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Idempotent Filters and Ultrafilters

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Contents

Introduction

In this dissertation we will investigate filters on semigroups and their properties regarding algebra in the Stone-Čech compactification.

The set of ultrafilters on a set S can be regarded as βS, the Stone-Čech compactification of S with the discrete topology. If S is a semigroup, we can define an associative operation on βS extending the operation on S. This can be done in such a way that the multiplication with a fixed right hand element is continuous; by the Ellis-Numakura Lemma there exist idempotent elements in βS, i.e., *idempotent ultrafilters*. Idempotent ultrafilters play the central role in the field of algebra in the Stone-Čech compactification especially because they allow for elegant proofs of Ramsey-type theorems such as Hindman's Finite Sums Theorem, the Hales-Jewett Theorem and the Central Sets Theorem.

Although ultrafilters are a natural topic of interest to set theorists, there are only few independence results regarding idempotent ultrafilters. One of the basic questions motivating this thesis was whether this is coincidence: are there other set theoretic constructions, maybe more similar to the different kinds of forcing reals?

In the first part of the thesis, we give a positive answer to this question. In Chapter 3, we develop a uniform approach to adjoin idempotent ultrafilters by means of the forcing method. We are also able to produce a way to discern non-equivalent forcing notions by associating each forcing for adjoining idempotent ultrafilters with an already established notion for adjoining set theoretically interesting, non-idempotent ultrafilters. For these constructions, we study the notion of *idempotent filter* in Chapter 2. This notion is based on the natural generalization of the multiplication of ultrafilters to arbitrary filters.

Idempotent filters are implicit in many applications throughout the field. Besides the usefulness for our forcing constructions, the notion of idempotent filter gives rise to a beautiful theory which we develop in Chapter 2. For example, idempotent filters correspond to subsemigroups of βS with strong closure properties and the notion is also a generalization of the well known concept of partial semigroups. The development of the theory of idempotent filters also enables us to give a simplified proof of Zelenyuk's Theorem on finite groups in the Stone-Čech compactification.

When we answer the above question positively, another question arises: what combinatorial and algebraic properties can these forcing constructions have? To make progress on this question it is natural to investigate the classical notions, i.e., the set theoretic and combinatorial properties of union ultrafilters on the one hand and the algebraic properties of summable ultrafilters on the other. Union ultrafilters were introduced by Andreas Blass and found to be equivalent to the already established summable ultrafilters in joint work with Neil Hindman in 1987.

In Chapter 4, we answer the open question negatively whether a union ultrafilter is already ordered union if certain images of it are selective ultrafilters. In Chapter 5, we investigate the algebraic properties of summable ultrafilters. In particular, we prove that a certain "special" property for summable ultrafilters is automatic and apply this fact to extend a theorem by Hindman and Strauss on writing summable ultrafilters as sums.

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Chapter 1

Preliminaries

1.1. Regarding the layout

The layout of this thesis is probably unusual at first. I designed it to enhance the reading experience as a whole as well as structuring the mathematical content for easier verification and understanding.

The page layout is inspired by Edward Tufte's classic [Tuf05]. The purpose of the design lies in structuring the information in a way suitable for mathematics without obstructing the flow of reading as much as is often accepted in mathematical publications. Most prominently a large margin is implemented. I use it for comments, footnotes and structural remarks as well as useful repetition of terminology so as to make all of this easily accessible to the reader. The actual LaTeX style is my own modification of the tufte-latex package developed at [Kle]; I thank Felix Breuer for many ideas and fruitful discussions on this topic.

In organizing my results and their proofs I base my presentation on the principles laid down in Uri Leron's inspiring paper "Structuring mathematical proofs", [Ler83]. There, he develops a structuring that is aimed to produce a detailed proof accessible to all readers while simultaneously enabling researchers well-versed in the field to skip the details that are unnecessary from their point of view.

Leron's idea of a hierarchy of levels within a proof was implemented using small indentations for the proofs of smaller claims. Additionally, I have organized those parts of a proof that might be deemed superfluous from a formal point of view in separate boxes using a sans serif font. I believe that these additional comments will be useful for comprehending the proofs by pointing out underlying principles, recapitulating strategies and pointing out important details so as to improve the mathematical reading fluency as a whole. With this approach I hope to adapt Leron's "in the elevator" approach in a fashion appropriate for a dissertation.

Since this method leads to proofs taking up additional space, I also colour code definitions, theorems etc. to ease visual sneak previews and to enable the reader to access quoted theorems and definitions faster while browsing back and forth.

For further simplicity, the digital version of this document heavily employs hyperlinks within the document thanks to the capacities of the Portable Document File format (PDF) and pdfTeX with the hyperref-package. Therefore, in the digital format, you will be able to simply click on a linked reference or label to automatically browse to the appropriate page; the back-button available in most PDF-viewers will bring you back to the page you visited before.

In my writing I will employ "we" instead of "I" or "the reader" because I wish to not just present my results but present them to the reader in an including fashion. The only exceptions to this convention will be the few personal remarks, e.g., concerning personal communication. The use of "we" must not be interpreted as an acclamation of joint work or references to results by other researchers. All results of others will be referenced properly.

1.2. *Basic notions*

In this section we wish to informally introduce our basic notation. We intend to keep everything standard and follow the classical books by Thomas Jech, Kenneth Kunen and Neil Hindman and Dona Strauss [Jec03], [Kun80] and [HS98] respectively. Since we are going to quote many papers authored or co-authored by Neil Hindman and/or Andreas Blass, it may be worthwhile to mention at this point that their personal homepages [Hin], [Blab] resp. offer digital copies of many of their published and unpublished papers.

We always work in *ZFC*, the Zermelo-Fraenkel set theory with choice; other axioms such as the continuum hypthesis *CH* will be specifically noted whenever we require them. We denote the natural numbers, i.e., the set of finite ordinals, by ω and let $\mathbb{N} := \omega \setminus \{0\}$; we frequently use the fact that, as an ordinal, a natural number is the set of its predecessors. The central notions of this thesis are "filter" and "ultrafilter".

Filters and ultrafilters

A *filter* on (a set) S is a subset of its power set that contains S and is closed under finite intersections and supersets. We will be concerned with filters on infinite sets. A filter is *proper* if $\emptyset \notin F$, and unless specifically noted we always assume filters to be proper; the power set $\mathfrak{P}(S)$ is called the *improper* filter.

A maximal (proper) filter on S is called an *ultrafilter*; equivalently, an ultrafilter is a *prime filter*, i.e., a (proper) filter such that $A \cup B \in p$ implies that $A \in p$ or $B \in p$. By the Boolean Prime Ideal Theorem [Jec03, Theorem 7.10] any (proper) filter can be extended to an ultrafilter.

The Fréchet filter and free filters

We define the *Fréchet (or cofinite) filter* on S by

$$Fr := Fr(S) := \{A \subseteq S \mid S \setminus A \text{ is finite}\}.$$

A filter is *free* if it contains the Fréchet filter or equivalently its intersection is empty. Otherwise, the filter is *fixed*. Free ultrafilters are the main object of our interest whereas fixed ultrafilters always have the form

$$\dot{s} := \{A \subseteq S \mid s \in A\}$$

for some $s \in S$ and thus embed the original set S into the set of its ultrafilters. We usually identify the fixed ultrafilters with the elements of S.

Infinite filters

As a potentially non-standard notation let us call a filter *infinite* if it contains only infinite sets. We sometimes need this notion to differentiate between free and infinite filters. Infinite filters are exactly those filters that can be extended to a free ultrafilter but may still be extendable to fixed ultrafilters if their intersection is non-empty.[1]

[1] For the simplest example consider any infinite subset and its supersets.

The Stone-Čech compactification

The set of all ultrafilters on S is denoted by βS. The natural topology on βS is generated by basic sets of the following form: for any $A \subseteq S$ let

$$\overline{A} := \{p \in \beta S \mid A \in p\}.$$

The generated topology yields a fascinating space with many strange and intricate properties. To name a few, it is compact and Hausdorff, zero-dimensional, extremally disconnected and S is dense in βS. We will, in general, assume that S carries the discrete topology, in which case βS is its Stone-Čech compactification, i.e., its maximal compactification. We denote the *Stone-Čech remainder* by $\beta S \setminus S =: S^*$ and note that it includes exactly those ultrafilters that extend the Fréchet filter. Note also that for $A \subseteq S$ we have $\overline{A} \cong \beta A$; hence we will identify the two sets. For an introduction to the Stone-Čech compactification see e.g. [HS98, Chapter 3].

(Ultra)filter notation

We usually denote filters by F, G, H and ultrafilters by p, q, u, often interpreting ultrafilters as points and filters as closed sets by the natural correspondences

Correspondence of filters and closed sets

$$F \text{ filter on } S \quad \longmapsto \quad \overline{F} := \bigcap_{A \in F} \overline{A}$$

$$C \subseteq \beta S \text{ closed, non-empty} \quad \longmapsto \quad \bigcap_{p \in C} p$$

between filters and non-empty closed subsets of βS (and of course the improper filter, i.e., the power set, maps to the empty set). These correspondences are inverse to each other and inclusion-reversing. Thus we will frequently switch between filters and closed, non-empty subsets. To minimize confusion, we will usually denote the closure operator for subsets of βS by $cl(\cdot)$.

FIP – the finite intersection property

Any subset of a power set that has the *finite intersection property (FIP)*, i.e., any intersection of finitely many elements is non-empty, generates a (proper) filter by closing it under intersections and supersets.

Filter base, subbase

A *base* of a filter is a subset of the filter such that any set of the filter contains a set from the base, i.e., closing the base under supersets yields the filter. Similarly a *subbase* of a filter is a subset of the filter such that closing it under finite intersections yields a base. We frequently confuse bases, subbases, families with the finite intersection property and filters they generate.

Two filters are said to be *coherent* if their union has the finite intersection property, i.e., their union generates a (proper) filter.

(Pre)image filters

Given any function $f : S \to T$ and a filter F on S let

$$\begin{aligned} f(F) :=& \{B \subseteq T \mid f^{-1}[B] \in F\} \\ =& \text{ the filter generated by } \{f[A] \mid A \in F\}. \end{aligned}$$

When there is no risk of confusion, we call this the *image filter* of F.

For a filter G on T we let

$$f^{-1}(G) := \text{the filter generated by } \{f^{-1}[B] \mid B \in G\}.$$

We call this filter the *preimage filter* of F.

The latter filter is proper if and only if $f[S]$ meets every set in G. We will encounter the latter kind of filters mostly with surjective or similarly well-behaved functions such that this condition is immediate; hence this should cause little confusion.

Rudin-Keisler order

With this we can recall the definition of the *Rudin-Keisler order* for ultrafilters, say p, q on S,

$$p \leq_{RK} q \text{ if there exists } f \in S^S \text{ such that } f(q) = p.$$

Topological limit

Given an ultrafilter p on S, a topological space X and $f : S \to X$, we call a point $x \in X$ a *(topological) limit of* f *along* p *(or* p*-limit)* if the preimage of every neighbourhood of x is in p.

If X is compact and Hausdorff, such a limit always exists and is unique; we denote it by

$$p\text{-}\lim_{s \in S} f(s).$$

It is easy to check that for $p \in \beta S$ and $A \in p$, we get $p\text{-}\lim_{s \in S} f(s) = p\text{-}\lim_{s \in A} f(s)$.

Cantor topology

Let us note that we consider the standard topology of $\mathfrak{P}(S)$, the power set of S, to be the *Cantor topology*, i.e., the topology derived from the product topology of 2^S. We denote the set of non-empty, finite subsets of S by $\mathfrak{P}_f(S)$.

For completeness let us note that the dual notion of a filter is the notion of an *ideal* , i.e., a subset of $\mathfrak{P}(S)$ closed under finite unions and subsets. We will discuss ideals only in Section 3.4. For any filter the set of the complements of the filter elements constitutes the dual ideal and vice versa for an ideal its dual filter. The terminology for filters such as base, generators etc. applies to ideals accordingly.

Now that we have quickly reviewed some of the foundations, we can begin to set the stage for the results of this thesis.

1.3. *Algebra in* βS *and* δS

Since we now begin with the notions more critical but less well known outside the field, we will try to be more formal and exact.

We are interested in ultrafilters with algebraic properties and for this we need some algebraic structure, namely we always work with a semigroup structure.

Definition: (partial) semigroup

Definition 1.1 We define as follows.

- A *semigroup* $(S, \cdot)$ is a set S with $\cdot$ an associative operation.
- A *partial semigroup* $(S, \cdot)$ is a set S with a partial operation $\cdot$ that fulfills the associativity law
$$s \cdot (t \cdot v) = (s \cdot t) \cdot v$$
in the sense that if one side is defined by the partial operation, then so is the other and they are equal. We call this *strong associativity*.

- For a partial semigroup S and $s \in S$ we denote the set of elements right-compatible with s by

$$\sigma(s) := \{t \in S \mid s \cdot t \text{ is defined}\},$$

whereas left-compatible elements of $t \in S$ are the elements of

$$\tau(t) := \{s \in S \mid t \in \sigma(s)\} = \{s \in S \mid s \cdot t \text{ is defined}\}.$$

When we consider partial subsemigroups we may use $\sigma_S(s)$ etc. to indicate in which partial semigroup we are working.

- A partial semigroup is called *adequate* if

$$\{\sigma(s) \mid s \in S\}$$

has the infinite finite intersection property, i.e., generates an infinite filter.[2] We always assume partial semigroups to be adequate. We denote the generated filter by $\sigma(S)$ and its corresponding closed subset of βS by

$$\delta S := \bigcap_{s \in S} \overline{\sigma(s)} = \{p \in \beta S \mid (\forall s \in S)\ \sigma(s) \in p\}.$$

- For a semigroup we denote the restriction of the multiplication[3] on one side by

$$(\forall s \in S)\ \rho_s : S \to S, t \mapsto t \cdot s$$
$$(\forall s \in S)\ \lambda_s : S \to S, t \mapsto s \cdot t.$$

[2] Adequate partial semigroups were first introduced in [BBH94] by Blass, Bergelson and Hindman. Although the filter $\sigma(S)$ is usually not required to be infinite, in every example in the literature it is since the interest lies in free ultrafilters. Note that we do not require that the filter is free.

[3] We do not introduce additional notation for the partial case since we have no use for it; cf. Remark 1.2

The restriction to partial semigroup is technically not necessary due to the following and later remarks. However, the restrictions of a partial semigroup help to focus on those algebraic properties that are of interest to us.

Remark 1.2 Any partial semigroup $(S, \cdot)$ can be extended to a semigroup.

For example, by adding a two-sided zero $\perp \notin S$ and extending the operation to $s, t \in S \cup \{\perp\} =: S_\perp$ by

$$t \notin \sigma_S(s) \Rightarrow s \cdot t = \perp .$$

It is easy to check that this operation is associative.

Note also that we can adjoin an identity to any (partial) semigroup.

This remark may seem a mere technical triviality, however it should also be noted that all examples of partial semigroup operations in the literature are restrictions of semigroup operations that are not as trivial as the above by means of restricting the operation to a subset of $S \times S$. In the first chapter we will, among other things, investigate this further.

Before we turn to an example, we use the opportunity to introduce a bit more terminology which we will use in Chapter 2, Theorem 2.17 and Chapter 4.

Definition 1.3 Let S, T be partial semigroups.

- Let S, T be semigroups. A map $\varphi : S \to T$ is a *(semigroup) homomorphism* if
$$(\forall s, s' \in S)\ \varphi(s \cdot s') = \varphi(s) \cdot \varphi(s').$$
As usual, we call a bijective homomorphism an *isomorphism*.[4]
- A map $\varphi : S \to T$ is a *partial semigroup homomorphism* if
$$(\forall s \in S)(\forall s' \in \sigma(s))\ \varphi(s \cdot s') = \varphi(s) \cdot \varphi(s').$$
In particular, the equality is meant to imply $t \in \sigma(s) \Rightarrow \varphi(t) \in \sigma(\varphi(s))$.
We call a bijective partial semigroup homomorphism a *partial semigroup isomorphism* if the inverse map is a partial semigroup homomorphism as well.[5]

[4] Bijectivity is enough to imply that the inverse map is a homomorphism.

[5] As opposed to the semigroup case, we need to differentiate between bijective homomorphisms and isomorphisms, since a bijection is not enough to guarantee that compatible images have compatible preimages.

One of the most important examples of a partial semigroup is the following.

Definition 1.4

- We define
$$\mathbb{F} := \{s \subseteq \omega \mid s \text{ finite}\}.$$
Note that we include $\emptyset$.
- We define a partial semigroup structure first on $\mathbb{F} \setminus \{\emptyset\}$ by
$$s \cdot t := s \cup t \text{ if and only if } \max(s) + 1 < \min(t).$$
We extend this to $\mathbb{F}$ simply by allowing $\emptyset$ to be a two-sided identity.

Since this semigroup is usually defined in a slightly different way, we remark the following.

Remark 1.5 Note that this definition deviates from the standard operation on $\mathbb{F}$ as found in the literature (cf. [Bla87b]) in two ways: $\emptyset$ is usually excluded and compatibility requires only $\max(s) < \min(t)$ or simply $s \cap t = \emptyset$.

However, our choice leads to nearly the same filter and hence nearly the same $\delta\mathbb{F}$ – the filter is not free but infinite, and only $\emptyset$ is added to $\delta\mathbb{F}$. This complication is compensated by certain advantages of the chosen definition which we will explore in Chapter 2.[6]

[6] We should note that our definition is also present in the literature; cf. [HS98, Chapter 5, Section 1]

Of course it should be noted that all three multiplications are restrictions of the Boolean group operation on $\mathbb{F}$, i.e., Δ, the symmetric difference of sets.

The theory of the Stone-Čech compactification allows for the (not necessarily unique) extension of any operation on the set to its compactification, in particular our semigroup operation. We will restrict this introduction to the following brute force definition with some additional remarks afterwards. This definition is the central one for the field. We cannot overemphasize how important it is to familiarize oneself with this notion – its technical formulation as well as its mathematical meaning.

For a complete introduction we refer to [HS98, Chapter 4].

Definition 1.6 Let $(S, \cdot)$ be a semigroup.

- For $p, q \in \beta S$ we define an operation

$$p \cdot q := \{A \subseteq S \mid (\exists V \in p)(\exists (W_v)_{v \in V} \text{ in } q) \bigcup_{v \in V} v \cdot W_v \subseteq A\}$$
$$= \{A \subseteq S \mid \{s \in S \mid s^{-1}A \in q\} \in p\},$$

  where $s^{-1}A := \{t \in S \mid st \in A\}$.[7]

  This notation is standard and should not cause confusion with the original operation; cf. the remark below.

- Given $A \subseteq S$ and $q \in \beta S$ we denote

$$A^{-q} := \{s \in S \mid s^{-1}A \in q\}.$$

- Given $A \subseteq S$ and $q \in \beta S$ we denote

$$A^* := A^{-q} \cap A.$$

  Note that the A^* notation will only be used when there is no confusion regarding the chosen ultrafilter. Similarly we use this notation only when there is no confusion with the Stone-Čech remainder $\beta A \setminus A$.

[7] Note that as usual this does not mean to imply any kind of inverse or cancellativity condition.

This definition seems to appear out of nowhere. However, underneath this definition lies the basic theory of the field – indispensable for our work yet in a way a ladder to be cast away after climbing it. Since it is too complex and long to introduce here, we restrict ourselves to quoting some important properties of this definition without any details. Instead, we wish to focus on those aspects that we consider essential for following the results of this thesis.

Remark 1.7 The operation is well defined, i.e., the product is in fact an ultrafilter. The operation is associative and ρ_p is continuous for any $p \in \beta S$; we call such a semigroup *right-* or more generally *semi-topological* (if instead the functions λ_p are continuous). Additionally, λ_s is continuous for $s \in S$.

The operation extends the operation on S, i.e., $\dot{s} \cdot \dot{t} = \dot{(st)}$. In fact, for any $p, q \in \beta S$

$$p \cdot q = p\text{-}\lim_{s \in S}(q\text{-}\lim_{t \in S} s \cdot t).$$

In particular, the operation arises from the operation on S by means of the Stone-Čech compactification's extension poperties.

For the introduced notation we can immediately check that for $A \subseteq S$ and $p, q \in \beta S$

$$A \in p \cdot q \Leftrightarrow A^{-q} \in p.$$

Among other algebraic aspects, βS always has a unique minimal ideal denoted by $K(\beta S)$.

For these and further technical aspects, we refer to [HS98, Chapters 3–4].

Let us quickly observe that in the case of a partial semigroup ultrafilters in δS in a way multiply as if the partial operation was total.

Proposition 1.8
Let $(S, \cdot)$ be a partial semigroup and $S_\perp$ its extension from Remark 1.2. Then

$$p \in \beta S, q \in \delta S \Rightarrow p \cdot q \neq \perp,$$

i.e., the multiplication on elements of p and q is essentially always well-defined.

Proof. Simply observe that for $V \in p$ and $(W_v)_{v \in V}$ in q we may assume (since $q \in \delta S$) that $W_v \subseteq \sigma(v)$. □

For a more thorough analysis, especially of the fact that δS is a semigroup, see Chapter 2 and Appendix B. Additionally, this also has a kind of converse. For a partial semigroup S there is a partial semigroup structure on βS that extends it; this can be found in Section B.1.

We will re-visit this definition and generalize it in Chapter 2. Since there is no specific notation in the literature for the sets we describe as A^{-q}, we wish to not only introduce this notation, but allow ourselves to digress and motivate why this notation is efficient and why we believe that it has the potential to be adopted more generally.

Proposition 1.9
Let $p, q, r \in \beta S$, $A \subseteq S$ and $s, t \in S$.

- $t^{-1}s^{-1}A = (st)^{-1}A$
- $s^{-1}A^{-p} = (s^{-1}A)^{-p}$
- $(s^{-1}A)^* = s^{-1}A^*$ *(with respect to the same ultrafilter).*
- $(A^{-q})^{-p} = A^{-pq}$
- *In particular, the multiplication on βS is associative.*

Proof. **Claim 1:** We can simply calculate

$$\begin{aligned} t^{-1}s^{-1}A &= \{x \in S \mid tx \in s^{-1}A\} \\ &= \{x \mid s(tx) \in A\} \\ &= \{x \mid (st)x \in A\} = (st)^{-1}A. \end{aligned}$$

Note that we only used associativity; hence this statement does not imply any cancellativity. This is a fortunate coincidence of notation rather than a meaningful observation (as compared to, say, the case of a group).

Claim 2: We can calculate

$$\begin{aligned} s^{-1}A^{-p} &= \{t \mid st \in A^{-p}\} \\ &= \{t \mid (st)^{-1}A \in p\} \\ &\overset{!}{=} \{t \mid t^{-1}(s^{-1}A) \in p\} \\ &= \{t \mid t \in (s^{-1}A)^{-p}\} \\ &= (s^{-1}A)^{-p}. \end{aligned}$$

Claim 3: We can now calculate for A^* (with respect to say p)

$$\begin{aligned} (s^{-1}A)^* &= s^{-1}A \cap (s^{-1}A)^{-p} \\ &= s^{-1}A \cap s^{-1}A^{-p} \\ &= s^{-1}(A \cap A^{-p}) \\ &= s^{-1}A^*. \end{aligned}$$

Claim 4: We can calculate (with the preceding remark in mind)

$$\begin{aligned} (A^{-q})^{-p} &= \{s \mid s^{-1}A^{-q} \in p\} \\ &= \{s \mid (s^{-1}A)^{-q} \in p\} \\ &= \{s \mid s^{-1}A \in pq\} \\ &= A^{-pq}. \end{aligned}$$

Claim 5: Finally, we can observe

$$A \in p(qr) \Leftrightarrow A^{-qr} \in p \Leftrightarrow (A^{-r})^{-q} \in p \Leftrightarrow A^{-r} \in pq \Leftrightarrow A \in (pq)r. \qquad \square$$

The proverbial big bang for the theory of ultrafilter semigroups is the following theorem.

Theorem 1.10 (Ellis-Numakura Lemma)
If $(S, \cdot)$ is a compact, semi-topological semigroup then there exists an idempotent *element in S, i.e., an element $p \in S$ such that*

$$p \cdot p = p.$$

Ellis-Numakura Lemma

The theorem was proved for topological, i.e., fully continuous, semigroups by K. Numakura [Num52][8] and A. Wallace [Wal52], but the first proof for semi-topological semigroups is attributed to R. Ellis [Ell58], although in that paper Ellis himself calls the lemma "probably well-known" – however nowhere in the literature seems to be any indication that this was known earlier and certainly no evidence that it had been published before Ellis.[9]

The proof is elegant and simple; we refer the reader to [Ell58], [HS98, Theorem 2.5] or even [org] Wikipedia (under the incorrect label Ellis-Nakamura Lemma).

For the central application of the Ellis-Numakura Lemma we introduce some more notation that we will heavily apply throughout this thesis.

[8] A link to a digital copy of the paper is included in the bibliographic entry.

[9] cf. also [HS98, notes on Chapter 5].

FP-set

Definition 1.11
- For any sequence $x = (x_n)_{n\in\omega}$ in a semigroup $(S, \cdot)$ define the set of finite products (the *FP-set*)

$$FP(x) := \{\prod_{i\in F} x_i \mid F \in \mathfrak{P}_f(\omega)\}$$

where products are always taken in increasing order of the indices.

FP^∞-filter

- In the above setting we let

$$FP_k(x) := FP(x'),$$

with $x'_n = x_{n+k}$ for all n. Additionally, let $FP^\infty(x)$ be the filter generated by $FP_k(x)$ $(k \in \mathbb{N})$.

FS-set

- For an abelian semigroup $(S, +)$ (written additively), we traditionally call the same sets $FS(x)$ etc. accordingly (for finite sums).

FU-set

- In $\mathbb{F}$ we additionally consider sequences $s = (s_i)_{i\in\omega}$ of disjoint elements of $\mathbb{F}$ and write similarly

$$FU(s) = \{\bigcup_{i\in F} s_i \mid F \neq \emptyset \text{ and } F \in [\omega]^{<\omega}\}.$$

These kinds of sets are of importance throughout Chapter 4. Note that FU-sets are not FP-sets with respect to our partial semigroup operation on $\mathbb{F}$, since that would additionally require the s_i to be ordered.

We should note that this notation might frequently cause confusion. On the one hand, we denote elements by s, x, y, on the other hand whole sequences of elements. However, since we are concerned with the algebraic operations of the members of such sequences, it would cause more confusion to introduce yet another notation for sequences; we will naturally try to minimize confusion by keeping these notations apart, e.g., speak about a sequence $s = (s_i)_{i\in\omega}$ and $x \in FP(s)$.

Let us quickly remark for future reference.

Remark 1.12 For any semigroup, FP-sets (including FU-sets) obtain a natural partial semigroup structure induced by the map

$$\mathbb{F} \to FP(x), s \mapsto \prod_{i\in s} x_i.$$

Similarly, this map induces a partial semigroup epimorphism between any two FP-sets.

Let us note for later reference one very important observation that is used in the proof of the Glavin-Glazer Theorem.

Galvin Fixpoint Lemma

Lemma 1.13 (Galvin Fixpoint Lemma)
Let $p \in \beta S$ be idempotent.

Then for $A \in p$ we have $A^* \in p$ and $(A^*)^* = A^*$.

Proof. **Claim 1:** $A^* \in p$ simply by Definition 1.6 and since $p \cdot p = p$.

Claim 2: $(A^*)^* = A^*$.

By definition, "$\subseteq$" holds. For the reverse inclusion pick any $a \in A^*$. We wish to show that $a \in (A^*)^*$.

By our choice already $a \in A^*$, so it remains to show that $a^{-1}A^* \in p$.

Since $a \in A^*$, we know that $a^{-1}A \in p$, which by Claim 1 implies

$$(a^{-1}A)^* \in p.$$

But by Proposition 1.9

$$(a^{-1}A)^* = a^{-1}A^*,$$

as desired. □

Now we can prove the central tool of the field, the Galvin-Glazer Theorem; for its history we refer to the notes at the end of this section. We begin with the central lemma for the proof.

Theorem 1.14 (Galvin-Glazer Theorem) Galvin-Glazer Theorem

Let $(S, \cdot)$ be a semigroup, $p \in \beta S$ idempotent and $A \in p$.

Then there exists $x = (x_i)_{i \in \omega}$ in A such that

$$FP(x) \subseteq A.$$

Additionally, if p is free, then x can be chosen without repitition.

Spoiler *We begin with two easily checked claims about sets of the form A^* and proceed to construct the sequence from such sets.*

Proof. Let p be idempotent, $A \in p$.

Step 1: $A^* \in p$ and $(A^*)^* = A^*$ by Lemma 1.13.

Now we are ready to describe the construction.

Step 2: To get the inductive construction started, we pick any $x_0 \in A^* = (A^*)^*$ and let $A_0 = A^*$ as well as

$$A_1 := x_0^{-1}A^* \cap A^*.$$

Note that $A_1 \in p$.

Step 3: After induction step n we have constructed elements $x_0, \ldots, x_n$ and sets

$$A_{i+1} = \bigcap_{y \in FP(x_0, \ldots, x_i)} y^{-1}A^* \cap A^* \qquad (i \leq n),$$

with $A_{i+1} \in p$ and $x_i \in A_i$.

Step 4: So we can pick $x_{n+1} \in A_{n+1}$ and set A_{n+2} as prescribed, hence satisfying the induction hypothesis.

It is very easy to check by induction on the length that the constructed sequence $x = (x_n)_{n \in \omega}$ is as desired.

Step 5: If p is free, all the A_i are infinite, so we can pick x_{n+1} to be distinct from the previously chosen x_i. If p is fixed, say $p = \dot{s}$, then $s \cdot s = s$ and $s \in A$; hence the constant sequence $x_i = s$ suffices – which would have to be the output anyway since quite possibly $A = \{s\}$. □

A most important corollary to this theorem is Hindman's Theorem which was originally proved combinatorially for the semigroup $(\mathbb{N}, +)$. We will discuss its history at the end of this section.

Hindman's Theorem

Theorem 1.15 (Hindman's Theorem)
Let $S = A_0 \dot\cup A_1$. Then there exists $i \in \{0,1\}$ and a sequence x such that

$$FP(x) \subseteq A_i.$$

Proof. By the Ellis-Numakura Lemma 1.10, pick $p \in \beta S$ idempotent. Since p is a prime filter, there exists $i \in 2$ with $A_i \in p$. Applying the Galvin-Glazer Theorem 1.14 we find the desired sequence in A_i. □

We note another useful and easily checked corollary which is folklore.

Corollary 1.16
Let $p \in \beta S$ be idempotent, $A \in p$ and $(B_i)_{i \in \omega}$ a sequence in p.

Then we can find a sequence $x = (x_i)_{i \in \omega}$ in A such that

$$(\forall i \in \omega)\ FP_i(x) \subseteq A \cap B_i.$$

Proof. In the proof of the Galvin-Glazer Theorem just intersect the sets called A_i with $\bigcap_{j \leq i} B_j^*$ – this finite intersection will, of course, still be in p. □

Idempotent ultrafilters are quite abundant and they allow for an interesting partial ordering.

Definition 1.17 For idempotent $p, q \in \beta S$ we define

$$p \leq q \text{ if } p \cdot q = q \cdot p = p.$$

The natural generalizations (with only one product equaling p) will not be relevant for this thesis. This partial order always has minimal elements and moreover there is a minimal below each idempotent. Minimal idempotents are extremely interesting objects and their elements are combinatorially rich; they always lie in the minimal ideals $K(\beta S)$. We also note the following useful lemma.

Lemma 1.18
For every idempotent $p \in (\beta\mathbb{N}, +)$ we have

$$(\forall n \in \mathbb{N})\ n \cdot \mathbb{N} \in p.$$

Proof. This is [HS98, Lemma 6.6] □

We end the technical observations with a few definitions of special kinds of ultrafilters which we will encounter throughout this thesis. We introduce only the basic definitions and discuss variants when they are needed.

Definition 1.19

Definition: Some Ultrafilters

- An ultrafilter $p \in \beta\omega$ is called *selective or Ramsey* if

$$(\forall f \in \omega^\omega)(\exists A \in p)\ f \restriction A \text{ is constant or injective.}$$

  We will use "selective" throughout this thesis.

- An ultrafilter $p \in \beta\omega$ is called a *P-point (or δ-stable)* if

$$(\forall f \in \omega^\omega)(\exists A \in p)\ f \restriction A \text{ is constant or finite-to-one.}$$

- An ultrafilter $p \in \beta\omega$ is called a *strong P-point* if for all sequences $(C_i)_{i\in\omega}$ with each $C_i \subseteq p$ a closed subset of $\mathfrak{P}(\omega)$, there exists $(n_i)_{i\in\omega}$ such that

$$(\forall X_i \in C_i) \bigcup_{i\in\omega} X_i \cap [n_i, n_{i+1}] \in p.$$

- An ultrafilter $p \in \beta\omega$ is called a *Q-point* if

$$(\forall f \in \omega^\omega \text{ finite-to-one})(\exists A \in p)\ f \restriction A \text{ is one-to-one.}$$

- An ultrafilter $p \in \beta\omega$ is called *rapid (or weak Q-point)* if the natural enumerations of its elements constitutes a dominating family in ω^ω, i.e.,

$$(\forall f \in \omega^\omega)(\exists A \in p)(\forall n \in \omega)\ |A \cap f(n)| \leq n.$$

- An ultrafilter on $\mathbb{F}$ with a base of FU-sets is called a *union ultrafilter*. (union ultrafilter)

- An ultrafilter on ω with a base of FS-sets is called a *(strongly) summable ultrafilter*. (summable ultrafilter)

We we will study these kinds of ultrafilters throughout this thesis. As noted earlier the definition of FU-set and hence of union ultrafilters at first seems different from that of FS-sets and summable ultrafilters. However, for multiple reasons which we will also extensively discuss in Chapter 4 this is the natural definition in the sense that union ultrafilters are in $\delta\mathbb{F}$ and idempotent with respect to our multiplication on $\mathbb{F}$.[10] Nevertheless, the different variants of union ultrafilters that we will discuss in Chapter 4 offer some subtle and intricate properties in that respect.

[10] cf. also Examples 2.6

Historically, before the connection between idempotency and Hindman's Theorem was known, ultrafilters in the closure of all idempotents were called *weakly summable utrafilters*. Since idempotency is the established and preferrable notion, we will never speak about weakly summables and hence speak only of summable ultrafilters whenever we mean strongly summable ultrafilters.

The history of Hindman's Theorem and what is called the Galvin-Glazer Theorem is intriguing and we would like to discuss it a little. The theorem itself was conjectured by Graham and Rothschild in [GR71]. According to

[HS98, notes on Chapter 5] the proof of what is now known as the Galvin-Glazer Theorem was developed (but never published) by Galvin around 1970, although he was not aware that idempotent ultrafilters existed (he called them "almost left translation invariant ultrafilters"). Hindman answered the reverse question consistently in [Hin72] by constructing a strongly summable ultrafilter under the assumption of the theorem as well as *CH*.[11] Hindman afterwards proved the Finite Sums Theorem purely combinatorially [Hin74], but as he charmingly put it in the abstract of his talk at the UltraMath conference in 2008:

[11] This assumption was later reduced to *MA*. Nowadays it is known that the question of the existence is closely related to the covering number of the meager ideal being 2^ω; cf. [Eis02]

> *I never understood the original complicated proof (no, I did not plagiarize it), so when I was made aware of the above facts, I began a career-long love affair with the algebraic structure of the set of ultrafilters on a discrete semigroup S.*
>
> *N. Hindman*

A shorter proof was published soon afterwards by Baumgartner in [Bau74]. In 1975, according to [HS98, notes on Chapter 5], Galvin could ask Glazer, whether the ultrafilters that Galvin had been looking for existed – whereupon Glazer almost immediately replied "yes", apparently leading to Galvin responding that it cannot be that simple. However, Glazer knew $\beta\mathbb{N}$ as the maximal semi-topological semigroup compactification of the natural numbers and he was well aware of the existence of idempotent elements – which he could easily check to be "almost translation invariant".

In [HS98, notes on Chapter 12] it is noted that it took another decade to renew interest in summable ultrafilters when van Douwen asked if summable ultrafilters could be constructed using *ZFC* alone. This turned out not to be the case, cf. [Bla87b] and [BH87]. We will discuss this fact extensively in Chapter 4.

1.4. Regarding the forcing method

In Chapter 3 we will work with the forcing method. However, our results are not very technical from a forcing point of view. Instead we will mostly work combinatorially, i.e., analyze partial orders and what kind of generic objects forcing with them would adjoin. Hence we will not deviate in any way from standard forcing notation and concepts. For an introduction we suggest [Kun80] and [Jec03].

