D^i\rho D^j\rho}{\varepsilon^2 + |D\rho|^2}\right)\Big(f'gD_{ij}d + fg'D_{ij}s\Big) - \varepsilon^2 - |D\rho|^2\\
&\quad + \left(\delta^{ij} - \frac{D^i\rho D^j\rho}{\varepsilon^2 + |D\rho|^2}\right)\Big(f''gD_idD_jd + f'g'(D_idD_js + D_isD_jd) + fg''D_isD_js\Big)\\
&= f'g\left(\delta^{ij} - \frac{f^2g'^{\,2}D^isD^js}{\varepsilon^2 + |D\rho|^2}\right)D_{ij}d + fg'\left(\delta^{ij} - \frac{f'^{\,2}g^2D^idD^jd}{\varepsilon^2 + |D\rho|^2}\right)D_{ij}s - \varepsilon^2 - |D\rho|^2\\
&\quad + f''g\left(1 - \frac{1}{\varepsilon^2 + |D\rho|^2}\Big[f'^{\,2}g^2 + 2\langle Dd, Ds\rangle ff'gg' + \langle Dd, Ds\rangle^2 f^2g'^{\,2}\Big]\right)\\
&\quad + fg''\left(1 - \frac{1}{\varepsilon^2 + |D\rho|^2}\Big[\langle Dd, Ds\rangle^2 f'^{\,2}g^2 + 2\langle Dd, Ds\rangle ff'gg' + f^2g'^{\,2}\Big]\right)\\
&\quad + 2f'g'\Big(\langle Dd, Ds\rangle - \frac{1}{\varepsilon^2 + |D\rho|^2}\Big[\langle Dd, Ds\rangle f'^{\,2}g^2 + ff'gg'(1 + \langle Dd, Ds\rangle^2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad + \langle Dd, Ds\rangle f^2g'^{\,2}\Big]\Big)\\
&= f'g\left(\delta^{ij} - \frac{f^2g'^{\,2}D^isD^js}{\varepsilon^2 + |D\rho|^2}\right)D_{ij}d + fg'\left(\delta^{ij} - \frac{f'^{\,2}g^2D^idD^jd}{\varepsilon^2 + |D\rho|^2}\right)D_{ij}s - \varepsilon^2 - |D\rho|^2\\
&\quad + f''g\left(1 - \frac{1}{\varepsilon^2 + |D\rho|^2}\Big[f'g + \langle Dd, Ds\rangle fg'\Big]^2\right)\\
&\quad + fg''\left(1 - \frac{1}{\varepsilon^2 + |D\rho|^2}\Big[\langle Dd, Ds\rangle f'g + fg'\Big]^2\right)\\
&\quad + \frac{2f'g'}{\varepsilon^2 + |D\rho|^2}\Big(\varepsilon^2\langle Dd, Ds\rangle + ff'gg'(\langle Dd, Ds\rangle^2 - 1)\Big)
\end{aligned}$$

and thus

$$\begin{aligned}
&\sqrt{\varepsilon^2+|D\rho|^2}Q^\varepsilon\rho \\
&= f'g\left(\delta^{ij}-\frac{f^2g'^{\,2}D^isD^js}{\varepsilon^2+|D\rho|^2}\right)D_{ij}d+fg'\left(\delta^{ij}-\frac{f'^{\,2}g^2D^idD^jd}{\varepsilon^2+|D\rho|^2}\right)D_{ij}s-\varepsilon^2-|D\rho|^2 \\
&\quad+\frac{f''g}{\varepsilon^2+|D\rho|^2}\Big(\varepsilon^2+f^2g'^{\,2}(1-\langle Dd,Ds\rangle^2)\Big) \\
&\quad+\frac{fg''}{\varepsilon^2+|D\rho|^2}\Big(\varepsilon^2+f'^{\,2}g^2(1-\langle Dd,Ds\rangle^2)\Big) \\
&\quad+\frac{2f'g'}{\varepsilon^2+|D\rho|^2}\Big(\varepsilon^2\langle Dd,Ds\rangle+ff'gg'(\langle Dd,Ds\rangle^2-1)\Big).
\end{aligned}$$

□

Remark 4.14. Note that in general $\partial_\Omega E_{0,\varepsilon}$ has to be extended beyond Σ^n in a small neighborhood of $\partial_\Omega E_0\cap\Sigma^n$ in order to use the distance function in a neighborhood of the corner. This extension can be constructed to have the same C^2-norm as $\partial_\Omega E_{0,\varepsilon}$ so the estimates will be independent of this extension.

Now we will construct upper and lower barriers on $\partial_\Omega E_{0,\varepsilon}$ of the type

$$\rho^\pm(x):=f^\pm\big(\operatorname{dist}(x,\partial_\Omega E_{0,\varepsilon})\big)\cdot g\big(\operatorname{dist}(x,\Sigma_\varepsilon)\big)$$

by defining appropriate functions $f^\pm$ and g. The function ρ is defined in a neighborhood Γ of $\partial_\Omega E_{0,\varepsilon}$. Therefore, we have to deal with an additional boundary part $\partial\Gamma_1$. Note that we will define $g\equiv 1$ far away from Σ_ε. This has the advantage that we have an easier barrier in the interior and ensures that whenever we use the distance functions they are at least C^2. We start with an estimate from below.

Lemma 4.15 (Gradient estimate on $\partial_\Omega \mathbf{E}_{0,\varepsilon}$ from below). *Suppose that $\partial_\Omega E_{0,\varepsilon}$ and Σ_ε are C^2-hypersurfaces. Suppose that $\varepsilon>0$ is sufficiently small. Let $u^{\varepsilon,\tau}$ be an admissible solution of $(\star)_{\varepsilon,\tau}$ which satisfies $u^{\varepsilon,\tau}\geq-\varepsilon^{1+\gamma}$ for some $\gamma\in(0,1)$. Then the gradient satisfies the estimate*

$$D_\nu u^{\varepsilon,\tau}\geq-2\varepsilon\qquad\text{on } \partial_\Omega E_{0,\varepsilon}$$

where ν is the exterior unit normal to $\partial_\Omega E_{0,\varepsilon}$ with respect to the set $E_{0,\varepsilon}$.

Proof. Let $d(x):=\operatorname{dist}(x,\partial_\Omega E_{0,\varepsilon})$ and $s(x):=\operatorname{dist}(x,\Sigma_\varepsilon)$. We restrict ourselves to the set $\Gamma:=\{x\in\Omega_\varepsilon\mid d(x)<d_{max}\}$. The boundary of Γ consists of $\partial_\Omega E_{0,\varepsilon}$, $\partial_\Sigma\Gamma$ and a new boundary part in the interior of Ω_ε which we call $\partial\Gamma_1$. We make the ansatz $\rho(x):=f(d(x))\cdot g(s(x))$ with

$$f(d):=\frac{\varepsilon}{A}\Big(\exp(-Ad)-1\Big)$$

and see that f, f' and f'' satisfy

$$-\frac{\varepsilon}{A}\leq f\leq 0,\qquad -\varepsilon\leq f'\leq-\frac{\varepsilon}{2},\qquad \frac{\varepsilon A}{2}\leq f''\leq\varepsilon A \tag{4.13}$$

where the upper bound on f' and the lower bound on f'' require $d_{max} \leq \ln(2)/A$. For g we choose

$$g(s) := \begin{cases} 1 + \exp\left(2 - 2\left(\frac{s_{max}}{s_{max} - s}\right)^2\right) & \text{for } 0 \leq s < s_{max} \\ 1 & \text{for } s \geq s_{max} \end{cases}$$

and a direct calculation shows that

$$1 \leq g \leq 2, \qquad -\frac{4}{s_{max}} \leq g' \leq 0, \qquad 0 \leq g'' \leq \frac{12}{s_{max}^2}. \tag{4.14}$$

The exact values d_{max}, s_{max} and A will be determined later. We see that ρ is a negative function which satisfies the Dirichlet boundary condition $\rho = 0$ on $\partial_\Omega E_{0,\varepsilon}$ since

$$\rho\Big|_{\partial_\Omega E_{0,\varepsilon}} = f(0) \cdot g(s(x)) = 0.$$

Next we want to show that ρ lies below $u^{\varepsilon,\tau}$ on $\partial\Gamma_1$. Using $u^{\varepsilon,\tau} \geq -\varepsilon^{1+\gamma}$ we see that

$$\rho\Big|_{\partial\Gamma_1} = f(d_{max}) \cdot g(s(x)) \leq -\frac{\varepsilon}{A}\Big(1 - \exp(-Ad_{max})\Big) \cdot 1 \leq -\varepsilon^{1+\gamma} \leq u^{\varepsilon,\tau}$$

for

$$\varepsilon \leq \left(\frac{1 - \exp(-Ad_{max})}{A}\right)^{1/\gamma}.$$

To prove that ρ is a subsolution we have to verify that $D_\mu\rho \leq 0$ on the remaining boundary part $\partial_\Sigma\Gamma$. Using (4.11) and the definition of g we obtain

$$D_\mu\rho \;=\; f'(d)g(0)D_{\mu(x)}d(x) - f(d)g'(0) \;=\; 2f'(d)|_{\partial_\Sigma\Gamma}D_\mu d + \frac{4}{d_{max}}f(d) \tag{4.15}$$

on $\partial_\Sigma\Gamma$. The second term is negative and therefore a good term for our estimate. The first term is a negative term if $D_\mu d$ is positive on $\partial_\Sigma\Gamma$. From (4.2) we know that this is possible in some small neighborhood of $\partial_\Omega E_{0,\varepsilon} \cap \Sigma_\varepsilon$ for all strictly positive ε. So the worst case is to consider the distance function to $\partial_\Omega E_0$ which only satisfies $D_\mu d = 0$ in the corner and therefore can become negative on $\partial_\Sigma\Gamma$. However, since $\partial_\Omega E_0$ and Σ^n meet at a non-zero angle and have bounded curvature there is some $C_1 > 0$ such that

$$D_{\mu(x)}d(x) \geq -C_1 d(x) \qquad \text{on } \partial_\Sigma\Gamma. \tag{4.16}$$

Furthermore, we use (4.13) to estimate $f'(d) \geq -\varepsilon$ and compute that $f(d) \leq -\varepsilon d/2$ for $d_{max} \leq A^{-1}$. Using (4.15) and (4.16) this yields

$$D_{\mu(x)}\rho(x)|_{\partial_\Sigma\Gamma} \leq 2\varepsilon\left(C_1 - \frac{1}{d_{max}}\right)d(x) \leq 0 \qquad \text{on } \partial_\Sigma\Gamma$$

for $d_{max} \leq \min\{C_1^{-1}, A^{-1}\}$. Finally, we have to make sure that $Q^\varepsilon(\rho) \geq 0$. Using (4.12) and the fact that $f \leq 0$, $f' \leq 0$, $f'' \geq 0$ and $g \geq 0, g' \leq 0, g'' \geq 0$ we get

$$\sqrt{\varepsilon^2 + |D\rho|^2}Q^\varepsilon\rho =$$

$$\begin{aligned}
&= f'g\left(\delta^{ij} - \frac{f^2 g'^{\,2} D^i s D^j s}{\varepsilon^2 + |D\rho|^2}\right) D_{ij}d + fg'\left(\delta^{ij} - \frac{f'^{\,2} g^2 D^i d D^j d}{\varepsilon^2 + |D\rho|^2}\right) D_{ij}s \\
&\quad - \varepsilon^2 - |D\rho|^2 + \frac{f''g}{\varepsilon^2 + |D\rho|^2}\Big(\varepsilon^2 + f^2 g'^{\,2}(1 - \langle Dd, Ds\rangle^2)\Big) \\
&\quad + \frac{fg''}{\varepsilon^2 + |D\rho|^2}\Big(\varepsilon^2 + f'^{\,2} g^2(1 - \langle Dd, Ds\rangle^2)\Big) \\
&\quad + \frac{2f'g'}{\varepsilon^2 + |D\rho|^2}\Big(\varepsilon^2\langle Dd, Ds\rangle + ff'gg'(\langle Dd, Ds\rangle^2 - 1)\Big)
\end{aligned}$$

$$\begin{aligned}
&\geq f'g\left(\delta^{ij} - \frac{f^2 g'^{\,2} D^i s D^j s}{\varepsilon^2 + |D\rho|^2}\right) D_{ij}d + fg'\left(\delta^{ij} - \frac{f'^{\,2} g^2 D^i d D^j d}{\varepsilon^2 + |D\rho|^2}\right) D_{ij}s - \varepsilon^2 \\
&\quad - |D\rho|^2 + \frac{f''g}{\varepsilon^2 + |D\rho|^2}\varepsilon^2 + \frac{fg''}{\varepsilon^2 + |D\rho|^2}\Big(\varepsilon^2 + f'^{\,2} g^2\Big) - \frac{2f'g'}{\varepsilon^2 + |D\rho|^2}\varepsilon^2
\end{aligned} \tag{4.17}$$

where the only positive term is the one which involves f''. If we are further than s_{max} away from Σ_ε the function g is identically one, $|D\rho|^2 = (f')^2$ and the estimate reads

$$\begin{aligned}
\sqrt{\varepsilon^2 + |D\rho|^2} Q^\varepsilon \rho &= f'\Delta d - \varepsilon^2 - (f')^2 + \frac{\varepsilon^2}{\varepsilon^2 + (f')^2} f'' \\
&\geq -\varepsilon n^2 |D^2 d| - \varepsilon^2 - \varepsilon^2 + \frac{\varepsilon^2}{\varepsilon^2 + \varepsilon^2}\frac{\varepsilon A}{2} \geq 0
\end{aligned}$$

for $\varepsilon \leq 1$ and $A \geq 4(2 + n^2|D^2 d|)$. Before we continue with the estimate close to Σ_ε we have to estimate $|D\rho|^2$ We use (4.10), (4.13),(4.14) and $As_{max} \geq 24$ to see that

$$\left(\frac{\varepsilon}{2} - \frac{4\varepsilon}{As_{max}}\right)^2 \leq \big(|f'g| - |fg'|\big)^2 \leq |D\rho|^2 \leq \big(|f'g| + |fg'|\big)^2 \leq \left(2\varepsilon + \frac{4\varepsilon}{As_{max}}\right)^2$$

and thus

$$\frac{1}{9}\varepsilon^2 \leq |D\rho|^2 \leq 9\varepsilon^2. \tag{4.18}$$

This estimate together with (4.13) and (4.14) allows us to estimate the maximal Eigenvalues of the matrices in front of the $D^2 s$ and $D^2 d$ terms

$$\begin{aligned}
&\left| f'g\left(\delta^{ij} - \frac{f^2 g'^{\,2} D^i s D^j s}{\varepsilon^2 + |D\rho|^2}\right)\xi_i\xi_j \right| \\
&\quad \leq |f'g|\left(1 + \frac{f^2 g'^{\,2}}{|D\rho|^2}\right)|\xi|^2 \leq 2\varepsilon\left(1 + \frac{(\varepsilon/A)^2(4/s_{max})^2}{\varepsilon^2/9}\right)|\xi|^2 \leq 4\varepsilon|\xi|^2
\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}
&\left| fg'\left(\delta^{ij} - \frac{f'^{\,2} g^2 D^i d D^j d}{\varepsilon^2 + |D\rho|^2}\right)\xi_i\xi_j \right| \\
&\quad \leq |fg'|\left(1 + \frac{f'^{\,2} g^2}{|D\rho|^2}\right)|\xi|^2 \leq \frac{\varepsilon}{A}\frac{4}{s_{max}}\left(1 + \frac{\varepsilon^2 2^2}{\varepsilon^2/9}\right)|\xi|^2 \leq 7\varepsilon|\xi|^2
\end{aligned} \tag{4.20}$$

where we used again $As_{max} \geq 24$. Now we put together (4.17), (4.18), (4.19) and (4.20) to prove the estimate for $Q^\varepsilon\rho$ away from Σ_ε

$$\begin{aligned}
&\sqrt{\varepsilon^2+|D\rho|^2}Q^\varepsilon\rho \\
&\geq -4n^2|D^2d|\varepsilon - 7n^2|D^2s|\varepsilon - \varepsilon^2 - 9\varepsilon^2 \\
&\quad + \frac{(\varepsilon A/2)\cdot 1}{\varepsilon^2+9\varepsilon^2}\varepsilon^2 - \frac{(\varepsilon/A)\cdot(12/s_{max}^2)}{0+\varepsilon^2/9}\Big(\varepsilon^2+\varepsilon^2 2^2\Big) - \frac{2\varepsilon\cdot(4/s_{max})}{0+\varepsilon^2/9}\varepsilon^2 \\
&= \varepsilon\left(\frac{A}{20} - 4n^2|D^2d| - 7n^2|D^2s| - 10\varepsilon - \frac{12\cdot 9\cdot(1+2^2)}{As_{max}^2}\varepsilon - \frac{2\cdot 4\cdot 9}{s_{max}}\varepsilon\right) \\
&\geq \frac{\varepsilon}{20As_{max}^2}\Big((As_{max})^2 - C_2(As_{max}) - C_2\Big)
\end{aligned}$$

for $\varepsilon \leq 1$, $s_{max} \leq 1$ and $C_2 := 10000(n^2|D^2d| + n^2|D^2s| + 1)$. Therefore, the expression is positive for $As_{max} \geq 2C_2$. Altogether we see that ρ is a subsolution of $(\star)_{\varepsilon,\tau}$ in Γ for the choice of parameters

$$s_{max} := \eta, \quad A := \frac{2C_2}{s_{max}}, \quad d_{max} := \min\{C_1^{-1}, A^{-1}, \eta\}, \quad \varepsilon \leq \left(\frac{1-e^{-Ad_{max}}}{A}\right)^{1/\gamma}$$

where $\eta \in (0,1)$ is chosen sufficiently small to guarantee that the distance functions are at least in C^2. Thus we get the desired estimate

$$D_\nu u^{\varepsilon,\tau} \geq D_\nu\rho = f'(0)gD_\nu d + f(0)g'D_\nu s = -\varepsilon g \geq -2\varepsilon \qquad \text{on } \partial_\Omega E_{0,\varepsilon}.$$

Here ν is the exterior unit normal to $\partial_\Omega E_{0,\varepsilon}$ with respect to the set $E_{0,\varepsilon}$. □

In the next step we prove the gradient estimate on $\partial_\Omega E_{0,\varepsilon}$ from above. In order to allow for arbitrary large Dirichlet boundary values we will first find a function ρ satisfying $Q^0\rho \leq 0$ for Dirichlet boundary values $0 \leq \tau \leq 1$. Then we deform ρ into a function $\tilde{\rho}$ which allows for arbitrary high boundary values. For this transformation it is useful to work with Q^0 since a sign on $Q^0\rho$ will imply a sign on $Q^0\tilde{\rho}$ which is not obvious when we consider Q^ε. Finally, we can argue that $\tilde{\rho}$ is also a supersolution for Q^ε.

Lemma 4.16 (**Gradient estimate on $\partial_{\boldsymbol{\Omega}}\mathbf{E}_{\mathbf{0},\varepsilon}$ from above**). *Let $\varepsilon > 0$ be sufficiently small. Suppose that $\partial_\Omega E_{0,\varepsilon}$ and Σ_ε are C^2-hypersurfaces. Let $u^{\varepsilon,\tau}$ be an admissible solution of $(\star)_{\varepsilon,\tau}$. Then the gradient of $u^{\varepsilon,\tau}$ satisfies the estimate*

$$D_\nu u^{\varepsilon,\tau} \leq C(n, \partial_\Omega E_0, \Sigma^n) \qquad \textit{on } \partial_\Omega E_{0,\varepsilon}$$

where ν is the exterior unit normal to $\partial_\Omega E_{0,\varepsilon}$ with respect to the set $E_{0,\varepsilon}$.

