Chapter 1

Vector Measures

In this chapter we present a survey of known results about vector measures. For the convenience of the readers some of the results are given with proofs, but neither results nor proofs pretend to be ours. The only exception is the material of Section 1.5 that covers the results of our joint paper with V. M. Kadets [18].

1.1 Elementary properties

The results of this section may be found in [7].

Definition 1.1.1 *A function μ from a field F of subsets of a set Ω to a Banach space X is called a finitely additive vector measure, or simply a vector measure, if whenever A_1 and A_2 are disjoint members of F then $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$.*

If in addition $\mu\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = \sum\limits_{n=1}^{\infty} \mu(A_n)$ in the norm topology of X for all sequences (A_n) of pairwise disjoint members of F such that $\bigcup\limits_{n=1}^{\infty} A_n \in F$, then μ is termed a countably additive vector measure, or simply μ is countably additive.

Example 1.1.2 *Finitely additive and countably additive vector measures.* Let Σ be a σ-field of subsets of a set Ω, λ be a nonnegative countably additive measure on Σ. Consider $1 \leq p \leq \infty$. Define

$\mu_p : \Sigma \to L_p(\Omega, \Sigma, \lambda)$ by the rule $\mu_p(A) = \chi_A$ for each set $A \in \Sigma$ (χ_A denotes the characteristic function of A). It is easy to see that μ_p is a vector measure, which is countably additive in the case $1 \le p < \infty$ and fails to be countably additive in the case $p = \infty$.

Definition 1.1.3 *A vector measure $\mu : F \to X$ is said to be nonatomic if for every set $A \in F$ with $\mu(A) \neq 0$ there exist $A_1, A_2 \subset F \setminus \emptyset$ such that $A_1 \cup A_2 = A$ and $\mu(A_i) \neq 0$ $(i = 1, 2)$.*

Definition 1.1.4 *Let $\mu : F \to X$ be a vector measure. The variation of μ is the nonnegative function $|\mu|$ whose value on a set $A \in F$ is given by*

$$|\mu|(A) = \sup_{\pi} \sum_{B \in \pi} \|\mu(B)\|,$$

where the supremum is taken over all partitions π of A into a finite number of pairwise disjoint members of F. If $|\mu|(\Omega) < \infty$, then μ will be called a measure of bounded variation.

The semivariation of μ is the extended nonnegative function $\|\mu\|$ whose value on a set $A \in F$ is given by

$$\|\mu\|(A) = \sup\{|x^*\mu|(A) : x^* \in X^*, \|x^*\| \le 1\},$$

where $|x^*\mu|$ is the variation of the real-valued measure $x^*\mu$.

Example 1.1.5 *A measure of bounded variation.* Let μ_1 be the measure discussed in Example 1.1.2. Since $\|\mu_1(A)\| \le \lambda(A)$, it is plain that $|\mu_1|(A) \le \lambda(A)$, so that μ_1 is of bounded variation.

The next proposition presents two basic facts about the semivariation of a vector measure.

Proposition 1.1.6 *Let $\mu : F \to X$ be a vector measure. Then for $A \in F$ one has*

(a) $\|\mu\|(A) = \sup_{\{\varepsilon_n\}, \pi} \sum_{B_n \in \pi} \|\varepsilon_n \mu(B_n)\|$, *where the supremum is taken over all partitions π of A into a finite number of pairwise disjoint members of F and over all finite collections $\{\varepsilon_n\}$ satisfying $|\varepsilon_n| \le 1$;*

(b) $\sup\{\|\mu(B)\|\colon\ A \supseteq B \in F\} \leq \|\mu\|(A) \leq 2\sup\{\|\mu(B)\|\colon\ A \supseteq B \in F\}$. *Consequently a vector measure is of bounded semivariation on* Ω *if and only if its range is bounded in* X.

It is easy to define the integral of a bounded measurable function with respect to a bounded vector measure. To this end, let F be a field of subsets of a set Ω and $\mu : F \to X$ be a bounded vector measure. If f is a scalar-valued simple function, say $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, where α_i are nonzero scalars and $A_1, ..., A_n$ are pairwise disjoint members of F, define $T_\mu(f) = \sum_{i=1}^{n} \alpha_i \mu(A_i)$. It is easy to show that this formula defines a linear operator T_μ from the space of simple functions of the above form into X. Moreover, if f is as above and $\beta = \sup\{|f(\omega)| : \omega \in \Omega\}$, then

$$\|T_\mu(f)\| = \left\|\sum_{i=1}^{n} \alpha_i \mu(A_i)\right\| = \beta \left\|\sum_{i=1}^{n} \frac{\alpha_i}{\beta} \mu(A_i)\right\| \leq \beta \|\mu\|(\Omega).$$

by Proposition 1.1.6 (a). Thus T_μ is a continuous linear operator, which acts on the space of simple functions on F, equipped with the supremum norm. Another look at the Proposition 1.1.6 (a) and the above calculations shows that

$$\|T_\mu\| = \|\mu\|(\Omega) \tag{1.1}$$

Then T_μ has a unique continuous linear extension, still denoted by T_μ, to $B(\mu)$, the space of all scalar-valued functions on Ω that are uniform limits of simple functions modeled on F. (Note that in the case F is a σ-field, $B(\mu)$ is precisely the familiar space of bounded F-measurable scalar-valued functions defined on Ω).

This discussion allows us to make

Definition 1.1.7 *Let* F *be a field of subsets of a set* Ω *and* $\mu : F \to X$ *be a bounded vector measure. For each* $f \in B(\mu)$, *the integral* $\int f d\mu$ *is defined by*

$$\int f d\mu = T_\mu(f),$$

where T_μ *is as above.*

It is easy to see that this integral is linear in f (and also in μ) and satisfies

$$\left\| \int f d\mu \right\| \leq \|f\|_{\infty} \|\mu\| (\Omega) .$$

Moreover, if $x^* \in X^*$, then $x^* \int f d\mu = \int f dx^* \mu$; indeed, for simple functions this equality is trivial and density of simple functions in $B(\mu)$ proves the identity for all $f \in B(\mu)$.

Definition 1.1.8 *A family $\{\mu_\tau : F \to X \mid \tau \in T\}$ of countably additive vector measures is said to be uniformly countably additive whenever for any sequence (A_n) of pairwise disjoint members of F, the series $\sum_{n=1}^{\infty} \mu_\tau(A_n)$ converges in norm uniformly in $\tau \in T$.*

There are many alternative and useful formulations of countable additivity. We present two of them.

Proposition 1.1.9 *Any one of the following statements about a vector measure μ defined on a field F implies all the others.*

(i) μ is countably additive.

(ii) The set $\{x^*\mu : x^* \in X^*, \|x^*\| \leq 1\}$ is uniformly countably additive.

(iii) The set $\{|x^*\mu| : x^* \in X^*, \|x^*\| \leq 1\}$ is uniformly countably additive.

1.2 Bartle-Dunford-Schwartz' theorem

The results of this section may be found in [7].

Definition 1.2.1 *Let F be a field of subsets of a set Ω, $\mu : F \to X$ be a vector measure, and λ be a finite nonnegative real-valued measure on F. If $\lim_{\lambda(A) \to 0} \mu(A) = 0$, then μ is called λ-continuous and this is denoted by $\mu \ll \lambda$. Sometimes we will say μ is absolutely continuous with respect to λ.*

It should be noted that writing $\mu \ll \lambda$ is not the same as saying μ vanishes on λ-null sets unless both λ and μ are countably additive and defined on a σ-field.

Theorem 1.2.2 (Pettis) *Let Σ be a σ-field of subsets of a set Ω, $\mu : \Sigma \to X$ be a countably additive vector measure, and λ be a finite nonnegative real-valued measure on Σ. Then μ is λ-continuous if and only if μ vanishes on sets of λ-measure zero.*

Proof. To prove the sufficiency, suppose μ vanishes on sets of λ-measure zero, but $\overline{\lim\limits_{\lambda(A)\to 0}}\ \mu(A) > 0$. Then there exist an $\varepsilon > 0$ and a sequence (A_n) in Σ, such that

$$\|\mu(A_n)\| \geq \varepsilon \text{ and } \lambda(A_n) \leq 2^{-n}$$

for all n. For each n select an $x_n^* \in X^*$ such that

$$\|x_n^*\| \leq 1 \text{ and } |x_n^* \mu(A_n)| \geq \frac{\varepsilon}{2}.$$

As μ is countably additive on Σ, then in accordance with Proposition 1.1.9 the set $\{|x^*\mu| : x^* \in X^*, \|x^*\| \leq 1\}$ is uniformly countably additive. Now set $B_n = \bigcup\limits_{i=n}^{\infty} A_i$. Evidently $\lambda\left(\bigcap\limits_{n=1}^{\infty} B_n\right) = 0$. Consequently μ vanishes on every set $D \in \Sigma$ that is contained in $\bigcap\limits_{n=1}^{\infty} B_n = B$. It follows that $|x^*\mu|(B) = 0$.

