Subalgebras of small Souslin Algebras

and

Maximal Chains in Souslin algebras

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Introduction

This dissertation addresses problems from infinitary combinatorics and the theory of complete Boolean algebras. As background theory we assume ZFC.

A subset K of a Boolean algebra B will be called a *chain* of B if it is totally ordered by the canonical partial ordering $<_B$ on B. The set of chains of B is inductively ordered by the subset-relation $\subset$, so a simple application of Zorn's Lemma gives the existence of maximal chains. We say that a Boolean algebra is *chain homogeneous* if all its maximal chains are pairwise isomorphic as linear orders.

The basic question which motivated this thesis was, whether there is under appropriate set theoretical assumptions an atomless and chain homogeneous, atomless and complete Boolean algebra $\mathbb{B}$, such that the maximal chains of $\mathbb{B}$ are not isomorphic to the real unit interval $[0, 1]$.

At the bottom of this question were two observations: Few, but very prominent complete Boolean algebras are chain homogeneous, such as the Cohen algebras and measure algebras. But their maximal chains are all isomorphic to $[0, 1]$. On the other hand, every maximal chain K of a chain homogeneous and atomless, complete Boolean algebra $\mathbb{B}$ is a complete linear order with endpoints that satisfies the countable antichain condition c.c.c., i.e., every family of pairwise disjoint, open intervals of K is countable.

Therefore, our problem is tightly related to Souslin's Hypothesis SH, which states that every complete and dense, linear order is already isomorphic to a real interval if it satisfies the c.c.c. If we assume SH, then our question is immediately answered to the negative.

However, in the 1960's years it has been proved that SH is independent from ZFC, i.e., if ZFC is consistent then so are the two theories $\mathsf{ZFC} + \mathsf{SH}$ and $\mathsf{ZFC} + \neg\mathsf{SH}$. A counter-example to Souslin's hypothesis is called a *Souslin line* and the corresponding complete Boolean algebras are *Souslin algebras*. We should also mention *Souslin-trees*, which are a manifestation of the same phenomenon and the main technical tool used in this context. For a Souslin algebra $\mathbb{B}$, all maximal chains of $\mathbb{B}$ are Souslin lines. So our basic question has the following reformulation:

> Is it consistent relative to ZFC that there is a chain homogeneous Souslin algebra?

The main result of this paper is the affirmative answer to this question, cf. Theorems 2.3.2, 2.4.2 and 2.4.3 in the chapter *Maximal chains in Souslin algebras*. Assuming $\diamondsuit^+$ (which is consistent relative to ZFC) we give three constructions of chain homogeneous Souslin algebras.

To lay the grounds for these constructions, we have extended the existing representation theory for Souslin algebras and their subalgebras. Furthermore we use this representation theory to deduce the following results.

1. In section 1.2-1.4 a structure theory for regular embeddings between Souslin algebras is developed.

2. For the class of so called *strongly homogeneous* Souslin trees we extend an existing decomposition result (cf. Theorem 1.5.10) and give a new one (Theorem 1.5.3).

3. We use these decompositions to get examples of Souslin trees that separate certain rigidity notions for Souslin trees. This answers a question of Fuchs and Hamkins [FH06, Question 4.1].

Editorial note: This thesis was conceived as consisting of two, more or less independent articles. For the submission they have been merged into one document. Now each of them is a large chapter, with its proper introductory and preliminaries sections. As a consequence, some of the notions are defined twice and some standard results are cited in the first as well as in the second chapter. We hope that this causes only a minor inconvenience to the reader.

Furthermore, an abstract, acknowledgements and a cv in German have been included.

Chapter 1

Subalgebras of small Souslin algebras

Introduction

In this article, we enhance the existing representation theory for atomless, complete subalgebras of small Souslin algebras (i.e. completely generated by a subset of cardinality $\aleph_1$) and apply this method to get a rough classification of possible types of subalgebra embeddings (Sections 1.1 through 1.4).

We also extend a known decomposition result for strongly homogeneous Souslin trees in Section 1.5.4. From this decomposition we obtain examples of Souslin trees which separate the strong rigidity notions considered in [FH06].

We only consider $\aleph_1$-Souslin algebras and $\aleph_1$-Souslin trees and leave aside the question to which extent our methods and results can be generalised to κ-Souslin algebras or trees.

Preliminaries and Notation

Though we often have to make additional assumptions which imply the existence of Souslin trees, our basic theory is ZFC, Zermelo-Fraenkel set theory with choice. We only consider $\aleph_1$-Souslin trees and -algebras, so we omit the parameter in the notions of κ-Souslin tree, κ-Souslin algebra etc. Concerning the tree notation, we follow [DJ74].

Souslin trees

A *tree* is a partial order $(T, <_T)$ with the additional property, that for every element $t \in T$, the set of its predecessors, $\{s \in T \mid s <_T t\}$, is well-ordered by the ordering $<_T$. Whenever possible, we omit the subscript $_T$ and denote the tree ordering just by $<$.

The elements of a tree are called its *nodes*, the minimal elements are *roots*. The *height* of a node, $\operatorname{ht} t$, is the order type of the well-order $(\{s < t\}, <)$. For every node t we define the set of its immediate successors,

$$\operatorname{succ} t := \{s \in T \mid t < s \text{ and } \operatorname{ht} s = \operatorname{ht} t + 1\}.$$

For a cardinal κ we say that T is *κ-branching* if every node has exactly κ immediate successors. For every ordinal α we define the αth level of T and denote it by $T_\alpha := \{t \in T \mid \operatorname{ht} t = \alpha\}$. The *height of* T is the minimal ordinal α such that T_α is empty. For a subset c of $\operatorname{ht} T$ we consider the tree

$$T\restriction c = \bigcup_{\alpha \in c} T_\alpha$$

with the ordering $<$ inherited from T and call this tree *the restriction of T to the levels from c*. If $t \in T_\alpha$ and $\gamma < \alpha$ then $t\lceil\gamma$ denotes the unique predecessor of t on level γ.

A subset b of a tree T is a branch, if it is closed downwards and linearly ordered by $<$. The *length* of a branch is just its order type with respect to $<$. A branch is *cofinal* if its length coincides with $\operatorname{ht} T$. An *antichain* of T is a subset that consists of pairwise incomparable nodes. We call branches or antichains *maximal* if they cannot be extended. Note, that every (non-empty) level of T is a maximal antichain.

A tree T is *normal* if the following hold: T has a unique root, every node t has at least two direct successors and successors on every level T_α with $\operatorname{ht} t < \alpha < \operatorname{ht} T$, and branches of limit length λ have at most one extension of length $\lambda + 1$. In this paper we only consider $\aleph_0$-branching, normal trees of height $\leq \omega_1$ with countable levels.

A map $f : S \to T$ between trees is called a *tree homomorphism*, if it transfers $<_S$ to $<_T$ and does not change the height: $\operatorname{ht}_T f(s) = \operatorname{ht}_S s$ for all $s \in S$.

A *Souslin tree* is a tree of height ω_1 that has neither antichains nor branches of size $\aleph_1$. Note, that a normal tree is Souslin if and only if it has no cofinal branches. A *subtree* is a subset which is a union of branches, i.e., it is closed downwards. Every Souslin tree has a normal subtree which is Souslin. So we only consider normal Souslin trees.

For every node $t \in T$ we let $T(t) := \{s \in t \mid t \leq s\}$ and call it the *tree T relativised to t*. So $\{s < t\} \cup T(t)$ is always a subtree of T (and cofinal if T is normal). This explains the name of the following basic observation.

Lemma 1.0.1 (Subtree Lemma, cf. [Lar99])**.** *Let T be a normal Souslin tree. Then every subtree of height ω_1 contains a set of the form $T(t)$ for some $t \in T$.*

If we consider a tree T with the topology generated by the basic open sets $T(t)$ for $t \in T$, then the conclusion of the lemma reads: Every nowhere dense subtree of T is countable.

Given a family $((T_i, <_i) \mid i \in I)$ of trees we define the *tree product* to be

$$\bigotimes_{i\in I} T_i := \left\{ s : I \to \bigcup_{i\in I} T_i \mid s_i \in T_i \text{ and } \operatorname{ht}_{T_i}(s_i) = \operatorname{ht}_{T_j}(s_j) \text{ for all } i, j \in I \right\}$$

with the ordering defined such that $s < t$ if and only if for all $i \in I$ we have $s_i <_i t_i$. Another way to describe the tree product is $\bigotimes T_i = \bigcup_{\alpha<\beta} \prod_{i\in I} (T_i)_\alpha$, where β is the minimal height of the T_i, $\prod$ denotes the Cartesian product and $(T_i)_\alpha$ is just the αth level of the tree T_i.

It is well known, that for a Souslin tree T, its tree square $T^{\otimes 2} = T \otimes T$ is no longer Souslin. This is easily seen with the aid of the Subtree Lemma, because $\{(t,t) \in T \otimes T \mid t \in T\}$ is a nowhere dense subtree of $T \otimes T$ with height ω_1.

Assuming that the trees T_i are pairwise disjoint we define the *tree sum* by

$$\bigoplus_{i \in I} T_i := \{\mathbf{root}\} \cup \bigcup_{i \in I} T_i,$$

where the node **root** is not in any of the T_i. The tree sum carries the ordering

$$(\{\mathbf{root}\} \times \bigcup_{i \in I} T_i) \cup \bigcup_{i \in I} <_i,$$

i.e., we just place all the T_i side by side on top of the new root.

Various construction methods for Souslin trees

Most commonly known are Souslin tree constructions which assume the combinatorial principle $\diamondsuit$, which states that there is a $\diamondsuit$-sequence, i.e. a sequence $(R_\nu \mid \nu < \omega_1)$ such that for every subset X of ω_1 the set $\{\alpha \mid R_\alpha = X \cap \alpha\}$ is stationary in ω_1 (cf. [Kun80, §II.7]). We will occasionally use this method here for the construction of examples. In Cahpter 2, a strengthening of $\diamondsuit$ is used to construct Souslin algebras with extremely strong homogeneity properties.

Other well-known Souslin trees are generic, i.e., they exist in the generic extensions of the universe by generic filters for a specific forcing partial order. Such are the p.o. of countable partial functions on ω_1, which adjoin a Cohen subset of ω_1 – this forcing is the equivalent to Jech's forcing with trees of countable height –, or the p.o. of finite partial functions from ω_1 to ω which adjoins $\aleph_1$ Cohen reals and is equivalent to Tennenbaums forcing with finite trees. In order to get a Souslin tree in a generic extension it is already sufficient to adjoin a single Cohen real, as Shelah has shown in [She]. Todorcevic has found a definition for a term with a free variable for a real, which yields a Souslin tree in V if the variable is replaced by a real r which is Cohen-generic over some inner model M of V, such that $M[r] = V$.

Boolean algebras

For the background on Boolean algebras we refer mainly to volume one of the *Handbook of Boolean Algebras*, [Kop89]. A Boolean algebra B is called

- homogeneous, if for every pair a, b of elements strictly between 0 and 1, there is an automorphism of B mapping a to b;

- rigid, if it does not admit any non-trivial automorphism;
- simple, if it has no proper atomless and complete subalgebra.

Since we are concerned with complete subalgebras, we briefly point out a fact that is important for our arguments but maybe not so well-known.

If A is a complete subalgebra of the complete Boolean algebra B, then we denote the canonical projection from B onto A by

$$h : B \to A, \quad b \mapsto \prod\{a \in \mathbb{A} \mid b \leq_B a\}.$$

(Here we follow [DJ74] instead of [Kop89].) The canonical projection respects arbitrary sums, but it is in general only sub-multiplicative, i.e. we only have $h(xy) \leq_B h(x)h(y)$ for sure, though if x is an element of A, then $h(xy) = xh(y)$.

For an element b of B we consider the algebra

$$b \cdot A := bA := \{ab \mid a \in A\},$$

which is a complete subalgebra of $B \restriction b$.

Proposition 1.0.2. *In the situation described above, the restriction of h is an isomorphism between bA and $A \restriction h(b)$. The converse isomorphism is given by multiplication with b.*

Proof. We first show that $\varphi := h\restriction(bA) : bA \to A\restriction h(b)$ is a Boolean homomorphism. Since h respects sums, it is sufficient to show that also complements are respected. For an arbitrary element x of bA we know that $x = h(x)b$ and therefore $b - x = b - h(x)$. But $b - x$ is the complement of x in bA, so

$$h(b - x) = h(b - h(x)) = h(b) - h(x),$$

which is the complement of $h(x)$ in $A\restriction h(b)$.

If for $x, y \in bA$ we have $h(x) = h(y)$, then $x = bh(x) = bh(y) = y$, so φ is 1-1. Finally, every element a of A below $h(b)$ is hit by $ba \in bA$. Since φ is thus also onto, it is an isomorphism. □

Souslin algebras

A *Souslin algebra* is an atomless and complete Boolean algebra that has only countable antichains and satisfies the $\aleph_0$-distributive law: for all families a_{ij} with $i \in \omega$ and $j \in J$, where J is arbitrary, the following equation holds:

$$\sum_{i\in\omega} \prod_{j\in J} a_{ij} = \prod \left\{ \sum_{i\in\omega} a_{if(i)} \mid f \in {}^{\omega}J \right\}.$$

We call a Souslin algebra $\mathbb{B}$ *small*, if it has a family of complete generators of size $\aleph_1$. In this case, we always find a dense subset T of $\mathbb{B}$, such that $(T, <)$ is a normal Souslin tree, where $<$ is $>_{\mathbb{B}}$, the reversed natural order of $\mathbb{B}$. (In [DJ74] this is used as a definition of "Souslin algebra". Since there can be non-small Souslin algebras, we stick to the general definition and always declare when smallness is needed.) Furthermore, this $\mathbb{B}$ is isomorphic to the algebra $\mathrm{RO}\,T = \mathrm{RO}(T, <)$ of regular open subsets of T, where T carries the (already mentioned) partial order topology generated by the sets $T(t)$ for $t \in T$ (cf. [Kop89, Theorem 14.20]). Following [DJ74], we call such a T a *Souslinization of* $\mathbb{B}$.

In this chapter we only consider small Souslin algebras. The following classical Restriction Lemma for isomorphisms of Souslin algebras implies that two Souslinizations of a Souslin algebra coincide on a club set of levels, cf. [DJ74, Lemma VIII.9].

Lemma 1.0.3 (Restriction Lemma). *If $\varphi : \mathbb{A} \to \mathbb{B}$ is an isomorphism between the small Souslin algebras $\mathbb{A}$ and $\mathbb{B}$, and S is a Souslinization of $\mathbb{A}$ and T of $\mathbb{B}$, then there is a club set C of ω_1 such that the restriction of φ to $S{\restriction}C$ is a tree isomorphism between $S{\restriction}C$ and $T{\restriction}C$.*

Concerning the operations of tree product, tree sum and relativisation, we remark, that the following isomorphisms are canonical:

1. the tree product corresponds to the (Dedekind-completion of the) free product of Boolean algebras as defined in [Kop89, §11]:

$$\mathrm{RO}(S \otimes T) \cong \overline{\mathrm{RO}\,S \oplus \mathrm{RO}\,T};$$

2. the tree sum corresponds to the Cartesian product of Boolean algebras:

$$\mathrm{RO}(S \oplus T) \cong \mathrm{RO}\,S \times \mathrm{RO}\,T;$$

3. for every $t \in T$ we have $\mathrm{RO}\,T(t) \cong (\mathrm{RO}\,T){\restriction}t$.

1.1 Representing subalgebras

Subalgebras of Souslin algebras have been considered e.g. in [Jec72b] or [BB89, §5] (both deal with the same result), [KM83] and more implicitely in [DJ74] or [Lar99, §8]. To represent a subalgebra $\mathbb{A}$ of the small Souslin algebra $\mathbb{B}$ with respect to some Souslinization T of $\mathbb{B}$, the first three sources define a *good equivalence relation* on the tree T, while the last two use maps between trees $T\restriction C$ (for some club set $C \subseteq \omega_1$) and a Souslinization S of $\mathbb{A}$.

We combine the two approaches in so far, that we will consider equivalence relations, which directly induce the relevant mappings between the Souslinizations. Recall that all trees under consideration are assumed to be normal, $\aleph_0$-branching and of height $\leq \omega_1$.

Definition 1.1.1. a) Let T be a normal, $\aleph_0$-branching tree of height $\mu \leq \omega_1$ with countable levels. An equivalence relation $\equiv$ on T is a *tree equivalence relation (t.e.r.)* if

i) $\equiv$ respects levels, i.e., $s \equiv t$ only if $\mathrm{ht}_T\, s = \mathrm{ht}_T\, t$;

ii) $\equiv$ is compatible with $<_T$, i.e., for $s <_T s'$ and $t <_T t'$ with s and t of the same height, $s' \equiv t'$ implies $s \equiv t$;

iii) the induced partial order on the set $T/\!\equiv$ of $\equiv$-cosets given by

$$a <_{T/\equiv} b \iff (\exists s \in a,\, t \in b) s <_T t$$

for $a, b \in T/\!\equiv$ is a normal, $\aleph_0$-branching tree order.

b) If T souslinizes $\mathbb{B}$ and $\mathbb{A}$ is an atomless and complete subalgebra of $\mathbb{B}$, we say that the t.e.r. $\equiv$ on T *represents* $\mathbb{A}$ on T if the sums over the $\equiv$-classes form a dense subset of $\mathbb{A}$:

$$\left\langle \sum s/\!\equiv \mid s \in T \right\rangle^{\mathrm{cm}} = \mathbb{A}.$$

Remark 1.1.2. A few remarks on Souslin subalgebras and t.e.r.s are in order.

1. Every complete and atomless subalgebra of a Souslin algebra is Souslin as well. This is why we also call them Souslin subalgebras.

2. If $\mathbb{A}$ is a complete and atomless subalgebra of the Souslin algebra $\mathbb{B}$, then there are a club C of ω_1 and a t.e.r. $\equiv$ on $T\restriction C$ which represents $\mathbb{A}$.

3. Every t.e.r. on a Souslin tree T represents some Souslin subalgebra.

4. On limit levels a t.e.r. is determined by its behaviour on the levels below, because of the requirement that the quotient tree be normal.

5. And finally: in the context of Souslin algebras the lower bound for the branching number is no restriction, since for every normal Souslin tree T the restriction $T\restriction C$ is $\aleph_0$-branching, if C only contains limit ordinals.

The concept of t.e.r. allows redundant information in the sense that s and t can be equivalent even though s and t are separated by an element of the subalgebra represented by the t.e.r. This kind of redundance cannot be completely eliminated, but we can consider t.e.r.s of higher accuracy.

Definition 1.1.3. a) A t.e.r. $\equiv$ on T is called *nice*, if for all s, s', t in T with $s <_T s'$ and $s \equiv t$ there is some $t' >_T t$ with $s' \equiv t'$.

b) A t.e.r. $\equiv$ on T is called *almost nice*, if for all s, s', t in T with $s <_T s'$ and $s \equiv t$ and $\mathrm{ht}(s) = \alpha + 1$ for some α there is some $t' >_T t$ with $s' \equiv t'$.

c) A t.e.r. $\equiv$ on T is called *decent*, if there is a tree T' and an almost nice t.e.r. $\equiv'$ on T' and there is an isomorphism $\varphi : T \to T'\restriction C$ with some club $C \subseteq \mathrm{ht}\, T'$ such that for all $s, t \in T$ we have $s \equiv t$ if and only if $\varphi(s) \equiv' \varphi(t)$.

Obviously every nice t.e.r. is almost nice and every almost nice t.e.r. is decent. If $\equiv$ is an almost nice t.e.r. then there is essentially only one kind of violation of the niceness condition: If $s \equiv t$ are limit nodes and there is some $s' > s$ such that no successor t' of t is equivalent to s', then the same applies to a *direct successor* of s already.

Unlike niceness, almost niceness is not handed down to restrictions of the form $T\restriction C$, so for general applications, decency is the right property (cf. Proposition 1.1.7). We will begin our studies of the different degrees of niceness with a result on the effects of the strongest of these properties. Recall the definition of the basic projection h of a Boolean algebra B onto a complete subalgebra A: $h(b) = \prod\{a \in A \mid b \leq_B a\}$, i.e., $h(b)$ is the minimal element of A above b.

The following definition is taken from [DJ74, p.85] while the above definition of a nice t.e.r. is extracted from the way the notion of a nice subalgebra is used in the same source.

Definition 1.1.4. Let $\mathbb{B}$ be a small Souslin algebra. We call a subalgebra $\mathbb{A}$ of $\mathbb{B}$ *nice* if it is atomless and complete and there are Souslinizations S of $\mathbb{A}$ and T of $\mathbb{B}$ s.t. $h''T_\alpha = S_\alpha$ for all $\alpha < \omega_1$.

Lemma 1.1.5. *Let $\mathbb{A}$ be a complete and atomless subalgebra of the small Souslin algebra $\mathbb{B}$. Then $\mathbb{A}$ is nice in $\mathbb{B}$ iff there is a nice t.e.r. $\equiv$ on some Souslinization T of $\mathbb{B}$ which represents $\mathbb{A}$.*

Proof. Suppose $\mathbb{A}$ is nice in $\mathbb{B}$ and let S, T be Souslinizations as in definition 1.1.4. Define $\equiv$ on T by

$$s \equiv t :\iff h(s) = h(t).$$

Then $\equiv$ clearly is a t.e.r. on T. Now given $s, s', t \in T$, s.t. $s \equiv t$ and $s' <_{\mathbb{B}} s$ we have to find below t a node $t' \in T$ which is $\equiv$-equivalent to s'. To reach a contradiction, assume that $s' \not\equiv t'$ for all $t' <_{\mathbb{B}} t$. Let $\gamma = \mathrm{ht}(s')$. Since $\mathbb{A}$ is a nice subalgebra as witnessed by T and S, we have

$$h(t') \cdot h(s') = 0$$

for all $t' <_{\mathbb{B}} t$ of level γ. But then, again by the niceness of $\mathbb{A}$,

$$h(s) = \sum\{h(r) \mid r \in T_\gamma,\, r <_{\mathbb{B}} s\} \neq \sum\{h(r) \mid r \in T_\gamma,\, r <_{\mathbb{B}} t\} = h(t)$$

which is the desired contradiction.

For the converse let $\equiv$ be a nice t.e.r. (without loss of generality assume $C = \omega_1$), such that the set S of sums over the $\equiv$-classes is a Souslinization of $\mathbb{A}$:

$$T/\!\equiv\, \cong S := \{\sum t/\!\equiv \mid t \in T\} \text{ souslinizes } \mathbb{A}.$$

We have to show that for all $t \in T$ we have $h(t) = \sum t/\!\equiv$.

Clearly we already have $t \leq_{\mathbb{B}} \sum t/\!\equiv\, \in \mathbb{A}$, so $h(t) \leq_{\mathbb{A}} \sum t/\!\equiv$. Now if $h(t) \neq \sum t/\!\equiv$ for $t \in T_\alpha$, then there is some $s' \in T_\gamma$, $\gamma > \alpha$, s.t.

$$\sum s'/\!\equiv\, \leq_{\mathbb{A}} \left(\sum t/\!\equiv\right) - h(t) \in \mathbb{A}$$

by the denseness of S in $\mathbb{A}$. Letting $s \restriction \alpha \in T_\alpha$ be the unique $s >_{\mathbb{B}} s'$ we have $s \equiv t$. But then our hypothesis on $\equiv$ implies the existence of some $t' \in T_\gamma$ below t and equivalent to s'. Therefore $(\sum s'/\!\equiv) \cdot h(t) > t'$ which contradicts our choice of s'. □

The last argument of the proof above is also applicable to almost nice t.e.r.s and successor nodes.

Corollary 1.1.6. *Let $\mathbb{B}$ be souslinized by T and $\mathbb{A}$ represented by the almost nice t.e.r. $\equiv$ on T. Then for all successor nodes $s \in T_{\alpha+1}$ we have $h(s) = \sum s/\!\equiv$.*

As will be shown in Theorems 1.3.5 and 1.3.9, every homogeneous Souslin algebra has both nice subalgebras and also atomless and complete subalgebras which are not nice.

But does every atomless and complete subalgebra of $\mathbb{B}$ have an almost nice representation? The affirmative answer (modulo the choice of an appropriate Souslinization of $\mathbb{B}$) is given by the next proposition, which is a variant of the well-known result that every Souslin algebra with a set of complete generators of size $\aleph_1$ contains a Souslin tree that is densely embedded (cf. [Kop89, Theorem 14.20]).

Proposition 1.1.7. *Let $\mathbb{A}$ be an atomless, complete subalgebra of $\mathbb{B}$ a Souslin algebras and T a Souslinization of $\mathbb{B}$. Then there is a club $C \subseteq \omega_1$ and a decent t.e.r. $\equiv$ on $T{\restriction}C$ that represents $\mathbb{A}$ on $T{\restriction}C$.*

Proof. We construct a Souslinization S of $\mathbb{B}$ on which there is an almost nice t.e.r. $\equiv'$ representing $\mathbb{A}$. Then by the Restriction Lemma for Isomorphisms 1.0.3 we know that there is a club C of ω_1 with $T{\restriction}C \cong S{\restriction}C$ which witnesses that the t.e.r. $\equiv$ on T that is induced by $\equiv'$ is decent.