Proof. Let $d(x) := \operatorname{dist}(x, \partial_\Omega E_{0,\varepsilon})$ and $s(x) := \operatorname{dist}(x, \Sigma_\varepsilon)$. We restrict ourselves to the set $\Gamma := \{x \in \Omega_\varepsilon \mid d(x) < d_{max}\}$. The boundary of Γ consists of $\partial_\Omega E_{0,\varepsilon}$, $\partial_\Sigma\Gamma$ and a new boundary part in the interior of Ω_ε which we call $\partial\Gamma_1$. We make the ansatz $\rho(x) := f(d(x)) \cdot g(s(x))$ with $f(d) := Ad$ for some $A > 0$. For g we choose again

$$g(s) := \begin{cases} 1 + \exp\left(2 - 2\left(\frac{s_{max}}{s_{max} - s}\right)^2\right) & \text{for } 0 \le s < s_{max} \\ 1 & \text{for } s \ge s_{max} \end{cases}$$

and remember that

$$1 \le g \le 2, \qquad -\frac{4}{s_{max}} \le g' \le 0, \qquad 0 \le g'' \le \frac{12}{s_{max}^2}. \tag{4.21}$$

The exact values d_{max}, s_{max} and A will be determined later. We see that ρ is a positive function which satisfies the Dirichlet boundary condition $\rho = 0$ on $\partial_\Omega E_{0,\varepsilon}$ since

$$\rho = f(0) \cdot g(s(x)) = 0 \qquad \text{on } \partial_\Omega E_{0,\varepsilon}.$$

Furthermore, ρ lies above $u^{\varepsilon,\tau}$ on $\partial\Gamma_1$ since

$$\rho = f(d_{max}) \cdot g(s(x)) \ge Ad_{max} \cdot 1 \ge \tau \ge u^{\varepsilon,\tau} \qquad \text{on } \partial\Gamma_1$$

for $Ad_{max} \ge \tau$. To show that ρ is a supersolution we have to verify that $D_\mu \rho \ge 0$ on $\partial_\Sigma\Gamma$. From (4.11) and the definition of f and g we obtain

$$D_\mu \rho \;=\; f'(d)g(0)D_\mu d - f(d)g'(0) \;=\; 2AD_\mu d + \frac{4}{d_{max}}Ad \qquad \text{on } \partial_\Sigma\Gamma.$$

This time the second term is positive and therefore a good term for our estimate. The first term is a positive term if $D_\mu d$ is positive on $\partial_\Sigma\Gamma$. From (4.2) we know that this is possible in a small neighborhood of $\partial_\Omega E_{0,\varepsilon} \cap \Sigma_\varepsilon$ for all strictly positive ε. So the worst case is again to consider the distance function to $\partial_\Omega E_0$ which only satisfies $D_\mu d = 0$ in the corner and therefore can become negative on $\partial_\Sigma\Gamma$. Using once more (4.16) we obtain

$$D_\mu \rho \ge 2A\left(-C_1 + \frac{2}{d_{max}}\right) d \ge 0 \qquad \text{on } \partial_\Sigma\Gamma$$

for $d_{max} \le 2C_1^{-1}$. In contrast to the lower bound we will first prove that $Q^0\rho \ge 0$. Using (4.12) and the fact that $f \ge 0$, $f' \equiv A$, $f'' \equiv 0$ and $g \ge 0, g' \le 0, g'' \ge 0$ we get

$$\begin{aligned} &|D\rho|Q^0\rho \\ &= f'g\left(\delta^{ij} - \frac{f^2 g'^{\,2} D^i s D^j s}{|D\rho|^2}\right) D_{ij} d + fg'\left(\delta^{ij} - \frac{f'^{\,2} g^2 D^i d D^j d}{|D\rho|^2}\right) D_{ij} s - |D\rho|^2 \\ &\quad + \frac{fg''}{|D\rho|^2}\Big(f'^{\,2} g^2 (1 - \langle Dd, Ds\rangle^2)\Big) + \frac{2f'g'}{|D\rho|^2}\Big(ff'gg'(\langle Dd, Ds\rangle^2 - 1)\Big) \\ &\le f'g\left(\delta^{ij} - \frac{f^2 g'^{\,2} D^i s D^j s}{|D\rho|^2}\right) D_{ij} d + fg'\left(\delta^{ij} - \frac{f'^{\,2} g^2 D^i d D^j d}{|D\rho|^2}\right) D_{ij} s - |D\rho|^2 \\ &\quad + \frac{fg''}{|D\rho|^2} f'^{\,2} g^2. \end{aligned} \tag{4.22}$$

Here the only good term is $-|D\rho|^2$. In the case that we are far from Σ_ε we have $g \equiv 1$ and $|D\rho| = |f'|$. Therefore, the estimate simplifies and we obtain

$$|D\rho|Q^0\rho \le f'\Delta d - |D\rho|^2 \le An^2|D^2 d| - A^2 \le 0$$

for $A \ge n^2|D^2 d|$. As in the previous lemma we proceed by estimating the gradient of ρ. We use again (4.10), (4.13), (4.14) and choose $d_{max} := s_{max}/8$ to see that

$$\begin{aligned} A^2\left(1 - \frac{4d_{max}}{s_{max}}\right)^2 &\le \left(|f'g| - |fg'|\right)^2 \le |D\rho|^2 \\ &\le \left(|f'g| + |fg'|\right)^2 \le 4A^2\left(1 + \frac{2d_{max}}{s_{max}}\right)^2 \end{aligned}$$

and thus

$$\frac{A^2}{4} \;\le\; |D\rho|^2 \;\le\; 7A^2. \tag{4.23}$$

This yields the following bounds

$$\left| f'g\left(\delta^{ij} - \frac{f^2 g'^{\,2} D^i s D^j s}{|D\rho|^2}\right)\xi_i\xi_j\right| \le 2A\left(1 + \frac{A^2 d^2\cdot(4/s_{max})^2}{(A^2/4)}\right)|\xi|^2 \le 6A|\xi|^2$$

and

$$\left| fg'\left(\delta^{ij} - \frac{f'^{\,2} g^2 D^i d D^j d}{|D\rho|^2}\right)\xi_i\xi_j\right| \le Ad\frac{4}{s_{max}}\left(1 + \frac{A^2\cdot 2^2}{(A^2/4)}\right)|\xi|^2 \le 9A|\xi|^2.$$

Now we can combine these bounds to obtain an estimate for $Q^0\rho$

$$\begin{aligned} &|D\rho|Q^0\rho \\ &\le f'g\left(\delta^{ij} - \frac{f^2 g'^{\,2} D^i s D^j s}{|D\rho|^2}\right)D_{ij}d + fg'\left(\delta^{ij} - \frac{f'^{\,2} g^2 D^i d D^j d}{|D\rho|^2}\right)D_{ij}s \\ &\quad - |D\rho|^2 + \frac{fg''}{|D\rho|^2}f'^{\,2}g^2 \\ &\le 6An^2|D^2 d| + 9An^2|D^2 s| - \frac{A^2}{4} + \frac{Ad\cdot\frac{12}{s_{max}^2}}{\frac{A^2}{4}}\cdot A^2\cdot 2^2 \\ &\le \frac{A}{4}\Big(1000(n^2|D^2 d| + n^2|D^2 s| + s_{max}^{-1}) - A\Big) \le 0 \end{aligned} \tag{4.24}$$

for $A \ge 1000(n^2|D^2 d| + n^2|D^2 s| + s_{max}^{-1}) =: 1000(C_3 + s_{max}^{-1})$. To summarize, we proved that ρ is a supersolution for $(\star)_{0,\tau}$ in Γ for the parameters

$$d_{max} := \min\{2C_1^{-1}, \eta\}, \quad s_{max} := 8d_{max}, \quad A_\tau := 1000(C_3 + s_{max}^{-1}) + \frac{\tau}{d_{max}}$$

where $\eta \in (0,1)$ is chosen sufficiently small to guarantee that the distance functions are at least in C^2.

So far, to match increasing boundary values τ on $\partial\Gamma_1$ we have to choose steeper functions ρ. This means that in the limit $\varepsilon \to 0$ ($L_\varepsilon \to \infty$) we loose the gradient estimate. To prevent this from happening we take the function ρ corresponding to $\tau := 1$. Then we consider the subdomain $\tilde\Gamma := \{0 \le \rho < 1\} \subset \Gamma$ and we define

$$\tilde\rho(x) := \frac{\rho(x)}{1-\rho(x)}, \qquad x \in \tilde\Gamma.$$

We see that $\tilde\rho = 0$ on $\partial_\Omega E_{0,\varepsilon}$ since $\rho = 0$ on $\partial_\Omega E_{0,\varepsilon}$. Furthermore,

$$D_i\tilde\rho = \frac{D_i\rho(x)}{(1-\rho(x))^2}, \qquad D_{ij}\tilde\rho = \frac{(1-\rho)D_{ij}\rho + 2D_i\rho D_j\rho}{(1-\rho)^3} \tag{4.25}$$

so in particular we get the same sign for $D_\mu\tilde\rho$ as for $D_\mu\rho$. The PDE is also satisfied with the same inequality since

$$\begin{aligned} Q^0\tilde\rho &= \operatorname{div}\left(\frac{D\tilde\rho}{|D\tilde\rho|}\right) - |D\tilde\rho| = \operatorname{div}\left(\frac{D\rho}{|D\rho|}\right) - |D\rho| + |D\rho| - \frac{|D\rho|}{(1-\rho(x))^2} \\ &= Q^0\rho + \frac{|D\rho|\rho\,(\rho-2)}{(1-\rho(x))^2} \le Q^0\rho \le -\frac{A_1}{4|D\rho|}(1000(C_3+s_{max}^{-1}) - A_1) \\ &\le -\frac{A_1}{4d_{max}|D\rho|} \overset{(4.23)}{\le} -\frac{1}{12d_{max}}. \end{aligned}$$

In contrast to ρ the function $\tilde\rho$ is a supersolution of $(\star)_{\varepsilon,\tau}$ on $\{0 \le \tilde\rho \le \tau\} \subset \tilde\Gamma$ for arbitrary large boundary values since the function blows up when it approaches the boundary $\{\rho = 1\}$.

Next, we observe that

$$\begin{aligned} &\left|Q^\varepsilon\tilde\rho - Q^0\tilde\rho\right| \\ &\quad\le \left|\sqrt{\varepsilon^2+|D\tilde\rho|^2} - |D\tilde\rho|\right| + \left|\operatorname{div}\left(\frac{\sqrt{\varepsilon^2+|D\tilde\rho|^2} - |D\tilde\rho|}{\sqrt{\varepsilon^2+|D\tilde\rho|^2}\cdot|D\tilde\rho|}Du\right)\right| \\ &\quad\le \left(1+\frac{3|D^2\tilde\rho|}{|D\tilde\rho|^2}\right)\varepsilon \overset{(4.25)}{\le} 7\left(1+\frac{|D^2\rho|}{|D\rho|^2}\right)\varepsilon \overset{(4.23)}{\le} 7\left(1+\frac{4|D^2\rho|}{A^2}\right)\varepsilon \le c_1\varepsilon. \end{aligned}$$

Therefore, by continuity we also have $Q^\varepsilon\tilde\rho < 0$ for ε sufficiently small. Thus, $\tilde\rho$ is also a supersolution of $(\star)_{\varepsilon,\tau}$ for small $\varepsilon > 0$ and arbitrary τ. This yields the estimate

$$D_\nu u \le D_\nu\tilde\rho = \frac{D_\nu\rho}{(1-\rho)^2} = D_\nu\rho \le 2D_\nu f \le 2A_1 = 2000(C_3+s_{max}^{-1}) + \frac{2}{d_{max}}$$

on $\partial_\Omega E_{0,\varepsilon}$. Note that we can estimate the C^2-norm of d independently of the approximation of $\partial_\Omega E_0$ by $\partial_\Omega E_{0,\varepsilon}$. Therefore C_3 and thus the estimate for $D_\nu u$ is independent of ε. ☐

The remaining boundary part of the domain Ω_ε is the Neumann boundary part Σ_ε. If the supporting hypersurface is convex the maximum principle tells us that a maximum of the gradient can not occur on Σ_ε.

Lemma 4.17 (Gradient estimate on Σ_ε). *Let Σ_ε be a convex C^3-hypersurface. Let $u^{\varepsilon,\tau}$ be an admissible solution of $(\star)_{\varepsilon,\tau}$. Then, $|Du^{\varepsilon,\tau}|$ can not attain a maximum on Σ_ε.*

Proof. Let $x_0 \in \Sigma_\varepsilon$. First we note that due to the regularity of Σ_ε, there is a neighborhood of x_0 in Ω_ε in which $u := u^{\varepsilon,\tau}$ is C^3. Let us define $v := |Du|^2/2$. Let $a^i(p) := p/\sqrt{\varepsilon^2 + |p|^2}$ and $a^{ij}(p) := \partial a^i(p)/\partial p^j$. We apply the operator $(D^j u)D_j$ to $Q^\varepsilon(u)$ defined in $(\star)_{\varepsilon,\tau}$. Here j runs from 1 to n. This yields

$$
\begin{aligned}
0 &= D^j u D_j \operatorname{div}\left(\frac{Du}{\sqrt{\varepsilon^2 + |Du|^2}}\right) - D^j u D_j \sqrt{\varepsilon^2 + |Du|^2} \\
&= D^j u D_i \left(a^{ik}(Du) D_{kj} u\right) - \frac{D^j u}{\sqrt{\varepsilon^2 + |Du|^2}} D^k u D_{kj} u \\
&= D_i \left(a^{ik}(Du) D^j u D_{kj} u\right) - a^{ik}(Du) D_i^j u D_{kj} u - \frac{D^j u}{\sqrt{\varepsilon^2 + |Du|^2}} D^k u D_{kj} u \\
&\overset{(*)}{\leq} D_i \left(a^{ik}(Du) D_k v\right) - \frac{D^j u}{\sqrt{\varepsilon^2 + |Du|^2}} D_j v =: Lv
\end{aligned}
\tag{4.26}
$$

where we used the negative sign of the second term in $(*)$ to obtain the inequality. Assume that the maximum of v is attained at x_0. In a neighborhood of x_0 we choose an orthonormal frame such that $e_1, ..., e_{n-1} \in T_{x_0}\Sigma_\varepsilon$ and $e_n = \mu$. At x_0 we have

$$
D_\mu v = \sum_{i=1}^n \mu_i D_i \frac{|Du|^2}{2} = \sum_{i=1}^n \mu_i \left(\sum_{j=1}^{n-1} D_j u D_{ij} u + D_n u D_{in} u\right) = \sum_{i=1}^n \mu_i \sum_{j=1}^{n-1} D_j u D_{ij} u.
$$

On the other hand, by applying $\sum_{j=1}^{n-1} (D_j u) D_j$ to the Neumann condition $D_\mu u = 0$ we get

$$
0 = \sum_{j=1}^{n-1} (D_j u) D_j D_\mu u = \sum_{j=1}^{n-1} (D_j u) \sum_{i=1}^n \Big((D_j \mu_i) D_i u + \mu_i D_{ij} u\Big).
$$

Comparing these two expressions we see that

$$
D_\mu v = -\sum_{i,j=1}^{n-1} (D_j \mu_i) D_i u D_j u = -\sum_{i,j=1}^{n-1} {}^{\Sigma_\varepsilon}h_{ij} D_i u D_j u \leq 0
$$

since Σ_ε is convex. The signs for $D_\mu v$ and Lv together with the maximum principle in Proposition A.10 tell us that v can not attain a maximum on Σ_ε. □

The last estimate which is needed is the interior gradient estimate. Once more we make use of the maximum principle.

Lemma 4.18 (Interior gradient estimate). *Let $u^{\varepsilon,\tau}$ be an admissible solution of $(\star)_{\varepsilon,\tau}$. Then, $|Du^{\varepsilon,\tau}|$ can not attain a maximum in the interior of Ω_ε. Additionally, the more precise estimate*

$$
|Du^{\varepsilon,\tau}(x)| \leq \sup_{\partial\Omega_\varepsilon \cap B_r(x)} |Du^{\varepsilon,\tau}| + \varepsilon + \frac{C(n)}{r}
$$

holds for $r > 0$. Note that $\partial\Omega_\varepsilon$ is the boundary of Ω_ε in $\mathbb{R}^{n+1}$. Thus the boundary consists of the Dirichlet boundary parts $\partial_\Omega E_{0,\varepsilon}$ and $\partial_\Omega F_{L_\varepsilon}$ and the Neumann boundary part $\Sigma_\varepsilon = \partial_\Sigma \Omega_\varepsilon$.

Proof. First we note that interior regularity implies that $u^{\varepsilon,\tau} \in C^3(\Omega_\varepsilon)$. From (4.26) and the maximum principle we see that $Du^{\varepsilon,\tau}$ can not attain an interior maximum. The more precise estimate follows from the interior estimate of H in the work of Huisken and Ilmanen [29], Lemma 3.4. Since admissible solutions are in particular in $C^{1,\beta}(\overline{\Omega_\varepsilon})$ we can allow $B_r(x)$ to intersect with the boundary. □

Recall from Definitions 4.5 and 4.6 the two angles between the Dirichlet boundary and the Neumann boundary.

$$\theta_1(\varepsilon) := \measuredangle\left(\nu_{\partial E_{0,\varepsilon}}, \mu\right), \qquad \theta_2(\varepsilon) := \measuredangle\left(-\nu_{\partial F_{L_\varepsilon}}, \mu\right).$$

Let us now collect all a priori estimates in the following Proposition.

Proposition 4.19. *Let $E_0, E_{0,\varepsilon}, F_{L_\varepsilon}$ and $(\star)_{\varepsilon,\tau}$ be as in Definitions 4.5 and 4.6. Let Σ^n be a $C^{2,\alpha}$-hypersurface. Assume that an admissible subsolution v^- of $(\star)_{\varepsilon,L_\varepsilon}$ exists such that $F_{L_\varepsilon} = \{v^- < L_\varepsilon\}$. Let u be an admissible solution of $(\star)_{\varepsilon,\tau}$ such that $u \geq -\varepsilon^{1+\gamma}$ for some $\gamma \in (0,1)$ and that $|Du|_{\Sigma_\varepsilon}$ can be controlled independently of ε. Then, u satisfies the following estimates*

(i) $-\varepsilon^{1+\gamma} \leq u \leq \tau$ *on* $\overline{\Omega_\varepsilon}$

(ii) $0 \leq D_\nu u \leq D_\nu v^-$ *on* $\partial_\Omega F_{L_\varepsilon}$, ($\nu$ *ext. unit normal to* F_{L_ε})

(iii) $-2\varepsilon \leq D_\nu u \leq C(n, \partial_\Omega E_0, \Sigma^n)$ *on* $\partial_\Omega E_{0,\varepsilon}$, ($\nu$ *ext. unit normal to* $E_{0,\varepsilon}$)

(iv) $$|Du(x)| \leq \sup_{\partial\Omega_\varepsilon \cap B_r(x)} |Du| + \varepsilon + \frac{C(n)}{r}$$

(v) $\|u\|^{(-1-\beta)}_{2,\alpha,\Omega_\varepsilon} \leq C(n, \partial_\Omega E_{0,\varepsilon}, \Sigma^n, L_\varepsilon, \varepsilon, |Dv^-|)$

for $\varepsilon > 0$ sufficiently small and $\beta = \beta(\theta_1, \theta_2)$. Note that $|Du| \leq |D_\nu u|$ on $\partial_\Omega F_{L_\varepsilon}$ and $\partial_\Omega E_{0,\varepsilon}$ due to the constant Dirichlet boundary values.

In particular, Proposition 4.19 holds in the following situation:

Corollary 4.20. *Let $n \geq 2$. Let $E_0, E_{0,\varepsilon}, F_{L_\varepsilon}$ and $(\star)_{\varepsilon,\tau}$ be as in Definitions 4.5 and 4.6. Furthermore, let Σ^n be given as the graph of a convex C^3-function which is asymptotic to a cone in the sense that* (4.4) *holds. Then any admissible solution of $(\star)_{\varepsilon,\tau}$ with ε as in* (4.9) *satisfies the estimates of Proposition 4.19. Additionally, we have $|Du|_{\partial_\Omega F_{L_\varepsilon}} \leq 2\varepsilon$.*

Proof. Under these assumptions a subsolution v^- can be constructed using Lemma 4.9 where F_{L_ε} and L_ε are chosen as in (4.7). The special lower bound for u follows from (4.9) and Lemma 4.10. Furthermore, the gradient estimate on Σ_ε is independent of ε since Σ^n is convex. This was shown in Lemma 4.17. Thus, all condition of Proposition 4.19 are satisfied. Finally, the more explicit estimate of $|Du|$ on $\partial_\Omega F_{L_\varepsilon}$ is contained in Lemma 4.12. □

Proof of Proposition 4.19. Estimate (i) follows from Lemma 4.8 and the assumption on the subsolution. Estimate (ii) follows from Lemma 4.12 and estimate (iii) follows from

Lemma 4.15 in conjunction with $u \geq -\varepsilon^{1+\gamma}$ and Lemma 4.16. We can use Lemma 4.18 to obtain (iv). Finally, the gradient estimate tells us that the elliptic equation in $(\star)_{\varepsilon,\tau}$ which is equivalent to

$$a^{ij}(Du)D_{ij}u := \frac{1}{\varepsilon^2 + |Du|^2}\left(\delta^{ij} - \frac{D^iuD^ju}{\varepsilon^2 + |Du|^2}\right)D_{ij}u = 1$$

can be regarded as a linear, uniformly elliptic equation with $\mu|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq |\xi|^2$ and bounded coefficients and right hand side. Therefore, interior Schauder estimates [37], Chapter 6, Section 1, Theorem 1.1 tell us that $u \in C^{1,\alpha}(\Omega_\varepsilon)$. More precisely, [37], Chapter 2, Section 6, Theorem 6.1 contains the explicit dependence on the distance d to the boundary which is $d^{-\alpha}$. This yields $Du \in H^{(0)}_{0,\alpha}(\Omega_\varepsilon)$ which implies $a^{ij}(Du) \in H^{(0)}_{0,\alpha}(\Omega_\varepsilon)$. Finally, θ_1 and θ_2 are both strictly less than $\frac{\pi}{2}$. Thus, the linear theory, i.e. Theorem A.14 is applicable which yields the estimate (v) for some $\beta = \beta(\theta_1, \theta_2) \in (0, 1)$. ☐

4.3 Existence for the approximating problems

Now we can use the a priori estimates from Section 4.2 to obtain a unique solution to the approximating problems $(\star)_{\varepsilon,\tau}$. Furthermore, we can use the uniform estimates on $|Du^{\varepsilon,\tau}|$ to obtain a converging subsequence of solutions as ε tends to zero.

Theorem 4.21 (Existence for the $(\star)_{\varepsilon,\tau}$ problem). *Let $E_0, E_{0,\varepsilon}, F_{L_\varepsilon}$ and $(\star)_{\varepsilon,\tau}$ be as in Definitions 4.5 and 4.6. Let Σ^n be a $C^{2,\alpha}$-hypersurface. Assume that for sufficiently small $\varepsilon > 0$ admissible subsolutions v^- of $(\star)_{\varepsilon,L_\varepsilon}$ exist such that $F_{L_\varepsilon} = \{v^- < L_\varepsilon\}$ and $L_\varepsilon \to \infty$. Furthermore, assume that any admissible solution $u^{\varepsilon,\tau}$ of $(\star)_{\varepsilon,\tau}$ satisfies $u^{\varepsilon,\tau} \geq -\varepsilon^{1+\gamma}$ for some $\gamma \in (0,1)$ and that $|Du^{\varepsilon,\tau}|_{\Sigma_\varepsilon}$ can be controlled independently of ε. Then there exists some $\beta = \beta(\theta_1, \theta_2) \in (0,1)$ and a unique solution $u^{\varepsilon,\tau} \in H^{(-1-\beta)}_{2,\alpha}(\Omega_\varepsilon)$ of $(\star)_{\varepsilon,\tau}$ for all $\tau \in [0, L_\varepsilon]$. Furthermore, there exist sequences $(\varepsilon_i)_{i\in\mathbb{N}}$, $(L_{\varepsilon_i})_{i\in\mathbb{N}}$, $(\Omega_{\varepsilon_i})_{i\in\mathbb{N}}$ and $(u^{\varepsilon_i,L_{\varepsilon_i}})_{i\in\mathbb{N}}$ such that for $\varepsilon_i \to 0$ we have*

$$L_{\varepsilon_i} \longrightarrow \infty, \qquad F_{L_{\varepsilon_i}} \setminus E_{0,\varepsilon} \longrightarrow \Omega \setminus E_0, \qquad \textit{and} \qquad u^{\varepsilon_i,L_{\varepsilon_i}} \longrightarrow u \in C^{0,1}_{loc}(\Omega \setminus E_0)$$

locally uniformly.