Let $C_1 = \Omega \setminus B_1$ and $C_{n+1} = B_n \setminus B_{n+1}$ for all n. Then (C_n) constitutes a sequence of pairwise disjoint members of Σ for which

$$B_{n-1} \setminus B = \bigcup_{i=n}^{\infty} C_i.$$

Also, since $|x_n^*\mu|(B) = 0$ for all n, one has

$$\lim_m |x_n^*\mu|(B_m) = \lim_m |x_n^*\mu|\left(\bigcup_{i=m}^{\infty} C_i\right) = \lim_m \sum_{i=m}^{\infty} |x_n^*\mu|(C_i) = 0$$

uniformly in n, by the uniform countable additivity of the family $\{|x_n^*\mu|\}$. But now

$$\left|x_{n-1}^*\mu\right|(B_{n-1}) \geq \left|x_{n-1}^*\mu\right|(A_{n-1}) \geq \left|x_{n-1}^*\mu(A_{n-1})\right| \geq \frac{\varepsilon}{2}.$$

This contradicts the last calculation and proves the sufficiency; the converse is transparent. ■

The following theorem is central to the theory of vector measures.

Theorem 1.2.3 (**Bartle-Dunford-Schwartz**) *Let μ be a bounded countably additive vector measure defined on a σ-field Σ. Then there exists a nonnegative real-valued countably additive measure λ on Σ such that μ is absolutely continuous with respect to λ. If μ is nonatomic, then λ can be chosen nonatomic. (λ can be chosen such that $0 \leq \lambda(A) \leq \|\mu\|(A)$ for all $A \in \Sigma$.)*

Corollary 1.2.4 *Let Σ be a σ-field, $\mu : \Sigma \to X$ be a bounded countably additive vector measure, λ be a measure from Theorem 1.2.3. Then there exist an operator $T : L_\infty(\lambda) \to X$, which is continuous for the weak* topology on $L_\infty(\lambda)$ and weak topology on X such that $T\chi_A = \mu(A)$ for any $A \in \Sigma$.*

Proof. Define $T : L_\infty(\lambda) \to X$ by $Tf = \int_\Omega f d\mu$. Then for each $x^* \in X^*$, one has

$$x^*Tf = \int f dx^*\mu = \int f \frac{dx^*\mu}{d\lambda} d\lambda,$$

where $dx^*\mu/d\lambda = g_{x^*} \in L_1(\lambda)$ is the Radon-Nikodym derivative of $x^*\mu$ with respect to λ. If (f_α) is a net in $L_\infty(\lambda)$ converging weak* to f_0, then for each $x^* \in X^*$,

$$\lim_\alpha x^*Tf_\alpha = \lim_\alpha \int f_\alpha g_{x^*} d\lambda = \int f_0 g_{x^*} d\lambda = x^*Tf_0,$$

i.e., (Tf_α) converges weakly to Tf_0. Hence T is a weak* to weak continuous linear operator. ■

Corollary 1.2.5 (**Bartle-Dunford-Schwartz**) *Let μ be a bounded countably additive vector measure on a σ-field Σ. Then μ has a relatively weakly compact range.*

Proof. Let T be the operator from Corollary 1.2.4. It follows that T maps the weak* compact set $\{f \in L_\infty(\lambda) : \|f\|_\infty \leq 1\}$ onto a weakly compact set $R \subseteq X$. But now

$$\{\mu(A) : A \in \Sigma\} = \{T(x_A) : A \in \Sigma\} \subseteq \{T(f) : \|f\|_\infty \leq 1\} \subseteq R,$$

and the proof is complete. ■

Corollary 1.2.6 *Let μ and λ be as in Corollary 1.2.4. Then*

$$\overline{\text{co}}\left(\mu\left(\Sigma\right)\right)=\left\{\int_{\Omega} f d\mu : 0\leq f\leq 1, f\in L_{\infty}\left(\lambda\right)\right\}.$$

Proof. Let

$$U=\left\{f\in L_{\infty}\left(\lambda\right):0\leq f\leq 1\right\} \text{ and } V=\left\{\int_{\Omega} f d\mu : f\in U\right\}.$$

Note that U is weak* compact and $\{1_A : A\in\Sigma\}=\operatorname{ex}U$ (where $\operatorname{ex}U$ is the set of the extreme points of U). By the Krein-Milman theorem $U=\overline{\text{co}}^{w^*}(\operatorname{ex}U)$. Then a glance at Corollary 1.2.4 proves that $V=\overline{\text{co}}\left(\mu\left(\Sigma\right)\right)$. ■

The following theorem shows that the measure λ from the Bartle-Dunford-Schwartz theorem may be taken of the form $|x^*\mu|$ for a certain $x^*\in X^*$.

Theorem 1.2.7 *(***Rybakov***) Let $\mu:\Sigma\to X$ be a countably additive vector measure. Then there is $x^*\in X^*$ such that $\mu\ll|x^*\mu|$.*

1.3 Lyapunov's convexity theorem

The results of this section may be found in [7].

One of the most important results in the theory of vector measures is the Lyapunov convexity theorem which states that the range of a nonatomic vector measures with values in a finite dimensional space is compact and convex. As it was mentioned above, this theorem fails in every infinite dimensional Banach space.

Example 1.3.1 *(***Uhl***) A nonatomic vector measure of bounded variation whose range is closed but nonconvex and noncompact.*

Let Σ be the Borel sets in $[0,1]$ and λ be Lebesgue measure. Define $\mu:\Sigma\to L_1\left(\lambda\right)$ by $\mu\left(A\right)=\chi_A$. If $\Pi\subset\Sigma$ is a partition of $[0,1]$, it is evident that $\sum_{A\in\Pi}\left\|\mu\left(A\right)\right\|_1=\sum_{A\in\Pi}\lambda\left(A\right)=1$. Since every L^1-convergent sequence contains an almost everywhere convergent subsequence, $\mu\left(\Sigma\right)$

is closed in $L_1(\lambda)$. To see that $\mu(\Sigma)$ is not a convex set, note that $\frac{1}{2}\chi_{[0,1]} \in \operatorname{co}\mu(\Sigma)$ while if $A \in \Sigma$

$$\left\|\mu(A) - \frac{1}{2}\chi_{[0,1]}\right\|_1 = [\mu([0,1] \setminus E) + \mu(E)]/2 = \frac{1}{2}.$$

To see that $\mu(\Sigma)$ is not compact let $A_n = \{t \in [0,1] : \sin(2^n \pi t) > 0\}$ for each positive integer n. A brief computation shows that

$$\|\mu(A_n) - \mu(A_m)\| = \frac{1}{2}$$

for $m \neq n$. Hence $\mu(\Sigma)$ is not compact.

Example 1.3.1 suggests that nonatomicity may not be a very strong property of vector measures from the point of view of the Lyapunov theorem in the infinite dimensional context. Let us attempt to understand what nonatomicity means.

Let $\mu : \Sigma \to \mathbf{R}^n$ have the form

$$\mu(A) = (\lambda_1(A), \ldots, \lambda_n(A)), A \in \Sigma,$$

where each λ_i is a countably additive finite signed scalar measure on Σ. Set $\lambda(A) = \sum_{k=1}^{n} |\lambda_k|(A)$. Then $\|\mu\|(A) \to 0$ if and only if $\lambda(A) \to 0$. Now if μ is nonatomic, then λ is nonatomic. Consequently, if $A \in \Sigma$ and $\lambda(A) > 0$ the mapping on the infinite dimensional subspace $\{f\chi_A : f \in L_\infty(\lambda)\}$ that takes f into $\int_A f d\mu$ is never one-to-one. We will see that in the infinite dimensional case, this latter condition is precisely what is needed to make the Lyapunov theorem work. Throughout, Σ is a σ-field of subsets of Ω and X is a Banach space.

Theorem 1.3.2 *(***Knowles; Lyapunov convexity theorem in the weak topology***) Let $\mu : \Sigma \to X$ be a countably additive vector measure and λ be a finite nonnegative countably additive measure on Σ from Theorem 1.2.3. The following statements are equivalent.*

(i) If $A \in \Sigma$ and $\lambda(A) > 0$, then the operator $f \mapsto \int_A f d\mu$ on $L_\infty(\lambda)$ is not one-to-one on the subspace of functions in $L_\infty(\lambda)$ vanishing off A.