Before constructing S, we describe a method of refining a given partition P of unity in $\mathbb{B}$ to a partition R in $\mathbb{B}$ with the property, that $h''R$ is a partition in $\mathbb{A}$. Let Q be the set of atoms of $\langle h''P\rangle^{\mathrm{cm}}$ and define

$$R = \{pq \mid p \in P,\, q \in Q\} \setminus \{0\}.$$

Then R refines P and for $pq \in R$ we have $h(pq) = qh(p) = q$ since q is an atom. So $h''R = Q$.

Fix a dense set $\{x_{\alpha+1} \mid \alpha \in \omega_1\}$ of $\mathbb{B}$ indexed by successor ordinals. Starting with $S_0 = \{1\}$ let P_α be any partition in $\mathbb{B}$ refining S_α in such a way that every $s \in S_\alpha$ is divided in infinitely many parts and $x_\alpha \in \langle P_\alpha\rangle^{\mathrm{cm}}$. Then let $S_{\alpha+1}$ be the refinement of P_α with respect to h as described above. So $h''S_{\alpha+1}$ is a partition in $\mathbb{A}$. The limit levels of S are canonically defined as

$$S_\alpha := \left\{\prod b \mid b \in [S{\restriction}\alpha]\right\} \setminus \{0\}.$$

Thus S is a Soulinization of $\mathbb{B}$. Next we define $\equiv'$ on successor levels. For $s, t \in S_{\alpha+1}$ let

$$s \equiv' t :\Longleftrightarrow\ h(s) = h(t).$$

Once again, for limit levels there is no freedom in the choice of $\equiv'$ since $S/{\equiv'}$ has to be a normal tree. For α a limit ordinal and $s, t \in S_\alpha$ let

$$s \equiv' t :\Longleftrightarrow\ (\forall \gamma < \alpha)\, s{\restriction}\gamma \equiv' t{\restriction}\gamma.$$

It remains to show that $\equiv'$ is almost nice. So let $s \equiv' t$ on some successor level $S_{\alpha+1}$ and $s' >_S s$. Without loss of generality we can assume that also s' is of successor height γ. Then we have

$$h(s') \leq_{\mathbb{B}} h(s) = h(t) = \Sigma\{h(t') \mid t' \in S_\gamma,\, t <_S t'\}.$$

Since $h''S_\gamma$ is a partition, we have $s' \equiv' t'$ for some $t' >_S t$ on level S_γ. □

By the next corollary, given a subalgebra $\mathbb{A}$ of $\mathbb{B}$ and an arbitrary Souslinization T of $\mathbb{B}$, we can test $\mathbb{A}$ for niceness on an arbitrary restriction of T to a stationary set of levels.

Corollary 1.1.8. *If $\mathbb{A}$ is a nice subalgebra of $\mathbb{B}$ and T souslinizes $\mathbb{B}$ and carries a decent t.e.r. $\equiv$ representing $\mathbb{A}$, then there is a club $C \subset \omega_1$, such that $\equiv$ is nice when restricted to $T{\restriction}C$.*

Proof. Since the intersection of two club sets of ω_1 is club again, we can assume that $\equiv$ is even almost nice on T. Denote the canonical mapping $T \to \mathbb{A}$ by

$$\pi : T \to \mathbb{A}, \quad t \mapsto \sum t/\!\equiv .$$

We know that $\pi(r) \geq_{\mathbb{B}} h(r)$ for all nodes $r \in T$ and for a successor node r even $\pi(r) = h(r)$ by the almost niceness of $\equiv$.

We have to find a club C such that π and the projection h coincide on $T{\restriction}C$. Now let S be a Souslinization of $\mathbb{B}$ such that $h''S$ souslinizes $\mathbb{A}$ and fix by the Restriction Lemma 1.0.3 a club D of ω_1 with $T{\restriction}D = S{\restriction}D$. Let C be the set of all limit points of D, i.e., $C := \{\gamma \in D \mid \bigcup(D \cap \gamma) = \gamma\}$. Now pick $\alpha \in C$ and $t \in T_\alpha$. Then

$$\pi(t) \geq_{\mathbb{B}} h(t) = \prod_{\gamma \in D \cap \alpha} h(t{\restriction}\gamma) \geq_{\mathbb{B}} \prod_{\gamma \in D \cap \alpha} h(t{\restriction}\gamma + 1) = \prod_{\gamma \in D \cap \alpha} \pi(t{\restriction}\gamma + 1) = \pi(t),$$

since π respects products along branches by the normalitiy requirement on $T/\!\equiv$. □

We can also improve our representation result of Proposition 1.1.7 a little bit, if $\mathbb{A}$ is not too bad. With the next definition we capture the property of being not even locally nice.

Definition 1.1.9. We say that $\mathbb{A}$ is *nowhere nice* in the small Souslin algebra $\mathbb{B}$ if $\mathbb{A}$ is an atomless and complete subalgebra of $\mathbb{B}$ and for every $b \in \mathbb{B}$ the algebra $b\mathbb{A} = \{ba \mid a \in \mathbb{A}\}$ is not nice in $\mathbb{B}{\restriction}b$.

Proposition 1.1.10. *Let $\mathbb{B}$ be a small Souslin algebra and $\mathbb{A}$ a complete and atomless subalgebra of $\mathbb{B}$. Let*

$$b := \sum \{x \in \mathbb{B} \mid x\mathbb{A} \textit{ is nice in } \mathbb{B}\restriction x\}.$$

Then $b\mathbb{A}$ is nice in $\mathbb{B}\restriction b$ and $(-b)\mathbb{A}$ is nowhere nice in $\mathbb{B}\restriction(-b)$.

Proof. It follows directly from the definitions that $(-b)\mathbb{A}$ is nowhere nice.

Clearly, the property "$x\mathbb{A}$ is nice in $\mathbb{B}\restriction x$" descends from x to $y \leq_{\mathbb{B}} x$, for if $T, \equiv$ is a nice representation for $x\mathbb{A}$ in $\mathbb{B}\restriction x$, then choose $\alpha < \omega_1$ such that y is in $\langle T_\alpha \rangle^{\mathrm{cm}}$, the complete subalgebra of $\mathbb{B}\restriction x$ generated by T_α. With $Y = \{t \in T_\alpha \mid t \leq_{\mathbb{B}} y\}$ the tree $R = \bigoplus_{t \in Y} T(t)$ souslinizes $\mathbb{B}\restriction y$ and $\equiv$ remains nice, when restricted to R.

We prove that this local niceness property is also preserved under taking arbitrary sums. So let M be a subset of $\mathbb{B}$, such that all elements of M have this property. We want to show that for $x := \sum M$ the subalgebra $x\mathbb{A}$ is nice in $\mathbb{B}\restriction x$. We can without loss of generality assume that M is an antichain. Then M is countable. Furthermore we can assume that also $h''M$ is an antichain by the argument used at the beginning of the proof of Proposition 1.1.7. We can furthermore assume that there is a Souslinization T of B such that M is a subset of T_1, the first nontrivial level of T, and T carries a decent t.e.r. $\equiv$ which represents $\mathbb{A}$.

Now for every element r of M there is by Corollary 1.1.8 a club C_r of ω_1, such that $\equiv$ is nice on $T(r)\restriction C_r$. Let C be the club intersection of all sets C_r for $r \in M$. We claim that $\equiv$ is nice on the subtree

$$S = \bigoplus_{r \in M} T(r)\restriction C$$

of T. So let $s \equiv t$ in S and $s' > s$. If there is a unique member r of M below both nodes s and t, then we can apply the hypothesis on r. Otherwise we still have $r_s := s\restriction 1 \equiv t\restriction 1 =: r_t$ and $h(r_s) = h(r_t)$ by our assumption. But then we have with Proposition 1.0.2, that $r_t h(s') > 0$. So there is a node $t^* > r_t$ equivalent to s'. Finally, by niceness above r_t, there also is $t' \equiv t^* \equiv s'$. $\square$

1.2 Automorphisms and large subalgebras

Definition 1.2.1. Let B be a complete Boolean algebra. We say that A is a *large subalgebra of* B, if A is a complete subalgebra of B and there is a countable subset X of B, such that $\langle A \cup X\rangle^{\mathrm{cm}} = B$.

Note first, that being a large subalgebra of B is not only a property of A but of the pair (B, A). For this reason we occasionally speak of large embeddings in this context.

Note also that large subalgebras of Souslin algebras are always atomless and therefore Souslin algebras, since for every atom a of $\mathbb{A}$, the countable set $X \cup \{a\}$ would have to generate the relative algebra $\mathbb{B}\restriction a$. But this is impossible, because $\langle X\rangle^{\mathrm{cm}}$ is atomic itself by the distributivity of $\mathbb{B}$ (cf. [Kop89, Proposition 14.8]). It also follows from distributivity that we can assume the set X from the definition to be an antichain of $\mathbb{B}$, if $\mathbb{A}$ is a large subalgebra of a Souslin algebra $\mathbb{B}$.

The most elementary example of a Souslin algebra with a large subalgebra is the following.

Example 1.2.2. Assume that $\mathbb{B}$ is Souslin and there is some b in $\mathbb{B}$, such that the relative algebras $\mathbb{B}\restriction b$ and $\mathbb{B}\restriction -b$ are isomorphic, say by the map $\varphi : \mathbb{B}\restriction b \to \mathbb{B}\restriction -b$. Then the set $\mathbb{A}$ of elements of the form $a + \varphi(a)$ for $a \leq b$ is a complete subalgebra of $\mathbb{B}$ and isomorphic to $\mathbb{B}\restriction b$ and therefore atomless. Here we even have $\langle \mathbb{A} \cup \{b\}\rangle = \mathbb{B}$, since every $c \in \mathbb{B}$ we be written as

$$c = b(cb\varphi(cb)) + (-b)(c - b)\varphi^{-1}(c - b),$$

which clearly lies in the subalgebra generated by $\mathbb{A}$ and b.

In the same manner we can construct more and more large subalgebras.

Example 1.2.3. Let X be an infinite partition of unity in $\mathbb{B}$ such that all the relative algebras $\mathbb{B}\restriction a$ for $a \in X$ are pairwise isomorphic. Fix a family of commuting isomorphisms $\varphi_{ab} : \mathbb{B}\restriction a \to \mathbb{B}\restriction b$ between these relative algebras, i.e., $\varphi_{ac} = \varphi_{bc} \circ \varphi_{ab}$ for all $a, b, c \in X$. (This can be done by first picking a member $a_0 \in X$ together with isomorphisms $\varphi_b : \mathbb{B}\restriction a_0 \to \mathbb{B}\restriction b$ for all $b \in X$ such that $\varphi_{a_0} = \mathrm{id}$ and then letting $\varphi_{ab} := \varphi_b \circ \varphi_a^{-1}$.) For every subset $Y \subseteq X$ we can assembly the corresponding relative algebras in the same manner as above:

$$\sum_{b \in Y} \varphi_{ab}(c) \quad \text{for } c \leq a$$

(here we need, that the isomorphisms φ_{ab} commute). Given $Y \subset X$ we get a large subalgebra $\mathbb{A}_Y$ of $\mathbb{B}\restriction(\sum Y)$. Considering Cartesian products, we get for every partition of X another large subalgebra of $\mathbb{B}$. And if Q is a partition of X refining P, then $\mathbb{A}_P$ is a subalgebra of $\mathbb{A}_Q$.

In particular, our Souslin algebra $\mathbb{B}$ has $2^{\aleph_0}$ large subalgebras.

The next example shows the possibility of the other extreme.

Example 1.2.4 (Only one non-trivial subalgebra)**.** Let $\mathbb{B}$ be a simple Souslin algebra, i.e., $\mathbb{B}$ has no proper atomless and complete subalgebra. (Such an algebra was constructed, assuming $\diamondsuit$, in [Jec72b]. Or take the regular open algebra of a full tree T, cf. Section 1.5.2 and Lemma 1.5.7.) We claim that the Souslin algebra $\mathbb{C} := \mathbb{B}\times\mathbb{B}$ has only one proper atomless and complete subalgebra, which is furthermore large in $\mathbb{C}$.

Clearly, $\mathbb{C}$ has the large subalgebra $\mathbb{A} := \{(b, b) \mid b \in \mathbb{B}\}$. But as we have $\mathbb{A} \cong \mathbb{B}$, there are no (atomless and complete) subalgebras of $\mathbb{C}$ below $\mathbb{A}$.

On the other hand we have $\mathbb{C}\restriction(1, 0) \cong \mathbb{C}\restriction(0, 1) \cong \mathbb{B}$. So if there was any other atomless and complete subalgebra $\mathbb{A}'$ of $\mathbb{C}$, then $(0, 1) \cdot \mathbb{A}'$ or $(1, 0) \cdot \mathbb{A}'$ would be a nontrivial subalgebra of the respective relative algebra of $\mathbb{C}$. But the latter are simple, and the existence of such a subalgebra $\mathbb{A}'$ is impossible.

In general, large subalgebras always occur when a Souslin algebra has non-trivial symmetries.

Theorem 1.2.5. *Let $\varphi \in \operatorname{Aut}\mathbb{B}$. Then the set of fixed points of φ is an atomless, complete algebra $\mathbb{A}$ and there is an antichain A of $\mathbb{B}$ such that $\langle \mathbb{A} \cup A\rangle^{\mathrm{cm}} = \mathbb{B}$.*

We give two proofs of this theorem, the first exploiting the fact that automorphisms restrict to certain Souslinizations, while the second is an application of Frolík's Theorem (cf. [Kop89, Theorem 13.23]), a deep result of the theory of complete Boolean algebras.

Proof 1. Since every automorphism of a complete Boolean algebra is complete, $\mathbb{A}$ is also a complete subalgebra of $\mathbb{B}$.

We now show that no $x \in \mathbb{A}^+$ is an atom. Let T be an ω-branching Souslinization of $\mathbb{B}\restriction x$ s.t. $\varphi\restriction T \in \operatorname{Aut} T$. It is clear that for all $t \in T$ the fixed point equation $\varphi(\sum \operatorname{orb}_\varphi t) = \sum \operatorname{orb}_\varphi t$ holds, so $\sum \operatorname{orb}_\varphi t \in \mathbb{A}$. If now for $t \in T_1$ we have $\sum \operatorname{orb}_\varphi t = x$, then $\operatorname{orb}_\varphi t = T_1$. But then for all $s \in T_2$ s.t. $t <_T s$ and for all $n \in \omega \setminus \{0\}$ we have $\varphi^n s \not<_T t$, so $\sum \operatorname{orb}_\varphi s \not\geq_{\mathbb{B}} t$ is a fixed point of φ below x.

To construct the antichain A with $\langle \mathbb{A} \cup A\rangle^{\mathrm{cm}} = \mathbb{B}$ we fix a Souslinization T of $\mathbb{B}$ s.t. φ restricts to an automorphism of T. We assume without loss of generality that the set of non-fixed points of φ is dense in $\mathbb{B}$. Otherwise consider the relative algebra $\mathbb{B}{\restriction}x$ where $x = \sum\{y \in \mathbb{B} \mid (\exists z \leq_{\mathbb{B}} y)\varphi z \neq z\}$.

Now for every $n \in \omega$ the set of nodes t of T that are fixed by φ^n is a subtree. By the subtree lemma (1.0.1) and our assumption we know that for $n = 1$ this subtree is countable.

For $n > 0$ we define the following open sets A_n of T

$$A_n = \{t \in T \mid \varphi^n{\restriction}T_t = \mathrm{id} \text{ and } (\forall 0 < k < n)\varphi^k{\restriction}T_t \neq \mathrm{id}\}$$

and for $n = 0$ we let

$$A_0 = \{s \in T \mid (\forall t)(\forall n > 0)s <_T t \Rightarrow t \notin A_n\}.$$

Again by the subtree lemma we see that for

$n = 0$ the set B_0 of the $s \in A_0$ with $\varphi^k s \neq s$ for every $k > 0$ is dense in A_0, and for

$n > 0$ the set B_n of the $t \in A_n$ with $\varphi^k t \neq t$ for all $k = 1, \ldots, n-1$ is also dense in A_n.

Finally we choose a maximal antichain A of $\bigcup_{n\in\omega} B_n$ and have

$$\sum A = \sum \bigcup B_n = \sum \bigcup A_n = 1_{\mathbb{B}}.$$

So if $t \in T$ lies below $s \in A_n$, then $x = \sum \mathrm{orb}_\varphi\, t \in \mathbb{A}$ and $sx = t$. □

Proof 2. Frolík's Theorem states that for every automorphism f of a complete Boolean algebra A, there is a partition of unity $\{a_0, a_1, a_2, a_3\}$ in A such that $f{\restriction}(A{\restriction}a_0)$ is the identity and for $i > 0$ we have $f(a_i) \cdot a_i = 0$.

We consider the countable family $(\varphi^n \mid n \in \mathbb{Z})$ of automorphisms of $\mathbb{B}$ and let $(a_{n0}, a_{n1}, a_{n2}, a_{n3})$ be a partition of unity given by Frolík's Theorem for φ^n, $n \in \mathbb{Z}$. Let X be the set of atoms of the complete subalgebra of $\mathbb{B}$ that is (completely) generated by the elements $\varphi^k(a_{ni})$ for $k, n \in \mathbb{Z}$ and $i < 4$. Note that $\varphi \restriction X$ is a permutation of X and if for some $x \in X$ and $n \in \omega$ we have $\varphi^n(x) = x$ then the restriction of φ^n to $\mathbb{B}{\restriction}x$ is the identical mapping.

We claim that $\langle \mathbb{A} \cup X\rangle^{\mathrm{cm}} = \mathbb{B}$. Since X is an antichain, it suffices to show that for all $x \in X$ and $b \in \mathbb{B}{\restriction}x$ there is a member $a \in \mathbb{A}$, a fixed point of φ, with $ax = b$. For all integers n we know that either $\varphi^n(b) = b$ or $\varphi^n(b)$ is disjoint from x. Let $a = \{\varphi^n(b) \mid n \in \mathbb{Z}\}$ and the proof is finished. □

In [Jec97, p. 266] Jech considers a complete Boolean algebra $\mathbb{B}$ and a complete subalgebra $\mathbb{A}$ of $\mathbb{B}$, such that the set $\{b \in \mathbb{B} \mid b\mathbb{A} = \mathbb{B}\restriction b\}$ is dense in $\mathbb{B}$. (The effect on forcing with such a subalgebra is stated in [Jec97, Lemma 25.4]: $\mathbb{A}$ and $\mathbb{B}$ give the same generic extensions.) The following technical lemma about *optimal witnesses of largeness* rests on the fact that we are in that situation when $\mathbb{A}$ is a large subalgebra of $\mathbb{B}$. With these witnesses at hand, we can easily deduce the main structural properties of large embeddings.

Lemma 1.2.6. *Let $\mathbb{A}$ be large subalgebra of $\mathbb{B}$ and let $X := \{x \in \mathbb{B} \mid \mathbb{B}\restriction x = \mathbb{A} \cdot x\}$.*

a) The set X is dense in $\mathbb{B}$ and for $x <_{\mathbb{B}} y \in X$ we have $x \in X$.

b) For every $x \in X$ the restriction of the canonical projection h to $\mathbb{B}\restriction x$ is an isomorphism between $\mathbb{B}\restriction x$ and $\mathbb{A}\restriction h(x)$. The inverse map of $h\restriction(\mathbb{B}\restriction x)$ is given by multiplication with x.

c) Every countable subset $M \subseteq X$ with $\sum M = 1$ (or even $1 - \sum M \in X$) witnesses that $\mathbb{A}$ is large.

d) For every $x \in X$ there is a maximal element y of X above x.

Assume that $\mathbb{A}\restriction a \neq \mathbb{B}\restriction a$ for all $a \in \mathbb{A}^+$ and let Y denote the set of maximal elements of X.

e) The image of Y under h is a maximal antichain of $\mathbb{A}$.

f) Every set M' of pairwise disjoint elements of Y is extendible to a maximal antichain $M \subset Y$ of $\mathbb{B}$.

g) For every maximal antichain $M \subseteq Y$ we have $h''M = h''Y$.

h) Let $a \in h''Y$. Then there is a cardinal $\kappa(a) \leq \aleph_0$, such that for all maximal antichains $M \subseteq Y$ the set $M \cap h^{-1}\{a\}$ has cardinality $\kappa(a)$:

$$|\{y \in M \mid h(y) = a\}| = \kappa(a).$$

The announced optimal witnesses of largeness are simply the partitions of unity A in $\mathbb{B}$ that are subsets of Y defined as above.

Proof. Let $N \subset \mathbb{B}$ be a maximal antichain of $\mathbb{B}$ witnessing that $\mathbb{A}$ is large in $\mathbb{B}$, i.e., such that $\mathbb{B} = \langle \mathbb{A} \cup N \rangle^{\mathrm{cm}}$. Then the denseness of X stated in a) is easily seen, since for every $b \in \mathbb{B}$ there is an element $a \in N$ comparable to b, and we clearly have $\mathbb{A} \cdot (ab) = \mathbb{B}\restriction(ab)$, so $ab \in X$. The second part of a) is trivial.

The proof of b) is a direct application of Proposition 1.0.2 and the definition of the set X.

Now let for the proof of c) $M \subset X$ with $\sum M = 1$. We want to show that every $b \in \mathbb{B}^+$ is of the form

$$b' = \sum\{xh(bx) \in M \mid xb >_{\mathbb{B}} 0\}.$$

It is clear that $b' \geq_{\mathbb{B}} b$, because $h(b), \sum M \geq b$. On the other hand we conclude from part b) that $xh(bx) = bx$ for $x \in X$, so $b' \leq_{\mathbb{B}} b$ as well. So we have $\langle \mathbb{A} \cup M\rangle^{\text{cm}} = \mathbb{B}$.

To prove the existence of maximal elements of X, it is enough to verify that sums over increasing sequences of length ω of elements of X lie in X. So let $x_n \in X$ and $x_{n+1} >_{\mathbb{B}} x_n$ for all n. Set $x = \sum x_n$. We prove that every x_n is in $\mathbb{A} \cdot x$ as follows. Fix n. For every $k > n$ pick an element $a_k \in \mathbb{A}$ that satisfies $x_k a_k = x_n$. Setting $a = \prod_{k>n} a_k$ we get $x_k a = x_n$ for all $k > n$ and therefore $xa = \sum x_k a = x_n$. But then we already have $\mathbb{A} \cdot x = \mathbb{B}\restriction x$, because every element $y \in \mathbb{B}\restriction x$ can be decomposed into $y_n := y(x_{n+1} - x_n)$, and by the above argument we have $y_n \in \mathbb{A} \cdot x$ for all $n \in \omega$.

For part e) of the lemma, we have that $1 = \sum Y$ by a) and d) and therefore $1 = \sum h''Y$. It remains to show that for all pairs $x, y \in Y$ with $h(x)h(y) >_{\mathbb{B}} 0$ we have $h(x) = h(y)$. To reach a contradiction we assume the existence of a pair $x, y \in Y$ with a non-empty intersection of the h-images, $h(x)h(y) >_{\mathbb{B}} 0$, yet $h(x) - h(y) >_{\mathbb{B}} 0$. This implies $x - h(y) >_{\mathbb{B}} 0$ for other wise $h(x)h(y) = 0$. We set $z := y + (x - h(y)) >_{\mathbb{B}} y$ and get $zh(y) = y$. This shows that y, $z - y \in \mathbb{A}^+ \cdot z$, so $z \in X$ (because $z - y <_{\mathbb{B}} x$ so $z - y \in X$), contradicting the maximality of y in X.

The proofs of f) and g) are trivial, so now for h). We assume without loss of generality that $h''Y = \{1\}$ and let $A, B \subseteq Y$ be maximal antichains of $\mathbb{B}$. We inductively transform A by virtue of a cut-and-paste-operation into maximal antichains A_i in $Y = h^{-1}\{1\}$ with the cardinality $|A_i| = |A|$ such that the elements $b_0, \ldots, b_{i-1}$ are in A_i where $b_0, \ldots$ is a fixed enumeration of B of order type $\kappa \leq \omega$.