In particular, Theorem 4.21 holds in the following situation:

Corollary 4.22. *Let $n \geq 2$. Let $E_0, E_{0,\varepsilon}, F_{L_\varepsilon}$ and $(\star)_{\varepsilon,\tau}$ be as in Definitions 4.5 and 4.6. Let Σ^n be given as the graph of a convex C^3-function which is asymptotic to a cone in the sense that* (4.4) *holds. Then the conditions of Theorem 4.21 are satisfied.*

Proof. Under these assumptions a subsolution v^- can be constructed using Lemma 4.9 where F_{L_ε} and L_ε are chosen as in (4.7). The definition of L_ε shows that $L_\varepsilon \to \infty$ as $\varepsilon \to 0$. The special lower bound for u follows from Lemma 4.10. Furthermore, the gradient estimate on Σ_ε is independent of ε since Σ^n is convex. This was shown in Lemma 4.17. Thus, all condition of Theorem 4.21 are satisfied. ☐

Proof of Theorem 4.21. We proceed in two steps. First we prove the existence of a solution for $\tau = 0$ and small $\varepsilon > 0$. In the second step we show that for all $\varepsilon > 0$ there exists a solution for $\tau \in [0, L_\varepsilon]$. So let us assume that $\tau = 0$ first. The operator occurring in $(\star)_{\varepsilon,0}$ is

$$Q^\varepsilon(u) := \operatorname{div}\left(\frac{Du}{\sqrt{\varepsilon^2 + |Du|^2}}\right) - \sqrt{\varepsilon^2 + |Du|^2}$$

For $\varepsilon > 0$ the equation $Q^\varepsilon(u) = 0$ is equivalent to $F(u/\varepsilon) = \varepsilon$ with

$$F(u) := \frac{1}{\sqrt{1 + |Du|^2}} \operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right).$$

Therefore, for $\varepsilon > 0$ the function u is a solution to $(\star)_{\varepsilon,0}$ if and only if $\hat{u} := u/\varepsilon$ solves

$$\widehat{(\star)_\varepsilon} \begin{cases} F(\hat{u}) & = & \varepsilon & \text{in} & \Omega_\varepsilon \\ D_\mu \hat{u} & = & 0 & \text{on} & \Sigma_\varepsilon \\ \hat{u} & = & 0 & \text{on} & \partial_\Omega E_{0,\varepsilon} \cup \partial_\Omega F_{L_\varepsilon}. \end{cases}$$

To prove the existence of a solution $\hat{u}$ we consider F as an operator $F : A \to B$ where

$$A := \left\{ w \in H^{(-1-\beta)}_{2,\alpha}(\Omega_\varepsilon) \;\middle|\; w = 0 \text{ on } \partial_\Omega E_{0,\varepsilon} \cup \partial_\Omega F_{L_\varepsilon},\ D_\mu w = 0 \text{ on } \Sigma_\varepsilon \right\}$$

and $B := H^{(1-\beta)}_{0,\alpha}(\Omega_\varepsilon)$. The spaces $H^{(b)}_{k,\alpha}(\Omega)$ are weighted Hölder spaces. They are Banach spaces when they are equipped with a weighted norm (see $(A.55)$ for the exact definition). The choice of $\beta = \beta(\theta_1, \theta_2)$ depends on the angle between the Dirichlet boundary and the Neumann boundary.

For $\varepsilon = 0$ the problem $\widehat{(\star)}_\varepsilon$ has the solution $\hat{u}_0 := 0$. Furthermore, the linearization of F around $\hat{u}_0$ is the Laplacian, since

$$\begin{aligned} DF_{\hat{u}_0}(w) &:= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} F(\hat{u}_0 + sw) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \left\{ \frac{1}{1 + s^2|Dw|^2} \left(s\Delta w - s^3 \frac{D^i w D^j w D_{ij} w}{1 + s^2 |Dw|^2} \right) \right\} = \Delta w. \end{aligned}$$

The linear theory for mixed boundary value problems Theorem A.14 guarantees the global invertibility of $DF_{\hat{u}_0}$, i.e. the existence of a unique solution $u \in A$ to

$$\begin{cases} \Delta w & = & f & \text{in} & \Omega_\varepsilon \\ D_\mu w & = & 0 & \text{on} & \Sigma_\varepsilon \\ w & = & 0 & \text{on} & \partial_\Omega E_{0,\varepsilon} \cup \partial_\Omega F_{L_\varepsilon} \end{cases}$$

for arbitrary $f \in B$. Therefore, the inverse function theorem implies the invertibility of F in a neighborhood of $F(\hat{u}_0) = F(0) = 0$. This means that for all $f \in B$ which are close to 0 (in the norm of B) the map F is invertible. Since in our case $f \equiv \varepsilon$ this proves the existence of a unique solution to $\widehat{(\star)_\varepsilon}$ for $\varepsilon > 0$ small enough, i.e. $\varepsilon \in (0, \bar{\varepsilon}]$.

Now we want to prove the existence of a solution to $(\star)_{\varepsilon,\tau}$. Therefore we fix $\varepsilon \in (0,\bar{\varepsilon}]$ and define the set

$$I_\varepsilon := \left\{ \tau \in [0, L_\varepsilon] \;\middle|\; \text{The problem } (\star)_{\varepsilon,\tau} \text{ has a unique solution in } H_{2,\alpha}^{(-1-\beta)}(\Omega_\varepsilon) \right\}.$$

We already know that $I_\varepsilon \neq \emptyset$ since $0 \in I_\varepsilon$ by the first step of the proof. If we can show that I_ε is open and closed we obtain the desired result, i.e. the existence of a unique solution to $(\star)_{\varepsilon,\tau}$ for all $\varepsilon \in (0,\bar{\varepsilon}]$ and all $\tau \in [0, L_\varepsilon]$. To show that I_ε is open we use once more the inverse function theorem. We modify the spaces A and B to allow other boundary values than zero on $\partial_\Omega F_{L_\varepsilon}$ and define

$$A := \left\{ w \in H_{2,\alpha}^{(-1-\beta)}(\Omega_\varepsilon) \;\middle|\; w = 0 \text{ on } \partial_\Omega E_{0,\varepsilon},\ D_\mu w = 0 \ on\ \Sigma_\varepsilon \right\}$$

$$B := B_1 \times B_2 := H_{0,\alpha}^{(1-\beta)}(\Omega_\varepsilon) \times H_{2,\alpha}^{(-1-\beta)}(\partial_\Omega F_{L_\varepsilon}).$$

We denote the projection on $\partial_\Omega F_{L_\varepsilon}$ by $\pi : A \to B_2 : w \mapsto \pi(w) := w|_{\partial_\Omega F_{L_\varepsilon}}$ and consider the operator

$$T : A \to B : w \mapsto Tw := \left(Q^\varepsilon(w), \pi(w)\right).$$

Its linearization around some $u_0 \in A$ is given by $DT_{u_0} w = (DQ^\varepsilon_{u_0} w, \pi(w))$. We write Q^ε as

$$Q^\varepsilon(u) = \frac{1}{\sqrt{\varepsilon^2 + |Du|^2}} \left(\delta^{ij} - \frac{D^i u D^j u}{\varepsilon^2 + |Du|^2} \right) D_{ij} u - \sqrt{\varepsilon^2 + |Du|^2}$$

$$=: a^{ij}(Du) D_{ij} u + b(Du)$$

and calculate the linearization of Q^ε:

$$DQ^\varepsilon_{u_0} w := \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} Q^\varepsilon(u_0 + sw)$$

$$= a^{ij}(Du_0) D_{ij} w + \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \left\{ a^{ij}(Du_0 + sDw) \right\} D_{ij} u_0 + \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \left\{ b(Du_0 + sDw) \right\}$$

$$= a^{ij}(Du_0) D_{ij} w + B^k(Du_0, D^2 u_0) D_k w =: L_{u_0} w.$$

To show that I_ε is open we assume that $\tau \in I_\varepsilon$ and we have to show that $\tau' \in I_\varepsilon$ for $|\tau - \tau'|$ sufficiently small. If $\tau \in I_\varepsilon$ then there exists a unique solution $u^{\varepsilon,\tau}$ to $(\star)_{\varepsilon,\tau}$. We linearize T around $u_0 := u^{\varepsilon,\tau}$. Since $u_0 \in A$ we see that $u_0 \in C^{1,\beta}(\overline{\Omega_\varepsilon})$. Therefore, a^{ij} is bounded and uniformly elliptic and we also have $a^{ij} \in H_{0,\alpha}^{(0)}(\Omega_\varepsilon)$. So a^{ij} satisfies the conditions of Theorem A.14. Furthermore, we deduce that $D^2 u_0 \in H_{0,\alpha}^{(1-\beta)}(\Omega_\varepsilon)$ and so also the B^k satisfy the conditions of Theorem A.14. Thus, the linear theory contained in Theorem A.14 tells us that DT_{u_0} is globally invertible, i.e. that the problem

$$\begin{cases} L_{u_0} w &= f_1 \quad \text{in } \Omega_\varepsilon \\ D_\mu w &= 0 \quad \text{on } \Sigma_\varepsilon \\ w &= 0 \quad \text{on } \partial_\Omega E_{0,\varepsilon} \\ w &= f_2 \quad \text{on } \partial_\Omega F_{L_\varepsilon} \end{cases}$$

has a unique solution in A for arbitrary $(f_1, f_2) \in B$. Therefore, T is invertible in a small neighborhood of $Tu_0 = (Q^\varepsilon(u^{\varepsilon,\tau}), \pi(u^{\varepsilon,\tau})) = (0, \tau)$. For $|\tau - \tau'|$ sufficiently small $(0, \tau')$ lies in a neighborhood of $(0, \tau)$ and so a unique solution to $(\star)_{\varepsilon,\tau'}$ exists. Thus, $\tau' \in I_\varepsilon$ and I_ε is open.

In order to show that I_ε is closed we take a sequence $(\tau_n)_{n\in\mathbb{N}} \subset I_\varepsilon$ which converges in $\mathbb{R}$ to some limit τ. We have to show that $\tau \in I_\varepsilon$. That means we have to use the fact that $(\star)_{\varepsilon,\tau_n}$ has a unique solution $u_n := u^{\varepsilon,\tau_n}$ and show that there exists a unique solution $u^{\varepsilon,\tau}$ of $(\star)_{\varepsilon,\tau}$. Let us first show that $(u_n)_{n\in\mathbb{N}}$ converges in $C^0(\overline{\Omega_\varepsilon})$ to some limit $u^{\varepsilon,\tau}$. To see this we first calculate

$$\begin{aligned}
0 &= Q^\varepsilon(u_n) - Q^\varepsilon(u_m) \\
&= \Big[a^{ij}(Du_n)D_{ij}u_n + b(Du_n)\Big] - \Big[a^{ij}(Du_m)D_{ij}u_m + b(Du_m)\Big] \\
&= a^{ij}(Du_n)D_{ij}(u_n - u_m) + \Big[a^{ij}(Du_n) - a^{ij}(Du_m)\Big]D_{ij}u_m + b(Du_n) - b(Du_m) \\
&= a^{ij}(Du_n)D_{ij}w - B^k(Du_n, Du_m, D^2u_m)D_k w =: \tilde{L}_{u_n} w.
\end{aligned}$$

In the last step we defined $w := u_n - u_m$ and used the fundamental theorem of calculus. The B^k are different to the ones we used before. This calculation tells us that w satisfies the linear problem

$$\begin{cases}
\tilde{L}_{u_n} w &= 0 & \text{in } \Omega_\varepsilon \\
D_\mu w &= 0 & \text{on } \Sigma_\varepsilon \\
w &= 0 & \text{on } \partial_\Omega E_{0,\varepsilon} \\
w &= \tau_n - \tau_m & \text{on } \partial_\Omega F_{L_\varepsilon}.
\end{cases}$$

The maximum principles in Propositions A.9, A.10 imply, that

$$\sup_{\Omega_\varepsilon} |u_n - u_m| \leq \sup_{\partial_\Omega E_{0,\varepsilon} \cup \partial_\Omega F_{L_\varepsilon}} |u_n - u_m| \leq |\tau_n - \tau_m|.$$

Since $(\tau_n)_{n\in\mathbb{N}}$ converges, the Cauchy criterion implies the convergence of the sequence of solutions $(u_n)_{n\in\mathbb{N}}$ in $C^0(\overline{\Omega_\varepsilon})$ to some function $u^{\varepsilon,\tau} \in C^0(\overline{\Omega_\varepsilon})$ satisfying $u^{\varepsilon,\tau} = 0$ on $\partial_\Omega E_{0,\varepsilon}$ and $u^{\varepsilon,\tau} = \tau$ on $\partial_\Omega F_{L_\varepsilon}$. Now we use the a priori estimates of Proposition 4.19 which are uniform in n:

$$\|u_n\|^{(-1-\beta)}_{2,\alpha;\Omega_\varepsilon} \leq C(\varepsilon).$$

Together with an Arzelà-Ascoli type theorem for these weighted spaces (see Proposition A.13) we obtain a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ which converges to $u^{\varepsilon,\tau}$ in $H^{(-1-\beta')}_{2,\alpha'}(\Omega_\varepsilon)$ for $\alpha' < \alpha$ and $\beta' < \beta$. In particular (see Proposition A.13) we have

$$u^{\varepsilon,\tau} \in C^{1,\beta'}(\overline{\Omega_\varepsilon}) \cap C^{2,\alpha'}(\Omega_\varepsilon)$$

which implies that $u^{\varepsilon,\tau}$ solves $(\star)_{\varepsilon,\tau}$ and by uniqueness $u^{\varepsilon,\tau} \in A$. Thus, $u^{\varepsilon,\tau} \in I_\varepsilon$ and I_ε is closed. Since we already showed that I_ε is open and not empty we proved that

$I_\varepsilon = [0, L_\varepsilon]$. Therefore we proved the existence of a unique solution to $(\star)_{\varepsilon,\tau}$ for all $\varepsilon > 0$ sufficiently small and $\tau \in [0, L_\varepsilon]$.

Using the a priori estimates for $|Du^{\varepsilon,L_\varepsilon}|$ which are independent of ε we see that $u^{\varepsilon,L_\varepsilon}$ is uniformly bounded and uniformly equicontinuous on compact subsets. Therefore, the Arzelà-Ascoli theorem yields the convergence of a sequence $u^{\varepsilon_i,L_{\varepsilon_i}}$ to a continuous function u. Finally, the Lipschitz estimate persists in the limit and so u is a locally Lipschitz continuous function. $\square$

4.4 Variational characterization of the limit

In the last section we obtained a function $u \in C^{0,1}_{loc}(\Omega \setminus E_0)$ as the limit of solutions $(u^{\varepsilon_i})_{i\in\mathbb{N}}$ of the approximating problems $(\star)_{\varepsilon_i,L_{\varepsilon_i}}$. The aim of this section is to show that this limit u is the unique weak solution of $(\star)$. For this section we follow the approach of Huisken and Ilmanen [29], Section 1 and 2. Most of the proofs presented in this section are the same as in [29] but we include them for the sake of completeness.

First we will define the notion of weak solutions of $(\star)$ and prove some geometric properties of the hypersurfaces $M^n_t := \partial_\Omega\{u < t\}$. Furthermore, we will show that classical solutions to $(\star)$ are weak solutions and that we have compactness and uniqueness for weak solutions. Having these properties at our disposal the argument will be the following: We will show that the u^{ε_i} allow us to define classical solutions $U^{\varepsilon_i}(x,z) := u^{\varepsilon_i}(x) - \varepsilon_i z$ of (IMCF) one dimension higher. Using the fact that they are also weak solutions together with the compactness result we conclude that the limit $U(x,z) := u(x)$ is a weak solution too. Finally, cut-off functions will allow us to prove that u is the unique weak solution of $(\star)$ in $\Omega \setminus \overline{E_0}$. This procedure yields existence and uniqueness for weak solutions to $(\star)$ in Theorem 4.47.

Remark 4.23. In this section we use the notation from Definition 4.1. In particular the set $\Omega \subset \mathbb{R}^{n+1}$ denotes all points above the supporting hypersurfaces Σ^n including Σ^n itself. Remember also the definitions for the different boundary parts, i.e.

$$\partial_\Omega A := \overline{\partial A \setminus \Sigma^n}, \qquad \partial_\Sigma := \partial A \setminus \partial_\Omega A$$

for $A \subseteq \Omega$. Furthermore, we will make use of sets which are open in Ω. So these sets are allowed to contain points on Σ^n. In the same way a (pre)compact subset K of $A \subseteq \Omega$ may contain points on Σ^n if $A \cap \Sigma^n \neq \emptyset$.

The definition of a weak solution requires the following functional

Lemma 4.24. *Let $A \subseteq \Omega$ be open in Ω. For $u \in C^{0,1}_{loc}(A)$ we consider the functional*

$$J^K_u : C^{0,1}_{loc}(A) \to \mathbb{R} : v \mapsto J^K_u(v) := \int_K \Big(|Dv| + v|Du|\Big)\,\mathrm{d}\lambda. \tag{4.27}$$

where $\{u \neq v\} \subset K$, K is a compact subset of A and $\lambda(\partial K) = 0$. The functional J^K_u is lower semicontinuous with respect to L^1_{loc}-convergence.

Proof. First we note that $v \mapsto \int_K v|Du|\,\mathrm{d}\lambda$ is continuous with respect to L^1_{loc}-convergence. Now we prove the lower semicontinuity of the first term of the functional. Let $B \subset A$ be bounded and open and consider a sequence $(v_n)_{n\in\mathbb{N}} \subset C^{0,1}_{loc}(A)$ converging to a function

$v \in C^{0,1}_{loc}(A)$ in L^1_{loc}. Since $f \mapsto \|Df\|(B) = \int_B |Df| \, \mathrm{d}\lambda$ is lower semicontinuous with respect to L^1_{loc}-convergence (see Definition A.22 and Lemma A.23) we obtain

$$\|Dv\|(K) = \|Dv\|(\mathring{K}) \leq \liminf_{n\to\infty} \|Dv_n\|(\mathring{K}) \leq \liminf_{n\to\infty} \|Dv_n\|(K)$$

if K is compact and $\lambda(\partial K) = 0$. □

Remark 4.25. In the following we will omit the set K and only write J_u instead of J_u^K. Furthermore, we always choose a compact set K which which satisfies $\lambda(\partial K) = 0$ without mentioning it explicitly. Note that it is enough for K to be a Cacciopoli set (see Definition A.25).

The definition of weak solutions is the following.

Definition 4.26. Let $A \subseteq \Omega$ be open in Ω.

(i) The function $u \in C^{0,1}_{loc}(A)$ is called a weak subsolution (supersolution) of $(\star)$ in A if

$$J_u(u) \leq J_u(v), \qquad v \text{ locally Lipschitz and } \{u \neq v\} \subset\subset A \tag{4.28}$$

for every v satisfying $v \leq u$ $(v \geq u)$. The integration is performed over any compact set K containing $\{u \neq v\}$.

(ii) The function $u \in C^{0,1}_{loc}(A)$ is called a weak solution of $(\star)$ in A if it is at the same time a weak subsolution and a weak supersolution of $(\star)$ in A.

(iii) The function $u \in C^{0,1}_{loc}(\Omega)$ is called a weak solution of $(\star)$ with initial condition $E_0 \subset \Omega$ if $E_0 = \{u < 0\}$ and u is a weak solution of $(\star)$ in $\Omega_0 := \Omega \setminus \overline{E_0}$.

Remark 4.27. The function $u \in C^{0,1}_{loc}(A)$ is a weak solution of $(\star)$ in A if and only if (4.28) holds. The integration is performed over any compact set K containing $\{u \neq v\}$.

Proof. Assume that (4.28) holds. Then in particular this is true for $v \leq u$ and $v \geq u$. So weak solutions are weak subsolutions and weak supersolutions. For the other direction we first note that

$$\begin{aligned} &J_u(\min(u,v)) + J_u(\max(u,v)) \\ &= \int_K \Big(|D\min(u,v)| + |D\max(u,v)| + \big(\min(u,v) + \max(u,v)\big)|Du|\Big) \, \mathrm{d}\lambda \\ &= \int_K \Big(|Du| + |Dv| + (u+v)|Du|\Big) \, \mathrm{d}\lambda \\ &= J_u(u) + J_u(v) \end{aligned} \tag{4.29}$$

whenever $\{u \neq v\}$ is precompact. Let u be a weak subsolution and a weak supersolution of $(\star)$ in A. Since $u \leq \max(u,v)$ and $u \geq \min(u,v)$ we can use $\min(u,v)$ and $\max(u,v)$ as competitors for weak supersolutions and weak subsolutions respectively. So we obtain

$$2J_u(u) \leq J_u(\min(u,v)) + J_u(\max(u,v)) \overset{(4.29)}{=} J_u(u) + J_u(v)$$

and thus $J_u(u) \leq J_u(v)$. □

It will be useful to have an alternative characterization of weak solutions. Therefore, we need another functional.