Chapter 2

Idempotent Filters

In this chapter we will introduce the central notion of this thesis, idempotent filters. These filters are implicitly used in many applications of the algebra in the Stone-Čech compactification. We begin by showing that idempotent filters induce closed subsemigroups and that the notion is a natural extension of many concepts of the field. We also develop a basic theory for this notion. From a set theoretic point of view, idempotent filters can be considered the natural conditions for forcing constructions when aiming at adjoining an idempotent ultrafilter; we will investigate that aspect in Chapter 3.

2.1. *Idempotent filters*

We begin with the definition of our main object of interest, idempotent filters. The definition for the product of filters is completely analogous to the product of ultrafilters in Definition 1.6.

Definition 2.1 Let $(S, \cdot)$ be a semigroup.

Definition: Filter Product, Idempotent Filter

- For filters F_1 and F_2 on S let

Combinatorial product

$$F_1 \cdot F_2 := \{A \subseteq S \mid (\exists V \in F_1)(\exists (W_v)_{v \in V} \text{ in } F_2) \bigcup_{v \in V} v \cdot W_v \subseteq A\}$$
$$= \{A \subseteq S \mid \{s \in S \mid s^{-1}A \in F_2\} \in F_1\}$$

  be the *combinatorial product* or the *filter product* (of F_1 and F_2).

- We call a filter F on S with $F \cdot F \supseteq F$ an *idempotent filter*.

Idempotent filter

The notion of idempotent filter is implicit in many important techniques of the field such as [HS98, Theorems 4.20, 4.21]. Explicitly, the only application seems to be in characterizing semigroup compactifications, cf. [HS98, Section 21.4].

The filter product was suggested to me as a potential starting point for my own research by Andreas Blass while I was visiting the University of Michigan, Ann Arbor, in the winter 2007/2008. The original motivation for this definition will be developed in Chapter 3, but the notion turned out to have many more applications and developed into a versatile tool for the analysis of the algebra in βS.

As a warm-up exercise we check that this kind of product does indeed lead to a filter.

Proposition 2.2
Let F_1 and F_2 be filters on a semigroup $(S, \cdot)$. Then $F_1 \cdot F_2$ is a filter.

Proof. Since $F_1 \cdot F_2$ is closed under supersets and clearly does not include the empty set, we only need to check that it is closed under finite intersections.

For simplicity, let us consider only two sets, $A, B \in F_1 \cdot F_2$.

We may assume that $\bigcup_{v \in V} v \cdot W_v \subseteq A$ and $\bigcup_{u \in U} u \cdot T_u \subseteq B$ for appropriate sets $V, U \in F_1$ and W_v, T_u from F_2. Then clearly

$$\emptyset \neq \bigcup_{v \in V \cap U} v \cdot (W_v \cap T_v) \subseteq A \cap B \in F_1 \cdot F_2,$$

as desired. □

Let us note a corollary with a similarly easy proof that will be extremely helpful.

Corollary 2.3
Any union of a family of coherent idempotent filters generates an idempotent filter.

Proof. Since the filters are coherent, their union generates a filter.

Hence we just have to check the idempotency. For simplicity, pick only two sets say $A \in F_1, B \in F_2$ from the union of filters.

By idempotency of F_1, F_2, we may assume that

$$\bigcup_{v \in V} v \cdot W_v \subseteq A \text{ and } \bigcup_{u \in U} u \cdot T_u \subseteq B$$

for appropriate sets $V, U \in F_1$ and W_v, T_u from F_2.

But then (confusing $F_1 \cup F_2$ with the filter it generates)

$$\bigcup_{v \in V \cap U} v \cdot (W_v \cap T_v) \subseteq A \cap B \in (F_1 \cup F_2) \cdot (F_1 \cup F_2).$$

Since the sets $V \cap U$, $W_v \cap T_v$ are in the generated filter, we have completed the proof. □

Since the concept of idempotent filters is not used in the literature, it might not be apparent that there are many examples implicitly available. We state the most important ones after a basic and important proposition.

The following topological characterization of the filter product suggested by Andreas Blass indicates in particular that idempotency is considerably stronger than to be corresponding to a closed subsemigroup.

Proposition 2.4
Let F_1, F_2 and F_3 be filters on S. The first two of the following statements are equivalent and imply the third one:

1. *For all $q \in \overline{F_1}$ and sequences $(p_s)_{s\in S}$ in $\overline{F_2}$ we have*
$$q\text{-}\lim_{s\in S}(s \cdot p_s) \in \overline{F_3}.$$
2. *$F_1 \cdot F_2 \supseteq F_3$, i.e., $\overline{F_1 \cdot F_2} \subseteq \overline{F_3}$.*
3. *$\overline{F_1} \cdot \overline{F_2} \subseteq \overline{F_3}$, i.e., for all $q \in \overline{F_1}$ and all $p \in \overline{F_2}$ we have*
$$q \cdot p = q\text{-}\lim_{s\in S}(s \cdot p) \in \overline{F_3}.$$

Proof. **Claim 1:** (1)→(2). (1)→(2)

For this we pick any $A \in F_3$. We want to show that $A \in F_1 \cdot F_2$, in other words we need to show that

$$\{s \in S \mid s^{-1}A \in F_2\} \in F_1.$$

Step 1: So let us assume to the contrary, that it is not.

Step 2: Since F_1 is a filter, this implies that its complement Fix a "bad" $q \in \overline{F_1}$

$$V := \{v \in S \mid v^{-1}A \notin F_2\}$$

is compatible with F_1. So fix some $q \in \beta S$ extending F_1 with $V \in q$.

Step 3: By choice of V, we can similarly find for every $v \in V$ some $p_v \in \beta S$ extending F_2 but including $W_v = S \setminus v^{-1}A$; additionally, let us fix some arbitrary $p_s \in \overline{F_2}$ for $s \notin V$. Fix "bad" $p_s \in \overline{F_2}$

Step 4: Now we can calculate the following: Towards a contradiction

$$\begin{aligned} V = &\{v \in V \mid v^{-1}A \notin p_v\} \in q \\ \Rightarrow &\{s \in S \mid s^{-1}A \notin p_s\} \in q \\ \text{iff } &\{s \in S \mid A \notin s \cdot p_s\} \in q \\ \text{iff } &A \notin q\text{-}\lim_{s\in S}(s \cdot p_s). \end{aligned}$$

Step 5: But by our construction of q and $(p_s)_{s\in S}$ and assumption (1) this limit must include A (since it includes all of F_3) ⚡⚡⚡ a contradiction.

Claim 2: (2)→(1). (2)→(1)

For this let us pick any $q \in \overline{F_1}$ and $(p_s)_{s\in S}$ in $\overline{F_2}$; we then define $x := q\text{-}\lim_{s\in S}(s \cdot p_s)$.

Claim: For every $A \in F_3$ we have $A \in x$.

Step 1: By definition $q\text{-}\lim_{s\in S}(s \cdot p_s) = \{X \subseteq S \mid \{s \mid s^{-1}X \in p_s\} \in q\}$.

Step 2: But by assumption (2),

$$\{v \in S \mid v^{-1}A \in F_2\} \in F_1.$$

Step 3: Since $F_1 \subseteq q$, $F_2 \subseteq p_v$ we can deduce $A \in x$.

(1) or (2)→(3)

Claim 3: Let us finish the proof by arguing that both (1) and (2) imply (3).

(3) follows from (1) immediately. Alternatively, (3) follows from (2): if $q \in \overline{F_1}$ and $p \in \overline{F_2}$, then it is trivially seen by our combinatorial definition of multiplication in βS, i.e., Definition 1.6, that $F_3 \subseteq F_1 \cdot F_2 \subseteq q \cdot p$ as desired. □

Let us repeat an essential corollary to the proposition.

Corollary 2.5
Every idempotent filter induces a closed subsemigroup; in particular, it can be extended to an idempotent ultrafilter.

Proof. Let F be idempotent. By the above proposition $\overline{F}$ is a closed subsemigroup of βS. Hence by the Ellis-Numakura Lemma 1.10 $E(\overline{F}) \neq \emptyset$ and such an idempotent by definition of the closure includes F – as desired. □

Now that we have the topological characterization available, we can easily identify many interesting examples as well as properties of this product. We will settle the natural question whether (3) of Proposition 2.4 implies (1) or (2) afterwards.

Example 2.6

FP filters

- For any sequence $x = (x_n)_{n\in\omega}$ in a semigroup $(S, \cdot)$, the filter $FP^\infty(x)$ (cf. Definition 1.11) is idempotent, since

$$\bigcup_{(\prod_{i\in f} x_i)\in FP_k(x)} (\prod_{i\in f} x_i) \cdot FP_{\max(f)+1}(x) \subseteq FP_k(x).$$

Partial semigroups

- In general, for any adequate partial semigroup $(S, \cdot)$, the filter $\sigma(S)$ corresponding to δS (cf. Definition 1.1) is idempotent, since (by strong associativity)

$$\bigcup_{t\in\sigma(s)} t \cdot \sigma(s \cdot t) \subseteq \sigma(s).$$

Closed left ideals

- Let $I \subseteq \beta S$ be a closed left ideal in βS.

  Then the corresponding filter $\bigcap I$ is idempotent – as is easily seen by Proposition 2.4, part (1). The products $s \cdot p_s$ are in the left ideal, as is their limit, since I is closed.

Idempotent from compactifications

- Let γS be another right-topological semigroup compactification, $e \in \gamma S$ idempotent. Then the set of preimages of e under the canonical map from βS corresponds to an idempotent filter; this follows from [HS98, Theorem 21.31].

As promised we also include two examples that show that Proposition 2.4 and Corollary 2.5 are optimal.

Example 2.7
Let $F_1 := p \in K(\beta\mathbb{N})$, $F_2 = \{\mathbb{N}\}$ and $F_3 = \bigcap cl(p + \beta\mathbb{N})$.

Proposition 2.4: (3) $\not\to$ (2)

It is well known, that[1]

$$\overline{F_3} = cl(p + \beta\mathbb{N}) \subseteq cl(K(\beta\mathbb{N})) \neq \mathbb{N}^*$$

and trivially

$$\overline{F_1} + \overline{F_2} = p + \beta\mathbb{N} \subseteq \overline{F_3}.$$

On the other hand, the filter product easily calculates to

$$p + F_2 = \{A \mid |\mathbb{N} \setminus A| < \omega\} = Fr(\mathbb{N}).$$

Thus these filters have property (3) but not property (2) of the preceding proposition.

[1] cf. [HS98, Chapter 4]

We now turn to an example of a closed subsemigroup not derived from an idempotent filter. It is a well known and natural example; a variant of this example has been studied in [BH87] and we will study our own variant in Chapter 4. For the example we work in the semigroup $\delta\mathbb{F}$, cf. Definition 1.4. We should recall that FU-sets are not to be confused with FP-sets with respect to our operation, since for FU-sets we only require the elements to be disjoint, not ordered.

Example 2.8
Let us denote the following set by H:

$$\{p \in \delta\mathbb{F} \mid (\forall A \in p)(\forall n \in \omega)(\exists s = (s_0, \dots, s_n) \text{ disjoint}) \; FU(s) \subseteq A \cap \sigma(n)\}.$$

Then H is a closed subsemigroup of $\delta\mathbb{F}$, but $\bigcap H$ is not an idempotent filter on $\mathbb{F}$.

Proof. **Claim 1:** $H \neq \emptyset$, since to include arbitrarily long finite FU-sets is partition regular by the Graham-Rothschild parameter-sets Theorem; cf. Theorem 4.20.

Claim 2: H is clearly closed since it is defined by a constraint on all members of its elements.

Claim 3: H is a semigroup.

Given $p, q \in H$, $V \in p$, $W_v \in q$ for $v \in V$. We need to show that $\bigcup_{v \in V} v \cdot W_v$ contains arbitrarily long finite FU-sets; so fix $n \in \omega$.

Step 1: Since $V \in p \in H$, we can find $s = (s_0, \dots, s_n)$ with

$$FU(s) \subseteq V.$$

Step 2: Since $W_v \in q \in H$ for all $v \in FU(s)$, we can find $t = (t_0, \dots, t_n)$ with

$$FU(t) \subseteq (\bigcap_{v \in FU(s)} W_x \cap \sigma(s_0 \cup \ldots \cup s_n))$$

Step 3: Then defining $z_i := s_i \cup t_i$ for $i < n$ we can easily check that

$$FU(z) \subseteq \bigcup_{v \in V} v \cdot W_v.$$

Claim 4: $\bigcap H$ is not an idempotent filter.

We know by Proposition 2.4 that being idempotent is equivalent to

$$(\forall q \in H)(\forall (p_s)_{s\in\mathbb{F}} \text{ in } H)\ q\text{-lim}(s \cdot p_s) \in H.$$

We will give a counterexample for this characterization.

Step 4: Let $(B_s)_{s\in\mathbb{F}}$ be a disjoint family of infinite subsets of ω with $\emptyset \notin B_\emptyset$.

Step 5: For $s \in \mathbb{F}$ we define $A_s = \mathfrak{P}_f(B_s)$, the finite subsets of B_s; clearly, these sets have arbitrarily long finite FU-sets and elements from different A_s are disjoint.

Step 6: For $s \in \mathbb{F}$ we pick $p_s \in H$ with $A_s \in p_s$; for better notation let $q := p_\emptyset$.

Step 7: Then $X := \bigcup_{s\in A_\emptyset} s \cdot (A_s \cap \sigma(s)) \in q\text{-}\lim_{s\in\mathbb{F}}(s \cdot p_s)$

We simply calculate

$$\begin{aligned} X \in q\text{-lim}(s \cdot p_s) &\text{ iff } \{s \in \mathbb{F} \mid X \in s \cdot p_s\} \in q \\ &\text{ iff } \{s \in \mathbb{F} \mid s^{-1}X \in p_s\} \in q. \end{aligned}$$

The latter is the case, since by construction $s^{-1}X \supseteq (A_s \cap \sigma(s)) \in p_s$.

Step 8: X does not contain a pair $v \neq w$ with $v \cup w \in X$.

Assume to the contrary that we have $v, w \in X$ with $v \cup w \in X$.

Then there exist $s_0, s_1, s_2 \in A_\emptyset$ and $t_i \in A_{s_i}$ such that $s_0 \cdot t_0 = v$, $s_1 \cdot t_1 = w$ and $s_2 \cdot t_2 = v \cdot w$.

It is easily seen that we must have $s_2 = s_0 \cdot s_1 = s_0 \cup s_1$, since $s_0, s_1, s_2 \in A_\emptyset$ are disjoint from $t_i \in A_{s_i}$ $(i \leq 2)$.

But then similarly $t_2 = t_0 \cup t_1$ ↯↯↯ contradicting that t_0 and t_1 are disjoint from $t_2 \in A_{s_0 \cdot s_1}$.

This concludes the proof. □

With this example we have established that idempotent filter is a stronger notion than (to be corresponding to a) closed subsemigroup. Additionally, we should note that the example is not artificial.

Remark 2.9 This example naturally gives rise to a multitude of similar examples. For example, in Chapter 4 we will encounter a subsemigroup of the above example where the same argument works.

Conversely, we could define H more generally having only sets in its elements fulfilling Schur's Theorem, i.e., including two elements and their product, instead of the finite version of Hindman's Theorem and the argument still works.

Before we continue, we wish to revisit Definition 1.6 that we introduced in the preliminaries for ultrafilters which – as promised – naturally generalizes to our setting.

Definition 2.10 Let F be a filter on a semigroup $(S, \cdot)$ and $A \subseteq S$.

- $A^{-F} := \{s \mid s^{-1}A \in F\}$.
- $A^* := A \cap A^{-F}$

This notation works just as well for filters as it does for ultrafilters.

Remark 2.11 Proposition 1.9 holds for filters as well.

In particular, for F, G be filters on ω,

$$A \in F \cdot G \Leftrightarrow A^{-G} \in F,$$

and if F is idempotent, then $A \in F$ implies $(A^*)^* = A^* \in F$.

Proof. We show the second part, the rest is just like the proof of Proposition 1.9.

Clearly, $A^* \in F$, so we only need to prove the equality.

Step: $(A^*)^* \subseteq A^*$ is true by definition of *.

Step 1: $(A^*)^* = (A^*)^{-F} \cap A^* = (A^*)^{-F} \cap (A^{-F} \cap A)$ by definition of *.

Step 2: $(A^*)^{-F} = (A \cap A^{-F})^{-F}$

Claim 1: $(A \cap B)^{-F} = A^{-F} \cap B^{-F}$ for any A, B and filter F.

$(A \cap B)^{-F} = \{c \mid c^{-1}(A \cap B) \in F\} = \{c \mid c^{-1}A \cap c^{-1}B \in F\}$. Now since F is a filter, this equals $\{c \mid c^{-1}A, c^{-1}B \in F\} = A^{-F} \cap B^{-F}$ – as desired.

Step 3: Just as in Proposition 1.9 we have $A^{-F} \cap (A^{-F})^{-F} = A^{-F} \cap A^{-(F \cdot F)}$.

Claim 2: $A^{-(F \cdot F)} \supseteq A^{-F}$.

$A^{-F} = \{c \mid c^{-1}A \in F\} \subseteq \{c \mid c^{-1}A \in F \cdot F\} = A^{-(F \cdot F)}$ – since $F \subseteq F \cdot F$.

Step 4: Therefore Steps 2 and 4 now yield $(A^*)^{-F} = A^{-F}$.

Step 5: In particular, Step 1 now yields $(A^*)^* = A^{-F} \cap A = A^*$ – as desired. □

To see that idempotent filters yield the essential part of the $()^*$ operation, let us remark the following.

Remark 2.12 Although it may happen for $F \subseteq F'$ that A^* with respect to F is different from A^* with respect to F', this never poses a problem since it is easy to check that A^* at most *increases* with respect to F'.

Hence if F, F' are idempotent, we may as well restrict ourselves to the smaller set A^* with respect to F (since this is in F' as well).

This observation is especially useful whenever we pass from idempotent filters via Corollary 2.5 to idempotent ultrafilters that extend them.

With this definition we can state maybe the most important example of an idempotent filter.

Galvin filters

Lemma 2.13 (Galvin filters)
Let F be an idempotent filter and $A \in F$. Then the sets A^* and $(a^{-1}A^*)_{a \in A^*}$ generate an idempotent filter (included in F). We call this kind of filter a Galvin filter.

Proof. **Step 1:** Since $A \in F$ and F is idempotent, we know from Remark 2.11 that

$$A^* = (A^*)^* = \{a \in A^* \mid a^{-1}A^* \in F\} \in F;$$

hence the sets $A^*, (a^{-1}A^*)_{a \in A}$ generate a filter, say $G \subseteq F$.

Step 2: Therefore it suffices to show that $A^*, (a^{-1}A^*)_{a \in A}$ are included in $G \cdot G$.

For A^* this is easy, since

$$\bigcup_{a \in A^*} a \cdot (a^{-1}A^*) \subseteq A^*;$$

so we are done.

Now fix some $a \in A^*$. Then for $b \in a^{-1}A^*$ we have $ab \in A^*$. So

$$\bigcup_{b \in a^{-1}A^*} b \cdot ((ab)^{-1}A^*) = \bigcup_{b \in a^{-1}A^*} b \cdot (b^{-1}a^{-1}A^*)) \subseteq a^{-1}A^*$$

as desired. □

Let us end this section with a remark regarding partial semigroups since we will focus on partial semigroups in later chapters.

Remark 2.14 Whenever we consider an adequate partial semigroup S, we always assume that idempotent filters are coherent with $\sigma(S)$ and we only consider extensions of idempotent filters that are coherent with $\sigma(S)$ as well – even though there may be other idempotent extensions, especially when our partial operation is the restriction of a semigroup operation. If we need to stress this fact, we say that a filter is *idempotent with respect to δS*.

2.2. *Countably generated idempotent filters*

We have already seen that partial semigroups induce idempotent filters. These filters are by definition generated by $|S|$-many sets – the sets $(\sigma(s))_{s \in S}$. Idempotent filters behave a lot like $\sigma(S)$, so it is natural to ask how similar they really are. Of course, we have already seen in Examples 2.6 an example of a very complicated idempotent filter, e.g., the filter corresponding to a closed left ideal, which will in general not be induced by a partial semigroup. In this section, we will however prove a kind of converse in the case of a countably generated idempotent filter.

The result we will discuss is related to [HS98, Theorem 6.32] and Yevhen Zelenyuk's famous result that there are no non-trivial finite groups in $\beta\mathbb{N}$. It also offers a first insight into the proof of Zelenyuk's Theorem A.9 which we give in the appendix.

To simplify our notation, we introduce a notion which is implicit in [HS98, Theorem 4.21, Definition 7.23].

Definition: Filter homomorphism

Definition 2.15 Let S, T be (partial) semigroups, F an idempotent filter on S and $X \in F$.

A map $h : X \to T$ is *an (idempotent) filter or F-homomorphism*, if there exist $(H_x)_{x \in X}$ in F such that

$$(\forall x \in X)(\forall y \in H_x \cap X)\ h(xy) = h(x)h(y).$$

Since F is idempotent, we may assume that $X^* = X$ and $H_x \subseteq x^{-1}X^*$ for all $x \in X$.

To see that this terminology is a sensible one, let us make a few simple observations.

Remark 2.16 First, it is easy to check that partial semigroup homomorphisms are always $\sigma(S)$-homomorphisms.

Second, whenever h is an F-homomorphism, then the continuous extension

$$\overline{h} : \overline{F} \to \beta T$$

is a semigroup homomorphism, since

$$\overline{h}(p \cdot q) = p\text{-}\lim_{x \in X} q\text{-}\lim_{y \in H_x} h(xy) = p\text{-}\lim_{x \in X} q\text{-}\lim_{y \in H_x} h(x)h(y) = \overline{h}(p)\overline{h}(q).$$

As usual, we will identify $\overline{h}$ with h.

Third, note for completeness that it is natural to extend this notion to arbitrary products of filters, i.e. $F \cdot G$-homomorphisms, but we have no use for this.

We can now proceed to the main result of this section which strengthens the intuition that countably generated idempotent filters are relatively simple, i.e., only as complex as partial semigroups.

For motivation let us approach the result naively. We wish to show that – given enough cancellativity – any countably generated idempotent filter can be refined to a filter that looks like $\sigma(\mathbb{F})$. Now idempotent filters behave a lot like partial semigroups: given a set A in an idempotent filter, we can look at A^* and we know that $a^{-1}A^*$ is in the filter for every $a \in A^*$.

If we were to define a partial operation on A^* simply by restricting the original operation by letting $\sigma(a) := a^{-1}A^*$ we could already discover many of the nice properties of partial semigroups. We always get a kind of weak associativity, i.e., for any $a \in A^*, b \in a^{-1}A^*$ there would be a filter set of c's where $(ab)c, a(bc)$ are both defined (and equal). However, this way we would still be missing strong associativity. To get a partial semigroup structure, and in fact a rich semigroup structure, we need to handle the construction more delicately.

The additional conclusion of the following theorem may be ignored for now since we will need it only for Zelenyuk's Theorem A.9 in the appendix. We strongly suggest to recall the definition of the partial semigroup $\mathbb{F}$ with the notational conventions of $\sigma(s)$, $\tau(t)$ for partial semigroups from Definition 1.1 as well as the special notation s^+, s^- etc. from Definition 1.4.

Theorem 2.17
Let $(S, \cdot)$ be a countable, cancellative[2] semigroup with identity e.

Let F be a free, countably generated, idempotent filter on S.

Then there exist $X \in F$ and a bijective partial semigroup homomorphism

$$\varphi : \mathbb{F} \to X \quad \text{with } F \subseteq \varphi(\sigma(\mathbb{F})).$$

Moreover, if there is an F-homomorphism $h : X \to \mathbb{Z}_z$ with

$$(\forall A \in F)\ h[A] = \mathbb{Z}_z,$$

then we can construct φ such that additionally

$$h = (|\cdot| \bmod z) \circ \varphi^{-1}.$$

Here $|\cdot|$ simply denotes the cardinality function on $\mathbb{F}$. In other words, φ allows for h to be calculated simply by cardinality $\bmod z$ of the preimage along φ.

[2] See the corollary for a generalization.

"There is a partial semigroup structure on S such that $\sigma(S)$ refines F."

Spoiler *In each step we will construct φ on $\mathfrak{P}(n)$. The strategy is as follows:*

For the homomorphic properties, we (just) have to pick appropriate images for the irreducible elements, i.e., the intervals – which can be done thanks to the idempotency at hand. To get injectivity we use the cancellativity.

Since at each step we have a new, "inert" element of $\mathbb{F}$, i.e., the interval $n = [0, n-1]$, we can ensure surjectivity by mapping it to whatever element we have not yet caught in the image.

The additional requests for h can be dealt with due to the abundance of arbitrary preimages in any filter set (and the above).

In the proof, we heavily use the notation for partial semigroups as introduced in Definition 1.1.

Preparations

Proof. **Step 1:** We require some preparations.

If no h is given, we trivially pick $z = 1$ and h constant.

We fix a descending base $(V_n)_{n\in\omega}$ of F and a well-ordering of S of order type ω.

Since h is an F-homomorphism, we can pick X and $(H_x)_{x\in X}$ in F as in the definition; as noted we may assume $X^* = X$ and additionally that $e \in X$ (changing $h(e)$ to 0 if necessary).

The plan

We construct a homomorphic bijection

$$\varphi : \mathbb{F} \to X$$

by induction on n, more specifically on $\mathfrak{P}(n)$, ensuring the "reconfiguration" of h. The induced partial semigroup structure on X will suffice for the claim of the theorem.[3]

[3] Again we note that this heavily relies on our choice of the operation on $\mathbb{F}$, especially the irreducible elements.

Step 2: To start, we have to define $\varphi(\emptyset) = e$.

Step 3: Assume that we have constructed $\varphi \restriction \mathfrak{P}(n-1)$.

> We can and must focus on the irreducible elements $n \setminus i$ and extend φ to the rest of $\mathfrak{P}(n)$ homomorphically.

We construct $\varphi(n \setminus i)$ for $i < n$ by induction over i.

The construction of φ

Step 4: For $i = 0$ and in the special case that $x^* := \min X \setminus \varphi[\mathfrak{P}(n-1)]$ has

Surjectivity of φ and "re-configuration" of h

$$h(x^*) \bmod z = n \bmod z,$$

we define $\varphi(n) := x^*$ for aforementioned induction at $i = 0$ – otherwise we skip this step and instead choose $\varphi(n)$ as in the next step.

> Note that this does not endanger the homomorphic properties of φ since[4]
>
> $$\tau(n) = \{\emptyset\} = \{\varphi^{-1}(e)\},$$
>
> and $e^{-1}X^* = X^*$.
> Also note that this fact implies that no initial segment is contained in any set $\sigma(s)$; hence these choices do not have any effect regarding our aim for $\sigma(\mathbb{F})$ to be refining F.

[4] Recall $\tau(s)$ from Definition 1.1.

Note for the surjectivity of φ, that we will have forced x^* to be in the range of φ after at most z steps, i.e., once in z steps we will choose $\varphi(n)$ as in this step, not skip it.

Step 5: Now let[5] $(0 <)i < n$ and assume we have constructed $\varphi(n \setminus j)$ for $j < i$. Then following sets are in F:

Getting ready to pick

[5] Include 0 if we skipped the last step.

X	(stay in $X = X^*$)
V_n	(refine F)
$X \setminus \varphi[\mathfrak{P}(n-1) \cup \{n, \ldots, n \setminus (i-1)\}]$	(injectivity)
For all $s \in \tau(n \setminus i)$	
$H_{\varphi(s)}$	(care for h)
$\varphi(s)^{-1}(X \setminus \varphi[\mathfrak{P}(n-1) \cup \{n, \ldots, n \setminus (i-1)\}])$	(compatible)
$X \setminus \varphi(s)^{-1}(\varphi[\mathfrak{P}(n-1) \cup \mathfrak{P}(n-1) \cdot \{n, \ldots, n \setminus (i-1)\}])$	(injectivity)

For the last two sets note that by cancellativity of S, the set

$$\varphi(s)^{-1}\varphi[\mathfrak{P}(n-1)]$$

is finite.[6]

[6] Cf. the corollary.

Step 6: So by our assumptions on h, we may pick $\varphi(n \setminus i)$ from the intersection of those sets and such that

Picking the image

$$h(\varphi(n \setminus i)) = n - i \bmod z.$$

Extending as needed

Step 7: Now we extend φ to $\mathfrak{P}(n)$ homomorphically as follows. For this we have to abide by the following rule

$$\mathfrak{P}(n) = \{n, n \setminus 1\} \cup \dot{\bigcup}_{i<n-1} \mathfrak{P}(i) \cdot n \setminus (i+2).$$

That is for $s \in \mathfrak{P}(n)$ we define[7]

$$\varphi(s) = \varphi(s^-) \cdot \varphi(s^+).$$

[7] Recall s^+ and s^- from Definition 1.4.

Checking injectivity

Step 8: We check that φ is injective.

So take $s, t \in \mathbb{F}$ with $\varphi(s) = \varphi(t)$.

Notice that $\max(s) \neq \max(t)$ is impossible – for if say $\max(s) < \max(t)$ we have $\varphi(s) \neq \varphi(t)$ by the choice of $\varphi(t^+)$, i.e., the last set in the list from Step 5.

Hence we can argue by induction on $\max(s) = \max(t)$, i.e., assume that the claim holds for elements in $\mathfrak{P}(\max(s) - 1)$.

But by our construction of the elements $\varphi(n \setminus i)$, we know that $\varphi(s) = \varphi(t)$ implies $s^+ = t^+$ (or rather, by construction the contraposition holds) due to the third set from Step 5.

Hence by right cancellativity, we may cancel this element and apply our inductive hypothesis.

The rest of the desired properties are immediate from the construction: φ is a partial semigroup homomorphism, it is surjective and $\sigma(n) \subseteq V_n$, hence we refine F. Additionally, we have ensured the connection with h. □

The proof actually yields a little bit more than what we have stated in our theorem.

Corollary 2.18
In Theorem 2.17 we can weaken the assumptions on S to being a countable, right-cancellative and weakly left-cancellative semigroup with identity e.

Proof. In Step 5 of the proof, we only need to require that λ_s is finite-to-one for $s \in S$ (since F is infinite); in Step 8 we required right cancellation. □

Compared to [HS98, Theorem 6.32], our proof is also interesting in terms of reverse mathematics; in particular, our proof can be carried out without the help of an ultrafilter. Unfortunately, there seems to be no analogue for filters generated by uncountably many sets. Although a natural candidate for a partial semigroup equally nice as $\mathbb{F}$ is a Boolean group of higher cardinality, this question is open, but a positive answer would be surprising.

Now that we have developed a basic theory of idempotent filters, we will continue by considering a special class of examples. These will be pivotal in the next chapter. The examples are generated using two very simple functions.

2.3. *Preimage filters under* bmin *and* bmax

One of the strengths of the partial semigroup $\mathbb{F}$ lies in the algebraically useful identification of the natural numbers with finite subsets of natural numbers by means of the binary representation; cf. [HS98, Chapter 6, Section 1].

For the rest of this chapter we will focus on the semigroups $(\omega, +)$, $(\mathbb{N}, +)$ and the partial semigroup $(\mathbb{F}, \cdot)$. We begin an analysis of two simple functions which have already proven useful for understanding summable ultrafilters in [Bla87b], [BH87]. We develop some connections to idempotent filters and other algebraic aspects which will not only complement our investigation of idempotent ultrafilters but also prepare for the forcing constructions in Chapter 3.

Definition 2.19

- We define the *binary support* as usual

$$\mathrm{bsupp}(n) : \omega \to [\omega]^{<\omega}, n \mapsto \mathrm{bsupp}(n)$$
$$\text{defined by} \sum_{i \in bsupp(n)} 2^i = n$$

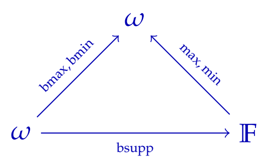
(with the convention that $\Sigma_{i\in\emptyset} = 0$).

- Define the *binary maximum* and *minimum* by

$$\mathrm{bmin} : \mathbb{N} \to \omega, n \mapsto \min(\mathrm{bsupp}(n))$$
$$\mathrm{bmax} : \mathbb{N} \to \omega, n \mapsto \max(\mathrm{bsupp}(n)).$$

- Similarly, we define the *maximum* and *minimum* by

$$\min : \mathbb{F} \setminus \{\emptyset\} \to \omega, s \mapsto \min(s)$$
$$\max : \mathbb{F} \setminus \{\emptyset\} \to \omega, s \mapsto \max(s)$$



- We define

$$\mathbb{H} := \bigcap_{n\in\mathbb{N}} \overline{2^n \cdot \mathbb{N}} = \bigcap_{n\in\mathbb{N}} \overline{FS_n((2^i)_{i\in\omega})}.$$

Let us remark some useful facts.

Remark 2.20 Note that bsupp^{-1} is a partial semigroup homomorphism from $(\mathbb{F}, \cdot)$ to $(\mathbb{N}, +)$. The induced partial semigroup structure on $\mathbb{N}$ is defined accordingly by

$$k \in \sigma(n) \text{ if } \mathrm{bmax}(n) + 1 < \mathrm{bmin}(k).$$

It is easily checked that the Stone-Čech extension of bsupp^{-1} is a semigroup isomorphism between $\delta\mathbb{F}$ and $\mathbb{H}$. By Lemma 1.18 we know that $\mathbb{H}$ contains all idempotent ultrafilters of $(\beta\mathbb{N}, +)$.

The functions bmax, bmin destroy so much information that the preimages tend to be very thick – thick enough to guarantee non-empty intersections and stronger algebraic properties.

We begin by observing the following simple yet important facts telling us that the preimages under these functions are always rich enough for our future purposes.

Proposition 2.21

- *For any* $A, B \subseteq \omega$, $A \neq \emptyset$, $|B| = \omega$
$$\mathrm{bmin}^{-1}[A] \cap \mathrm{bmax}^{-1}[B] \neq \emptyset.$$
- *Accordingly, for free*[8] *filters* F_1, F_2 *on* ω *the union*
$$\mathrm{bmin}^{-1}(F_1) \cup \mathrm{bmax}^{-1}(F_2)$$
generates a filter.
- *For a free filter* F *on* ω *both* $\mathrm{bmin}^{-1}(F)$ *and* $\mathrm{bmax}^{-1}(F)$ *are free idempotent filters (with respect to addition).*
- *Accordingly, for free filters* F_1, F_2 *on* ω *the union*
$$\mathrm{bmin}^{-1}(F_1) \cup \mathrm{bmax}^{-1}(F_2)$$
generates a free idempotent filter.

[8] Infinite filters, i.e., filters containing only infinite sets, would suffice here; cf. the preliminaries.

Proof. We prove the items in order.

Claim 1: By the assumption we find $x \in A$ and $y \in B$ with $x < y$. Then $2^x + 2^y \in \mathrm{bmin}^{-1}[A] \cap \mathrm{bmax}^{-1}[B]$.

Claim 2: Claim 1 implies that the union $\mathrm{bmin}^{-1}(F_1) \cup \mathrm{bmax}^{-1}(F_2)$ has the finite intersection property – as desired.

Claim 3: First we show that $\mathrm{bmin}^{-1}(F)$ is idempotent.

Given $Y \in F$ we set

$$\begin{aligned} V &:= \mathrm{bmin}^{-1}[Y] \\ W_v &:= \mathrm{bmin}^{-1}[Y \setminus \mathrm{bmax}(v) + 1] \qquad \text{(for } v \in V\text{)}. \end{aligned}$$

Since F is free, these sets are in $\mathrm{bmin}^{-1}(F)$; note that the sets W_v imply that $\mathrm{bmin}^{-1}(F)$ is free.

Then for every $w \in W_v$ we have $\mathrm{bmin}(w) > \mathrm{bmax}(v)$.

This implies $\mathrm{bmin}(v + w) = \mathrm{bmin}(v) \in Y$.

Therefore $\bigcup_{v \in W_v}(v + W_v) \subseteq \mathrm{bmin}^{-1}[Y]$ as desired.

Now we show that $\mathrm{bmax}^{-1}(F)$ is idempotent.

Given $Y \in F$ we set

$$V := \text{bmax}^{-1}[Y]$$
$$W_v := \text{bmax}^{-1}[Y] \setminus 2^{\text{bmax}(v)+1} \qquad (\text{for } v \in V).$$

Since F is free and bmax is finite-to-one, these sets are in $\text{bmax}^{-1}(F)$; note that the sets W_v imply that $\text{bmax}^{-1}(F)$ is free.

Then for $w \in W_v$ we have $\text{bmin}(w) > \text{bmax}(v)$.

This implies $\text{bmax}(w + v) = \text{bmax}(w) \in Y$.

Therefore $\bigcup_{v \in V}(v + W_v) \subseteq \text{bmax}^{-1}[Y]$.