(*ii*) *For each* $A \in \Sigma$, $\{\mu(B \cap A) : B \in \Sigma\}$ *is a weakly compact convex set in* X.

(*iii*) *If* $0 \neq f \in L_\infty(\lambda)$, *there exists a function* $g \in L_\infty(\lambda)$ *such that* $\|fg\|_\infty > 0$ *but* $\int_\Omega fg d\mu = 0$.

Proof. Note that (*iii*) implies (*i*). To show that (*i*) implies (*iii*) let $f \in L_\infty(\lambda)$, $\|f\|_\infty > 0$. Then there is an $\varepsilon > 0$ and $A \in \Sigma$ such that $|f\chi_A| > \varepsilon$ and $\mu(A) > 0$. According to (*i*), there is an $h \in L_\infty(\lambda)$ such that $\|h\chi_A\|_\infty > 0$ and $\int_A h d\mu = 0$. Set $g = h/f$ on A and $g = 0$ off A. Then $fg = h$ on A and so $\|fg\chi_A\|_\infty > 0$. Also $\int_\Omega fg d\mu = \int_A h d\mu = 0$. Hence (*iii*) holds.

To check that (*ii*) implies (*i*), suppose (*i*) is false. Without loss of generality, we can assume that $A = \Omega$, so that the operator $f \mapsto \int_A f d\mu$ ($f \in L_\infty(\lambda)$) is one-to-one. It is not difficult to see that this means that $\mu(\Sigma) = \left\{\int_\Omega \chi_A d\mu : A \in \Sigma\right\}$ is a proper subset of the set $V = \left\{\int_\Omega f d\mu : f \in L_\infty(\lambda), 0 \leq f \leq 1\right\}$. But by Corollary 1.2.6, $V = \overline{\text{co}}(\mu(\Sigma))$; thus $\mu(\Sigma)$ cannot be both closed and convex. Hence (*ii*) is false. To complete the proof, we shall verify that (*iii*) implies (*ii*) again. It is enough to show that $\mu(\Sigma)$ is convex and weakly compact. Let $f \in L_\infty(\lambda)$ be such that $0 \leq f \leq 1$. Since by Corollary 1.2.4 the operator $\int_\Omega (\cdot) d\mu$ is continuous for the weak* topology on $L_\infty(\lambda)$ and the weak topology on X, the set

$$H = \left\{g \in L_\infty(\lambda) : 0 \leq g \leq 1, \int_\Omega f d\mu = \int_\Omega g d\mu\right\}$$

is a weak* compact convex set in $L_\infty(\lambda)$ and therefore has extreme points. If we can show that the extreme points of H, denoted by $\text{ext}(H)$, are all in $\{\chi_A : A \in \Sigma\}$ it will follow that there exists $A \in \Sigma$ such that $\mu(A) = \int_\Omega f d\mu$. Then an appeal to Corollary 1.2.6 will yield the equalities

$$\overline{\text{co}}\left(\mu\left(\Sigma\right)\right) = \left\{\int_{\Omega} f d\mu : f \in L_{\infty}\left(\lambda\right), 0 \leq f \leq 1\right\} = \mu\left(\Sigma\right)$$

and prove that $\mu\left(\Sigma\right)$ is weakly compact and convex.

To this end, suppose $f_0 \in \text{ext}\left(H\right)$ but $\|f_0 - \chi_A\| > 0$ for each $A \in \Sigma$. A simple calculation shows that there exists $f_1 \in L_{\infty}\left(\lambda\right)$ with $\|f_1\|_{\infty} > 0$ such that $0 \leq f_0 \pm f_1 \leq 1$. An appeal to (iii) gives us a $g_1 \in L_{\infty}\left(\lambda\right)$, that may be selected with $\|g_1\|_{\infty} \leq 1$ such that $\|f_1 g_1\|_{\infty} > 0$ but $\int_{\Omega} f_1 g_1 d\mu = 0$. Then $f_0 \pm f_1 g_1 \in H$; thus f_0 is not an extreme point of H. This completes the proof. ■

Note that in light of the remarks before Theorem 1.3.2 the classical Lyapunov theorem is contained in this theorem; in fact, the above proof of $(iii) \Rightarrow (ii)$ in the case $X = \mathbf{R}^n$ is Lindenstrauss's argument [23] to prove the Lyapunov theorem.

Theorem 1.3.3 *(***Lyapunov***) Let Σ be a σ-field of subsets of Ω, X be a finite dimensional Banach space and $\mu : \Sigma \to X$ be a countably additive vector measure. If μ is nonatomic, then the range of μ is a compact convex subset of X.*

Corollary 1.3.4 *Let Σ be the σ-field of Borel subsets of $[0,1]$. If X is an infinite dimensional Banach space, then there is a countably additive vector measure of bounded variation $\mu : \Sigma \to X$ and a set $A \in \Sigma$ such that $\{\mu\left(B \cap A\right) : B \in \Sigma\}$ is not a weakly compact convex set in X.*

Proof. Let λ be Lebesgue measure on Σ. Select a sequence (f_n) in $L_1\left(\lambda\right)$ such that $\|f_n\|_1 = 1$ and the only $g \in L_{\infty}\left(\lambda\right)$ with $\int_{[0,1]} f_n g d\lambda = 0$ for all n is $g = 0$. Choose a sequence of pairs (x_n^*, x_n) such that $x_n^* \in X^*$, $x_n \in X$, $x_m^*\left(x_n\right) = 0$ if $m \neq n$ and $x_n^*\left(x_n\right) = 1$ for all n. Define $T : L_{\infty}\left(\lambda\right) \to X$ by

$$T\left(g\right) = \sum_{n=1}^{\infty} x_n \left(2^n \|x_n\|^{-1}\right) \int_{[0,1]} f_n g d\lambda.$$

If $T(g) = 0$, then $x_n^* T(g) = \left(2^n \|x_n\|^{-1}\right) \int_{[0,1]} f_n g d\lambda = 0$ for all n. Hence T is one-to-one on $L_\infty(\lambda)$. To produce the advertised measure, define $\mu(A) = T(\chi_A)$ for $A \in \Sigma$. It is not hard to see that μ is countably additive and $T(.) = \int_{[0,1]} (.)\, d\mu$. Hence μ violates (i) of Theorem 1.3.2. By Theorem 1.3.2, μ is as advertised. ■

1.4 l_p-valued measures

The main results of this section are contained in [17].

Another way of generalizing the Lyapunov convexity theorem is considering the closure of the range of the vector measure in the norm topology.

Further by an X-valued vector measure we mean a countably additive vector measure μ defined on a σ-field Σ of subsets of a set Ω.

Definition 1.4.1 *A nonatomic X-valued vector measure μ is said to be Lyapunov if the closure of its range is convex.*

In 1969 Uhl [7] proved the following theorem.

Theorem 1.4.2 *Let Σ be a σ-field of subsets of Ω and suppose X has the Radon-Nikodym property. If $\mu : \Sigma \to X$ is a nonatomic and countably additive measure of bounded variation, then μ is a Lyapunov measure.*

If we waive the requirement that the variation be bounded, then Uhl's theorem no longer holds, in particular there exists a nonatomic measure with values in a Hilbert space such that the closure of its range is not convex. (It is enough to consider the measure μ_2 from Example 1.1.2.)

In 1991 V. M. Kadets and M. M. Popov [16] obtained the following strengthening of this result.

Theorem 1.4.3 *The following conditions are equivalent for the space X:*

1. *Each nonatomic X-valued measure of bounded variation is a Lyapunov measure;*

2. *There does not exist a sign-embedding of the space $L_1[0,1]$ in X.*

Definition 1.4.4 *A Banach space X is said to have the Lyapunov property if every nonatomic countably additive vector measure with values in X is a Lyapunov measure.*

As already mentioned above, an infinite dimensional Hilbert space does not have the Lyapunov property. Consequently, all spaces containing isomorphic copies of the space l_2 do not have the Lyapunov property; in particular, $L_p[0,1]$, $C[0,1]$, l_∞ do not have the Lyapunov property. Nevertheless, spaces having the Lyapunov property exist. In 1992 V. M. Kadets and G. Schechtman proved that l_p ($1 \le p < \infty$, $p \neq 2$) and c_0 have the Lyapunov property [17]. Before proving this result we need some lemmas.