First enumerate the elements of A with order type $k \leq \omega$ and let $c_i = a_0 h(a_i b_0)$ for $i < k$. For distinct indices i, j we have $h(a_i b_0) h(a_j b_0) = 0$, because $h\restriction(\mathbb{B}\restriction b_0)$ is an isomorphism. So the c_i form a partition of a_0, where $c_i = 0$ is allowed and we have $c_0 = a_0 b_0$ by b). Then clearly the set $A_1 := \{b_0\} \cup \{c_i + (a_i - b_0) \mid i < k\}$ is a partition in $\mathbb{B}$. Finally we have

$$h(c_i + (a_i - b_0)) = h(c_i) + h(a_i - b_0) = h(a_0)h(a_i b_0) + h(b_0 - a_i)$$

$$= h(a_i b_0 + (b_0 - a_i)) = h(b_0) = 1$$

so indeed $A_i \subset h^{-1}\{1\}$. For the next step, ignore b_0 in A and B and start with an enumeration of A such that $a_1 b_1 >_{\mathbb{B}} 0$. Iterating this procedure we either get $A_k = B$ if A and is finite or we see that both sets, A and B are infinite. □

Corollary 1.2.7. *If $\mathbb{B}$ is homogeneous and $\mathbb{A}$ is large in $\mathbb{B}$, then $\mathbb{A}$ and $\mathbb{B}$ are isomorphic.*

Proof. If, in the notation of the last lemma, we let $A := h''Y$ and $\kappa = |h''Y| \leq \aleph_0$, then we clearly see that $\mathbb{A} \cong \mathbb{B}^\kappa \cong \mathbb{B}$ where the second isomorphism holds by homogeneity and the chain condition. □

Once we have enough local isomorphisms as in part b) of Lemma 1.2.6, it is easy to glue them together in order to obtain a Boolean automorphism. We obtain a converse to Theorem 1.2.5 stating that every large subalgebra of a small Souslin algebra $\mathbb{B}$ is the set of fixed points of some automorphism φ of $\mathbb{B}$.

Theorem 1.2.8. *Given a large subalgebra $\mathbb{A}$ of $\mathbb{B}$, there is an automorphism φ of $\mathbb{B}$ with $\mathbb{A} = \{x \in \mathbb{B} \mid \varphi(x) = x\}$. If Y defined as in Lemma 1.2.6 has an infinite subset which is an antichain, then there are $2^{\aleph_0}$ such automorphisms.*

Proof. We only consider the case that $h(y) = 1$ for all $y \in Y$. Choose a maximal antichain $A \subset Y$. In the case, that A is finite enumerate its elements by $a_0, \ldots, a_n$. For $k < n$ and $x \in \mathbb{B}\restriction a_k$ let $\varphi_k(x) := a_{k+1}h(x)$ and for $x \in \mathbb{B}\restriction a_n$ let $\varphi_n(x) := a_0 h(x)$. Then

$$\varphi : \mathbb{B} \to \mathbb{B}, \quad x \mapsto \sum_{k \leq n} \varphi_k(a_k x)$$

is as required.

Enumerate A by a_k, $k \in \mathbb{Z}$ if infinite. For $k \in \mathbb{Z}$ define φ_k as above in the case $k < n$ and modify the definition of φ in the obvious way.

The cardinality statement is then trivial by the number of possible choices to make. □

We now return to small Souslin algebras and deduce a nice representation for a large subalgebra above an optimal witness of largeness.

Proposition 1.2.9. *Every large subalgebra $\mathbb{A}$ of a small Souslin algebra $\mathbb{B}$ is nice in $\mathbb{B}$.*

Proof. We construct a Souslinization of T which is nice for $\mathbb{A}$. This is similar to the proof of Proposition 1.1.7. Let T_1 be a maximal antichain of $\mathbb{B}$ consisting of maximal elements x with $\mathbb{A} \cdot x = \mathbb{B}\restriction x$, i.e. $T_1 \subset Y$ in the notation used above. Then $A := h''T_1$

is an antichain. Now fix for each $a \in A$ an inverse image $b_a \in h^{-1}(a) \cap T_1$. In order to construct the higher successor levels, we first refine the nodes above b_a for each $a \in A$ and then copy these refinements by virtue of the isomorphisms

$$\psi_{a,b} : \mathbb{B}{\restriction}b_a \to \mathbb{B}{\restriction}b,\ x \mapsto bh(x)$$

for all $b \in h^{-1}(a) \cap T_1$. This automatically transfers to limit levels and guarantees that $h''T_\alpha$ is an antichain also for limit α. We refine T_α such that every $t \in T_\alpha$ is split in infinitely many parts.

Finally, we claim that the relation

$$s \equiv t :\iff \operatorname{ht} s = \operatorname{ht} t \text{ and } h(s) = h(t)$$

is a nice t.e.r., so let $s \equiv t$ on level T_α and let s' be a T-successor of s. Then $t' = (t{\restriction}1) \cdot h(s')$ is the witness of this instance of niceness. □

The representation given above can also be read in the other direction, as a representation of $\mathbb{B}$ with respect to $\mathbb{A}$, which is a very simple case of the sheaf representation of a Boolean algebra over a subalgebra, as in [Kop89, §8.4].

Corollary 1.2.10. *Assume that $\mathbb{A}$ is large in the Souslin algebra $\mathbb{B}$. In the notation of Lemma 1.2.6, we have the following representation of $\mathbb{B}$:*

$$\mathbb{B} \cong \prod_{a \in h''Y} (a\mathbb{A})^{\kappa(a)}.$$

1.3 Nowhere large subalgebras

Definition 1.3.1. Let T be a Souslinization of $\mathbb{B}$ and $\mathbb{A}$ a complete subalgebra of $\mathbb{B}$.

a) $\mathbb{A}$ is *nowhere large* (in $\mathbb{B}$) if for all $b \in \mathbb{B}^+$ we have $b\mathbb{A} \neq \mathbb{B}\restriction b$.

b) A t.e.r. $\equiv$ on T is ∞*-nice* if it is nice and for all $\alpha < \beta < \mathrm{ht}(T)$ and for all $s \in T_\beta$, the projections $t \mapsto t\restriction\alpha$, when restricted to the $\equiv$-class of s, are ∞-to-one, i.e.,

$$\text{for all } r \in (s\restriction\alpha)/{\equiv}: \quad |\{t \in s/{\equiv} \mid t\restriction\alpha = r\}| = \aleph_0.$$

c) $\mathbb{A}$ is ∞*-nice* in $\mathbb{B}$ if there is a club C of ω_1, such that $T\restriction C$ carries an ∞-nice t.e.r. $\equiv$ that represents $\mathbb{A}$, i.e., $\{\sum s/{\equiv} \mid s \in T\}$ is dense in $\mathbb{A}$.

Example 1.3.2. Let S and T be $\aleph_0$-branching trees such that their tree product $S \otimes T$ is Souslin. (E.g., the principle $\diamondsuit$ implies, that for every given Souslin tree S there is a Souslin tree T, such that $S \otimes T$ is c.c.c., cf. [Lar99, Lemma 7.3].) Set $\mathbb{B} := \mathrm{RO}(S \otimes T)$ and $(s,t) \sim (u,v)$ if and only if $s = u$. Then $\sim$ is an ∞-nice t.e.r.: If $s <_S s'$ and $\mathrm{ht}_S(s) = \mathrm{ht}_T(t) = \mathrm{ht}_T(r)$ and $t' >_T t$ then for any $r' > r$ we have that $(s',t') \sim (s',r')$, so $\sim$ is nice. The ∞-part follows from the branching assumption on T. The quotient tree $(S \otimes T)/{\sim}$ is obviously isomorphic to S. And the subalgebra $\mathbb{A}$ represented by $\sim$ is ∞-nice in $\mathbb{B}$.

Proposition 1.3.3. *Let $\mathbb{A}$ be a nice subalgebra of $\mathbb{B}$. Then $\mathbb{A}$ is ∞-nice iff it is nowhere large.*

Proof. Let T souslinize $\mathbb{B}$, and let the nice t.e.r. $\equiv$ on T represent $\mathbb{A}$. We start from left to right. We show for every node s of T, that $\mathbb{A} \cdot s \neq \mathbb{B}\restriction s$. Pick any node t above s in T. Since $\equiv$ is ∞-nice, there is a node r above s and equivalent to t, so $t \notin \mathbb{A} \cdot s$. (So what we actually have shown, is $\mathbb{A} \cdot s = \{0, s\}$.)

For the other implication let $\mathbb{A}$ be nowhere large. We define a club set $C \subset \omega_1$, such that the restriction of $\equiv$ to $T\restriction C$ is ∞-nice. The inductive definition of C is trivial once we have proven the following fact. Given any countable ordinal α there is a $\beta \in (\alpha, \omega_1)$ such that for all nodes $t \in T_\beta$ the set

$$\{r \in t/{\equiv} \mid r\restriction\alpha = t\restriction\alpha\},$$

i.e., the set of nodes above $t\restriction\alpha$ and equivalent to t, is infinite.

To prove the claim, fix $\alpha < \omega_1$ and $s \in T_\alpha$. For pairs of nodes $r <_T t$ define $n(r,t)$ to be the number of successors of r that are equivalent to t. Now for all nodes r, t we have by the niceness of $\equiv$

$$s <_T r <_T t \Rightarrow n(s,r) \leq n(s,t).$$

Now assume towards a contradiction that in each level T_β above T_α there is some node $r \in T_\beta$ for which $n(s,r)$ is finite. Then there is a natural number N such that

$$S := \{r \in T \mid r \leq_T s \text{ or } (s <_T r \text{ and } n(s,r) \leq N)\}$$

is a subtree of T of height ω_1. Assume furthermore that N is minimal with this property. Then there is, by virtue of the Subtree Lemma 1.0.1, a node $r >_T s$ such that for all $t >_T r$ the value $n(s,t)$ is exactly N. Together with $\equiv$'s niceness, this implies that for all $t >_T r$ the $\equiv$-class of t contains only one node above r which is t. But then $\mathbb{A} \cdot r = \mathbb{B}\restriction r$, contradicting the hypothesis on $\mathbb{A}$ to be nowhere large in $\mathbb{B}$. □

1.3.1 ∞-nice subalgebras and homogeneity

Proposition 1.3.4. *Every small and homogeneous Souslin algebra has a homogeneous Souslinization.*

Proof. Let $\mathbb{B}$ be homogeneous and T be any Souslinization of $\mathbb{B}$. Our task is to find a club $C \subset \omega_1$ such that $T\restriction C$ is a homogeneous Souslin tree. By the homogeneity of $\mathbb{B}$ we can choose for every pair $s, t \in T$ of the same height $\alpha < \omega_1$ a Boolean isomorphism $\psi_{st} : \mathbb{B}\restriction s \to \mathbb{B}\restriction t$. By the Restriction Lemma for Isomorphisms between Souslin algebras, there is also a club C_{st} containing α, such that $\psi_{st}\restriction(T(s)\restriction C_{st})$ is an isomorphism onto $T(t)\restriction C_{st}$.

Finally, we define C to be the range of the normal sequence (γ_ν) defined as follows: Set $\gamma_0 = 0$ and let for $\nu < \omega_1$

$$\gamma_{\nu+1} := \min \bigcap_{s,t \in T_{\gamma_\nu}} C_{st} \setminus (\gamma_\nu + 1).$$

□

Theorem 1.3.5. *Every small, homogeneous Souslin algebra has a nice and nowhere large subalgebra.*

Proof. Let T be a homogeneous and $\aleph_0$-branching Souslin tree, i.e., for every pair s, t of nodes on the same level of T there is a tree isomorphism between $T(s)$ and $T(t)$. We show that T carries an ∞-nice t.e.r. $\equiv$ using the homogeneity of T.

After construction stage α we will have the t.e.r. $\equiv$ on $T \restriction \alpha+1$, sets $I_\gamma \subset T_\gamma$ of representatives of the $\equiv$-classes for $\gamma \leq \alpha$ and a family of isomorphisms $\{\varphi_{st} : T(s) \cong T(t) \mid s \equiv_\gamma t, \gamma \leq \alpha\}$. These isomorphisms commute, i.e. $\varphi_{tt} = \mathrm{id}_{T(t)}$ and $\varphi_{st} = \varphi_{rt} \circ \varphi_{sr}$ for all $r \equiv_\gamma s \equiv_\gamma t$, and they have the following coherence property: for $s, t \in T_\alpha$ and $r = s \restriction \gamma$, $u = t \restriction \gamma$ where $\gamma < \alpha$ we have $\varphi_{st} = \varphi_{ru} \restriction T(s)$.

The existence of the φ_{st} will assure that $\equiv$ is a nice t.e.r., and so the $\equiv$-classes on limit levels are not thinned out with respect to their predecessing classes.

In the successor case of $\alpha + 1$, we consider the equivalence relation $\equiv$ on T_α, the set of representatives $I_\alpha \subset T_\alpha$ and the isomorphisms φ_{st} for $s \equiv_\alpha t$, all given by the inductive hypothesis. In order to define $\equiv$ on $T_{\alpha+1}$, we first choose for each $s \in I_\alpha$ a partition of $\mathrm{succ}(s)$ into $\aleph_0$ infinite sets P_n, $n \in \omega$. Then for all $r, t \equiv s$ and $u \in \mathrm{succ}(r)$ and $v \in \mathrm{succ}(t)$ we let $u \equiv v$ if and only if $\varphi_{rs}(u)$ and $\varphi_{ts}(v)$ are in the same partition set P_n. (Of course, if $r, t \in T_\alpha$ are not equivalent, then neither their successors are.) We pick a set of representatives $I_{\alpha+1} \subset \bigcup_{s \in I_\alpha} \mathrm{succ}(s)$. Finally we have to choose the tree isomorphisms φ_{st} for all equivalent pairs $s, t \in T_{\alpha+1}$ such that the above coherence requirement is satisfied. Fix $s \in I_{\alpha+1}$ and choose for T-successors r, t of $s^- := s \restriction \alpha$, both equivalent but unequal to s, isomorphisms φ_{st} and φ_{sr} respectively and let $\varphi_{rt} = \varphi_{st} \circ \varphi_{sr}^{-1}$. For r, t, both equivalent to s, but not necessarily successors of s^-, define

$$\varphi_{rt} := (\varphi_{s^- t^-} \restriction T(v)) \circ \varphi_{uv} \circ (\varphi_{r^- s^-} \restriction T(r)),$$

where $u := \varphi_{r^- s^-}(r)$ and $v := \varphi_{t^- s^-}(t)$.

For a countable limit ordinal, there is only one choice for the equivalence relation $\equiv$ on T_α. For $s, t \in T_\alpha$ we let $s \equiv t$ if and only if $s \restriction \gamma \equiv t \restriction \gamma$ for all $\gamma < \alpha$. Fix $s \in T_\alpha$. For every $\gamma < \alpha$ and $r \equiv s \restriction \gamma$ there is some $t \in T_\alpha$ equivalent to s and above r, namely $t = \varphi_{s \restriction \gamma, r}(s)$. So niceness is maintained up to level α and for these equivalent pairs (s, t) we already have the isomorphisms $\varphi_{st} = \varphi_{s \restriction \gamma, r} \restriction T(s)$ at hand. But there can be equivalent nodes r and t on level α, such that for all their pairs u, v of respective predecessors on the same level we have $\varphi_{uv}(r) \neq t$. However, the $\equiv$-class of s splits into a partition such that for all $r, t \equiv s$, the nodes r and t have such an inherited isomorphism if and only if they are members of the same element of the partition. After choosing a set of representatives J for this partition of the $\equiv$-class of s and fixing isomorphisms φ_{rt} for representatives $r, t \in J$ we can construct the still missing isomorphisms in the same manner as above.

We finally choose a set I_α of representatives for the $\equiv$-classes of T_α without any further restriction. This finishes the construction, and we hope, that it is clear from this construction that the result is an ∞-nice t.e.r. on T. □

Remark 1.3.6. The chain homogeneous Souslin algebras constructed in Chapter 2 have ∞-nice subalgebras by the last theorem. If $\mathbb{B}$ is chain homogeneous, then $\mathbb{B}$ is in particular isomorphic to each of its atomless and complete subalgebras. As a consequence, a chain homogeneous Souslin algebra cannot have pairs of independent subalgebras as the Souslin algebra of the type $\mathrm{RO}(S \otimes T)$ from Example 1.3.2 does: (isomorphic copies of) $\mathrm{RO}\,S$ and $\mathrm{RO}\,T$.

1.3.2 Hidden symmetries and nowhere nice subalgebras

The following is an example of a Souslin algebra without large subalgebras (by rigidity, cf. Theorem 1.2.8) but with an ∞-nice subalgebra. Lemma 1.3.8 shows that in such a case rigidity does not reflect to subalgebras. We will then use the main idea of the proof of Lemma 1.3.8 to deduce the existence of non-nice subalgebras once $\mathbb{B}$ has a non-large subalgebra.

Example 1.3.7 (A rigid Souslin algebra with a nice subalgebra under $\diamondsuit$)**.** We aim at constructing a Souslin tree T with an ∞-nice t.e.r. $\equiv$. Rigidity of $\mathbb{B} = \mathrm{RO}\,T$ is obtained by designing T such that for all club sets C of ω_1 the restricted tree $T{\restriction}C$ is rigid.

Let $(R_\nu)_{\nu<\omega_1}$ be a $\diamondsuit$-sequence. We inductively construct T as an $\aleph_0$-branching tree on the supporting set ω_1 along with the t.e.r. $\equiv$. So in successor steps we appoint to each maximal node $\aleph_0$ direct successor nodes and extend $\equiv$ in any way that maintains ∞-niceness.

In the limit step α we have so far constructed $T{\restriction}\alpha$ and $\equiv$ on this tree. We consider the Polish space $X = [T{\restriction}\alpha]$ of cofinal branches through $T{\restriction}\alpha$ and the equivalence relation $\sim$ on X induced by $\equiv$ via

$$x \sim y \iff (\forall \gamma < \alpha)\, x{\restriction}\gamma \equiv y{\restriction}\gamma.$$

(For a more detailed treatment of the topological terminology use here, we refer to Sections 2.1.3 and 2.2.2.) The $\sim$-classes are perfect (and non-empty) subsets of X. The level under construction, T_α, corresponds to a countable and dense subset Q of X. In order to obtain a nice extension of $\equiv$ we have to choose this countable set Q in a way that guarantees, that for every $\sim$-class $a \subset X$ the set $a \cap Q$ is either empty or dense in a.

Every automorphism φ of $T{\restriction}C$ for some club C of ω_1 induces an auto-homeomorphism $\overline{\varphi}$ on X. In order to achieve a rigid algebra we have to choose some limit levels in a way that prevents some automorphisms (proposed by the $\diamondsuit$-sequence) from extending. This is done by first choosing a branch $x \in X$ and then the dense and countable set $Q \subset X$ such that $x \in Q$ but $\overline{\varphi}(x) \notin Q$.

Now for the choice of Q in the following three cases:

1) If $\alpha < \omega \cdot \alpha$ or R_α is neither a maximal antichain of $T \upharpoonright \alpha$ nor does it code an automorphism of $T \upharpoonright C$ for some club C of α, then we first choose a countable and dense set Q_0 of X. Then let for $x \in Q_0$ be Q_x a countable and dense subset of $x/\sim$ with $x \in Q_x$. Finally set $Q = \bigcup_{x \in Q_0} Q_x$.

2) If $\alpha = \omega\alpha$ and R_α codes an automorphism φ, then we choose $x_0 \in X$ and continue as in the first case, only that we require $x_0 \in Q_0$ and $\overline{\varphi}(x_0) \notin Q_0, Q_x$ for all $x \in Q_0$. This is always possible.

3) If R_α is a maximal antichain of $T \upharpoonright \alpha$, we want, as in classical Souslin tree constructions under $\diamondsuit$, that every node of T_α lies above some node of R_α. A proof of the fact that this can be achieved together with the denseness of Q in the $\sim$-classes of the extended branches can be found in Chapter 2, cf. the Second Reduction Lemma 2.2.15.

Note that we can arrange the coding such that we do not have to consider a coincidence of cases 2) and 3).

Lemma 1.3.8. *Let $\mathbb{A}$ be a nice and nowhere large subalgebra of $\mathbb{B}$. Then there is an nice and nowhere large subalgebra $\mathbb{C}$ of $\mathbb{B}$, such that $\mathbb{A}$ is large in $\mathbb{C}$:*

$$\mathbb{A} \leq_{large} \mathbb{C} \leq_{\infty\text{-}nice} \mathbb{B}.$$

Proof. Let T souslinize $\mathbb{B}$ and $\equiv$ on T be ∞-nice and represent $\mathbb{A}$ in T. Choose a countable limit λ and let $\sim$ coincide with $\equiv$ on $T \upharpoonright (\lambda + 1)$. Now we divide every $\sim$-class a of T_λ in two indexed parts $a = a_0 \dot{\cup} a_1$ such that for every pair $s \in a_0$ and $r <_T s$ there is a node $t \in a_1$ above r and vice versa, i.e., for $r < t \in a_1$ there exists $s \in a_0 \cap T(r)$. This can be done after choosing an enumeration of length ω of the predecessor set $\bigcup \{s \in T \mid (\exists t \in a) s <_T t\}$. These partitions define a map $f : T_\lambda \rightarrow 2$, associating to every node s the index i of its partition member $s \in a_i$.

Now let for $\alpha > \lambda$ and $s, t \in T_\alpha$

$$s \sim t :\Leftrightarrow s \equiv t \text{ and } f(s \upharpoonright \lambda) = f(t \upharpoonright \lambda).$$

Then $\sim$ is clearly ∞-nice when restricted to $T \upharpoonright C$ where $C = \{0\} \cup \omega_1 \setminus \lambda + 1$. This shows that $\mathbb{C} := \langle \sum s/{\sim} \mid s \in T \rangle^{\text{cm}}$ is nice and nowhere large in $\mathbb{B}$.

Furthermore, for every $s \in T$ above level $\lambda + 1$ the $\equiv$-class of s is partitioned into exactly two $\sim$-classes. So we can define the automorphism φ of $(T{\restriction}C)/{\sim}$ that interchanges for each $\equiv$-class the corresponding $\sim$-classes. Then φ naturally extends to $\mathbb{C}$ and has $\mathbb{A}$ as its fixed point algebra which is by Theorem 1.2.5 large in $\mathbb{C}$. □

The main idea of the last proof, that of dividing the classes on a limit level in two "dense" subsets, can be used to construct non-nice subalgebras.

Theorem 1.3.9. *If a small Souslin algebra $\mathbb{B}$ has a nice and nowhere large subalgebra $\mathbb{A}$ then there is a nowhere nice (cf. Definition 1.1.9) subalgebra $\mathbb{A}'$ of $\mathbb{B}$ (and $\mathbb{A}$ is a nice subalgebra of $\mathbb{A}'$).*

Proof. Let T souslinize $\mathbb{B}$ and $\equiv$ represent $\mathbb{A}$ in T. We inductively construct an almost nice, but non-nice refinement $\equiv'$ of $\equiv$, which yields $\mathbb{A}'$ as stated in the theorem. Up to level T_ω the new relation coincides with $\equiv$. Limit levels have to be treated canonically, and on double successor steps $\alpha + 2$ we choose the minimal possible refinement by meeting

$$s \equiv' t :\iff s \equiv t \text{ and } s{\restriction}(\alpha+1) \equiv' t{\restriction}(\alpha+1).$$

To define $\equiv$ on $T_{\alpha+1}$ for a countable limit α, we first refine $\equiv'$ on T_α to the equivalence relation $\sim$ in a way such that every $\equiv'$ class splits in infinitely many $\sim$-classes and for every $s \equiv' t \in T_\alpha$ and $u <_T t$, there is a successor of u in $s/{\sim}$. (One could say that the $\sim$-classes lie densely in the sense of $[T{\restriction}\alpha]$ in the $\equiv'$-classes.) Then let for $s, t \in T_{\alpha+1}$:

$$s \equiv' t :\iff s \equiv t \text{ and } s{\restriction}\alpha \sim t{\restriction}\alpha.$$

We finally show, that no Souslinization S of $\mathbb{B}$ admits a nice t.e.r. representing $\mathbb{A}'$. We only need to consider restrictions $S = T{\restriction}C$ of T for a club set $C \subset \omega_1$. So let $\alpha \in C$ be a limit ordinal and choose $s, t \in T_\alpha$, such that $s \equiv' t$ but $s \not\sim t$. Then for every $r \in T{\restriction}C$ above s there is no successor of t which is $\equiv'$-equivalent to r. So s, t, r witness that $\mathbb{A}'$ is not nice.

If we now let C be the set of all countable limit ordinals joined by 0, and defining on $T/{\equiv'}$ the t.e.r. $\simeq$ by

$$(s/{\equiv'}) \simeq (t/{\equiv'}) :\iff s \equiv t,$$

it is easy to see, that the ∞-niceness decends from $\equiv$ to $\simeq$. □

Corollary 1.3.10. *a) Every small Souslin algebra with an independent pair of complete and atomless subalgebras as well as every homogeneous and small Souslin algebra have non-nice subalgebras.*

b) If there is a non-large, complete and atomless subalgebra of $\mathbb{B}$ then there is also one which is not nice.

Proof. For part a), let $\mathbb{A}, \mathbb{A}'$ be an independent pair of atomless and complete subalgebras of $\mathbb{B}$. If any of $\mathbb{A}$ or $\mathbb{A}'$ is not nice, there is nothing to prove. If $\mathbb{A}$ and $\mathbb{A}'$ are nice subalgebras of $\mathbb{B}$, then so is $\langle \mathbb{A} \oplus \mathbb{A}' \rangle^{\mathrm{cm}}$ by independence of $\mathbb{A}$ and $\mathbb{A}'$. But $\mathbb{A}$ and $\mathbb{A}'$ are also nowhere large, so they are ∞-nice. By Theorem 1.3.9, $\mathbb{B}$ also has a non-nice subalgebra.