Claim 4: This follows from Claims 2 and 3 and the fact that the union of coherent, idempotent filters generates an idempotent filter by Corollary 2.3.□

The analogous proposition holds for $\mathbb{F}$.

Proposition 2.22

- *For any $A, B \subseteq \omega$, $A \neq \emptyset$, $|B| = \omega$*

$$\min^{-1}[A] \cap \max^{-1}[B] \neq \emptyset.$$

- *Accordingly, for infinite filters F_1, F_2 on ω the union*

$$\min^{-1}(F_1) \cup \max^{-1}(F_2)$$

generates a filter.

- *For a free filter F on ω both $\min^{-1}(F)$ and $\max^{-1}(F)$ are free idempotent filters (with respect to $\delta\mathbb{F}$).*
- *Accordingly, for free filters F_1, F_2 on ω the union*

$$\max^{-1}(F_1) \cup \max^{-1}(F_2)$$

generates an idempotent filter (with respect to $\delta\mathbb{F}$).

Proof. The proof is completely analogous to, in fact easier than the proof of the preceding proposition; we include the proof of the third claim.

First we show that $\min^{-1}(F)$ is idempotent.

Given $Y \in F$ we set

$$V := \min^{-1}[Y]$$
$$W_v := \min^{-1}[Y \setminus (\max(v) + 2)] \qquad (\text{for } v \in V).$$

Since F is free, these sets are in $\min^{-1}(F)$; note that the sets W_v imply that $\min^{-1}(F)$ extends $\sigma(\mathbb{F})$.[9]

Then for every $w \in W_v$ we have $\min(w) > \max(v) + 1$.

This implies $\min(v \cdot w) = \min(v \cup w) = \min(v) \in Y$.

Therefore $\bigcup_{v \in W_v}(v \cdot W_v) \subseteq \min^{-1}[Y]$ as desired.

[9] We need $(\max(v) + 2)$ due to our definition of compatibility in $\mathbb{F}$; cf. the discussion following Definition 1.4.

Now we show that $\max^{-1}(F)$ is idempotent.

Given $Y \in F$ we set

$$V := \max^{-1}[Y]$$
$$W_v := \max^{-1}[Y \setminus (\max(v) + 2)] \qquad \text{(for } v \in V\text{)}.$$

Since F is free and max is finite-to-one, these sets are in $\max^{-1}(F)$; note that the sets W_v imply that $\min^{-1}(F)$ extends $\sigma(S)$.

Therefore $\bigcup_{v \in V}(v \cdot W_v) \subseteq \max^{-1}[Y]$ – in the sense of our partial operation, i.e., $v \cdot W_v = v \cdot (W_v \cap \sigma(v))$. □

With this we have completed the basic development that is needed for the subsequent chapters. We will use the remainder of this chapter to study further algebraic properties of these functions. These results are not necessary for understanding the other results of this thesis, but they are interesting in their own right since they give us valuable insight into the algebraic structures of preimage filters for our functions.

Proposition 2.23
Given infinite $A \subseteq \mathbb{N}$ we have the following.

$$(\mathrm{bmin}^{-1}[A])^* \text{ is a (closed) right ideal in } \mathbb{N}^*,$$
$$\mathrm{bmax}^{-1}[A^*] \text{ is a (closed) left ideal in } \mathbb{N}^*.$$

Note that here $()^*$ denotes the Stone-Čech remainder, i.e., the free ultrafilters on the set, and in the lower line bmax *is identified with its Stone-Čech extension.*

Proof. **Claim 1:** $(\mathrm{bmin}^{-1}[A])^*$ is a right ideal.

For the claim we simply observe that

$$\bigcup_{a \in \mathrm{bmin}^{-1}[A]} (a + (\mathbb{N} \setminus (\min(a) + 1))) \subseteq \mathrm{bmin}^{-1}[A].$$

This implies that $(\mathrm{bmin}^{-1}[A])^*$ is a right ideal of $\mathbb{N}^*$.

Claim 2: $\mathrm{bmax}^{-1}[A^*]$ is a left ideal.

Let $p \in \mathbb{N}^*$, $V \in p$. Then

$$\bigcup_{v \in V} v + (\mathrm{bmax}^{-1}\{a \in A \mid \mathrm{bmax}(v) < \mathrm{bmin}(a)\}) \subseteq \mathrm{bmax}^{-1}[A].$$

Since these sets of preimages are almost included in A, this shows that $\mathrm{bmax}^{-1}[A^*]$ is a left ideal of $\mathbb{N}^*$. □

Again we get the analogous result for $\mathbb{F}$.

Proposition 2.24
Given infinite $A \subseteq \mathbb{N}$ we have the following.

$$\overline{\min^{-1}[A]} \cap \delta\mathbb{F} \text{ is a (closed) right ideal in } \delta\mathbb{F}.$$
$$\max^{-1}[\overline{A}] \cap \delta\mathbb{F} \text{ is a (closed) left ideal in } \delta\mathbb{F},$$

After this algebraic warm-up it is interesting to see just how rich such preimages are. For this we recall a standard definition.

Definition 2.25 For any semigroup S and $A \subseteq S$ we say that A is *piecewise syndetic (pws)* if there exists $G \in \mathfrak{P}_f(S)$ such that

$$\{a^{-1}(\bigcup_{s \in G} s^{-1}A) \mid a \in S\}$$

has the FIP.

Note that $A \subseteq \mathbb{N}$ is pws if there exists a bound c such that

$$(\forall n \in \mathbb{N})(\exists i)\ A \cap [i, i+n] \text{ has gaps of size } \leq c.$$

In other words, for $(\mathbb{N}, +)$ pws sets have "locally small gaps".

Piecewise syndetic sets are very important in the analysis of the minimal ideal of βS, however we will only encounter this notion in this section; for an introductory discussion see [HS98, Section 4.4]. We continue by observing the following.

Proposition 2.26
For infinite $A, B \subseteq \mathbb{N}$

$$\mathrm{bmin}^{-1}[A] \cap \mathrm{bmax}^{-1}[B] \textit{ is pws.}$$

Spoiler *We get a bound from any $a \in A$. To find an arbitrarily long interval with gaps of this bound, we just pick a large element $b \in B$, such that adding N, a large multiple of our bound, to does not change the binary minimum and maximum of $2^a + N + 2^b$.*

Proof. We start by fixing an element $a_0 \in A$ as well as some $c > a_0$ and proceed to show that

Claim: the bound 2^c is as desired.[10]

[10] In fact, $A \neq \emptyset$ suffices.

Given arbitrary $n \in \mathbb{N}$ we need to choose an interval of length n such that our set has small gaps on that interval.

Step 1: First we pick x such that

$$n \leq x \cdot 2^c.$$

Step 2: To guarantee that we stay in our set, we want to fix a large binary maximum. For this we simply choose $b \in B$ such that

$$x \cdot 2^c < 2^b.$$

Step 3: Then $[2^{a_0} + 2^b, 2^{a_0} + x \cdot 2^c + 2^b]$ has length at least n.

Step 4: Finally, we can observe that

$$\mathrm{bmin}^{-1}[A] \cap \mathrm{bmax}^{-1}[B] \cap [2^{a_0} + 2^b, 2^{a_0} + x \cdot 2^c + 2^b]$$

contains at least

$$2^{a_0}+2^b, 2^{a_0}+2^c+2^b, \dots, 2^{a_0}+x\cdot 2^c+2^b.$$

So

$$\mathrm{bmin}^{-1}[A]\cap \mathrm{bmax}^{-1}[B]\cap[2^{a_0}+2^b, 2^{a_0}+x\cdot 2^c+2^b]$$

is of length at most n and has gaps of size at most 2^c – as desired. □

We proceed to get an even stronger result. The following definition is very technical, but is essentially an extension to families of pws sets where the pws property is very uniform.

Definition 2.27 Let $(S,\cdot)$ be a semigroup. We call $\mathfrak{A}\subseteq\mathfrak{P}(S)$ *collectionwise piecewise syndetic (cwpws)* if there exists $\theta:\mathfrak{P}_f(\mathfrak{A})\to\mathfrak{P}_f(S)$ such that

$$\{y^{-1}(\theta(F))^{-1}\bigcap F \mid y\in S, F\in\mathfrak{P}_f(\mathfrak{A})\}$$

has the FIP – with the convention

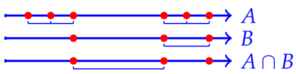
$$y^{-1}(\theta(F))^{-1}\bigcap F := \bigcup_{t\in\theta(F)} y^{-1}t^{-1}\bigcap F = \bigcup_{t\in\theta(F)}(ty)^{-1}\bigcap F.$$

Note that this implies that $\mathfrak{A}$ generates a filter of pws sets.

The function θ picks witnesses for the piecewise syndeticity of (finite intersections of) members of $\mathfrak{A}$ to be uniform in the sense that finitely many sets locally cover the same area; for $(\mathbb{N},+)$ this means that we can choose the bound for gaps of finitely many sets in such a way that such gaps appear at common intervals, i.e., the bound of their finite intersection has locally bounded gaps on intervals where all of the sets have locally bounded gaps as well.

Given the comment at the end of the definition, one might suspect "cwpws" to mean nothing more than "generating a filter of pws sets". However, this seems to be an open question which we do not pursue here. Its difficulty certainly lies in the following problem: even if we have a filter of pws sets, it is not apparent how to choose θ. Of course, we can pick arbitrary witnesses for the individual sets, but these might not work well together; cf. the diagram.

Figure 2.1: Intersecting pws sets.



Even though we know that all finite intersections are pws, we cannot hope to choose a witness for a given set by ''waiting'' until all finite intersections with other sets from the filter have been considered. These problems are closely related to the difficult questions surrounding the minimal ideal and its closure.

The following theorem explains the importance of this notion for the algebra of βS.

Hindman, Lisan

Theorem 2.28 (Hindman, Lisan)
Let $(S,\cdot)$ be a discrete semigroup. Then $\mathfrak{A}\subseteq\mathfrak{P}(S)$ is cwpws if and only if there exists $p\in K(\beta S)$ with $\mathfrak{A}\subseteq p$.

Proof. This is [HS98, Theorem 14.21]; the proof is not difficult, but repeating it here is not necessary for our purpose. □

We can now prove the following.

Theorem 2.29
For free filters F, G on $\mathbb{N}$ the set

$$\{A \cap B \mid A \in \mathrm{bmin}^{-1}(F), B \in \mathrm{bmax}^{-1}(G)\}$$

is cwpws.

Spoiler *The idea of the proof is quite simple, its only difficulty lies in understanding the definition of cwpws. We use the bound from the proof of Proposition 2.26, but we have to show more than before: the bound from that proof works uniformly, i.e., given finitely many members of the filter, the bounded gaps for the intersection appear at intervals where the original sets have gaps of their chosen bound.*

Proof. For convenience let us abbreviate

$$\mathfrak{A} := \{A \cap B \mid A \in \mathrm{bmin}^{-1}(F), B \in \mathrm{bmax}^{-1}(G)\}.$$

Step 1: First we define

$$\begin{aligned} \theta : \mathfrak{A} &\to \mathfrak{P}_f(\mathbb{N}) \\ A \cap B &\mapsto [1, 2^{\min(A)+1}]. \end{aligned}$$

Since the choice of A might not be unique we fix one such choice.

Step 2: Since $\mathfrak{A}$ is closed under intersections, we can extend θ to $\mathfrak{P}_f(\mathfrak{A})$ by

$$\theta(\{C_0, \ldots, C_k\}) := \theta(C_0 \cap \ldots \cap C_k).$$

Step 3: For simplicity consider just two sets $C_0 = A_0 \cap B_0, C_1 = A_1 \cap B_1$ in $\mathfrak{A}$ and a given length n.

Step 4: Fix $x, y \in \mathbb{N}$ such that

$$\begin{aligned} n &\leq x \cdot 2^{\theta(C_0)} = x \cdot 2^{\min(A_0)+1} \\ n &\leq y \cdot 2^{\theta(C_1)} = y \cdot 2^{\min(A_1)+1}. \end{aligned}$$

Step 5: We can find $a \in A_0 \cap A_1$ large enough such that

$$x \cdot 2^{\min(A_0)+1}, y \cdot 2^{\min(A_1)+1} \leq 2^a,$$

and we also fix $b \in B_0 \cap B_1$ with $a < b$.

Step 6: Then just as in the proof of 2.26

$$\begin{aligned} &C_0 \cap [2^a + 2^b, 2^a + 2^b + x \cdot 2^{\min(A_0)+1}] \\ &C_1 \cap [2^a + 2^b, 2^a + 2^b + y \cdot 2^{\min(A_1)+1}] \end{aligned}$$

have gaps of size at most $2^{\min(A_0)+1}$ and $2^{\min(A_1)+1}$ respectively.

Step 7: In particular, both C_0 and C_1 have bounded gaps (of size $\theta(C_1)$ and $\theta(C_0)$ respectively) on

$$[2^a + 2^b, 2^a + 2^b + n]$$

Hence $\mathfrak{A}$ is cwpws. □

This result is quite surprising since F, G are arbitrary. We give an interesting example that motivates our final observations.

Example 2.30
Consider p, an idempotent ultrafilter in $\beta\mathbb{N}$, and its images $p_1 = \text{bmin}(p)$, $p_2 = \text{bmax}(p)$.

We know that the filter generated by $\text{bmin}^{-1}(p_1) \cup \text{bmax}^{-1}(p_2)$ is cwpws by the above theorem; hence its closure intersects the minimal ideal; in particular, this intersection is a semigroup, although not necessarily closed.

However, the same holds true using the closure of the minimal ideal instead of the minimal ideal; hence there exists an idempotent ultrafilter $q \in cl(K(\beta\mathbb{N}))$, such that

$$\text{bmin}^{-1}(p_1) \cup \text{bmax}^{-1}(p_2) \subseteq p \cap q$$

This scenario indicates just how much information is lost with bmin and bmax – preimage filters are not even capable of differentiating between say a right-maximal idempotent (cf. [HS98, Theorem 9.10]) and an idempotent close to the minimal ideal. The obvious question is, if we can even find a minimal idempotent in the preimage filter.

We can in fact generalize this special case a little for which we require the following well-known lemma.

Lemma 2.31
Let $p \in \beta\mathbb{N}, q \in \mathbb{H}$. Then

$$\text{bmin}(p+q) = \text{bmin}(p)$$
$$\text{bmax}(q+p) = \text{bmax}(p).$$

Proof. This is [HS98, Lemma 6.8]. □

Our generalization of the last example is as follows. Recall the partial order of idempotent elements from Definition 1.17.

Proposition 2.32
Given $p \in \beta\mathbb{N}$ idempotent and its images $p_1 = \text{bmin}(p)$, $p_2 = \text{bmax}(p)$, for every idempotent q with $q \leq p$ or $p \leq q$ we have

$$\text{bmin}^{-1}(p_1) \cup \text{bmax}^{-1}(p_2) \subseteq q.$$

Proof. We prove the statement for $q \leq p$; the other case is symmetrical. Remember that all idempotent ultrafilters lie in $\mathbb{H}$; hence we can apply Lemma 2.31.

For the proof we just have to calculate

$$\text{bmin}(q) \overset{q \leq p}{=} \text{bmin}(p+q) \overset{2.31}{=} \text{bmin}(p)$$
$$\text{bmax}(q) \overset{q \leq p}{=} \text{bmax}(q+p) \overset{2.31}{=} \text{bmax}(p).$$

□

A corollary yields even more information about the sets in preimage filters.

Corollary 2.33
For free filters F, G on $\mathbb{N}$, the filter

$$\mathrm{bmin}^{-1}(F) \cup \mathrm{bmax}^{-1}(G)$$

can be extended to a minimal idempotent ultrafilter. In particular, it consists only of central sets.

Proof. By Proposition 2.21, we know that the preimage is idempotent, hence it can be extended to an idempotent ultrafilter by Corollary 2.5.

By the above Proposition, the preimage filter is also included in the minimal idempotent below any idempotent ultrafilter extending the filter – as desired. □

This proposition yields another point of view for the proof of a well-known theorem which can be found as [HS98, Theorem 6.7].

Corollary 2.34
In $\beta\mathbb{N}$ there exist $2^{\mathfrak{c}}$ many disjoint minimal left ideals, $2^{\mathfrak{c}}$ many disjoint minimal right ideals.

Proof. For all free $p \in \beta\mathbb{N}$, the preimage filter

$$\mathrm{bmin}^{-1}(p) \cup \mathrm{bmax}^{-1}(p)$$

extends to a minimal idempotent ultrafilter; with the above lemma, no two minimal ideals generated by such minimal idempotents can intersect. □

2.4. *Synopsis*

In this chapter we have encountered the central objects of this thesis, idempotent filters. We have seen that these filters are abundant, yield closed semigroups and offer a generalization of many important concepts in the field. Additionally, we were able to clarify the similarity between "small" idempotent filters and partial semigroups.

We have also introduced an important tool for later chapters, preimage filters for minima and maxima. For the functions bmin and bmax we have studied their rich algebraic properties. Finally, we have seen that such preimage filters (and other variants using similarly nice functions) might also prove helpful for the analysis of the partial order of idempotent ultrafilters and quite generally to discern different kinds of (idempotent) ultrafilters. We will revisit these aspects shortly in Chapter 3.

Chapter 3

Forcing Idempotent Ultrafilters

The forcing method has proven to be one of the most versatile tools made available to mathematicians by the development of modern set theory in the second half of the twentieth century. However, its applications have been restricted mostly to those areas naturally close to set theory, such as model theory, topology and functional analysis.

There are many set theoretic results, especially forcing results, known with respect to those ultrafilters (on ω) that are classically studied by logicians and set theorists in general (such as selective ultrafilters and P-points). However, these ultrafilters have few interesting algebraic properties, or rather "activity", i.e., they usually cannot even be written as sums of other ultrafilters.[1]

[1] For a lonely exception see PS-ultrafilters and weakly Ramsey ultrafilters in Section B.2.

In this chapter we will establish a general approach to adjoining idempotent ultrafilters on ω by means of the forcing method. Even though, for now, these results may be considered to be a proof of concept which we hope lays a basis for further applications of set theory in the field of algebra in the Stone-Čech compactification.

Throughout this chapter we will work in the semigroup $(\omega, +)$. Most of the results naturally transfer to other semigroups similarly to the generalization of summable ultrafilters in [HPS98], but we have found no evidence for any deeper application of this fact.

3.1. *Forcing summable ultrafilters*

As we have seen the notion of idempotent filters is implicitly present in many concepts of the field. An important example is the most prominent independence result in the field: the existence of summable ultrafilters. In the following section we will sketch the construction in forcing terminology. Compared to the other versions that can be found in the literature, we formulate the result in our own terminology so as to gain some more practice in dealing with idempotent filters before we go on to develop our new results.

We repeat the definition for completeness. We will investigate variants of this definition in the form of union ultrafilters in Chapter 4.

Definition 3.1 An ultrafilter on ω with a base of FS-sets is called *summable ultrafilter*.

Even though forcing ideas are implicit already in [Hin72] and [Bau74] the following result can be considered classical. It was studied first in [Bla87b], although in a different form, cf. the remark after the proof of the theorem.

Forcing summable ultrafilters (Blass)

Theorem 3.2 (Forcing summable ultrafilters (Blass))
Consider

$$\mathbb{P} := \{F \mid F \text{ free, countably generated, idempotent filter on } \omega\},$$

partially ordered by $\supseteq$.

Then forcing with $\mathbb{P}$ *adjoins a summable ultrafilter.*

Spoiler *We apply a standard argument using the Galvin-Glazer Theorem 1.14 to show that* FS_∞*-filters are dense in the partial ordering and decide all subsets.*

Proof. **Claim 1:** The partial order is σ-closed, in fact, every countable chain has an infimum.

On the one hand, the union of countably many countably generated filters is countably generated – and a filter since comparable filters are coherent.

On the other hand, we noted in Corollary 2.3 that the union of coherent idempotent filters is always idempotent.

Hence the union of a given countable chain generates (in fact is) a suitable infimum.

Claim 2: For any $A \subseteq \omega$ the set

$$D_A := \{FS^\infty(x) \mid A \in FS^\infty(x) \text{ or } \omega \setminus A \in FS^\infty(x)\}$$

is dense in $\mathbb{P}$.

First we recall that any FS_∞-filter is idempotent as seen in the examples 2.6, so $D_A \subseteq \mathbb{P}$.

Second we take any $F \in \mathbb{P}$. Then by Corollary 2.5 F can be extended to an idempotent ultrafilter, say $p \in \beta\omega$; of course p is free since F was and we may assume without loss that $A \in p$.

Then fix $(A'_n)_{n\in\omega}$, a (decreasing) base of F and define

$$A_n := A \cap A'_n \qquad (\text{for } n \in \omega).$$

Now applying Corollary 1.16 we find $x = (x_n)_{n\in\omega}$ such that

$$FS_k(x) \subseteq A_k \qquad \text{for all } k \in \omega.$$

Then $FS^\infty(x)$ is in D_A and includes $F \cup \{A\}$, i.e., D_A is dense in $\mathbb{P}$.

Claim 3: The union over a generic object $\mathfrak{G} \subseteq \mathbb{P}$ is a summable ultrafilter.

Since all elements of the generic are compatible, the union over the generic is a union of coherent filters, hence a filter.

By Claim 1 the forcing does not adjoin any countable sets, hence the union over the generic is an ultrafilter if it decides all ground model subsets of ω – but this follows from Claim 2 with the addition that every set in the generic object is included in some FS_∞-filter, i.e., includes an FS-set that is also a member of the union over the generic object. □

We will continue to investigate this proof in more detail since we are are going to generalize the result over the next few sections. But we briefly remark how our version is equivalent to the original approach in [Bla87b].

Remark 3.3 The partial order $\mathbb{P}$ in the last theorem is forcing equivalent to the original notion developed in [Bla87b].

The partial order considered in [Bla87b] is defined by

$$\{s \mid s = (s_i)_{i\in\omega} \text{ disjoint sequence in } \mathbb{F}\},$$

partially ordered by $s \leq t$ if $s \subseteq^* FU(t)$.

By our above proof we can equivalently restrict our forcing $\mathbb{P}$ to its dense set of FS^∞-filters; it is easy to see that

$$FS^\infty(x) \supseteq FS^\infty(y) \iff x \subseteq^* FS(y),$$

so we can identify the FS^∞-filters with their sequences partially ordered similarly to Blass' forcing.

To complete this remark it suffices to see that there is a dense set of sequences with disjoint binary support – then bsupp yields an isomorphism to Blass' forcing.

For this recall that idempotent ultrafilters contain all sets of multiples, i.e., all sets of the form $n \cdot \mathbb{N}$. With this we can modify the proof of Corollary 1.16 to pick x_{n+1} ensuring $2^{x_n} | x_{n+1}$, in particular guaranteeing disjoint binary support. Using this to modify our proof of the above theorem, we get the desired result.

From a set theorist's point of view, the generic object that the above forcing adjoins, carries very interesting properties. Most prominently, although such an ultrafilter is far from being selective, the images under bmin and bmax are selective ultrafilters. The investigation of this relationship in [Bla87b] led to the generalization that we are about to investigate. In the remark we have already seen that the summable ultrafilter adjoined in Theorem 3.2 carries stronger properties than simply being summable. In fact there is an explicit, combinatorial description of this kind of ultrafilter which we will investigate in Chapter 4; this description is also optimal in the sense of complete combinatorics.[2]

[2] This is discussed at the end of [Bla87b, Section 4]. For the terminology of "complete combinatorics" cf. also [Laf89].

3.2. *Daguenet-Teissier's topological families*

Many questions naturally arise from the classical results in the last section. Is the relation between the generic object and selective ultrafilters accidental? Is there a connection in forcing terms? What other idempotent ultrafilters have interesting images? What kind of algebraic properties do they have? Our main goal in the remainder of this chapter is to lift the forcing construction to a general setting. For this we introduce a variant of a definition by Maryvonne Daguenet-Teissier (née Daguenet) originally found in [Dag75]. We investigate the situation in purely forcing-theoretic ways discussing more direct interpretations in terms of [Dag75] in Section 3.4.

Definition: Topological family

Closure under union

Closure under (pre)images

σ-closed

Definition 3.4 A non-empty family Ξ of filters on ω is called a *topological family (of filters)*, if

1. For coherent $F, G \in \Xi$ the filter generated by $F \cup G$ is again in Ξ.
2. Ξ is closed under images and preimages for every $f \in \omega^\omega$ (but only including the proper preimage filters).
3. For any increasing sequence $(F_n)_{n\in\omega}$ in Ξ
$$\bigcup_{n\in\omega} F_n \in \Xi,$$
in other words, Ξ is σ-closed.

We should note that in [Dag75] Daguenet-Teissier's original definition does not include the closure under images. However, the classical examples that we will discuss later share this additional property. For our results this addition is very useful.

We chose the name for this definition because Daguenet-Teissier defines a Ξ-topology to be the topology on $\beta\omega$ having $\{\overline{F} \mid F \in \Xi\}$ as a base of open sets for some similarly defined family Ξ; then the definition guarantees that this is a topology that has many continuous (open) maps. For these topologies Daguenet-Teissier proves a strong version of the Baire-category Theorem and uses the topology to "globally" construct certain kinds of ultrafilters on ω.

Let us begin our investigation with some additional basic properties of topological families.

Proposition 3.5
Let Ξ be a topological family.

Given $F \in \Xi$ and an infinite $A \subseteq \omega$ compatible with F, the filter generated by $\{A\} \cup F$ is again in Ξ.

Spoiler *We use the closure under maps in ω^ω to adjoin A to the filter.*

Proof. Let $f : \omega \to A$ be any bijection; of course $f \in \omega^\omega$.

Claim: $f(f^{-1}(F))$ is the filter we are looking for – and in Ξ by the closure under preimages and images.

To see this, note that for any $X \subseteq \omega$ we have
$$f[f^{-1}[X]] = f[f^{-1}[A \cap X]] = A \cap X$$
by our choice of f; in particular, the preimage filter is proper.

Recall that $f^{-1}(F)$ is the filter generated by $\{f^{-1}[X] \mid X \in F\}$.

Accordingly, $f(f^{-1}(F))$ is the filter generated by
$$\{f[f^{-1}[X]] = A \cap X \mid X \in F\},$$
which is exactly the filter generated by $F \cup \{A\}$ – as desired. □

Let us add a simple consequence of this proposition.

Corollary 3.6
Every topological family contains

- *The fixed ultrafilters.*
- *The countably generated filters.*

Proof. The first part follows from being closed under images of constant maps in ω^ω.

For the second note that reversely the preimage of a fixed ultrafilter under its constant map produces the filter $\{\omega\}$. Hence the above proposition yields that for any $A \subseteq \omega$ the filter

$$\{B \subseteq \omega \mid A \subseteq B\}$$

is in every topological family. With the third property of topological families we can then derive the conclusion. □

The following fact is well known, cf. [Dag75] and [Laf89]. Recall from the preliminaries that we endow $\mathfrak{P}(\omega)$, the power set of ω, with the usual Cantor topology, i.e., the product topology of 2^ω.

Remark 3.7 The following sets are topological families of filters.

- The set of countably generated filters
- The set of F_σ-filters, i.e., filters that are F_σ subsets of $\mathfrak{P}(\omega)$
- The set of $\mathbf{\Sigma}^1_1$-filters, i.e., filters that are analytic subsets of $\mathfrak{P}(\omega)$.
- The set of meager filters, i.e., filters that are meager subsets of $\mathfrak{P}(\omega)$.
- The set of completely meagre filters, i.e., filters whose images under every $f \in \omega^\omega$ are meager or fixed.

We have seen that topological families contain many filters. Since we are concerned with forcing, it is necessary to extract the relevant parts from such families.

We are interested in the following subsets of topological families.

Definition 3.8 Given a topological family Ξ we define

Definition: (Idempotent) Filter Forcing

- $\mathbb{P}_\Xi := \{F \in \Xi \mid F \text{ free}\}$,
- $\mathbb{P}^+_\Xi := \{F \in \Xi \mid F \text{ is idempotent and free}\}$.

and partially order both sets by $\supseteq$.

With this we can conlude the introduction to topological families. The partial order $\mathbb{P}_\Xi$ has been studied by Daguenet-Teissier in [Dag75] and Laflamme in [Laf89]. We will build on their research when investigating the partial order $\mathbb{P}^+_\Xi$ in the next section.

3.3. *The forcing construction*

We are now ready to formulate and prove our results in this generalized setting. For the remainder of this section fix a topological family Ξ.

Forcing Idempotent Ultrafilters

Theorem 3.9 (Forcing Idempotent Ultrafilters)
Forcing with $\mathbb{P}^+_\Xi$ adjoins a free idempotent ultrafilter on $\beta\omega$.

Spoiler *To show that we adjoin an ultrafilter, we use the Galvin filters from Lemma 2.13 as well as the fact that idempotent filters can be extended to idempotent ultrafilters. The rest follows from the properties of topological families.*

Proof. We begin with three simple observations.

The generic is a filter

Claim 1: Just as we have already seen in Theorem 3.2 the union over the generic object will be a filter on ω since all elements are compatible in the partial ordering, hence coherent filters.

The generic is idempotent

Claim 2: Additionally, the union over the generic object will be idempotent since all elments are idempotent, i.e., by Corollary 2.3.

$\mathbb{P}^+_\Xi$ is σ-closed

Claim 3: By property (3) of topological families and again Corollary 2.3, the forcing is countably closed; in particular it does not add new subsets of ω.

Claim 4: Therefore, as in Theorem 3.2, it suffices to show that for any $A \subseteq \omega$ the set

$$D_A := \{F \in \mathbb{P}^+_\Xi \mid A \in F \text{ or } \omega \setminus A \in F\}$$

is dense in $\mathbb{P}^+_\Xi$.

We already know that we could add one set to F by Proposition 3.5. However, this extension will usually not be idempotent; for this we need to add more.

To prove the claim take any $F \in \mathbb{P}^+_\Xi$.

Pick an idempotent ultrafilter

Since F is free and idempotent, we can extend F to a free idempotent ultrafilter $e \in \overline{F}$ (by Corollary 2.5).

We may assume without loss that $A \in e$, hence also $A^* \in e$ by Lemma 1.13.

Help from Galvin Filters

Now let G be the Galvin filter generated by A^* and the sets $\{-a + A^* \mid a \in A^*\}$.

Since every topological family contains the countably generated filters, $G \in \Xi$ and by Lemma 2.13 it is idempotent.

But note that F and G are coherent – since $(F \cup G) \subseteq e$.

Finally, since topological families are closed under compatible unions, $F \cup G$ generates an idempotent filter in D_A. □

Now that we know that every "idempotent filter forcing" $\mathbb{P}^+_\Xi$ does what we wanted all along – add a new idempotent ultrafilter – we are bound to consider the question: is this forcing really new? Indeed, as we noticed early on, any topological family contains the countably generated filters. Therefore we might have introduced nothing more than a very complicated partial order with a separative quotient being isomorphic to the FS-filter forcing from the classical approach that we discussed after Theorem 3.2.

Therefore we need to investigate if different forcings actually yield different results. For now we are ready to give the most general argument which is the main result of this chapter. This theorem enables us to make the connection from $\mathbb{P}_\Xi^+$ to the partial ordering $\mathbb{P}_\Xi$ in a very strong way, namely the images (of the elements) under bmin and bmax are mutually generic for $\mathbb{P}_\Xi$.

We will investigate the examples again afterwards which will give us some understanding of how the forcings differ from each other.

Theorem 3.10
Let $\mathfrak{G}$ be a generic filter for $\mathbb{P}_\Xi^+$. Then

$$\begin{aligned}\mathfrak{F} :=& \mathrm{bmin}[\mathfrak{G}] \times \mathrm{bmax}[\mathfrak{G}] \\ =& \{(F, F') \in \mathbb{P}_\Xi \times \mathbb{P}_\Xi \mid F \in \mathrm{bmin}[\mathfrak{G}], F' \in \mathrm{bmax}[\mathfrak{G}]\}\end{aligned}$$

is generic for $\mathbb{P}_\Xi \times \mathbb{P}_\Xi$ (with the usual partial order of product forcing).[3]

[3] Note that $\mathrm{bmin}[\mathfrak{G}]$ is meant to contain the filters of the form $\mathrm{bmin}(G)$ for all $G \in \mathfrak{G}$; similarly for $\mathrm{bmax}[\mathfrak{G}]$.

Spoiler *The difficult part is to prove that $\mathfrak{F}$ intersects every dense set. Given a dense set in the product forcing we can give a predense set in the idempotent forcing. As a generic, $\mathfrak{G}$ contains a witness from every predense set. This witness will guarantee a non-empty intersection of $\mathfrak{F}$ with the original set.*

The essential part of the argument utilizes the immense loss of information under bmin *and* bmax *that we identified in Section 2.3. From the reverse point of view, the essence lies in the fact that we can have and manipulate much more subtle structures in the preimage without changing the image.*

Proof. We begin with the easier parts.

Claim 1: $\mathfrak{F}$ is a filter on $\mathbb{P}_\Xi \times \mathbb{P}_\Xi$.

$\mathfrak{F}$ is a filter

First, $\mathfrak{F} \subseteq \mathbb{P}_\Xi \times \mathbb{P}_\Xi$ since topological families are closed under images. Second,

$$\begin{aligned}\mathrm{bmin}(Fr(\omega)) &= Fr(\omega)\\ \mathrm{bmax}(Fr(\omega)) &= Fr(\omega),\end{aligned}$$

so the largest element of the partial order, $(Fr(\omega), Fr(\omega))$ is in $\mathfrak{F}$.

Third, since each component is derived from elements of $\mathfrak{G}$, any two elements of $\mathfrak{F}$ are coherent, hence compatible.

To finish the proof of the claim, we check that $\mathfrak{F}$ is also closed upwards.

Take any $(F, F') \in \mathfrak{F}$ and (H, H') with $(F, F') \leq (H, H')$.

Since $(F, F') \in \mathfrak{F}$, we can find $G, G' \in \mathfrak{G}$ such that

$$\mathrm{bmin}(G) = F,\ \mathrm{bmax}(G') = F'.$$

Next we can easily check that

a simple computation

$$\mathrm{bmin}(G) = F, F \supseteq H \quad \Rightarrow \quad G \supseteq \mathrm{bmin}^{-1}(F) \supseteq \mathrm{bmin}^{-1}(H).$$

Of course, the same holds for G', F' and H' with respect to bmax.[4]

[4] Of course, this holds for all filters and any other function in ω^ω.

Since $\mathfrak{G}$ was a filter, we can conclude from the above that

$$\mathrm{bmin}^{-1}(H),\ \mathrm{bmax}^{-1}(H') \in \mathfrak{G}.$$

But
$$H = \mathrm{bmin}(\mathrm{bmin}^{-1}(H)),\ H' = \mathrm{bmax}(\mathrm{bmax}^{-1}(H')),$$
so $(H, H') \in \mathfrak{F}$.

Therefore $\mathfrak{F}$ is a filter on $\mathbb{P}_\Xi \times \mathbb{P}_\Xi$.

We now check the genericity of the filter $\mathfrak{F}$.

$\mathfrak{F}$ is generic

Claim 2: $\mathfrak{F}$ intersects every dense set from the ground model.

The obvious candidate D

Given any dense subset of $\mathbb{P}_\Xi \times \mathbb{P}_\Xi$, say D', we define

$$D := \{\mathrm{bmin}^{-1}(F) \cup \mathrm{bmax}^{-1}(F') \mid (F, F') \in D'\}.$$

As usual, we confuse $\mathrm{bmin}^{-1}(F) \cup \mathrm{bmax}^{-1}(F')$ and the filter this set generates.

If we can show that D is predense in $\mathbb{P}_\Xi^+$, we are done. For then, by genericity of $\mathfrak{G}$, there exists $G \in \mathfrak{G} \cap D$ and hence

$$(\mathrm{bmin}(G), \mathrm{bmax}(G)) \leq (H, H') \quad \text{for some } (H, H') \in D'.$$

But since $\mathfrak{F}$ is a filter, this yields $(H, H') \in \mathfrak{F}$.

We will use the techniques for preimage filters that we have developed in Section 2.3 to show that D is predense in $\mathbb{P}_\Xi^+$. Recall that by Proposition 2.21 $\mathrm{bmin}^{-1}(F) \cup \mathrm{bmax}^{-1}(F')$ generate an idempotent filter which is in $\mathbb{P}_\Xi^+$ thanks to Ξ being a topological family.

Therefore it suffices to prove the following claim.

Claim: D is predense in $\mathbb{P}_\Xi^+$.

Take any $G \in \mathbb{P}_\Xi^+$. Since D' is dense there exists

$$(F, F') \in D' \text{ with } (F, F') \leq (\mathrm{bmin}(G), \mathrm{bmax}(G)).$$

We wish to show that $\mathrm{bmin}^{-1}(F) \cup \mathrm{bmax}^{-1}(F')$ and G are compatible (in $\mathbb{P}_\Xi^+$).