Lemma 4.28. *Let $A \subseteq \Omega$. For $u \in C^{0,1}_{loc}(A)$ we consider the functional*

$$J_u^K : Ca(A) \to \mathbb{R} : F \mapsto J_u^K(F) := |\partial^*_\Omega F \cap K| - \int_{F \cap K} |Du| \,\mathrm{d}\lambda \tag{4.30}$$

where K is a compact set such that $|\partial^*_\Omega F \cap \partial K| = 0$. Here $Ca(A)$ denotes the set of all Caccioppoli sets (see Definition A.25) in A, $\partial^*_\Omega F$ denotes the reduced boundary (see Definition A.28) of the set F in Ω and $|.|$ applied to sets denotes the n-dimensional Hausdorff measure. The functional J_u is lower semicontinuous with respect to L^1_{loc}-convergence.

Proof. First we note that $F \mapsto \int_{F\cap K} |Du| \,\mathrm{d}\lambda$ is continuous with respect to L^1_{loc} convergence (of $\mathbb{1}_F$). Now we prove the lower semicontinuity of the first term of the functional. Let $B \subset A$ be bounded and open. Let $(F_n)_{n\in\mathbb{N}} \subset Ca(A)$ be a sequence of Caccioppoli sets which converges to the set $F \in Ca(A)$ in L^1_{loc}, i.e. $\mathbb{1}_{F_n} \to \mathbb{1}_F$ in L^1_{loc}. Since $F \mapsto \|D\mathbb{1}_F\|(B) = |\partial^*_\Omega F \cap B|$ is lower semicontinuous with respect to L^1_{loc}-convergence (see Lemma A.23 and Theorem A.29) we obtain

$$|\partial^*_\Omega F \cap K| = |\partial^*_\Omega F \cap \mathring{K}| \leq \liminf_{n\to\infty} |\partial^*_\Omega F_n \cap \mathring{K}| \leq \liminf_{n\to\infty} |\partial^*_\Omega F_n \cap K|$$

if K is compact and $|\partial^*_\Omega F \cap \partial K| = 0$. □

Remark 4.29. In the following we will omit the set K and only write J_u instead of J_u^K. Furthermore, we always choose a compact set K which which satisfies $|\partial^*_\Omega F \cap \partial K| = 0$ without mentioning it explicitly. Note that here it is not (!) enough for K to be a Caccioppoli set.

With the help of this functional we can give an alternative definition of weak solutions.

Definition 4.30. Let $A \subset \Omega$.

(i) Let $u \in C^{0,1}_{loc}(A)$ and let $E \in Ca(A)$. The set E minimizes J_u on the outside (inside) of A if

$$J_u(E) \leq J_u(F), \qquad \text{F Caccioppoli and } E\Delta F \subset\subset A \tag{4.31}$$

for every F with $F \supseteq E$ ($F \subseteq E$). The integration is performed over any compact set K containing $E\Delta F$.

(ii) Let $u \in C^{0,1}_{loc}(A)$. Let E have locally finite perimeter. We say that E minimizes J_u in A if E minimizes J_u on the outside and inside of A.

(iii) Let $(E_t)_{t>0} \subset \Omega$ be a nested family of open sets with locally finite perimeter, closed under ascending union. Let u be defined by $E_t = \{u < t\} \subset \Omega$. The family $(E_t)_{t>0}$ is called a weak solution of ($\star$) with initial condition $E_0 \subset \Omega$ if $u \in C^{0,1}_{loc}(\Omega)$ and E_t minimizes J_u in $\Omega_0 = \Omega \setminus E_0$ for each $t > 0$.

Remark 4.31. Let $u \in C^{0,1}_{loc}(A)$ and let E have locally finite perimeter. The set E minimizes J_u in A if and only if (4.31) holds for every F having locally finite perimeter. The integration is performed over any compact set K containing $E\Delta F$.

Proof. Assume that (4.31) holds. Then in particular this is true for $F \supseteq E$ and $F \subseteq E$. So if E minimizes J_u in A it also minimizes J_u on the outside and inside of A. For the other direction we first note that the inequality for the Hausdorff measure (see Lemma A.30) yields

$$\begin{aligned} &J_u(E \cup F) + J_u(E \cap F) \\ &= |\partial^*_\Omega(E \cup F) \cap K| + \int_{E\cup F} |Du| \,\mathrm{d}\lambda \ + \ |\partial^*_\Omega(E \cap F) \cap K| + \int_{E\cap F} |Du| \,\mathrm{d}\lambda \\ &\leq |\partial^*_\Omega E \cap K| + \int_E |Du| \,\mathrm{d}\lambda \ + \ |\partial^*_\Omega F \cap K| + \int_F |Du| \,\mathrm{d}\lambda \\ &= J_u(E) + J_u(F) \end{aligned} \tag{4.32}$$

whenever $E\Delta F$ is precompact. Let E minimize J_u in A. Since $E \subset E\cup F$ and $E \supset E\cap F$ we can use $E\cup F$ and $E\cap F$ as competitors for sets minimizing J_u on the outside and on the inside respectively. So we obtain

$$2J_u(E) \leq J_u(E \cup F) + J_u(E \cap F) \overset{(4.32)}{\leq} J_u(E) + J_u(F)$$

and thus $J_u(E) \leq J_u(F)$. □

Since we want to work with both definitions we have to show that they are equivalent. First we prove the result for the parts (i) and (ii).

Lemma 4.32. *Let $A \subseteq \Omega$ be open in Ω. Let $u \in C^{0,1}_{loc}(A)$. Then the following statements are equivalent*

(1) *For each $t > 0$, $E_t := \{u < t\}$ minimizes J_u in (outside of, inside of) A.*

(2) *u is a weak solution (subsolution, supersolution) of $(\star)$ in A.*

Proof. (1) ⇒ (2): Let $E_t := \{u < t\}$ minimize J_u in A. Let $v \in C^{0,1}_{loc}(A)$ with $\{u \neq v\} \subset K$ and K compact. We define $F_t := \{v < t\}$ and note that $F_t\Delta E_t \subset K$ for every t. For $a < b$ with $a \leq u, v \leq b$ on K the co-area formula yields

$$\begin{aligned} J_u(v) &= \int_K \Big(|Dv| + v|Du|\Big) \,\mathrm{d}\lambda \\ &= \int_a^b \left(\int_{K\cap\{v=t\}} 1 \ \mathrm{d}\mathcal{H}^n\right) \mathrm{d}t + \int_K v|Du| \,\mathrm{d}\lambda \\ &= \int_a^b |\partial^*_\Omega F_t \cap K| \,\mathrm{d}t - \int_K (b-v)|Du| \,\mathrm{d}\lambda + b\int_K |Du| \,\mathrm{d}\lambda \\ &= \int_a^b |\partial^*_\Omega F_t \cap K| \,\mathrm{d}t - \int_K \left(\int_a^b \mathbb{1}_{\{v<t\}} \,\mathrm{d}t\right) |Du| \,\mathrm{d}\lambda + b\int_K |Du| \,\mathrm{d}\lambda \\ &= \int_a^b \left(|\partial^*_\Omega F_t \cap K| - \int_{K\cap F_t} |Du| \,\mathrm{d}\lambda\right) \mathrm{d}t + b\int_K |Du| \,\mathrm{d}\lambda \\ &= \int_a^b J_u(F_t) \,\mathrm{d}t + b\int_K |Du| \,\mathrm{d}\lambda. \end{aligned} \tag{4.33}$$

The same calculation can be done for $J_u(u)$. Thus, if each E_t minimizes J_u then

$$J_u(u) = \int_a^b J_u(E_t)\,\mathrm{d}t + b\int_K |Du|\,\mathrm{d}\lambda \le \int_a^b J_u(F_t)\,\mathrm{d}t + b\int_K |Du|\,\mathrm{d}\lambda = J_u(v)$$

i.e. u is a weak solution of $(\star)$ in A. The same argument treats weak supersolutions and subsolutions separately.

(2) $\Rightarrow$ (1): We will first prove that if u is a weak supersolution of $(\star)$ in A then E_t minimizes J_u on the inside of A. Therefore, we fix some t_0. For a set F such that

$$F \subset E_{t_0}, \qquad E_{t_0} \setminus F \subset\subset A$$

we have to show that $J_u(E_{t_0}) \le J_u(F)$. Since J_u is lower semicontinuous and u is fixed we can minimize J_u and thus assume that

$$J_u(F) \le J_u(G), \qquad \forall\, G \quad \text{s.t.} \quad F \subset G \tag{4.34}$$

with $G\Delta E_{t_0} \subset F\Delta E_{t_0}$. Now we define the nested family

$$F_t := \begin{cases} F \cap E_t, & t \le t_0, \\ E_t & t > t_0. \end{cases}$$

Using (4.32) and (4.34) we obtain

$$J_u(F_t) = J_u(F \cap E_t) \overset{(4.32)}{\le} J_u(F) + J_u(E_t) - J_u(F \cup E_t) \overset{(4.34)}{\le} J_u(E_t). \tag{4.35}$$

Defining

$$v : A \to \mathbb{R} : x \mapsto v(x) := \begin{cases} t_0 & x \in E_{t_0} \setminus F \\ u(x) & x \notin E_{t_0} \setminus F \end{cases}$$

we see that $F_t = \{v < t\}$ and $\{u \neq v\} = E_{t_0} \setminus F \subset\subset A$.

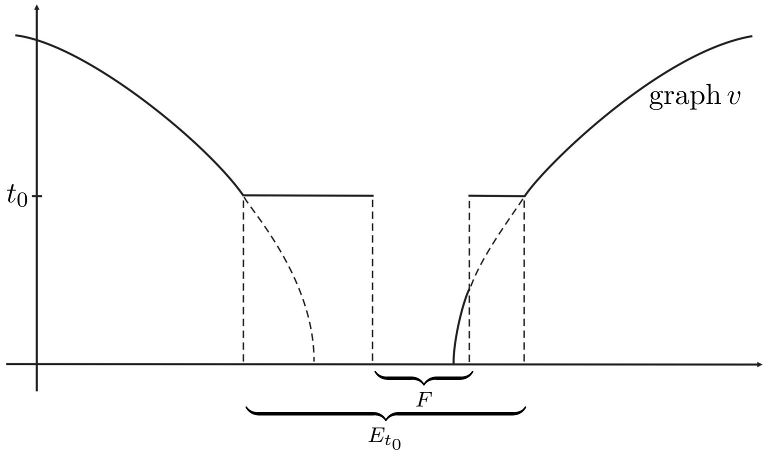

Figure 4.5: Construction of the competitor v.

Because of the jump at $\partial_\Omega F$ (see Figure 4.5) we only have $v \in BV_{loc}(A) \cap L^\infty_{loc}(A)$. Therefore, we approximate v by a sequence $(v_k)_{k\in\mathbb{N}} \subset BV_{loc}(A) \cap C^\infty(A)$. By Lemma A.27 we see that $v_k \to v$ in L^1_{loc} and

$$\int_{E_{t_0}} |Dv_k| \, \mathrm{d}\lambda = \|Dv_k\|(E_{t_0}) \to \|Dv\|(E_{t_0})$$

as Radon measures. Since $u \leq v$ we obtain $J_u(u) \leq J_u(v)$ in the limit as $k \to \infty$. Furthermore, (4.33) is valid for v. This yields

$$\int_a^b J_u(E_t) \, \mathrm{d}t \leq \int_a^b J_u(F_t) \, \mathrm{d}t.$$

Together with (4.35) we see that the integrals are equal and $J_u(F_t) \leq J_u(E_t)$ which implies $J_u(F_t) = J_u(E_t)$ for almost every t. Finally, (4.32) shows that

$$J_u(E_t \cup F) \overset{(4.32)}{\leq} J_u(E_t) + J_u(F) - J_u(F_t) = J_u(F)$$

for almost every $t \leq t_0$. Passing $t \nearrow t_0$ and using the lower semicontinuity of J_u we obtain the desired result

$$J_u(E_{t_0}) \leq J_u(F), \qquad \text{for} \qquad F \subset E_{t_0},\ E_{t_0} \setminus F \subset\subset A.$$

So for every t_0 the set E_{t_0} minimizes J_u on the inside of A.

It is left to show that for a subsolution u and some t_0 the sets E_{t_0} minimize J_u on the outside of A. To do so, one shows that the sets $\{u \leq t\}$ minimize J_u on the outside of A and then chooses a sequence $t \nearrow t_0$ and notes that $\{u \leq t_i\}$ converges to E_{t_0} in L^1_{loc}. Using lower semicontinuity of J_u and a standard replacement argument, it follows that E_{t_0} minimizes J_u on the outside of A. □

From Lemma 4.32 we obtain the equivalence for the initial value problems.

Lemma 4.33. *Let $u \in C^{0,1}_{loc}(\Omega)$. Then the following statements are equivalent*

(†) *For each $t > 0$, $E_t := \{u < t\}$ minimizes J_u in $\Omega \setminus E_0$.*

$(\dagger)^+$ *For each $t \geq 0$, $\{u \leq t\}$ minimizes J_u in $\Omega \setminus E_0$.*

(††) *$E_0 = \{u < 0\}$ and u is a weak solution of $(\star)$ in $\Omega \setminus \overline{E_0}$.*

Proof. The equivalence of (†) and (††) follows from Lemma 4.32 and approximation up to the boundary. The equivalence of (†) and $(\dagger)^+$ follows by approximating $s \searrow t$. □

For minimizers of the functional we obtain the following regularity.

Lemma 4.34. *Let $u \in C^{0,1}_{loc}(A)$. Let $E \subset \Omega$ be a minimizer of the functional J_u defined in (4.27). Then $\partial^*_\Omega E$ is a subset of a $C^{1,\frac{1}{2}}$-hypersurface and*

$$\mathcal{H}^k(\partial_\Omega E \setminus \partial^*_\Omega E) = 0 \quad \forall k > n - 8.$$

Note that this is a regularity result for $M^n := \partial_\Omega E$ which does not yet include an information about the regularity of ∂M^n.

Proof. Since $u \in C^{0,1}_{loc}(A)$ we see that minimzers of J_u are almost minimal in the sense that for balls of radius R we have

$$|\partial^*_\Omega E \cap B_R| \leq |\partial^*_\Omega F \cap B_R| + C(\|Du\|_\infty, n)R^{n+1}, \qquad \text{for} \quad E\Delta F \subset\subset B_R. \tag{4.36}$$

Thus [63], Theorem 1 yields the result. See also [47]. □

For classical solutions of (IMCF) the mean curvature H of the evolving hypersurface can be calculated using the level-set function u which solves $(\star)$, i.e. $H = |Du|$. Next we want to show that this equality still holds in a weak sense for minimizers of J_u. Therefore, we first define a notion of weak mean curvature guided by the classical equality

$$\int_{M^n} \Big(\operatorname{div}_{M^n} X - H\nu \cdot X \Big) \,\mathrm{d}\mu = -\int_{\partial M^n} X \cdot \eta \,\mathrm{d}s$$

which is valid for C^2-submanifolds M^n of $\mathbb{R}^{n+1}$ with $(n-1)$-dimensional C^1-boundary ∂M^n and C^1-vectorfields X (see [57], Chapter 2, §7, (7.6)). Here η is the inward pointing unit co-normal of ∂M^n. Note, that if M^n and Σ^n met orthogonally we would have $\eta = -\mu$ and thus the right hand side would vanish for variations X which are tangential along Σ^n.

Definition 4.35. We say that the hypersurface M^n possesses a weak mean curvature in L^p if there exists a vector valued function $\vec{H} \in L^p_{loc}(M^n, \mathbb{R}^{n+1})$ such that

$$\int_{M^n} \Big(\operatorname{div}_{M^n} X - \vec{H} \cdot X \Big) \,\mathrm{d}\mu = 0 \tag{4.37}$$

for all $X \in C^\infty_c(TM^n)$ with $\operatorname{spt} X \cap \partial M^n = \emptyset$. Furthermore, we say that M^n is weakly orthogonal to Σ^n if (4.37) holds for all $X \in C^\infty_c(TM^n)$ which are tangential along Σ^n, i.e. $X(x) \in T_x\Sigma^n$ for $x \in \Sigma^n$.

The next lemma shows that in the sense of Definition 4.35 we have $H = |Du|$.

Lemma 4.36 (Weak mean curvature). *Let $a, b \in \mathbb{R}_+$, $a < b$ and let $E_t := \{u < t\}$ minimize J_u in $A := E_b \setminus E_a$ where $u \in C^{0,1}_{loc}(A)$. Then up to a set of dimension less than or equal to $n-8$, $M^n_t := \partial_\Omega E_t$ is a $C^{1,\frac{1}{2}}$-hypersurface which possesses a weak mean curvature in L^∞ given by*

$$\vec{H}(x) = |Du(x)|\nu(x) \qquad \text{where} \qquad \nu(x) := \frac{Du(x)}{|Du(x)|}$$

for almost every[1] *$t \in (a, b)$ and almost every $x \in M^n_t$. Furthermore, for those values of t, M^n_t is orthogonal to Σ^n in the classical sense in any neighborhood of points $x \in \partial^*_\Omega E_t \cap \Sigma^n$.*

Proof. Let $U \subset \mathbb{R}^{n+1}$ be open such that $U \cap A \neq \emptyset$. Let $K \subset U$ be compact and $K \cap M^n_t \neq 0$. We consider a family of diffeomorphisms

$$\Phi : (-1, 1) \times U \to U : (x, s) \mapsto \Phi(s, x) =: \Phi_s(x)$$

satisfying

$$\Phi_0 = \mathrm{id}, \qquad \Phi_s\big|_{U \setminus K} = \mathrm{id}\big|_{U\setminus K}, \qquad \frac{\partial \Phi(s,x)}{\partial s}\Big|_{s=0} = X(\Phi_0(x)) = X(x)$$

---

[1] In particular the values of t where u develops a plateau, i.e. $|Du| = 0$ are excluded.

where X is a smooth vector field with support in K. Furthermore, X should be tangential to Σ^n if $K \cap \Sigma^n \neq \emptyset$. Note that

$$\frac{\partial \Phi_s^{-1}(y)}{\partial s}\Big|_{s=0} = -X(\Phi_0^{-1}(y)) = -X(y). \tag{4.38}$$

By Lemma 4.33 the function u minimizes J_u in $E_b \setminus \overline{E_a}$. Therefore, the first varition of J_u vanishes. Now we use the area and co-area formula (see Lemma A.31 and Lemma A.32), (4.38) and the dominant convergence theorem to compute

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} J_u(u \circ \Phi_s^{-1}) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \int_{\Phi_s(U)} \Big(|D(u \circ \Phi_s^{-1})(y)| + (u \circ \Phi_s^{-1})(y)|Du(y)|\Big)\, \mathrm{d}\lambda(y) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \Big(\int_U |Du(x)| \cdot |\det D\Phi_s(x)|\, \mathrm{d}\lambda(x) + \int_a^b \int_{M_t^n \cap \Phi_s(U)} (u \circ \Phi_s^{-1})(y)\, \mathrm{d}\mathcal{H}^n(y)\, \mathrm{d}t\Big) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \Big(\int_a^b \int_{M_t^n \cap U} |\det D\Phi_s(x)|\, \mathrm{d}\mathcal{H}^n(x)\, \mathrm{d}t + \int_a^b \int_{M_t^n \cap U} (u \circ \Phi_s^{-1})(y)\, \mathrm{d}\mathcal{H}^n(y)\, \mathrm{d}t\Big) \\
&= \int_a^b \int_{M_t^n \cap U} \Big(\operatorname{div}_{M_t^n} X(x) - Du(\Phi_0^{-1}(x)) \cdot X(\Phi_0^{-1}(x))\Big)\, \mathrm{d}\mathcal{H}^n(x)\, \mathrm{d}t \\
&= \int_a^b \int_{M_t^n \cap U} \Big(\operatorname{div}_{M_t^n} X(x) - Du(x) \cdot X(x)\Big)\, \mathrm{d}\mathcal{H}^n(x)\, \mathrm{d}t.
\end{aligned}$$

The Lebesgue differentiation theorem (see Lemma A.18) implies that the inner intergal vanishes for almost every $t \in (a, b)$. Thus, a comparison with (4.37) yields the result. Note that the values of t where u develops a plateau are automatically excluded by the co-area formula. The regularity result is contained in Lemma 4.34.

The fact that we obtained (4.37) for all vector fields which are tangential to Σ^n shows that M_t^n is weakly orthogonal to Σ^n. Combining the fact that E_t is almost minimal, i.e. (4.36) with the existence of a weak mean curvature in L^∞ one can argue as in [24] or [23] and apply the results of [25] to prove the regularity result of Lemma 4.34 up to the boundary of M_t^n. This implies that M_t^n meets Σ^n orthogonally in the classical sense in any neighborhood of points $x \in \partial^*_\Omega E_t \cap \Sigma^n$. □

Now we come to a geometric characterization of the jumps of the hypersurfaces which occure under the weak flow. The jumping time is controled by the property of the surface to be a strictly minimizing hull.

Definition 4.37. Let $A \subseteq \Omega$ be open in Ω. The set $E \subset \Omega$ is called a minimizing hull in A if for all sets $F \subset \Omega$ and all compact sets $K \subset A$ containing $F \setminus E$ we have

$$|\partial^*_\Omega E \cap K| \leq |\partial^*_\Omega F \cap K|, \qquad \text{for } F \supseteq E.$$

Furthermore, E is called a strictly minimizing hull in A if E is a minimizing hull in A and in addition

$$|\partial^*_\Omega E \cap K| = |\partial^*_\Omega F \cap K| \ \Rightarrow \ E \cap A = F \cap A.$$

We use this definition to define the strictly minimizing hull of a certain set.

Definition 4.38. Let $E \subseteq \Omega$ be some measurable set and let $A \subseteq \Omega$ be open. We consider the family $(E_\iota)_{\iota \in J}$ of the Lebesgue points of strictly minimizing hulls in A which contain E. Using this family we define the strictly minimizing hull of E in A as

$$E'_A := \bigcap_{\iota \in J} E_\iota.$$

Note that up to a set of measure zero E'_A may be realized by a countable intersection and therefore E'_A is a strictly minimizing hull and open (compare with [4], Definition 2.1).