Since every homogeneous, small Souslin algebra has an ∞-nice subalgebra, the same argument applies here.

Now assume for the proof of part b), that $\mathbb{A}$ is a nice subalgebra of $\mathbb{B}$ but neither nowhere large nor large. Let

$$b := \sum \{a \in \mathbb{B} \mid a\mathbb{A} = \mathbb{B}\restriction a\}$$

be the sum of all elements in which $\mathbb{A}$ is large. Then $\mathbb{A} \cdot (-b)$ is nowhere large and nice in $\mathbb{B}\restriction(-b)$. But then there is by Theorem 1.3.9 a nowhere nice subalgebra $\mathbb{C}$ of $\mathbb{B}\restriction(-b)$ above $\mathbb{A} \cdot (-b)$. Finally $\langle \mathbb{C} \cup \mathbb{B}\restriction b \rangle$ is complete, atomless and not nice in $\mathbb{B}$. □

In Example 1.2.4 we have seen a Souslin algebra $\mathbb{B}$ with a unique subalgebra which is furthermore large in $\mathbb{B}$ and therefore nice. Now we have proven that we always find a non-nice subalgebra unless there are only large subalgebras in $\mathbb{B}$.

A natural question to ask is, whether there is always a nice subalgebra if there is any (atomless and complete). We do not know the answer. But we can vary Jech's construction of a simple Souslin algebra (using $\diamondsuit$) in order to get a Souslin algebra $\mathbb{B}$ with a nowhere nice subalgebra $\mathbb{A}$, such that $\mathbb{A}$ itself is a simple Souslin algebra and is therefore of another type as the one constructed in the last proof. In particular, below $\mathbb{A}$ there cannot be a *nice* subalgebra of $\mathbb{B}$, since there is none at all.

Nevertheless, it is impossible to use the same trick in order to destroy subalgebras above $\mathbb{A}$.

1.4 Summary of the structure theory

This section collects the structural results we have found so far.

1.4.1 Localisation of embedding properties

The general complete subalgebra $\mathbb{A}$ of a Souslin algebra has not yet been classified by the previous sections. Consider for example a Souslin algebra $\mathbb{B}$ with a nowhere nice subalgebra $\mathbb{A}$. Then $\mathbb{A} \times \mathbb{B}$ has the complete subalgebra $\mathbb{C} := \{(a, a) \mid a \in \mathbb{A}\}$, which is not nice but is large below $(1, 0)$, i.e., $(1, 0) \cdot \mathbb{A}$ is large in $\mathbb{B}\restriction(1, 0)$.

The next theorem states that we always can decompose a Souslin algebra with respect to a given complete subalgebra into relative algebras, where the properties considered appear in their pure form.

Theorem 1.4.1. *Given a small Souslin algebra and a complete subalgebra $\mathbb{A}$ of $\mathbb{B}$, there are uniquely determined and pairwise disjoint elements a, b, c, d of $\mathbb{B}$, such that*

1.) $a + b + c + d = 1_{\mathbb{B}}$*;*

2.) $a\mathbb{A}$ *is atomic;*

3.) $b\mathbb{A}$ *is large in* $\mathbb{B}\restriction b$*;*

4.) $c\mathbb{A}$ *is* ∞*-nice in* $\mathbb{B}\restriction c$*;*

5.) $d\mathbb{A}$ *is nowhere nice in* $\mathbb{B}\restriction d$*.*

Proof. The element a is simply the sum of the atoms of $\mathbb{A}$. So from now on we can assume that $\mathbb{A}$ is atomless. By Proposition 1.1.10 we have to let d be the sum of all elements $x \in \mathbb{B}$ such that $x\mathbb{A}$ is nowhere nice in $\mathbb{B}\restriction x$. And Lemma 1.2.6 gives us, that $b = \{x \in \mathbb{B} \mid x\mathbb{A} = \mathbb{B}\restriction x\}$. It is finally trivial that for $c := 1 - (a + b + c)$ the algebra $c\mathbb{A}$ is nice and nowhere large in $\mathbb{B}\restriction c$. □

1.4.2 Implications between structural properties

We try to give an overview of the structural implications we have found in the form of a diagram. This diagram should be read as: "If the small Souslin algebra $\mathbb{B}$ has a ..., then it also has a ...", where "$\mathbb{B}$ has a homogeneity" has to be translated with "$\mathbb{B}$ is homogeneous".

Except for the places marked by question marks, the diagram contains all the implications there are, as argued below.

		non-trivial automorphism
		$\updownarrow$
homogeneity	$\longrightarrow$	large subalgebra
$\downarrow$		
∞-nice subalgebra	$\longleftarrow$	indep. pair of al., cm. subalgebras whose sum is dense
$\downarrow$?		$\downarrow$?
nowhere large subalgebra	$\longleftarrow$	independent pair of atomless, complete subalgebras
$\updownarrow$		
nowhere nice subalgebra		

Table 1.1: Implication digram for subalgebra and symmetry properties

The only item missing in this diagram is the result of Lemma 1.3.8 on hidden symmetries, i.e., the fact that a Souslin algebra that has an ∞-nice subalgebra also has a non-rigid (and ∞-nice) subalgebra.

We have not included the references to the proofs of the implications, but we hope that the reader can quickly find them in the previous two sections.

We conclude this section with a list of counter-examples for the non-implications. Concerning the hypotheses needed for their existence, we do not know whether the negation of Souslin's hypothesis SH is sufficient. But in the known models of set theory with only few yet any Souslin algebras, e.g. as constructed in [AS93, Section 6], they can all be found except for the (chain) homogeneous ones.

- In Example 1.2.4 we gave a Souslin algebra $\mathbb{B}$ with exactly one proper atomless and complete subalgebra, which is furthermore large. This algebra has therefore no nowhere large subalgebra and is not homogeneous, for it admits only one non-trivial automorphism.

- In the proof of Proposition 1.6.2 we will construct a rigid Souslin algebra $\mathbb{C}$ which is (the completion of) the free product of two Souslin algebras $\mathbb{A}$ and $\mathbb{B}$. These two are therefore independently embedded in $\mathbb{C}$. This shows that a Souslin algebra of the form $\overline{\mathbb{A} \oplus \mathbb{B}}$ does not have to admit any non-trivial automorphisms.

- The main results of Chapter 2 of this thesis are $\diamondsuit^+$-constructions of chain homogeneous Souslin algebras. These are homogeneous and therefore have nowhere large subalgebras, but Lemma 2.3.4 states that they do not admit independent pairs of atomless, complete subalgebras.

As the question marks in the diagram are quite dissatisfactory, we would like to know whether or not the existence of a nowhere nice subalgebra implies that of a nice one, and similarly for the pairs independent subalgebras.

1.5 Strongly homogeneous and full Souslin trees

We take a closer look at two classes of Souslin trees, that are widely known among set theorists, although often under different names. So in each of the sections introducing one of these two classes we give a listing of the names we have found in the literature.

1.5.1 Strong homogeneity

Strongly homogeneous Souslin trees occur quite often in set theoretic literature. In [LT01] they are called coherent Souslin trees and play a central role in the solution of Katetov's Problem on the metrizability of certain compact spaces. Shelah and Zapletal show in [SZ99, Theorem 4.12] that Todorcevic's term for a Souslin tree in one Cohen real is strongly homogeneous, Larson gives a direct $\diamondsuit$-construction ([Lar99, Lemma 1.2]), and also Jensen's construction under the same hypothesis of a 2-splitting, homogeneous tree, as carried out in [DJ74, Chapter IV], is easily seen to be strongly homogeneous.

Definition 1.5.1. A Souslin tree T is called *strongly homogeneous* if there is a family $(\psi_{st} \mid s,t \in T,\ \mathrm{ht}\, s = \mathrm{ht}\, t)$ which has the following properties:

1) ψ_{st} is an isomorphism between the tree $T(s)$ of nodes above s and the tree $T(t)$ of nodes above t and ψ_{ss} is the identity.

2) (commutativity) For all nodes r, s, t of the same level of T we have $\psi_{rt} = \psi_{st} \circ \psi_{rs}$.

3) (coherence) For nodes r, s from the same level, t above r and $u = \psi_{rs}(t)$ we require that ψ_{tu} is the restriction of ψ_{rs} to the tree $T(t) \subset T(r)$.

4) (transitivity) If t and u are nodes on the same limit level T_α, then there is a level T_γ below such that for the corresponding predecessors r of t and s of u we have $\psi_{rs}(t) = u$.

The crucial property of a coherent family is that of *transitivity*, which means that every limit level is a minimal extension of of the initial segment below with respect to the coherent family on that initial segment. Given any homogeneous tree, it is easy to define a family on T with the properties 1-3) above.

One feature of Jensen's tree mentioned above is that it has exactly $\aleph_1$ automorphisms (cf. [DJ74, Theorem IV.4.ii]). As stated in the next proposition, this number of automorphisms (weakened to the continuum for the sake of consistency with $\neg$CH) is a common feature of strongly homogeneous Souslin trees and their regular open algebras. This also shows that the chain homogeneous Souslin algebras constructed in Chapter 2 are not strongly homogeneous, since they are constructed under $\diamondsuit^+$ (which implies CH), and they have $2^{\aleph_1} > 2^{\aleph_0}$ automorphisms. (Here we call a Souslin algebra *strongly homogeneous* if it has a strongly homogeneous Souslinisation.) In general, the size of the automorphism group of a Souslin tree is either finite or between (and including) the continuum and $2^{\aleph_1}$. (This was shown by Jech in [Jec72a].)

Proposition 1.5.2. *Let T be a strongly homogeneous Souslin tree. Then $|\operatorname{Aut} T| = |\operatorname{Aut}(\operatorname{RO} T)| = 2^{\aleph_0}$.*

Proof. It is easily seen (and also follows form the result of Jech just cited), that a homogeneous tree has at least $2^{\aleph_0}$ automorphisms and so has its regular open algebra. On the other hand we show, that for any club $C \subseteq \omega_1$ and $\varphi \in \operatorname{Aut}(T\restriction C)$ we find a maximal antichain A of T, such that $\varphi(t) = \psi_{s\varphi(s)}(t)$ for $t > s \in A$, i.e., φ above A is given by the countably many maps $\psi_{s\varphi(s)}$ for $s \in A$.

To reach a statement contradicting the transitivity of the family (ψ_{st}), we assume that there is a node $r \in T$ such that for each successor s of r there is a node $t \geq s$, such that $\varphi(t) \neq \psi_{s\varphi(s)}(t)$. Without loss of generality we can now assume that $C = \omega_1$ and that r is the root of T. We find inductively an increasing sequence of ordinals α_n such that for all nodes $t \in T_{\alpha_{n+1}}$ we have

$$\varphi(t) \neq \psi_{t\restriction\alpha_n \varphi(t\restriction\alpha_n)}(t).$$

Let α be the supremum of the α_n and $t \in T_\alpha$. Since α is a limit ordinal we by transitivity have an $n \in \omega$ such that $\varphi(t) = \psi_{t\restriction\alpha_n \varphi(t\restriction\alpha_n)}(t)$ which is impossible by the choice of the α_n. □

The next result is our first on the existence of certain decompositions of strongly homogeneous Souslin trees. (A second one follows in Section 1.5.4.) This again gives a proof of the fact that strongly homogeneous Souslin algebras cannot be chain homogeneous, since by Lemma 2.3.4 chain homogeneous Souslin algebras cannot have an independent pair of atomless, complete subalgebras.

Theorem 1.5.3. *Every $\aleph_0$-branching, strongly homogeneous Souslin tree T is (isomorphic to) the tree product of n strongly homogeneous Souslin trees for any given natural number $n > 0$.*

Proof. We give the proof for $n = 2$, from which the general result follows by induction. (It is also easy to draw a direct generalisation to arbitrary $n \geq 2$.) Given the strongly homogeneous Souslin tree T with the coherent family $(\varphi_{s,t} : (\exists\alpha) s, t \in T_\alpha)$ of tree isomorphisms, we inductively define two ∞-nice t.e.r.s

$\equiv$ and $\sim$ on T, which are independent in the following sense: for every pair s, t of nodes of the same height the intersection $s/{\equiv} \cap t/{\sim}$ is non-empty of size one.

We use a partition pattern of rows (for $\equiv$) and columns (for $\sim$) of the Cartesian product $\omega \times \omega$. In the case of T_1 where all nodes are direct successors of a single one, the root, choose a bijection f_0 between T_1 and $\omega \times \omega$. Define for $s, t \in T_1$

$$s \equiv t :\Longleftrightarrow \text{ the first coordinates of } f_0(s) \text{ and } f_0(t) \text{ coincide}$$

and

$$s \sim t :\Longleftrightarrow \text{ the second coordinates of } f_0(s) \text{ and } f_0(t) \text{ coincide.}$$

In successor stage $\alpha + 1$ of the construction the same will be done for the direct successors of a single anchor node $r \in T_\alpha$: Choose a bijection f_α between the $\operatorname{succ}(r)$ and $\omega \times \omega$ and define $\equiv$ and $\sim$ for $s, t \in \operatorname{succ}(r)$ as above. For the copying of this pattern from the set of successors of the anchor node to the remaining nodes of level $\alpha + 1$ we use the automorphisms φ_{ru} where $u \in T_\alpha$, i.e., for $s, t \in T_{\alpha+1}$ with $s \in \operatorname{succ}(u)$ and $t \in \operatorname{succ}(v)$ we have

$$s \equiv t :\Longleftrightarrow u \equiv v \text{ and the first entries of } f_\alpha \circ \varphi_{ur}(s) \text{ and } f_\alpha \circ \varphi_{vr}(t) \text{ coincide}$$

and

$$s \sim t :\Longleftrightarrow u \sim v \text{ and the second entries of } f_\alpha \circ \varphi_{vr}(s) \text{ and } f_\alpha \circ \varphi_{ur}(t) \text{ coincide.}$$

On limit levels of T the relations $\equiv$ and $\sim$ are already determined. It follows directly that $\equiv$ and $\sim$ are (∞-)nice: If $r \equiv s$ and t is above r, then $\varphi_{rs}(t) \equiv t$ is above s.

The coherent families of isomorphisms on $T/{\equiv}$ and $T/{\sim}$ are easily defined using the coherent family on T. For r, s on the same level ant $t \in T(r)$ we simply let

$$\psi_{r/\equiv, s/\equiv}(t/{\equiv}) = \varphi_{rs}(t)/{\equiv} \ .$$

The independence from the choice of the representants follows from the construction and the properties commutativity, coherence and transitivity from the corresponding properties of (φ_{st}). Since the definitions of $\equiv$ and $\sim$ are totally symmetric, the same applies to $\sim$, and so the quotient trees are indeed strongly homogeneous.

To finish the proof we inductively show that on all levels T_α for $0 < \alpha < \omega_1$ each $\equiv$-class meets every $\sim$-class in exactly one node. For T_1 this is clear from the construction, and for a successor $T_{\alpha+1}$ it follows with the same argument from the inductive assumption. So let s, t be nodes of countable limit height α. We search for $r \in T_\alpha$ with $s \equiv r \sim t$ (and then show that it is unique). By transitivity there is $\gamma < \alpha$ such that $\varphi_{uv}(s) = t$ for $u = s{\upharpoonright}\gamma$ and $v = t{\upharpoonright}\gamma$. The inductive assumption for γ yields a unique node $w \in T_\gamma$ with $u \equiv w \sim v$. Set $r := \varphi_{uw}(s) = \varphi_{vw}(t)$. An easy induction then gives $s \equiv r \sim t$. To prove uniqueness, assume that there is a second node r' with this property. Let $\gamma < \alpha$ be large enough that s, t, r, r' are mapped to each other by the automorphisms corresponding to their predecessors on level γ and such that the distinction between r and r' is revealed below level γ. So $r\restriction\gamma \neq r'\restriction\gamma$ yet both lie in the intersection of $u/\equiv$ and $v/\sim$ (with u, v defined as above) – contradiction! □

Though, of course, not every tree product of two strongly homogeneous Souslin trees is Souslin again (e.g. take $T \otimes T$), there is a converse to the last theorem.

Proposition 1.5.4. *If S and T are strongly homogeneous Souslin trees and the tree product $S \otimes T$ satisfies the c.c.c., then $S \otimes T$ is a strongly homogeneous Souslin tree as well.*

Proof. Let (φ_{rs}) and (ψ_{tu}) be the coherent families of S and T respectively. Then it is easily verified that the maps

$$\rho_{rt,su} : (S \otimes T)(r,t) \to (S \otimes T)(s,u), \quad (x,y) \mapsto (\varphi_{rs}(x), \psi_{tu}(y))$$

form a coherent family of isomorphisms for the product tree. □

1.5.2 Fullness

Also full Souslin trees are quite ubiquitous, though they often appear under different names. E.g. the well known two types of generic Souslin trees as constructed by Jech and Tennenbaum to show the consistency of the negation of Souslin's hypothesis are full. The term "full" is taken from Jensen's handwritten notes [Jen], where he shows, that his tree as constructed under $\diamondsuit$ in [DJ74, Theorem V.1.1] is full. Abraham and Shelah describe the same property in [AS93] as being Souslin and all derived trees being Souslin, too. Full trees are called free trees in [SZ99, §4.0] and [Lar99, §8]. Fuchs and Hamkins in [FH06] introduce the properties of being n-fold Souslin off the generic branch and the n-absolute UBP for all $n \in \omega$. We introduce the parametrised notion of n-fullness to demonstrate the uniform equivalence of these two concepts.

Definition 1.5.5. a) Let n be a natural number. We say that a Souslin tree T is *n-full* if for every subset P of size n of some T_α, $\alpha < \omega_1$, the tree product $\bigotimes_{s \in P} T(s)$ satisfies the countable antichain condition. The tree T is *full* if it is n-full for all $n \in \omega$.

b) A Souslin tree is said to be *n-fold Souslin off the generic branch*, if for any sequence $\vec{b} = (b_0, \dots, b_{n-1})$ generic for the n-fold forcing product of (the inverse partial order of) T and any node $s \in T \setminus \bigcup_{i \in n} b_i$, the subtree T_s of all nodes of T comparable to s has the c.c.c. (and is therefore a Souslin tree) in the generic extension $M[\vec{b}]$.

It is trivial that the properties just defined all are handed down to subtrees. They also can be viewed as properties of the regular open algebras as shown below. Note that the n-fold tree product $T^{\otimes n} := T \otimes \dots \otimes T$ of a single tree T is a dense subset of T^n, the n-fold Cartesian (or forcing-) product of T. So their regular open algebras are canonically isomorphic. We sometimes identify one with the other to simplify notation.

We start our consideration of full trees by proving the equivalence of iterated Souslinity off the generic branch with the parametrised fullness condition we introduced above. Though it seems likely that the construction in [FH06] mentioned above can be modified to yield an n-full but not $n+1$-full tree, the separation between the different degrees of n-fullness has not been established yet. We will do this in Corollary 1.5.13.

Lemma 1.5.6. *For a positive natural number n, a normal Souslin tree T and its regular open algebra $\mathbb{B}$ the following statements are equivalent.*

a) T is n-fold Souslin off the generic branch.

b) T is $n+1$-full.

c) For every antichain $P \subset \mathbb{B}$ of size $n+1$, the free product of the relative algebras corresponding to the elements of P is a Souslin algebra, i.e., $\bigoplus_{b \in P} \mathbb{B}\restriction b$ satisfies the countable chain condition.

Proof. We start with the implication (b→a). So assume that T is $n+1$-full and let $\vec{b} = (b_0, \dots, b_{n-1})$ generic for $T^{\otimes n}$. Choose $\alpha < \omega_1$ big enough, such that the nodes $t_i := b_i(\alpha)$ are pairwise incompatible. Finally, pick a node $t_n \in T_\alpha$ distinct from all the $b_i(\alpha)$. By our fullness assumption on T, the product tree $\bigotimes_{i \in n+1} T(t_i)$ satisfies the countable antichain condition. But then $M[\vec{b}] \models$"$T(t_n)$ is Souslin" by a standard argument concerning chain conditions in forcing iterations. Now it is easy to see that T is n-fold Souslin off the generic branch.

For the other direction we inductively show that T is m-full for $m \leq n+1$, assuming that T is n-fold Souslin off the generic branch. The inductive claim is trivial for $m = 1$, so let $m \geq 1$ and let $s_0, \dots, s_m$ be pairwise distinct nodes of the same height. Then for any generic sequence $\vec{b} = (b_0, \dots, b_{m-1})$ for $\bigotimes_{i\in m} T(s_i)$ we know that $T(s_m)$ is Souslin in the generic extension $M[\vec{b}]$. Finally the two-step iteration $\bigotimes_{i\in m} T(s_i) * \check{T}(s_m)$ is isomorphic to $\bigoplus_{i\in m+1} T(s_i)$ and satisfies the countable antichain condition.

Since it is trivial that c) implies b), we finally show that n-fullness is really a property of the regular open algebra. It is easy to see that for every n-full tree T and every club set C in ω_1, the restriction $T\restriction C$ is n-full again. The converse holds as well: Every tree S with $T \cong S\restriction C$ is also n-full. To prove the latter, let $\varphi : T \cong S\restriction C$ and $s_0, \dots, s_{n-1} \in S_\alpha$. Let $\gamma < \omega_1$ such that $\varphi'' T_\gamma = S_\beta$ for some $\beta > \alpha$ and define

$$Q := \{(t_0, \dots, t_{n-1}) \in T_\gamma^{\otimes n} \mid \varphi(t_i) >_S s_i\}.$$

Then clearly Q is a countable set, so the tree-sum of the tree-products corresponding to the elements of Q, i.e.,

$$\bigoplus_{\vec{t}\in Q} \bigotimes_{i<n} T(t_i)$$

satisfies the c.c.c. and is densely embeddable in $\bigoplus_{i<n} S(s_i)$. So n-fullness is transferred to every other Souslinisation of $\mathrm{RO}\, T$. $\square$

This last lemma implies that a full tree T is also full off the generic branch in the sense that in the generic extension obtained by adjoining a cofinal branch b through T, for every node $t \in T \setminus b$, the tree $T(t)$ is still full.

From now on we say that $\mathbb{B}$ is an *n-full Souslin algebra* if any of its Souslinisations is n-full. For Souslin algebras fullness has strong structural consequences.

Lemma 1.5.7. *The regular open algebra of a 2-full tree is simple, i.e., it has no proper atomless and complete subalgebras. In particular: Forcing with a 2-full tree yields a minimal generic extension of the ground model.*

Proof. Suppose that $\mathbb{B}$ is 2-full and $\mathbb{A}$ is a proper, atomless and complete subalgebra. Then there are a Souslinisation T of $\mathbb{B}$ and an almost nice t.e.r. $\equiv$ on T which represents $\mathbb{A}$. Since $\mathbb{A}$ is proper, there must by distinct nodes $s \equiv t$ on a successor level of T. On successor levels of T the canonical projection $h : \mathbb{B} \to \mathbb{A}$ is induced by the canonical map associated to $\equiv$, call it $\rho : T \to T/{\equiv}$, via

$$h(s) = \sum s/{\equiv} = h(t),$$

cf. Corollary 1.1.6. Then Proposition 1.0.2 implies that $\mathbb{B}\restriction s$ and $\mathbb{B}\restriction t$ both have a subalgebra which is an isomorphic copy of $\mathbb{A}\restriction(\sum s/{\equiv})$, while s and t are disjoint in $\mathbb{B}$. This contradicts the fact that $\mathbb{B}\restriction s \otimes \mathbb{B}\restriction t$ satisfies the c.c.c. $\square$

The last result implies in particular, that n-full trees (for $n > 1$) cannot be decomposed as a product of trees. On the other hand, not every simple Souslin algebra is 2-full: In [AS93, Section 2.2] Abraham and Shelah give a $\Diamond$-construction of two full Souslin trees whose tree product is special, i.e. it can be covered by countably many antichains. The tree sum of these two full trees has a simple regular open algebra which is not 2-full.

1.5.3 Optimal matrices of partitions

In order to isolate the main combinatorial ingredient for the proof of the main result in Section 1.5.4, we introduce a technical notion and prove the associated existence result.

Definition 1.5.8. For $n \in \omega$, an *n-optimal matrix of partitions* is a family $(P_{k,m} \mid k \in \omega,\, m < n)$ of infinite partitions $P_{k,m} = (a_i^{k,m} \mid i \in \omega)$ of ω with the following properties.

(i) *Column-wise consensus:* For all $m < n$ and all $i : k \to \omega$ where $k \in \omega$, the intersection $\bigcap_{\ell<k} a_{i(\ell)}^{\ell,m}$ is infinite.