We already know that compatibility in the partial order only depends on the finite intersection property, since idempotency is not the issue by Corollary 2.3 (and neither is membership in the topological family).

Therefore it is enough to prove that the union of these filters has the finite intersection property.

Step 1: Let $A \in \mathrm{bmin}^{-1}(F)$, $B \in \mathrm{bmax}^{-1}(F')$ and $C \in G$.

We may assume that

$$A = \mathrm{bmin}^{-1}[\mathrm{bmin}[A]], B = \mathrm{bmax}^{-1}[\mathrm{bmax}[B]],$$

since $\mathrm{bmin}^{-1}(F)$ and $\mathrm{bmax}^{-1}(F')$ have a base of such sets.

Step 2: Since $\mathrm{bmin}[C] \in \mathrm{bmin}(G) \subseteq F$ we have

The easy part – not yet enough

$$\mathrm{bmin}[C] \cap \mathrm{bmin}[A] \in F,$$

and hence this intersection is non-empty. Together with the assumptions on A from Step 1 this implies

$$A \cap C \neq \emptyset.$$

Step 3: It is completely analogous to see that $B \cap C \neq \emptyset$ (using $\mathrm{bmax}[C]$ and F').

But we need to show that $C \cap A \cap B \neq \emptyset$ – and it is very easy to come up with an example that $C \cap A$ and $C \cap B$ and $A \cap B$ are all non-empty while the common intersection is empty. However, this is not the case here thanks to the idempotency of G.

Step 4: Since $C \in G$ and G is idempotent, we have $C^* \in G$.[5]

[5] with respect to G but cf. Remark 2.12.

Therefore just as in Step 2, $C^* \cap A \neq \emptyset$; so we can pick

The first summand of the witness

$$a \in C^* \cap A.$$

Then of course $-a + C \in G$.

Step 5: As in Step 3, $(-a + C) \cap B \neq \emptyset$.

So we could find $b \in B$ with $a + b \in C$. But in no way does this guarantee that $a + b \in A \cap B$. However, we still know a little bit more.

The extra ingredient

Additionally, $G \cup \mathrm{bmax}^{-1}(F')$ generates an idempotent filter. Since any idempotent filter can be extended to an idempotent ultrafilter, it is compatible with every set of multiples, cf. Corollary 2.5 and Lemma 1.18.

So in fact, we can find

The second summand of the witness

$$b \in (-a + C) \cap B \cap 2^{\mathrm{bmax}(a)+1} \cdot \mathbb{N}.$$

Step 6: Then $a + b \in C$ and we can calculate that

Finally, the witness

$$\mathrm{bmin}(a + b) = \mathrm{bmin}(a), \ \mathrm{bmax}(a + b) = \mathrm{bmax}(b).$$

Therefore $a + b \in A \cap B \cap C$ as desired. □

Thanks to this theorem, we now know how the forcing notions are related: the idempotent filter forcing $\mathbb{P}^+_\Xi$ will always yield two independently generic objects for the more general filter forcing $\mathbb{P}_\Xi$. So we can analyze the question posed after the first theorem – are these forcings new? – by analyzing the forcing $\mathbb{P}_\Xi$; for the latter we are able to build upon the results by Daguenet and Laflamme.

We state the results due to Laflamme in [Laf89] which will enable us to generalize the classical result of Theorem 3.2.

Filter forcing (Laflamme)

Theorem 3.11 (Filter forcing (Laflamme))
The following holds.

- *(Mathias [Mat77]) Forcing with countably generated filters adjoins a selective ultrafilter.*
- *Forcing with $\mathbb{P}_{F_\sigma}$ adjoins a strong P-point with no rapid P-point $\leq_{RK}$-below it.*
- *Forcing with $\mathbb{P}_{\Sigma^1_1}$ adjoins a an ultrafilter with no $\leq_{RK}$-predecessor (including itself) being a P-point or a rapid ultrafilter.*
- *Forcing with meager filters adjoins an ultrafilter which is neither a P-point nor rapid but $\leq_{RK}$-above all ground model ultrafilters.*
- *Forcing with completely meager filters adjoins an ultrafilter not $\leq_{RK}$-above any P-point.*

Proof. The first result is classical. The other results can be found in Sections 6,7 and 8 of [Laf89]. □

So we can deduce for our three examples that our forcing notions are indeed quite different. For completeness we repeat our original motivation – our new ultrafilters are indeed nothing like the stable ordered union ultrafilters.

Corollary 3.12
No two of the idempotent filter forcing notions corresponding to the forcings in Theorem 3.11 are equivalent.

Additionally, the question whether the geneneric is a summable ultrafilter has a an affirmative answer for $\mathbb{P}^+_{ctbl}$, is open for $\mathbb{P}^+_{F_\sigma}$ and has a negative answer for the other examples from Remark 3.7.

Proof. The first part is immediate from the above theorem since the partial orders $\mathbb{P}_\Xi$ are not equivalent (or otherwise their bmin and bmax images would be, too).

With Lemma 5.18 we will prove that for every summable ultrafilter the image under bmin is a P-point; the additional claim then follows from Theorems 3.2, 3.10 and 3.11. □

This completes the general forcing construction. We have established a general method to consistently add new idempotent ultrafilters. Additionally, we have identified how to connect these new forcing notions to the general filter forcings derived from topological families. However, we are left with an open question whether the idempotent filter forcing with F_σ filters adjoins a summable.

3.4. A note on F_σ filters

We ended the last section with the knowledge that we have achieved our main goal to consistently create new kinds of idempotent ultrafilters with interesting set theoretical properties. However, we were left with the open question, whether the generic ultrafilter for $\mathbb{P}_{F_\sigma}$ is summable.

If it is not, then general forcing considerations imply that there is a dense subset of $\mathbb{P}^+_{F_\sigma}$ consisting of filters that cannot be extended to summable ultrafilters. In that case, we would hope for a more hands-on construction of such a filter. Although we are unable to do this for summable ultrafilters, the aim of the section is to make progress on this question by studying these phenomena for union ultrafilters; this also serves as preparation for the investigations in Chapter 4.

It was shown in [BH87] that max maps union ultrafilters to rapid P-points and we know from the previous section that the $\mathbb{P}_{F_\sigma}$ forcing adjoins a P-point that is not rapid. Therefore we aim to exploit this connection. Let us recall from the preliminaries the following part of Definition 1.19.

Definition 3.13 An ultrafilter p is called *rapid*, if the natural enumerations of its sets form a dominating family in ω^ω, i.e.,

$$(\forall f \in \omega^\omega)(\exists A \in p)(\forall n \in \omega)|A \cap f(n)| \leq n.$$

Definition: Rapid Ultrafilters

It is obvious that a dominating family only needs to dominate the strictly monotone functions.

With this in mind it is sometimes easier to consider the reverse point of view – instead of strictly monotone functions we look at the "reverse" finite-to-one functions projecting a number to the immediately smaller number in the image.

The following proposition is folklore; we include its proof since we will argue with this characterization.

Proposition 3.14
An ultrafilter p is rapid iff

$$(\forall f \in \omega^\omega \text{ finite-to-one})(\exists A \in p)(\forall n \in \omega)|A \cap f^{-1}[n]| \leq n.$$

Proof. **Step 1:** First we prove the latter characterization assuming that p is rapid.

Given any finite-to-one $f \in \omega^\omega$ we define $g_f \in \omega^\omega$ by

$$g_f(n) := 1 + \max f^{-1}[n].$$

By assumption, there is $A \in p$ such that $|A \cap g_f(n)| \leq n$ for $n \in \omega$.

But $k \in f^{-1}[n]$ implies $k < g_f(n)$.

Therefore $A \cap f^{-1}[n] \subseteq A \cap g_f(n)$ holds, implying $|A \cap f^{-1}[n]| \leq n$.

Step 2: Next assuming that p has the latter property we prove that p is rapid.

Given $f \in \omega^\omega$ we may assume that f is strictly increasing.

Define $h \in \omega^\omega$ by $n \in (f(h(n)), f(h(n)+1)]$, i.e.,

$$h(n) := f^{-1}(\max(f[\omega] \cap n)).$$

Then h is finite-to-one since f is injective.

By assumption we can find $A \in p$ such that $|A \cap h^{-1}[n]| \leq n$.

Now $k < f(n)$ implies $h(k) < n$.

Hence $A \cap f(n) \subseteq A \cap h^{-1}[n]$ and $|A \cap f(n)| \leq n$. □

After this useful characterization, we are able to describe those F_σ-filters that cannot be extended to rapid ultrafilters.

Definition 3.15 Fix a finite-to-one function $f \in \omega^\omega$ such that $(|f^{-1}[n]|)_{n\in\omega}$ grows faster (with respect to $\leq^*$) than any linear function; let us recall the function g_f from the last proof.

The dominating ideal

- We define

$$I_f := \{A \subseteq \omega \mid (\forall n \in \omega)|A \cap f^{-1}[n]| \leq n\}.$$

  Note that by assumptions on f the ideal generated by I_f is proper and the sets in I_f are those that dominate the function g_f.

The dual filter F_f

- We define F_f to be the dual filter of the ideal generated by I_f, i.e., the filter generated by the complements of sets in I_f.

  Note that F_f cannot be extended to an ultrafilter that includes a set dominating the function g_f.

We should note that the restrictions on f are necessary to guarantee I_f to generate a proper ideal.

The following proposition is folklore; it can be found in [Dag75].

Proposition 3.16
The filter F_f is F_σ.

In particular, there are many F_σ filters that cannot be extended to a rapid filter.

Proof. Let f be as in the statement.

Claim 1: By looking at the definition, the set I_f is clearly a closed (hence compact) subset of $\mathfrak{P}(\omega)$ and it is already closed under subsets.

Claim 2: The ideal generated by I_f is F_σ (and proper by assumptions on f)

To generate the ideal we only apply countably many continuous maps on $\mathfrak{P}(\omega)$, i.e., the maps of "n-fold union", mapping a compact set to a compact set. Therefore, the generated ideal is the countable union of closed sets.

Claim 3: The dual filter F_f is F_σ as well since

$$c : \mathfrak{P}(\omega) \to \mathfrak{P}(\omega), A \mapsto \omega \setminus A$$

is a homeomorphism of $\mathfrak{P}(\omega)$.

Claim 4: As noted in the definition, F_f is not compatible with a set dominating the function g_f from the last proof. Therefore, F_f cannot be extended to a rapid ultrafilter. □

We can now state an immediate though rather abstract result regarding idempotent F_σ filters.

Proposition 3.17
Given any finite-to-one function $f \in \omega^\omega$ such that $(|f^{-1}(n)|)_{n\in\omega}$ is strictly increasing, the idempotent F_σ filter

$$\mathrm{bmax}^{-1}(F_f)$$

cannot be extended to an (idempotent) ultrafilter with bmax*-image being rapid.*

Proof. Clearly, the preimage under the continuous map is again F_σ and idempotent by Proposition 2.21. Trivially, any ultrafilter extending has a bmax-image refining F_f; hence the image cannot be a rapid ultrafilter. □

It is analogous to describe an F_σ filter on $\mathbb{F}$ that cannot be extended to a union ultrafilter.

Proposition 3.18
Given any finite-to-one function $f \in \omega^\omega$ such that $(|f^{-1}(n)|)_{n\in\omega}$ is strictly increasing, the idempotent F_σ filter

$$\max^{-1}(F_f)$$

cannot be extended to a union ultrafilter.

Proof. Assume to the contrary that there is a union ultrafilter u extending $\max^{-1}(F_f)$.

Then $\max(u)$ is a rapid ultrafilter by [BH87, Theorem 2],[6] but of course $\mathrm{bmax}(u) \supseteq F_f$ – a contradiction to the previous proposition ↯↯↯ . □

[6] This is also [HS98, Theorem 12.36]. We will discuss union ultrafilters in more detail in Chapter 4.

These two results are of course rather unsatisfying. Instead we would like a practical, combinatorial explanation of this phenomenon. While this seems more problematic with summable ultrafilters, we can give such a characterization for union (ultra)filters.

We construct a filter on $\mathbb{F}$ that cannot even be extended to a union filter, i.e., a filter with a base of FU-sets, let alone a union ultrafilter.

Proposition 3.19
Given any finite-to-one function $f \in \omega^\omega$ such that $(|f^{-1}(n)|)_{n\in\omega}$ is strictly increasing, the idempotent F_σ filter

$$\max^{-1}(F_f) \cup \min^{-1}(Fr)$$

cannot be extended to a union filter.

Spoiler *We modify the classical partition argument that is used to show that the* bmax *image of a union ultrafilter is rapid, i.e., we give a partition where one part cannot be in any idempotent filter and the other cannot be refined by an FU-set since this would dominate the function.*

Proof. Given f we abbreviate the relevant filter

$$G := \max^{-1}(F_f) \cup \min^{-1}(Fr).$$

Recall from Proposition 2.21 that G is idempotent. We will prove the stronger statement that G cannot be refined to an union filter, i.e., a filter with a base of FU-sets.

Step: Consider the partition $\mathbb{F} = A \dot{\cup} B$ with

$$A := \{s \in \mathbb{F} \mid f(\max(s)) \leq \min(s)\}$$
$$B := \{s \in \mathbb{F} \mid f(\max(s)) > \min(s)\}.$$

Claim 1: A cannot be in any idempotent filter; in particular, A is not in any union filter extending G.

Assume to the contrary that there is an idempotent filter G' with $A \in G'$.

Then also $A^* \in G'$. So pick any $s \in A^*$

Since G' is idempotent, $\sigma(s)$ is compatible with G'.[7]

[7] Recall that we only consider idempotent filters with respect to $\delta\mathbb{F}$; cf. Remark 2.14.

Therefore

$$s^{-1}A^* \cap \mathbb{F} \cap \sigma(s) \neq \emptyset,$$

so we can pick some t from that intersection.

We can then calculate

$$f(\max(t)) = f(\max(s \cup t)) \leq \min(s \cup t) = \min(s).$$

But f is finite-to-one, hence

$$\max[s^{-1}A^* \cap \sigma(s)]$$

is finite – a contradiction for any filter compatible with $\delta\mathbb{F}$. ↯↯↯

Therefore every idempotent filter extending G is compatible with B.

Claim 2: B cannot be refined by an FU-set compatible with G, i.e., every FU-set contained in B has its complement in G.

For this let us take some (pairwise disjoint) $s = (s_i)_{i \in \omega}$ with $FU(s) \subseteq B$.

Claim: $\max[FU(s)] \in I_f$, i.e., for every $n \in \omega$ we have

$$|\max[FU(s)] \cap f^{-1}[n]| \leq n.$$

In particular, $\omega \setminus \max[FU(s)] \in F_f$, hence $\mathbb{F} \setminus FU(s) \in G$.

For the claim fix $n \in \omega$ and consider an arbitrary

$$k \in \max[FU(s)] \cap f^{-1}[n].$$

Then we can find some j such that $\max(s_j) = k$; in particular,

$$f(\max(s_j)) = f(k) < n.$$

Since $FU(s) \subseteq B$, we get

$$\min(s_j) < f(\max(s_j)) < n.$$

But since they are disjoint, there can only be at most n-many s_j with $\min(s_j) < n$.

Hence there are at most n-many s_j with $f(\max(s_j)) < n$ – as desired.□

It seems natural to extend this proof to summable ultrafilters: just replace max with bmax in the argument. Unfortunately, we still could not assume that a summable ultrafilter contains an FS-set for a sequence with disjoint binary support. Although there is an identification of union and summable ultrafilters using support with respect to some sequence, this identification does not allow us to extend the above argument. We will discuss this in detail in Chapter 4 and Chapter 5.

3.5. *A note on FS-filters*

We have seen that there are many idempotent F_σ filters that cannot be extended to union ultrafilters. We wish to conclude this chapter with some observations for the opposite case. Again this is also useful to prepare for the investigations of Chapter 4 and Chapter 5. We begin with a standard definition.

Definition 3.20 Given filters F, G on $\mathbb{N}$ we say that G *dominates* F if the natural enumerating functions of members of G dominate those of F.

Naturally, this definition is closely related to the notion of rapid ultrafilters. We can make the following observation.

Proposition 3.21
Given free filters F, G on $\mathbb{N}$ such that G dominates F, the filter

$$\mathrm{bmin}^{-1}(F) \cup \mathrm{bmax}^{-1}(G)$$

can be extended to an FS-filter, i.e., a filter with a base of FS-sets.

Proof. By assumption on F and G we can choose for any $A \in F$ and $B \in G$ some $B_A \in G$ such that the natural enumeration of $B \cap B_A$, say

$$(b_n^A)_{n \in \omega}$$

dominates the enumeration $(a_n)_{n\in\omega}$ of A. Then it is readily seen that

$$\{FS(2^{a_n} + 2^{b_n^A}) \mid A \in F, B \in G\}$$

refines $\mathrm{bmin}^{-1}(F) \cup \mathrm{bmax}^{-1}(G)$ – as desired. □

We can extend this a little further. For the definition of stability cf. Definition 4.2.

Corollary 3.22
Let $p_1, p_2 \in \beta\mathbb{N}$ be P-points and assume that p_2 dominates p_1. The FS-filter extending $\mathrm{bmin}^{-1}(p_1) \cup \mathrm{bmax}^{-1}(p_2)$ *described in the proof of the above proposition is stable.*

Proof. **Step 1:** Given countably many FS-sets in the filter described in the proof of the last proposition, we can find without loss of generality sequences $(A_\alpha)_{\alpha<\omega}$ in p_1 and $(B_\alpha)_{\alpha<\omega}$ in p_2 such that the FS-sets are those from the last proof corresponding to A_α and B_α.

Step 2: Since p_1 and p_2 are P-points, we can find $A \in p_1$ and $B \in p_2$ an almost intersection of the respective sequence.

Step 3: Then it's easy to see that the FS-set described in the previous proof corresponding to A and B is an almost condensation of the given FS-sets (and in the filter) – as desired. □

There is another way of constructing a coarser union filter in the preimage of two ultrafilters.

Remark 3.23 Given $A_0, B_0 \in [\omega]^\omega$ with B_0 dominating A_0 we define inductively $s = (s_i)_{i\in\omega}$ by

$$s_i := (A_i \cap \min B_i) \cup (B_i \cap \min(A_i \setminus \min B_i)),$$

where

$$B_{i+1} = B \setminus s_i, \quad A_{i+1} = A_i \setminus s_i.$$

Then $FU(s) \subseteq \min^{-1}[A_0] \cap \max^{-1}[B_0]$.

We close the chapter with the following natural question.

Question 3.24
When does a (stable, ordered) union filter extend to a (stable, ordered) union ultrafilter?

The following example offers some insight into this problem.

Example 3.25
Assuming CH, Zelenyuk in [Zel96] constructs a finite semigroup in $\delta\mathbb{F}$ such that none of its elements is a union ultrafilter, but its intersection is an FU-filter. An English translation of the relevant parts of [Zel96] can be found in Appendix C.

Naturally, there is a cardinal invariant related to this question which has been implicitly studied by Eisworth in [Eis02], where it is shown that $cov(\mathcal{M}) = \mathfrak{c}$ is equivalent to "every union filter of size $< \mathfrak{c}$ can be extended to a stable ordered union ultrafilter".

3.6. Synopsis

In this chapter we have established our first major result, the general forcing constructions of idempotent ultrafilters using idempotent filter forcings. We have seen that this approach does not only allow for idempotent ultrafilters to be adjoined, but by the connection to the classical forcing constructions of Daguenet-Teissier and Laflamme we have established rich combinatorial properties for our examples and a base for the analysis of the adjoined ultrafilters.

Although the question remains open, our progress with respect to union filters suggests the conjecture that the forcing with idempotent F_σ filters does not adjoin a summable ultrafilter; if this conjecture is true then only the example of countable idempotent filter forcing would be able to adjoin a summable ultrafilter. If it is false it would be more than interesting to study the combinatorics of such a summable.

We have also established a direct combinatorial description of filters that cannot be extended to FU-filters as well as some natural constructions of non-trivial FS-filters.

Chapter 4

Union Ultrafilters

In this chapter we study the different notions of union ultrafilters. Our motivation for this is twofold. On the one hand, our results will help to clarify the relationships between the different kinds of union and summable ultrafilters as initially studied in [Bla87b] and further developed in [BH87]. On the other hand, we also investigate these ultrafilters hoping to better understand the combination of set theoretic, combinatorial and algebraic properties of union and summable ultrafilters. In this respect, it will also further our understanding of the forcing constructions from Chapter 3 and the directions we can and cannot hope to develop these results in the future.

We begin this chapter with some classical definitions related to union ultrafilters. The motivation for the first and lengthy definition is as follows. By the Galvin-Glazer Theorem 1.14 every set in an idempotent ultrafilter in $\delta\mathbb{F}$ contains the FU-set for some disjoint sequence. Conversely, every such FU-set is contained in an idempotent ultrafilter by Corollary 2.5 together with the fact that FU-sets are natural partial subsemigroups of $\mathbb{F}$. In particular, given a partition of an FU-set, we can find a new FU-set in one part of the partition. To be able to speak about this situation smoothly, we introduce the following definitions.

Definition: Condensation, ordered, meshed

Definition 4.1 Given a disjoint sequence $s = (s_i)_{i<N}$ in $\mathbb{F} \setminus \{0\}$ with $N \leq \omega$ and given some $K \leq \omega$ we define as follows.

- A disjoint sequence $t = (t_j)_{j<K}$ in $\mathbb{F} \setminus \{0\}$ is called a *condensation* of $(s_i)_{n<N}$ if
$$\{t_j \mid j < K\} \subseteq FU(s).$$
In the case $N = K = \omega$ we call t an *almost condensation* if
$$\{t_j \mid j < K\} \subseteq^* FU(s).$$
- The sequence s is called *ordered* if
$$\max(s_i) < \min(s_j) \qquad \text{for all } i < j < N.$$
We abbreviate the latter relation by $s_i < s_j$. We will also write $s_i \ll s_j$ to indicate that $\max(s_i)$ and $\min(s_j)$ are very far apart, where "very far" arises from the specific context.
- For $v, w \in \mathbb{F}$ we say that *v meshes with w or $v \sqcap w$* if
$$\min(v) < \min(w) < \max(v)$$
$$\text{or } \min(w) < \min(v) < \max(w).$$

- We define the *s-support of $v \in FU(s)$* as

$$s\text{-supp}(v) := \{s_i \mid s_i \subseteq v\}.$$

- and (allowing confusion) the *s-support of $A \subseteq FU(s)$* as

$$s\text{-supp}(A) := \{s_i \mid (\exists x \in A)\ s_i \subseteq x\}.$$

Note that the s-support of a condensation and its FU-set are the same.

Now that we can easily speak about FU-sets, we introduce the three variants of union ultrafilters that were first described in [Bla87b].

Definition: union, ordered union, stable union ultrafilters

Definition 4.2 We define as follows.

- An ultrafilter u on $\mathbb{F}$ is called *union* if it has a base of FU-sets.
- A union ultrafilter u on $\mathbb{F}$ is called *ordered* if it has a base of FU-sets from ordered sequences.
- A union ultrafilter u on $\mathbb{F}$ is called *stable* if for any sequence $(FU(s^\alpha))_{\alpha<\omega}$ in u there exists $FU(t) \in u$ such that

$$(\forall \alpha < \omega)\ t \sqsubseteq^* s^\alpha,$$

i.e., t almost condenses all the sequences s^α at once.

We should remember that even though our semigroup operation on $\mathbb{F}$ requires gaps between compatible elements, union ultrafilters are still idempotent elements in $\delta\mathbb{F}$.

Finally, let us define one more notion which was introduced in [BH87] to help differentiate union ultrafilters; it is a special form of equivalence in the Rudin-Keisler order, but one might argue it is the relevant notion for union ultrafilters.

Definition: Additive isomorphism

Definition 4.3 Given semigroups S, T, we say that two ultrafilters $p \in \beta S, q \in \beta T$ are *additively isomorphic* if there exist $FP(s) \in p, FU(t) \in q$ such that the natural partial semigroup homomorphism (cf. Remark 1.12) maps the ultrafilters to each other.

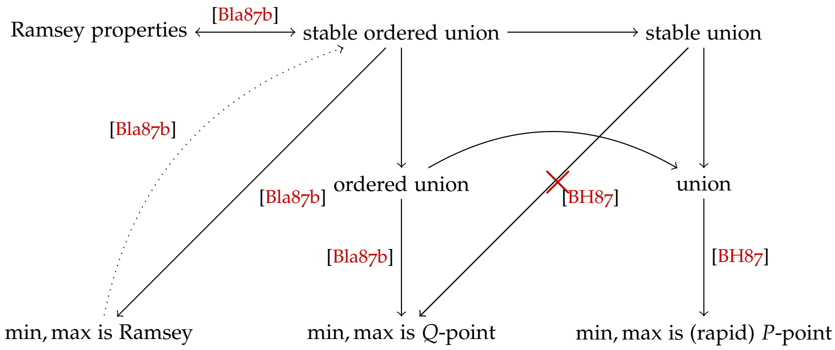
4.1. A short literature review

We will use this first section to give an overview of the established results regarding the different notions of union ultrafilters. We will exclude the proofs except for two results by Andreas Blass, one of which motivates the main result of this chapter, Theorem 4.29, whereas the other is unpublished.

Let us begin with a diagram of the known implications with the appropriate references, [Bla87b] and the later [BH87].

Let us quickly note the exact references.

Figure 4.1: Union ultrafilers



- For ordered union ultrafilters being stable is equivalent to interesting Ramsey properties; this is [Bla87b, Theorem 4.2] and Theorem 4.12 below.
- For a stable ordered union ultrafilter the images under min and max are selective ultrafilters which are not $\leq_{RK}$-equivalent; this is [Bla87b, Theorem 4.2] and [Bla87b, Corollary 4.3].
- Reversely, under *CH*, given two selective ultrafilters not $\leq_{RK}$-equivalent, there exists a stable union ultrafilter with min and max being the prescribed Ramsey ultrafilters; this is [Bla87b, Theorem 2.4] and Theorem 4.6 below.
- For an ordered union ultrafilter the images under min and max are Q-points which are not $\leq_{RK}$-equivalent; this is [Bla87b, Proposition 3.9].
- For a union ultrafilter the image under min is a P-point and the image under max is a rapid P-point; this is [BH87, Theorem 2]. Blass and Hindman attribute the discovery of rapidity of the max image to Pierre Matet, cf. [Mat88], but their own proof immediately implies rapidity as well.
- Assuming *CH* there exists a stable union ultrafilter with images under min and max not being Q-points; in particular, such a union ultrafilter cannot be ordered. This is [Bla87b, Theorem 4'].
- Additionally, the forcing notion on $\mathbb{F}$ consisting of *FU*-sets partially ordered by the condensation relation was introduced in [Bla87b]; this is the forcing that we have already discussed in the remark after Theorem 3.2. Blass remarks in [Bla87b] that stable ordered union ultrafilters have "complete combinatorics" for this forcing, i.e., in the Lévy collapse of a Mahlo cardinal the stable ordered union ultrafilters are generic over the Solovay model, i.e., the submodel consisting of the sets that are hereditarily ordinal definable from the reals. Therefore, from the point of view of the Solovay model, all stable ordered union ultrafilters look alike and are equally complex vis-a-vis generic.

From these results a couple of questions arise naturally.

- Is there a similar characterization of stability in terms of Ramsey properties for union ultrafilters?
- For stable ordered union ultrafilters having non-isomorphic selective ultrafilters as min, max images is optimal in the sense that any two such

Ramsey ultrafilters are the respective images of a stable ordered union ultrafilter (assuming CH). Is there similarly a (stable) union ultrafilter for each pair of (rapid) P-points?

- Can we identify orderedness of a union ultrafilter by the properties of min and max, i.e., can there be unordered union ultrafilters with images under min, max being Ramsey?
- Can there be (ordered) union ultrafilters that are not stable?

We will make some progress on these questions in the remainder of this chapter. We begin, however, with the proof of the existence of ordered union ultrafilters with prescribed min and max non-isomorphic selective ultrafilter from [Bla87b]; this also serves as a motivation for our main result in this chapter, Theorem 4.29. We are able to give a shorter proof using the following result from [Bla88].

Homogeneity (Blass)

Theorem 4.4 (Homogeneity (Blass))
Let $p_0 \not\cong_{RK} p_1$ be selective ultrafilters and let $f \in 2^\omega$.

If $\mathfrak{P}(\omega)$ is partitioned into an analytic and coanalytic part, we can find sets $X_i \in p_i$ $(i = 0, 1)$ such that every increasing sequence

$$(x_n)_{n\in\omega} \text{ with } x_n \in X_{f(n)}$$

Definition: f-alternating

*is homogeneous. We call such sequences f-*alternating.

Proof. This is [Bla88, Theorem 7]; we omit the proof since it would take us too far away from our interests. □

Let us recall the following useful observation from [Bla88].

Remark 4.5 Given any $f \in 2^\omega$ and selective ultrafilters $p_0 \not\cong_{RK} p_1$, there exists an f-alternating sequence such that its alternating parts are sets in p_0 and p_1 respectively.

This homogeneity result is the key in proving the following theorem from [Bla87b]. Note that the result is optimal in that the min and max images of an ordered union ultrafilter are $\not\cong_{RK}$ as we will show afterwards.

Prescribing min and max (Blass)

Theorem 4.6 (Prescribing min and max (Blass))
Assume CH. Given selective ultrafilters $p_0 \not\cong p_1$, there exists a stable ordered union ultrafilter u with $\min(u) = p_0$, $\max(u) = p_1$.

Spoiler *The proof is done by transfinite induction. We construct a sequence of almost condensations of length ω_1. Given countably many constructed FU-sets of our future ultrafilter, we can write down the conditions to continue our induction in the form of an analytic set. Then we use the Galvin-Glazer Theorem 1.14 to show that the homogeneous set we can get from Theorem 4.4 allows us to continue our induction.*

Proof. Let $(X_\alpha)_{\alpha<\omega_1}$ be an enumeration of $\mathfrak{P}(\mathbb{F})$. We will construct u by transfinite induction adding an FU-set included in or disjoint to X_α at each step α.

Inductive **Step:** For the induction assume that we have $\beta < \omega_1$ and $(s^\alpha)_{\alpha<\beta}$ such that the following holds for all $\gamma < \alpha < \beta$:

inductive assumptions

- s^α is an ordered sequence in $\mathbb{F}$
- $s^\alpha \sqsubseteq^* s^\gamma$
- $FU(s^\alpha) \subseteq X_\alpha$ or $FU(s^\alpha) \cap X_\alpha = \emptyset$
- $\min[FU(s^\alpha)] \in p_0$
- $\max[FU(s^\alpha)] \in p_1$.

Claim: There exists an ordered sequence s_β to continue this induction.

Obviously, the resulting FU-sets will generate a stable ordered union ultrafilter as desired.

Step 1: To begin we choose some cofinal subsequence in β, say $(\alpha(n))_{n\in\omega}$.

A countable set of generators

Then the sets $(FU_{>k}(s^{\alpha(n)}))_{n,k<\omega}$ generate an idempotent filter – which is easily seen to be equal to the one generated by all of $(FU(s^\alpha))_{\alpha<\beta}$.

Step 2: By inductive assumption this filter and $\min^{-1}(p_0) \cup \max^{-1}(p_1)$ are coherent.

Since both collections generate an idempotent filter compatible with $\delta\mathbb{F}$, we can find an idempotent ultrafilter e containing all these sets, i.e.,

$$\begin{aligned} &FU(s^\alpha) \in e && (\alpha < \beta) \\ &\min(e) = p_0, \\ &\max(e) = p_1. \end{aligned}$$

Without loss of generality, we may assume that $X_\beta \in e$.

Step 3: Consider the following set

The analytic set

$$\begin{aligned} \{X \subseteq \omega \mid (\exists s^\beta \text{ ordered}) \\ &X = \min[FU(s^\beta)] \cup \max[FU(s^\beta)], \\ &(\forall n \in \omega)\ s^\beta \sqsubseteq^* s^{\alpha(n)}, \\ &FU(s^\beta) \subseteq X_\beta\}. \end{aligned}$$

It is easily checked that the set defined above is analytic.[1]

[1] For example, "ordered" is Π^0_1 and "condensation" is Σ^0_2 and there are many recursive bijections from $\mathbb{F}$ to ω.

Step 4: Consider the parity function, i.e., $f \in 2^\omega$ with

Applying Theorem 4.4

$$f(n) = n \bmod 2.$$

So for our choice of f indeed f-alternating actually means alternating.

Applying Theorem 4.4 we can find

$$X_i \in p_i \quad (i = 0, 1)$$

such that the set of alternating sequences is either disjoint from or included in the above set.

Step 5: But it cannot be disjoint!

Applying Galvin-Glazer

By our choice of e, we have

$$\min^{-1}[X_0] \cap \max^{-1}[X_1] \cap X_\beta \in e.$$

Then we can apply Corollary 1.16 to the Galvin-Glazer Theorem with $(FU(s^\alpha))_{\alpha<\beta}$ to find an ordered sequence s^β in X_β that almost condenses the sequences $(s^\alpha)_{\alpha<\omega}$.

Therefore, this sequence satisfies the conditions of the analytic set; in particular, orderedness implies alternating minima and maxima.

Since $\min[FU(s)] \cup \max[FU(s)] \subseteq X_0 \cup X_1$, we have found a subset of $X_0 \cup X_1$ that is included in the analytic set.

Therefore we get the "include" case from applying the above theorem 4.4.

Step 6: Finally, we have remarked that since $p_0 \not\cong_{RK} p_1$ we may assume X_0 and X_1 themselves to be alternating; in particular such $X_0 \cup X_1$ is in fact a member of the analytic set itself.

But this means that we can find s^β to continue our inductive construction as desired. □

In the above result we presumed that the two images of a stable ordered union ultrafilter have to be non-isomorphic. As mentioned, it was noticed already in [Bla87b] that this is necessary.

However, when it comes to arbitrary union ultrafilters, the situation becomes more complicated. Since the images under min and max are not necessarily selective,[2] the strategy from [Bla87b] will not work. To be able to settle the question whether such a reverse can be established, it is natural to ask what additional properties hold for the images of a union ultrafilter. While I visited Ann Arbor in the winter 2007/2008 Andreas Blass developed an elegant proof of such a strengthening.

[2] Cf. Figure 4.1

For this we need to introduce a concept from [Bla86].

Definition: NCF

Definition 4.7 Two filters F, G on ω are called *near coherent*, if there exists a finite-to-one function $f \in \omega^\omega$, such that $f(F) = f(G)$. Free filters F and G are near coherent if there exists a partition of ω into intervals such that for any two sets from F and G respectively there is one interval that both sets intersect.

The statement that all filters on ω are near coherent is abbreviated by *NCF*; it is independent from *ZFC* but for example contradicts *CH*.

The given equivalent reformulation for free filters is easily checked. It is trivial that being not near coherent implies not being equivalent in the Rudin-Keisler order; it is obvious that for Q-points, in particular selective ultrafilters, the converse holds as well.

The question whether any two filters are near coherent is connected to many fascinating concepts in set theory and beyond it. It was first formulated by Andreas Blass in [Bla86] and developed further in [Bla87a]; for a simplified consistency proof of *NCF* see [BS89]. We should note that the model of *NCF* in [BS89] does not contain a union ultrafilter, since it does not contain a rapid

P-point. However, it does contain a P-point which answers a question from Neil Hindman if one P-point is enough to have a union ultrafilter.

The following result gives a much stronger answer to this question. It is due to Andreas Blass and will appear in [Blaa].

Theorem 4.8 (Union ultrafilters and *NCF* (Blass))
Let u be a union ultrafilter. Then $\min(u)$ and $\max(u)$ are not near coherent.

Union ultrafilters and *NCF* (Blass)

Spoiler *We argue indirectly and apply a parity argument for union ultrafilters towards a contradiction.*

Proof. Assume to the contrary that $\min(u)$ and $\max(u)$ are near coherent.

Step 1: Accordingly, we can find intervals $(I_n)_{n\in\omega}$ such that

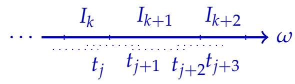
$$FU(s) \in u \Rightarrow (\exists k \in \omega) \begin{cases} \max[FU(s)] \cap I_k \neq \emptyset, \\ \min[FU(s)] \cap I_k \neq \emptyset. \end{cases}$$

A parity argument

Step 2: We can now partition $\mathbb{F}$ into

$$A_i := \{t \in \mathbb{F} \mid |\{k \mid I_k \cap t \neq \emptyset\}| = i \bmod 2\} \qquad (i = 0, 1).$$

Figure 4.2: Counting Intersections.



Since u is a union ultrafilter we can pick a homogeneous $FU(s) \in u$.

Step 3: On the one hand, a homogeneous set $FU(s)$ can only be included in A_0.