Lemma 1.4.5 *If the measure μ is not a Lyapunov measure, then there exist a number $\varepsilon > 0$ and a set $V \in \Sigma$ such that $\left\|\mu(U) - \frac{1}{2}\mu(V)\right\| \ge \varepsilon$ for any subset $U \in \Sigma|_V$ (here and below, $\Sigma|_V$ denotes the family of elements of the σ-algebra Σ that are subsets of the set V : the "restriction of Σ to V").*

Proof. Assume the contrary, i.e., assume that for any $\varepsilon > 0$ and any $V \in \Sigma$ there exists a $U \subset V$ such that $\left\|\mu(U) - \frac{1}{2}\mu(V)\right\| < \varepsilon$. Let x and y be arbitrary elements of $\mu(\Sigma)$, $x = \mu(A)$, $y = \mu(B)$. We prove that $(x+y)/2 \in \overline{\mu(\Sigma)}$. This means that μ is a Lyapunov measure, contrary to assumption. We choose $U_1^\varepsilon \subset A \setminus B$ and $U_2^\varepsilon \subset B \setminus A$ such that

$$\left\|\frac{1}{2}\mu(A \setminus B) - \mu(U_1^\varepsilon)\right\| < \varepsilon, \qquad \left\|\frac{1}{2}\mu(B \setminus A) - \mu(U_2^\varepsilon)\right\| < \varepsilon.$$

Then $\left\|\frac{1}{2}(x+y) - \mu(W^\varepsilon)\right\| \le 2\varepsilon$ for the set $W^\varepsilon = U_1^\varepsilon \cup U_2^\varepsilon \cup (A \cap B)$, i.e., $(x+y)/2$ can be approximated by values of the measure to any degree of accuracy. The lemma is proved. ∎

Lemma 1.4.6 *Suppose that λ is a nonatomic positive numerical measure on Σ, and $\mu : \Sigma \to X$ is a vector valued measure that is absolutely continuous with respect to λ but is not a Lyapunov measure. Then there exist a number $a > 0$ and a subset $\Omega_1 \in \Sigma$ with $\lambda(\Omega_1) > 0$ having the following property : for any subsets $A_1, A \in \Sigma|_{\Omega_1}, A_1 \subset A$,*

$$\left\| \mu(A_1) - \frac{1}{2}\mu(A) \right\| \geq a\lambda(A). \tag{1.2}$$

Proof. Suppose that $\varepsilon > 0$ and $V \in \Sigma$ are as in the statement of Lemma1.4.5. Take $a = \varepsilon/\lambda(V)$ as the required number a. A set $A \in \Sigma|_V$ is said to be a *narrow* if there exists a subset $A_1 \subset A$ such that the inequality converse to (1.2) holds. By construction, V is not a narrow set. It suffices for us to prove the existence of a set $\Omega_1 \in \Sigma|_V$ with $\lambda(\Omega_1) > 0$ that does not have narrow subsets. Assume the contrary, i.e., assume that the quantity

$$\operatorname{narr}(A) = \sup \{\lambda(A_1) : A_1 \in \Sigma|_A, \ A_1 \text{ is narrow}\}$$

is positive for any $A \in \Sigma|_V$ with $\lambda(A) \neq 0$. We use induction to construct a chain $V = V_0 \supset V_1 \supset V_2 \supset ...$ of sets such that at each step the set $W_n = V_n \setminus V_{n+1}$ is narrow and "large": $\lambda(W_n) \geq \frac{1}{2}\operatorname{narr}(V_n)$. Then $W = \bigcup_{n=0}^{\infty} V_n$ is a narrow set. Consequently, since V is not narrow, $\lambda(\Omega_1) \neq 0$ holds for the set $\Omega_1 = V \setminus W = \bigcap_{n=0}^{\infty} V_n$. At the same time, the sets W_n are disjoint and $\sum_{n=1}^{\infty} \lambda(W_n) = \lambda(W) < \infty$; therefore, $\lambda(W_n) \to 0$. We get that $\operatorname{narr}(\Omega_1) \leq \operatorname{narr}(V_n) \leq 2\lambda(W_n) \to 0$; i.e., $\operatorname{narr}(\Omega_1) = 0$, although $\lambda(\Omega_1) > 0$. This contradiction proves the lemma. ■

Lemma 1.4.7 *The following two conditions are equivalent for a Banach space X:*

(i) X does not have the Lyapunov property.

(ii) There exist a triple $(\Omega, \Sigma, \lambda)$ with a nonatomic numerical measure λ on Σ and an operator $T : L_\infty(\Omega, \Sigma, \lambda) \to X$ such that

(a) T is continuous for the weak topology on $L_\infty(\Omega, \Sigma, \lambda)$ and the weak topology on X, and*

(b) *there exists an* $\varepsilon > 0$ *such that* $\|Tf\| \geq \varepsilon\lambda(\operatorname{supp} f)$ *for any "sign"* $f \in L_\infty(\Omega, \Sigma, \lambda)$*, i.e., any function taking only the values* 0 *and* ± 1.

Proof. We first show that (i) implies (ii). Let $(\Omega, \Sigma, \lambda)$ be a triple $(\Omega_1, \Sigma|_{\Omega_1}, \lambda)$ as in Lemma 1.4.6, we get the existence of a measure $\mu : \Sigma \to X$ and a number $a > 0$ such that the inequality (1.2) holds for any $A_1 \subset A \in \Sigma$. According to Corollary 1.2.4, there exists a weak*-weak continuous operator $T : L_\infty(\Omega, \Sigma, \lambda) \to X$ connected with the measure μ by the relation

$$T(\chi_A) = \mu(A), \qquad A \in \Sigma. \tag{1.3}$$

It remains to verify the condition (b). Suppose that f is a "sign"; then $f = \chi_{A_1} - \chi_{A_2}$, where $A_1, A_2 \in \Sigma$ and $A_1 \cap A_2 = \emptyset$. Let $A = A_1 \cup A_2$ and $\varepsilon = 2a$. We get

$$\begin{aligned} \|Tf\| &= \|\mu(A_1) - \mu(A_2)\| = 2\left\|\mu(A_1) - \tfrac{1}{2}\mu(A)\right\| \\ &\geq 2a\lambda(A) = \varepsilon \cdot \lambda(\operatorname{supp} f). \end{aligned}$$

(ii) implies (i). Suppose that the operator T satisfies the conditions (a) and (b). The measure μ is given by (1.3). It is easy to verify that μ is nonatomic and not a Lyapunov measure. The lemma is proved. ■

Remark 1.4.8 *Since* $\mu : \Sigma \to X$, $\mu(A) = T\chi_A$ *is absolutely continuous with respect to* λ*, the operator* T *will also have the following property:*

(c) *for every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that* $\|Tf\| \leq \varepsilon\|f\|$ *for any* $f \in L_\infty(\Omega, \Sigma, \lambda)$ *with* $\lambda(\operatorname{supp} f) \leq \delta$.

Let us introduce some additional symbols for the sequel. Let $r_i = r_i(\omega)$ be a sequence of independent random variables taking values $+1$ and -1 with probability $\frac{1}{2}$. Fix a number $N \in \mathbf{N}$ and denote the random variable

$$T(\omega) = \inf\left\{ j : \left| \sum_{i=1}^{j} r_i(\omega) \right| \geq \sqrt{N} \right\}.$$

Further define the stopped martingale $\sum_{i=1}^{k} s_i$ by the rule

$$s_i(\omega) = \begin{cases} r_i(\omega) & \text{if} \quad i \le T(\omega) \\ 0 & \text{if} \quad i > T(\omega) \end{cases}$$

The constructed martingale is evidently symmetric and its absolute value is bounded by $\sqrt{N} + 1$.

Lemma 1.4.9 *Let k be a function of N. If $N = o(k)$, then*

$$\lim_{N\to\infty} P\left(\left|\sum_{i=1}^{k} s_i\right| < \sqrt{N}\right) = 0.$$

If $k = o(N)$, then

$$\lim_{N\to\infty} P\left(\left|\sum_{i=1}^{k} s_i\right| < \sqrt{N}\right) = 1.$$

Proof. We have,

$$\begin{aligned} P\left(\left|\sum_{i=1}^{k} s_i\right| < \sqrt{N}\right) &= P\left(\max_{1\le j\le k}\left|\sum_{i=1}^{j} r_i\right| < \sqrt{N}\right) \\ &\le P\left(\left|\sum_{i=1}^{k} r_i\right| < \sqrt{N}\right) = P\left(\left|\frac{1}{\sqrt{k}}\sum_{i=1}^{k} r_i\right| < \sqrt{\frac{N}{k}}\right). \end{aligned}$$

By the central limit theorem, the last expression tends to zero if $N = o(k)$. On the other hand, if $k = o(N)$, then

$$P\left(\left|\sum_{i=1}^{k} s_i\right| < \sqrt{N}\right) = 1 - P\left(\left|\sum_{i=1}^{k} s_i\right| \ge \sqrt{N}\right) \ge 1 - e^{-N/2k} \underset{N\to\infty}{\longrightarrow} 1.$$

The lemma is proved. ■

Lemma 1.4.10 *Suppose that $x, x_n \in l_p$, $1 \le p < \infty$ and $x_n \xrightarrow{w} 0$. Then for any $\varepsilon > 0$ there exists an index $n \in \mathbf{N}$ such that*

$$\|x\|^p + \|x_n\|^p - \varepsilon < \|x + x_n\|^p < \|x\|^p + \|x_n\|^p + \varepsilon.$$

But if $x, x_n \in c_0$, then under the same assumptions it is possible to ensure that

$$\max\{\|x\|, \|x_n\|\} - \varepsilon < \|x + x_n\| < \max\{\|x\|, \|x_n\|\} + \varepsilon.$$

The method in the following theorem due to V. M. Kadets and G. Schechtman will often be used in the main results of the thesis.