Using the notion of minimizing hulls and strictly minimizing hulls we can state the following geometric properties of weak solution.

Proposition 4.39 (Minimizing hull property). *Let* $u \in C^{0,1}_{loc}(\Omega)$ *satisfy* (††). *Then*

(i) For $t > 0$, $E_t := \{u < t\}$ *is a minimizing hull in* Ω.
(ii) For $t \geq 0$, $E^+_t := int\{u \leq t\}$ *is a strictly minimizing hull in* Ω.
(iii) For $t \geq 0$, $E'_t = E^+_t$, *provided that* E^+_t *is precompact.*
(iv) For $t > 0$, $|\partial^*_\Omega E_t| = |\partial^*_\Omega E^+_t|$ *provided that* E^+_t *is precompact.*
The same holds for $t = 0$ *if and only if* E_0 *is a minimizing hull.*

Proof. (*i*) By Lemma 4.33 (††) is equivalent to (†), i.e. for $t > 0$ the sets $E_t := \{u < t\}$ minimize J_u in $\Omega \setminus E_0$. That means for $E_t \Delta F \subset\subset \Omega \setminus E_0$ and K a compact set containing $E_t \Delta F$ we have

$$|\partial^*_\Omega E_t \cap K| - \int_{E_t \cap K} |Du| \, \mathrm{d}\lambda \leq |\partial^*_\Omega F \cap K| - \int_{F \cap K} |Du| \, \mathrm{d}\lambda.$$

Now suppose that $F \supset E_t$ and $E_t \Delta F \subset\subset \Omega$. We see that $E_t \Delta F \subset\subset \Omega \setminus E_0$ and

$$|\partial^*_\Omega E_t \cap K| \ \leq \ |\partial^*_\Omega E_t \cap K| + \int_{(F \setminus E_t) \cap K} |Du| \, \mathrm{d}\lambda \ \leq \ |\partial^*_\Omega F \cap K|. \tag{4.39}$$

for those competitors F. This shows that E_t is a minimizing hull in Ω.

(*ii*) By Lemma 4.33 (††) is equivalent to $(\dagger)^+$, i.e. for $t \geq 0$ the sets $\{u \leq t\}$ minimize J_u in $\Omega \setminus E_0$. That means for $\{u \leq t\}\Delta F \subset\subset \Omega \setminus E_0$ and K a compact set containing $\{u \leq t\}\Delta F$ we have

$$|\partial^*_\Omega \{u \leq t\} \cap K| - \int_{\{u \leq t\} \cap K} |Du| \, \mathrm{d}\lambda \leq |\partial^*_\Omega F \cap K| - \int_{F \cap K} |Du| \, \mathrm{d}\lambda. \tag{4.40}$$

Since E^+_t and $\{u \leq t\}$ only differ by the set $\partial\{u \leq t\}$ we can replace $\{u \leq t\}$ by E^+_t. For F with $F \Delta E^+_t \subset\subset \Omega \setminus E_t$ we observe that

$$|\partial^*_\Omega E^+_t \cap K| \ \leq \ |\partial^*_\Omega E^+_t \cap K| + \int_{(F \setminus E^+_t) \cap K} |Du| \, \mathrm{d}\lambda \ \leq \ |\partial^*_\Omega F \cap K|. \tag{4.41}$$

In particular we can choose F such that $F \supset E_t^+$ and $E_t^+ \Delta F \subset\subset \Omega$. This shows that E_t^+ is a minimizing hull in Ω.

To prove that E_t^+ is a strictly minimizing hull we assume that $|\partial_\Omega^* E_t^+ \cap K| = |\partial_\Omega^* F \cap K|$. First we see that this assumption together with (4.41) implies that $Du = 0$ almost everywhere on $(F \setminus E_t^+) \cap K$. Furthermore, the equality tells us that F is itself a minimizing hull. Since the Lesbesgue points of a minimizing hull form an open set we can modify F on a set of measure zero and thus assume that F is open. Then $Du = 0$ almost everywhere on the open set $F \setminus \overline{E_t^+}$. Therefore u is constant on each connected component. But since F is a minimizing hull no such component can have closure disjoint from $\overline{E_t^+}$. Thus $u = t$ on $F \setminus E_t^+$ which tells us that $F \subseteq E_t^+ := \{u \leq t\}$. On the other hand $E_t^+ \subseteq F$. Thus, $E_t^+ = F$.

(iii) We see that $E_t^+ := int\{u \leq t\} \supseteq \{u < t\} =: E_t$. Furthermore, by *(ii)* E_t^+ is a strictly minimizing hull. Since E_t' is defined as the intersection (of the Lebesgue points) of all minimizing hulls which contain E_t we intersect with E_t^+ as well. This shows that $E_t' \subseteq E_t^+$. To prove the other inclusion we assume that E_t^+ is precompact and $E_t' \not\supseteq E_t^+$. Then $E_t^+ \Delta E_t' \subset\subset \Omega$ and since E_t' is a strictly minimizing hull either $|\partial_\Omega^* E_t' \cap K| = |\partial_\Omega^* E_t^+ \cap K|$ which implies $E_t' = E_t^+$ or

$$|\partial_\Omega^* E_t' \cap K| \ < \ |\partial_\Omega^* E_t^+ \cap K|$$

which contradicts (4.41) by using $F := E_t'$. Thus $E_t' \supseteq E_t^+$.

(iv) If E_t^+ is precompact then E_t is also precompact. So we can use $F := E_t^+$ as a competitor in (4.39) to obtained

$$|\partial_\Omega^* E_t \cap K| \ \leq \ |\partial_\Omega^* F \cap K| = |\partial_\Omega^* E_t^+ \cap K|$$

and we can use $F := E_t$ as a competitor in (4.40) to obtain

$$|\partial_\Omega^* E_t^+ \cap K| \ \leq \ |\partial_\Omega^* F \cap K| = |\partial_\Omega^* E_t \cap K|.$$

This implies the statement for $t > 0$ and for $t = 0$ if E_0 is a minimizing hull. ☐

Remark 4.40. Note that E_t minimizes J_u in $\Omega \setminus E_0$ for all $t \geq 0$ if and only if the same holds for $t > 0$ (that is (†) holds), E_0 is a minimizing hull and E_0^+ is precompact. This follows from Proposition 4.39 (iv) which shows that

$$\begin{aligned} J_u(E_0) &= |\partial_\Omega^* E_0 \cap K| - \int_{E_0 \cap K} |Du| \, \mathrm{d}\lambda \overset{(iv)}{=} |\partial_\Omega^* E_0^+ \cap K| - \int_{\{u<0\} \cap K} |Du| \, \mathrm{d}\lambda \\ &= |\partial_\Omega^* \{u \leq 0\} \cap K| - \int_{\{u \leq 0\} \cap K} |Du| \, \mathrm{d}\lambda = J_u(\{u \leq 0\}) \end{aligned}$$

and $(\dagger)^+$ which states that $\{u \leq 0\}$ minimizes J_u in $\Omega \setminus E_0$.

As for the classical flow the rescaled surface area is constant.

Lemma 4.41 (Exponential growth Lemma). *Let $(E_t)_{t>0}$ solve (†) with initial condition E_0. As long as E_t remains precompact, we have*

$$|\partial_\Omega^* E_t| = ce^t, \qquad c \in \mathbb{R}, \quad t > 0 \tag{4.42}$$

If E_0 is a minimizing hull, then $c = |\partial_\Omega^* E_0|$.

Proof. Assume that E_t solves (†) and remains precompact for all $t > 0$. Then we can use E_{t_1} as a competitor for E_t in J_u. This shows that for $t > 0$ and E_t precompact the value of $J_u(E_t)$ is independent of t. Therefore, the co-area formula yields

$$\begin{aligned}J_u(E_t) &= |\partial^*_\Omega E_t \cap K| - \int_{E_t\cap K} |Du| \,\mathrm{d}\lambda = |\partial^*_\Omega E_t \cap K| - \int_0^t \int_{\partial^*_\Omega E_s \cap K} \mathrm{d}\mathcal{H}^n \,\mathrm{d}s \\ &= |\partial^*_\Omega E_t \cap K| - \int_0^t |\partial^*_\Omega E_s \cap K| \,\mathrm{d}s = c \in \mathbb{R}, \qquad \text{for } t > 0.\end{aligned}$$

For K containing E_T this implies (4.42) for $t \in (0, T]$. Since K can be taken arbitrary large (4.42) holds for all $t > 0$. If E_0 is a minimizing hull then the Remark 4.40 implies that E_t minimizes J_u for all $t \geq 0$. Thus, we can evaluate (4.42) at $t = 0$ which gives $c = |\partial^*_\Omega E_0|$. ☐

The next Proposition tells us that the limit of a converging sequence of weak solutions is itself a weak solution.

Proposition 4.42 (Compactness of weak solutions). *Let* $(A_i)_{i\in\mathbb{N}}, A \subset \Omega$ *be open in* Ω. *Let* $(u_i)_{i\in\mathbb{N}} \subset C^{0,1}_{loc}(A_i)$ *be a sequence of weak solutions to* $(\star)$ *such that*

$$A_i \longrightarrow A, \qquad u_i \longrightarrow u \in C^{0,1}_{loc}(A)$$

locally uniformly for $i \to \infty$. *If for each compact set* $K \subset A$ *and* i *large enough*

$$\operatorname*{ess\,sup}_K |Du_i| \leq C(K).$$

Then u *is a weak solution of* $(\star)$ *in* A.

Proof. We have to proof that $J_u(u) \leq J_u(v)$ for $\{u \neq v\} \subset\subset A$. We will prove this statement for $v < u + 2^k$ by induction with respect to k and start with $k = 0$, i.e. $v < u + 1$. We consider a cutoff function $\Phi \in C^1_c(A, [0, 1])$ such that $\Phi = 1$ on $\{u \neq v\}$ and define

$$v_i := \Phi v + (1 - \Phi)u_i.$$

Since u_i is a weak solution to $(\star)$ in A_i we deduce that

$$\begin{aligned}&\int_U \Big(|Du_i| + u_i|Du_i|\Big) \,\mathrm{d}\lambda \leq \int_U \Big(|Dv_i| + v_i|Du_i|\Big) \,\mathrm{d}\lambda \\ &= \int_U \Big(|\Phi Dv + (1 - \Phi)Du_i + D\Phi(v - u_i)| + (\Phi v + (1 - \Phi)u_i)|Du_i|\Big) \,\mathrm{d}\lambda\end{aligned}$$

for appropriate U. This implies

$$\int_U \Phi|Du_i|(1 + u_i - v) \,\mathrm{d}\lambda \leq \int_U \Phi|Dv| \,\mathrm{d}\lambda + \sup_U |v - u_i| \int_U |D\Phi| \,\mathrm{d}\lambda.$$

The last term converges to zero as i tends to infinity. By assumption $1 + u_i - v$ is positive for i sufficiently large. Therefore, the lower semicontinuity of J_u implies

$$\int_U \Phi|Du|(1 + u - v) \,\mathrm{d}\lambda \leq \liminf_{i\to\infty} \int_U \Phi|Du_i|(1 + u_i - v) \,\mathrm{d}\lambda \leq \int_U \Phi|Dv| \,\mathrm{d}\lambda.$$

This yields $J_u(u) \leq J_u(v)$ for $k = 0$. Now we assume that the inequality holds for all $w < u + 2^k$ and we have to show that this implies the inequality for all $v < u + 2^{k+1}$. For such a v and for $\eta > 0$ we define

$$v_1 := \min\{v, u + 2^k - \eta\}, \qquad v_2 := \max\{v - 2^k, u\}.$$

Obviously $v_1 < u + 2^k$. Thus $J_u(u) \leq J_u(v_1)$, i.e.

$$\begin{aligned}&\int_U \Big(|Du| + u|Du|\Big)\,\mathrm{d}\lambda \leq \int_U \Big(|Dv_1| + v_1|Du|\Big)\,\mathrm{d}\lambda \\ &= \int_{U\cap\{v\leq u+2^k-\eta\}} \Big(|Dv| + v|Du|\Big)\,\mathrm{d}\lambda + \int_{U\cap\{v>u+2^k-\eta\}} \Big(|Du| + (u+2^k)|Du|\Big)\,\mathrm{d}\lambda.\end{aligned}$$

Since $v < u + 2^{k+1}$ also $v_2 < u + 2^k$ and as thus $J_u(u) \leq J_u(v_2)$, i.e.

$$\begin{aligned}&\int_U \Big(|Du| + u|Du|\Big)\,\mathrm{d}\lambda \leq \int_U \Big(|Dv_2| + v_2|Du|\Big)\,\mathrm{d}\lambda \\ &= \int_{U\cap\{v\leq u+2^k\}} \Big(|Du| + u|Du|\Big)\,\mathrm{d}\lambda + \int_{U\cap\{v>u+2^k\}} \Big(|Dv| + (v-2^k)|Du|\Big)\,\mathrm{d}\lambda.\end{aligned}$$

Adding these two inequalities and taking the limits $\eta \to 0$ yields $2J_u(u) \leq J_u(v) + J_u(u)$ and therefore the desired result. □

The next Lemma shows that we can not expect to obtain a unique weak solution in general.

Lemma 4.43. *Let $u \in C^{0,1}_{loc}(\Omega)$ satisfy (††). Then, for every $t > 0$ the function $\widehat{u}(x) := \min(u(x), t)$ satisfies (††) as well.*

Proof. Using Lemma 4.33 we have to show that $\widehat{E}_s := \{\widehat{u} < s\}$ minimizes $J_{\widehat{u}}$ in $\Omega \setminus E_0$ for all $s > 0$. Let F have locally finite perimeter and suppose that $\hat{E}_s \Delta F \subset\subset \Omega \setminus E_0$. For $0 < s \leq t$ we use the fact that u is a solution of (†) to obtain

$$\begin{aligned}J_{\widehat{u}}\left(\widehat{E_s}\right) &= \left|\partial^*_\Omega \widehat{E_s} \cap K\right| - \int_{\widehat{E_s}\cap K} |D\widehat{u}|\,\mathrm{d}\lambda \;=\; |\partial^*_\Omega E_s \cap K| - \int_{E_s\cap K} |Du|\,\mathrm{d}\lambda \\ &\leq |\partial^*_\Omega F \cap K| - \int_{F\cap K} |Du|\,\mathrm{d}\lambda \;\leq\; |\partial^*_\Omega F \cap K| - \int_{(F\cap E_t)\cap K} |Du|\,\mathrm{d}\lambda \\ &= |\partial^*_\Omega F \cap K| - \int_{F\cap K} |D\widehat{u}|\,\mathrm{d}\lambda \;=\; J_{\widehat{u}}(F)\,.\end{aligned}$$

For $s > t$ we have

$$\begin{aligned}J_{\widehat{u}}\left(\widehat{E_s}\right) \;&=\; J_{\widehat{u}}(\Omega) \;=\; |\partial^*_\Omega \Omega \cap K| - \int_{\Omega\cap K} |D\widehat{u}|\,\mathrm{d}\lambda \\ &= 0 - \int_K |D\widehat{u}|\,\mathrm{d}\lambda \;\leq\; |\partial^*_\Omega F \cap K| - \int_{K\cap F} |D\widehat{u}|\,\mathrm{d}\lambda \;=\; J_{\widehat{u}}(F)\,.\end{aligned}$$

Therefore the inequality holds for all $s > 0$. □

Proposition 4.44 (Uniqueness of weak solutions). *Let $A \subset \Omega$ be open in Ω.*

(i) Let $u, v \in C^{0,1}_{loc}(A)$ be weak solutions of $(\star)$ in A and $\{v > u\} \subset\subset A$. Then $v \le u$ on A.

(ii) If $(E_t)_{t>0}$ and $(F_t)_{t>0}$ satisfy $(\dagger)$ in Ω and the initial conditions satisfy $E_0 \subseteq F_0 \subset \Omega$. Then $E_t \subseteq F_t$ as long as E_t is precompact.

(iii) For a given $E_0 \subset \Omega$, there exists at most one solution $(E_t)_{t>0} \subset \Omega$ of $(\dagger)$ such that each E_t is precompact.

Proof. (i) We will prove the statement in two steps. First we assume that u is a strict weak supersolution. At the end we will discuss the general case. So, let u be a strict weak supersolution in the sense that for $w \in C^{0,1}_{loc}(\Omega)$ with $\{u \neq w\} \subset\subset \Omega$ there exists some $\varepsilon > 0$ such that

$$J_u(u) + \varepsilon \int_K |Du|(w-u)\,\mathrm{d}\lambda \le J_u(w), \qquad \{u \neq w\} \subset K.$$

As a competitor we use $w := u + (v-u)_+$ and since w only differs from u on $\{v > u\}$ where $w = v$ we obtain

$$\begin{aligned}\int_{\{v>u\}} \Big(|Du| + u|Du|\Big)\,\mathrm{d}\lambda &+ \varepsilon \int_{\{v>u\}} |Du|(v-u)\,\mathrm{d}\lambda \\ &\le \int_{\{v>u\}} \Big(|Dv| + v|Du|\Big)\,\mathrm{d}\lambda. \qquad (4.43)\end{aligned}$$

By assumption v is also a subsolution. Thus, $J_v(v) \le J_v(w)$ and this time we choose $w := v - (v-u)_+$. Again the subsolution and the competitor w only differ on the set $\{v > u\}$ where this time $w = u$. This yields

$$\int_{\{v>u\}} \Big(|Dv| + v|Dv|\Big)\,\mathrm{d}\lambda \le \int_{\{v>u\}} \Big(|Du| + u|Dv|\Big)\,\mathrm{d}\lambda. \qquad (4.44)$$

Adding (4.43) and (4.44) we get

$$\int_{\{v>u\}} (v-u)(|Dv| - |Du|)\,\mathrm{d}\lambda + \varepsilon \int_{\{v>u\}} |Du|(v-u)\,\mathrm{d}\lambda \le 0. \qquad (4.45)$$

Now we make use of the minimizing property of u once more, i.e. $J_u(u) \le J_u(w_s)$ where we choose $w_s := u + (v - s - u)_+$ for $s > 0$. The subsolution and the competitor differ on the set $\{v - s > u\}$ where $w_s = v - s$. Addidional integration over s yields

$$\int_0^\infty \int_{\{v-s>u\}} \Big(|Du| + u|Du|\Big)\,\mathrm{d}\lambda\,\mathrm{d}s \le \int_0^\infty \int_{\{v-s>u\}} \Big(|Dv| + (v-s)|Du|\Big)\,\mathrm{d}\lambda\,\mathrm{d}s.$$

Changing the order of integration, we have

$$\int_\Omega |Du| \int_{s=0}^{v-u} \Big(1 + u - v + s\Big)\,\mathrm{d}s\,\mathrm{d}\lambda \le \int_\Omega |Dv| \int_{s=0}^{v-u} \mathrm{d}s\,\mathrm{d}\lambda$$

which is the same as

$$\int_{\{v>u\}} |Du| \left((1 + u - v)(v-u) + \frac{(v-u)^2}{2} \right) \mathrm{d}\lambda \le \int_{\{v>u\}} (v-u)|Dv|\,\mathrm{d}\lambda$$

and thus

$$\int_{\{v>u\}} -|Du|\frac{(v-u)^2}{2}\,\mathrm{d}\lambda \leq \int_{\{v>u\}} (v-u)\big(|Dv|-|Du|\big)\,\mathrm{d}\lambda.$$

Together with (4.45) we obtain

$$\int_{\{v>u\}} |Du|\left(-\frac{(v-u)^2}{2}+\varepsilon(v-u)\right)\mathrm{d}\lambda \leq 0.$$

Without loss of generality we may assume that $v \leq u+\varepsilon$ since otherwise we substract a constant from v to arrange that $0 < \sup(v-u) \leq \varepsilon$. Then $v \leq u+\varepsilon$ implies $|Du| = 0$ almost everywhere on $\{v > u\}$. Using this information together with inequality (4.44) we see that

$$\int_{\{v>u\}} |Dv|(1+v-u)\,\mathrm{d}\lambda \leq 0$$

and therefore also $|Dv| = 0$ almost everywhere on $\{v > u\}$. This shows that u and v are constant on each component of $\{v > u\}$ and since $\{v > u\}$ is precompact and Ω has no compact components we can conclude that $u = c_1$, $v = c_2$ on $\{v > u\}$. Thus, $\varepsilon \geq v-u = c_2-c_1$ for arbitrary small ε. Taking $\varepsilon := (c_2-c_1)/2$ causes a contradiction unless $v \leq u$. This proves the statement for strict weak subsolutions.

For an arbitrary weak supersolution u we reduce the problem to the first step by defining

$$u^\varepsilon := \frac{u}{1-\varepsilon}$$

which is a strict weak supersolution and $\{v > u^\varepsilon\}$ is precompact. By the previous argument we have $v \leq u^\varepsilon$ and thus $v \leq u$ in the limit as $\varepsilon \to 0$.

(*ii*) Let u and v be the level-set functions of $(E_t)_{t>0}$ and $(F_t)_{t>0}$, i.e.

$$E_t = \{u < t\}, \qquad F_t = \{v < t\}.$$

By Lemma 4.43 we know that $v^t := \min(v,t)$ minimizes J_u in $\Omega \setminus \overline{F_0}$. We define the set $W := E_t \setminus \overline{F_0}$. Since $E_0 \subseteq F_0$ the set W has the boundary parts $\partial_\Sigma W$ and $\partial_\Omega W = A \cup B$ where

$$A := \partial_\Omega W \cap \partial_\Omega F_0, \qquad B := \partial_\Omega W \cap \partial_\Omega E_t.$$

We observe that for all $\delta > 0$

$$v^t = v = 0 < u+\delta \quad \text{on } A, \qquad v^t \leq t = u < u+\delta \quad \text{on } B$$

and thus $v^t < u+\delta$ near $\partial_\Omega W$. Therefore, $\{v^t > u+\delta\} \subset\subset W$ precompact and (*i*) implies $v^t \leq u+\delta$ on W. Taking the limits $\delta \to 0$ yields $v^t \leq u$ on W and since $u < t$ on W we see that $v \leq u$ on W, i.e. $E_t \subseteq F_t$.