(ii) *n-optimality:* For all maps $(i, k, m) : n \to \omega \times \omega \times n$ with $(k(j), m(j)) \neq (k(\ell), m(\ell))$ for all $j < \ell < n$ and $m(j) \neq m(\ell)$ for at least one pair $j, \ell < n$ the intersection
$$\bigcap_{j<n} a_{i(j)}^{k(j),m(j)} \quad \text{is a singleton.}$$

Note that if in (ii) the domain of (i, k, m) is restricted to a proper subset of n, then the corresponding intersection has to be infinite. When we draw the matrix as

$$\begin{pmatrix} P_{0,0} & \dots & P_{0,m} & \dots & P_{0,n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{k,0} & \dots & P_{k,m} & \dots & P_{k,n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

then (i) says, that every finite intersection of sets taken from partitions along a single column is infinite, as long as it does not have to be empty for the trivial reason, that two distinct sets from the same partition occur among these sets. By (ii), intersections of less then n sets, each coming from another partition, are infinite, and if n sets come from pairwise distinct partitions from at least two different columns, then they meet in a unique natural number.

Lemma 1.5.9. *There is an n-optimal matrix of partitions for every natural number $n > 1$.*

We will give two different proofs for this lemma. The first is a direct construction with nested inductions and somewhat cumbersome book-keeping details, while the second, the main idea of which is due to Sabine Koppelberg, uses a more sophisticated forcing style argument.

Proof 1. To start we fix a bijective enumeration $h = (h_0, \ldots, h_{n-1}) : \omega \to \omega^n$ and define $a_i^{0,m}$ to be the pre-image of i under h_m. Let $P_{0,m} := \{a_i^{0,m} \mid i \in \omega\}$.

The rest of the proof consists of a three-fold induction. The outer loop is indexed with $(k, m) \in \omega \times n$, and goes row by row, from the left to the right. One could also say that the progression of the indices follows the lexicographic order of $\omega \times n$, i.e., m grows up to $n-1$ and then drops down to 0 while k increases to $k+1$. (The first n stages, where $k = 0$, of the outer induction have been included in the induction anchor in the first line of the proof.)

The inner inductions are common ω-inductions. In each stage of the middle one we define one element $a_i^{k,m}$ of the partition $P_{k,m}$, and the innermost induction consists of a choice procedure for the elements of that set $a_i^{k,m}$.

So assume that the partitions $P_{\ell,m} = \{a_i^{\ell,m} \mid i \in \omega\}$ have already been defined for $(\ell, m) <_{\text{lex}} (k, n)$ and also the i first sets $a_0^{k,m} = a_0, \ldots, a_{i-1}^{k,m} = a_{i-1}$ of $P_{m,k}$ have been fixed. Assume also, that the family constructed so far has the properties (i) and (ii) from Definition 1.5.8. We inductively choose three sequences x_ℓ, y_ℓ and z_ℓ of members of $\omega \setminus \bigcup_{h<i} a_h$ and afterwards set $a_i := \{x_\ell, y_\ell \mid \ell \in \omega\}$. The members of the x sequence will satisfy requirement (i) while the y_ℓ guarantee that the intersections for (ii) are non-empty. The z_ℓ go back to the stack for the construction of the later members of $P_{k,m}$.

The following definitions of f, b, I, c, d and τ have been fixed at the start of the definition of the members of $P_{m,k}$, before the construction of a_0.

Let $f : \omega \to {}^k\omega$ be onto and $\aleph_0$-to-1 and set for $\ell \in \omega$

$$b(\ell) := \bigcap_{j<k} a_{f(\ell)(j)}^{j,m}.$$

These sets have to be met by a_i infinitely many times. So we will choose x_ℓ from $b(\ell)$. Let I be the set of subsets σ of $(\omega \times k \times n) \cup (\omega \times (k+1) \times m)$ of cardinality $n-1$ such that $\mathrm{pr}_{2,3}\restriction\sigma$ is injective and $\mathrm{pr}_3\restriction\sigma$ is not constant, where $\mathrm{pr}_{i,j}$ is the projection to the ith and jth component. So if $\sigma \in I$ then distinct members of σ are indices for $n-1$ members of pairwise distinct partitions and these partitions lie in at least two distinct columns of the n-optimal matrix. For $\sigma \in I$ let

$$c(\sigma) := \bigcap_{(j,p,q)\in\sigma} a_j^{p,q}$$

and note that this set is infinite, because if $P_{p,q}$ is a partition which is not involved in σ then $c(\sigma)$ meets every element of $P_{p,q}$ in exactly one natural number.

Also the sets $c(\sigma)$ have to be met by a_i, so we fix an injective enumeration τ of I and choose y_ℓ from $c(\tau(\ell))$ unless that set has already been hit by earlier members of a_i.

Condition (ii) imposes that each set $c(\sigma)$ be only met one time. So once the intersection is non-empty, that particular set $c(\sigma)$ has to be avoided in later choices. We thus define for every natural number x the set

$$d(x) := \bigcup \{c(\sigma) \mid (\forall\,(j,p,q) \in \sigma) x \in a_j^{p,q}\}$$

and choose x_ℓ and y_ℓ from outside $\bigcup_{j<\ell} d(x_j) \cup d(y_j)$.

We now turn to the formal definition of our three sequences and argue afterwards, why these choices always are possible. Set $e = \bigcup_{h<i} a_h$ and let inductively

$$\begin{aligned} x_\ell &:= \min\left(b(\ell) \setminus \left(e \cup \bigcup_{j<\ell} (d(x_j) \cup d(y_j) \cup \{z_j\})\right)\right) \\ y_\ell &:= \begin{cases} x_\ell, & \text{if } c(\tau(\ell)) \cap \bigcup_{j<\ell}\{x_j, y_j, x_{j+1}\} \neq \emptyset \\ \min c(\tau(\ell)) \setminus (e \cup \{z_0, \dots, z_{\ell-1}\}), & \text{otherwise,} \end{cases} \\ z_\ell &:= \min\left(b(\ell) \setminus \left(e \cup \bigcup_{j\le\ell} \{x_j, y_j, z_j, x_{j+1}, y_{j+1}\}\right)\right). \end{aligned}$$

There are always candidates for x_ℓ, because the set $b(\ell)\setminus e$ is non-empty (by induction on i — this is the reason for the choice of the corresponding z_ℓ in earlier steps a_h), and for $\sigma \in I$ and $\ell \in \omega$ the intersection $b(\ell) \cap c(\sigma)$ is either empty or a singleton. So we have to avoid only finitely many elements of $b(\ell) \setminus e$ (d is a finite union for all x_j and y_j) which is of course possible. The same argument shows that z_ℓ is well-defined. Finally for y_ℓ, the set $c(\tau(\ell))$ has met every set a_h for $h < i$ in a unique element, so once again we only have to delete finitely many elements from an infinite set.

We hope that it has now become clear that the result of this threefold induction is an n-optimal matrix of partitions. □

Proof 2. To start this second proof, read the first two paragraphs of the first proof, where the inductive anchor and the progression of the indices (k,m) during the induction are described. Now assume that we want to construct $P_{k,m}$ and that the part of the matrix of partitions which has already been constructed has the properties of column-wise consensus and n-optimality. We denote the set of indices for that part by

$$L := \{\sigma \mid \sigma \subseteq (k \times n) \cup (k+1 \times m)\}.$$

We now define a forcing partial order $\mathbb{P}$ and a countable family $\mathcal{F}$ of dense subsets of $\mathbb{P}$, such that every $\mathcal{F}$-generic filter in $\mathbb{P}$ gives rise to a partition P which fulfils our inductive claim. The partial order $\mathbb{P}$ will consist of finite partial functions on the natural numbers, such that for every $\mathcal{F}$-generic filter G, the union $f_G := \bigcup G$ is a total function on ω. P will then be defined as

$$P := \{f^{-1}(j) \mid j \in \omega\}.$$

In order to define $\mathbb{P}$, let

$$J := \{\sigma \in L \mid \mathrm{pr}_2''\sigma = \{m\} \text{ or } |\sigma| < n-1\}$$

and

$$K := \{\sigma \in L \mid \mathrm{pr}_2''\sigma \neq \{m\} \text{ and } |\sigma| = n-1\}.$$

(Here pr_2 again denotes the projection on the second component.) For every $\sigma \in L$ let P_σ denote the canonically defined common refinement of the partitions $P_{k,m}$ for $(k,m) \in \sigma$. The properties of column-wise consensus and n-optimality determine for which σ the set P_σ is still a partition. E.g., if σ intersects more than one row of the partition and has more than n elements, then $P_\sigma = \{\emptyset\} \cup \{\{i\} \mid i \in \omega\}$ which is clearly not what one would call a partition. However, P_σ *is* a partition and has only infinite elements whenever σ comes from J or K. Choose for all $\sigma \in J \cup K$ an injective enumeration $(a_i^\sigma \mid i \in \omega)$ of P_σ.

Now let

$$\mathbb{P} := \{f : \mathrm{dom} f \to \omega \mid \mathrm{dom} \subset \omega \text{ is finite and for all } \sigma \in K, i \in \omega : f\restriction a_i^\sigma \text{ is 1-1}\}.$$

Our family $\mathcal{F}$ of dense subsets of $\mathbb{P}$ consists of the sets defined as follows. For $\ell \in \omega$ let

$$U_\ell := \{f \in \mathbb{P} \mid \ell \in \mathrm{dom} f\},$$

for $\sigma \in J$ and $i, j, \ell \in \omega$ let

$$V_{ij\ell}^\sigma := \{f \in \mathbb{P} \mid j \in f''(a_i^\sigma \setminus \ell)\},$$

and for $\sigma \in K$ and $i, j \in \omega$ let

$$W^\sigma_{ij} := \{f \in \mathbb{P} \mid j \in f''a^\sigma_i\}.$$

We first show that every $\mathcal{F}$-generic filter indeed induces a partition as we want it and show afterwards that the members of $\mathcal{F}$ are dense in $\mathbb{P}$.

So let $G \subset \mathbb{P}$ be an $\mathcal{F}$-generic filter, $f := \bigcup G$ and $P := \{f^{-1}(j) \mid j \in \omega\}$. Then $f : \omega \to \omega$ is total, because G intersects all sets U_ℓ. The sets $V^\sigma_{ij\ell}$ care about column-wise consensus and the implicit part of n-optimality, i.e., the fact that less than n sets coming from distinct partitions always intersect in an infinite set. Finally the sets W^σ_{ij} are responsible for the explicit part of n-optimality.

It is trivial that U_ℓ is dense for every ℓ. To show the same for W^σ_{ij}, let $f \in \mathbb{P}$, $\sigma \in K$ and $i, j \in \omega$. If $j \in f''a^\sigma_i$ we are done. If not, then consider the set

$$X = \{\tau \in K \mid \text{ there is } a \in P_\tau \text{ such that } j \in f''a\}.$$

For the choice of our new pre-image of j, we have to avoid the union of all witnesses a for $\tau \in X$ for all $\tau \in K$. But as σ is distinct from all $\tau \in X$ and each of them has $n-1$ members not all from the same column, each of the intersections $a^\sigma_i \cap a$ is at most a singleton. Since K and $\mathrm{dom}(f)$ are finite, we can always find an extension g of f in W^σ_{ij}.

The argument for the density of $V^\sigma_{ij\ell}$ is similar, yet a bit more complicated. Pick $f \in \mathbb{P}$, $\sigma \in J$ and $i, j \in \omega$ and define X and a^τ for $\tau \in X$ as above. For those τ in X with $\sigma \not\subset \tau$ the argument given above applies. So let $X_0 := \{\tau \in X \mid \sigma \subset \tau\}$ and assume without loss of generality that $|\sigma| = n-2$. Then the union of two distinct members τ and τ' of X is of size n and by n-optimality we have $P_{\tau\cup\tau'} = \{\{n\} \mid n \in \omega\}$. If $\tau \in X_0$ and $a \in P_\tau$, then we either a is contained in a^σ_i, or the two sets are disjoint. So for $\tau \in X_0$ the set

$$P_\tau \cap \mathcal{P}(a^\sigma_i)$$

is a partition of a^σ_i into $\aleph_0$ infinite sets. Fix a member τ_0 of X_0 and let $A := \{a \in P_{\tau_0} \mid j \in f''a\}$ be the set of all witnesses for $\tau_0 \in X_0$. Since $\mathrm{dom}(f)$ is finite, also A is. Let $b := \bigcap A$. Then clearly $a^\sigma_i \setminus b$ is infinite, and for every pair a_0, a with $a_0 \in P_{\tau_0} \setminus \mathcal{P}(b)$ and $a \in P_\tau$ with $\tau \in X_0 \setminus \{\tau_0\}$ the intersection of a_0 with a is a singleton. Now fix also a_0 and collect the finitely many sets

$$B := \{a \mid \text{ there is } \tau \in X_0 \setminus \{\tau_0\} : a \in P_\tau \text{ and } j \in f''a\}.$$

Then clearly $a_0 \setminus \bigcup B$ is infinite (cofinite in a_0) and a subset of $a^\sigma_i \setminus (b \cup \bigcup B)$. So we can find our new suitable pre-image for j above any given $\ell \in \omega$. This finishes the proof. □

1.5.4 Decompositions with full factors

The following theorem is stated in [SZ99, p.246] for the case $n = 2$ without proof. Larson gives the construction of a single full subalgebra of a strongly homogeneous Souslin algebra in the proof of Theorem 8.5 in his paper [Lar99].

Theorem 1.5.10. *For every natural number $n > 1$ and every $\aleph_0$-branching, strongly homogeneous Souslin tree T there are n full Souslin trees $S_0, \ldots, S_{n-1}$ such that $T \cong \bigotimes_{m<n} S_m$.*

Proof. Let T be a strongly homogeneous Souslin tree and denote by $\varphi_{s,t}$ the members of the coherent family of T. We inductively define level by level n t.e.r.s $\equiv_0, \ldots, \equiv_{n-1}$ with the following properties:

- $T/{\equiv_m}$ is a full Souslin tree for $m < n$.
- For any sequence $(s_0, \ldots, s_{n-1}) \in T_\alpha$ the intersection of the classes $s_m/{\equiv_m}$ for $m < n$ is a singleton: $\bigcap_{m<n} s_m/{\equiv_m} = \{r\}$ for some $r \in T_\alpha$.

The isomorphism between $\bigotimes_{m<n} T/{\equiv_m}$ and T is then given by mapping $(s_m/{\equiv_m} \mid m < n)$ to the unique element of the intersection of the coordinates and the inverse map is just the product of the canonical maps for the equivalence relations $\equiv_m$.

Let $(P_{k,m} \mid m < n, k \in \omega)$ be an n-optimal matrix of partitions, where we view each $P_{k,m}$ as enumerated by $a_i^{k,m}$, $i \in \omega$. In order to define t.e.r.s we transfer the whole matrix of the $P_{k,m}$ to every set $\mathrm{succ}(s)$ for $s \in T$ in a coherent way: Choose for every $\alpha < \omega_1$ an anchor node $r_\alpha \in T_\alpha$ and a bijection $\sigma_\alpha : \omega \to \mathrm{succ}(r_\alpha)$, and define for $s \in T_\alpha$ and all indices i, k, m the sets

$$a_i^{k,m}(s) := (\varphi_{r_\alpha,s} \circ \sigma_\alpha)'' a_i^{k,m}.$$

Then clearly for every $s \in T$, $k \in \omega$ and $m < n$, the set $P_{k,m}(s) := \{a_i^{k,m}(s) \mid i \in \omega\}$ forms a partition of $\mathrm{succ}(s)$, and these partitions are linked by the coherent family in a coherent way, i.e., $\varphi_{s,t}$ transfers $P_{k,m}(s)$ to $P_{k,m}(t)$.

Fix $m < n$ in order to define $\equiv_m$ on T by recursion on the height. We will also enumerate the $\equiv_m$-classes of each level with order type ω, i.e., we define a map $h : T \to \omega$, such that for $s, t \in T_\alpha$ we have $s \equiv_m t$ if and only if $h(s) = h(t)$.

Choose $P_{0,m}(\mathrm{root})$ as the partition of the set $T_1 = \mathrm{succ}(\mathrm{root})$ and let $\equiv_m$ on T_1 be the equivalence relation with classes $a_i^{0,m}(\mathrm{root})$ for $i \in \omega$. Let $h(\mathrm{root}) = 0$ and choose h on T_1 in a way, such that nodes s and t are $\equiv_m$-equivalent just in case that their h-values coincide.

Next we consider the case where α is a successor ordinal, $\alpha = \gamma + 1$ for some $\gamma < \omega_1$. Let $s, t \in T_\alpha$ and let $s^- <_T s$ and $t^- <_T t$ be their direct predecessors on level γ. We let $s \equiv_m t$ if and only if their direct predecessors are $\equiv_m$-equivalent, $s^- \equiv_m t^-$ (so in particular $h(s^-, m) = h(t^-, m)$), and if there is $i \in \omega$ such that

$$s \in a_i^{h(s^-),m}(s^-) \text{ and } t \in a_i^{h(t^-),m}(t^-).$$

In words, the $\equiv_m$-equivalence of the direct predecessors gives us a natural number $h(s^-)$ and we apply $P_{h(s^-),m}$ on level α to enquire whether s and t are $\equiv_m$-equivalent. Extend h to the T_α as described above.

On limit stages λ the relation $\equiv_m$ is already determined by its behaviour below, and we choose the $h{\restriction}T_\lambda$ once more in any way such that $h(s) = h(t)$ is equivalent to $\equiv_m$-equivalence for nodes $s, t \in T_\lambda$.

Having finished the construction of the relation $\equiv_m$, we show that it is ∞-nice, where the ∞-part follows easily from the fact that $P_{k,m}$ partitions ω in infinitely many sets. So we deduce niceness. Letting $s \equiv_m r$ on level α and t above s we claim that $\varphi_{s,r}(t) \equiv_m t$ and show this by induction on the height of t above s. For successor stages the claim follows directly from the construction and the inductive hypothesis, since the relevant partition $P_{j,m}$ is transferred via $\varphi_{s,r}$ by the coherence of the coherent family. The limit case follows directly from the inductive assumption. (The property of $\equiv_m$, that $\equiv_m$-equivalence lifts from s and r to preimages and images of $\varphi_{s,r}$ will be used again in the proof of the Fact below.)

It remains to prove the two claims stated before the construction. We start with the fullness of $T/{\equiv_m}$. Let $s_0, \dots, s_{k-1}$ be pairwise non-m-equivalent nodes of the same level T_α. We write S_i for $(T/{\equiv_m})(s_i/{\equiv_m})$ and claim that for every antichain A of $\bigotimes_{i<k} S_i$ we can find an antichain B of T with the same cardinality. We give a hint for where to look for the members of B in the following

Fact. Fix $m < n$ and pairwise non-m-equivalent nodes $s_0, \dots, s_{k-1}$. For any sequence $(t_0, \dots, t_{k-1})$ of nodes in T_β, with $\alpha < \beta$ and $s_i < t_i$ for $i < k$, the intersection of the classes $t_i/{\equiv_m} \cap T(s_i)$ above the nodes s_i, shifted above s_0 by φ_{s_i,s_0}, i.e. the set $\bigcap_{i<k} \varphi''_{s_i,s_0}(t_i/{\equiv_m})$, is infinite and therefore non-empty.

Proof of the fact. By induction on the height β of the nodes t_i, starting with $\beta = \alpha + 1$. In this minimal case we have $t_i^- = s_i$, so the sets $\varphi''_{s_i,s_0}(t_i/{\equiv_m})$ belong to distinct partitions $P_{h(s_i),m}(s_0)$, $i < k$ and therefore have an infinite intersection by property (i) of the n-optimal matrix.

For the higher successor case $\beta = \gamma + 1$, $\alpha < \gamma$, we simulate this initial situation. By the inductive hypothesis pick $r_0 \in \bigcup_{i<k} \varphi''_{s_i,s_0}(t_i^-/{\equiv_m}) > s_0$, and let $r_i := \varphi_{s_0,s_i}(r_0) > s_i$ for $i < k$. We then know that $r_i \equiv_m t_i^-$, so $t_i/{\equiv_m}$ has elements above r_i. Consequently $\bigcup_{i<k} \varphi''_{r_i,r_0}(t_i/{\equiv_m})$ is infinite by the same argument as above and furthermore a subset of $\bigcup_{i<k} \varphi''_{s_i,s_0}(t_i/{\equiv_m})$.

For the case where β is a limit ordinal, we choose $\gamma < \beta$ large enough, such that letting $q_i = t_i \restriction \gamma$ for all $i, j < k$ we have $\varphi_{q_i,q_j}(t_i) = t_j$. This is possible due to the transitivity of the coherent family. We also require $\alpha < \gamma$. The inductive hypothesis gives us a node $r_0 \in \bigcup_{i<k} \varphi''_{s_i,s_0}(q_i/{\equiv_m})$, which we copy to $r_i := \varphi_{s_0,s_i}(r_0)$, so we also have $r_i \equiv_m q_i$. We consider $u = \varphi_{q_i,r_0}(t_i)$. By the commutativity of the coherent family this definition is independent from the choice of $i < k$. But then

$$\varphi_{s_0,s_i}(u) = \varphi_{r_0,r_i}(u) = \varphi_{r_0,r_i} \circ \varphi_{q_i,r_0}(t_i) = \varphi_{q_i,r_i}(t_i)$$

where the first equation follows from coherence, the second one from the definition of u and the third one from commutativity. So the property stated above right after the construction of $\equiv_m$ implies that $\varphi_{s_i,s_0}(t_i) \equiv_m u$ since $r_i \equiv_m t_i$ for all $i < k$. This completes the proof of the Fact. □

By virtue of the Fact we can pick for every element $(t_0/{\equiv_m}, \dots, t_{k-1}/{\equiv_m})$ of our antichain $A \subset T/{\equiv_m}$ a node $u \in \bigcap_{i<k} \varphi''_{s_i,s_0}(t_i/{\equiv_n})$ and collect these nodes in B. Then B is clearly an antichain of T with the same cardinality as A. So we have shown that $T/{\equiv_m}$ is indeed a full tree.

Now for the second claim. Let $(s_0, \dots, s_{n-1})$ be any sequence of nodes of some T_α. We need to show that $\bigcap_{m<n} s_m/{\equiv_m}$ has a unique element. This is done by induction on $\alpha > 0$. Starting with $\alpha = 1$ we know that $s_m/{\equiv_m} = a_{i_m}^{k_m,m}(root)$ for some i_m and k_m. So property (ii) of our n-optimal matrix is all we need here. For $\alpha = \gamma + 1$ we assume that the classes $s_m^-/{\equiv_m}$ meet in a single node, say $r \in T_\gamma$. The set of elements of $s_m/{\equiv_m}$ which lie above r is then just $a_{i_m}^{h_m(r),m}(r)$ and again property (ii) of the matrix proves the claim. In the limit case we once more use the transitivity of the coherent family. So let α be a limit and $\gamma < \alpha$ large enough such that $\varphi_{q_m,q_\ell}(s_m) = s_\ell$ where we abbreviate $s_m \restriction \gamma = q_m$. For a last time in this proof we use the commutativity of the coherent family: Let r be the unique element of the intersection of the classes $q_m/{\equiv_m}$. Then $t = \varphi_{q_m,r}(s_m)$ is well defined and independent from the choice of $m < n$. By the lifting property of the equivalence relations stated above, it follows from $q_m \equiv_m r$, that $s_m \equiv_m t$. □

Readers familiar with [Lar99, §8] know how much the final part of the last proof owes to Larson's proof of his Theorem 8.5. For the sake of completeness we note the following.

Corollary 1.5.11. *For every strongly homogeneous Souslin tree T and every pair m, n of natural numbers satisfying $2m + n \geq 2$ there are sequences R_i, $i < m$ of* *strongly homogeneous trees and S_j, $j < n$ of full trees such that T can be decomposed into the tree product of all these trees:*

$$T \cong \bigotimes_{i<m} R_i \otimes \bigotimes_{j<n} S_j.$$

Proof. By the Theorems 1.5.3 and 1.5.10, the only case left that needs proof is $m = n = 1$. Let $P_{k,0}$ and $P_{k,1}$ for $k < \omega$ be the members of a 2-optimal matrix of partitions. Set $P := P_{0,0}$ and $Q_k := P_{k,1}$ for $k < \omega$ and construct the t.e.r. $\sim$ on the base of P along the lines of the proof of Theorem 1.5.3 and $\equiv$ on the base of the Q_k as for Theorem 1.5.10. □

The proof of Lemma 1.5.9 on n-optimal matrices of partitions becomes very simple when the statement is restricted to the case $n = 2$. Furthermore, the decomposition of a strongly homogeneous tree in only two full factors together with Theorem 1.5.3 and the simple argument of the last proof would give a less complicated, inductive proof of Theorem 1.5.10.

But the decomposition as carried out in our proof of Theorem 1.5.10 allows for the two following remarkable consequences which cannot be deduced in the same straight forward fashion from the simpler decomposition sketched above.