Theorem 1.4.11 *The spaces l_p with $p \neq 2$, $1 \leq p < \infty$, and the space c_0 have the Lyapunov property.*

Proof. The case $1 \leq p < 2$. Fix $N \in \mathbf{N}$ for the present, and let $k = [N/\ln N]$. We assume that $X = l_p$ fails the Lyapunov property, and we use Lemma 1.4.7. Let $\Omega, \Sigma, \lambda, \varepsilon$, and $T : L_\infty \to X$ be as in that lemma, with $\lambda(\Omega) = 1$. We prove by induction on j that there exist functions $\{t_j\}_{i=1}^{\infty} \in L_\infty(\Omega, \Sigma, \lambda)$ such that for each j the functions $\{t_i\}_{i=1}^{j}$ are jointly equidistributed with the $\{s_i\}_{i=1}^{j}$ in Lemma 1.4.9 and

$$\left\| T\left(\sum_{i=1}^{j} t_i\right)\right\|^p > \sum_{i=1}^{j} \|T(t_i)\|^p - 1. \tag{1.4}$$

Indeed, everything is obvious for $j = 1$. Suppose that the assertion is already proved for $j = m$, and that $\{t_i\}_{i=1}^{m}$ have been constructed that satisfy (1.4) with $j = m$. We consider the set $A = \left\{\omega \in \Omega : \left|\sum\limits_{i=1}^{m} t_i\right| < \sqrt{N}\right\}$ and a sequence $\{z_n\}_{n=1}^{\infty}$ of functions independent on A, equal to zero off A, and taking the values ± 1 with probability $\lambda(A)/2$. For any n the system $\{\{t_i\}_{i=1}^{m}, z_n\}$ is equidistributed with $\{s_i\}_{i=1}^{m+1}$. The sequence z_n tends to zero in the weak* topology; hence $Tz_n \xrightarrow{w} 0$, and by Lemma 1.4.10 it is possible to choose n such that

$$\left\| T\left(\sum_{i=1}^{m} t_i\right) + T(z_n)\right\|^p > \left\|\sum_{i=1}^{m} T(t_i)\right\|^p + \|T(z_n)\|^p - \delta,$$

where δ is arbitrarily small. Choosing δ sufficiently small and letting $t_{m+1} = z_n$, we get the required inequality (1.4) with $j = m + 1$.

We now use the fact just proved. Suppose that $\{t_i\}_{i=1}^{k}$ is equidistributed with $\{s_i\}_{i=1}^{k}$ and subject to the condition (1.4) with $j = k$. Then

$$\|T\|^p \cdot \left\|\sum_{i=1}^{k} t_i\right\|^p \geq \left\| T\left(\sum_{i=1}^{k} t_i\right)\right\|^p \geq \sum_{i=1}^{k} \|T(t_i)\|^p - 1.$$

In view of the condition (ii) of Lemma 1.4.7, which T satisfies by construction, we have $\|T(t_i)\| \geq \varepsilon \cdot \lambda(\operatorname{supp} t_i) \geq \varepsilon \cdot \lambda(\operatorname{supp} t_k)$ for $i \leq k$.

Since $\{t_i\}_{i=1}^k$ and $\{s_i\}_{i=1}^k$ are equidistributed, this gives us by the choice of k that

$$\left(\sqrt{N}+1\right)^p \|T\|^p + 1 \geq \sum_{i=1}^k \|T(t_i)\|^p \geq k\varepsilon^p \left(P\left(s_k \neq 0\right)\right)^p \geq$$
$$\geq (N/\ln N - 1)\cdot \varepsilon^p \cdot \left(P\left(\left|\sum_{i=1}^{k-1} s_i\right| < \sqrt{N}\right)\right)^p.$$

By Lemma 1.4.9, the last factor tends to 1; hence this inequality cannot hold for large N. This contradiction completes the analysis of the first case.

The case $2 < p < \infty$. Assume that $X = l_p$ fails the Lyapunov property. By analogy with the first case, we fix an $N \in \mathbf{N}$, let $k = [N \ln N]$, and construct functions $\{t_i\}_{i=1}^k$ equidistributed with $\{s_i\}_{i=1}^k$ and satisfying the requirement

$$\left\|T\left(\sum_{i=1}^k t_i\right)\right\|^p < \sum_{i=1}^k \|T(t_i)\|^p + 1 \leq \|T\|^p \cdot k + 1. \tag{1.5}$$

We define the auxiliary function f_N by

$$f_N = \begin{cases} 1 & \text{if } \sum\limits_{i=1}^k t_i \geq \sqrt{N} \\ -1 & \text{if } \sum\limits_{i=1}^k t_i \leq -\sqrt{N} \\ 0 & \text{for the rest} \end{cases}.$$

and denote $\operatorname{supp} f_N$ by A. Since $\left|\sum\limits_{i=1}^k t_i\right| < \sqrt{N}+1$,

$$\left|f_N - \frac{1}{\sqrt{N}}\sum_{i=1}^k t_i\right| \leq \frac{1}{\sqrt{N}} \text{ on } A.$$

Consequently,

$$\left\|\chi_A \cdot \left(f_N - \frac{1}{\sqrt{N}}\sum_{i=1}^k t_i\right)\right\|_\infty \underset{N\to\infty}{\longrightarrow} 0. \tag{1.6}$$

Further, according to Lemma 1.4.9, $\lambda(\Omega \backslash A) \to 0$, and by Remark 1.4.8

$$\left\|T\left(\chi_{\Omega/A} \cdot \frac{1}{\sqrt{N}}\sum_{i=1}^k t_i\right)\right\| \underset{N\to\infty}{\longrightarrow} 0. \tag{1.7}$$

Comparing (1.6) and (1.7), we get that

$$\left\| T\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{k} t_i\right)\right\| \geq \|T(f_N)\| + o(1). \tag{1.8}$$

By (1.5) and the property (1.8) of T,

$$[\varepsilon \cdot \lambda(A)]^p \leq \|T\|^p \cdot \frac{k}{\left(\sqrt{N}\right)^p} + o(1) = o(1).$$

This contradicts Lemma 1.4.9, which asserts that $\lim\limits_{N\to\infty} \lambda(A) = 1$.

The case $X = c_0$ is analysed just like the preceding case. The theorem is proved. ■

There are simpler proofs for c_0 and l_1, due to V. Kadets, which were never published before. We use the chance to present (with the kind permission of V. Kadets) these proofs too.

Proposition 1.4.12 *c_0 has the Lyapunov property.*

Proof. Let $\mu : \Sigma \to c_0$ be a nonatomic countably additive vector measure. According to Corollary 1.2.4 there is an operator $T : L_\infty(\Omega, \Sigma, \lambda) \to c_0$ such that $\mu(A) = T\chi_A$ for all $A \in \Sigma$ and which is continuous for the weak* topology on $L_\infty(\Omega, \Sigma, \lambda)$ and the weak topology on c_0, where λ is a nonnegative nonatomic real-valued countably additive measure on Σ such that μ is absolutely continuous with respect to λ. By Lemma 1.4.5, it is sufficient to show that for every A and every $\varepsilon > 0$ there exists a set $B \in \Sigma|_A$ such that

$$\left\|\mu(B) - \frac{1}{2}\mu(A)\right\| \leq \varepsilon. \tag{1.9}$$

Let $\{e_i\}_{i=1}^{\infty}$ be the unit vector basis of c_0 and $\{f_i\}_{i=1}^{\infty}$ be the coordinate functionals on c_0. Then $Tx = \sum_{i=1}^{\infty} f_i(Tx)\, e_i$. Denote $T_i x = f_i(Tx)$. Take an arbitrary $A \in \Sigma$ and an $\varepsilon > 0$. Fix $N \in \mathbf{N}$ for the present. As μ is a nonatomic measure we can decompose A into the union of mutually disjoint sets $A_k \in \Sigma|_A$ $(k = 1, \ldots, N)$ with $\mu(A_k) \neq 0$ for all k, $\lambda(A_k) = \lambda(\Omega)/N$. Let r_1^1 be a Rademacher function on A_1. Put $n_1 = 1$ and $m_0 = 0$. Select $m_1 \in \mathbf{N}$ such that $\left\|Tr_1^1 - \sum\limits_{i=1}^{m_1} T_i r_1^1\right\| \leq$