Parity of FU-sets must always be even.

Since the members of s are disjoint, we can always find $s_i \ll s_j$ such that

$$|\{k \mid (s_i \cup s_j) \cap I_k \neq \emptyset\}| = |\{k \mid s_i \cap I_k \neq \emptyset\}| + |\{k \mid s_j \cap I_k \neq \emptyset\}|.$$

We simply pick any s_i and choose some s_j with a minimum after the last interval I_k that meets s_i.

But then the parity for $s_i \cup s_j$ is 0, i.e., $s_i \cup s_j \in A_0$; in particular, $FU(s)$ is not disjoint from A_0.

Step 4: On the other hand – by the assumption of near coherence – we can find $k \in \omega$ such that

Near coherence implies odd parity!

$$I_k \cap \max[FU(s)] \neq \emptyset, I_k \cap \min[FU(s)] \neq \emptyset.$$

So choose s_i and s_j with $\max(s_i), \min(s_j) \in I_k$.

Step 5: Now $s_i = s_j$ is impossible, for otherwise $s_i \subseteq I_k$, so s_i's parity is odd.

Generally

I_k is the last interval meeting s_i and the first meeting s_j – no other interval meets both.

$$\begin{aligned} &|\{n \mid (s_i \cup s_j) \cap I_n \neq \emptyset\}| \\ = &|\{n \mid s_i \cap I_n \neq \emptyset\}| + |\{n \mid s_j \cap I_n \neq \emptyset\}| - 1 \\ = &1 \bmod 2, \end{aligned}$$ □

since by $\max(s_i), \min(s_j) \in I_k$ in fact k but only k will be counted twice here – leading again to the desired contradiction. ↯↯↯

This result immediately implies that for a union ultrafilter to exist, we must at least have two non-isomorphic P-points.

To end this section we give a proof of a strong partition theorem for not near coherent P-point. The theorem is in the spirit of Theorem 4.4. For this we require a small lemma from [Bla87a].

Lemma 4.9
Let F and G be not near coherent filters on ω. Assume that ω is partitioned into infinitely many intervals. Then there are $X \in F$, $Y \in G$ such that no union of two consecutive intervals meets both X and Y.

Proof. Let h be the monotone, finite-to-one function corresponding to the partition of ω. Let h^+ (h^-) be $\frac{h(x)}{2}$ rounded up (down); note that they are both monotone as well. Then both functions are finite-to-one and hence by assumption we find $X \in F, Y \in G$ such that $h^+[X] \cap h^+[Y] = h^-[X] \cap h^-[Y] = \emptyset$. It is easy to check that these sets have the desired properties. □

The following theorem shows that not near coherent P-points yield nearly as strong a partition theorem as non-isomorphic selective ultrafilters.

Theorem 4.10
Let p_0, p_1 be not near coherent P-points on ω. Let $[\omega]^\omega$ be partitioned into an analytic and a coanalytic part. Then there exist $X_i \in p_i$ ($i = 0, 1$) such that all $\{x_n \mid n \in \omega\}$ with $x_n < x_{n+1}$ and $x_n \in X_{(n \bmod 2)}$ are homogeneous.

Proof. **Step 1:** By [Bla88, Theorem 6'] there exists $X'_i \in p_i$ $(i = 0, 1)$ and a strictly monotone function $g \in \omega^\omega$ such that all alternating sequences $(x_n)_{n\in\omega}$ with the additional restriction of $g(x_n) < x_{n+1}$ are homogeneous.

Step 2: Let $I_0 := [0, g(0)]$ and $I_{n+1} := [g^n(0) + 1, g^{n+1}(0)]$ and choose $X_i \subseteq X'_i$ still in p_i $(i = 0, 1)$ according to the preceding lemma. We claim that these sets are as desired.

For let $(x_n)_{n\in\omega}$ be an increasing sequence alternating between X_0 and X_1. Since $x_n < x_{n+1}$ and say $x_n \in I_k$ for some $k \in \omega$ we know by the preceeding lemma that $x_{n+1} > \max(I_{k+1}) = g^{k+1}(0)$. But by choice of k and monotonicity of g we have $g(x_n) \leq g(g^k(0)) = g^{k+1}(0) < x_{n+1}$. Hence $\{x_n \mid n \in \omega\}$ is homogeneous. □

This result is not quite as strong as Theorem 4.4 since only selectives will yield ultrafilter sets alternating themselves; with arbitrary P-points this cannot be expected.

To generalize the construction of stable ordered union ultrafilters with prescribed min and max to arbitrary (stable) union ultrafilters with prescribed (rapid) P-points, the above partition theorem could prove to be a useful tool.

4.2. Characterizing stability

As noted before Andreas Blass laid the foundation for all further research regarding union and hence strongly summable ultrafilters in [Bla87b]. The final theorem from that paper gives a potent characterization of the strongest notion – stable ordered union ultrafilters. The theorem allows the reader to take a deep look into the core of the involved notions unifying combinatorial and model theoretic characteristics.

Our interest lies for this section in the question if there could be similar characterizations for the more general notion of stable union ultrafilters. This could help to clarify the difference between ordered and unordered union ultrafilters, as well as identify the general consequences of stability, especially with respect to a possible construction of a union ultrafilter that is not stable. We require some definitions that will be relevant for this section only.

Definition 4.11 Consider $u \in \delta\mathbb{F}$.

1. We say that u has the *canonical partition property* if u-prod(ω) has exactly 5 constellations, i.e., there are five functions in ω^ω such that for any function $f \in \omega^\omega$ there exists bijective $g \in \omega^\omega$ such that $g \circ f$ is equal (mod u) to exactly one of the five.
2. We say that u has the *Ramsey property for pairs* if for any finite partition of $\mathbb{F}^2_< := \{(s,t) \in \mathbb{F}^2 \mid s < t\}$ there exists a set $A \in u$ such that $A^2_<$ is homogeneous.
3. We say that u has the *Ramsey property* if for any n and any finite partition of $\mathbb{F}^n_< := \{(s_0, \ldots, s_{n-1}) \in \mathbb{F}^n \mid s_i < s_{i+1}\}$ there exists a set $A \in u$ such that $A^n_<$ is homogeneous.
4. We say that u has the *infinitary partition property* if for every analytic $X \subseteq \mathbb{F}^\omega_< = \{s \in \mathbb{F}^\omega \mid s \text{ is ordered }\}$[3] there exists a set $A \in u$ such that $A^\omega_<$ is included or disjoint from X.

[3] As usual, we speak of analytic sets with respect to the product topology of $\mathbb{F}^\omega$.

We are now ready to state part of the original theorem.

Theorem 4.12 (Stable ordered union characterization (Blass)) *For ordered union ultrafilters the following properties are equivalent.*

Stable ordered union characterization (Blass)

1. *Stability*
2. *The Ramsey property for pairs*
3. *The non-standard element* $[\min]_u$ *generates an initial segment of* u*-prod*(ω)
4. *The canonical partition property*
5. *The Ramsey property*
6. *The infinitary partition property.*

Proof. This is [Bla87b, Theorem 4.2]. We skip the proof since we wish to generalize it. □

The generalization we are about to formulate may seem nearly trivial at first – we only drop the "ordered" condition and the proof is nearly identical. However, it settles the first, albeit small, open question: does the Ramsey property for a union ultrafilter already imply that it is ordered. Additionally, in the case of ordered union ultrafilters, the orderedness helps simplify the proof. So although we "only" show that characterization of stability in terms of the Ramsey property is prima facie the same, there are subtle differences and implications that arise from it. Of course, we stated the Ramsey properties in such a way that they naturally generalize to union ultrafilters – we should keep in mind, though, that stable ordered union ultrafilters can additionally get the homogeneous set to be an *FU*-set of an ordered sequence. This suggests (at least for the Ramsey properties) a considerable strengthening, since ordered sequences decrease the complexity of the set of excluded pairs dramatically. Nevertheless, this similarity is quite important as it offers common ground towards a construction of a union ultrafilter that is not stable – be it ordered or not; even though we cannot distinguish ordered stable from unordered stable union ultrafilters by means of the Ramsey property, this is not a completely negative observation. It is also an open question whether an analogous characterization in terms of the ultrapower of ω could be found for stable union ultrafilters.

In the following theorem we only prove the equivalence between stability and the Ramsey property for pairs, but the different Ramsey properties are in fact equivalent; cf. the remark after the theorem.

Theorem 4.13
For union ultrafilters the following properties are equivalent.

1. *Stability*
2. *The Ramsey property for pairs.*

Spoiler *Our strategy is the same as for Theorem 4.12. To show that stability implies the Ramsey property we argue similarly to the proof of Ramsey's Theorem using an ultrafilter, i.e., we (implicitly) use the fact that the tensor product of ultrafilters yields an ultrafilter. To get a homogeneous set in our ultrafilter we use stability and a refined parity argument. The reverse conclusion is just as in the original proof by Blass.*

$(2 \to 1)$ *Proof.* **Claim 1:** The Ramsey property for pairs implies stability. This is done just as in the original proof of Theorem 4.12; we repeat it here for completeness.

Step 1: Given any sequence $(FU(s^\alpha))_{\alpha<\omega}$ in u we consider the following set of ordered pairs

$$\{(v,w) \in \mathbb{F}^2_< \mid w \in \bigcap_{\alpha<\max(v)} FU(s^\alpha)\}.$$

Step 2: Any $FU(t) \in u$ will yield some pairs that are in the above set – pick any $v \in FU(t)$ and then take $w > v$ from the appropriate intersection with $FU(t)$.

Step 3: Therefore by the Ramsey property for pairs, there must be a set $FU(s) \in u$ such that all ordered pairs are included in the above set.

Claim: $s \sqsubseteq^* s^\alpha$ for all $\alpha < \omega$.

Given $\alpha < \omega$, pick s_i with $\max(s_i) > \alpha$; then all but finitely many s_j have $s_j > s_i$.

For such s_j we have $(s_i, s_j) \in FU(s)^2_<$, hence

$$s_j \in \bigcap_{\beta < \max(s_i)} FU(s),$$

in particular $s_j \in FU(s_\alpha)$ – as desired.

We now turn to the main part of the proof.

Claim 2: Stability implies the Ramsey property for pairs. $(1 \to 2)$

Assume that $A_0 \dot\cup A_1 = \mathbb{F}^2_<$.

Step 1: Since u is an ultrafilter (in $\delta\mathbb{F}$) we have

We always pick one colour

$$(\forall x \in \mathbb{F})(\exists i)\{y \in \mathbb{F} \mid (x < y) \in A_i\} \in u.$$

Step 2: Since u is an ultrafilter, we may assume that the colour i is always the same; without loss it is 0, i.e., we find $A \in u$ such that

C_x – We almost always pick the same colour.

$$(\forall x \in A)\ C_x := \{y \in \mathbb{F} \mid (x < y) \in A_0\} \in u.$$

Step 3: Since u is union we can pick $FU(s^\alpha) \in u$ for each $\alpha < \omega$ such that

$$FU(s^\alpha) \subseteq \bigcap_{\max(x) \leq \alpha} C_x.$$

Note that for $x \in A$ we now have (by choice of s^α)

$$FU(s^{\max(x)}) \subseteq C_x.$$

Step 4: Since u is stable by assumption, we find $FU(s) \in u$ such that

Stability – we "nearly" almost always pick from the same set

$$s \sqsubseteq^* s^\alpha \text{ for all } \alpha < \omega.$$

We want to check how many members of s are not included in s^α.

Step 5: Let us consider the following function

j – Counting where s fails

$$j : \omega \to \omega, \alpha \mapsto \max\{\max(s_i) \mid s_i \notin \bigcap_{\beta \leq \alpha} FS(s^\beta)\}.$$

Note that we may assume that j is strictly increasing since without loss $s^{\alpha+1} \sqsubseteq s^\alpha$ for all $\alpha < \omega$.

Step 6: We can now observe that for all $x \in FU(s)$

$$\min(x) > j(\alpha) \Rightarrow x \in FU(s^\alpha).$$

For s_i this follows from the definition of j by contraposition. But if $\min(x) > j(\alpha)$, then by this argument all $s_i \subseteq x$ are in $FU(s^\alpha)$, hence their union, i.e., x.

After this observation we will now aim to construct some in $A \in u$ for which $v < w$ in A implies $\min(w) > j(\max(v))$. For then $w \in FU(s^{\max(v)}) \subseteq C_v$ by Steps 6 and 3 respectively.

Thinning out 1 – bounding j

Step 7: $\{x \in FU(s) \mid j(\min(x)) < \max(x)\} \in u$.

In any condensation of s we find some x, x' and $x' \cup x$ with $x < x'$ and $j(\min(x)) < \max(x')$. Then we calculate

$$j(\min(x \cup x')) = j(\min(x)) < \max(x') = \max(x \cup x').$$

Hence any set in u will intersect the above set; so it lies in u.

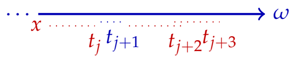
In particular, we find $FU(t) \in u$ included in the above set.

Thinning out 2 – splitting pairs

Step 8: For $x \in FU(t)$ call t_i, t_j a *splitting pair (of x)*, whenever

$$x = (x \cap \max(t_i)) \dot\cup (x \setminus \min(t_j)) \text{ and } (\exists t_k)\ t_i < t_k < t_j.$$

Figure 4.3: Splitting x at $(j, j+2)$.



Let $\pi(x)$ be the number of splitting pairs of x, i.e.,

$$\pi(x) := |\{(i,j) \mid t_i, t_j \text{ a splitting pair of } x\}|$$

The splitting pairs tell us how often we can split v into two ordered parts (the one up to t_i and the one beyond t_j) with a gap in between.

Step 9: $\{x \in FU(t) \mid \pi(x) = 1 \bmod (2)\} \in u$.

As in Step 7, any condensation of t will contain some $x < y < z$ and $x \cup z$. So we calculate

$$\pi(x \cup z) = \pi(x) + \pi(z) + 1.$$

Therefore the number must be odd.

In particular, we may pick $FU(v) \in u$ contained in the above set.

Step 10: For any $w_0 < w_1$ in $FU(v)$, there exists t_j with

$$w_0 < t_j < w_1$$

Or else we would have an even number of splitting pairs for $w_0 \cup w_1$.

The conclusion

Claim: $FU(v)$ is homogeneous, i.e., $FU(v)^2_< \subseteq A_0$.

For this pick any $w, w' \in FU(v)$ with $w < w'$.

By the last step, there exists some t_j with $w < t_j < w'$. Therefore we can calculate

$$\max(w') > \max(t_j) > j(\min(t_j)) > j(\max(w)),$$

where the last inequality holds since j is strictly increasing.

But we already noted just after Step 6 that this implies $w' \in C_w$, i.e., $(w, w') \in A_0$ – as desired. □

After this theorem we might be misled to think that the Ramsey property for pairs is somewhat special, but of course the Ramsey property for pairs and the other Ramsey properties are closely connected.

Remark 4.14 We should note that in the proof of Theorem 4.12 the proof of the equivalence of the Ramsey property for pairs, the Ramsey property and the infinitary Ramsey property does not require the witnessing sequence to be ordered.[4] Therefore stability for union ultrafilters is equivalent to these characterizations as well.

[4] But of course we get homogeneity only for ordered pairs, tuples and condensations respectively.

The question whether there can be an (ordered) union ultrafilter that is not stable remains a difficult open question. The construction that we develop in the next section does seem to adapt to such a construction. However, after a considerable amount of research in this direction I have not found any indication towards the contrary either. Therefore I would still conjecture that such ultrafilters consistently exist.

4.3. *Unordered union ultrafilters*

We now turn to the question of whether the properties of the min and max can indicate orderedness of a union ultrafilter. We answer this question negatively assuming *CH*.

How to approach the problem?

To construct such an unordered union ultrafilter with (possibly prescribed) selective images, we are bound to analyze if an argument similar to that in the proof of Theorem 4.6 is possible while preventing the constructed ultrafilter from becoming ordered. This also arises from the "complete combinatorics" of selective ultrafilters for the countably generated filter forcing; if a construction works for one pair, it should work for all pairs. Examining the proof of Theorem 4.6 we can see that the essential tool besides the Galvin-Glazer Theorem lies in Theorem 4.4: whenever we have a pair of non-isomorphic selective ultrafilters we are able to "decide" an analytic subset of the powerset by an alternating sequence from two sets from the ultrafilters; in our case, the analytic set corresponded to all *FU*-sets continuing an inductive construction of a stable ordered union ultrafilter.

At first sight however, this idea does not seem to extend to a construction of an unordered union ultrafilter due to the sequences being alternating; we can easily check that any *FU*-set with alternating min and max image must be ordered.[5] But to have some hope it is already enough to realize that in fact two non-isomorphic selective ultrafilters do not concentrate on alternating sets – or else mapping two alternating sets to each other by picking the next alternate element would be an isomorphism. And luckily Theorem 4.4 offers f-alternating sets for every function $f \in \omega^\omega$. Therefore we have plenty of sets available that do not necessarily force an *FU*-set coming from a homogeneity argument to be ordered.

[5] The smallest min must come from the same s_i with the smallest max – or else some other s_j would have a greater min than max ↯↯↯ and so forth inductively.

What are unordered union ultrafilters?

But what does an unordered union ultrafilter actually look like? Abstractly speaking, to be unordered means that there must be some "special" *FU*-set in the ultrafilters that will not be refined to an ordered *FU*-set in the ultrafilter. But of course, any disjoint sequence has an ordered subsequence, in particular an ordered condensation, since only finitely many members can mesh with any one member. So not refining the "special" *FU*-set to an ordered *FU*-set means that no ordered condensation ends up in our union ultrafilter.[6] Of

[6] And we should note that we always have to include some sets coming from ordered sequences, e.g., $\mathbb{F}$ coming from the sequence of singletons.

course, we know by the construction from [BH87] that there exist plenty of unordered sequences with unordered condensations for any partition, but we cannot really get a practical idea regarding this. Since the aim of that construction lies in producing a union ultrafilter with no Q-point as min or max, the tools that can be used focus on this question, i.e., preventing finite-to-one functions from becoming injective; it is such an indirect argument, that there isn't even the need for consideration of such a "special" set, its existence only follows indirectly. The question is, can we find unordered sequences that do not prevent the min and max from being Q-points?

What kind of meshing?

But if a sequence is not ordered, it is meshed in the sense that some of its members must mesh. Since any union ultrafilter is in $\delta\mathbb{F}$, there must be "arbitrarily late" meshing. It is also easy to see that union ultrafilters concentrate on condensations that contain unions of many members of the sequence, e.g., because the sequence itself will not be in the union ultrafilter; therefore we cannot have a bound on the number of elements that mesh. Finally, by parity arguments we can see that we cannot have meshing only of e.g. the form $s_{2i} \sqcap s_{2i+1}$, since a union ultrafilter will concentrate on those with an even number of indices – so any union ultrafilter will condense such a sequence to an ordered sequence. In short, whatever we do, we will have to get an idea of what an appropriate form of meshing could be.

4.3.1 *Formulating the result*

We want to construct a union ultrafilter that is not ordered, but has selective ultrafilters as images under min and max. Let us dig a little deeper into the problem by formulating a first version of our result.

Unordered Union Ultrafilters

Theorem (Unordered Union Ultrafilters)
Assume CH. There exists a union ultrafilter u with $\min(u)$ *and* $\max(u)$ *selective, but u is not additively isomorphic to an ordered union ultrafilter.*

In fact, we can prescribe any two non-isomorphic selective ultrafilters for min *and* max.

We will need a series of results for the proof, but first let us specify our result to which the above theorem is a corollary. This should help clarify what we intend to prove.

Stable Unordered Union Ultrafilters

Theorem (Stable UnorderedUnion Ultrafilters)
Assume CH. There exists a stable union ultrafilter u with $\min(u)$ *and* $\max(u)$ *selective, but there exists $FU(s) \in u$ such that for any ordered sequence t*

$$t \sqsubseteq s \Rightarrow FU(t) \notin u.$$

In fact, we can prescribe any two non-isomorphic selective ultrafilters for min *and* max.

At first this does not seem to be imply the first theorem; although the first part specifies that u is not ordered we replace the rigidity under additive isomorphism by stabililty for union ultrafilters.

To see that this suffices for the additional requirement that no additive copy of u happens to be ordered union, we prove the following lemma.

Lemma 4.15 (Stability and Additive Isomorphisms) *Every additively isomorphic image of a stable union ultrafilter is a stable union ultrafilter and every additively isomorphic image of a stable ordered union ultrafilter is a stable ordered union ultrafilter.*

In particular, if u is as in the above theorem, then u is not additively isomorphic to an ordered union ultrafilter.

Stability and Additive Isomorphisms

Spoiler *Stability is straightforward; for orderedness we use the Ramsey property of stable ordered union ultrafilters.*

Proof. Let u' and u be additively isomorphic union ultrafilters, i.e., there exist $FU(s) \in u$, $FU(x) \in u'$ such that

$$\pi : FU(s) \to FU(x), \prod_{i \in F} s_i \mapsto \prod_{i \in F} x_i$$

additionally has $\pi(u) = u'$.

Claim 1: If u is stable, so is u'.

Given a sequence of condensations with FU-sets in u' we can intersect every FU-set with $FU(x)$ and from this get a sequence of condensations below x.

But then the preimage of the sequence under π is a sequence of condensations in u. So we can apply u's stability to get a common almost condensation of these images in u – but its image under π is exactly the desired common almost condensation in u'.

Claim 2: If u is stable ordered, so is u'.

By Claim 1 we only need to show that u' is ordered. Pick any $A \in u'$; we may assume without loss that $A \subseteq FU(x)$ and that s was ordered.

Step 1: Consider

Working in u: a partition for orderedness.

$$X := \{\{v < w\} \in [\pi^{-1}[A]]^2 \mid \max(\pi(v)) < \min(\pi(w))\}.$$

Step 2: By the Ramsey property from Theorem 4.12, we can find an ordered sequence t such that $FU(t) \in u$ and

$$FU(t)^2_< = \{\{v < w\} \mid v, w \in FU(t)\}$$

is either included in or disjoint from X.

Step 3: But it cannot be disjoint from X.

Since π is injective, we can find $t_i \ll t_j$ such that

$$\pi(t_i) < \pi(t_j).$$

As usual, given any t_i all but finitely many t_j have this property.

Then of course $\{t_i, t_j\} \in X \cap FU(t)^2_<$.

Step 4: But this implies that $\pi[FU(t)] = FU(\pi[t])$ is ordered – and of course in u' and refining A, i.e., u' is ordered union. □

We are now ready to begin the series of lemmas that we need for the construction.

4.3.2 *The construction*

The critical issue – a special set $FU(s)$

Let's recall our goal: however we construct u, we must include a set $FU(s)$, such that for *any ordered condensation* $t \sqsubseteq s$ we can guarantee $FU(t) \notin u$.

Now it would be easy to just inductively put together a union ultrafilter with a base of not-ordered FU-sets. But this does not suffice, since there might be a different base of ordered FU-sets.

To prevent this, we must make sure that no sequence that we pick in our inductive construction will accidentally be, at the same time, a condensation of some other, ordered condensation of the fixed $FU(s)$ (thus including that ordered condensation of $FU(s)$ in our ultrafilter u as well).

This means that every sequence we choose must eventually have a high degree of meshing not just in itself but due to the s_i that appear in its support. The following definition prepares us for the right notion of meshing.

Definition: The meshing graph

Definition 4.16 For a disjoint sequence $s = (s_i)_{i<N}$ in $\mathbb{F}$ (for some $N \leq \omega$) and some condensation $t = (t_j)_{j<K}$ of s (for some $K \leq N$) we define the following:

The meshing graph G_t is the graph on the vertices $\{t_j \mid j < K\}$ with edges

$$E(G_t) = \{\{t_i, t_j\} \mid (\exists s_n \subseteq t_i, s_m \subseteq t_j)\ s_n \sqcap s_m\},$$

i.e., there is an edge whenever two t_j are meshed and this meshing is caused by single elements from s.

We can now talk about meshing combinatorially in terms of the connectedness of this graph. On the one hand, it is an obvious advantage to connect to graph theory; instead of partitions on $\mathbb{F}$ we can speak about graph colourings. On the other hand, we do not know how strongly or weakly connected the graph should be – and it is not trivial to get Ramsey-type theorems for graphs that allow a flexible degree of connectedness. Fortunately, it will turn out that we can work with complete graphs.

To begin our construction, we need a sequence that is thoroughly meshed.

Remark: **Fix the meshed sequence** s

Remark 4.17 (Fix the meshed sequence s) From now on fix a sequence $s = (s_i)_{i\in\omega}$ such that for any n, k there exist $k < i_0 < \ldots < i_n$ such that

$$G_{(s_{i_0},\ldots,s_{i_n})}$$

is a complete graph with $n+1$ vertices.[7]

[7] Here the meshing graph is computed with respect to s itself.

This simply means that our sequence includes arbitrarily large subsequences that have the best meshing. For now we shall assume that we pick any such sequence. After the main theorem we will discuss how to find such a sequence with respect to two prescribed selective ultrafilters.

The following definition is trying to capture the right kind of meshing that is needed for sets that might be suitable for our ultrafilter.

Definition 4.18 A set $A \subseteq \mathbb{F}$ is called *FU-meshed (with respect to s)* if for any $n \in \omega$ there exist (disjoint) $t = (t_i)_{i<n}$ such that

Definition: FU-meshed

- $FU(t) \subseteq (FU(s) \cap A)$
- The meshing graph G_t is a complete graph.

We call such a finite sequence an *n-witness* of A.[8]

[8] Note that an FU-meshed set is compatible with $\delta\mathbb{F}$ since for any $v \in \mathbb{F}$ any disjoint sequence of length $\max(v) + 2$ must have an element in $\sigma(v)$.

A set A is FU-meshed if there are members of A that have a high degree of meshing and additionally the witnesses for the meshing are given by arbitrarily large, finite FU-sets where the members of the s-support mesh very much.

The following observation should support the claim that this is the right notion for our setting, i.e., such sets do not force us to add ordered condensations to an ultrafilter.

Proposition 4.19
If A is FU-meshed, then it is not included in $FU(t)$ for any ordered $t \sqsubseteq s$.

Proof. To be an ordered condensation $t \sqsubseteq s$ means that G_t has no edges, hence $FU(t)$ cannot include an FU-meshed set. □

To be able to link the new notion with ultrafilters we need to show that it is partition regular. For this we require a classical result that we have already discussed in Example 2.8.

Theorem 4.20 (Graham, Rothschild)
For any n there exists $h(n)$ such that for any disjoint sequence $x = (x_i)_{i<h(n)}$ in $\mathbb{F}$ the following holds:

Graham, Rothschild

Whenever $FU(x)$ is finitely partitioned, there exists a homogeneous condensation of length n.

Finite Hindman's Theorem

Proof. This is a corollary of the Graham-Rothschild Parameter Sets Theorem, [GR71, Corollary 3]. For a proof from Hindman's Theorem by a compactness argument cf. [HS98, Theorem 5.15]; for a more recent overview on the combinatorial aspects cf. [PV90]. □

We are now ready to prove the first piece of our puzzle.

Lemma 4.21 (*FU*-meshed partition regular)
The notion of being FU-meshed is partition regular.

FU-meshed partition regular

In particular, any FU-meshed set is included in an ultrafilter consisting only of sets that are FU-meshed.

Spoiler *Given a partition of an FU-meshed set, the theorem by Graham and Rothschild allows us to find large homogeneous condensations. For n-witnesses we will find that the homogeneous condensation inherits a complete meshing graph.*

Proof. Clearly, $\mathbb{F}$ is FU-meshed since it contains $FU(s)$. So let us fix an arbitrary FU-meshed set $A \subseteq FU(s)$ and let $A = A_0 \dot\cup A_1$.

Let us assume to the contrary that both A_0 and A_1 are not FU-meshed.

Step 1: Then we can find n_0, n_1 such that there are no n_i-witnesses in A_i.

Let us fix $n := \max(n_0, n_1)$.

The theorem by Graham and Rothschild implies that we cannot fail to satisfy the finite version of Hindman's Theorem.

Applying "Finite Hindman"

Step 2: Since A is FU-meshed, we can apply the theorem by Graham and Rothschild to find an $h(n)$-witness $t = (t_i)_{i<h(n)}$ in A.

Naturally, the partition of A induces a partition of $FU(t)$, i.e.,

$$FU(t) = A'_0 \dot\cup A'_1,$$

with $A'_i := FU(t) \cap A_i$.

Since we chose $h(n)$ to be large enough using Theorem 4.20, there exists a condensation $x = (x_i)_{i<n}$ such that $FU(x)$ is included in one $A'_i \subseteq A_i$.

We analyze the meshing graph to see that the finite FU-set must have a complete graph.

Claim 1: But then $FU(x)$ is an n-witness for A_i.

G_t was complete. Since x is a condensation, its members are disjoint unions of members of t.

Clearly, the edges between the members of t induce edges between the members of x that include them. Hence the meshing graph of $x = (x_i)_{i<n}$ is also complete.

But this contradicts the fact that neither A_0 nor A_1 had an (n, v)-witness ⚡⚡⚡ as desired. □

Now we know that there are ultrafilters containing only FU-meshed sets. But we need to show that this set is algebraically rich, so that we can get some kind of induction going akin to the proof of Theorem 4.6. There we just needed the partial semigroups induced by FU-sets, but for our purpose we need to be more subtle.

The meshing semigroup

Lemma 4.22 (The meshing semigroup)
The set

$$H := \{p \in FU^\infty(s) \cap \delta\mathbb{F} \mid (\forall A \in p)\ A \text{ is } FU\text{-meshed}\}$$

is a closed subsemigroup of $\delta\mathbb{F}$.

Spoiler *We argue just like Example 2.8.*

Proof. **Claim 1:** H is a closed subset of $\delta\mathbb{F}$ since it is defined by a constraint on all members of its elements.

Claim 2: We just showed in Lemma 4.21 that it is non-empty.

Claim 3: H is a subsemigroup.

Step 1: Pick arbitrary $p, q \in H$ and $V \in p$, $(W_v)_{v\in V}$ in q; in particular all these sets are FU-meshed.

We need to show that

$$\bigcup_{v\in V} (v \cdot W_v) \text{ is } FU\text{-meshed.}$$

So let $n \in \omega$.

Step 2: By assumption on p we can find an n-witness $t = (t_i)_{i<n}$ such that

$$FU(t) \subseteq V.$$

Step 3: Similarly by assumption on q, we can find an n-witness $t' = (t'_i)_{i<n}$ such that

$$FU(t') \subseteq \bigcap_{x\in FU(t_0,\dots,t_n)} W_v \cap \sigma(\bigcup_{i\leq n} t_i).$$

Step 4: But then for $z = (z_i)_{i<n}$ with $z_i := t_i \cup t'_i$ we have

$$FU(z) \subseteq \bigcup_{v\in V} v \cdot W_v \cap \sigma(a).$$

Of course G_v is complete since G_t was (or since $G_{t'}$ was) – making the sets "fatter" only increases the chance of being meshed.

In particular, our set is FU-meshed – as desired. □

In fact, we have shown a bit more.

Corollary 4.23
H is a two sided ideal in the closed semigroup from Example 2.8.

The next important step is to show that the preimage filters for certain ultrafilters are compatible with H, i.e., contain FU-meshed sets. Remembering Theorem 2.29 it comes as no surprise that we can show more.

Lemma 4.24
If $A \subseteq \min[FU(s)]$, $B \subseteq \max[FU(s)]$ are both infinite, then

$$\min{}^{-1}[A] \cap \max{}^{-1}[B]$$

is FU-meshed.

Spoiler *We pick three sets of members of s: one set to get the correct minimum, another set (past those chosen) to get the meshing, and finally (past those chosen) a set to get the allowed maximum.*

Proof. Given $n \in \mathbb{N}$ we pick three times n-many elements of the sequence $s = (s_i)_{i \in \omega}$.

Step 1: First, since A is infinite, we can pick $(s_{i_k})_{k<n}$ with $\min(s_{i_k}) \in A$.

Step 2: Second, by the meshing of s, we can pick $(s_{j_k})_{k<n}$ with a complete meshing graph but beyond everything chosen so far.[9]

[9] In other words, with minima greater than the greatest maximum so far.

Step 3: Third, since B is infinite, we can pick $(s_{l_k})_{k<n}$ with $\max(s_{l_k}) \in B$, again beyond everything chosen so far.

Then $(t_k)_{k<n}$ defined by $t_k := s_{i_k} \cup s_{j_k} \cup s_{l_k}$ is an n-witness for $\min^{-1}[A] \cap \max^{-1}[B]$. □

For completeness, we include the following standard proposition about partition regular notions.

Proposition 4.25
Let ψ be a partition regular property on X.

If F is a filter on X with all sets in F having property ψ, then there exists an ultrafilter including F with all sets having property ψ.

Proof. **Step 1:** Consider

$$G := \{A \subseteq X \mid X \setminus A \text{ does not have property } \psi\}.$$

Since ψ is partition regular, G is a filter.

Clearly $X \in G$ and G is closed under supersets. Additionally, if $(X \setminus A) \cup (X \setminus B) = X \setminus (A \cap B)$ had property ψ, then so would one part, which is impossible for $A, B \in G$.

Step 2: F and G are coherent.

If $A \in G$ and $B \in F$ had $A \cap B = \emptyset$, then $B \subseteq (X \setminus A) \in F$, which is impossible since F only contains sets with property ψ.

Step 3: Any ultrafilter p extending $F \cup G$ is as desired.

If $A \in p$ did not have property ψ, then $X \setminus A \in G \subseteq p$, which is absurd. □

After this technicality we can state its corollary needed for the construction.

Corollary 4.26
Let p_1 and p_2 be ultrafilters including $\min[FU(s)]$, $\max[FU(s)]$ *respectively. Then*

$$\overline{\min{}^{-1}(p_1)} \cap \overline{\max{}^{-1}(p_2)} \cap H \neq \emptyset.$$

Proof. By Lemma 4.24 all elements of the preimage filter are FU-meshed. Applying Proposition 4.25 we get an ultrafilter in H extending the preimage filter. □

4.3.3 *Main lemma and theorem*

Now we are ready for our final technical lemma that is needed for an inductive construction in the spirit of the proof of Theorem 4.6.

Main Lemma

Lemma 4.27 (Main Lemma)
Assume we are given selective ultrafilters $p_1 \not\cong_{RK} p_2$ *with* $\max[FU(s)] \in p_1$ *and* $\min[FU(s)] \in p_2$ *and we are also given an arbitrary* $X \subseteq \mathbb{F}$.

For every $\alpha < \omega$ *let* $t^\alpha = (t_i^\alpha)_{i\in\omega}$ *be a sequence such that*

$$t^{\alpha+1} \sqsubseteq^* t^\alpha$$
$$FU(t^\alpha) \text{ is } FU\text{-meshed}$$
$$FU(t^\alpha) \in \min{}^{-1}(p_1) \cap \max{}^{-1}(p_2).$$

Then there exists $z = (z_i)_{i\in\omega}$ *such that*

$$z \sqsubseteq^* t^\alpha \text{ for every } \alpha < \omega,$$
$$FU(z) \text{ is } FU\text{-meshed},$$
$$FU(z) \subseteq X \text{ or } FU(z) \cap X = \emptyset,$$
$$\min[FU(z)] \in p_1 \text{ and } \max[FU(z)] \in p_2.$$

Spoiler *With Corollary 1.16 to the Galvin-Glazer Theorem , we can find a common almost condensation of the given FU-sets and X (or its complement – whichever set is FU-meshed and compatible). Since all of these are FU-meshed, we can find a sequence that is FU-meshed. To get an almost condensation with* min *and* max *images actually elements of* p_1 *and* p_2 *respectively, we apply Theorem 4.4.*

A helpful idempotent e

Proof. **Step 1:** By our assumptions

$$H \cap \overline{\min{}^{-1}(p_1)} \cap \overline{\max{}^{-1}(p_2)} \cap \bigcap_{\alpha<\omega} \overline{FU(t^\alpha)} \neq \emptyset.$$

As an intersection of closed semigroups it is a closed semigroup. So we can pick an idempotent $e \in \delta\mathbb{F}$ therein. We may assume that $X \in e$; in particular X is FU-meshed.

> We will now apply the homogeneity result 4.4. Instead of the alternating parity function used in Theorem 4.6, we will[10] be considering another function.

[10] and have to – cf. the introduction to this section.

An analytic set

Step 2: Consider the following analytic set in $\mathfrak{P}(\omega)$.

$$\{Y \subseteq \omega \mid (\exists z = (z_i)_{i\in\omega})\ Y = \min[FU(z)] \cup \max[FU(z)],$$
$$(\forall \alpha < \omega)\ z \sqsubseteq^* t^\alpha,$$
$$FU(z) \subseteq X,$$
$$FU(z) \text{ is } FU\text{-meshed}\}.$$

$0,1,0,0,1,1,0,0,0,1,1,1,\ldots$

Let us define $f \in 2^\omega$ inductively to have n-many 0's followed by n-many 1's for each n in increasing order of n's.