Corollary 1.5.12. *Let the strongly homogeneous Souslin tree T be decomposed as the tree product of n full trees S_i for $i < n$ as carried out in the proof of Theorem 1.5.10. Let $\{a, b\}$ be a partition of the set n with $a, b \neq \varnothing$. Then*

$$\Vdash_{\bigotimes_{i\in a} S_i} \text{“} \bigotimes_{i\in b} \check{S}_i \textit{ is strongly homogeneous.”}$$

Proof. We proceed by induction on the size of a. Fix $i < n$, let c a generic branch through S_i and work in the generic extension by c. Then the tree product

$$R := \bigotimes_{\substack{j<n \\ j\neq i}} S_j$$

is canonically isomorphic to

$$c \otimes \bigotimes_{\substack{j<n \\ j\neq i}} S_j \subset T$$

(here we take the branch c to be a degenerate tree factor in the tree product). Denote the canonical isomorphism by ρ and the canonical projection $T \to R$ by π.

We define the tree isomorphisms ψ_{rs} for nodes r and s of R_α, $\alpha < \omega_1$ from the members $\varphi_{\rho(r)\rho(s)}$ of the coherent family of T. For this, we refer to the maps $h(\cdot, j) : T \to \omega$ used in the construction of the t.e.r.s $\equiv_j$ in the proof of Theorem 1.5.10. We collect them and define $h : T \to \omega^{n-1}$ by simply concatenating the values $h(t, j)$ for $t \in T$ and $j < n$, $j \neq i$.

If r, $s \in R_\alpha$ and $h(\rho(r)) = h(\rho(s))$, then we let

$$\psi_{rs} := \pi \circ \varphi_{\rho(r)\rho(s)} \circ \rho : R(r) \to R(s).$$

It follows from the fact that the n-optimal partition matrices are transported between the (sets of direct successors of the) nodes by the members φ_{tu} of the coherent family of T that this definition is sound and indeed yields an isomorphism.

Now let $r, s \in R_\alpha$ with $h(\rho(r)) \neq h(\rho(s))$. In order to define ψ_{rs} we compose the tree isomorphisms that we have already defined for the direct successors of r and s. For every direct successor $u \in \operatorname{succ}(r)$ there is exactly one $v \in \operatorname{succ}(s)$ with $h(\rho(u)) = h(\rho(v))$. This follows from the n-optimality of the partition matrix. Let $\psi_{rs}(u)$ be just this v. If x is a non-immediate successor of r then first find the direct successor u of r below x and the image $v = \psi_{rs}$, and set

$$\psi_{rs}(x) := \psi_{uv}(x).$$

It remains to prove, that the family of tree isomorphisms just defined is coherent, commutative and transitive. Commutativity and coherence are inherited from the coherent family of T. (Note, that $\psi_{rs}(x) = y$ implies that $h(\rho(x)) = h(\rho(y))$, so the two cases do not interfere.) As for transitivity, let $x, y \in R_\lambda$ for some countable limit ordinal λ. Find Then by the transitivity of the family of the φ_{tu} for T, there are $t < \rho(x)$ and $u < \rho(y)$ with $\varphi_{tu}(\rho(x)) = \rho(y)$. But then t and u lie in $b \otimes R$, so there are $r, s \in R$ such that $\rho(r) = t$ and $\rho(s) = u$ and thus $\psi_{rs}(x) = y$.

We finally argue, why the step carried out above also serves as the general step in our induction on the power of a. By all the coherence properties of the family φ_{tu} of T, the subtree $b \otimes R$ of T (in the generic extension, of course) carries an $n-1$-optimal matrix of partitions: $P_{k,j}$ for $j < n$ and $j \neq i$. From this matrix, a family of $n - 1$ t.e.r.s is defined with exactly the same properties: $\equiv_j$ restricted to $b \otimes R$ for $j < n$ and $j \neq i$. Using ρ^{-1} to transfer all the structure to R, our argument can now be applied to R and a member i' of $a \setminus \{i\}$. □

So, e.g. in the case $n = 2$, forcing with one full tree can not only destroy the fullness of another one, but even turn the latter into a strongly homogeneous Souslin tree.

Our last corollary of (the proof of) Theorem 1.5.10 gives the announced separation of the finite degrees of fullness. This shows, that the family of parametrised fullness conditions is properly increasing in strength.

Corollary 1.5.13. *If there is a strongly homogeneous Souslin tree, then there is an n-full, but not $n+1$-full tree.*

Proof. Let the strongly homogeneous Souslin tree T be decomposed as the tree product of n full trees S_i for $i < n$ as carried out in the proof of Theorem 1.5.10. We show, that the tree sum of the factors,

$$R := \bigoplus_{i<n} S_i$$

is an n-full but not $n+1$-full Souslin tree. The Fact used in the proof of Theorem 1.5.10 remains true in the following variant:
Fact'. For any pair of sequences $(s_0, \ldots, s_{n-1})$ in T_α and $t_i > s_i$ in T_β and any sequence $m : n \to n$ the intersection

$$\bigcap_{i<n} \varphi''_{s_i,s_0} t_i / \equiv_{m(i)}$$

is not empty.

Modulo the obvious changes in the notation, the proof remains completely the same as before, using the n-optimality of the matrix. And also with the same argument as above we can derive an antichain of T from any given antichain of R maintaining the cardinality. So R is n-full.

We argue that R is not n-fold Souslin off the generic branch. If b_i is a cofinal branch through S_i then in the generic extension obtained by adjoining $\vec{b} = (b_0, \ldots, b_{n-1})$, the strongly homogeneous tree T has a cofinal branch as well, thus destroying the Souslinity of all subtrees of R. □

1.6 Separating high degrees of rigidity

In Sections 1-3 of [FH06] Fuchs and Hamkins consider different notions of rigidity for Souslin trees: (ordinary) rigidity, total rigidity and the unique branch property and their absolute counterparts. In this context they also introduced the stronger notion of being (n-fold) Souslin off the generic branch which we already considered in the last section, cf. Lemma 1.5.6. The present section gives examples that witness the independence between 2-fullness and the n-absolute UBP which was asked for in [FH06].

Definition 1.6.1. a) A Souslin tree T is called *n-absolutely rigid*, if T is a rigid tree in the generic extension obtained by forcing with T^n (or equivalently $T^{\otimes n}$).

b) A Souslin tree is *totally rigid*, if the trees $T(s)$ and $T(t)$ are non-isomorphic for all pairs of distinct nodes s and t of T. It is *n-absolutely totally rigid* if it is totally rigid after forcing with T^n.

c) A Souslin tree T has the *unique branch property (UBP)*, if forcing with T adjoins only a single cofinal branch to T. For $n > 0$ we say, that T has the *n-absolute UBP*, if forcing with T^{n+1} adjoins exactly $n + 1$ cofinal branches to T.

As well as fullness, the UBP is handed down to subtrees and transfers to RO-equivalent trees. The latter is clear from the fact that the generic branches of two Souslinizations of a Souslin algebra $\mathbb{B}$ are in canonical 1-to-1 correspondence. Fuchs

2-full	$\longleftarrow$	3-full	$\longleftarrow$	4-full	$\longleftarrow$	…
$\downarrow$		$\downarrow$		$\downarrow$		
UBP	$\longleftarrow$	absolutely UBP	$\longleftarrow$	2-absolutely UBP	$\longleftarrow$	…
$\downarrow$		$\downarrow$		$\downarrow$		
totally rigid	$\longleftarrow$	absolutely totally rigid	$\longleftarrow$	2-absolutely totally rigid	$\longleftarrow$	…
$\downarrow$		$\downarrow$		$\downarrow$		
rigid	$\longleftarrow$	absolutely rigid	$\longleftarrow$	2-absolutely rigid	$\longleftarrow$	…

and Hamkins prove the implications as well as some independencies between these rigidity notions. They also give in [FH06, Section 4] a diagram of implications between the degrees of rigidity that we have approximately reconstructed here for the convenience of the reader.

In [FH06] Fuchs and Hamkins have shown that the part of the diagram to the left and below "absolutely UBP" is complete in the sense that there are no further implications between these rigidity properties. They ask whether the rest of the diagram is complete as well, cf. [FH06, Question 4.1]. We will show below that there are neither implications from left to the right including diagonals (cf. Theorem 1.6.4, nor from the second to the first row (Theorem 1.6.3).

1.6.1 Fullness and the unique branching property

We start with an easy result deduced from the elementary properties of finitely full trees for the second column of the diagram.

Proposition 1.6.2. *If there is a 3-full Souslin tree, then there is also a Souslin tree which has the UBP and is not 2-full.*

Proof. Let T be 3-full and pick distinct nodes s, t of the same height from T. We show, that the Souslin tree $S = T(s) \otimes T(t)$ has the UBP. Let $b \otimes c$ be a generic, cofinal branch in S (viewing b and c as trees). By the 2-fold Souslinity off the generic branch of T, every tree of the form $T(r)$ with $r \in T \setminus (b \cup c)$ is Souslin in the generic extension by $b \otimes c$. On the other hand, if there was a second cofinal branch through S in the generic extension, then one of its components would have to pass through such a node $r \notin b \cup c$, which yields a contradiction.

RO S is not simple, because it has an isomorphic copy of RO $T(s)$ as a proper subalgebra. So S cannot be 2-full by Lemma 1.5.7. □

This result cannot be improved by simply requiring T to be full, because by iterating the forcing with a tree product $n+1$ times, we always get at least 2^n cofinal branches.

We do have the following non-implication result for the n-absolute UBP and 2-fullness under the stronger assumption of $\diamondsuit$.

Theorem 1.6.3. *Assume $\diamondsuit$. Then there is a Souslin tree which is not simple but has the n-absolute UBP for all $n \in \omega$.*

Sketch of proof. We construct an n-absolutely UBP Souslin tree T along with an ∞-nice t.e.r. $\equiv$ on T. To achieve the n-absolute UBP, we use the forcing machinery with the partial orders $(T\restriction\alpha)^{\otimes n}$. Fix a $\diamondsuit$-sequence $(R_\nu \mid \nu < \omega_1)$, and let R_α either guess a maximal antichain of $T\restriction\alpha$ or a pair of the form $(\vec{s}, \dot{b})$, where $\vec{s} = (s_0, \ldots, s_{n-1}) \in (T\restriction\alpha)^{\otimes n}$ and $\dot{b}$ is a $\bigotimes_{i<n} T(s_i)$-name for a cofinal branch in $T\restriction\alpha$ passing by besides all the s_i, i.e.,

$$\vec{s} \Vdash_{(T\restriction\alpha)^{\otimes n}} \check{s}_i \notin \dot{b} \text{ for all } i < n.$$

If the latter is the case, then the name $\dot{b}$ gives rise to a mapping

$$\varphi : \left[\bigotimes_{i<n} T(s_i)\right] \to [T_\alpha]$$

sending the sequence $\vec{c}$ of α-branches to the set of all nodes that are forced into $\dot{b}$ by $\vec{c}$. Now we simply have to seal the branch $\varphi(\vec{c})$ for every $\vec{c}$ through $\vec{s}$ which consists of branches $c_0, \ldots, c_{n-1}$ that have extensions in T_α.

If R_α is a maximal antichain, then, as in Example 1.3.7, the Second Reduction Lemma 2.2.15 guarantees that $\equiv$ can be maintained in an ∞-nice fashion. □

For the other non-implication we can work with a weaker hypothesis again.

Theorem 1.6.4. *Let $n > 1$. If there is a strongly homogeneous Souslin tree, then there is an n-full tree which is not $(n-1)$-absolutely rigid.*

Proof. We fix $n > 1$ and use the tree R from the proof of Corollary 1.5.13 obtained from a strongly homogeneous tree T as the tree sum $R = \bigoplus_{i<n} S_i$ of the full factors S_i, $i < n$ of T.

From Corollary 1.5.13 we know, that R is n-full.

To show, that R is not $(n-1)$-absolutely rigid we refer to Corollary 1.5.12. It follows directly, from the case that $a = n \setminus \{i\}$ for some $i < n$, that R is not rigid in the generic extension obtained by adjoining a cofinal branch through the trees S_j for $j < n$ and $j \neq i$. But this generic extension can also be reached by forcing with $R^{\otimes n-1}$. □

Chapter 2

Maximal chains in Souslin algebras

Introduction

In this chapter we develop a method for constructing Souslin algebras that abound with automorphisms and use it to give an affirmative answer to the question whether or not it is consistent relative to ZFC, that there is a chain homogeneous Souslin algebra. In our representation we strongly exploit the tight connection between the combinatorics of normal trees of countable height and Polish spaces.

The first section reviews the notation and known results used in our constructions. In Section 2 we define several technical notions and prepare them for their use in the Souslin tree constructions of the last two Sections. Section 3 contains the first main result, the construction of a Souslin tree whose algebra of regular open subsets is chain homogeneous, assuming the combinatorial principle $\diamondsuit^+$. In Section 4 we enhance this construction and get, again under ZFC + $\diamondsuit^+$ a chain homogeneous Souslin algebra with homogeneity properties that seemingly cannot be strengthened any further. This algebra serves as the starting point of an iterative construction of a big, chain homogeneous Souslin algebra.

2.1 Preliminaries

Here we collect the basic definitions and results to fix the notation and for later reference. The **Definition**-environment is omitted, because almost everything is to be understood as definitions. We work in ZFC, Zermelo-Fraenkel set theory with choice.

2.1.1 Boolean algebras

Boolean algebras and the most elementary related notions, such as subalgebras, completeness, atoms will not be defined here, though we will mention some more or less subtle relationships between them. The main reference for looking up undefined notions is the first volume of the *Handbook of Boolean Algebras*, [Kop89].

Let B be a Boolean algebra. We call the binary relation on B

$$x \leq_B y \iff xy = x \iff x - y = 0$$

the natural or the canonical order of B. A subset $X \subset B$, which is totally ordered by $\leq_B$ is a *chain* of B. A chain X is a *maximal chain* of B if X is furthermore $\subset$-maximal amongst the chains of B. We call a Boolean algebra all of whose maximal chains are pairwise order isomorphic *chain homogeneous.* We let

$$\operatorname{mc} B = \{K \subset B \mid K \text{ is a maximal chain of } B\}.$$

An *antichain* of B is a family $X \subset B$ of pairwise disjoint (i.e. $xy = 0$) elements of B. We say that B satisfies (or has) the *κ-chain condition* (or short *κ-c.c.*) if every antichain of B has cardinality less than κ. The $\aleph_1$-c.c. is also called *c.c.c.* for countable chain condition.

We can already state the first lemma concerning our search for complete and chain homogeneous Boolean algebras.

Lemma 2.1.1. *If B is complete and all maximal chains are pairwise order isomorphic, then B has the c.c.c.*

Proof. Given an uncountable antichain X of B it is easy to construct well-ordered chains with supremum 1, $K_0 = \{x_0 <_B x_1 <_B \ldots\}$ of order type ω and $K_1 = \{y_0 <_B y_1 <_B \ldots\}$ of order type ω_1. Now for any pair K, $K' \in \operatorname{mc} B$ with $K_0 \subset K$ and $K_1 \subset K'$ there should be an isomorphism $\varphi : K \to K'$ but then $n \mapsto \min\{\alpha \mid \varphi(x_n) \leq_B y_\alpha\}$ would give a cofinal countable sequence in ω_1. $\square$

Now let B be a complete Boolean algebra. By saying that A is a *complete subalgebra of B* we mean that A is a subalgebra of B which is a complete Boolean algebra, and it computes the same infinite sums and products as B does. I.e., for all $M \subseteq A$ we have

$$\sum\nolimits^A M = \sum\nolimits^B M$$

and the analogous equation for products holds as well. For any subset X of B we call

$$\langle X \rangle^{\mathrm{cm}} := \bigcap \{A \mid X \subseteq A \text{ and } A \text{ is a complete subalgebra of } B\}$$

the *subalgebra of B that is completely generated by X* and maybe write $\langle X \rangle^{\mathrm{cm}}_B$ if B is not clear from the context. If the superscript cm is omitted, i.e., by $\langle X \rangle_B$, we denote the intersection of all subalgebras containing X as a subset (and not only the complete subalgebras) and call $\langle X \rangle$ the subalgebra of B, that is *finitarily* generated by X.

Note that given an arbitrary Boolean algebra A which is a subalgebra of a complete Boolean algebra B, the *(Dedekind) completion* $\overline{A} = \operatorname{RO} A^+$ (cf. [Kop89, Section 4.3]), which is the unique complete Boolean algebra containing A as a dense subalgebra, is not necessarily isomorphic to $\langle A \rangle^{\mathrm{cm}}_{\mathbb{B}}$ (cf. *regular subalgebras* below).

A Boolean algebra B is *$\aleph_0$-distributive* if for every family $(a_{ij})_{i\in\omega, j\in J}$ with an index set J of arbitrary size, the following equation holds:

$$\prod_{i\in\omega} \sum_{j\in J} a_{ij} = \sum \left\{ \prod_{i\in\omega} a_{if(i)} \mid f \in {}^{\omega}J \right\}.$$

We do not need more specific concepts of distributivity.

An antichain is a *maximal antichain* or a *partition (of unity)* if it is $\subset$-maximal, and if X, Y are maximal antichains we say that X *refines* Y if for all $x \in X$ there is an $y \in Y$ above, i.e. $x \leq_B y$.

A useful characterisation of of $\aleph_0$-distributivity is this: A Boolean algebra B is $\aleph_0$-distributive if and only if for every countable family (Y_n) of maximal antichains of B there is a common refinement X, i.e. a maximal antichain that refines all the Y_n (cf. [Kop89, Proposition 14.9]).

A *Souslin algebra* is a complete, atomless, c.c.c. and $\aleph_0$-distributive Boolean algebra. We use letters $\mathbb{A}, \mathbb{B}, \mathbb{C}$ to denote Souslin algebras. In other contexts than ours, κ-Souslin algebras are defined as complete, atomless, κ-c.c. and $(< \kappa)$-distributive Boolean algebras and κ can be any uncountable cardinal. In this notation the objects of our consideration are called $\aleph_1$-Souslin algebras. But by Lemma (2.1.1) above, these higher Souslin algebras will always have maximal chains of distinct order types and are therefore not chain homogeneous.

In a Souslin algebra $\mathbb{B}$, a complete subalgebra $\mathbb{A}$ is atomic if and only if $\mathbb{A}$ is completely generated by some countable subset X of $\mathbb{B}$ (cf. [Kop89, Proposition 14.8]). In the other extreme, since distributivity and the c.c.c. are handed down to complete subalgebras, the atomless, complete subalgebras of a Souslin algebra $\mathbb{B}$ are Souslin algebras as well.

A *regular subalgebra* A of B is a subalgebra, such that for all $M \subset A$ we have $\sum^B M = \sum^A M$ if the latter sum exists. Here A need not be complete and only the sums that exist in A are considered. A well-known fact is, that A is a regular subalgebra of B if and only if A is a dense subset of $\langle A \rangle_B^{\mathrm{cm}}$. In this case $\langle A \rangle_B^{\mathrm{cm}}$ is indeed isomorphic to $\overline{A} := \mathrm{RO}(A^+, <_B)$, the Dedekind completion of A (cf. [Kop93, Proposition 4]).

Lemma 2.1.2. *Let $\mathbb{B}$ be a Souslin algebra and $K \in \mathrm{mc}\,\mathbb{B}$. Then $\langle K \rangle$, the subalgebra of $\mathbb{B}$ that is finitarily generated by K, is a regular subalgebra of $\mathbb{B}$.*

Proof. Let $A := \langle K \rangle$ and M be a subset of A such that $\Sigma^A M$ exists. We have to show that $\sum^A M = \sum M$. Products and sums without superscript are to be taken in $\mathbb{B}$. Without loss of generality we can assume that M is a partition of unity in A and therefore countable.

To get a better representation for $\sum^A M$ we apply the Normal Form Theorem [Kop89, Proposition 4.4] to the elements $x \in M$:

$$x = \sum_{i=0}^{n} a_{i0} a_{i1}, \qquad \text{where } -a_{i0}, a_{i1} \in K,$$

where we assume that $0, 1 \in K$. The elementary products over K have at most two factors since K is linearly ordered by $\leq_{\mathbb{B}}$. By splitting up the sums we can assume that each $x \in M$ has the form $(-a)b$, where $a, b \in K$.

So we enumerate

$$M = \{a_{i0} \cdot a_{i1} : i \in \omega\}, \text{ where } (-a_{i0}), a_{i1} \in K \cup \{1\}$$

and apply the (dual) $(\aleph_0, \infty)$-distributivity law satisfied by $\mathbb{B}$:

$$\sum M = \sum_{i\in\omega} \prod_{j\in 2} a_{ij} = \prod \left\{ \sum_{i\in\omega} a_{if(i)} : f \in {}^{\omega}2 \right\}.$$

We have to show that for each $f \in {}^{\omega}2$ we have $\sum_{i\in\omega} a_{if(i)} = 1$. So we fix f, divide ω in two parts, $N_0 := f^{-1}(0)$ and $N_1 := f^{-1}(1)$, and consider

$$\sum a_{if(i)} = \sum_{i\in N_0} a_{i0} + \sum_{i\in N_1} a_{i1} = -\underbrace{\prod_{i\in N_0} -a_{i0}}_{=:b_0\in K} + \underbrace{\sum_{i\in N_1} a_{i1}}_{=:b_1\in K}.$$

The inequality $\sum a_{if(i)} < 1$ would therefore imply that $b_0 > b_1$ and finally the element $b_0 - b_1$ of A would be disjoint from every member of M – contradiction. □

Note that we only used that $(K, \leq_{\mathbb{B}})$ is completely embedded in $(\mathbb{B}, \leq_{\mathbb{B}})$, both viewed as complete lattices. I.e. for the conclusion of the lemma it suffices that $(K, <_{\mathbb{B}})$ is complete and the infima and suprema in L are the same as in $\mathbb{B}$.

We frequently consider the *regular open algebra* $\mathrm{RO}\, X$ of some topological space X. A subset U of X is regular open if the interior of the closure of U is equal to U and $\mathrm{RO}\, X$ is the set of all regular open subsets of X. The regular open algebra of any space X is a complete Boolean algebra but it is in general not a subalgebra of $\mathcal{P}(X)$ from which the operations are modified by taking the regularisations (cf. [Kop89, Theorem 1.37]). If the topology of X is not specified, then one of following applies:

- if X is a linear order, then X carries the order topology (as in the following proposition),
- if $X = T$ is a tree, then T carries the partial order topology which is generated by all subsets of the form $T(t)$ for some $t \in T$ (a declaration of our tree notation follows below).

Corollary 2.1.3. *For every $K \in \operatorname{mc}\mathbb{B}$ we have*

$$\langle K\rangle^{\mathrm{cm}} \cong \overline{\langle K\rangle} \cong \mathrm{RO}\,K,$$

where K carries the order topology induced by $\leq_{\mathbb{B}}$.

Proof. The first isomorphism follows with [Kop93, Proposition 4] by the last lemma. Now for the second: Every element of $\mathrm{RO}\,K$ contains some basic open interval (u, v) of K (or $[u, v)$ if $u = 0$ respective $(u, v]$ if $v = 1$, the regularisation), so the canonical homomorphism

$$\phi : \langle K\rangle \to \mathrm{RO}\,K,\ \phi\left(\sum_{i=0}^{n} u_i \cdot (-v_i)\right) = \sum_{i=0}^{n}{}^{\mathrm{RO}\,K} \operatorname{reg}(u_i, v_i) = \operatorname{reg}\left(\bigcup_{i=0}^{n}(u_i, v_i)\right)$$

(with $u_i, v_i \in K \cup \{0\}$) is an embedding whose range is a dense subset of $\mathrm{RO}\,K$. □

2.1.2 Souslin lines

A *Souslin line* is a complete, dense linear order that satisfies the countable chain condition but is not separable, i.e., it has no countable dense subset, but all families of pairwise disjoint open intervals are countable.

In this definition the possibility of a Souslin line having a separable non-trivial interval is included. We will, however, never consider a Souslin line with a non-trivial separable interval.

It is a standard result, that every Souslin line L has a dense subset of cardinality $\aleph_1$ and therefore the cardinality $|L| = 2^{\aleph_0}$.

Lemma 2.1.4. *The maximal chains of a Souslin algebra $\mathbb{B}$ are Souslin lines with endpoints and without non-trivial separable intervals.*

Proof. Let K be a maximal chain of $\mathbb{B}$. Completeness and denseness are trivial. We can associate each open interval (u, v) of K with the product $v \cdot (-u)$ in $\mathbb{B}$. So every set of pairwise disjoint open intervals of K corresponds to an antichain of $\mathbb{B}$ with the same cardinality and must thus be at most countable.

We finally show that K is non-separable. Let H be any countable subset of K. If H was a dense subset of K then the subalgebras of $\mathbb{B}$ completely generated by H and by K respectively would coincide, but by distributivity of $\mathbb{B}$, the subalgebra $\langle H\rangle^{\mathrm{cm}}$ is atomic by [Kop89, Proposition 14.8] while the complete subalgebra $\langle K\rangle^{\mathrm{cm}} \cong \mathrm{RO}(K, <_{\mathbb{B}})$, is atomless, because K is densely ordered by $<_{\mathbb{B}}$.