$\frac{1}{2^{N+1}}$. Consider a sequence $\{r_n^2\}_{n=1}^{\infty}$ of Rademacher functions on A_2. By Corollary 1.2.5, $Tr_n^2 \underset{n\to\infty}{\longrightarrow} 0$ in the weak topology. Hence there are a number n_2 such that $\left\|\sum_{i=1}^{m_1} T_i r_{n_2}^2\right\| \leq \frac{1}{2^{N+3}}$ and a number m_2 such that $\left\|\sum_{i=m_2+1}^{\infty} T_i r_{n_2}^2\right\| \leq \frac{1}{2^{N+3}}$. Consequently

$$\left\|Tr_{n_2}^2 - \sum_{i=m_1+1}^{m_2} T_i r_{n_2}^2\right\| \leq \frac{1}{2^{N+2}}.$$

Continuing this process we obtain the sequences of numbers $\{n_k\}_{k=1}^N$ and $\{m_k\}_{k=0}^N$ and the set of functions $\left\{r_{n_k}^k\right\}_{k=1}^N$ such that $r_{n_k}^k$ is a Rademacher function on A_k and

$$\left\|Tr_{n_k}^k - \sum_{i=m_{k-1}+1}^{m_k} T_i r_{n_k}^k\right\| \leq \frac{1}{2^{N+k}}$$

for $k = 1, \ldots, N$. Consequently

$$\left\|T\left(\sum_{k=1}^N r_{n_k}^k\right)\right\| \leq \left\|\sum_{k=1}^N \sum_{i=m_{k-1}+1}^{m_k} T_i r_{n_k}^k\right\| + \frac{1}{2^N} = \max_{1\leq k\leq N}\left\|\sum_{i=m_{k-1}+1}^{m_k} T_i r_{n_k}^k\right\| + \frac{1}{2^N}.$$

If the maximum is reached at k_0 then applying the equation (1.1) we have

$$\left\|T\left(\sum_{k=1}^N r_{n_k}^k\right)\right\| \leq \left\|\sum_{i=m_{k_0-1}+1}^{m_{k_0}} T_i r_{n_{k_0}}^{k_0}\right\| + \frac{1}{2^N} \leq \left\|T|_{A_{k_0}}\right\| + \frac{1}{2^N} \leq \|\mu\|\left(A_{k_0}\right) + \frac{1}{2^N}.$$

As μ is absolutely continuous with respect to λ and $\lambda\left(A_k\right) = \lambda\left(\Omega\right)/N$ for $k = 1, ..., N$ and by Proposition 1.1.6, the last sum tends to zero as $N \to \infty$. Thus choosing N sufficiently large and taking $B = \bigcup_{k=1}^N \left\{\omega \in \Omega : r_{n_k}^k = 1\right\}$ we obtain the required inequality (1.9). The proposition is proved. ■

Before we prove that l_1 has the Lyapunov property, let us recall that Schur has proved that in l_1 the norm-convergence and the weak-convergence coincide. We will show that all Banach spaces, which have this property, have the Lyapunov property.

Definition 1.4.13 *A Banach space X is said to have the Schur property if a sequence (x_n) of elements of X converges in the weak topology if and only if it converges in the norm topology.*

Proposition 1.4.14 *If a Banach space X has the Schur property, then it has the Lyapunov property.*

Proof. Let μ be a nonatomic countably additive vector measure on a σ-field Σ and $T : L_\infty(\lambda) \to X$ be the operator from Corollary 1.2.4. Take an arbitrary $A \in \Sigma$ and consider on A the sequence of functions $f_n = \frac{1+r_n}{2}$, where (r_n) are Rademacher functions on A. As $f_n \underset{n\to\infty}{\longrightarrow} \frac{1}{2}\chi_A$ in the weak* topology, then by Corollary 1.2.5 we have $Tf_n \underset{n\to\infty}{\longrightarrow} \frac{1}{2}\mu(A)$ in the weak topology. It follows that $\left\|Tf_n - \frac{1}{2}\mu(A)\right\| \underset{n\to\infty}{\longrightarrow} 0$. But $f_n = \chi_{A_n}$, where $A_n \in \Sigma$. Thus we have

$$\left\|\mu(A_n) - \frac{1}{2}\mu(A)\right\| \underset{n\to\infty}{\longrightarrow} 0.$$

By Lemma 1.4.5, the proposition is proved. ■

1.5 The three-space problem

The results of this section have been published in [18], [5].

The "three-space problem" arises for every Banach space property. Namely, do a subspace Y and the quotient space X/Y have the property if X does and does X have the property if Y and X/Y do? Some of these problems can be solved easily in the case of the Lyapunov property. Indeed, if X has the Lyapunov property and $Y \subset X$, then Y has the Lyapunov property, but X/Y needs not have the Lyapunov property (for instance, $X = l_1$ because the set of its quotient spaces contains all separable Banach spaces [24, p.108]). At the same time the answer to the last problem runs into difficulties. The purpose of this section is the positive solution of the above mentioned problem : let Y be a subspace of a Banach space X. If Y and X/Y have the Lyapunov property, then X has the Lyapunov property.

The following three lemmas are of technical nature.

Lemma 1.5.1 *Let X have the Lyapunov property and $\mu : \Sigma \to X$ be a nonatomic measure. Then there is a nonatomic nonnegative measure $\lambda : \Sigma \to \mathbf{R}$ such that for every $A \in \Sigma$ and $n \in \mathbf{N}$ there exists $B_n \in \Sigma|_A$ satisfying the following inequalities:*

$$\left\| \mu(B_n) - \frac{1}{2}\mu(A) \right\| \leq \frac{1}{2^n} \textit{ and } \left| \lambda(B_n) - \frac{1}{2}\lambda(A) \right| \leq \frac{1}{2^n}. \tag{1.10}$$

Proof. By Theorem 1.2.7 there is a functional $x^* \in X^*$ with $\|x^*\| = 1$ for which $\mu \ll |x^*\mu|$. Put $\lambda = |x^*\mu|$. Let $A \in \Sigma$. In accordance with the Hahn decomposition theorem let us denote by $\Omega^+$and $\Omega^-$the positivity and negativity sets for $x^*\mu$ respectively. Then $\lambda(C) = x^*\mu(C \bigcap \Omega^+) - x^*\mu(C \bigcap \Omega^-)$ for any $C \in \Sigma$. Put $A^+ = A \bigcap \Omega^+$and $A^- = A \bigcap \Omega^-$. Since X has the Lyapunov property, we can choose $B_n^+ \in \Sigma|_{A^+}$ and $B_n^- \in \Sigma|_{A^-}$such that

$$\left\| \mu\left(B_n^+\right) - \frac{1}{2}\mu\left(A^+\right) \right\| \leq \frac{1}{2^{n+1}} \text{ and } \left\| \mu\left(B_n^-\right) - \frac{1}{2}\mu\left(A^-\right) \right\| \leq \frac{1}{2^{n+1}}.$$

Then

$$\left| x^*\mu\left(B_n^+\right) - \frac{1}{2}x^*\mu\left(A^+\right) \right| \leq \frac{1}{2^{n+1}} \textit{ and } \left| x^*\mu\left(B_n^-\right) - \frac{1}{2}x^*\mu\left(A^-\right) \right| \leq \frac{1}{2^{n+1}}.$$

Define $B_n = B_n^+ \bigcup B_n^-$. It is easy to check that for B_n the inequalities (1.10) are true. ■

Lemma 1.5.2 *Let X have the Lyapunov property, $\mu : \Sigma \to X$ be a nonatomic measure, λ be the measure from Lemma 1.5.1. Then for every $A \in \Sigma$ with $\lambda(A) \neq 0$ and $\varepsilon > 0$ there exist $G' \in \Sigma|_A$, $G'' = A \backslash G'$such that*

(i) $\lambda\left(G'\right) = \lambda\left(G''\right) = \lambda(A)/2$,

(ii) $\left\| \mu\left(G'\right) - \frac{1}{2}\mu(A) \right\| < \varepsilon$.