Using homogeneity

Step 3: We can apply Theorem 4.4 to find $Y_1 \in p_1$, $Y_2 \in p_2$ such that

$$\{A \subseteq Y_1 \cup Y_2 \mid A \ f\text{-alternating}\}$$

is either contained in or disjoint from the analytic set.

f-alternating sets from p_1 and p_2.

Note that since p_1 and p_2 are not isomorphic, we can find $Y_1 \in p_1$, $Y_2 \in p_2$ being f-alternating already; cf. Remark 4.5.

Step 4: But it can only be included!

> We will show that this homogeneity can only be "right", i.e., all f-alternating sets are included in our analytic set. In particular, given the remarks just above, we are then able to find a sequence $(z_i)_{i\in\omega}$ for Y_1 and Y_2 themselves; in other words, we find an FU-set with all the properties desired for the lemma.
> Let us recapitulate: We want to show that for every Y_1, Y_2 as above, the homogeneous set intersects our analytic set. For this we require an FU-set with min and max not only mapping it "into" the analytic set but also mapping it to *subsets* of Y_1 and Y_2 respectively.

Applying Galvin-Glazer to find t

Claim: For every $Y_1 \in p_1$, $Y_2 \in p_2$ we can construct $t = (t_i)_{i\in\omega}$ a common almost condensation of $(t^\alpha)_{\alpha<\omega}$ with

$$FU(t) \subseteq X \cap \min^{-1}[Y_1] \cap \max^{-1}[Y_2] \cap FU(s).$$

Additionally, $FU(t)$ is FU-meshed and the minima and maxima are f-alternating.

First note that the intersection of the four sets is in e; let us abbreviate it by Z. As usual, we may assume that $Z^* = Z$ (with respect to e). We construct the desired sequence by induction.

At the inductive step, having constructed $t_0, \ldots, t_{k-1}$[11] we pick some FU-meshing n-witness $t_k, \ldots, t_{n+k-1}$ from the following set in e

$$\bigcap_{x\in FU(t_0,\ldots,t_k)} x^{-1}Z^* \cap \sigma(\bigcup_{i<k} t_i) \cap \bigcap_{\alpha<n} FU(t^\alpha).$$

[11] To be exact, $k = \frac{1}{2}n(n-1)$ (for an induction on n).

The resulting sequence is FU-meshed by construction.

For the f-alternation note that for a sequence of length n with a complete meshing graph, all minima must come before all maxima. Since we chose our witnesses in an ordered fashion, this implies that our entire sequence has f-alternating minima and maxima.

Therefore homogeneity can only ever be "right"; in particular, we find a sequence z for Y_1 and Y_2 themselves — and with all the desired properties. □

We can now describe the CH-construction.

Putting it all together

Theorem 4.28
Assume CH and let $p_1 \not\cong_{RK} p_2$ be selective ultrafilters including $\min[FU(s)]$ and $\max[FU(s)]$ respectively.

Then there is a stable union ultrafilter u including $FU(s)$ with $\min(u) = p_1$, $\max(u) = p_2$ such that no ordered condensation t of s has $FU(t) \in u$.

Proof. Somce we assume *CH* we can fix $(X_\alpha)_{\alpha<\omega_1}$, an enumeration of $\mathfrak{P}(\mathbb{F})$.

In the inductive step we have constructed $(FU(t^\alpha))_{\alpha<\beta}$ for some ordinal $\omega \leq \beta < \omega_1$ [12] such that for all $\gamma < \alpha < \beta$

$$
\begin{aligned}
& t^\alpha \sqsubseteq s \\
& t^\alpha \sqsubseteq^* t^\gamma \\
& FU(t^\alpha) \subseteq X_\alpha \;\vee\; FU(t^\alpha) \cap X_\alpha = \emptyset \\
& \min[FU(t^\alpha)] \in p_1 \wedge \max[FU(t^\alpha)] \in p_2 \\
& FU(t^\alpha)\ FU\text{-meshed}
\end{aligned}
$$

[12] fill this sequence with $FU(s)$ for $\alpha \leq \omega$

We simply pick a cofinal sequence $(\alpha(n))_{n\in\omega}$ in β. Then we can apply Lemma 4.27 to $X := X_\beta$ and $(FU(t^{\alpha(n)}))_{n\in\omega}$ to get t^β sufficient to continue the induction.

It should not cause any difficulties to check that the resulting sets will generate a union ultrafilter as desired. □

Finally let us remark that the choice of the sequence s is not all that special, so that we can indeed claim to have proved the theorem stated at the beginning of this section.

Corollary 4.29 (The main theorem)
Assume CH. For any p_1, p_2 non-isomorphic selective ultrafilters, there exists a stable union ultrafilter u that is not ordered, such that $\min(u) = p_1$ and $\max(u) = p_2$.

Corollary (The main theorem)

Spoiler *We use Theorem 4.4 to make sure that we always find a sequence with sufficient meshing to apply the last theorem.*

Proof. **Step 1:** We just need to find a suitably meshed sequence s with

$$\min[FU(s)] \in p_1, \max[FU(s)] \in p_2,$$

be able to invoke the last theorem.

Step 2: We consider the analytic set

$$\{X \subseteq \omega \mid (\exists s)\ X = \max[FU(s)] \cup \min[FU(s)] \text{ and } FU(s) \text{ is } FU\text{-meshed}\}.$$

Here we mean FU-meshed with respect to s itself.

We argue similarly to the last proof, but keep it a bit shorter.
To apply Theorem 4.4 as in the last proof, we just check that any

$$\min{}^{-1}[A] \cap \max{}^{-1}[B]$$

must include $FU(s)$ for some suitably meshed sequence (which as in the the proof of the theorem will guarantee f-alternation).

Step 3: Recall that

$$\min^{-1}(p_1) \cup \max^{-1}(p_2)$$

generates an idempotent filter, so we can extend it to an idempotent ultrafilter.

So given any $A \in p_1$, $B \in p_2$ there exists $FU(v) \subseteq \min^{-1}[A] \cap \max^{-1}[B]$ by the Galvin-Glazer Theorem 1.14.

Claim: We can condense v to a meshed sequence s, i.e., with $FU(s)$ being FU-meshed (with respect to itself).

For the inductive step $n \in \omega$ we assume that for $k = \frac{1}{2}n(n-1)$ we have picked $(s_i)_{i<k}$ with increasingly meshed graphs of sizes 1 through $n-1$.

We pick $2n$-many elements from $FU(v)$ as follows.

First pick $(v_{i_j})_{j<n}$ past everything so far and then pick $(v_{i_j})_{n-1<j<2n}$ past additionally the ones just chosen and define

$$s_{k+j} := v_{i_j} \cup v_{i_j+n}.$$

Then $s_k, \ldots, s_{k+n}$ is an $(n, \bigcup_{i<n} s_i)$-witness, so $s = (s_i)_{i\in\omega}$ will be as desired. □

Andreas Blass suggested another proof for this last corollary; we include a sketch.

Remark 4.30 Given selectives p_1, p_2 we can find a permutation of ω that maps both p_1 and p_2 simultaneously to ultrafilters p_1', p_2' with $\min[FU(s)] \in p_1'$ and $\max[FU(s)] \in p_2'$ (which are again selective). Then we can apply Lemma 4.27 to get a suitable u' for p_1', p_2'. But the extension of the permutation to $\mathbb{F}$ yields an additive isomorphism on $FU(s) \in u'$ mapping u' to a union ultrafilter u with $\min(u) = p_1$ and $\max(u) = p_2$; since additive isomorphisms preserve all the desired properties, this completes the proof.

To conclude this chapter let us remark upon the use of the continuum hypothesis in our proof.

Remark 4.31 In [BH87, Theorem 5], it is shown that union ultrafilters with min and max not selective can be constructed under MA. In [Eis02] ordered union ultrafilters are constructed using only $cov(\mathscr{M}) = 2^\omega$, i.e., using Cohen reals.

If we drop the prescribed selectives, Lemma 4.27 can be derived using Cohen forcing in the form of FU-sets of finite condensations of s; using Lemma 4.24 some additional bookkeeping is possible to ensure that min and max will be selective.

4.4. *Synopsis*

In this chapter we have achieved several insights into the different notions of union ultrafilters. We found characterizations for stable union ultrafilters in terms of Ramsey properties and although this result may be considered negative from the point of view in that we have not helped to differentiate stable ordered union from stable union, we have established that differences must be sought elsewhere.

We have also constructed union ultrafilters mapping by min and max to selective ultrafilters but not being ordered. To do this we had to fuse together the arguments of 4.6 and [BH87]. Again this may be deemed a negative result, since we have eliminated one idea of differentiating ordered and unordered union ultrafilters. However, this is not the case. In fact, it would have been surprising to have such a strong connection "backwards" from the min and max image considering how much information is lost with these function. Indeed, our result is particularly beautiful, becaused we have established an understanding of the kind of meshing that may occur in unordered union ultrafilters.

This research is additionally of interest for future research to understand possible combinatorial algebraical properties of our new forcing constructions from Chapter 3.

Chapter 5

Summable Ultrafilters as Sums

After having studied the set theoretic properties of union and therefore summable ultrafilters[1] in the preceding chapter we now turn to the algebraic aspects.

[1] Recall Definition 1.19

Since the idempotent ultrafilters are so incredibly abundant the summable ultrafilters could have easily been forgotten, since they were originially constructed in [Hin72] as an example of an idempotent ultrafilter. However, Eric van Douwen became interested in their special property of having a base of FS-sets. According to [HS98, Notes on Chapter 5] this caused him to ask Neil Hindman the question if such ultrafilters can be constructed assuming ZFC alone. This turned out not to be the case, as was first answered by Andreas Blass with the construction of stable ordered union ultrafilters in [Bla87b]; in [BH87], a collaboration of Blass and Hindman, they generalized this independence result to all summable ultrafilters.

However, besides their historical role concerning the development towards Hindman's Theorem, summable ultrafilters turned out to carry extraordinary algebraic properties. They were the first known example of strongly right maximal idempotent elements, i.e.,$p \in \beta\mathbb{N}$ such that the equation $p = x + p$ has the unique solution p.[2] They still are the only known examples of idempotents such that their maximal group, i.e.,the maximal subgroup of $\beta\mathbb{N}$ with neutral element being the idempotent, is isomorphic to $\mathbb{Z}$.[3]

[2] Later Protasov proved the existence of such ultrafilters under ZFC alone; cf. [HS98, Theorem 9.10].

[3] This is also the minimal case, since by centrality of $\mathbb{Z}$ in $\beta\mathbb{N}$, every idempotent is contained a copy of $\mathbb{Z}$.

Hindman and Strauss also showed in [HS95][4] that there is a special kind of summable ultrafilters having another fascinating and so far unique quality: they can only be written as a sum in a trivial fashion, i.e., in the form $(p+z)+(p-z)$ for some $z \in \mathbb{Z}$ (which due to the centrality of $\mathbb{Z}$ is always possible). In this chapter we will generalize these results.

[4] cf. [HS98, Chapter 12]

My personal motivation for the results of this chapter was the same as for the set theoretic analysis that led me back to union ultrafilters in the preceding chapter. To gain more insight into the forcing constructions from Chapter 3 it was only natural to study the possible algebraic properties of a generic idempotent ultrafilter. Even though I was not able to derive consequences for the arbitrary forcing constructions, this process led me to the results of this chapter regarding summable ultrafilters.

5.1. *Summable ultrafilters are special*

To investigate what was called "special strongly summable" ultrafilters in [HS95] and [HS98, Chapter 12], we connect this notion with union ultrafilters.

Most of Definition 4.1 for FU-sets[5] transfers immediately to arbitrary FP-sets. Therefore we will skip repeating these notions for FS-sets. But just as was the case with FU-sets, we require some extra terminology to acommodate the particularities of summable ultrafilters.

[5] Notably, the notion of a condensation and the notion of support with respect to a given sequence

Sequences with sufficient growth

Definition 5.1 1. We say that $(x_n)_{n\in\omega}$ (in $\mathbb{N}$) has *sufficient growth* if for some $c > 4$

$$(\forall n \in \omega)\ x_n > c \sum_{i<n} x_i.$$

Allowing confusion, we often say that $FS(x)$ has sufficient growth instead.

Special summable ultrafilter

2. A summable ultrafilter $p \in \beta\mathbb{N}$ is called *special* if there exists $FS(x) \in p$ with sufficient growth such that

$$(\forall L \in [\omega]^{\omega})(\exists y = (y_n)_{n\in\mathbb{N}}) FS(y) \in p \wedge |L \setminus \bigcup_{n\in\mathbb{N}} x\text{-supp}(y_n)| = \omega.$$

As noted before the definition x-supp(y_n) denotes the finite set F such that $\sum_{i\in F} x_i = y_n$. Given the sequence x we also say that p is special *with respect to* x.

Special union ultrafilter

3. A union ultrafilter $u \in \beta\mathbb{F}$ is called *special* if

$$(\forall L \in [\omega]^{\omega})(\exists X \in u)|L \setminus \bigcup X| = \omega.$$

In [HS95] and [HS98, Chapter 12] the notion of "special" is stronger; we would say: special with respect to $(n!)_{n\in\omega}$ and *divisible*, i.e., there is a base of sets $FS(x)$ with $x_n|x_{n+1}$ (for $n \in \omega$).

The need for sufficient growth may seem arbitrary. The reason for this technical condition lies in several practical consequences that we list in the next remark.

Remark 5.2 Every summable ultrafilter has a base of FS-sets with sufficient growth, in fact, for any prescribed growth constant c; cf. [HS98, Lemma 12.20].

For a sequence $x = (x_n)_{n\in\omega}$, to have growth with constant $c \in \mathbb{N}$ implies the following:

- $\sum_{i\in F} x_i = \sum_{i\in G} x_i$ iff $F = G$
- $\sum_{i\in F} x_i + \sum_{i\in G} x_i \in FS(x)$ iff $F \cap G = 0$
- In fact, any linear combination of the x_i with factors $1, \ldots, c$ is unique.
- If for some r, G and y we have $y < x_r$ and $\min(G) > r$, then

$$y + \sum_{i\in G} x_i \in FS(x) \Rightarrow y \in FS(x).$$

This is checked (inductively on $|G|$); see e.g. [HS98, Chapter 12].

It is not difficult to see that summable ultrafilters additively isomorphic to stable ordered union ultrafilters allow for dynamic growth assumptions such as exponential growth.

The first observation is that special summable and special union ultrafilters are in fact equivalent notions. Recall Definition 4.3 of additive isomorphism.

Proposition 5.3
Every special summable ultrafilter is additively isomorphic to a special union ultrafilter (and vice versa).

Proof. Given some special summable ultrafilter, let $x = (x_n)_{n\in\omega}$ be a sequence in $\mathbb{N}$ with sufficient growth witnessing the specialness.

Consider

$$\varphi : FS(x) \to \mathbb{F}, \sum_{i\in F} x_i \mapsto F.$$

Then φ is bijective and maps special summable to special union ultrafilters and vice versa.

By Remark 5.2, φ is well defined (and bijective).

It is clear that φ maps the given special summable ultrafilters to some special union ultrafilters and vice versa any special union ultrafilter to some special summable. □

Next we show that union ultrafilters are already special.

Theorem 5.4 (Union ultrafilters are special)
Every union ultrafilter is special. Accordingly, all summable ultrafilters are special.

Union ultrafilters are special

Spoiler *The main argument of the proof is a parity argument. Assuming that some set covers the whole of L the parity argument will yield a condensation that misses a lot of L – this is seen by closely investigating the actual unions and their parity.*

Proof. Let $L \in [\omega]^\omega$.

Step 1: We may assume that $\{s \in \mathbb{F} \mid s \cap L \neq \emptyset\} \in u$

A simple partition

Otherwise, since u is an ultrafilter,

$$\{s \in \mathbb{F} \mid s \cap L = \emptyset\} \in u.$$

So there exists $X \in u$ such that $(\bigcup X) \cap L = \emptyset$; in particular $L \setminus \bigcup X = L$ is infinite.

Since u is a union ultrafilter we find $FU(s) \in u$ such that $s_i \cap L \neq \emptyset$ for all $i \in \omega$.

We need to find $FU(t) \subseteq FU(s)$, $FU(t) \in$ such that $L \setminus \bigcup_{i\in\omega} t_i$ is infinite.

Adjusting L if necessary

Step 2: If $L \setminus \bigcup_{i\in\omega} s_i$ is infinite, we are done. Otherwise it is finite; we may without loss of generality delete those finitely many elements from L – if we manage to leave out infinitely many from the remaining set, we will do just fine.

Therefore we assume that

$$L \setminus \bigcup_{i\in\omega} s_i = \emptyset.$$

A parity argument

Step 3: Let us consider the function

$$\pi : FU(s) \to 2, x \mapsto |\{i \mid s_i, s_{i+1} \subseteq x\}| \bmod 2.$$

Step 4: An FU-set can only be included in $\{t \in FU(s) \mid \pi(t) = 0\}$.

Given a condensation $t = (t_i)_{i\in\omega}$ of s, we can find $t_j \ll t_k$ such that

$$\pi(t_i \cup t_k) = \pi(t_i) + \pi(t_k).$$

In fact, for any t_i all but finitely many t_k are past the successor of the last s_j included in t_i.

Therefore $\pi(t_j \cup t_k) = 0 \bmod 2$ and $FU(t)$ intersects the above set

So the above set is included in u and we find $FU(t) \in u$ refining it.

Final step by contradiction.

Claim: $|L \setminus \bigcup_{j\in\omega} t_j| = \omega$.

Assume to the contrary that it is finite.

In particular, since the s_i are disjoint and

$$s_i \cap L \neq \emptyset (\text{for } i \in \omega$$
$$\bigcup_{i\in\omega} s_i \supseteq L$$
$$t \sqsubseteq s,$$

we can conclude that only finitely many s_i are disjoint from $\bigcup_{i\in\omega} t_i$. In other words the s-supp of t "covers" almost all the s –

$$|\{s_i \mid (\forall j \in \omega) s_i \notin s\text{-supp}(t_j)\}| < \omega.$$

In particular, we find some index $k \in \omega$ such that all later s_i are covered, i.e.,

$$(\forall i \geq k)(\exists j_i) s_i \subseteq t_{j_i}.$$

Now consider t_{j_k}, i.e., the t_j that covers s_k, the member of s from which point on we cover all of s.

Let us pick the "last" element of s-supp(t_{j_k}), i.e., s_v with $\max(s_v) = \max(t_{j_k})$.

But now s_{v+1} is contained in some t_j; so we can pick $t^+_{j_k}$ such that $s_{v+1} \subseteq t_{j_k}$.

Now let

$$t = \bigcup \{t_l \mid t_l \neq t^+_{j_k}, t_l \text{ contains some } s_i \text{ with } k \leq i \leq v\}.$$

In other words, $t \cup t^+_{j_k}$ contains all $(s_i)_{k\leq i\leq v+1}$.

Now remember that $\pi(t_{j_k} \cup t)$ is even by our choice of $FU(t_j)$. But count the number of consecutive pairs of s_i in this union:

$\pi(t_{j_k}^+)$ and $\pi(t)$ are even; but the pair s_v, s_{v+1} (from $t_{j_k}, t_{j_k}^+$ resp.) ends up in $t \dot\cup t_{j_k}^+$. Hence $\pi(t_{j_k} \cup t) = \pi(t_{j_k}) + \pi(t) + 1 = 1 \bmod 2$ ↯↯↯ a contradiction. □

Note that if u had been ordered union, this would have been much easier: then the t_i were ordered, hence only t_{i+1} can include s_{v+1} – but then $\pi(t_i \cup t_{i+1}) = \pi(t_i) + \pi(t_{i+1}) + 1 = 1 \bmod 2$.

Now that we have eleminated the necessity of the specialness condition, we can proceed to the main result.

5.2. *Summable ultrafilters as sums*

We need one more Definition for pursuing our main goal.

Definition 5.5

- A sequence $x = (x_n)_{n \in \omega}$ (in $\mathbb{N}$) has *disjoint binary support* if its elements do; allowing confusion we say that $FS(x)$ has disjoint binary support.
- We call a summable ultrafilter *with disjoint binary support* if it contains the FS-set for a sequence with disjoint binary support and sufficient growth. In other words, if bsupp maps it to a union ultrafilter.
- We say that an idempotent ultrafilter p can only be *written as a sum trivially* if

$$(\forall q, r \in \beta\mathbb{N})\ q + r = p \Rightarrow q, r \in (\mathbb{Z} + p)$$

We are ready to state our main result for this chapter.

Theorem (Summable ultrafilters with disjoint support as sums) *Every summable ultrafilter with disjoint binary support can only be written as a sum trivially.*

Summable ultrafilters with disjoint support as sums

We will need a series of technical propositions before we can prove this theorem, but we can immediately deduce the following convenient corollary.

Corollary 5.6
Every summable ultrafilter is additively isomorphic to a summable ultrafilter that can only be trivially written as a sum.

Proof. Any summable ultrafilter p is additively isomorphic to one as in the assumptions of the above theorem – just pick $FS(x) \in p$ with sufficient growth witnessing that p is special; in particular, x has unique sums. Then the additive isomorphism between $FS(x)$ and $FS(2^n)$ maps p to a summable ultrafilter with disjoint binary support. □

This improves the original result from [HS95]. There it is shown that summable ultrafilters that are divisible and special with respect to $(n!)_{n\in\omega}$ can only be written as a sum trivially; however, by [HS95, Theorem 5.8] there exist (under *CH*) summable ultrafilters that are not divisible.[6]

[6] cf. the comment after Definition 5.1.

We now begin our work towards proving the above theorem, first dealing with the left summand of a sum equal to a summable ultrafilter. For this part we do not yet require disjoint binary support, just growth.

We start with some results from [HS95] and [HS98, Chapter 12]. We show them in detail since we will often argue in a similar fashion.

Proposition 5.7
Given a sequence $x = (x_n)_{n\in\omega}$ with sufficient growth we have the following.

$$(\forall r, q \in \beta\mathbb{N})\ r, q + r \in FS^\infty(x) \Rightarrow q \in FS^\infty(x).$$

Spoiler *This is e.g. [HS95, Theorem 2.3].*

Proof. Assume $r, q + r \in FS^\infty(x)$.

Claim: For $k \in \omega$ we have $FS_k(x) \in q$.

Step 1: $FS_k(x) \in q + r$, so by definition of addition in $\beta\mathbb{N}$

$$FS_k(x)^{-r} = \{a \mid -a + FS_k(x) \in r\} \in q.$$

Step 2: Then this set is included in $FS_k(x)$.

To see this, let us pick $a \in FS_k(x)^{-r}$ and choose $x_l > a$.

By choice of r, we can pick

a shifts many elements to $FS(x)$

$$y \in (-a + FS_l(x)) \cap FS_l(x) \in r.$$

In particular, $y = \sum_{i\in F} x_i$ with $\min(F) \geq l$.

Using uniqueness of sums

But also $a + y \in FS(x)$ – so by Remark 5.2, $a \in FS_k(x)$ – as desired. □

We can extend this a a little bit, which we will use later.

Corollary 5.8
Given a sequence $x = (x_n)_{n\in\omega}$ with sufficient growth we have the following.

$$(\forall r, q \in \beta\mathbb{N})\ r, q + r \in \mathbb{Z} + FS^\infty(x) \Rightarrow q \in \mathbb{Z} + FS^\infty(x).$$

Proof. Take $z \in \mathbb{Z}$ with

$$r' = r - z \in FS^\infty(x).$$

Since $FS^\infty(x)$ is a semigroup, so is $\mathbb{Z} + FS^\infty(x)$; in particular,

$$r' + q = (r + q) - z \in \mathbb{Z} + FS^\infty(x).$$

Therefore, we can find $y \in \mathbb{Z}$ with

$$r' + (q - y) \in FS^\infty(x).$$

By the above Proposition, $q - y \in FS^\infty(x)$, i.e.,

$$q \in \mathbb{Z} + FS^\infty(x),$$

as desired. □

Another easy corollary is the following.

Corollary 5.9
Every summable ultrafilter p is a strongly right maximal idempotent*, i.e., the equation $q + p = p$ has the unique solution p (in $\beta\mathbb{N}$).*

Spoiler *This is e.g. [HS98, Theorem 12.39].*

Proof. Any summable ultrafilter p has a base of sets $FS(x)$ with additionally $FS_k(x) \in p$ for all $k \in \omega$; this is e.g. [HS98, Lemma 12.18].

Therefore we can apply the above Proposition to each $FS(x) \in p$ and deduce from $q + p = p$ that $FS(x) \in q$ – so $q = p$ as desired. □

We have now practically dealt with the left summand of a sum equal to a summable ultrafilter. We turn to the right summand.

Similar to the proof in [HS95], our proof for the right summand consists of two parts. First we show that if one of the summands is "very close" to the summable ultrafilter, it is already equal. Second we show that writing a summable ultrafilter with disjoint support as a sum can only be done with the right summand "very close" to it.

For the first part, we begin with a technical lemma that essentially reflects the property we desire: under certain conditions, we can write elements of an FS-set only trivially as sums.

Lemma 5.10 (Trivial sums for FS-sets) Trivial sums for FS-sets
Let $x = (x_n)_{n\in\mathbb{N}}$ be a sequence with disjoint support and enumerated with increasing bmin*, $a \in \mathbb{N}$ and*

$$m := \min\{i \mid \mathrm{bmax}(a) < \mathrm{bmin}(x_i)\},$$

Then for every $b \in \mathbb{N}$ with $\mathrm{bmax}(x_m) < \mathrm{bmin}(b)$ we have

$$a + b \in FS(x) \Rightarrow a, b \in FS(x).$$

Spoiler *The main idea of the proof is that due to our assumptions neither the sums of the x_i nor the sum $a + b$ will have any (binary) carrying over.*

Proof. Assume we have been given appropriate x, a and b.

Step 1: Since $a + b \in FS(x)$, we can find some finite, non-empty $H \subseteq \mathbb{N}$

$$(\circledast) \qquad a + b = \sum_{i\in H} x_i.$$

Step 2: Let us define

$$H_a := \{j \in H \mid \mathrm{bsupp}(x_j) \cap \mathrm{bsupp}(a) \neq \emptyset\}$$

and H_b similarly.

Claim: $H = H_a \dot\cup H_b$.

By our assumptions on b we have

$$\mathrm{bsupp}(a) \cap \mathrm{bsupp}(b) = \emptyset.$$

Of course, x has disjoint support, so we have no carrying over on either side of the equation ⊛, i.e.,

$$H = H_a \cup H_b.$$

On the other hand, if $\mathrm{bsupp}(x_i) \cap \mathrm{bsupp}(a) \neq \emptyset$, then $i \leq m$ by choice of m. This in turn implies $\mathrm{bsupp}(x_i) \cap \mathrm{bsupp}(b) = \emptyset$ by the choice of b; in other words,

$$H_a \cap H_b = \emptyset.$$

But by this claim $\sum_{i \in H_a} x_i = a$ and $\sum_{i \in H_b} x_i = b$ – just as desired. □

Essentially, the proof of the above lemma can be seen as switching to the equivalent union ultrafilter and realizing that there will always be some orderedness. We will discuss the analogue for union ultrafilter at the end of this chapter.

We now show that this lemma takes us nearly all the way, i.e., if one of the summands is "very close" to our summable ultrafilter, we are already done.

Trivial sums for $\mathbb{H}$

Lemma 5.11 (Trivial sums for $\mathbb{H}$)
For any summable ultrafilter p with disjoint binary support we have

$$(\forall q \in \beta\mathbb{N})(\forall r \in \mathbb{H})\ q + r = p \Rightarrow q = r = p.$$

Spoiler *The proof is basically a reflection argument. Arguing indirectly, the addition on $\beta\mathbb{N}$ reflects to elements in the sets of the ultrafilters in such a way that non-trivial sums of ultrafilters lead to non-trivial sums of the usual FS-set, contradicting Lemma 5.10.*

Proof. **Step 1:** Since any summable ultrafilter is strongly right maximal by Corollary 5.9, it suffices to show that $r = p$.[7]

[7] In other words, we have already dealt with the left summand. This is the reason why we only need to ask for r to be "close" to p.

Let us assume to the contrary that $r \neq p$.

Step 2: Let us pick a witness for p, i.e., $x = (x_n)_{n \in \mathbb{N}}$ with sufficient growth and disjoint binary support; we may assume without loss that $FS(x) \in p \setminus r$.

Step 3: Since $r + q = p$, there exists a such that $-a + FS(x) \in r$.; let us pick m as in the Lemma 5.10, i.e., such that all $(x_n)_{n>m}$ have $\mathrm{bmax}(a)) < \mathrm{bmin}(x_n))$.

Step 4: Let us define $M := \mathrm{bmax}(x_m)) + 1$; note that the multiples of 2^M have binary support past the support of x_m.

Step 5: Now

$$(-a + FS(x)) \cap (\mathbb{N} \setminus FS(x)) \cap 2^M\mathbb{N} \in r.$$

So pick b from this intersection.

Step 6: Then $a + b \in FS(x)$. But we can apply Lemma 5.10 to get $a, b \in FS(x)$ ↯↯↯ contradicting $b \notin FS(x)$. □

Before we approach our final step, let us quickly note one detail.

Lemma 5.12
A summable ultrafilter with disjoint binary support is also special with respect to $(2^n)_{n\in\omega}$.

Proof. Let us pick x as a witness for the disjoint binary support of a summable ultrafilter p; without loss of generality x also witnesses that p is special, since the members of a condensation have disjoint x-supp by Remark 5.2; in particular, they have disjoint binary support.

Given $L \subseteq \omega$ we consider

$$L' := \{n \mid (\exists i \in L)\ i \in \mathrm{bsupp}(x_n)\}.$$

Then by specialness we find $y \sqsubseteq x$ with $FS(y)$ with $L' \setminus x\text{-supp}(y)$ infinite – but this implies $L \setminus \mathrm{bsupp}(y)$ is infinite by choice of L' and the disjoint x-supp of members of y. □

As a final step, we prove that if a summable ultrafilter is written as a sum, then the first component is "close by".

Lemma 5.13 (Nearly trivial sums)
For any summable ultrafilter p with disjoint binary support we have

$$(\forall q, r \in \beta\mathbb{N})\ q + r = p \Rightarrow r \in \mathbb{Z} + \mathbb{H}.$$

Nearly trivial sums

Spoiler *We basically argue as before, i.e., we show that if $r \notin \mathbb{Z} + \mathbb{H}$, there will always be a sum $a + b$ that cannot end up in a certain FS-set. Naturally, we will need a more subtle argument than before. For this we will consider the binary support as element of the Cantor space 2^ω and the image of q under the continuous extension. Analyzing this image and using that summable ultrafilters are "special", we will find that there cannot not be enough carrying over available to always end up in the FS-set.*

Proof. Let us assume to the contrary that $r \notin \mathbb{Z} + \mathbb{H}$.

Step 1: By contraposition of Corollary 5.8[8] also $q \notin \mathbb{Z} + \mathbb{H}$.

[8] Applied to $\mathbb{H} = \bigcap_{k\in\omega} \overline{FS_k(2^n)}$.

Step 2: Let us consider the mapping

Def. α

$$\alpha : \mathbb{N} \to 2^\omega, \text{ defined by } x = \sum_{i\in\omega} \alpha(x)(i) \cdot 2^i.$$

As usual, we also denote the continuous extension to $\beta\mathbb{N}$ by α.[9]

[9] Here we consider 2^ω as the Cantor Space with the product topology; in particular, this is a compact space, so the (unique) continuous extension exists.

Step 3: We define the following subset of ω.

Def. Q_0, Q_1 infinite

$$Q_0 := \{i \in \omega \mid \alpha(q)(i) = 0\}$$
$$Q_1 := \{i \in \omega \mid \alpha(q)(i) = 1\}.$$

Step 4: Q_0 and Q_1 are infinite.

Q_1 finite contradicts Step 1.

Claim 1: Let us assume to the contrary that Q_1 is finite, i.e., we can pick $k \in \omega$ such that $\alpha(q)(n) = 0$ for $n > k$.

Using the non-trivial part of $\alpha(q)$

Considering $z := \sum_{i \leq k} \alpha(q)(i)2^i$ we show that

$$(\forall n > k)\ z + 2^n\mathbb{N} \in q,$$

↯↯↯ contradicting $q \notin \mathbb{Z} + \mathbb{H}$.

Given $n > k$ we define

$$U_{z,n} := \{s \in 2^\omega \mid s \upharpoonright n = \alpha(q) \upharpoonright n = \alpha(z) \upharpoonright n\}.$$

Obviously, $U_{z,n}$ is a neighbourhood of $\alpha(q)$, hence $\alpha^{-1}(U_z) \in q$. But we can easily check that $\alpha^{-1}[U_z] = z + 2^n\mathbb{N}$, as desired.

Q_0 finite contradicts Step 1.

Claim 2: Now let us assume to the contrary that Q_0 is finite, i.e., we can pick k such that $\alpha(q)(n) = 1$ for $n > k$.

Using the non-trivial part of $\alpha(q)$

This time considering $z := 2^{k+1} - \sum_{i<k} \alpha(q)(i)2^i$ we show that

$$(\forall n > k)\ -z + 2^n\mathbb{N} \in q.$$

↯↯↯ contradicting $q \notin \mathbb{Z} + \mathbb{H}$.

Again, given $n > k$, we consider $\alpha^{-1}[U_{z,n}]$. This time we check that $\alpha^{-1}[U_{z,n}] = -z + 2^n\mathbb{N}$.

Let $w \in \alpha^{-1}[U_{z,n}]$. Then for some a we have

$$w = a \cdot 2^{n+1} + \sum_{i>k}^{n} 2^i + \sum_{i \leq k} \alpha(q)(i)2^i,$$

since by assumption that M is finite, all of $\alpha(q)(i)$ beyond k is 1.

Telescope sums

But this implies

$$w + z = a \cdot 2^{n+1} + \sum_{i>k}^{n} 2^i + 2^{k+1} = a \cdot 2^{n+1} + 2^{n+1} = (a+1)2^{n+1},$$

as desired.

Choosing $FS(x)$, X

Step 5: Since u is summable with disjoint binary support, we pick a sequence $x = (x_n)_{n\in\omega}$ with disjoint binary support, sufficient growth and $FS(x) \in u$. We abbreviate

$$X := \bigcup_{n\in\omega} \mathrm{bsupp}(x_n).$$

Step 6: By Lemma 5.12 and Theorem 5.4, we may assume without loss that $Q_0 \setminus X$ is infinite.

We now have to choose a couple of natural numbers in a clever fashion. Let us preview the reason for our choices and their role in the final calculation.
First we pick some a, then beyond $\mathrm{bmax}(a)$ elements from the $Q_i \setminus X$. With this we pick b with $\alpha(b) \subseteq \alpha(q)$ up to those elements from the Q_i. Then any y translating both a and b into $FS(x)$ cannot translate b to a multiple of a high power of 2, because $b - a$ will have a "hole" from Q_0 just everything in $FS(x)$ does.
We will recapitulate this motivation after we made the choices.

Step 7: By $q + r = p$ we have $FS(x)^{-r} \in q$; so let us pick a with $-a + FS(x) \in r$. Choosing a

Step 8: Next, we pick $s_1 \in Q_1$ and $s_0 \in Q_0 \setminus X$ with $s_0 > s_1 > \mathrm{bmax}(a)$. $s_0 > s_1 > \mathrm{bmax}(a)$

> We choose s_1 only so that $\sum_{i \le s_0} \alpha(q)2^i - a > 0$.
> Note that $\alpha(q)(s_0) = 0$, $\alpha(q)(s_1) = 1$, but every element from $z \in FS(x)$ has $\alpha(z)(s_0) = \alpha(z)(s_1) = 0$.

Step 9: By $q + r = p$ we have $(2^{s_0+1}\mathbb{N})^{-r} \in q$, so let us pick b from Choosing b

$$(2^{s_0+1}\mathbb{N})^{-r} \cap (2^{s_0+1}\mathbb{N} + \sum_{i \le s_0} \alpha(q)(i)2^i) \in q.$$

where the latter set is in q since it is equal to $U_{q\restriction(s_0+1),s_0+1}$; cf. Step 4.

> Note that $\alpha(b)(s_i + 1) = \alpha(q)(s_i + 1) = i$, by choice of the s_i $(i < 2)$. Therefore, $b - a > 0$, but more specifically $\mathrm{bmax}(\sum_{i \le s_0} \alpha(b)2^i - a) < s_0$ (and this number is positive by choice of s_1!).

Step 10: Finally, we can also choose $y \in (-b + 2^{s_0+1}\mathbb{N}) \cap (-a + FS(x)) \in r$. Choosing y

> Note that since $s_0 \notin X$ and $a + y \in FS(x)$, we have $\alpha(a + y)(s_0) = 0$. But also $y + b \in 2^{s_0+1}\mathbb{N}$.