(*iii*) Assume there are two precompact families $(A_t)_{t>0}, (B_t)_{t>0} \subset \Omega$ solving (†) with initial condition E_0, i.e. $A_0 = E_0 = B_0$. Then, by (*ii*) $A_t \subseteq B_t$ and $B_t \subseteq A_t$ for all $t \geq 0$. □

The next proposition shows that smooth solutions are weak solutions.

Proposition 4.45 (**Classical $\Rightarrow$ weak**). *Let $(N_t)_{c\le t\le d}\subset\Omega$ be a family of compact surfaces of positive mean curvature that solve* (IMCF) *classically. Let $u=t$ on N_t, $u<c$ in the region bounded by N_c, and $E_t:=\{u<t\}\subset\Omega$. Then for $c<t<d$, E_t minimizes J_u in $E_d\setminus\overline{E_c}$.*

Proof. Let $t\in(c,d)$. We have to show that $E_t:=\{u<t\}$ minimizes J_u in $E_d\setminus\overline{E_c}$, i.e.

$$|\partial^*_\Omega E_t\cap K|-\int_{E_t\cap K}|Du|\,\mathrm{d}\lambda\le|\partial^*_\Omega F\cap K|-\int_{F\cap K}|Du|\,\mathrm{d}\lambda \tag{4.46}$$

for all F having locally finite perimeter and satisfying $\overline{E_t\Delta F}\subset\subset E_d\setminus\overline{E_c}$. We choose $r,s\in\mathbb{R}$ such that $c<r<t<s<d$ and use $K:=\overline{E_s\setminus E_r}$. Then inequality (4.46) reads

$$|\partial^*_\Omega E_t|-\int_{E_t\setminus E_r}|Du|\,\mathrm{d}\lambda\le|\partial^*_\Omega F|-\int_{F\setminus E_r}|Du|\,\mathrm{d}\lambda.$$

Let us consider the vector field $X:=Du/|Du|$ which is C^1 away from $\partial_\Omega E_c\cap\partial_\Sigma E_c$ and $\partial_\Omega E_d\cap\partial_\Sigma E_d$. The divergence theorem and the fact that u is a solution of $(\star)$ yield

$$\int_{\partial A}\nu_{\partial A}\cdot X\,\mathrm{d}s=\int_A\operatorname{div}(X)\,\mathrm{d}\lambda=\int_A|Du|\,\mathrm{d}\lambda. \tag{4.47}$$

Furthermore, for any set $A\subset\Omega$ we have

$$\int_{\partial_\Sigma A}\nu_{\partial_\Sigma A}\cdot X\,\mathrm{d}s=\int_{\partial_\Sigma A}\mu\cdot\frac{Du}{|Du|}\,\mathrm{d}s=0. \tag{4.48}$$

These two equalities help us to calculate

$$\begin{aligned}
&|\partial^*_\Omega E_t|-\int_{E_t\setminus E_r}|Du|\,\mathrm{d}\lambda\\
&=\int_{\partial^*_\Omega E_t}\nu_{\partial^*_\Omega E_t}\cdot X\,\mathrm{d}s-\int_{E_t\setminus E_r}|Du|\,\mathrm{d}\lambda\\
&\overset{(4.48)}{=}\int_{\partial^*(E_t\setminus E_r)}\nu_{\partial^*_\Omega(E_t\setminus E_r)}\cdot X\,\mathrm{d}s-\int_{E_t\setminus E_r}|Du|\,\mathrm{d}\lambda-\int_{\partial^*_\Omega E_r}\nu_{\partial^*_\Omega E_r}\cdot X\,\mathrm{d}s\\
&\overset{(4.47)}{=}-\int_{\partial^*_\Omega E_r}\nu_{\partial^*_\Omega E_r}\cdot X\,\mathrm{d}s\\
&\overset{(4.47)}{=}\int_{\partial^*(F\setminus E_r)}\nu_{\partial^*(F\setminus E_r)}\cdot X\,\mathrm{d}s-\int_{F\setminus E_r}|Du|\,\mathrm{d}\lambda-\int_{\partial^*_\Omega E_r}\nu_{\partial^*_\Omega E_r}\cdot X\,\mathrm{d}s\\
&\overset{(4.48)}{\le}\int_{\partial^*_\Omega F}\nu_{\partial^*_\Omega F}\cdot X\,\mathrm{d}s-\int_{F\setminus E_r}|Du|\,\mathrm{d}\lambda\\
&\le|\partial^*_\Omega F|-\int_{F\setminus E_r}|Du|\,\mathrm{d}\lambda.
\end{aligned}$$

This shows that E_t minimizes J_u in $E_d\setminus\overline{E_c}$. □

Now we are able to prove that the limit u which was obtained in the previous section is a weak solution of $(\star)$ in Ω_0.

Proposition 4.46 (Criterion for Existence). *Let* $(u_i)_{i\in\mathbb{N}} \subset H_{2,\alpha}^{(-1-\beta)}(\Omega_{\varepsilon_i})$ *be a sequence of classical solutions of* $(\star)_{\varepsilon_i, L_{\varepsilon_i}}$ *with*

$$F_{L_{\varepsilon_i}} \setminus E_{0,\varepsilon_i} \longrightarrow \Omega \setminus E_0, \qquad u_i \to u \in C^{0,1}_{loc}(\Omega \setminus E_0)$$

locally uniformly for $i \to \infty$. *If for each compact set* $K \subset \Omega \setminus E_0$ *and* i *large enough*

$$\sup_K |Du_i| \leq C(K).$$

Then u *is a weak solution of* $(\star)$ *in* $\Omega_0 := \Omega \setminus \overline{E_0}$ *with initial condition* E_0.

Proof. Note that $\overline{\Omega_{\varepsilon_i}} = \overline{F_{L_{\varepsilon_i}}} \setminus E_{0,\varepsilon_i}$. We define

$$U_i : \overline{\Omega_{\varepsilon_i}} \times \mathbb{R} \to \mathbb{R} : (x,z) \mapsto U_i(x,z) := u_i(x) - \varepsilon_i z,$$

$$U : (\Omega \setminus E_0) \times \mathbb{R} \to \mathbb{R} : (x,z) \mapsto U(x,z) := u(x).$$

Then $U_i \to U$ locally uniformly in $(\Omega \setminus E_0) \times \mathbb{R}$. For fixed $i \in \mathbb{N}$ we consider the sets

$$\begin{aligned} M_t^i &:= \left\{ (x,z) \in \overline{\Omega_{\varepsilon_i}} \times \mathbb{R} \;\middle|\; U_i(x,z) = t \right\} \\ &= \left\{ (x,z) \in \overline{\Omega_{\varepsilon_i}} \times \mathbb{R} \;\middle|\; z = \frac{u_i}{\varepsilon_i} - \frac{t}{\varepsilon_i} \right\} = \operatorname{graph}\left(\frac{u_i}{\varepsilon_i} - \frac{t}{\varepsilon_i} \right). \end{aligned}$$

To see that these graphs are classical solutions to invers mean curvature flow one dimension higher we can argue that

$$\operatorname{div}_{\mathbb{R}^{n+2}} \left(\frac{DU_i}{|DU_i|} \right) = \operatorname{div}_{\mathbb{R}^{n+1}} \left(\frac{Du_i}{\sqrt{\varepsilon_i^2 + |Du_i|^2}} \right) = \sqrt{\varepsilon_i^2 + |Du_i|^2} = |DU_i|$$

which is equivalent to the classical formulation of inverse mean curvature flow since $|DU_i| = H > 0$. The Neumann condition is satisfied as well since the normal to $\Sigma^n \times \mathbb{R}$ is given by $\widehat{\mu} = (\mu, 0)$ where μ is the unit normal to Σ^n. This yields

$$D_{\widehat{\mu}} U_i = \left\langle \begin{pmatrix} \mu \\ 0 \end{pmatrix}, \begin{pmatrix} Du_i \\ -\varepsilon_i \end{pmatrix} \right\rangle = D_\mu u_i = 0.$$

on $\partial_\Sigma \Omega_{\varepsilon_i} \times \mathbb{R}$. Another way to verify the PDE is to compute the speed of the graphs in normal direction, i.e.

$$\left\langle -\frac{1}{\varepsilon_i} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{\varepsilon_i^2 + |Du_i|^2}} \begin{pmatrix} Du_i \\ -\varepsilon_i \end{pmatrix} \right\rangle = \frac{1}{\sqrt{\varepsilon_i^2 + |Du_i|^2}} = \frac{1}{H}$$

where we used that the speed in z-direction is $-\varepsilon^{-1}$. Also for the verification of the Neumann condition we can use the graph setting. There the calculation reads

$$\left\langle \widehat{\mu}, \nu(M_t^i) \right\rangle = \left\langle \begin{pmatrix} \mu \\ 0 \end{pmatrix}, \frac{1}{\sqrt{\varepsilon_i^2 + |Du_i|^2}} \begin{pmatrix} Du_i \\ -\varepsilon_i \end{pmatrix} \right\rangle = \frac{D_\mu u_i}{\sqrt{\varepsilon_i^2 + |Du_i|^2}} = 0$$

on $\partial_\Sigma \Omega_{\varepsilon_i} \times \mathbb{R}$. Altogether, Proposition 4.45 implies that U_i is a weak solution in $(F_{L_{\varepsilon_i}} \setminus \overline{E_{0,\varepsilon_i}}) \times \mathbb{R}$ and thus the compactness result, Proposition 4.42 tells us that U is a weak solution in $(\Omega \setminus \overline{E_0}) \times \mathbb{R}$. To deduce that u is a weak solution in $\Omega_0 := \Omega \setminus \overline{E_0}$ we use the following cutoff functions

$$\Phi_s : \mathbb{R} \to \mathbb{R} : z \mapsto \Phi_s(z) := \begin{cases} 1 & \text{for } z \in [0, s] \\ \Phi(s) & \text{for } z \in [-1, 0] \\ \Phi(s - z) & \text{for } z \in [s, s+1] \\ 0 & \text{for } z \in \mathbb{R} \setminus [-1, s+1] \end{cases}$$

where Φ is chosen such that $\Phi_s \in C^1(\mathbb{R})$ with $\Phi_s(z) \in [0,1]$ and $|\Phi_s'(z)| \le 2$ for all $z \in \mathbb{R}$. As competitor to $U(x,z) = u(x)$ we use

$$V : \Omega_0 \times \mathbb{R} : (x,z) \mapsto V(x,z) := \Phi_s(z)v(x) + (1 - \Phi_s(z))u(x)$$

where $v \in L^{0,1}_{loc}(\Omega_0)$ with $\{u \neq v\} \subset K$ and K a compact subset of Ω_0. We compute that

$$\begin{aligned} |D_{x,z}V| &= \left(\sum_{i=1}^n |D_{x_i}V|^2 + |D_z V|^2 \right)^{1/2} \\ &= \left(\sum_{i=1}^n |\Phi_s D_{x_i} v + (1 - \Phi_s) D_{x_i} u|^2 + |\Phi_s'|^2 \cdot |v - u|^2 \right)^{1/2} \\ &\le \Phi_s |D_x v| + (1 - \Phi_s)|D_x u| + |\Phi_s'||v - u|. \end{aligned}$$

Since $\{U \neq V\} \subset K \times [-1, s+1] \subset\subset \Omega_0 \times \mathbb{R}$ we use $J_U(U) \le J_U(V)$ to obtain

$$\begin{aligned} &\int_{K \times [-1,s+1]} \Big(|D_x u| + u|D_x u| \Big) \, \mathrm{d}\lambda(x,z) \\ &\le \int_{K \times [-1,s+1]} \Big(\Phi_s |D_x v| + (1 - \Phi_s)|D_x u| \\ &\qquad + |\Phi_s'||v - u| + \Phi_s v |D_x u| + (1 - \Phi_s) u |D_x u| \Big) \, \mathrm{d}\lambda(x,z). \end{aligned} \tag{4.49}$$

This implies

$$\begin{aligned} sJ_u(u) &= s \int_K \Big(|D_x u| + u|D_x u| \Big) \, \mathrm{d}\lambda(x) \\ &\le \int_{K \times [-1,s+1]} \Phi_s \Big(|D_x u| + u|D_x u| \Big) \, \mathrm{d}\lambda(x,z) \\ &\overset{(4.49)}{\le} \int_{K \times [-1,s+1]} \Phi_s \Big(|D_x v| + v|D_x u| \Big) \, \mathrm{d}\lambda(x,z) + \int_{K \times [-1,s+1]} |\Phi_s'||v - u| \, \mathrm{d}\lambda(x,z) \\ &\le (s+2) \int_K \Big(|D_x v| + v|D_x u| \Big) \, \mathrm{d}\lambda(x) + \int_{K \times ([-1,0] \cup [1,2])} |\Phi_s'||v - u| \, \mathrm{d}\lambda(x,z) \end{aligned}$$

$$\leq (s+2)J_u(v) + 4\int_K |v-u|\,\mathrm{d}\lambda(x).$$

Dividing by s and passing $s \to \infty$ proves that $J_u(u) \leq J_u(v)$. Finally, we extend u negatively to E_0 in order to satisfy $E_0 = \{u < 0\}$. ☐

We can summarize our existence result and the properties of weak solutions by stating our main theorem.

Theorem 4.47 (Existence and uniqueness of weak solutions). *Let $E_0, E_{0,\varepsilon}, F_{L_\varepsilon}$ and $(\star)_{\varepsilon,\tau}$ be as in Definitions 4.5 and 4.6. Let Σ^n be a $C^{2,\alpha}$-hypersurface. Assume that for sufficiently small $\varepsilon > 0$ admissible subsolutions v^- of $(\star)_{\varepsilon,L_\varepsilon}$ exist such that $F_{L_\varepsilon} = \{v^- < L_\varepsilon\}$ and $L_\varepsilon \to \infty$. Furthermore, assume that any admissible solution $u^{\varepsilon,\tau}$ of $(\star)_{\varepsilon,\tau}$ satisfies $u^{\varepsilon,\tau} \geq -\varepsilon^{1+\gamma}$ for some $\gamma \in (0,1)$ and that $|Du^{\varepsilon,\tau}|_{\Sigma_\varepsilon}$ can be controlled independently of ε. Then there exists a weak solution $u \in C^{0,1}_{loc}(\Omega)$ of $(\star)$ with initial condition E_0 such that for all $t > 0$ the set $E_t := \{u < t\}$ is the unique precompact minimizer of J_u in $\Omega\backslash E_0$. Up to a set of dimension less than or equal to $n-8$, $M^n_t := \partial_\Omega E_t$ is a $C^{1,\frac{1}{2}}$-hypersurface which possesses a weak mean curvature in L^∞ given by*

$$H(M^n_t) = |Du| \geq 0 \qquad \text{for a.e. } t \in \mathbb{R}_+ \text{ and a.e. } x \in M^n_t$$

and for those values of t, ∂M^n_t is orthogonal to Σ^n in the classical sense in any neighborhood of points $x \in \partial^*_\Omega E_t \cap \Sigma^n$. Furthermore, E_t is a minimizing hull and the strictly minimizing hull of E_t is given by $E'_t = int\{u \leq t\}$ as long as $int\{u \leq t\}$ is precompact. In this case we have

$$|\partial^*_\Omega E'_t| = |\partial^*_\Omega E_t| = ce^t$$

If E_0 is a minimizing hull then $c = |\partial_\Omega E_0|$.

In particular, Theorem 4.47 holds in the following situation:

Corollary 4.48. *Let $n \geq 2$. Let $E_0, E_{0,\varepsilon}, F_{L_\varepsilon}$ and $(\star)_{\varepsilon,\tau}$ be as in Definitions 4.5 and 4.6. Let Σ^n be given as the graph of a convex C^3-function which is asymptotic to a cone in the sense that* (4.4) *holds. Then the conditions of Theorem 4.47 are satisfied.*

Proof. Under these assumptions a subsolution v^- can be constructed using Lemma 4.9 where F_{L_ε} and L_ε are chosen as in (4.7). The special lower bound for u follows from Lemma 4.10. Furthermore, the gradient estimate on Σ_ε is independent of ε since Σ^n is convex. This was shown in Lemma 4.17. Thus, all conditions of Theorem 4.47 are satisfied. ☐

Proof of Theorem 4.47. Theorem 4.21 provides a sequence of unique solutions $(u_i)_{i\in\mathbb{N}} \subset H^{(-1-\beta)}_{2,\alpha}(\Omega_i)$ of $(\star)_{\varepsilon_i,L_{\varepsilon_i}}$ which converges locally uniformly to a function $u \in C^{0,1}_{loc}(\Omega \setminus E_0)$. Proposition 4.46 tells us that u is a weak solution of $(\star)$ in $\Omega_0 := \Omega \setminus \overline{E_0}$ with initial condition E_0. Proposition 4.44 implies that u is the unique solution as long as E_t is precompact. The formula for the weak mean curvature and the orthogonality follow from Lemma 4.36. The minimizing hull property and the characterization of E'_t were proven in Proposition 4.39. Finally, the exponential growth of the surface area and the value of c are due to Lemma 4.41. ☐

4.5 Outlook : Monotonicity of the Hawking mass

The classical IMCF for closed surfaces was put forward by Geroch [20] and Jang and Wald [33] in the seventies as an approach to the proof of the positive mass theorem. The positive mass theorem states that the so-called ADM-mass m_{ADM} for an asymptotically flat[2] 3-manifold is non-negative. This concept of mass was developed by Arnowitt, Deser and Misner in [3]. Geroch showed that as long as IMCF remains smooth it can be used to prove the Riemannian Penrose inequality and therefore, the positive mass theorem. The Riemannian Penrose inequality states that an asymptotically flat, complete, connected 3-manifold with non-negative scalar curvature, with one (to keep things simple here) compact minimal surface M_0^2 as its compact boundary, satisfies the inequality

$$m_{\mathrm{ADM}} \geq \sqrt{\frac{|M_0^2|}{16\pi}}.$$

In a nut shell Geroch's argument was the following. He combined Hawking's observation that the so called Hawking quasi-local mass

$$m_{\mathrm{Haw}}(M^2) := \frac{|M^2|^{1/2}}{(16\pi)^{3/2}}\left(16\pi - \int_{M^2} H^2 \,\mathrm{d}\mu\right)$$

calculated for $m_{\mathrm{Haw}}(M_t^2)$ converges to m_{ADM} if the surfaces M_t^2 converge to a sphere at infinity with his observation that $m_{\mathrm{Haw}}(M_t^2)$ is monotone increasing in t for smooth solutions of IMCF. Thus if the initial hypersurface for IMCF is the minimal surface M_0^2, $m_{\mathrm{Haw}}(M_t^2) \to m_{\mathrm{ADM}}$ and the flow remains smooth one obtains

$$\sqrt{\frac{|M_0^2|}{16\pi}} \;=\; m_{\mathrm{Haw}}(M_0^2) \;\leq\; m_{\mathrm{Haw}}(M_t^2) \;\to\; m_{\mathrm{ADM}}$$

if the surfaces M_t^2 become round in the limit.

Remark 4.49. Note that the flow does not remain smooth in general. Therefore, a key ingredient in the proof of the Riemannian Penrose inequality by Huisken and Ilmanen [29] was to develop a weak formulation for inverse mean curvature flow which exists for all time and keeps $m_{\mathrm{Haw}}(M_t^2)$ monotone.

Now we want to understand what kind of results we can expect in our case where the hypersurfaces possess a boundary. Therefore, we assume that the flow remains smooth and investigate under which conditions the Hawking mass is monotone. We will need the following lemma.

Lemma 4.50. *Let $M^2, \Sigma^2 \subset \mathbb{R}^3$ be orientable $C^{2,\alpha}$-surfaces and μ be the unit normal to Σ^2 pointing away from M^2. Assume that M^2 has a boundary which is a subset of Σ^2 such that M^2 touches Σ^2 orthogonally. Let*

$$\gamma : I \to M^2 \cap \Sigma^2 : s \mapsto \gamma(s), \qquad \dot\gamma := \frac{\mathrm{d}\gamma}{\mathrm{d}s}, \qquad |\dot\gamma| = 1. \tag{4.50}$$

Then the geodesic curvature of ∂M^2 in M^2 is given by $k_g = {}^{\Sigma^2}h_{\dot\gamma\dot\gamma}$ with $\dot\gamma \in T\Sigma^2 \cap TM^2$.

---

[2] Roughly speaking a manifold $M = C \cup D$ is asymptotically flat if C is compact and D is diffeomorphic to $\mathbb{R}^n \setminus K$ for some compact set K. See e.g. [29] for an exact definition.

Proof. Let γ satisfy (4.50). The geodesic curvature k_g of the boundary curve γ bounding the region M^2 is

$$k_g := \Big\langle D_{\dot\gamma}\dot\gamma, \eta \Big\rangle.$$

where $\eta \in TM^2 \cap N\partial M^2$ is the normal to ∂M^2 pointing towards M^2. Since M^2 touches Σ^2 orthogonally we have $\eta = -\mu$. Furthermore, $0 = \langle \dot\gamma, -\mu \rangle$ which implies

$$0 = \Big\langle D_{\dot\gamma}\dot\gamma, -\mu \Big\rangle - \Big\langle \dot\gamma, D_{\dot\gamma}\mu \Big\rangle.$$

This yields

$$k_g = \Big\langle D_{\dot\gamma}\dot\gamma, -\mu \Big\rangle = \Big\langle \dot\gamma, D_{\dot\gamma}\mu \Big\rangle = {}^{\Sigma^2}h_{\dot\gamma\dot\gamma}$$

which is the desired result. □

Proposition 4.51 (Monotonicity of $\mathbf{m}_{\mathrm{Haw}}$ - smooth case). *Let $\Sigma^2, M_0^n \subset M^2$ be orientable $C^{2,\alpha}$-surfaces such that M_0^2 touches Σ^2 orthogonally. Let $(M_t^2)_{t\in\mathbb{R}_+} \subset \mathbb{R}^3$ be a family of smooth, connected solutions to* (IMCF) *which exist for all time. If Σ^2 is mean-convex, i.e. $H(\Sigma^2) \geq 0$, then the Hawking mass*

$$\widetilde{m}_{\mathrm{Haw}}(M^2) := \frac{|M^2|^{1/2}}{(8\pi)^{3/2}}\left(8\pi - \int_{M^2} H^2 \,\mathrm{d}\mu\right)$$

is monotone in t.