Proof. Let us choose B_n as in Lemma 1.5.1. We can select $C_n \in \Sigma|_{B_n}$ if $\lambda(B_n) \geq \frac{1}{2}\lambda(A)$ or $C_n \in \Sigma|_{A \backslash B_n}$if $\lambda(B_n) < \frac{1}{2}\lambda(A)$ for which $\lambda(C_n) = \left|\frac{1}{2}\lambda(A) - \lambda(B_n)\right|$ (because λ is a nonatomic real-valued measure). By (1.10), $\lambda(C_n) \underset{n\to\infty}{\to} 0$. Put $G_n' = B_n \triangle C_n$, $G_n'' = A \backslash G_n'$. Then we obtain $\lambda\left(G_n^{'}\right) = \lambda\left(G_n^{''}\right) = \frac{1}{2}\lambda(A)$, and $\lambda\left(G_n^{'}\backslash B_n\right) \underset{n\to\infty}{\to} 0$, and $\lambda\left(B_n\backslash G_n^{'}\right) \underset{n\to\infty}{\to} 0$. Since μ is absolutely continuous with respect to λ the last condition implies that $\mu\left(B_n\backslash G_n^{'}\right) \underset{n\to\infty}{\to} 0$ and $\mu\left(G_n^{'}\backslash B_n\right) \underset{n\to\infty}{\to} 0$. Together with inequality (1.10) this gives us

$$\left\|\mu\left(G_n^{'}\right) - \frac{1}{2}\mu(A)\right\| \underset{n\to\infty}{\to} 0.$$

So for sufficiently large n the sets $G^{'} = G_n^{'}$ and $G^{''} = G_n^{''}$ will satisfy the conditions (i) and (ii). ■

Lemma 1.5.3 *Under the conditions of Lemma 1.5.2 for every $A \in \Sigma$ with $\lambda(A) \neq 0$ and $\varepsilon > 0$ there exists a σ-algebra $\Sigma^{'} \subset \Sigma|_A$ such that for every $B \in \Sigma^{'}$ we have*

$$\left\|\mu(B) - \lambda(B)\frac{\mu(A)}{\lambda(A)}\right\| \leq \varepsilon\lambda(B) \tag{1.11}$$

and the measure λ is nonatomic on $\Sigma^{'}$.

Proof. Take $A \in \Sigma$ and $\varepsilon > 0$. Employing Lemma 1.5.2, we choose sets $A_1 \in \Sigma|_A$, $A_2 = A\backslash A_1$ with $\lambda(A_1) = \lambda(A_2) = \frac{1}{2}\lambda(A)$ and $\left\|\mu(A_1) - \frac{1}{2}\mu(A)\right\| \leq \frac{1}{4}\varepsilon$ (note that

$$\left\|\mu(A_1) - \frac{1}{2}\mu(A)\right\| = \frac{1}{2}\left\|\mu(A_1) - \mu(A_2)\right\| = \left\|\mu(A_2) - \frac{1}{2}\mu(A)\right\| \leq \frac{1}{4}\varepsilon).$$

Employing Lemma 1.5.2 twice (for $A = A_1$ and $A = A_2$) we obtain $A_{1,1} \in \Sigma|_{A_1}, A_{1,2} = A_1\backslash A_{1,1}; A_{2,1} \in \Sigma|_{A_2}, A_{2,2} = A_2\backslash A_{2,1}$ with $\lambda(A_{1,1}) = \lambda(A_{1,2}) = \lambda(A_{2,1}) = \lambda(A_{2,2}) = \frac{1}{4}\lambda(A)$ and

$$\left\|\mu(A_{1,1}) - \frac{1}{2}\mu(A_1)\right\| \leq \frac{\epsilon}{16}, \left\|\mu(A_{2,1}) - \frac{1}{2}\mu(A_2)\right\| \leq \frac{\epsilon}{16}.$$

When we continue this process, we obtain a tree of sets $A_{i_1,i_2,\ldots,i_n}, i_k \in \{1,2\}, n \in \mathbf{N}$ with

$$A_{i_1,i_2,\ldots,i_{n+1}} \subset A_{i_1,i_2,\ldots,i_n},\ A_{i_1,i_2,\ldots,i_n,2} = A_{i_1,i_2,\ldots,i_n}\backslash A_{i_1,i_2,\ldots,i_n,1},$$
$$\left\|\mu(A_{i_1,i_2,\ldots,i_n}) - \tfrac{1}{2}\mu\left(A_{i_1,i_2,\ldots,i_{n-1}}\right)\right\| \leq \tfrac{1}{4^n}\varepsilon,\ \lambda(A_{i_1,i_2,\ldots,i_n}) = \tfrac{1}{2^n}\lambda(A)$$

Let Σ' be a σ-algebra generated by the sets $A_{i_1,i_2,...,i_n}$. We are going to show that the algebra Σ' has the required property.

Let $B = A_{i_1,i_2,...,i_n}$. Then

$$\begin{aligned}\left\|\mu(B) - \tfrac{\lambda(B)}{\lambda(A)}\mu(A)\right\| &= \left\|\mu\left(A_{i_1,i_2,...,i_n}\right) - \tfrac{1}{2^n}\mu(A)\right\| \\ &\le \left\|\mu\left(A_{i_1,i_2,...,i_n}\right) - \tfrac{1}{2}\mu\left(A_{i_1,i_2,...,i_{n-1}}\right)\right\| \\ &+\tfrac{1}{2}\left\|\mu\left(A_{i_1,i_2,...,i_{n-1}}\right) - \tfrac{1}{2}\mu\left(A_{i_1,i_2,...,i_{n-2}}\right)\right\| + ... \\ &+\tfrac{1}{2^{n-1}}\left\|\mu\left(A_{i_1}\right) - \tfrac{1}{2}\mu(A)\right\| \\ &\le \tfrac{1}{4^n}\varepsilon + \tfrac{1}{2}\tfrac{1}{4^{n-1}}\varepsilon + ... + \tfrac{1}{2^{n-1}}\tfrac{1}{4}\varepsilon \\ &= \varepsilon\left(\tfrac{1}{2^{2n}} + \tfrac{1}{2^{2(n-1)}} + ... + \tfrac{1}{2^{n+1}}\right) \le \tfrac{1}{2^n}\varepsilon \\ &= \varepsilon\lambda\left(A_{i_1,i_2,...,i_n}\right).\end{aligned}$$

Hence by the triangle inequality and the σ-additivity of μ and λ we get (1.11) for $B = \bigcup_{n=1}^{\infty} B_n$, where the B_n are disjoint sets of the form $A_{i_1,i_2,...,i_k}$. Then we obtain (1.11) for every $B \in \Sigma'$ using approximation of $\lambda(B)$ by bigger sets of the form $B = \bigcup_{n=1}^{\infty} B_n$. ■

Lemma 1.5.4 *The statements of Lemmas 1.5.2 and 1.5.3 are valid for an arbitrary nonatomic measure* λ.

Proof. Let $\mu : \Sigma \to X,\ \lambda : \Sigma \to \mathbf{R}_+$ be nonatomic measures. Let ν be the measure which played the role of λ in Lemma 1.5.1. Consider two cases.

Case 1: $\lambda \ll \nu$.Take $A \in \Sigma$, $n \in \mathbf{N}$. In view of Lemma 1.5.3, there is $\Sigma' \subset \Sigma|_A$ such that for any $B \in \Sigma'$

$$\left\|\mu(B) - \frac{\nu(B)}{\nu(A)}\mu(A)\right\| \le \frac{1}{2^n}\nu(B), \tag{1.12}$$

ν is a nonatomic measure with respect to Σ'. Applying the Lyapunov theorem to the measure $\sigma : \Sigma' \to \mathbf{R}^2 : \sigma(A) = (\nu(A), \lambda(A))$, we obtain sets $G', G'' \in \Sigma'$ such that $\lambda\left(G'\right) = \lambda\left(G''\right) = \frac{1}{2}\lambda(A)$ and $\nu\left(G'\right) = \nu\left(G''\right) = \frac{1}{2}\nu(A)$. Then (1.12) implies inequality (ii) which we need.

Case 2: $\lambda \not\ll \nu$. We decompose λ into the sum of absolutely continuous and strictly singular measures with respect to $\nu : \lambda = \lambda_1 + \lambda_2$. Then λ_2 is concentrated on a ν-negligible set S. Now we consider $A \backslash S$ and $S \bigcap A$ separately. By the case 1 chose $G_1^{'} \subset A \backslash S$ so that $\lambda_1 \left(G_1^{'} \right) = \frac{1}{2} \lambda_1 (A)$, $\left\| \mu \left(G_1^{'} \right) - \frac{1}{2} \mu (A) \right\| \leq \frac{1}{2^n}$ and $G_2^{'} \subset S \bigcap A$ with $\lambda_2 \left(G_2^{'} \right) = \frac{1}{2} \lambda (S \bigcap A)$. Because $\lambda_1 \left(G_2^{'} \right) = 0$, $\mu \left(G_2^{'} \right) = 0$ and $\lambda_2 \left(G_1^{'} \right) = 0$ it is clear that $G^{'} = G_1^{'} \bigcup G_2^{'}$ satisfies (ii). The proof of the Lemma 1.5.3 shows that if Lemma 1.5.2 is valid for an arbitrary λ then Lemma 1.5.3 is valid too for an arbitrary nonatomic measure λ. ■

The following statement is evident.