Step 11: Let us recapitulate what we have right now:

(i) $s_i \in Q_i \quad \Rightarrow \quad \alpha(q)(s_i) = i \ (i < 2)$

(ii) $s_0, s_1 > \mathrm{bmax}(a) \quad \Rightarrow \quad 0 < \mathrm{bmax}(\sum_{i \le s_0} \alpha(q)(i)2^i - a) < s_0.$

(iii) $b + y \in 2^{s+1}\mathbb{N} \quad \Rightarrow \quad \sum_{i \in F} 2^i = b + y$ for some $F > s$

(iv) $a + y \in FS(x) \quad \Rightarrow \quad \sum_{i \in G} 2^i$ for some G with $s_0 \notin G$.

> Let us explain the reason for our choices and their role in the final calculation.
> The basic idea is that since y translates such a small a into $FS(x)$, it cannot simultaneously translate elements like b, i.e., elements that agree with $\alpha(q)$ up to s_0, to be divisible by 2^{s_0+1}.
> This is due to the “hole” of both $(y + a)$ and $(b - a)$ at s_0 which simply does not allow for enough carrying over in their sum $(y + b)$ to get a multiple of 2^{s_0+1}.
> For this reason we had picked s_1 to guarantee that $(b - a) > 0$ because $\sum_{i<s_0} \alpha(q)(i)2^i - a > 0$; so we really do get “stuck” at some lower power of 2.

Step 12: We first calculate the following:

$$\begin{aligned}\sum_{i \in F} 2^i &= (y + a) + (b - a) \\ &= \sum_{i \in G} 2^i + d \cdot 2^{s_0+1} + \sum_{i<s_0} \epsilon_i 2^i,\end{aligned}$$

for suitable $d \in \mathbb{N}$ (by our choice of b) as well as some $\epsilon_i \in 2$ $(i < s_0)$ which are not all 0 by our choice for s_1 – but $s_0 \notin \mathrm{bsupp}(b - a)$ as noted before.

Step 13: Now we can rearrange this equation and calculate further that

$$\begin{aligned}\sum_{i\in F} 2^i - \sum_{i\in G\setminus(s_0+1)} 2^i - d\cdot 2^{s_0+1} &= \sum_{i\in G\cap(s_0+1)} 2^i + \sum_{i<s_0} \epsilon_i 2^i \\ &= \sum_{i\in G\cap s_0} 2^i + \sum_{i<s_0} \epsilon_i 2^i \qquad (s_0 \notin G) \\ &< 2^{s_0} + 2^{s_0} = 2^{s_0+1}.\end{aligned}$$

Since 2^{s+1} divides the left-hand side of the first line, it must be 0, but by Equation (ii) the middle line is not ↯↯↯ we finally have our contradiction. □

After this complicated proof, our main result almost immediately follows.

Trivial sums

Theorem 5.14 (Trivial sums)
A summable ultrafilter with disjoint binary support can only be written as a sum trivially.

Proof. Assume that p is a summable ultrafilter with disjoint binary support and $q, r \in \beta\mathbb{N}$ with

$$q + r = p.$$

By the above Lemma 5.13, we deduce that $r \in \mathbb{Z} + \mathbb{H}$.

Therefore we can find $k \in \mathbb{Z}$ such that $-k + r \in \mathbb{H}$; in particular

$$(k + q) + (-k + r) = p.$$

But now we can apply Lemma 5.11 with $k + q$ and $-k + r$ to get $k + q = -k + r = p$ as desired. □

Let us make a short remark regarding the tools that we have actually applied throughout the proofs of this section.

Remark 5.15 In no part of the proof did we use more than the fact that some $FS(x) \in u$ had disjoint (binary) support and unique sums and that every natural number has a binary represantation. Therefore, the same proof works for any other support.

We can therefore claim to have shown the following, more general theorem.

Theorem 5.16
Let u be summable and some $FS(x) \in u$ has disjoint support (with respect to some sensible sequence). Then $q + r = u$ implies $q, r \in \mathbb{Z} + u$.

We should note that there is a related result by different means which was discovered by Lakeshia Legette. In [Leg08] she constructs (among other things) ultrafilters on a free semigroup using ordered union ultrafilters; these idempotents p can only be written as $p + \ldots + p$.[10]

[10] I thank Neil Hindman for a copy of Lakeshia Legette's dissertation.

5.3. *Revisiting the forcing construction*

We include two more results regarding summable ultrafilters. First we show that "stable ordered" summables have disjoint binary support and second we complete the proof of Corollary 3.12.

Lemma 5.17
Let p be a summable ultrafilter additively isomorphic to a stable ordered union ultrafilter. Then p has disjoint binary support and $\mathrm{bmin}(p)$, $\mathrm{bmax}(p)$ *are selective.*

Spoiler *We use the Ramsey property for pairs.*

Proof. **Step 1:** The latter follows from the former, since then bsupp maps p to a stable ordered union ultrafilter; in particular, $\mathrm{bmin}(p) = \min(\mathrm{bsupp}(p))$, $\mathrm{bmax}(p) = \max(\mathrm{bsupp}(p))$.

Step 2: For the former claim consider φ an additive isomorphism on a suitable $FS(x) \in p$ such that $\varphi(p)$ is stable ordered union.

Step 3: Consider the following set

$$\{(v, w) \in \varphi[FS(x)]^2_< \mid \mathrm{bmax}(\varphi^{-1}(v)) < \mathrm{bmin}(\varphi^{-1}(w))\}.$$

Step 4: By Theorem 4.12 we find $FU(s) \in \varphi(p)$ such that $FU(s)^2_<$ is included or disjoint from the above set.

Step 5: But it cannot be disjoint.

Given $FU(s) \in \varphi(p)$, we have $\varphi^{-1}[FU(s)] = FS(y)$ for some $y \sqsubseteq x$.

But for any $z \in FS(y)$ and pick $z' \in FS(y) \cap 2^{\mathrm{bmax}(z)}\mathbb{N}$ the pair $(\varphi(z), \varphi(z')$ is included in the above set.

Step 6: Therefore the homogeneous $FU(s) \in \varphi(p)$ yields $\varphi^{-1}[FU(s)] = FS(y)$ where y must have disjoint (in fact ordered) binary support. □

We end this chapter providing the lemma needed for Corollary 3.12, i.e., we prove that bmin of a summable is a P-point.

Lemma 5.18
For every summable ultrafilter p the ultrafilter $\mathrm{bmin}(p)$ *is a P-point.*

Spoiler *We modify the proof for union ultrafilters.*

Proof. **Step 1:** Given any $f \in \omega^\omega$ let us define

$$\begin{aligned}\varphi :& \mathbb{N} \to \mathbb{N},\\ & a \mapsto |\{(i,j) \in \mathrm{bsupp}(a)^2 \mid j \text{ immediate succ. of } i \text{ and } f(j) < i\}|.\end{aligned}$$

Note that if $\mathrm{bmax}(a) < \mathrm{bmin}(b)$, then

$$\varphi(a+b) = \varphi(a) + \varphi(b) + \begin{cases} 1 & \text{if } f(\mathrm{bmin}(b)) < \mathrm{bmax}(a) \\ 0 & \text{if } f(\mathrm{bmin}(b)) \geq \mathrm{bmax}(a) \end{cases}$$

Step 2: We know that on some set $FS(x) \in p$ the parity of elements of $\varphi[FS(x)]$ is constant.

Claim 1: If the parity is 1, then f is constant on a set in $\mathrm{bmin}(p)$.

Let us assume that the parity if 1; we consider some fixed x_k. Then

$$FS_k(x) \cap 2^{\mathrm{bmax}(x_k)}\mathbb{N} \in p,$$

so we find $FS(y) \in p$ included in this set.

But for $a \in FS(y)$ we know that $x_k + a \in FS(x)$, so we can calculate

$$\varphi(x_k + a) = 1 \bmod 2,$$

By our above calculation for φ this is only possible if

$$f(\mathrm{bmin}(a)) < \mathrm{bmax}(x_k).$$

In other words, f is bounded on $\mathrm{bmin}[FS(y)]$ by $\mathrm{bmax}(x_k)$.

Claim 2: If the parity is 0, then f is finite-to-one on $\mathrm{bmin}[FS(x)]$.

Given $k \in \omega$ fix some $b \in FS(x) \cap 2^{k+1}\mathbb{N}$, so $\mathrm{bmax}(b) > k$.

Then for any $a \in FS(x)$ we have

$$f(\mathrm{bmin}(a)) = k \Rightarrow f(\mathrm{bmin}(a)) < \mathrm{bmax}(b).$$

This time our calculation for $\varphi(a + b)$ implies $\mathrm{bmin}(a) \leq \mathrm{bmax}(b)$ – and there can only be finitely many such $\mathrm{bmin}(a)$. □

This concludes the chapter.

5.4. *Synopsis*

In this chapter we have seen that the "special" condition introduced in [HS95] holds for essentially all summable ultrafilters. We have used this result to extend the Trivial Sums Theorem considerably; we have identified many more summable ultrafilters satisfying the Trivial Sums Theorem and every summable is now additively isomorphic to one. Of course, the open question remains whether every summable ultrafilter satisfies the theorem, but it is a definite improvement to reduce this question to the question of disjoint support for some suitable sequence.

We have also seen that besides forcing with FS-filters only the F_σ idempotent filter forcing could possibly adjoin a summable, thereby concluding our investigations from Chapter 3.

Appendix

Chapter A

A short proof of Zelenyuk's Theorem

In this chapter we will present a shorter proof of Zelenyuk's Theorem using idempotent filters. The original theorem is as follows.

Zelenyuk's Theorem

Theorem (Zelenyuk's Theorem)
Let G be a countable group without non-trivial finite subgroups.

Then there are no non-trivial finite subgroups in βG.

We have already studied one important techique in Theorem 2.17. But let's begin by getting some idea what we are about and have to prove.

First, we migth ask for stronger results. On the one hand, Protasov [Pro98] proved that for a countable group G every finite group in βG is of the form $p + H$ for $p \in \beta G$ idempotent and H a finite subgroup of G. On the other hand, Theorem 2.17 might suggest another generalization, namely to switch from a group to a right-cancellative, weakly left cancellative semigroup without non-trivial finite groups. However, this is not as easy as it seems since e.g. $T = \mathbb{N} \oplus \mathbb{Z}_2$ is a cancellative semigroup without any finite groups and it is easily seen that any idempotent in βT gives rise to copies of $\mathbb{Z}_2$. Hence the theorem seems rather sharp in this respect, too.[1]

[1] Similarly, for $\mathbb{N} \oplus_{n \in \mathbb{N}} \mathbb{Z}_n$ we find copies of all $\mathbb{Z}_n$; I thank a referee of Fundamenta Mathematica for this example.

Finite cyclic groups suffice.

Second, we will argue indirectly and for this it is obviously enough to consider copies of $\mathbb{Z}_z$ for some prime z that we would find contained in any given non-trivial finite group.

Just as in the proof that can be found in [HS98, Chapter 7], our proof consists of three major steps.

The first step will be to observe that a supposedly simple example is impossible, i.e., to find a copy of $\mathbb{Z}_z$ in $\delta\mathbb{F}$ having (the continuous extension of) $|\cdot| \bmod z$ as isomorphism to $\mathbb{Z}_z$.[2] Although this is technically not necessary, we include it for better understanding.

[2] For the definitions related to $\mathbb{F}$ see Definition 1.4.

In the second step we will identify an idempotent filter with strange properties included in all elements of the finite group. In the third and final step we will show that this filter is still rich enough to include a copy of $\delta\mathbb{F}$. With this at our disposal, we will be able to translate the proof of the simple case to the general one, i.e., we are able to prove Zelenyuk's Theorem.

A.1. The simple case

This first section deals with what could be considered to be the simplest scenario for a finite group to occur. It is based on taking a very simple configuration for a finite group to occur in $\mathbb{Z}^*$, i.e., assume that a copy of the finite group $\mathbb{Z}_z$ has a very simple isomorphism. However, we do not want to do this directly in $\mathbb{Z}^*$.

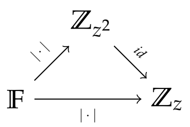
Instead of working in $\mathbb{Z}^*$, it is easier to work in the countable boolean group. But this is impossible, since we can find non-trivial finite subgroups there. To evade this fact we are going to work in our favourite adequate partial semigroup, $\mathbb{F}$.

This section will only contain the following theorem about the "easy" case.

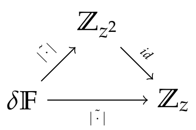
Lemma A.1
Consider $\gamma : \mathbb{F} \to \mathbb{Z}_z, s \mapsto |s| \bmod z$.

Then there is no $C \subseteq \delta\mathbb{F}$ *such that* $\tilde{\gamma} \restriction C$ *isomorphic, unless* $z = 1$.

Proof. Assuming that there is such C for $z > 1$ we first check that the following diagram of partial semigroup homomorphisms commutes (with everything taken appropriately $\bmod z$ or $\bmod z^2$).



Hence so does the following commutative diagram of semigroup homomorphisms.



But then the existence of $C \subseteq \delta\mathbb{F}$ such that $|\cdot| \bmod z$ is isomorphic on C implies $\mathbb{Z}_{z^2}$ contains an element of order z but equal $1 \bmod z$ – which is absurd, unless $z = z^2 = 1$. □

A.2. Two idempotent filters

We will now describe two idempotent filters that are important for the proof.

Definition A.2 For the rest of this chapter let G be a countable group with identity e without finite non-trivial subgroups and $C = \{c_0, \ldots, c_{z-1}\} \subseteq \beta G$ a copy of $\mathbb{Z}_z$ (with the obvious isomorphism).

We set $C' := \{x \in \beta G \mid xC = C\}$, $F := \bigcap C$, $F' := \bigcap C'$.

Let us develop some powerful properties of F'.

Proposition A.3
F and F' are idempotent.

Spoiler *We apply the topological characterization of idempotency.*

Proof. We prove this for F'.

Step 1: Assume $q, (p_s)_{s \in S} \in C'$. We want to show that $q\text{-lim}(s \cdot p_s) \in C'$.

Step 2: Pick $c \in C$. Then $q\text{-lim}(sp_s) \cdot c = q\text{-lim}(sp_s c)$.

Step 3: Since $p_s c \in C$ and C is finite, there exists c_q such that $q\text{-lim}(sp_s c) = q\text{-lim}(s \cdot c_q) = qc_q \in C$ as desired. □

We can now identify a strange, yet powerful base for the filter F' derived from F.

Proposition A.4
The sets $U^{-F} (U \in F)$ form a base of F', i.e., $\bigcap_{U \in F} \overline{U^{-F}} = C'$.

Proof. We simply calculate for $x \in \beta G$

$$\begin{aligned} x \in C' &\Leftrightarrow (\forall c \in C)\ xc \in C \Leftrightarrow (\forall U \in F)(\forall c \in C)\ xc \in \overline{U} \\ &\Leftrightarrow (\forall U \in F)(\forall c \in C)\ U^{-c} \in x \Leftrightarrow (\forall U \in F) \bigcap_{c \in C} U^{-c} = U^{-F} \in x. \end{aligned}$$ □

The next lemma enables us to get an almost prime property for F'.

Lemma A.5
Let $U \in F$. Then $s \notin U^{-F} (\in F')$ implies $S \setminus s^{-1}U^{-F} \in F'$.

Proof. Pick U, s as assumed.

Step 1: Since $s \notin U^{-F} = \bigcap_{c \in C} U^{-c}$, there exists $c \in C$ with $sc \notin \overline{U}$, i.e., $U \notin sc$.

Step 2: Now let any $c' \in C'$ be given.

Step 3: Since $c'C = C$ by assumptions on C, $sc'C = sC$. Therefore we find some $d \in C$ with $sc'd = sc$.

Step 4: In particular $U \notin sc = sc'd$, i.e., $U^{-d} \notin sc'$.

Step 5: A fortiori, $U^{-F} \notin sc'$, i.e., $s^{-1}U^{-F} \notin c'$.

Step 6: Since c' was arbitrary, $S \setminus s^{-1}U^{-F} \in F'$. □

A.3. Zelenyuk's Theorem

We begin the final section by extending the isomorphism between C and $\mathbb{Z}_z$ to C'. For this we need to isolate certain sets in the elements of C.

Splitting up C

Definition A.6

- For $c \in C$ pick pairwise disjoint $U_c \in c$.

1st Approximation of C

- For $c \in C$ let

$$A_c := \bigcap_{d \in C} (U_{cd})^{-d} (\in c).$$

- $X := \dot{\bigcup}_{c \in C} A_c$ (disjoint since $A_c \subseteq U_{cc_0}^{-c_0} = U_c$).
- $f : X \to C, a \mapsto c$ with $a \in A_c$; in other words by $a \in A_{f(a)}$.

2nd approximation of C

- For $c \in C$, $a \in X$ let $V(a) := \bigcup_{d \in C} a^{-1} A_{f(a)d} \cap A_d$

The A_c approximate $\overline{A_c} \cdot d \subseteq \overline{U_{cd}}$ for $d \in C$; the $V(a)$ take this approximation to the next level, i.e., the elements of X.

The important observation is that the $V(a)$ are included in F'.

Proposition A.7
The sets $(V(a))_{a \in X}$ are in F'.
Additionally, f is an F'-homomorphism[3] and $f[V(a)] = C$ for all $a \in X$.

[3] cf. Definition 2.15

Proof. **Step 1:** $V(a) \in F$

By definition, $a \in A_{f(a)} \subseteq U_{f(a)d}^{-d}$; hence $a^{-1} A_{f(a)d} (\cap A_d) \in d$ for any $d \in C$. Therefore $V(a) \in F$.

Step 2: $V(a) \in F'$.

Since $V(a) \in F$ and $(U_{cd}^{-d})^{c_0} = U_{cd}^{-c_0 d} = U_{cd}^{-d}$, we can easily check

$$V(a)^{-F} \subseteq V(a)^{-c_0} = \bigcup_{d \in C} a^{-1} A_{f(a)d}^{-c_0} \cap A_d^{c_0} = \bigcup_{d \in C} a^{-1} A_{f(a)d} \cap A_d.$$

By Step 1 and the last proposition of the previous section, we conclude $V(a) \in F'$.

Step 3: f is an F'-homomorphism.

By definition, $b \in V(a)$ implies $ab \in V(f(a)f(b))$, so $f(ab) = f(a)f(b)$.

Richness of f

Step 4: For all $a \in X$ we have $f[V(a)] = C$.

By the previous claims $V(a) \in F$, in particular

$$(\forall c \in C)\ V(a) \cap A_c \neq \emptyset.$$

By definition, $f[V(a) \cap A_c] = \{c\}$ – as desired. □

We are ready for the pivotal step.

Lemma A.8
There exists $\varphi : \mathbb{F} \to X$ bijective, (partially) homomorphic such that

$$\varphi(\sigma(\mathbb{F})) \subseteq F'.$$

Additionally, $f \circ \varphi = |\cdot| \bmod z$.

Spoiler *We argue just like for Theorem 2.17. However, this time we can find the copy of $\sigma(\mathbb{F})$ in F' using the powerful observations from the previous section.*

Proof. **Introduction:** We construct φ inductively on n (or more precisely on $\mathfrak{P}(n)$). For this fix some well ordering on G of order type ω. At each step we will fix a set in $A_n \subseteq F'$ which will end up being equal to $\varphi[\sigma(n)]$ (note that $\sigma(s) = \sigma(\max(s))$ for $s \in \mathbb{F}$, so these sets suffice for the second part of the lemma).

Step 1: Naturally, $\varphi(0) := e$ and $A_0 := X = V(e)$.

Note that $\bigcap F' = C' \cap G$ is a finite[4] subgroup of G and thus trivial. Hence we have now dealt with the elements common to all sets in F'.

[4] By [HS98, Lemma 6.28].

Step 2: For the induction let us assume that we have constructed a homomorphic $\varphi[\mathfrak{P}(n)]$ and $(A_i)_{i \leq n}$ in F' such that

$$s \neq t \implies (\varphi(s) \cdot A_{\max(s \cup t)}) \cap (\varphi(t) \cdot A_{\max(s \cup t)}) = \emptyset.$$

and $A_i \subseteq V(\varphi(s))$ for all $s \subseteq n$.

Step 3: Inductively for $i < n+1$ and assuming we have constructed $rng(\varphi) := \varphi[\mathfrak{P}(n) \cup \bigcup_{j<i} \mathfrak{P}(j-1) \cdot ((n+1) \setminus j)]$ we choose

$$\varphi((n+1) \setminus i) := \min(A_i \setminus rng(\varphi)) \text{ with } f(\varphi((n+1) \setminus i)) = c_{(n+1 \setminus i) \bmod z}$$

which is possible by the last proposition. Finally, we extend φ homomorphically to $\mathfrak{P}(i-1) \cdot ((n+1) \setminus i)$.

Step 4: Now for A_{n+1}. Pick $U \in F$ with $\varphi(s)^{-1}\varphi(t) \notin U^{-F} \subseteq A_n$ for all $s \neq t$ (which is possible, since $\bigcap F' = \{e\}$ and there are only finitely many such pairs s, t) and define

$$A_{n+1} := A_n \cap U^{-F} \cap \bigcap_{s \neq t \in \mathfrak{P}(n+1)} S \setminus (\varphi(s)^{-1}\varphi(t))^{-1} U^{-F} \cap \bigcap_{s \in \mathfrak{P}(n+1)} V(\varphi(s))$$

which is in F' by the last lemma from the previous section.

Step 5: We show that A_{n+1} is as requested for Step 3.

Take $s \neq t$; we only need to show this for $\max(s \cup t) = n+1$.

Then by construction of A_{n+1}

$$\begin{aligned} \varphi(t) A_{n+1} &\subseteq \varphi(t) \cdot S \setminus \varphi(t)^{-1}\varphi(s) U^{-F} \\ &\subseteq S \setminus \varphi(s) U^{-F} \subseteq S \setminus \varphi(s) \cdot A_{n+1}. \end{aligned}$$

Step 6: We show that such a φ is as desired.

Claim 1: Surjectivity follows from Step 3 (for $i = 0$), since after at most z steps, we will be able to pick the actual minimum; in fact, the second part of the lemma is immediate from Step 3 as well, i.e., $A_i = \varphi[\sigma(i)]$.

Claim 2: Clearly, φ is a partial homomorphism by construction.

Claim 3: We claim that φ is injective.

Consider $s \neq t$; then there must exist a minimal i such that

$$s \cap i = s \cap i + 1 \neq t \cap i + 1 = t \cap i.$$

Then $\varphi(s \cap i) \cdot A_i \cap \varphi(t \cap i) \cdot A_i = \emptyset$ by Steps 2 and 5 – in particular, $\varphi(s) \neq \varphi(t)$.

This completes the proof of the claim.

This completes the proof of the lemma. □

We can now conclude.

Zelenyuk's Theorem

Theorem A.9 (Zelenyuk's Theorem)
Let G be a countable group without non-trivial finite subgroups.

Then there are no non-trivial finite subgroups in βG.

Proof. Arguing indirectly, we assume we can find a copy of $\mathbb{Z}_z$ in βG. But then the preceding lemma tells us that we find a copy of $\mathbb{Z}_z$ in $\delta\mathbb{F}$ – with isomorphism $|\cdot| \bmod z$ ↯↯↯ but this is impossible as we saw in the first section. □

Chapter B

Miscellaneous results

B.1. The partial semigroup βS

We present a short observation that for a partial semigroup S its Stone-Čech compactification βS is also a partial semigroup. This is probably folklore but we have found no presentation in the literature.

Definition B.1 Let $(S, \cdot)$ be a partial semigroup.

We define a partial operation on βS as usual by

$$p \cdot q := \{A \subseteq S \mid (\exists V \in p, (W_v)_{v \in V} \text{ in } q) \bigcup_{v \in V} v \cdot W_v \subseteq A\},$$

if this is an ultrafilter – with the convention that $v \cdot W_v = v \cdot (W_v \cap \sigma(v))$ (which of course might be empty).

We begin with a simple observation.

Proposition B.2
Let $(S, \cdot)$ be a partial semigroup and $p, q \in \beta S$. Then $p \cdot q$ is defined if and only if there exists $A \in p$ such that $\sigma(a) \in q$ for all $a \in A$.

Proof. **Step 1:** First let us assume that for every $A \in p$ there is $a \in A$ with $\sigma(a) \notin q$.

Since p and q are ultrafilters we find $A \in p$ such that for all $a \in A$ in fact $S \setminus \sigma(a) \in q$.

But then by the definition of $p \cdot q$ we have $\emptyset = \bigcup_{a \in A} a \cdot (S \setminus \sigma(a)) \in p \cdot q$; hence $p \cdot q$ is not a filter.

Step 2: Now assume that there exists $A \in p$ such that for all $a \in A$ we have $\sigma(a) \in q$.

Then $p \cdot q$ is easily checked not to include the empty set and, in fact, to be a filter.

So take $A \in p \cdot q$; we may assume that $A = \bigcup_{v \in V} v \cdot W_v$ for $V \in p$ and some $W_v \in q$.

To see that $p \cdot q$ is a prime filter, let $A^0 \dot{\cup} A^1 = A$, and define $W_v^i := \{w \in W_v \mid v \cdot w \in A^i\}$ (for $i = 0, 1$). Since q is prime, there is a function $\epsilon : V \to 2$ with $W_v^{\epsilon(v)} \in q$. Then let $V^i = \epsilon^{-1}(i)$ and since p is prime, for one $i \in 2$ we get $V^i \in p$. We conclude that $\bigcup_{v \in V^i} v \cdot W_v^i \subseteq A_i \in p \cdot q$ and that $p \cdot q$ is prime. □

This observation is enough to get the desired result.

Proposition B.3
If $(S, \cdot)$ is partial semigroup, so is $(\beta S, \cdot)$.

Proof. We only need to show that the strong associativity of the multiplication on S is transferred to βS as well.

Step 1: For this let $p, q, r \in \beta S$. By the above proposition we can calculate the following equivalences.

$p \cdot (q \cdot r)$ def.
$\iff (\exists V \in p)(\forall v \in V) : \sigma(v) \in q \cdot r$ and $(\exists W \in q)(\forall w \in W) : \sigma(w) \in r$
$\iff (\exists V \in p)(\forall v \in V)(\exists W_v \in q, Z_w^v \subseteq \sigma(w) \text{ in } r) : \bigcup_{w \in W_v} w \cdot Z_w^v \subseteq \sigma(v)$.

and

$(p \cdot q) \cdot r$ def.
$\iff (\exists V \in p)(\forall v \in V) : \sigma(v) \in q$ and $(\exists U \in p \cdot q)(\forall u \in U) : \sigma(u) \in r$
$\iff (\exists V \in p, W_v \subseteq \sigma(v) \text{ in } q)(\forall v \in V, w \in W_v) : \sigma(v \cdot w) \in r$.

Step 2: First assume that $p \cdot (q \cdot r)$ is defined and let $V, (W_v)_{v \in V}$ and $(Z_w^v)_{w \in W_v}$ as in the second equivalence for this case, i.e., $\bigcup_{w \in W_v} w \cdot Z_w^v \subseteq \sigma(v)$. In particular, $v \cdot (w \cdot z)$ is defined for every $w \in W_v$, $z \in Z_w^v$.

By the strong associativity of $(S, \cdot)$ we can conclude that $(v \cdot w) \cdot z$ is defined (and equal to $v \cdot (w \cdot z)$), hence $Z_w^v \subseteq \sigma(v \cdot w) \in r$ for every $v \in V, w \in W_v$.

By the second set of equivalances we conclude that $p \cdot (q \cdot r)$ is defined.

Step 3: Second, assume that $(p \cdot q) \cdot r)$ is define and let V, $(W_v)_{v \in V}$ (each $W_v \subseteq \sigma(v)$) be as in the last equivalence for $(p \cdot q) \cdot r$ and set $Z_w^v = \sigma(v \cdot w) \in r$.

Then $\bigcup_{w \in W_v} w \cdot Z_w^v \subseteq \sigma(v)$ since for $w \in W_v, z \in Z_w^v$ we know $(v \cdot w) \cdot z$ is defined, hence by strong associativity $v \cdot (w \cdot z)$ is defined (and equal to it), which in turn implies $w \cdot z \in \sigma(v)$.

This concludes the proof. □

B.2. *PS*-Ultrafilters

We present an example of a set-theoretically interesting ultrafilter that lies in $\mathbb{N}^* + \mathbb{N}^*$. This example is due to I. Protasov in [Pro01]; we give a slight reformulation.

Definition B.4 We define as follows.

- For $A \subseteq S$ let $PS(A) := \{x \cdot y \mid x \neq y \text{ both in } A\}$
- An ultrafilter p on a semigroup $(S, \cdot)$ is called PS-ultrafilter if for every colouring of S there exists $A \in p$, such that $PS(A)$ is homogeneous.

Igor Protasov conjectured in [Pro01] that every PS-ultrafilter is either selective or the translation of a strongly summable ultrafilter on a boolean subgroup of S.

Theorem B.5
Let u be a selective ultrafilter on $\mathbb{N}$. Then $u + u$ is generated by $\{PS(A) \mid A \in u\}$.

Proof. **Step 1:** We need to show that for $V, W_V \in u$, the set $\bigcup_{v \in V} v + W_v$ can be refined by some PS-set in $u + u$.

We will find as suitable $X \in u$ and use the fact that $PS(X) = \bigcup_{x \in X} x + X \setminus \{x\}$ is clearly in $u + u$. So we aim to assure that

$$\circledast \qquad \bigcup_{x \in X} x + X \setminus \{x\} \subseteq \bigcup_{v \in V} v + W_v.$$

Step 2: So fix $V, (W_v)_{v \in V}$ in u. Since u is a P-point, we can find $X \subseteq V$ such that $X \subseteq^* W_v$ for all $v \in V$, furthermore since u is a Q-point, $|X \setminus W_v| \leq 1$. Since $X \in u$ we may assume that $X = V$.

> To see what might go wrong in our approach, define $x : V \to V, v \mapsto x_v =$ the element in $V \setminus W_x$ (or else 0). The element x_v is the problematic one, since $v + V \setminus \{v\}$ will include the additional sum $v + x_v$, which $v + W_v$ did not. We will show that this is never a problem.

Step 3: Since u is selective we can assume (by going to a subset of V still in u) that x is 1-to-1 or constant on V.

Step 4: Assume x is constant on V, i.e., there exists a unique w such that $V \subseteq W_v \cup \{w\}$. But then $V \setminus \{w\} \subseteq W_v$ and of course $V \setminus \{w\} \in u$, so

$$PS(V \setminus \{w\}) \subseteq \bigcup_{v \in V \setminus \{w\}} v + V \setminus \{v, w\} \subseteq \bigcup_{v \in V} v + W_v \subseteq A.$$

Therefore we have $\circledast$.

Step 5: Now consider the case that x is 1-to-1 on V. We have three subcases. By going to a subset of V still in the u, we may assume that $x = id$ or $x > id$ or $x < id$ on V.

Step 6: If $x = id$, then we're done, since $v + v$ is never in $PS(V)$. More specifically, $v + V \setminus \{v\} = v + W_v$. Therefore $\bigcup_{v \in V} v + V \setminus \{v\} = \bigcup_{v \in V} v + W_v$, and we have $\circledast$.

Step 7: If $x > id$ let us consider some $v \in V$. Then $v < x_v$ but of course $x_v \in V$, so let us look at W_{x_v}. Of course again $x_{x_v} \notin W_{x_v}$ by choice of x – but: $v < x_v < x_{x_v}$, so $v \in W_{x_v}$. Hence $v + x_v = x_v + v \in W_{x_v}$. In total $v + V \setminus \{v\} \subseteq (v + W_v \cup x_v + W_{x_v})$, so $\bigcup_{v \in V} v + V \setminus \{v\} \subseteq \bigcup_{v \in V} v + W_v$ and we have $\circledast$ again.

Step 8: The case $x < id$ is symmetrical to $x > id$.

This concludes the proof. □

Note that this result can be seen from another angle.

Remark B.6 For a selective ultrafilter u consider the ultrafilter $u \otimes u \in$ on $\mathbb{N}^2$. It is well known and essentially what we proved above that this ultrafilter has a base of sets $X \times X$ with $X \in u$. Then the addition maps $u \otimes u$ to $u + u$ and it is not difficult to show that they are isomorphic.

Additionally, $u \otimes u$ fulfils a weak Ramey property for $\mathbb{N}^2$ (reducing colourings with six colours to five colours) which makes it set theoretically interesting.

Chapter C

On a result by Y. Zelenyuk

In this chapter we wish to partially reproduce an English translation of [Zel96] with some additional comments.

When I first became interested in this result, Yevhen Zelenyuk was kind enough to send me a copy of the paper by mail, since there was no digital copy available. I sincerely thank Lutz Heindorf who was kind enough to let me record his on-the-fly translation of the Russian original which I later typset using LaTeX. The complete translation is available from me or Yevhen Zelenyuk.

In the first section we give the reproduction, to which we added only some minor comments. We skip the introductory comments to jump right in.

C.1. A translated extract from [Zel96]

C.1.1 The approximation Theorem

Let G be an infinite Group, βG its Stone-Čech-Compactification (taking G discretely), the elements of βG are the ultrafilters over G, as usual we identify the elements of G with the principal ultrafilers. The set $\{\overline{A} \mid A \subseteq G\}$ with $\overline{A} = \{p \in \beta G \mid A \in p\}$ yields a (zero-dimensional) base for the topology of βG. The multiplication on G extends to a multiplication on βG; this multiplication can be characterized by letting $\{\bigcup_{a \in A} a \cdot B_a \mid A \in p, B_a \in q\}$ be a base for $p \cdot q$, being rightcontinuous.

Remark C.1 There is an inclusion reversing bijection between filters on G and closed, nonempty subspaces of βG, namely

$$\begin{array}{lll} \varphi & \mapsto & \overline{\varphi} := \bigcap_{A \in \varphi} \overline{A} = \{p \in \beta G \mid \varphi \subseteq p\} \\ \emptyset \neq C = \overline{C} \subseteq \beta G & \mapsto & \bigcap_{p \in C} p. \end{array}$$

Thus when we write "$\overline{\varphi}$ a closed subspace of βG" we assume φ to be a filter on G.
Obviously the character of such a space is the same as the character of the corresponding filter φ, i.e., the minimal cardinality of a filter base. We are mostly concerned with countably generated filters, i.e., filters with a countable

base. We call a filter free, if it contains the filter of cofinite subsets of G. The closed subsemigroups of βG are called *ultrafilter semigroups* of G. As compact right-topological semigroups they contain idempotent elements.

Lemma C.2
Let $S = \{p_i \mid i < m\}$ a finite subsemigroup of ultrafilters on G, $F_n(p_i)$ $(n < \omega)$ a descending sequence of elements of p_i, so that $F_0(p_i) \cap F_0(p_j) = \emptyset$ for $i \neq j$. Let $p(a) := p_i$ whenever $a \in F_n(p_i)$ for some n.
Then there is a sequence $(a_n)_{n\in\omega}$ in G, such that

1)	$a_n \in F_n(p_i)$	*whenever $i = n \bmod m$*
2)	$a_{n_0} \cdot \ldots \cdot a_{n_k} \in F_{n_0}(p(a_{n_0}) \cdot \ldots \cdot p(a_{n_k}))$	*whenever $n_0 < \ldots < n_k$.*

Proof. We construct the sequence recursively. Let $A_0 \in p_0$ be such that for $p \in S$

$$\overline{A_0} \cdot p \subseteq \overline{F_0(p_0 \cdot p)} \qquad \text{(since } \rho_p \text{ is continuous)}$$

Let $a_0 \in A_0 \cap F_0(p_0)$.
Let $l \in \omega$ and assume we have constructed $\{a_n \mid n < l\}$ such that:

$a_n \in F_n(p_i)$	for $i = n \bmod m$,
$a_{n_0} \cdot \ldots \cdot a_{n_k} \in F_{n_0}(p(a_{n_0}) \cdot \ldots \cdot p(a_{n_k}))$	for $n_0 < \ldots < n_k < l$,
$a_{n_0} \cdot \ldots \cdot a_{n_k} \cdot p \in \overline{F_{n_0}(p(a_{n_0}) \cdot \ldots \cdot p(a_{n_k}) \cdot p)}$	for $n_0 < \ldots < n_k < l$ and $p \in S$.