Proof. Remember that the evolution equation for H given in (3.3) reads

$$\frac{\partial H}{\partial t} = \frac{\Delta H}{H^2} - \frac{|A|^2}{H} - \frac{2|DH|^2}{H^3} \tag{4.51}$$

and the Neumann condition for H derived in (3.6) is

$$D_\mu H = -H\, {}^{\Sigma^2}h_{\nu\nu}. \tag{4.52}$$

Furthermore, we make use of the Gauss-equations which for $M^2 \subset \mathbb{R}^3$ has the special form

$$K := \lambda_1\lambda_2 = \frac{1}{2}\Big((\lambda_1+\lambda_2)^2 - (\lambda_1^2+\lambda_2^2)\Big) = \frac{1}{2}\Big(H^2 - |A|^2\Big) \tag{4.53}$$

where λ_1 and λ_2 are the principal curvatures of a surface M^2 and K is its Gauss-curvature. Finally, we will use the Gauss-Bonnet theorem (see [39], Theorem 9.3). It states that for a 2-dimensional, orientable C^2-surface which is homeomorphic to a disc we have

$$\int_{M^2} K \,\mathrm{d}\mu + \int_{\partial M^2} k_g \,\mathrm{d}s = 2\pi$$

where k_g is the geodesic curvature of ∂M^2 in M^2. Lemma 4.50 tells us that in our case this reads

$$\int_{M^2} K \,\mathrm{d}\mu = 2\pi - \int_{\partial M^2} {}^{\Sigma^2}h_{\tau\tau} \,\mathrm{d}s \quad \text{for} \quad \tau \in TM^2 \cap T\Sigma^2, \quad |\tau| = 1. \tag{4.54}$$

Putting everything together we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{M_t^2} H^2\,\mathrm{d}\mu_t &= \int_{M_t^2}\left(H^2 + 2H\frac{\partial H}{\partial t}\right)\mathrm{d}\mu_t \\
&\overset{(4.51)}{=} \int_{M_t^2}\left(H^2 - |A|^2 + \frac{2\Delta H}{H} - \frac{2|DH|^2}{H^2} - |A|^2 - \frac{2|DH|^2}{H^2}\right)\mathrm{d}\mu_t \\
&\overset{(4.53)}{=} \int_{M_t^2}\left(2K + \frac{2\Delta H}{H} - \frac{2|DH|^2}{H^2} - \frac{H^2}{2} - \frac{(\lambda_1-\lambda_2)^2}{2} - \frac{2|DH|^2}{H^2}\right)\mathrm{d}\mu_t \\
&\overset{(*)}{\leq} 2\int_{M_t^2}\left(K - \frac{H^2}{4} - \left\langle DH, D(H^{-1})\right\rangle - \frac{|DH|^2}{H^2}\right)\mathrm{d}\mu_t + 2\int_{\partial M_t^2} H^{-1}D_\mu H\,\mathrm{d}s_t \\
&\overset{(4.52)}{=} 2\int_{M_t^2}\left(K - \frac{H^2}{4}\right)\mathrm{d}\mu_t - 2\int_{\partial M_t^2} {}^{\Sigma^2}h_{\nu\nu}\,\mathrm{d}s_t \\
&\overset{(4.54)}{=} 4\pi - \frac{1}{2}\int_{M_t^2} H^2\,\mathrm{d}\mu_t - 2\int_{\partial M_t^2}\left({}^{\Sigma^2}h_{\tau\tau} + {}^{\Sigma^2}h_{\nu\nu}\right)\mathrm{d}s_t \\
&= \frac{1}{2}\left(8\pi - \int_{M_t^2} H^2\,\mathrm{d}\mu_t\right) - 2\int_{\partial M_t^2} H(\Sigma^2)\,\mathrm{d}s_t \\
&\leq \frac{1}{2}\left(8\pi - \int_{M_t^2} H^2\,\mathrm{d}\mu_t\right)
\end{aligned}
$$

where we threw away the last two terms in $(*)$ and performed an integration by parts on the term involving the Laplacian. This yields the desired result

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{m}_{\mathrm{Haw}}(M_t^2) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{|M_t^2|^{1/2}}{(8\pi)^{3/2}}\left(8\pi - \int_{M_t^2} H^2\,\mathrm{d}\mu_t\right)\right) \\
&= \frac{1}{(8\pi)^{3/2}}\left(\frac{1}{2}|M_t^2|^{-1/2}\frac{\mathrm{d}|M_t^2|}{\mathrm{d}t}\left(8\pi - \int_{M_t^2} H^2\,\mathrm{d}\mu_t\right) - |M_t^2|^{1/2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{M_t^2} H^2\,\mathrm{d}\mu_t\right) \geq 0
\end{aligned}
$$

since $|M_t^2| = ce^t$. □

Remark 4.52. Notice that for convex supporting surfaces we get $\frac{\mathrm{d}}{\mathrm{d}t}\int_{M_t^2} H^2\,\mathrm{d}\mu_t \leq 0$. Furthermore, comparing our calculation with the calculation for closed 2-surfaces in a Riemannian 3-manifold we see that the monotonicity formula also holds if we replace $\mathbb{R}^3$ by a Riemannian 3-manifold with positive scalar curvature.

Remark 4.53. Proposition 4.51 shows that the most general case in which we can expect the monotonicity of m_{Haw} to hold is the case of mean-convex supporting hypersurfaces Σ^2. To make use of this property we first have to prove the existence of weak solutions in that situation. The only missing part in this procedure is the gradient estimate on Σ^2 for mean-convex (instead of convex) supporting hypersurfaces. If this is done, one has to carry over the smooth calculation we presented in the proof of Proposition 4.51 to the ε-level as in the work of Huisken and Ilmanen [29]. This project is ongoing research.

Appendix

A.1 Parabolic Neumann problems

We start with a definition of the domain and the Hölder norms.

Definition A.1. Let Ω be an open, bounded, connected subset of $\mathbb{R}^n$. We denote with $S := \partial\Omega$ the boundary of Ω. For some $T > 0$ we define

$$Q_T := \Omega \times (0,T), \qquad S_T := \partial\Omega \times (0,T), \qquad \Gamma_T := S_T \cup \Omega \times \{0\}.$$

Analogous to Hölder spaces for functions depending on $x \in \Omega$ we define Hölder spaces for functions depending on $(x,t) \in \overline{\Omega} \times [0,T]$ by:

$$C^{k+\alpha,\frac{k+\alpha}{2}}(\overline{Q}_T) := \{u : \overline{Q}_T \to \mathbb{R} : \mid \|u\|_{k+\alpha,\frac{k+\alpha}{2},Q_T} < \infty\}$$

with

$$\begin{aligned} \|u\|_{k+\alpha,\frac{k+\alpha}{2},Q_T} := & \sum_{j=0}^{k} \sum_{2\gamma_t+|\gamma_x|=j} \sup_{Q_T} |D_t^{\gamma_t} D_x^{\gamma_x} u| \\ & + \sum_{2\gamma_t+|\gamma_x|=k} [D_t^{\gamma_t} D_x^{\gamma_x} u]_{x,\alpha} + \sum_{0<k+\alpha-2\gamma_t-|\gamma_x|<\frac{1}{2}} [D_t^{\gamma_t} D_x^{\gamma_x} u]_{t,\beta} \end{aligned}$$

where $2\beta := k + \alpha - 2\gamma_t - |\gamma_x|$. Here γ_x is a multi-index and the brackets $[h]_{z,\rho}$ denote ρ-Hölder coefficients of the function h with respect to z.

Remark A.2. By definition a function $u \in C^{2,\alpha}(\overline{Q}_T)$ is continuous and has continuous derivatives up to second order in x and up to first order in t. Additionally the following Hölder coefficients are defined: $[D_t u]_{x,\alpha}$, $[D_t u]_{t,\frac{\alpha}{2}}$, $[D_x^2 u]_{x,\alpha}$, $[D_x^2 u]_{t,\frac{\alpha}{2}}$, $[D_x u]_{t,\frac{1+\alpha}{2}}$.

Remark A.3. The above definition can be extended to the case where $\Omega = M^n$ is a compact manifold. In this case one uses locally the Euclidean definition from above and constructs global norms with the help of a finite partition of the unity.

Note that without this localized definition it is not obvious what it means to calculate the Hölder norm of Du when u is a function defined on a manifold. One way to obtain a useful definition would be to involve the push forward to compare the two vectors $Du(x_1)$ and $Du(x_2)$ as it is described in [5], Chapter 1.4. Another interesting way to make a chart-independent definition of Hölder norms on manifolds is given in [17] Chapter 11.8.18 but the same author remarks in [18], Chapter 2.5 that a local definition via the partition of the unity is reasonable and completely sufficient.

We consider a linear parabolic problem with Neumann boundary condition

$$(1) \begin{cases} Lu = f_1 & \text{in } M^n \times (0,T) \\ Nu = f_2 & \text{on } \partial M^n \times (0,T) \\ u(\,.\,,0) = u_0 & \text{on } M^n \end{cases}$$

where L and N are linear operators of the form

$$Lu := \frac{\partial u}{\partial t} - a^{ij}D_{ij}u + b^k D_k u + cu, \qquad Nu := \mu^k D_k u + \eta u$$

with coefficients $a^{ij}, b^k, c, \mu^k, \eta \in L^\infty(Q_T)$. Furthermore, we assume L to be uniformly parabolic, i.e. for some $0 < \lambda < \Lambda$ we have

$$\lambda\xi^2 \leq a^{ij}\xi_i\xi_j \leq \Lambda\xi^2 \qquad \text{in } \overline{Q}_T, \qquad \forall\, \xi \in \mathbb{R}^n.$$

Additionally we impose the *transversality condition*

$$g^{ij}\nu_i\mu_j \neq 0 \qquad \text{on } \partial M^n \times [0,T] \tag{TC}$$

where ν is the outward unit normal to ∂M^n and the 0^{th}-order compatibility condition

$$Nu_0 = f_2 \qquad \text{on } \partial M^n. \tag{CC}$$

In this situation the following theorem holds.

Theorem A.4. *Let* $0 < \alpha < 1$. *Let* M^n *be a smooth, compact, manifold with smooth, compact boundary. Suppose that the coefficients of* L *belong to* $C^{\alpha,\frac{\alpha}{2}}(\overline{Q_T})$ *and* $\mu \in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{S_T})$ *satisfies* (*TC*). *Furthermore suppose that* $f_1 \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q_T})$ *and that* $f_2 \in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{S_T})$ *and* $u_0 \in C^{2+\alpha}(M^n)$ *satisfy* (*CC*). *Then the problem* (1) *has a unique solution* $u \in C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q_T})$. *Furthermore, the estimate*

$$\|u\|_{2+\alpha,\frac{2+\alpha}{2},Q_T} \leq C\left(\|f_1\|_{\alpha,\frac{\alpha}{2},Q_T} + \|f_2\|_{1+\alpha,\frac{1+\alpha}{2},S_T} + \|u_0\|_{2+\alpha,M^n}\right)$$

holds.

Proof. The proof from [38], Chapter IV, Theorem 5.3. can be adjusted to work in the case where Ω is replaced by the compact, smooth manifold M^n. □

The most important tool for second order parabolic equations is the maximum principle. Before we mention it we have to define sub- and supersolutions.

Definition A.5. Let $v^+, v^- \in C^{2,1}(M^n \times (0,T)) \cap C^0(M^n \times [0,T])$. We say that v^+ is a supersolutions to (1) if it satisfies

$$\begin{cases} Lv^+ \;\geq\; f_1 & \text{in } M^n \times (0,T) \\ Nv^+ \;\geq\; f_2 & \text{on } \partial M^n \times (0,T) \\ v^+(\,.\,,0) \;\geq\; u_0 & \text{on } M^n. \end{cases}$$

The function v^- is called subsolution if the opposite inequalities hold.

Now we can state the version of the maximum principles which we use in this work.

Theorem A.6. *Let* $u \in C^0(M^n \times [0,T]) \cap C^{2,1}(M^n \times (0,T))$ *be a solution to* (1). *Assume that* L *and* N *have bounded coefficients, that* L *is uniformly parabolic and that the transversality condition* (*TC*) *is satisfied. If* v^+ *and* v^- *are super- and subsolutions to* (1) *the* $v^- \leq u \leq v^+$ *in* $\overline{Q}_T$.

Proof. Note that for $w := v^+ - u$ and $w := u - v^-$ we have $Lw \geq 0$, $Nw \geq 0$ and $w(\,.\,,0) \geq 0$. So we can reduce the proof to the case of the upper bound for $f_1 = 0, f_2 = 0$ and $u_0 = 0$. This proof is contained in [52] Chapter 3, Section 3, Theorem 5,6 and 7. Furthermore Stahl proved in [59] the generalization which in particular allows for the more general operator N which occurs here. $\square$

Corollary A.7. *If $f_1 \equiv 0$ and $f_2 \equiv 0$, then $v^+ := \max_{M^n} u_0$ is a supersolution if*

$$c \max_{M^n} u_0 \geq 0 \qquad \textit{and} \qquad \eta \max_{M^n} u_0 \geq 0.$$

Furthermore $v^- := \min_{M^n} u_0$ is a subsolution if

$$c \min_{M^n} u_0 \leq 0 \qquad \textit{and} \qquad \eta \min_{M^n} u_0 \leq 0.$$

In particular these inequalities are all satisfied for $c \equiv 0$ and $\eta \equiv 0$.

Corollary A.8. *Assume that $f_1 \equiv 0$, $f_2 \equiv 0$, $\eta = 0$ and $c(x,t) = c(t)$. Then v^+ given as a solution to*

$$(\mathrm{ODE}) \begin{cases} \dfrac{\partial v^+}{\partial t} + cv^+ \geq 0 & \textit{on} \;\; M^n \times (0,T) \\ v^+(0) = \max\limits_{M^n} u_0 \end{cases}$$

is a supersolution. Furthermore, the function v^- satisfying the same ODE *with the reverse inequality and the initial value $\min_{M^n} u_0$ is a subsolution.*

A.2 Elliptic mixed boundary value problems

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We denote by Σ a relatively open part of $\partial\Omega$ and write $\sigma = \partial\Omega \setminus \overline{\Sigma}$. Let ν be the outward unit normal to Ω on Σ. We consider the following mixed Dirichlet-Neumann boundary value problem

$$(2) \begin{cases} Lu := a^{ij} D_{ij} u + b^k D_k u = f & \text{in } \Omega \\ \nu^k D_k u = 0 & \text{on } \Sigma \\ u = v & \text{on } \overline{\sigma}. \end{cases}$$

We assume uniform ellipticity in the form $\mu|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq |\xi|^2$ for all $\xi \in \mathbb{R}^n$ and some $\mu > 0$. Since the outward unit normal occurs in the Neumann condition the problem is uniformly oblique. Before we come to the existence and uniqueness results we want to state some more classical maximum principles.

Proposition A.9. *Let $u \in C^2(\Omega)$. Assume that the coefficients of L are locally bounded. If $Lu \geq 0$ then u can not attain a non-negative maximum M at an interior point unless $u \equiv M$.*

Proof. See [52], Chapter 2, Section 3, Theorem 6. $\square$

Proposition A.10. *Assume that* Σ *is at least* C^1. *Let* $u \in C^2(\Omega) \cap C^1(\Omega \cup \Sigma) \cap C^0(\overline{\Omega})$ *and assume that* $u \leq M$ *in* Ω *and* $u(x_0) = M$ *for some* $x_0 \in \Sigma$. *If* $D_\nu u \leq 0$ *then* u *can not attain a non-negative maximum at* x_0 *unless* $u \equiv M$.

Proof. See [52], Chapter 2, Section 3, Theorem 7. $\square$

For the next result we have to define sub- and supersolutions.

Definition A.11. Assume that $v^+, v^- \in C^2(\Omega) \cap C^1(\Omega \cup \Sigma) \cap C^0(\overline{\Omega})$. If v^+ satisfies

$$Lv^+ \leq f \quad \text{in } \Omega, \qquad \nu^k D_k v^+ \geq 0 \quad \text{on } \Sigma, \qquad v^+ \geq v \quad \text{on } \overline{\sigma}$$

then v^+ is called a supersolution to (2). If v^- satisfies the reverse inequalities it is called a subsolution to (2).

Proposition A.12. *Let* Σ *be at least* C^1. *Let* $u, v^-, v^+ \in C^2(\Omega) \cap C^1(\Omega \cup \Sigma) \cap C^0(\overline{\Omega})$. *Assume that* u *is a solution to* (2) *and that* v^+, v^- *are super- and subsolutions to* (2). *Then* $v^- \leq u \leq v^+$ *in* $\overline{\Omega}$.

Proof. See [52], Chapter 2, Section 6. $\square$

We want to state an existence and regularity result for elliptic mixed problems in domains with corners $V := \overline{\sigma} \cap \overline{\Sigma}$. Therefore, we have to introduce weighted Hölder spaces. Similar to [43] we set $\Omega_\delta := \{x \in \Omega \mid \operatorname{dist}(x, V) > \delta\}$ where δ is a sufficiently small positive number. Using the well known Hölder norms $\|.\|_{k,\alpha;\Omega}$ as they appear in [21] we define for $k \in \mathbb{N}, \alpha \in (0,1)$ and $b > -k - \alpha$

$$\|u\|^{(b)}_{k,\alpha;\Omega} := \sup_{\delta > 0} \delta^{b+k+\alpha} \|u\|_{k,\alpha;\overline{\Omega_\delta}}, \qquad H^{(b)}_{k,\alpha}(\Omega) := \left\{ u \;\middle|\; \|u\|^{(b)}_{k,\alpha;\Omega} < \infty \right\}. \tag{A.55}$$

These norms have the following useful properties.

Proposition A.13. *Let* $k_1, k_2, k, l \in \mathbb{N}$ *and* $\alpha, \beta \in (0,1)$. *If* $k + \alpha \geq l + \beta$ *then*

$$H^{(-l-\beta)}_{k,\alpha}(\Omega) \subset C^{l,\beta}(\overline{\Omega}) \cap C^{k,\alpha}(\Omega). \tag{A.56}$$

Let $k_1 + \alpha \geq b > 0$. *If* $(u_n)_{n \in \mathbb{N}} \subset H^{(-b)}_{k_1,\alpha}(\Omega)$ *is bounded. Then there is a subsequence* $(u_{n_k})_{k \in \mathbb{N}}$ *such that*

$$u_{n_k} \overset{H^{(-b')}_{k_2,\beta}(\Omega)}{\longrightarrow} u \qquad (k \to \infty) \tag{A.57}$$

for $0 < b' < b$, $0 < k_2 + \beta < k_1 + \alpha$ *and* $k_2 + \beta \geq b'$.

Proof. See [41], Section 1 and the introduction of [42]. $\square$

Now we can state the main existence and regularity result for mixed elliptic boundary value problems which is due to Lieberman [43, 44].

Theorem A.14. *Let Σ and σ be subsets of $C^{2,\alpha}$-hypersurfaces. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with boundary $\partial\Omega = \overline{\sigma} \cup \overline{\Sigma}$ where σ and Σ are relatively open in $\partial\Omega$. Assume that a^{ij} is uniformly continuous in Ω and that L is uniformly elliptic. Furthermore, assume that for all $x \in V := \overline{\sigma} \cap \overline{\Sigma}$ the boundary parts σ and Σ enclose the domain at an angle $0 < \theta(x) \leq \theta_{max} < \frac{\pi}{2}$. Then there exists some $\beta(\theta_{max}) \in (0,1)$ such that if*

$$a^{ij} \in H^{(0)}_{0,\alpha}(\Omega), \qquad b^i \in H^{(1-\beta)}_{0,\alpha}(\Omega), \qquad f \in H^{(1-\beta)}_{0,\alpha}(\Omega), \qquad v \in C^{1,\beta}(\overline{\Omega})$$

then there exists a unique solution $u \in C^2(\overline{\Omega} \setminus V) \cap C^0(\overline{\Omega})$ of (2). Furthermore, each such solution of (2) satisfies the estimate

$$\|u\|^{(-1-\beta)}_{2,\alpha;\Omega} \leq C(\|f\|^{(1-\beta)}_{0,\alpha;\Omega} + \|v\|_{1,\beta;\overline{\Omega}}).$$

Proof. The existence and uniqueness result can be found in [43], Theorem 2. This theorem requires a so called wedge condition on Σ as well as an interior and exterior cone condition on $\overline{\sigma}$. Both are satisfied since Σ and σ are C^2-hypersurfaces which meet at an non-zero angle. The regularity of the coefficients is satisfied by assumption as well. Note that in Lieberman's notation $c \equiv 0$ and $\gamma_0 \equiv 0$ so we have to make use of his remark that in this case a Fredholm alternative applies. Furthermore, L is uniformly elliptic and $\beta = \nu$ and so the operator which occurs in the Neumann condition is uniformly oblique. Finally, note that we have

$$\lim_{\delta \to 0} \delta^{1+\alpha} \|b^i\|_{0,\alpha;\Omega_\delta} \leq \lim_{\delta \to 0} \delta^{\beta} \|b^i\|^{(1-\beta)}_{0,\alpha;\Omega_\delta} = 0$$

since $b^i \in H^{(1-\beta)}_{0,\alpha}(\Omega)$ and so the convergence to zero as required in [43], Theorem 2 holds too. Altogether we obtain a unique solution $u \in C^2(\overline{\Omega} \setminus V) \cap C^0(\overline{\Omega})$.