Lemma 1.5.5 *If X is a Banach space, Y is a subspace of X, $\mu : \Sigma \to X$ is a nonatomic measure, then $\bar{\mu} : \Sigma \to X/Y$ ($\bar{\mu}(A)$-the equivalence class of $\mu(A)$) is a nonatomic measure too.*

Theorem 1.5.6 *Let Y be a subspace of a Banach space X. If Y and X/Y have the Lyapunov property, then X has the Lyapunov property too.*

Proof. Let $\mu : \Sigma \to X$, $\lambda : \Sigma \to \mathbf{R}_+$ be nonatomic measures, $\lambda(\Omega) = 1$. Fix $A \in \Sigma$, $\lambda(A) \neq 0$, and $\varepsilon > 0$. By Lemmas 1.5.4 and 1.5.5 there is a σ-algebra $\Sigma^{'} \subset \Sigma|_A$ such that

$$\left\| \bar{\mu}(B) - \frac{\lambda(B)}{\lambda(A)} \bar{\mu}(A) \right\| < \frac{1}{2} \varepsilon \lambda(B) \tag{1.13}$$

for all $B \in \Sigma^{'}$ and λ is nonatomic on $\Sigma^{'}$. Define $\sigma : \Sigma^{'} \to X$ by the rule $\sigma(B) = \mu(B) - \frac{\lambda(B)}{\lambda(A)} \mu(A)$. Now we show that there exist a σ-algebra $\widetilde{\Sigma} \subset \Sigma^{'}$ and a nonatomic measure $\beta : \widetilde{\Sigma} \to Y$ such that

$$\| \sigma(B) - \beta(B) \| < 2\varepsilon$$

for all $B \in \widetilde{\Sigma}$. Indeed, by inequality (1.13) and nonatomicity of λ we can choose sets $A_1, A_2 \in \Sigma^{'}$ such that $A = A_1 \bigcup A_2$, $\lambda(A_1) = \lambda(A_2) = \frac{1}{2} \lambda(A)$, and $\| \bar{\sigma}(A_1) \| < \varepsilon \frac{1}{2} \lambda(A)$. Clearly, there is $x \in \bar{\sigma}(A_1)$ such that $\|x\| \leq \varepsilon \frac{1}{2} \lambda(A)$. Denote

$$\alpha(A_1) = x, \alpha(A_2) = -x,$$

Applying step by step Lemma 1.5.2 and Lemma 1.5.4, we get sets $A_{i_1,...,i_k} \in \Sigma^{'}$, $k = 2, 3, ...;$ $i_1, ..., i_k = 1, 2$, such that $A_{i_1,...,i_{k-1}} = A_{i_1,...,i_k,1} \bigcup A_{i_1,...,i_k,2}$, $\lambda\left(A_{i_1,...,i_k}\right) = \frac{1}{2}\lambda\left(A_{i_1,...,i_{k-1}}\right) = \frac{1}{2^k}\lambda\left(A\right)$, and $\left\|\bar{\sigma}\left(A_{i_1,...,i_{k-1},1}\right) - \frac{1}{2}\,\bar{\sigma}\left(A_{i_1,...,i_{k-1}}\right)\right\| < \varepsilon\frac{1}{2^{2k}}\lambda\left(A\right)$. It is readily seen that in every equivalence class $\bar{\sigma}\left(A_{i_1,...,i_{k-1},1}\right) - \frac{1}{2}\,\bar{\sigma}\left(A_{i_1,...,i_{k-1}}\right) \subset X$ there exists an element $x_{i_1,...,i_{k-1}}$ such that $\left\|x_{i_1,...,i_{k-1}}\right\| \leq \varepsilon\frac{1}{2^{2k}}\lambda\left(A\right)$. Put

$$\alpha\left(A_{i_1,...,i_{k-1},1}\right) = \tfrac{1}{2}\alpha\left(A_{i_1,...,i_{k-1}}\right) + x_{i_1,...,i_{k-1}},$$
$$\alpha\left(A_{i_1,...,i_{k-1},2}\right) = \tfrac{1}{2}\alpha\left(A_{i_1,...,i_{k-1}}\right) - x_{i_1,...,i_{k-1}}.$$

Evidently, $\alpha\left(A_{i_1,...,i_{k-1},1}\right) \in \bar{\sigma}\left(A_{i_1,...,i_{k-1},1}\right)$ and $\alpha\left(A_{i_1,...,i_{k-1},2}\right) \in \bar{\sigma}\left(A_{i_1,...,i_{k-1},2}\right)$. By $\widetilde{\Sigma}$ denote σ-algebra generated by the sets $A_{i_1,...,i_k}$. Iterating the inequality $\left\|\alpha\left(A_{i_1,...,i_k}\right)\right\| \leq \frac{1}{2}\left\|\alpha\left(A_{i_1,...,i_{k-1}}\right)\right\| + \varepsilon\frac{1}{2^{2k}}\lambda\left(A\right)$, we obtain $\left\|\alpha\left(A_{i_1,...,i_k}\right)\right\| \leq \varepsilon\frac{1}{2^{k-1}}\lambda\left(A\right) = 2\varepsilon\lambda\left(A_{i_1,...,i_k}\right)$. Let us extend α to $\widetilde{\Sigma}$ and show that

$$\left\|\alpha\left(B\right)\right\| \leq 2\varepsilon\lambda\left(B\right) \tag{1.14}$$

for any $B \in \widetilde{\Sigma}$. For this purpose we take $B = \bigcup\limits_I A_{i_1,...,i_k}$, where I is a finite set of indices and $A_{i_1,...,i_k}$ are mutually disjoint sets. Put $\alpha\left(B\right) = \sum\limits_I \alpha\left(A_{i_1,...,i_k}\right)$. Clearly,

$$\left\|\alpha\left(B\right)\right\| \leq \sum_I \left\|\alpha\left(A_{i_1,...,i_k}\right)\right\| \leq 2\varepsilon\sum_I \lambda\left(A_{i_1,...,i_k}\right) = 2\varepsilon\lambda\left(B\right).$$

This proves that inequality (1.14) is valid for elements of the algebra S, generated by the sets $A_{i_1,...,i_k}$. Now applying the Kluvanek-Uhl extension theorem [6, Chapter 6, Section 8, Addition 4] we obtain an extension of α to $\widetilde{\Sigma}$. Thus, we have constructed a measure $\alpha : \widetilde{\Sigma} \to X$ such that $\alpha\left(B\right) \in \overline{\sigma}\left(B\right)$ for any $B \in \widetilde{\Sigma}$ and $\left\|\alpha\left(B\right)\right\| \leq 2\varepsilon\lambda\left(B\right)$. α is a nonatomic measure because λ is nonatomic by construction. It is clear that $\beta = \alpha - \sigma$ has the required property. The statement is proved.

Let us complete the proof of the theorem. Since $\beta : \widetilde{\Sigma} \to Y$ is a nonatomic measure and Y has the Lyapunov property, we see that by Lemma 1.5.2 there is $B \in \widetilde{\Sigma}$ such that $\lambda\left(B\right) = \frac{1}{2}\lambda\left(A\right)$ and

$$\left\|\beta\left(B\right) - \frac{1}{2}\beta\left(A\right)\right\| \leq \varepsilon.$$

Thus we have

$$\begin{aligned}\left\|\sigma(B)-\tfrac{1}{2}\sigma(A)\right\| &\le \|\sigma(B)-\beta(B)\|+\left\|\tfrac{1}{2}\sigma(A)-\tfrac{1}{2}\beta(A)\right\| \\ &+\left\|\beta(B)-\tfrac{1}{2}\beta(A)\right\| \le 3\varepsilon.\end{aligned}$$

Since $\sigma(B)-\frac{1}{2}\sigma(A)=\mu(B)-\frac{1}{2}\mu(A)$, the proof is completed. ■

Let us stress an important particular case of this theorem.

Corollary 1.5.7 *Let $X = X_1 \oplus X_2$ where X_1, X_2 have the Lyapunov property. Then X has the Lyapunov property too.*

It is clear that by induction we can extend this property to any finite sum of Banach spaces.

Corollary 1.5.8 *The spaces X_p from the work [19] ("twisted sums" of l_p spaces) have the Lyapunov property for $p \neq 2$.*