Let $j = l \bmod m$. We can choose $A_l \in p_j$ such that

$$a_{n_0} \cdot \ldots \cdot a_{n_k} \cdot A_l \subseteq F_{n_0}(p(a_{n_0}) \cdot \ldots \cdot p(a_{n_k}) \cdot p_j),$$
$$\overline{A_l} \cdot p \subseteq \overline{F_l(p_j \cdot p)},$$
$$a_{n_0} \cdot \ldots \cdot a_{n_k} \cdot \overline{A_l} \cdot p \subseteq \overline{F_{n_0}(p(a_{n_0}) \cdot \ldots \cdot p(a_{n_k}) \cdot p_j \cdot p_l)},$$

whenever $n_0 < \ldots < n_k < n$, $p \in S$ – this is again since all ρ_p, ρ_{p_i} are continuous and we only have a finite number of combinations to check.

Then we can choose $a_l \in A_l \cap F_l(p_j) \setminus \{a_n \mid n < l\}$. Then the three conditions are fulfilled for $l + 1$. Obviously this inductively chosen sequence is as required. □

Definition C.3

1. A filter φ on G is called *elementary FP-filter*, if there is a sequence $(a_n)_{n\in\omega}$ in G, such that

$$FP_m(a_n) = \{a_{n_0} \cdot \ldots \cdot a_{n_k} \mid n \leq n_0 < \ldots < n_k\}$$

   is a base of φ. We also call φ *the FP-Filter induced by* $(a_n)_{n\in\omega}$.

2. The union of an increasing sequence of elementary FP-filters is called *FP-filter*.

3. A subgroup of ultrafilters $\overline{\varphi}$ of G is called *(elementary) FP-semigroup*, if φ is an (elementary) FP-filter. Obviously every elementary FP-semigroup has countable character.

If the sequence is unambiguous we sometimes write FP_m instead of $FP_m(a_n)$. If G is abelian, we write FS instead of FP.

Remark C.4 Note that if G is a boolean group, φ FS-Filter than $\varphi_0 = \{F \cup \{0\} \mid F \in \varphi\}$ a base of a topology.

The following lemma shows, that (3) of the last definition is indeed justified.

Lemma C.5
If φ is FP-filter, then for every $U \in \varphi$ there is a $V \in \varphi$, such that

$$\overline{V} \cdot \overline{\varphi} \subseteq \overline{U}.$$

So $\overline{\varphi}$ is a semigroup.

Proof. Let ψ be an elementary FP-filter such that $U \in \psi \subseteq \varphi$, and $(a_n)_{n\in\omega}$ an increasing sequence in G, such that $(FP_m)_{m\in\omega}$ is a base for ψ, and finally let k with $FP_k \subseteq U$. It suffices to show, that $\overline{FP_k} \cdot \overline{\psi} \subseteq \overline{U_k} \subseteq \overline{U}$.
To see this let $F \subseteq \omega$ be finite and consider $\Sigma_{i\in F} a_i \in FP_m$. Then for all $i > \max F$ we have $\Sigma_{i\in F} \cdot FP_i \subseteq FP_m$. It follows $\Sigma_{i\in F} \cdot \overline{\psi} \subseteq FP_{\min F}$, so $\overline{FP_m} \cdot \overline{\psi} \subseteq \overline{FP_m}$. □

Now we are ready for the approximation theorem.

Theorem C.6
Let $S = \{p_i \mid i < N\}$ be a finite subsemigroup of G^*, φ_i countably generated filters, such that $\varphi_i \subseteq p_i$ and $\overline{\varphi_i} \cap \overline{\varphi_j} = \emptyset$ for $i \neq j$.
Then there exist countably generated filters $\psi_i \supseteq \varphi_i$, such that $\psi := \bigcap_{i<m} \psi_i$ is an elementary FP-filter and additionally $\overline{\psi_i} \cdot \overline{\psi_j} \subseteq \overline{\psi_l}$ whenever $p_i \cdot p_j = p_l$.

Proof. From every filter φ_i choose a decreasing base $\{F_n(p_i) \mid n < \omega\}$, such that $F_0(p_i) \cap F_0(p_j) = \emptyset$ for $i \neq j$. Let $(a_n)_{n\in\omega}$ be as in Lemma C.2, let ψ be the elementary FP-filter induced by this sequence, i.e., with base FP_m, and finally let ψ_i be the filter with base $FP_m \cap F_0(p_i) =: V_m(p_i)$. Obviously $\psi_i \supseteq \varphi_i$, $\psi := \bigcap_{i<m} \psi_i$. We need only to show, that $\overline{\psi_i} \cdot \overline{\psi_j} = \overline{\psi_l}$ for applicable i, j, l. It suffices to show, that $\overline{V_m(p_i)} \cdot \overline{\psi_j} \subseteq \overline{V_m(p_l)}$.
So let $F \subseteq \omega$ be finite, $\Sigma_{k\in F} a_k \in V_m(p_i)$. Then for $t > \max F$ we have $\Sigma_{k\in F} a_k \cdot V_t(p_j) \subseteq V_{\min F}(p_l)$ by lemma 1. Thus follows $\Sigma_{k\in F} a_k \cdot \overline{\psi_j} \subseteq \overline{V_{\min F}(p_l)}$, hence $\overline{V_M(p_i)} \cdot \overline{\psi_j} \subseteq \overline{V_m(p_l)}$. □

Corollary C.7
Every semigroup of free ultrafilters of countable character includes an elementary FP-semigroup.

Proof. Since such a semigroup is again a compact and rightcontinuous, we can find an idempotent element in it and apply the theorem with $m = 1$. □

C.1.2 Finite semigroups of ultrafilters

Definition: and Remark

Definition C.8 On the group G consider some FP-filter φ.
For every $a \in G$ we define

$$\varphi_e := \{F \cup \{e\} \mid F \in \varphi\}$$
$$\varphi_a := \{a \cdot F \mid F \in \varphi_e\};$$

this is again a filter and clearly $\varphi_a = a \cdot \varphi_e$. Additionally for every $U \in \varphi_e$ there is $V \in \varphi_e$ such that $U \in \varphi_a$ for all $a \in V$.
Therefore there is a topology on G, such that the multiplication is rightcontinuous and φ_e the filter of neighbourhoods of e. We denote this topology by $\langle\varphi\rangle$.

Proof. This follows from Lemma C.5 as follows: choose $V \in \varphi$ such that $\overline{V} \cdot \overline{\varphi_e} \subseteq \overline{U}$, then $a \cdot \overline{\varphi_e} = \overline{\varphi_a} \subseteq \overline{U}$ for alle $a \in V$, i.e., $U \in \varphi_a$.

Lemma C.9
Every semigroup of free filters $\overline{\varphi}$ of countable character on G includes an elementary FP-semigroup $\overline{\psi}$, such that $\langle\psi\rangle$ is regular.

Proof. By the corollary after Theorem C.6 we may assume, that φ is already an elementary FP-filter generated by, say, $(a_n)_{n\in\omega}$. It suffices to construct a subsequence $(a_{n_k})_{k\in\omega}$ such that

1. For every $e \neq g \in G$ there is $j \in \omega$ such that $g \notin U_j := \{e\} \cup FP_j(a_{n_r})$.
2. For all $j \in \omega$, $e \neq g \in G$ with $g \notin U_j$, there exists $i \in \omega$ with $g \cdot U_i \cap U_j = \emptyset$.

(Firstly it suffices to be regular at e. Then 1. shows, that the antecedens of 2. can be fulfilled for any $g \neq e$, consequently the topology has a clopen basis.)

Note that it suffices to show 1. and 2. for the subgroup H generated by $(a_n)_{n\in\omega}$, since 1. holds trivially for $g \notin H$ and the negation of the consequence of (2) implies $g \in H$.
So let $(g_i)_{i\in\omega}$ be an enumeration of H. We will construct the subsequence inductively.
For $n = 0$, we can choose $n_0 \in \omega$ with $g_0 \cdot \{e, a_{n_0}\} \cap \{e, a_{n_0}\} = \emptyset$ (since H is cancellable).
For the inductive step define

$$U(i,k) := \{e\} \cup FP(a_{n_i}, \ldots, a_{n_{k-1}})$$
$$j(i) := \begin{cases} i & \text{if } g_i \in U(j,i) \text{ for all } j < i \\ min\{j \mid g_i \notin U(j,i)\} & \text{else,} \end{cases}$$

and assume we have already constructed $\{n_i \mid i < k\}$ with

$$\circledast_k \qquad \forall i < k: \; g_i \cdot U(i,k) \cap U(j(i),k) = \emptyset.$$

But then we can find $n_k > n_{k-1}$ with $\circledast_{k+1}$. For assume otherwise, then there by the pigeon hole principle there exists $i < k+1$ and infinitely many n_k such that $g_i \cdot U(i,k) \cap U(j(i),k) \neq \emptyset$. But then we have three cases to check:

$$\begin{aligned} i = k, j(k) = k &\Rightarrow g_i \cdot \{e, a_{n_k}\} \cap \{e, a_{n_k}\} \neq \emptyset, \\ i = k, j(k) < k &\Rightarrow g_i \cdot \{e, a_{n_k}\} \cap U(j(i),k) \cdot \{e, a_{n_k}\} \neq \emptyset, \\ i < k &\Rightarrow g_i \cdot U(i,k) \cdot \{e, a_{n_k}\} \cap U(j(i),k) \cdot \{e, a_{n_k}\} \neq \emptyset. \end{aligned}$$

Of course, the first case leads to a contradiction. Consider the second case. Again by the pigeon hole principle, there exists an $x \in U(j(i),k)$ such that

$$g_i \cdot y_1 = x \cdot y_2,$$

where $y_1, y_2 \in \{e, a_{n_k}\}$ are fixed (for infinitely many n_k). Then

$$x^{-1} \cdot g_i = y_2 \cdot y_1^{-1}.$$

But by induction hypotheses and the choice of $j(i)$ we have $y_1 \neq y_2$, so wlog for infinitely many n_k we have

$$x^{-1} \cdot g_i = a_{n_k}.$$

But this is of course a contradiction.
For the third case, the argument is basically the same. □

Lemma C.10 (Protasov) Protasov
Let $\overline{\varphi} \leq \beta G$, $p \in K(\overline{\varphi})$, the minimal ideal of $\overline{\varphi}$. Then for all $A \in p$, $U \in \varphi$ there exists finite $K \subseteq U$ such that $K^{-1}AA^{-1} \in \varphi$.

Proof. Let $p \in K(\overline{\varphi})$. Then there exists L min. left ideal (of $\overline{\varphi}$!) with $p \in L$. Let $r \in \overline{\varphi}$ arbitrary. Then $rp \in L$, so $L \cdot rp = L \ni p$. Thus there exists $t_r \in L$ with $t_r \cdot r \cdot p = p$. Then of course $A \in t_r \cdot r \cdot p$ and $U \in t_r (\in \overline{\varphi})$. Therefore we can choose $x_r \in U$ with $A \in x_r \cdot r \cdot p$, i.e., $r \cdot p \in \overline{x_r^{-1} \cdot A}$. Finally choose $A_r \in r$ with

$$\circledast \qquad \overline{A_r} \cdot p \subseteq \overline{x_r^{-1} \cdot A}.$$

Since $\overline{\varphi}$ is compact and $\overline{\varphi} \subseteq \bigcup_{r \in \overline{\varphi}} \overline{A_r}$, there are $r_1, \ldots, r_n$ with

$$\overline{\varphi} \subseteq \overline{A_{r_1}} \cup \ldots \cup \overline{A_{r_n}}.$$

Rename $x_{r_i} := x_i$. To see that $K := \{x_1, \ldots, x_n\}$ suffices, let $q \in \overline{\varphi}$ and show that $K^{-1}AA^{-1} \in q$.
Since $q \in \overline{\varphi}$, choose i with $q \in A_{r_i}$. By $\circledast$ we have $x_i^{-1}A \in q \cdot p$, so $X := \bigcup_{i=1}^n x_i^{-1}A \in q \cdot p$. So there exists $V \in q$, $(W_v)_{v \in V}$ in p with $\bigcup_{v \in V} v \cdot W_v \subseteq X$; wlog we may assume $W_v \subseteq A$ for $v \in V$. Then

$$X \cdot A^{-1} \supseteq \bigcup_{v \in V} v \cdot W_v \cdot A^{-1} \supseteq \bigcup_{v \in V} v \cdot e = V.$$

So we have: $\forall q \in \overline{\varphi} : \bigcup_{i=1}^n x_i^{-1} \cdot A \cdot A^{-1} \in q$, i.e., $K \cdot A \cdot A^{-1} \in \varphi$. □

Lemma C.11
Every countably generated filter φ with $\overline{\varphi} \leq G^*$ extends to an elementary FP-filter ψ with $\overline{\psi} \cap K(\overline{\varphi}) = \emptyset$.

Proof. By the corollary of Theorem C.6 there exists an elementary FP-filter ϕ extending φ; let ϕ be generated by $(a_n)_{n\in\omega}$.
By Lemma C.10 it suffices to construct a subsequence $(a_{n_l})_{l\in\omega}$ such that

$$\exists V \in \varphi \forall U \in \varphi, K \subseteq V \text{ finite: } U \setminus K^{-1}AA^{-1} \neq \emptyset,$$

where $A = FP(a_{n_l})$.

Then choose ψ as the elementary FP-filter generated by this sequence. Then clearly no $U \in \varphi$ has $U \subseteq K^{-1}AA^{-1}$, so $K^{1-}AA^{-1} \notin \varphi$. Since $A \in p$ for every $p \in \overline{\psi}$, the contraposition of lemma 4 applies.)

If φ consists only of uncountable sets, let $n_l = l$; then $|K^{-1}AA^{-1}| \leq \omega$, so for $U \in \varphi$ necessarily $U \setminus K^{-1}AA^{-1} \neq \emptyset$.
Otherwise choose $V \in \varphi$ countable. Let $\{\alpha_n \mid n \in \omega\}$ the set of all pairs $\alpha = (U_\alpha, K_\alpha)$, where U_α is in a fixed countable base of φ and $K_\alpha \subseteq V$ finite. We construct the subsequence inductively.
To start off, fix (by an argument similar to the proof of Lemma C.9) some $n_0 \in \omega$ and some $b_0 \in U_{\alpha_0} \setminus K_{\alpha_0}^{-1} \cdot \{e, a_{n_0}, a_{n_0}^{-1}\}$.
Assume we have chosen n_i, b_i for $i < l$ such that with $A_l = FP(a_{n_0}, \ldots, a_{n_{l-1}})$

$$b_i \in U_{\alpha_i} \setminus K_{\alpha_i}^{-1} A_l A_l^{-1}.$$

Then we can (again by an argument similar to the proof of Lemma C.9) choose some $n_l > n_{l-1}$ with

$$b_i \in U_{\alpha_i} \setminus K_{\alpha_i}^{-1} A_{l+1} A_{l+1}^{-1} \quad (\text{for } i < l).$$

Then simply choose $b_l \in U_{\alpha_l} \setminus K_{\alpha_l}^{-1} A_{l+1} A_{l+1}^{-1}$.
From this construction we have $b_l \in U_{\alpha_l} \setminus K_{\alpha_l}^{-1} AA^{-1} \neq \emptyset$ for all $l < \omega$. □

Let's recall some well known semigroup definitions.

Definition C.12
1. A semigroup S is called *right(left)-zero semigroup*, if $pq = q$ ($pq = p$) for all $p, q \in S$, i.e., every element of S is left(right)-neutral.
2. The semigroup S is called *chain of idempotent elements*, if $pq = qp = p$ or $qp = pq = q$ for every $p, q \in S$, i.e., S is a chain in the usual partial order of idempotents given by

$$p \leq q \iff pq = qp = p \iff pS \subseteq qS, Sp \subseteq Sq.$$

Note that all these types of semigroups consist entirely of idempotent elements.

Theorem C.13
For every countably generated filter φ with $\overline{\varphi} \leq G^*$ and every $m \in \omega$, we find a right-zero, a left-zero subsemigroup and a chain of idempotents each of cardinality m contained in $\overline{\varphi}$.

Proof. From the theory of complete simple semigroups ($K(\overline{\varphi})$ is such a semigroup) it follows: to have a right-zero semigroup of cardinality m we need to find m disjoint left ideals in $\overline{\varphi}$.

(Take any minimal right ideal R. Then $E(R)$ is right zero, since $p \cdot R = R$ for all $p \in E(R)$, thus for $q \in E(R)$ there is $t \in R$ with $p \cdot t = q$, and so $p \cdot q = p \cdot p \cdot t = p \cdot t = q$. Now by e.g. [HS98, 1.61] we have $E(R \cap L) \neq \emptyset$ for every minimal Leftideal L, so we can find m many idempotents in R.)

Of course, the analogue for left zero-holds as well.
By Lemma C.9 φ extends to a FP-filter ψ such that the topology $\langle \psi \rangle$ is regular. Consider G with that topology. We can choose an infinite injective sequence of elements converging to e; denote the set of these elements by A.
Note that $|A^* \cap \overline{\psi}| = 2^c$.
Claim 1: $p \neq q$ in $A^* \cap \overline{\psi} \Rightarrow \beta G \cdot p \cap \beta G \cdot q = \emptyset$
To prove this let $H := \{h_n \mid n \in \omega\}$ be the subgroup generated by A and let $\{g_\alpha \mid \alpha \in \mathfrak{A}\}$ be representatives of the left-equivalence classes of G mod H, i.e.,

$$\dot{\bigcup}_{\alpha \in \mathfrak{A}} g_\alpha \cdot H = G.$$

Then we can inductively construct $(P_n)_{n \in \omega}$ in p and $(Q_n)_{n \in \omega}$ in q such that

$$P_n, Q_n \subseteq A,\ P_n \cap Q_n = \emptyset,\ R_n := h_n \cdot P_n \cup Q_n \subseteq H \setminus \bigcup_{i<n} R_i.$$

(Having constructed P_i, Q_i for $i < n$ we can choose P_n, $Q_n \subseteq A$ accordingly, since $\bigcup_{i<n} R_i \in h_i \cdot p, h_i \cdot q$ and $h_i \cdot p \neq h_n \cdot p$, $h_i \cdot q \neq h_n \cdot q$.)

Now let

$$P := \bigcup \{g_\alpha h_n P_n \mid n < \omega, \alpha \in \mathfrak{A}\},\ Q := \bigcup \{g_\alpha h_n Q_n \mid n < \omega, \alpha \in \mathfrak{A}\}.$$

Then $P \cap Q = \emptyset$.

(Else we find $g_\alpha h_n p_n = g_\beta h_k q_k$, so $g_\alpha \cdot (h_n p_n q_k^{-1} h_k^{-1}) = g_\beta$, therefore $\alpha = \beta$, so by the choice of the R_i follows $n = k$ and thus $p_n = q_k$ – contradicting $P_n \cap Q_n = \emptyset$.)

Additionally $\beta G \cdot p \subseteq \overline{P}$ and $\beta G \cdot q \subseteq \overline{Q}$ (which finishes the proof).

(To see this, let $g \in G$ and choose g_α, h_n such that $g = g_\alpha \cdot h_n$. Then $g \cdot p = g_\alpha h_n p \in \overline{g_\alpha h_n P_n} \subseteq \overline{P}$; for q and Q this follows similarly.)

To find the left zero semigroup, we show that $p \cdot \overline{\psi} \cap q \cdot \overline{\psi} = \emptyset$. Since $\langle \psi \rangle$ is regular there are disjoint open sets $U \in p$, $V \in q$. For every $a \in A \cap U$ choose $W_a \in \psi$ with $a \cdot W_a \subseteq U$, and for $b \in V$ choose $W_b \in \psi$ such that $b \cdot W_b \subseteq V$ (cf. beginning of this section). If we let $P := \bigcup \{a \cdot W_a \mid a \in A \cap U\}$, $Q := \bigcup \{b \cdot W_b \mid b \in A \cap V\}$, then $P \cap Q = \emptyset$, $p \cdot \overline{\psi} \subseteq \overline{P}$, $q \cdot \overline{\psi} \subseteq \overline{Q}$.

(To see $p \cdot \overline{\psi} \subseteq \overline{P}$: $r \in \overline{\psi} \Rightarrow A \cap U \in p \wedge (W_a)_{a \in A \cap U} \subseteq \psi \subseteq r \Rightarrow P \in p \cdot r$ by the above choice. $q \cdot \overline{\psi} \subseteq \overline{Q}$ follows similarly.)

To finish the theorem we show that the semigroup $\overline{\varphi}$ has a chain of idempotents of length m. By Lemma C.11 we can construct a decreasing sequence of subgroups $\overline{\varphi} = \overline{\varphi_1} \supseteq \overline{\varphi_2} \supseteq \ldots \supseteq \overline{\varphi_m}$ all of countable character such that $I(\overline{\varphi_i}) \cap \overline{\varphi_{i+1}} = \emptyset$ for $i < m$. Let p_m an idempotent in $\overline{\varphi_m}$, L minimal left ideal of φ_{m-1} included in $\overline{\varphi_{m-1}} \cdot p_m$. Furthermore let R be a minimal right ideal of $\overline{\varphi_{m-1}}$ included in $p_m \cdot \overline{\varphi_{m-1}}$, and let p_{m-1} an idempotent element in $R \cap L$.

Then $p_{m-1} < p_m$. We repeat this to gain in $\overline{\varphi_{m-1}}$ the appropriate $p_{m-2} < p_{m-1}$ and so forth inductively. □

Question C.14
Is there a finite semigroup of ultrafilters on $\mathbb{Z}$ that is neither right-zero nor left-zero nor chain of idempotetents?

C.1.3 A construction of topological groups with finite semigroups of ultrafilters

Definition C.15 Let (G, τ) be a topological group. We call the set of free ultrafilters converging to 0, i.e., $\{p \in G^* \mid \forall U \in \tau : \; 0 \in U \Rightarrow U \in p\}$, *the ultrafilter semigroup of* (G, τ) or *the ultrafilter semigroup of* τ; since G is topological, this is indeed a semigroup.
If G is boolean, than every FS-semigroup $\overline{\varphi}$ of free ultrafilters on G defines a group topology for which it is conversely the semigroup of ultrafilters.

Theorem C.16
Assume CH. Let φ a free filter on G with countable base and $\overline{\varphi} \leq G^*$, let $m \in \omega$. Then we can extend φ to three FP-Filters whose associated subspaces are of cardinality m and a left-zero semigroup, a right zero semigroup and a chain of idempotents respectively.

Proof. We prove the theorem for right-zero semigroups.
By the corollary to Theorem C.6 we may assume that G is countable by switching to the subgroup generated by the elementary FP-filter.
Let $\{X_\alpha \mid \alpha < \omega_1\}$ be an enumeration of the subsets of G. We construct inductively.
For $\xi = 0$: By Theorem C.13 we find a right-zero semigroup of cardinality m in $\overline{\varphi}$, say $S_0 = \{p_i^0 \mid i < m\}$.
Choose pairwise disjoint $F_i^0 \in p_i^0$ with either $F_i^0 \subseteq X_0$ or $F_i^0 \subseteq G \setminus X_0$; let ϕ_i^0 denote the filter generated from $\varphi \cup \{F_i^0\}$.
By Theorem C.6 there exist filters $\varphi_i^0 \supseteq \phi_i^0$ such that $\varphi^0 := \bigcap_{i<m} \varphi_i^0$ is an elementary FP-filter with $\overline{\varphi_i^0} \cdot \overline{\varphi_j^0} \subseteq \overline{\varphi_j^0}$ (since S_0 was right-zero).
Fix $\xi < \omega_1$ and assume that we have constructed $\{\varphi_i^\alpha \mid \alpha < \xi\}$ (for $i < m$) increasing chains of length ξ consisting of countably generated filters such that for $\alpha < \xi$ the space $\varphi^\alpha := \bigcap_{i<m} \varphi_i^\alpha$ is an elementary FP-filter satisfying $\overline{\varphi_i^\alpha} \cdot \overline{\varphi_j^\alpha} \subseteq \overline{\varphi_j^\alpha}$. Define

$$\psi_i^\xi := \bigcup_{\alpha<\xi} \varphi_i^\alpha, \; \psi^\xi := \bigcap_{i<m} \psi_i^\xi.$$

Since we had chains, these are all filters and still $\overline{\psi^\xi} \leq G^*$. So let R be a minimal right ideal in $\overline{\psi^\xi}$.
Then $R \cap \overline{\psi_i^\xi} \neq \emptyset$ right ideal for $\overline{\psi_i^\xi}$.

(Obviously $R \cdot \overline{\psi_i^\xi} \subseteq R$ since R is a right ideal. On the other hand let $x \in R \subseteq \overline{\psi^\xi}$, then there are j, α with $x \in \overline{\varphi_j^\beta}$ for all $\xi > \beta \geq \alpha$, so in fact since multiplication behaves nicely and we consider chains we get $x \cdot \overline{\varphi_i^\beta} \subseteq \overline{\varphi_i^\beta}$ for all $\beta < \xi$, so $\emptyset \neq R \cdot \overline{\psi_i^\xi} \subseteq \overline{\psi_i^\xi}$.)

So there is an idempotent $p_i^\xi \in R \cap \overline{\psi_i^\xi}$. And then $S_\xi := \{p_i^\xi \mid i < m\}$ is a right-zero semigroup.
We then choose for $i < m$ some $F_i^\xi \in p_i^\xi$ with either $F_i^\xi \subseteq X_\xi$ or $F_i^\xi \subseteq G \setminus X_\xi$; let ϕ_i^ξ denote the filter generated by $\psi_i^\xi \cup \{F_i^\xi\}$. By Theorem C.6 there are countably generated filters $\varphi_i^\xi \subseteq \phi_i^\xi$ such that $\varphi^\xi := \bigcap_{i<m} \varphi_i^\xi$ is an elementary FP-filter with $\overline{\varphi_i^\xi} \cdot \overline{\varphi_j^\xi} \subseteq \overline{\varphi_j^\xi}$.
Now for $i < m$ we define $p_i := \bigcup_{\alpha<\omega_1} \varphi_i^\alpha$. This finishes the construction.
Then by construction p_i is an ultrafilter and $S := \{p_i \mid i < m\}$ is a right-zero FP-semigroup. ($\varphi_i^{\beta+1}$ decides X_β and by choice of S_β it remains a semigroup.)
For left-zero the proof is symmetric.
For chains it is still very similar, so we only shortly note some important points in that proof. Assuming we have increasing sequences $\{\varphi_i^\alpha \mid \alpha < \xi\}$ for $i < m$, such that φ^α (definded as above) is an elementary FP-filter but now with $\overline{\varphi_i^\alpha} \cdot \overline{\varphi_j^\alpha} \subseteq \overline{\varphi_{min\{i,j\}}^\alpha}$ (since in this case we take the S_ξ to be chains of idempotents). Define $\overline{\psi_i^\xi}$ as above and $\mu_i^\xi := \bigcap_{i \leq j < m} \overline{\psi_j^\xi}$. Then $\{\overline{\mu_i^\xi} \mid i < m\}$ a decreasing sequence of semigroups such that

$$K(\overline{\mu_i^\xi}) \cap \overline{\mu_{i+1}^\xi} = \emptyset \quad (i < m-1).$$

So we have an increasing chain of idempotents $S_\xi := \{p_i^\xi \mid i < m\}$ such that $p_i^\xi \in \overline{\psi_i^\xi}$.

□

Corollary C.17
Assume CH. Let B be a countable boolean group, then for every $m \in \omega$ there exist group topologies (on B) ρ_m, λ_m, κ_m such that the appropriate ultrafilter semigroups are left-zero, right-zero and chains of idempotents respectively.

Remark C.18 Theorem C.16 can be proven under *MA*, with increased technical work.

C.2. *A standard generalization*

In this section we shortly describe how to prove Remark C.18, i.e., how we can construct finite semigroups with intersection being an *FP*-filter using *MA* instead of *CH*.

All we additionally need is the following lemma, the *MA*-analogue of Lemma C.1.

Lemma C.19
Let $S := \{p_i \mid i < m\}$ HG in G^* and $(F_\alpha(p_i))_{\alpha<\kappa}$ a sequence in p_i for some $\kappa < 2^\omega$ with $F_0(p_i) \cap F_0(p_j) = \emptyset$ for $i \neq j$. We let $p(a) := p_i$, if there is some $\alpha < \kappa$ with $a \in F_\alpha(p_i)$. Assume $MA(\kappa)$. Then there exists $(a_n)_{n\in\omega}$ in G such that

1. $p(a_n) = p_{n \mod m}$ *(so $a_n \in F_\alpha(p_{n \mod m})$ for some α).*
2. $p(a_{n_0} \cdot \ldots \cdot a_{n_k}) = p(a_{n_0}) \cdot \ldots \cdot p(a_{n_k})$ *for $n_0 < \ldots < n_k$.*

Proof. Let $\mathbb{P} := \{(s, A_0, \ldots, A_{m-1}) \mid s \in \omega^{<\omega}, A_i \in p_i\}$ and partially order $\mathbb{P}$ by $(t, \vec{B}) \leq (s, \vec{A})$ iff $t \supseteq s$, $B \subseteq A$, $rng(t \setminus s) \subseteq B$ and $\forall x = t_{n_0} \cdot \ldots \cdot t_{n_k} \in FS_{le(s)}(t) : p(x) = p(t_{n_0}) \cdot \ldots \cdot p(t_{n_k})$.

Then clearly $\mathbb{P}$ is ccc, since any incompatible elements can only be incompatible in the first coordinate. We define

$$D_{F_\alpha(p_i)} := \{(s, \vec{A}) \mid A_i \subseteq F_\alpha(p_i)\}$$
$$D_n := \{(s, \vec{A}) \mid |s| \geq n\}.$$

Then $(D_{F_\alpha(p_i)})_{\alpha<\kappa}$ is dense in $\mathbb{P}$ for $i < m$ since p_i is a filter. Also $(D_n)_{n\in\omega}$ is dense – this is what the proof of Lemma C.2 tells us. So by $MA(\kappa)$ we can take a filter H on $\mathbb{P}$ intersecting all these sets; it is not difficult to see, that the first coordinates of H yield the required sequence. □

And finally a new version of Theorem C.6

Theorem C.20
Let $S = \{p_i \mid i < N\}$ be a finite subsemigroup of G^*, φ_i be κ generated filters for some $\kappa < 2^\omega$, such that $\varphi_i \subseteq p_i$ and $\overline{\varphi_i} \cap \overline{\varphi_j} = \emptyset$ for $i \neq j$.
Then there exist countably generated filters $\psi_i \supseteq \varphi_i$, such that $\psi := \bigcap_{i<m} \psi_i$ is an elementary FP-filter and additionally $\overline{\psi_i} \cdot \overline{\psi_j} \subseteq \overline{\psi_l}$ whenever $p_i \cdot p_j = p_l$.

Proof. The proof is the analogue of the proof of Theorem C.6 using the above Lemma instead of Lemma C.2. □

Now we can turn to our theorem.

Theorem C.21
Assume MA. Let φ be a free filter on G with base less than 2^ω and fix $m \in \omega$. Then there are three FP-filters whose associated subspaces are of cardinality m and a left-zero semigroup, a right-zero semigroup and a chain of idempotents respectively.

Proof. Proceed as in Theorem C.16. At uncountable limit stages simply apply the above Theorem instead of Theorem C.6 and proceed as before. □

Let us end this section with one further remark.

Remark C.22 Recall that $\mathbb{F}$ with symmetric difference is a countable Boolean group, which is a topological group with the natural Cantor topology on $\mathbb{F}$. In particular, converging to 0 means to be in $\delta\mathbb{F}$. Hence finite subsemigroups as in the above theorem can be found in $\mathbb{H}$.

Then in fact the (non-trivial) right-zero finite semigroups as in the above theorem consist of right maximal[1] but not strongly right maximal idempotents.

They cannot be strongly right maximal since it is a (non-trivial) right-zero semigroup, i.e. $x + p = p$ for all x and p from the semigroup.

Right-maximality follows from the additional fact that FS-semigroups are "nearly prime", cf. Corollary 5.8: Any $\geq_R$ idempotent for a given element of the semigroup must contain all of the FS-sets, hence it is a member of the finite semigroup, which in a right-zero semigroup immediately implies $\leq_R$.

In particular, assuming MA we can find right maximal idempotent ultrafilters in $\beta\mathbb{N}$ which are not strongly right maximal.

[1] p is right maximal, if $p \leq_R x$ (i.e., $x + p = p$), implies $x \leq_R p$ (i.e., $p + x = x$).

List of theorems

Index

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Zusammenfassung

Gegenstand der vorliegenden Arbeit sind Filter auf Halbgruppen und deren Eigenschaften bezüglich Algebra in der Stone-Čech Kompaktifizierung.

Die Menge der Ultrafilter auf einer Menge S kann mit βS identifiziert werden, der Stone-Čech Kompaktifizierung von S mit diskreter Topologie. Ist S eine Halbgruppe, so lässt sich eine assoziative Operation auf βS definieren, die die ursprüngliche Operation auf S fortsetzt. Dies ist dergestalt möglich, dass die Operation stetig ist, solange das rechtsseitge Element fixiert ist; nach dem Lemma von Ellis-Numakura gibt es daher idempotente Elemente in βS, d.h. *idempotente Ultrafilter*. Idempotente Ultrafilter stehen im Zentrum des Forschungsgebiets der Algebra in der Stone-Čech Kompaktifizierung vor allem, weil sie elegante Beweise für ramseytheoretische Resultate wie den Satz von Hindman, den Satz von Hales-Jewett und den Satz von zentralen Mengen ermöglichen.

Obwohl Ultrafilter für Mengentheoretiker von natürlichem Interesse sind, gibt es nur wenige Unabhängigkeitsresultate zu idempotenten Ultrafiltern. Eine der grundlegenden Fragen dieser Dissertation besteht deshalb darin, ob dies Zufall ist oder nicht: Sind jene Ergebnisse isoliert oder gibt es vielfältige mengentheoretische Konstruktionen analog zu den unterschiedlichen Forcingkonstruktionen reeller Zahlen.

Im ersten Teil dieser Arbeit geben wir eine positive Antwort auf diese Frage. In Kapitel 3 entwickeln wir eine allgemeine Herangehensweise, um idempotente Ultrafilter mit Hilfe der Forcingmethode zu adjungieren. Außerdem sind wir in der Lage, die Forcingkonstruktionen zu unterscheiden, indem wir sie mit bekannten Konstruktionen für mengentheoretisch interessante Ultrafilter assoziieren. Zu diesem Zweck studieren wir in Kapitel 2 den Begriff des *idempotenten Filters*. Dieser Begriff basiert auf der natürlichen Verallgemeinerung der Multiplikation von Ultrafiltern zu einer Multiplikation von Filtern.

Idempotente Filter finden sich implizit in vielerlei Anwendungen des Gebietes. Neben ihrer Nützlichkeit für die Forcingkonstruktionen besitzen idempotente Filter eine elegante Theorie, die wir in Kapitel 2 entwickeln. So induzieren idempotente Filter zum Beispiel Halbgruppen mit sehr starken Abschlusseigenschaften und sind gleichzeitig eine Verallgemeinerung des Konzepts der partiellen Halbgruppe. Die Theorie der idempotenten Filter emöglicht uns außerdem eine vereinfachten Beweis einer Verallgemeinerung des Satz von Zelenyuk über endliche Gruppen in βS zu formulieren.

Da wir die obige Frage positiv beantworten, stellt sich automatisch eine weitere: Welche kombinatorischen und algebraischen Eigenschaften können unsere Forcingkonstruktionen besitzen? Diese Frage motiviert die Analyse der Mischung von mengentheoretischen und kombinatorischen Eigenschaften der sogenannten Union Ultrafilter einerseits, sowie der algebraischen Eigenschaften des verwandten Begriffs der summierbaren Ultrafilter andererseits.

Das Konzept der Union Ultrafilter wurde von Andreas Blass 1987 entwickelt; in seiner Arbeit mit Neil Hindman wurde die Äquivalenz von Union Ultrafiltern mit den schon bekannten summierbaren Ultrafiltern sowie die Unabhängigkeit ihrer Existenz etabliert.

In Kapitel 4 beantworten wir die offene Frage negativ, ob ein Union Ultrafilter schon Ordered Union ist, falls bestimmte Bilder des Ultrafilters Ramsey Ultrafilter sind. In Kapitel 5 untersuchen wir die algebraischen Eigenschaften der summierbaren Ultrafilter. Insbesondere zeigen wir, dass die „special“ Bedingung von allen Summierbaren erfüllt wird, und verallgemeinern mit Hilfe dieses Resultats einen Satz von Hindman und Strauss über summierbare Ultrafilter als Summen.

Lebenslauf

Aus datenschutzrechtlichen Gründen enthält die digitale Kopie keinen Lebenslauf.

Eidestattliche Erklärung

Hiermit erkläre ich an Eides statt, die vorliegende Arbeit selbstständig verfasst zu haben. Alle verwendeten Hilfsmittel sind aufgeführt. Des Weiteren versichere ich, dass diese Arbeit nicht in dieser oder ähnlicher Form an einer weiteren Universität im Rahmen eines Prüfungsverfahrens eingereicht wurde.

Peter Krautzberger.