To see that K has no separable intervals replace K in the preceding argument by any closed interval $[u, v]$ of K and $\mathbb{B}$ by the relative algebra $\mathbb{B}{\restriction}(-u) \cdot v$ which is Souslin and has an isomorphic copy of the interval $[u, v]$ of K as a maximal chain. □

Lemma 2.1.5. *Let L be a Souslin line without separable intervals. The regular open algebra of L is a Souslin algebra. Furthermore, if L has endpoints then* $\mathrm{RO}\, L$ *has a maximal chain K which is isomorphic to L such that $\langle K\rangle^{\mathrm{cm}} = \mathrm{RO}\, L$.*

Proof. As above, completeness, absence of atoms and the antichain condition are easy. We now show that if we are given a family $(a_{nm})_{n,m\in\omega}$ of open intervals a_{nm} of L, such that for all $n \in \omega$ the set $\{a_{nm} \mid m \in \omega\}$ is a maximal antichain, we have

$$\sum \left\{ \prod_{n\in\omega} a_{n,f(n)} \mid f \in {}^{\omega}\omega \right\} = 1.$$

By [Kop89, Theorem 14.9.a)] this suffices in order to prove $\aleph_0$-distributivity. The set A of the boundaries of the a_{nm} is countable and therefore nowhere dense in L. Any open interval $(x, y) \subset L \setminus A$ lies below the product $b_f = \prod_{n\in\omega} a_{n,f(n)}$ for some $f \in {}^{\omega}\omega$ by our assumption on the family of the a_{nm}, and the sum over all such intervals is clearly 1. □

2.1.3 Normal trees

A *tree* is a partial order $(T, <_T)$ where the set of predecessors $\{s \mid s <_T t\}$ is well-ordered by $<_T$ for all $t \in T$. The elements of a tree are called *nodes*. For a node $t \in T$ we let $\mathrm{succ}(t)$ be the set of t's immediate successors. The *height* of the node t in T is the order type of the set of its predecessors under the ordering of T, $\mathrm{ht}_T(t) := \mathrm{ot}(\{s \mid s <_T t\}, <_T)$. For an ordinal α we let T_α denote the set of nodes of T with height α. If $\mathrm{ht}_T(s) > \alpha$ we let $s{\restriction}\alpha$ be the unique predecessor of s in level α.

The *height of a tree* T, $\mathrm{ht}\, T$, is the minimal ordinal α such that T_α is empty. An *antichain* is a set of pairwise incomparable nodes of T, so for $\alpha < \mathrm{ht}\, T$, the level T_α is an antichain of T.

If c is a subset of the height of a tree T, let $T{\restriction}c$ denote the tree that consists of all the nodes of T whose height lies in c together with the inherited tree order $<_T$:

$$T{\restriction}c = \bigcup_{\alpha\in c} T_\alpha, \quad s <_{T\restriction c} t \iff s <_T t.$$

In this context it is worth noting that $\mathrm{ht}_{T\restriction c}(t) = \mathrm{ot}(c \cap \mathrm{ht}_T(t))$ for all $t \in T{\restriction}c$ which is possibly smaller than $\mathrm{ht}_T(t)$.

Nodes, that do not have $<_T$-successors, are called *leaves*, and T is called κ-branching, κ a cardinal, if all nodes of T have exactly κ immediate successors, except for the leaves.

A *branch* is a subset b of T that is linearly ordered by $<_T$ and closed downwards, i.e. if $s <_T t \in b$ then $s \in b$. For $\alpha < \mathrm{ot}(b, <_T)$ we let $b \restriction \alpha$ be the unique element of $T_\alpha \cap b$ and so extend the similar notation for nodes and predecessors to branches and their elements. (For a normal tree this is just natural, since the nodes can be identified with the branches leading to them.) A branch is *maximal* in T if it is not properly contained in any other branch of T.

A maximal branch b of a tree T is called *cofinal* if its order type with respect to $<_T$ coincides with the height of T. We let

$$[T] = \{b \mid b \text{ is a cofinal branch of } T\}.$$

For $s \in T$ set $\hat{s} := \{b \in [T] \mid s \in b\}$. We consider $[T]$ as a topological space with the base $\{\hat{s} \mid s \in T\}$.

Under the notion of a *normal* tree we subsume the following four conditions:

a) there is a single minimal node called the *root*;

b) each node has at least two immediate successors;

c) each node has successors in every higher non-empty level;

d) branches of limit length have unique limits (if they are extended in the tree), i.e., if s, t are nodes of T of limit height whose sets of predecessors coincide, then $s = t$.

A striking property of normal trees is formulated in Kurepa's Isomorphism Lemma for normal trees, cf. [Kur35, p.102]. The result and its proof are well-known, but since we will use some variations of the argument later on, we also state the proof.

Lemma 2.1.6. *Let S, T be two normal κ-branching trees, $\kappa \leq \omega$, of the same height $\alpha < \omega_1$ with countable levels only. Then $S \cong T$.*

Proof. For trees of finite height, an easy inductive choice gives the isomorphism. So let the height of S and T be a countable limit ordinal α. Choose countable and dense sets $X \subset [S]$ and $Y \subset [T]$ and enumerate them by $(x_i \mid i \in \omega)$ and $(y_i \mid i \in \omega)$ respectively.

We give a back-and-forth-construction of a bijective mapping $f : X \to Y$ which lifts to a tree isomorphism $\varphi : S \to T$. Define $f(x_0) = y_0$ and $\varphi(x_0 \restriction \gamma) = y_0 \restriction \gamma$ for all $\gamma < \alpha$.

We now give the back-argument, the forth-argument is completely analogous. Let i be minimal such that $f^{-1}(y_i)$ has not yet been fixed, and pick the minimal γ such that $\varphi^{-1}(y_i \restriction \gamma)$ has not yet been defined.

This γ is a successor ordinal, say $\gamma = \delta + 1$, because S and T are assumed to be normal trees.

Now choose an immediate successor s of $\varphi^{-1}(y_i \restriction \delta)$ such that $\varphi(s)$ has not yet been defined. Such a node s exists by the choice of γ. Finally let j be minimal such that $s \in x_j$. Then $f(x_j)$ has not yet been defined, but we set $f(x_j) = y_i$ and $\varphi(x_j \restriction \gamma) = y_i \restriction \gamma$. This is consistent with the choices met for f and φ so far.

After every step of the construction φ is a partial isomorphism between S and T. By the choice of X and Y, the union of these partial isomorphisms is bijective, so in the end $\varphi : S \to T$ is an isomorphism.

If the trees are of infinite successor height $\alpha + n$, then find $\varphi : S \restriction \alpha \to T \restriction \alpha$ as above with the choices $X = \{\{r <_S s\} \mid s \in S_\alpha\}$ and $Y = \{\{r <_T t\} \mid t \in T_\alpha\}$. Then f gives the extension of φ to level α. The final n steps are then easy. □

The last argument shows that for any $\gamma < \alpha$ and every isomorphism φ_0 between $S\restriction(\gamma + 1)$ and $T\restriction(\gamma + 1)$ there is an extension $\varphi_0 \subset \varphi : S \cong T$.

The following observation turns out to be very useful in the construction of homogeneous Souslin algebras, cf. Example (2.1.13).

Proposition 2.1.7. *Let T be a normal tree of countable limit height α with only countable levels. Then $[T]$ (with the topology defined above) is a perfect Polish space, i.e. a separable and completely metrizable space without isolated points.*

Proof. Choose a cofinal sequence (γ_n) of length ω in α with range c. Then $[T]$ is homeomorphic to $[T\restriction c]$ via

$$\varphi : [T] \to [T\restriction c], \quad b \mapsto b \cap T\restriction c.$$

If $T\restriction c$ is $\aleph_0$-branching, then $[T\restriction c]$ is nothing less than the Baire space ${}^\omega\omega$. In general, $[T\restriction c]$ can be embedded onto a perfect subspace of the Baire space and so is Polish as well. □

Whenever we consider a mapping $\varphi : T \to S$ between trees and call this mapping a *tree homomorphism* we mean that φ carries $<_T$ to $<_S$ and respects the height function: $\mathrm{ht}_S(\varphi(s)) = \mathrm{ht}_T(s)$ for all $s \in T$.

A tree T is said to be *homogeneous*, if for all pairs $s, t \in T$ of the same height there is a tree isomorphism between $T(s)$ and $T(t)$, the trees of nodes in T above s and t respectively.

It is clear that every tree isomorphism $\varphi : T \to S$ induces a homeomorphism $\overline{\varphi} : [T] \to [S]$ and that for every club $C \subseteq \mathrm{ht}\, T$ we have a natural homeomorphism between $[T]$ and $[T\restriction C]$ since every cofinal branch of T is uniquely determined by its intersection with $T\restriction C$.

2.1.4 Souslin trees

In general, a *Souslin tree* is a tree T of height ω_1 such that every family of pairwise incomparable nodes and every branch of T are at most countable. We will only consider normal Souslin trees, where absence of uncountable antichains already implies that the tree has no cofinal branch.

It is well known that for a normal Souslin tree T, its regular open algebra $\mathrm{RO}\,T$ is a Souslin algebra and T can be embedded densely in $\mathrm{RO}\,T$. On the other hand, in every Souslin algebra $\mathbb{B}$ that has a family of complete generators of cardinality $\aleph_1$ (this is what we will call a *small* Souslin algebra), there is a dense subset T of $\mathbb{B}\setminus\{0\}$ such that $(T, >_{\mathbb{B}})$ is a normal Souslin tree (note that $<_T$ is $>_{\mathbb{B}}$) whose regular open algebra is isomorphic to $\mathbb{B}$ (see e.g. [Kop89, Theorem 14.20]).

We will use the following convention established in [DJ74]. Let T be a subset of a Souslin algebra $\mathbb{B}$. Then T is said to *souslinise* $\mathbb{B}$ or to be a *Souslinisation* of $\mathbb{B}$ if T is dense in $\mathbb{B}$ and becomes a Souslin tree under the reverse Boolean order of $\mathbb{B}$. Every level T_α of a Souslinisation T of $\mathbb{B}$ is a partition of unity in $\mathbb{B}$ and taking limits in T is simply the evaluation of the corresponding infinite product in $\mathbb{B}$: If $t \in T_\alpha$ is a limit node, let t_γ for $\gamma < \alpha$ be the unique $>_{\mathbb{B}}$-predecessor of t of height γ. Then $t = \prod\{t_\gamma \mid \gamma < \alpha\}$.

The Souslinisation is unique up to the elimination of a non-stationary set of levels. For a proof of this well-known result, cf. [DJ74, Lemma VIII.9].

Lemma 2.1.8 (Restriction Lemma)**.** *Let $\mathbb{B}_0$, $\mathbb{B}_1$ be Souslin algebras with Souslinisations T_0 and T_1 respectively. Then a maping $\varphi : \mathbb{B}_0 \to \mathbb{B}_1$ is an isomorphism just in case that there is a closed unbounded subset C of ω_1 such that the restriction of φ to $T_0{\restriction}C$ is a tree isomorphism onto $T_1{\restriction}C$.*

Remark 2.1.9. It is easily seen that each tree automorphism of the Souslinisation T extends to a Boolean automorphism on $\mathbb{B}$, yet not every automorphism of $\mathbb{B}$ needs to restrict to T. Anyway, the last theorem implies that there must be some c club in ω_1 such that the Boolean automorphism restricts to $T{\restriction}c$. Therefore, we call Lemma 2.1.8 *the Restriction Lemma (for isomorphisms between Souslin algebras).*

A Souslinisation T of $\mathbb{B}$ provides a natural stratification of $\mathbb{B}$ by countably generated, complete and therefore atomic subalgebras. Let for $\alpha < \omega_1$

$$\mathbb{B}_\alpha := \langle T_\alpha \rangle^{\mathrm{cm}}.$$

Note that for all $\alpha < \omega_1$ we have $\mathbb{B}_\alpha \cong \mathcal{P}(\omega)$ and $T{\restriction}(\alpha + 1) \subset \mathbb{B}_\alpha$. Clearly the sequence of the $\mathbb{B}_\alpha$ is increasing. To show $\mathbb{B} = \bigcup_{\alpha<\omega_1} \mathbb{B}_\alpha$, pick $a \in \mathbb{B} \setminus \{0\}$. There is a maximal pairwise disjoint subset A of T of elements $\leq_{\mathbb{B}} a$, so $\sum A = a$. By the countable chain condition A must be countable and therefore a subset of $\mathbb{B}_\alpha$ for some $\alpha < \omega_1$. We finally note, that the sequence of the $\mathbb{B}_\alpha$ is not continuous. For a countable limit ordinal α we have

$$\bigcup_{\gamma<\alpha} \mathbb{B}_\gamma \lneq \left\langle \bigcup_{\gamma<\alpha} \mathbb{B}_\gamma \right\rangle^{\mathrm{cm}} = \mathbb{B}_\alpha.$$

For the representation of a given maximal chain K of a Souslin algebra $\mathbb{B}$ with Souslinisation T we will use the approximations $K_\alpha = K \cap \mathbb{B}_\alpha$ which are chains in $\mathbb{B}_\alpha$ (but not necessarily maximal). In the simple case of power set algebras (or their isomorphic copies, such as the $\mathbb{B}_\alpha$) we can give the following characterisation for maximal chains.

Proposition 2.1.10. *Let X be a set. Given a $\subset$-chain K of $\mathcal{P}(X)$, define the quasi-ordering $\leq_K$ on X by*

$$x \leq_K y \leftrightarrow (\forall u \in K)(y \in u \Rightarrow x \in u)$$

and let $\overline{K} := \{\sum M, \prod M \mid M \subseteq K\}$ be the completion of the linear order $(K, \subset)$ in $\mathcal{P}(X)$. Then the following statements are equivalent:

(i) $\overline{K}$ *is a maximal chain of* $\mathcal{P}(X)$*;*

(ii) $\leq_K$ *is a linear ordering of* X*;*

(iii) $\langle K \rangle^{\mathrm{cm}}_{\mathcal{P}(X)} = \mathcal{P}(X)$.

Proof. We prove (i→ii) by contraposition. Assume that there are distinct $x, y \in X$ such that $x \leq_K y$ and $y \leq_K x$, so for all $u \in K$ we have $x \in u$ if and only if $y \in u$. But then the set

$$v := \bigcap \{u \in K \mid x, y \in u\} \setminus \{x\}$$

is $\subset$-comparable with every $u \in \overline{K}$ yet not a member of $\overline{K}$ itself. So K cannot be a maximal chain.

If X is totally ordered by $\leq_K$, then $\overline{K}$ coincides with the set of all initial segments of $\leq_K$ which is a maximal chain of $\mathcal{P}(X)$. So we have (i↔ii).

Now we show the equivalence between (ii) and (iii). We know that

$$\langle K \rangle^{\mathrm{cm}} = \left\langle \overline{K} \right\rangle^{\mathrm{cm}} \supseteq \left\langle \overline{K} \right\rangle.$$

The algebra to the right is a regular subalgebra of the middle one which is regular in $\mathcal{P}(X)$ itself. If we assume (iii) this implies that $\langle\overline{K}\rangle$ is a dense subalgebra, i.e., every atom $\{x\}$, $x \in X$ of $\mathcal{P}(X)$ is a member of $\langle\overline{K}\rangle$ and therefore admits a representation

$$\{x\} = \bigcap M \setminus \bigcup N \qquad \text{for some } M,\ N \subseteq K$$

which in turn implies that $\leq_K$ separates all members of X, i.e., $\leq_K$ is total. Now for the converse, given (ii), for each $x \in X$,

$$\{x\} = \bigcap\{u \in \overline{K} \mid x \in u\} \setminus \bigcap\{u \in \overline{K} \mid x \notin u\} \in \langle K\rangle^{\mathrm{cm}}.$$

□

Remark 2.1.11. Of course, if in the last lemma, K is a complete sub-lattice of $\mathcal{P}(X)$, then $K = \overline{K}$. So we have a bijective association between the set $\mathrm{mc}(\mathcal{P}(X))$ of maximal chains of a power set algebra $\mathcal{P}(X)$ and the set of linear orders of the underlying set X. The order types of $(K, \subset)$ and $(X, <_K)$ are tightly related. This also shows that for infinite X the power set algebra $\mathcal{P}(X)$ has many non-isomorphic maximal chains.

2.1.5 $\diamondsuit$-principles

As for most Souslin tree constructions in the literature, we will assume diamond-principles.

Definition 2.1.12. a) The sequence $(R_\alpha)_{\alpha<\omega_1}$ of sets $R_\alpha \subseteq \alpha$ for $\alpha < \omega_1$ is a $\diamondsuit$-sequence if for all $X \subseteq \omega_1$ there is some stationary set $s \subseteq \omega_1$ such that, for all $\alpha \in s$ we have $X \cap \alpha = R_\alpha$. The statement"There is a $\diamondsuit$-sequence." will be denoted by $\diamondsuit$, say *diamond*.

b) The sequence $(S_\alpha)_{\alpha<\omega_1}$ of countable sets $S_\alpha \subseteq \mathcal{P}(\alpha)$ is a $\diamondsuit^+$-sequence if for all $X \subseteq \omega_1$ there is some club set $c \subseteq \omega_1$ s.t., for all $\alpha \in c$ we have $X \cap \alpha,\ c \cap \alpha \in S_\alpha$. The statement "There is a $\diamondsuit^+$-sequence." will be denoted by $\diamondsuit^+$, say *diamond-plus*.

If we delete the requirement that $c \cap \alpha$ be in S_α for $\alpha \in c$ in the definition of the $\diamondsuit^+$-sequence, then we define a $\diamondsuit^*$-sequence. This will not be used here. Since $\diamondsuit^+ \to \diamondsuit^* \to \diamondsuit$ we only need to assume $\diamondsuit^+$ in our statements even though for the sake of convenience we will use both, a $\diamondsuit^+$-sequence and a $\diamondsuit$-sequence, at the same time.

To illustrate the use of Proposition 2.1.7 we will construct under the assumption of $\diamondsuit$ a homogeneous Souslin tree, i.e., a Souslin tree T with an automorphism mapping s to t for every pair of nodes s, t of the same height in T.

Example 2.1.13 (a homogeneous Souslin tree)**.** Assume $\diamondsuit$ and denote the members of the $\diamondsuit$-sequence by R_α for $\alpha < \omega_1$. As in the classical constructions (e.g. [Kun80, Theorem 7.8]) we inductively construct a tree order $<$ on ω_1. The tree automorphisms are constructed alongside with the tree.

We choose 0 as the root node, and for construction step $\alpha + 1$ we provide every node of level T_α with $\aleph_0$ direct successors. The inductive hypothesis gives for each pair $s, t \in T_\beta$ with $\beta < \alpha$ an automorphism $\varphi_{u,v}$ of $T\restriction(\alpha + 1)$ sending u to v. Now pick s and t from the new level $T_{\alpha+1}$, write s^- and t^- for $s\restriction\alpha$ and $t\restriction\alpha$ respectively, and denote by $\varphi_{s,t}$ any automorphism of $T\restriction\alpha + 1$ extending φ_{s^-,t^-} and sending s to t. Since we gave every node of T_α the same number of direct successors, this is always possible. For the same reason we can extend the $\varphi_{u,v}$ for $u, v \in T_\beta$ where $\beta \leq \alpha$ to automorphisms of $T\restriction(\alpha + 2)$.

Now for the limit stage α. We have constructed so far the tree $T\restriction\alpha$ and automorphisms $\varphi_{s,t}$ of $T\restriction\alpha$ for each pair of nodes of the same height. Proposition 2.1.7 states that the space $[T\restriction\alpha]$ is Polish. The level T_α we are about to choose has to correspond (branches equal sets of predecessors) to a countable, dense subset of $[T\restriction\alpha]$ by the normality requirements posed on T. Now every automorphism φ of $T\restriction\alpha$ naturally induces a homeomorphism $\overline{\varphi}$ of $[T\restriction\alpha]$. It is easy to see that for any choice of φ and T_α there is an automorphism φ^+ of $T\restriction(\alpha+1)$ extending φ if and only if the subset of $[T\restriction\alpha]$ corresponding to T_α is closed under the application of $\overline{\varphi}$.

Our task is to choose a countable, dense subset of $[T\restriction\alpha]$ in such a way, that the tree automorphisms $\varphi_{s,t}$ of $T\restriction\alpha$ constructed so far can be extended to the new level T_α.

We distinguish two cases. In the first case, α satisfies the ordinal arithmetic equation $\alpha = \omega \times \alpha$ and the set R_α is a maximal antichain of $T\restriction\alpha$. Denote by X the set of all cofinal branches b of $T\restriction\alpha$ which have a common member with R_α. Since R_α is a maximal antichain, X is dense and open in $[T\restriction\alpha]$. Consider the dense and open sets $h''X$ where h is a composition of finitely many homeomorphisms $\overline{\varphi_{s,t}}$ or their inverse mappings. In particular, the set H of all these maps h is countable, so the intersection

$$N = \bigcap_{h \in H} h''X$$

is comeagre in $[T\restriction\alpha]$ and therefore dense by the Baire Category Theorem. Furthermore, N is closed under the applications of $\overline{\varphi_{s,t}}$ and its inverse mapping for all

$s, t \in T_\beta$, $\beta < \alpha$. Finally choose a countable, dense set Z of N and let T_α correspond to the closure of Z under the application of all the homeomorphisms $h \in H$ and which is a countable and dense subset of X again. By this choice we have ensured that R_α is also maximal as an antichain of $T{\restriction}(\alpha + 1)$.

In the second case, there is no need for the preparatory reduction step from $[T{\restriction}\alpha]$ to N. We just choose the countable, dense set Z from $[T{\restriction}\alpha]$ and continue as in the first case.

In both cases, by the unique limits property, there is only one choice for the extensions of the $\varphi_{s,t}$ for $s, t \in T_\beta$, $\beta < \alpha$.

The automorphisms $\varphi_{s,t}$ for $s, t \in T_\alpha$ can be chosen using a variant of Kurepa's Lemma 2.1.6.

Remark 2.1.14. A variation of the construction above, where in each limit step, instead of the infinite set Z a singleton b is chosen from N or $[T{\restriction}\alpha]$ respectively, yields a tree with a property called *strong homogeneity*, cf. Section 1.5.1. In order to obtain a dense subset of $[T{\restriction}\alpha]$ it suffices to choose a single branch b, because the family of automorphisms then produces branches that go through all nodes not in b.

The crucial property of a strongly homogeneous tree T, respectively of its family of automorphisms is called *transitivity* and requires that the limit levels are minimal with respect to the possibility of extending certain automorphisms.

2.2 Generating chains and complete subalgebras

In this section $\mathbb{B}$ always denotes a small Souslin algebra, i.e., $\mathbb{B}$ has a set of complete generators of size $\aleph_1$. We fix a Souslinisation T of $\mathbb{B}$ and aim at representing the maximal chains of $\mathbb{B}$ as relations on the Souslinisation T. Since this is easy for a maximal chain K if $\langle K\rangle^{\text{cm}} = \mathbb{B}$ and cumbersome if not, we only represent the generating chains and find something easier for the non-generating chains: we represent the complete and atomless subalgebras of $\mathbb{B}$ as equivalence relations on T. In our constructions in the next section, we arrange the subalgebras to be isomorphic to $\mathbb{B}$, so every maximal chain will be isomorphic to a generating chain and if the latter are all of the same order type, our primary goal is achieved.

2.2.1 Windscreen wipers

Small Souslin algebras are completely generated by some of their maximal chains (cf. Lemma 2.2.5). And these generating chains can be represented on a Souslinisation in a particularly simple manner. These representations, coherent families of total orders on levels of the Souslinisation, will be called wipers. We will use the wipers to make all *generating chains* pairwise isomorphic.

Definition 2.2.1. A maximal chain K of $\mathbb{B}$ that completely generates $\mathbb{B}$ will be called a *generating chain.*

Being a generating chain of $\mathbb{B}$ clearly is not only a property of the chain itself, but of the pair of the Souslin algebra and its maximal chain. We emphasise that a chain K that satisfies $\langle K\rangle^{\text{cm}} = \mathbb{B}$ will not be called generating unless it is maximal. Since, for any Souslin line L without separable intervals, $\operatorname{RO} L$ is a Souslin algebra with a generating chain isomorphic to L, the last definition is non-void. It will soon be clear that every small Souslin algebra has many generating chains (while, of course, non-small Souslin algebras have no generating chains at all).

Lemma 2.2.2. *Given a generating chain $K \subseteq \mathbb{B}$ the set*

$$C := \{\alpha < \omega_1 : K \cap \mathbb{B}_\alpha \in \operatorname{mc}(\mathbb{B}_\alpha)\} = \{\alpha \in \omega_1 : \langle K \cap \mathbb{B}_\alpha\rangle^{\text{cm}} = \mathbb{B}_\alpha\}$$

is closed and unbounded in ω_1.