The optimal regularity result is contained in [44], Theorem 4. This Theorem makes some requirements on the contact angle between Σ and σ as well as on the vector occuring in the Neumann condition. In our case these conditions are satisfied as long as the contact angle is strictly less than $\frac{\pi}{2}$. ☐

Remark A.15.

(i) Note that the weighted norms for the existence result [43] contain a weight with respect to the Dirichlet boundary whereas the weighted norms which are used for the regularity statement [44] have a weight with respect to the whole boundary of the domain. Since our boundary parts σ and Σ are both $C^{2,\alpha}$ we decided to use a weight which only affects $V := \overline{\sigma} \cap \overline{\Sigma}$. So we use slightly more restrictive norms to be able to obtain existence and regularity in the same weighted spaces.

(ii) In general a solution $u \in C^{2,\alpha}(\Omega)$ of (2) will only be in $C^{1,\beta}(\overline{\Omega})$ if the angle between the Dirichlet and the Neumann boundary parts is strictly less than $\pi/2$. See also the review article of Lieberman [46].

(iii) Note that we only stated the result in the form which is needed in this work. Lieberman's result holds under more general assumptions. In particular one can include a linear term cu in the operator L and one can treat other oblique derivative boundary conditions such as $\beta^i D_i u = f_2$ on Σ.

A.3 Geometric measure theory

We start with the definitions and properties of measures. Especially, we consider Radon measures and Hausdorff measure. For the next definitions we follow [15], Section 1.1.

Definition A.16 (Borel regular measure)**.** Let X be a set. We denote by $\mathcal{P}(X)$ the set of all subsets of X. The map $\mu : \mathcal{P}(X) \to [0, \infty]$ which satisfies

$$\mu(\emptyset) = 0, \qquad \mu(A) \leq \sum_{k \in \mathbb{N}} \mu(A_k), \qquad \forall\, A, A_k \subset X \;\text{ s.t. }\; A \subset \bigcup_{k \in \mathbb{N}} A_k$$

is called measure. The sets $A \subset X$ which satisfy

$$\mu(B) = \mu(A \cap B) + \mu(B \setminus A), \qquad \forall\, B \subset X$$

are called μ-measurable. The family $\mathcal{F} \subset \mathcal{P}(X)$ of all μ-measurable subsets of X forms a σ-algebra. The smallest σ-algebra of $X = \mathbb{R}^n$ which contains all open sets is called Borel σ-algebra and is denoted by $\mathcal{B}(\mathbb{R}^n)$. A measure μ is called Borel regular if all sets $B \in \mathcal{B}(\mathbb{R}^n)$ are μ-measurable and if

$$\forall\, A \subset \mathbb{R}^n,\ \exists\, B \in \mathcal{B}(\mathbb{R}^n) \text{ s.t. } A \subset B \text{ and } \mu(A) = \mu(B).$$

Definition A.17 (Radon measure)**.** Let $\mu : \mathbb{R}^n \to [0, \infty]$ be a Borel regular measure. If additionally we have $\mu(K) < \infty$ for all compact sets $K \subset \mathbb{R}^n$. Then μ is called a Radon measure.

Theorem A.18 (Lebesgue-Besicovitch differentiation theorem)**.** *Let μ be a Radon measure on $\mathbb{R}^n$ and $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Then*

$$f(x_0) = \lim_{r \to 0} \fint_{B(x_0,r)} f(x)\,\mathrm{d}\mu(x) \qquad \mu\text{-a.e. } x_0 \in \mathbb{R}^n.$$

In particular for $f \in L^p_{loc}(\mathbb{R}^n, \mu)$ we have

$$0 = \lim_{r \to 0} \fint_{B(x_0,r)} |f(x) - f(x_0)|^p\,\mathrm{d}\mu(x) \qquad \mu\text{-a.e. } x \in \mathbb{R}^n.$$

The points $x_0 \in \mathbb{R}^n$ where this holds are called Lebesgue points of f.

Proof. See [15], Section 1.7, Theorem 1 and Corollary 1. □

Theorem A.19 (Riesz representation theorem)**.** *Let $L : C_c(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}$ be a bounded and linear functional.Then there exists a Radon measure μ on $\mathbb{R}^n$ and a μ-measurable*[3] *function $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ such that $|\sigma(x)| = 1$ for μ-a.e. $x \in \mathbb{R}^n$ and L can be represented as*

$$L(f) = \int_{\mathbb{R}^n} f(x)\sigma(x)\,\mathrm{d}\mu(x)$$

for all $f \in C_c(\mathbb{R}^n, \mathbb{R}^m)$.

Proof. See [15], Section 1.8, Theorem 1. □

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[3] A map $f : X \to Y$ is called μ-measurable, if $f^{-1}(U)$ is μ-measurable for all $U \subset Y$ open.

Definition A.20 (Weak convergence of Radon measures)**.** Let $k \in \mathbb{N}$ and μ, μ_k be Radon measures on $\mathbb{R}^n$. We say that μ_k converges weakly to μ, denoted by $\mu_k \rightharpoonup \mu$ if and only if one of the two equivalent statements hold

$$\text{(i)} \lim_{k\to\infty} \int_{\mathbb{R}^n} f(x)\,\mathrm{d}\mu_k(x) = \int_{\mathbb{R}^n} f(x)\,\mathrm{d}\mu(x), \qquad \forall\, f \in C_c(\mathbb{R}^n).$$

$$\text{(ii)} \lim_{k\to\infty} \mu_k(B) = \mu(B), \qquad \forall\, B \in \mathcal{B}(\mathbb{R}^n),\ B \text{ bounded},\ \mu(\partial B) = 0.$$

Proof. The equivalence of (i) and (ii) is proved in [15], Section 1.9, Theorem 1. □

Next we define Hausdorff measures. The definition and properties can be found in [15], Chapter 2.

Definition A.21 (Hausdorff measure)**.** Let $0 < \delta \leq \infty$ and $0 \leq k < \infty$. The Hausdorff measure $\mathcal{H}^k : \mathbb{R}^n \to [0, \infty] : A \mapsto \mathcal{H}^k(A)$ is defined by

$$\begin{aligned}\mathcal{H}^k(A) &:= \lim_{\delta\to 0} \mathcal{H}^k_\delta(A) \\ &:= \lim_{\delta\to 0} \inf \left\{ \frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}+1\right)} \sum_{j\in\mathbb{N}} \left(\frac{\operatorname{diam} C_j}{2}\right)^k \,\middle|\, A \subset \bigcup_{j\in\mathbb{N}} C_j,\ \operatorname{diam} C_j \leq \delta \right\}\end{aligned}$$

where $\Gamma(s) := \int_0^\infty e^{-x} x^{s-1}\,\mathrm{d}x$ for $s \in (0, \infty)$. The Hausdorff measure is a Borel regular measure with the following properties

(i) $\mathcal{H}^0$ is the counting measure.

(ii) $\mathcal{H}^k = \lambda^k$ on $\mathbb{R}^k$ where λ^k is the k-dimensional Lebesgue measure.

(iii) $\mathcal{H}^k \equiv 0$ on R^n for $k > n$.

Note also that for general $k \in \mathbb{N}$ the measure $\mathcal{H}^k$ is not a Radon measure.

In order to define sets of finite perimeter and the reduced boundary we have to consider functions of bounded variations. The following definition can be found in [15], Section 5.1.

Definition A.22 (Functions of bounded variation)**.** Let $U \subseteq \mathbb{R}^n$ be open and let $f \in L^1(U, \lambda^n)$. We define the symbol

$$\|Df\|(U) := \sup \left\{ \int_U f(x) \operatorname{div} \varphi(x)\,\mathrm{d}\lambda^n(x) \,\middle|\, \varphi \in C^1_c(U, \mathbb{R}^n),\ |\varphi| \leq 1 \right\}.$$

and $\|f\|_{BV(U)} := \|f\|_{L^1(U)} + \|Df\|(U)$. The set

$$BV(U) := \left\{ f \in L^1(U) \,\middle|\, \|f\|_{BV(U)} < \infty \right\}$$

is called the space of functions of bounded variation. The map $\|\cdot\|_{BV(U)}$ is a norm and $BV(U)$ equipped with this norm is a Banach space (see [22], Remark 1.12). Furthermore, the set

$$BV_{loc}(U) := \left\{ f \in L^1_{loc}(U) \,\middle|\, \|f\|_{BV(V)} < \infty\ \forall\, V \subset\subset U \text{ open} \right\}$$

is called the space of functions of locally bounded variation. Note that for $U \subset \mathbb{R}^n$ open and $f \in W^{1,1}(U)$ we have $\|Df\|(U) = \int_U |Df| \,\mathrm{d}\lambda$ (see [22], Example 1.2).

Lemma A.23 (Lower semicontinuity in BV)**.** *Let $U \subset \mathbb{R}^n$ be open. If a sequence $(f_n)_{n\in\mathbb{N}} \subset BV(U)$ converges in $L^1_{loc}(U)$ to $f \in BV(U)$ then*

$$\|Df\|(U) \leq \liminf_{n\to\infty} \|Df_n\|(U).$$

Proof. See [22], Theorem 1.9. □

Lemma A.24. *The following inclusions hold*

$$W^{1,1}(U) \subset BV(U), \qquad W^{1,1}_{loc}(U) \subset BV_{loc}(U).$$

Note that we do not have equality. This can be seen by considering the characteristic function of a bounded set $E \subset \mathbb{R}^n$ with C^2-boundary and finite boundary length, i.e. $\mathcal{H}^{n-1}(\partial E \cap U) < \infty$. It turns out that

$$\|\mathbb{1}_E\|_{BV(U)} = \|\mathbb{1}_E\|_{L^1(U)} + \|D\mathbb{1}_E\|(U) = |E \cap U| + \mathcal{H}^{n-1}(\partial E \cap U) < \infty$$

but $\mathbb{1}_E$ is not a Sobolev function. Thus $W^{1,1}(U) \neq BV(U)$ and $W^{1,1}_{loc}(U) \neq BV_{loc}(U)$.

Proof. See [15], Section 5.1. □

Sets for which $\mathbb{1}_E$ is a function of locally bounded variation are given a special name.

Definition A.25. Let $E \subset \mathbb{R}^n$ be a λ^n-measurable set. If $\mathbb{1}_E \in BV_{loc}(U)$ we say that E has locally finite perimeter in $U \subset \mathbb{R}^n$. If $\mathbb{1}_E \in BV_{loc}(U)$ for every bounded, open set $U \subset \mathbb{R}^n$, then E is called a Caccioppoli set.

Furthermore, we have the following structure theorem.

Theorem A.26 (Structure theorem for BV_{loc})**.** *Let $U \subset \mathbb{R}^n$ be open and let $f \in BV_{loc}(U, \lambda^n)$. Then there exists a Radon measure μ on U and a μ-measurable function $\sigma : U \to \mathbb{R}^n$, such that $|\sigma(x)| = 1$ for μ-a.e. $x \in U$ and*

$$\int_U f(x) \operatorname{div} \varphi(x) \,\mathrm{d}\lambda(x) = -\int_U \varphi(x)\sigma(x) \,\mathrm{d}\mu(x) = -\int_U \varphi(x)\sigma(x) \,\mathrm{d}\|Df\|$$

for all $\varphi \in C^1_c(U, \mathbb{R}^n)$. In the case that $f = \mathbb{1}_E$ where E is a set of locally finite perimeter in U, we define $\|\partial E\| := \|D\mathbb{1}_E\|$ and $\nu_E := -\sigma$. This allows us to rewrite the statement as

$$\int_E \operatorname{div} \varphi(x) \,\mathrm{d}\lambda(x) = \int_U \varphi(x)\nu_E(x) \,\mathrm{d}\|\partial E\|$$

for all $\varphi \in C^1_c(U, \mathbb{R}^n)$. $\|Df\|$ is the variation measure of f, $\|\partial E\|$ is the perimeter measure of E and $\|\partial E\|(U)$ is called the perimeter of E in U.

Proof. See [15], Section 5.1, Theorem 1 and the Remarks of Section 5.1. □

The following approximation result holds.

Lemma A.27 (Approximation of BV-functions)**.** *Let A be open. Assume $f \in BV_{loc}(A)$. Then there exists a sequence $(f_k)_{k\in\mathbb{N}} \subset BV_{loc}(A) \cap C^\infty(A)$ such that $f_k \to f$ in $L^1_{loc}(A)$. Furthermore, for the Radon measures $(\mu_k)_{k\in\mathbb{N}}$ and μ defined by*

$$\mu_k(B) := \int_{B\cap A} Df_k \,\mathrm{d}\lambda^n, \qquad \mu(B) := \int_{B\cap A} \sigma \,\mathrm{d}\|Df\| \qquad B \in \mathcal{B}(\mathbb{R}^n)$$

we have $\mu_k \rightharpoonup \mu$ (see Definition A.20). In particular

$$\int_A |Df_k| \,\mathrm{d}\lambda^n = \|Df_k\|(A) \to \|Df\|(A)$$

as $k \to \infty$.

Proof. See [15], Section 5.2, Theorem 2 and Theorem 3. ☐

Next we follow [15], Section 5.7 and define the reduced boundary.

Definition A.28 (Reduced boundary)**.** Let $\subset \mathbb{R}^n$ be a set of locally finite perimeter in $\mathbb{R}^n$. We call $\partial^* E$ the reduced boundary of E. A point x belongs to the reduced boundary if the following conditions hold

(i) $\|\partial E\|(B(x,r)) > 0 \qquad \forall\, r > 0.$

(ii) $\nu_E(x) = \lim_{r\to 0} \fint_{B(x,r)} \nu_E(x) \,\mathrm{d}\|\partial E\|.$

(iii) $|\nu_E(x)| = 1.$

The structure of $\partial^* E$ is characterized by the following result.

Theorem A.29 (Structure theorem for the reduced boundary)**.** *Assume that $E \subset \mathbb{R}^n$ has locally finite perimeter in $\mathbb{R}^n$. Then*

$$\partial^* E = \bigcup_{k\in\mathbb{N}} K_k \cup N$$

where $\|\partial E\|(N) = 0$ and K_k are compact subsets of C^1-hypersurfaces S_k. Furthermore $\nu_E|_{S_k}$ is the classical normal to S_k and

$$\|\partial E\| = \|D\mathbb{1}_E\| = \mathcal{H}^{n-1}\lfloor\partial^* E$$

where $\|\partial E\|(A) = \mathcal{H}^{n-1}(A \cap \partial E)$.

Proof. See [15], Section 5.7, Theorem 2. ☐

Lemma A.30. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $E, F \subset \mathbb{R}^n$. Then*

$$\|\partial(E \cup F)\|(\Omega) + \|\partial(E \cap F)\|(\Omega) \leq \|\partial(E)\|(\Omega) + \|\partial(F)\|(\Omega)$$

Note that $\|\partial A\| = \mathcal{H}^{n-1}\lfloor\partial^* A$.

Proof. See [2], Section 3.3, Proposition 3.38. ☐

Lemma A.31 (Area formula)**.** *Let* $n \leq m$ *and* $f : \mathbb{R}^n \to \mathbb{R}^m$ *be Lipschitz. Then for each* λ^n*-summable function* $g : \mathbb{R}^n \to \mathbb{R}$ *(i.e. a function satisfying* $\int_{\mathbb{R}^n} |g| \, \mathrm{d}\lambda^n < \infty$*) we have*

$$\int_{\mathbb{R}^n} g(x) Jf(x) \, \mathrm{d}\lambda(x) = \int_{\mathbb{R}^m} \sum_{x \in f^{-1}(y)} g(x) \, \mathrm{d}\mathcal{H}^n(y)$$

where $Jf := |\det((Df)^* \circ (Df)|^{1/2}$ *and* $(Df)^*$ *is the adjoint map*[4] *to* Df*. Especially,*

$$\int_U g(x) |\det(Df(x))| \, \mathrm{d}\lambda(x) = \int_{f(U)} g(f^{-1}(y)) \, \mathrm{d}\mathcal{H}^m(y)$$

if $n = m$ *and* $f : U \subset \mathbb{R}^n \to f(U)$ *is injective.*

Proof. Jf is the Jacobian of f defined in [15], Subsection 3.2.2. The formula of Jf that we used is contained in [15], Section 3.2. Theorem 3. The area formula itself is stated in [15], Section 3.3, Theorem 2. ☐

Lemma A.32 (Co-area formula)**.** *Let* $n \geq m$ *and* $f : \mathbb{R}^n \to \mathbb{R}^m$ *be Lipschitz. Then for each* λ^n*-summable function* $g : \mathbb{R}^n \to \mathbb{R}$ *(i.e. a function satisfying* $\int_{\mathbb{R}^n} |g| \, \mathrm{d}\lambda^n < \infty$*) we have*

$$\int_{\mathbb{R}^n} g(x) Jf(x) \, \mathrm{d}\lambda^n(x) = \int_{\mathbb{R}^m} \int_{f^{-1}(\hat{y})} g(\check{y}) \, \mathrm{d}\mathcal{H}^{n-m}(\check{y}) \, \mathrm{d}\lambda^m(\hat{y})$$

where $Jf := |\det((Df) \circ (Df)^*|^{1/2}$ *and* $(Df)^*$ *is the adjoint map to* Df*. Especially*

$$\int_{\mathbb{R}^n} g(x) |Df(x)| \, \mathrm{d}\lambda^n(x) = \int_{\mathbb{R}} \int_{f^{-1}(\hat{y})} g(\check{y}) \, \mathrm{d}\mathcal{H}^{n-1}(\check{y}) \, \mathrm{d}\lambda^1(\hat{y})$$

if $m = 1$*. Note that during this lemma we denote the* k*-dimensional Lebesgue measure by* $\mathrm{d}\lambda^k$ *to prevent misunderstandings. In the rest of this work we always use* $\mathrm{d}\lambda$ *to denote the Lebesgue measure of the appropriate dimension.*

Proof. Jf is the Jacobian of f defined in [15], Subsection 3.2.2. The formula of Jf that we used is contained in [15], Section 3.2. Theorem 3. The co-area formula itself is stated in [15], Section 3.4, Theorem 2. ☐

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[4] For a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ we denote by A^* its adjoint which is by the relation $\langle A^* y, x \rangle_{\mathbb{R}^n} = \langle Ax, y \rangle_{\mathbb{R}^m}$.

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German thesis summary

Zusammenfassung der Arbeit

Diese Arbeit befasst sich mit Hyperflächen, welche sich in Richtung der Einheitsnormalen mit der Geschwindigkeit reziprok zur mittleren Krümmung bewegen. Diese Evolutionsgleichung heisst Fluß entlang der inversen mittleren Krümmung (engl. inverse mean curvature flow, kurz IMCF). Die hier betrachteten Hyperflächen besitzen einen Rand. Dieser soll senkrecht auf einer festen Stützfläche aufsitzen und sich entlang dieser bewegen.

In Kapitel 1 wird ein Überblick über geometrische Evolutionsgleichungen im Allgemeinen und IMCF für geschlossene Flächen im Speziellen gegeben. Der dritte Abschnitt des ersten Kapitels beschreibt das Evolutionsproblem für Hyperflächen mit Rand und stellt somit den Startpunkt für die folgenden Untersuchungen dar.

Die erste Frage, die man sich stellen muss ist, ob die Evolutionsgleichung wenigstens für eine kurze Zeitspanne eine Lösung besitzt. Dieses Resultat über Kurzzeitexistenz erhalten wir im Kapitel 2, Theorem 2.12, indem wir die Hyperflächen für kleine Zeiten als Graphen über der Anfangsfläche darstellen. Dadurch lässt sich die Evolutionsgleichung auf ein skalares, parabolisches Neumannproblem reduzieren. Dieser Zugang wurde auch von Stahl [59] für den Fluß entlang der mittleren Krümmung (engl. mean curvature flow) verwendet.

Die natürliche Frage, die sich als nächstes stellt, ist die der Langzeitexistenz. Das Gegenbeispiel eines Halb-Torus, welcher sich auf einer Ebene bewegt zeigt, dass man für den klassischen Fluß im Allgemeinen keine Langzeitexistenz erwarten kann. Daher betrachten wir im Kapitel 3 den Spezialfall eines konvexen Kegels als feste Stützfläche und betrachten Anfangsflächen positiver mittlerer Krümmung, welche bezüglich der Kegelspitze sternförmig sind. In Kapitel 3, Theorem 3.21 beweisen wir unter diesen Voraussetzungen Langzeitexistenz und Konvergenz zu einer sphärischen Kappe. Für geschlossene Flächen geht dieses Resultat auf Gerhardt [16] zurück.

Um Aussagen für allgemeinere Stützflächen zu erhalten, folgen wir im Kapitel 4 den Ideen von Huisken und Ilmanen [29] und definieren schwachen Lösungen. Dafür führen wir eine Niveauflächenformulierung des Evolutionsproblems ein. Dies führt zu einem degenerierten elliptischen Problem mit gemischten Randwerten in einem Gebiet mit Kanten. Dieses Problem lässt sich durch elliptische Regularisierung zunächst approximativ lösen. Die approximativen Lösungen erlauben es, eine Folge von schwache Lösungen in einer höheren Dimension zu konstruieren. Zusammen mit einem Kompaktheitsresultat erhält man schließlich eine Grenzfunktion, die der eindeutige Minimierer eines mit dem Evolutionsproblem zusammenhängenden Funktionals ist. Dies führt in Kapitel 4, Theorem 4.47 zu einem Existenz- und Eindeutigkeitssatz für schwache Lösungen des IMCF für Hyperlächen mit Rand. Dieses Theorem ist das Hauptergebnis dieser Arbeit.