Proof. We set $K_\alpha := K \cap \mathbb{B}_\alpha$ for $\alpha < \omega_1$ and start by showing that the two definitions of C coincide. Since the subalgebras $\mathbb{B}_\alpha$ are all isomorphic to $\mathcal{P}(T_\alpha)$ we have by Remark 2.1.11 on maximal chains in power set algebras a bijective mapping between the maximal chains of $\mathbb{B}_\alpha$ and the linear orders of T_α. Given $K_\alpha \in \mathrm{mc}(\mathbb{B}_\alpha)$ on one side, for every $t \in T_\alpha$ the difference

$$\{s \in T_\alpha \mid s \leq_{K_\alpha} t\} \setminus \{s \in T_\alpha \mid s <_{K_\alpha} t\}$$

is the singleton $\{t\}$. So $\langle K_\alpha \rangle^{\mathrm{cm}}$ contains the generators of $\mathbb{B}_\alpha$ and the two complete subalgebras are equal. For the other direction we just note that $\langle K_\alpha \rangle^{\mathrm{cm}} = \mathbb{B}_\alpha$ implies that $<_{K_\alpha}$ separates each pair of nodes $s, t \in T_\alpha$ and is therefore total, and the induced chain, which is K_α, is maximal.

Now let $\alpha_0 < \omega_1$. To show that C is unbounded we inductively define

$$\alpha_{i+1} = \min\{\gamma \geq \alpha_i : T_{\alpha_i} \subseteq \langle K_\gamma \rangle^{\mathrm{cm}}\}.$$

We show that this minimum is well-defined. The assumption on K to completely generate $\mathbb{B}$ implies that each of the countably many $u \in T_\alpha$ lies in some $\langle K_\gamma \rangle^{\mathrm{cm}}$, because we can represent u as an infinite sum over countably many products $-x \cdot y$ with x, y in some K_β, $\beta < \omega_1$.

It remains to show that $T_\alpha \subset \langle K_\alpha \rangle^{\mathrm{cm}}$ for $\alpha = \sup \alpha_i$. So let $u \in T_\alpha$ and for $i \in \omega$ let $x_i \in T_{\alpha_i}$ be the unique T-predecessor of x. So we have $x_i \in \langle K_{\alpha+1} \rangle^{\mathrm{cm}}$ and

$$x = \prod\{x_i : i \in \omega\} \in \left\langle \bigcup_{i \in \omega} K_{\alpha_{i+1}} \right\rangle^{\mathrm{cm}} \subseteq \langle K_\alpha \rangle^{\mathrm{cm}}.$$

By a similar argument we see that C is closed in ω_1. □

The last theorem and its proof indicate how our representation scheme for the generating chains works.

Definition 2.2.3. Let T be a normal tree with countable levels of arbitrary height $\alpha \leq \omega_1$ and let C be a subset of α. A *wiper of total orders* (or more convenient a *wiper*) on $T\restriction C$ is a family $W = \langle <_\gamma \mid \gamma \in C \rangle$ of total orders $<_\gamma$ on T_γ such that

(i) W respects the tree order of T: for all $\beta, \gamma \in C$, $\beta < \gamma$ and $s, t \in T_\beta$, $s', t' \in T_\gamma$ we have

$$s <_T s' \wedge t <_T t' \wedge s <_\beta t \Rightarrow s' <_\gamma t',$$

(ii) if $\beta, \gamma \in C$, $\beta < \gamma$, and $s \in T_\beta$, then the set $I_{s,\gamma} = \{t \in T_\gamma \mid s <_T t\}$ of successors of s in level γ is ordered densely without endpoints by $<_\gamma$.

If T of height ω_1 is a Souslinisation of $\mathbb{B}$ and W is a wiper on T, we say that

$$K_W := \left\{ \sum M \mid (\exists \alpha \in \omega_1) M \subseteq T_\alpha,\ M \text{ is an initial segment of } <_\alpha \right\}$$

is the subset of $\mathbb{B}$ which is *induced* by W.

Only the first condition in this definition is inspired by Lemma 2.2.2 while the second is a useful standardisation requirement whose technical importance will become clear in the proof of Proposition 2.2.6 where we aim at building isomorphisms between wipers. To illustrate the definition and justify its definiendum imagine the Souslin tree $T{\restriction}C$ printed on the windscreenof your car, the levels on horizontal lines and if $s, t \in T{\restriction}C$ are of height α then s stands to the left of t if and only if $s <_\alpha t$. In this picture each state of a windscreen wiper, which has its axis fixed in the root of T, corresponds to a member of the generating chain induced by the wiper.

The following lemma formally establishes this relationship between generating chains and wipers.

Lemma 2.2.4. *Let $\mathbb{B}$ be a Souslin Algebra with Souslinisation T.*

a) If W is a wiper on T, then

$$K_W := \left\{ \sum M \mid (\exists \alpha < \omega_1) M \subseteq T_\alpha,\ M \text{ is an initial segment of } <_\alpha \right\}$$

is a generating chain of $\mathbb{B}$.

b) Let K be a generating chain of $\mathbb{B}$. Then there is a club $C \subseteq \omega_1$, s.t. there is a wiper W on $T{\restriction}C$, inducing K, i.e. with $K = K_W$ as above.

Proof. To see that K_W is a chain in $\mathbb{B}$, let $\beta < \gamma$, both in C. Let $M \subset T_\gamma$ and $N \subset T_\beta$ be initial segments with respect to $<_\gamma$ and $<_\beta$ respectively. Then

$$P := \{s \in T_\gamma \mid s{\restriction}\beta \in N\}.$$

defines an initial segment of $<_\gamma$ and we have $\sum N = \sum P$ and $M \subseteq P$ or $P \subseteq M$.

To show maximality of K_W let $x \in \mathbb{B} \setminus K_W$ and find $\alpha \in C$ with $x \in \mathbb{B}_\alpha$. Then $M = \{s \in T_\alpha \mid s \leq_{\mathbb{B}} x\}$ is not an initial segment of $<_\alpha$ and it is easy to construct one which witnesses that $K_W \cup \{x\}$ is no chain of $\mathbb{B}$.

We show that K_W is generating. For the node t of height α, fix the initial segments

$$M_< := \{s \in T_\alpha \mid s <_\alpha t\} \quad \text{and} \quad M_\leq := \{s \in T_\alpha \mid s \leq_\alpha t\}$$

of $<_\alpha$. It is then clear that $t = \sum M_\leq - \sum M_<$. So $\langle K_W \rangle^{\text{cm}}$ contains a dense subset of $\mathbb{B}$.

Let now K be a generating chain of $\mathbb{B}$. By Lemma 2.2.2 we know that on a club set $C' \subseteq \omega_1$ of levels, the chains $K \cap \mathbb{B}_\alpha$ are maximal chains of $\mathbb{B}_\alpha \cong \mathcal{P}(T_\alpha)$. Each maximal chain of $\mathcal{P}(T_\alpha)$ corresponds to a total order of the set of atoms T_α and the elements of that maximal chain are simply the initial segments of this total order.

It is trivial that these induced total orders of T_α, $\alpha \in C'$ respect the tree order. So we only have to show that the denseness requirement in the definition of wipers is fulfilled on some club subset of C'.

Recall that $I_{s,\gamma}$ is the set of all successors of s with height γ. For all $s \in T_\beta$ there is some $\gamma > \beta$ such that $I_{s,\gamma}$ has no minimum, because T has no cofinal branch. The same holds for every $\gamma' \in C'$ above γ, since if $t \in I_{s,\gamma'}$ was $<_{\gamma'}$-minimal then $t \upharpoonright \gamma$ would have to be $<_\gamma$-minimal.

The analogous argument works of course for maximality. For $s \in T_\beta$ let γ_s^0 be some countable ordinal in C', such that I_{s,γ_s^0} has neither minimum nor maximum with respect to $<_{\gamma_s^0}$. Let

$$\gamma_s^{n+1} := \sup\{\gamma_t^0 \mid t \in I_{\gamma_s^n,s}\}$$

and $\alpha_s = \sup \gamma_s^n$. Then I_{s,α_s} is densely ordered by $<_{\alpha_s}$. We inductively define a normal sequence $(\delta_i)_{i<\omega_1}$ in C' whose range is the desired club set C:

$$\delta_{i+1} := \bigcup_{s \in T_{\delta_i}} \alpha_s$$

and $\delta_i = \bigcup_{j<i} \delta_j$ for i a countable limit. Then clearly for every $s \in T\upharpoonright C$ and $\delta_i > \mathrm{ht}(s)$ the linear order $(I_{s,\delta_i}, <_{\delta_{i+1}})$ is dense by construction. □

Every Souslin line has a dense subset of cardinality $\aleph_1$. Thus a small Souslin algebra has at most $2^{\aleph_1}$ maximal chains, because every maximal chain is uniquely determined be a dense subset (as a linear order).

By the next lemma we can find as many generating chains as we like.

Lemma 2.2.5. *Every small Souslin algebra has exactly $2^{\aleph_1}$ generating chains.*

Proof. Each maximal chain K of $\mathbb{B}$ is a Souslin line and therefore uniquely determined by a dense subset of power $\aleph_1$, so there cannot be more than $2^{\aleph_1} = |\mathbb{B}|^{\aleph_1}$ maximal chains. Now let T be a Souslin tree and consider the club set C of all countable limit ordinals and the family F of all wipers on $T\upharpoonright C$. It is clear that F has cardinality $2^{\aleph_1}$ and distinct members of F induce distinct generating chains. □

The second requirement in Defintion 2.2.3 of *wipers* enables us to combine the argument from proof of Kurepa's Isomorphism Lemma with that of Cantor for the $\aleph_0$-categoricity of the dense linear orders: We construct isomorphisms between wipers. This will be a major step in the construction of a Souslin algebra that is homogeneous for all generating chains.

Proposition 2.2.6. *Let T and S be two countable, $\aleph_0$-branching, and normal trees of the same height $\alpha < \omega_1$ and let $W_0 = \langle \prec_\gamma | \ \gamma < \alpha \rangle$ be a wiper on T and $W_1 = \langle <_\gamma | \ \gamma < \alpha \rangle$ a wiper on S. Furthermore let $\beta < \alpha$ and φ' be an isomorphism from $T\restriction(\beta+1)$ onto $S\restriction(\beta+1)$, such that for all $\gamma \leq \beta$ and $s, t \in T_\gamma$ we have*

$$s \prec_\gamma t \Leftrightarrow \varphi'(s) <_\gamma \varphi'(t).$$

Then there is an isomorphism φ between T and S extending φ', such that for all $\gamma < \alpha$ and $s, t \in T_\gamma$ we have

$$s \prec_\gamma t \Leftrightarrow \varphi'(s) <_\gamma \varphi'(t).$$

Proof. We refer to the proof of Kurepa's Isomorphism Lemma 2.1.6 and describe the only manipulation: When it comes to choosing of $\varphi^{-1}(s)$, this choice has to respect the wipers, which is always possible by the denseness requirement. □

2.2.2 The Reduction Lemmata

We will now present a sequence of results, which lay the technical grounds for our treatment of subalgebras in the construction of chain homogeneous Souslin algebras. In Section 1.1 we introduced tree equivalence relations (t.e.r.s) in order to study complete subalgebras of Souslin algebras. We briefly repeat the relevant definitions and results for the reader's convenience.

Let T be a normal, $\aleph_0$-branching tree. We say that an equivalence relation $\equiv$ on T is a *tree equivalence relation* if $\equiv$ respects levels (i.e., $\equiv$ refines $T \otimes T$), is compatible with the tree order of T (i.e., $\operatorname{ht} s = \operatorname{ht} r$ and $s < t \equiv u > r$ imply $s \equiv r$) and the quotient $T/{\equiv}$ is a normal and $\aleph_0$-branching tree with the induced order.

For a Souslin tree T, every t.e.r. $\equiv$ on T represents an atomless and complete subalgebra $\mathbb{A}_\equiv = \langle \sum t/{\equiv} | \ t \in T \rangle^{\mathrm{cm}}$ of $\mathbb{B} = \operatorname{RO} T$. (We denote the $\equiv$-class of $t \in T$ by $t/{\equiv}$.) A t.e.r. $\equiv$ on T is said to be

- *nice* if for all triples $r, s, t \in T$ with $r \equiv s$ and $s < t$ there is some $u > r$ equivalent to t;
- *almost nice* if the same condition holds for all triples such that $\operatorname{ht} s = \operatorname{ht} r$ is a successor ordinal;
- *decent* if there is a tree S carrying an almost nice t.e.r. $\sim$, and a club C of $\operatorname{ht} S$ such that $(S\restriction C, \sim) \cong (T, \equiv)$, i.e., $\equiv$ is the restriction of an almost nice t.e.r. to a club subset of levels.

A subalgebra of $\mathbb{B}$ is called nice, if it is represented by a nice t.e.r. on some Souslinization of $\mathbb{B}$. Since niceness descends to the restriction to a club set of levels, this is independent from the choice of the Souslinisation. Nice subalgebras are much easier to handle than general subalgebras. But as chain homogeneous Souslin algebras are necessarily homogeneous (cf. Proposition 2.3.3) and every homogeneous, small Souslin algebra also has non-nice, atomless and complete subalgebras (by Theorems 1.3.5 and 1.3.9) we also have to consider this general case in our construction. Keep in mind, that *we aim at rendering all atomless, complete subalgebras isomorphic to the Souslin algebra we construct*.

The main result of Section 1.1 is Proposition 1.1.7 which states that for every atomless and complete subalgebra $\mathbb{A}$ of $\mathbb{B}$, there is a Souslinisation T with an almost nice t.e.r. $\equiv$ that represents $\mathbb{A}$. So for every Souslin algebra with a Souslinisation T and every complete and atomless subalgebra $\mathbb{A}$ of $\mathbb{B}$ there is a decent t.e.r. $\equiv$ that represents $\mathbb{A}$ on $T{\upharpoonright}C$ for some club C of ω_1.

Recall that a tree homomorphism is a mapping that preserves the tree order and levels, i.e., if $\varphi : S \to T$ is a tree homomorphism and $\mathrm{ht}_S\, s = \mathrm{ht}_T\, \varphi(s)$. We say that a tree homomorphism $\varphi : S \to T$ *carries* the t.e.r. $\sim$ in S to the t.e.r. $\equiv$ on T, if for all $r, s \in S$ we have $r \sim s$ if and only if $\varphi(r) \equiv \varphi(s)$.

Later on, we will use the following proposition to rule out unwanted subalgebras. It gives a "necessary denseness condition" for decent t.e.r.s which we can design to fail, if a given t.e.r. does not fit in our method for extending tree isomorphisms. But before we state Proposition 2.2.8 we introduce a notion of suitability that allows us to state this and later results in a compact way.

Definition 2.2.7. Let X be a topological space and $\equiv$ an equivalence relation on X. We say that a subset $N \subseteq X$ is *suitable for* $\equiv$ if for every element $x \in X$ the intersection of its $\equiv$-class $x/{\equiv}$ with N is either empty or dense in $x/{\equiv}$ (with the subspace topology).

For a mapping $h : X \to Z$ we say $N \subset X$ is *suitable for* h if N is suitable for the equivalence relation given by $x \equiv y :\Longleftrightarrow h(x) = h(y)$.

Proposition 2.2.8. *Let T be a normal, $\aleph_0$-branching tree of height $\beta \leq \omega_1$, and let $\equiv$ be a decent t.e.r. on T. Let $\alpha < \beta$ be a limit and consider the equivalence relation $\cong$ on the perfect Polish space $[T{\upharpoonright}\alpha]$ induced by the restriction of $\equiv$ to $T{\upharpoonright}\alpha$.*

a) For every $x \in [T{\upharpoonright}\alpha]$, the class $x/{\cong}$ is nowhere dense and closed in $[T{\upharpoonright}\alpha]$

b) Consider the natural embedding $b : T_\alpha \to [T{\upharpoonright}\alpha]$ given by $b(t) := \{s \in T{\upharpoonright}\alpha \mid s < t\}$. Then the set $b''T_\alpha$ is suitable for $\cong$.

Proof. Part a) follows directly from the unique-limits-instance of the normality requirement we posed on t.e.r.s . Here we do not even need that $\equiv$ is decent.

On the other hand, decency is crucial for the second statement. For its proof suppose there is a tree T' of height $\lambda+1$ (where λ is a limit) equipped with an almost nice t.e.r. $\equiv'$, there is a club C of λ and a tree isomorphism $\varphi : T\restriction\alpha+1 \to T'\restriction C\cup\{\lambda\}$ which carries $\equiv$ to $\equiv'$. Then the induced homeomorphism $\overline{\varphi}$ between $[T\restriction\alpha]$ and $[T'\restriction C]$ is compatible with the induced equivalence relations $\cong$ and $\cong'$ on $[T\restriction\alpha]$ and $[T'\restriction C]$ respectively. It therefore suffices to show that for $t \in T'_\lambda$ the corresponding set $\{b(r) \mid r \in T'_\lambda, r \equiv' t\}$ is dense. So let $x \cong' b(t)$ and $\gamma < \lambda$. We have to show that there is some $u \in T'_\lambda$ above $x\restriction\gamma$ with $u \equiv' t$. But this follows easily by the almost niceness of $\equiv'$, because we have $x\restriction(\gamma+1) \equiv' t\restriction(\gamma+1)$.

Finally the tree isomorphism φ translates all we have done with the almost nice t.e.r. on T' to the decent t.e.r. on T. □

We use two theorems by Sierpiński and Choquet respectively to deduce the following lemmata.

1. A dense subset of a Polish space is comeagre if and only if it contains a Polish space as a subset (Choquet, cf. [Kec95, 8.17.ii]).
2. A criterion for Polish spaces: If Y is a seperable and metrizable space which is the image of a Polish space under a continuous and open surjection, then Y is Polish (Sierpiński, cf. [Kec95, 8.19]).

Lemma 2.2.9. *Given a normal, countable tree T of limit height equipped with a decent t.e.r. $\equiv$ consider the canonical tree epimorphism*

$$\pi : T \to T/{\equiv}, \quad \varphi(t) = t/{\equiv} .$$

a) Let the set $Y := \{y \in [T/{\equiv}] \mid (\exists x \in [T])\, y = \pi''x\}$ be equipped with the subspace topology inherited from $[T/{\equiv}]$. Then the induced map

$$\overline{\pi} : [T] \to Y,\ x \mapsto \pi''x$$

is an open mapping.

b) Let $\varphi : T/{\equiv} \to S$ be a tree isomorphism, $\rho := \varphi \circ \pi$ and $Z := \{\rho''x \mid x \in [T]\}$. Then

$$\overline{\rho} : [T] \to Z, b \mapsto \rho''b$$

is a continuous and open surjection, and Z is comeagre in $[S]$.

Proof. As in the proof of Proposition 2.2.8, since it does not infect the behaviour of the mappings, we can assume, that $\equiv$ is indeed almost nice. We use this to show that $\overline{\pi}$ is open. So let $t \in T$ be a successor node and $y \in [T]$ such that $y{\restriction}\gamma \equiv t$. We will find $x \in \hat{t}$ such that $x \equiv y$ thus showing that $\pi''\hat{t} = \widehat{\pi(t)} \cap Y$.

Fix an increasing sequence γ_n with limit $\operatorname{ht} T$ such that $\gamma_0 = \gamma = \operatorname{ht} t$ and all the higher γ_n are successor ordinals as well. Set $t_0 := t$ and find by almost niceness of $\equiv$ a node $t_{n+1} >_T t_n$ such that $t_{n+1} \equiv y{\restriction}\gamma_{n+1}$. Finally let $x = \{s \in T \mid (\exists n \in \omega) s <_T t_n\}$. So we have shown that the image of a basic open set is (the intersection of Y with] basic open set.

For the first part of b), note that $\overline{\rho} = \overline{\varphi} \circ \overline{\pi}$ and $\overline{\varphi}$ is a homeomorphism. So $\overline{\rho}$ is clearly continuous and open, while surjectivity is trivial by the definition of Z. Denseness of Z in $[S]$ follows from the surjectivity of ρ, so Z is Polish by Sierpiński's theorem and thus comeagre in $[S]$ by that of Choquet. □

It is crucial for our applications that the images of comeagre sets under mappings like $\overline{\rho}$ considered in the last lemma are comeagre again. This general fact is provided by the following proposition.

Proposition 2.2.10. *Let X and Y be Polish spaces and $h : X \to Y$ a continuous mapping, such that $h''X$ is comeagre in Y and the right-hand side restriction of h to its image, i.e. $h : X \to h''X$ is an open mapping.*

a) If $D \subset Y$ is meagre, then $h^{-1}{}''D$ is meagre as well.

b) If $M \subset X$ is comeagre, then $h''M$ is comeagre as well.

Proof. We start by showing that dense sets have dense images. This is clear, since $h''X$ is dense in Y, and the image of a dense subset of X is dense in $h''X$.

Next we show that nowhere dense subsets have nowhere dense pre-images. Let $D \subset Y$ be nowhere dense and $U \subseteq X$ open. Then $h''U$ is open in $h''X$ by our hypothesis on h, and, of course, $D \cap h''U$ is nowhere dense along with D. If $h^{-1}{}''D$ was dense in U, then by the argument above $h''(U \cap h^{-1}{}''D)$ would be dense in $h''U$. But on the other hand, $h''(U \cap h^{-1}{}''D)$ is a subset of $D \cap h''U$, which is nowhere dense.

Since the operations of taking unions and taking pre-images commute, we directly get statement a) of the proposition.

To prove part b) we assume without loss of generality, that M is a dense G_δ-subset of X. Then clearly $h''M$ is an analytic subset of Y. So, by a theorem of Lusin and Sierpinski (cf. [Jec03, Theorem 11.18.b] or [Kec95, Theorem 21.6]), $h''M$ has the Baire property, i.e. there is an open subset U of Y, such that the symmetric difference of $h''X$ and U is meagre. In particular the sets $h''X \setminus U$ and $U \setminus h''X$ are meagre. If we can show, that U is dense in Y, then the proof is finished.

So assume to the contrary that there is an open subset V of Y which is disjoint from U. Then $V \cup h''X$ is meagre and so is $M \cap h^{-1''}V = h^{-1''}(V \cap h''M)$ by part a). But since M is comeagre and $h^{-1''}V$ is open, their intersection cannot be meagre — contradiction! □

In order to extend isomorphisms between our Souslin algebra and its subalgebras on the level of Souslinisations during the constructions to come, we will use the following First Reduction Lemma, while Lemma 2.2.9 will etsablish that the hypotheses hold.

Lemma 2.2.11 (First Reduction Lemma)**.** *Consider the perfect Polish space X and and a countable set H such that every $h \in H$*

i) is a continuous function $h : X \to X$,

ii) has a comeagre image $h''X$ in X,

iii) becomes an open mapping when restricted to its range $h''X$ on the right hand side.

Let furthermore M be a comeagre subset of X. Then there is a comeagre subset $N = N(M, H)$ of M, such that for all $h \in H$ we have either $h''N = N$ or N is not suitable for h.

Note, that for every $h \in H$ and $y \in h''X$ the preimage $h^{-1}(y)$ is a closed and nowhere dense subset of X by the hypotheses. It can be discrete (and thus at most countable) or contain a perfect subset. For $x \in N$ the last property of N implies that the discrete part of $A = h^{-1}(h(x))$ is contained in N and that N intersects the perfect part of A in a comeagre subset — comeagre in the relative topology of the perfect part of A which is a perfect Polish space. So we can simply say that N intersects A in a comeagre subset of A.

Proof. By Proposition 2.2.10 we know that for every comeagre subset M of X and every $h \in H$ the set $h''M$ is comeagre in X.

In the other direction, i.e. for $h^{-1''}M$ for comeagre $M \subset X$, we can assume that M is the intersection of the open, dense sets U_n for $n \in \omega$. Then, as we are taking pre-images, $h^{-1''}M$ is the intersection of the open sets $h^{-1''}U_n$. We have shown that $h^{-1''}M$ is comeagre, when we show that these sets are dense in X as well. Pick any open subset $V \subset X$. Then $h''V$ is open in $h''X$ by the third condition on the members of H. But $U_n \cap h''X$ is open and dense in $h''X$. So the sets $U_n \cap h''V$ and $h^{-1''}U_n \cap V$ are not empty.

Now choose an enumeration $(h_n \mid n \in \omega)$ of H such that every $h \in H$ appears cofinally on the sequence. We will define our set N as the intersection over a countable family $(M_n^k \mid k, n \in \omega)$ of comeagre subsets of M. We start with $M_0^0 := M$, and given the comeagre set M_n^k the next set M_{n+1}^k is chosen as follows. If there is a comeagre subset $M' \subseteq M_n^k$ which is suitable for h_n, then set $M_{n+1}^k := M'$. If not, then set $M_{n+1}^k := M_n^k$. If M_n^k has been defined for all $n \in \omega$, we let

$$M_\omega^k := \bigcap_{n\in\omega} M_n^k$$

and

$$H_k := \{h \in H \mid M_{n+1}^k \text{ is suitable for } h \text{ for all } n \text{ with } h_n = h\}.$$

Note, that M_ω^k is suitable for all $h \in H_k$, for if $x \in M_\omega^k$ $A = h^{-1}(h(x))$ then

$$M_\omega^k \cap A := \bigcap_{h_n = h} M_{n+1}^k \cap A,$$

which is a countable intersection of comeagre subsets of A and therefore comeagre. Let finally

$$M_0^{k+1} := M_\omega^k \cap \bigcap_{h\in H_k} (h''M_\omega^k \cap h^{-1''}M_\omega^k)$$

and

$$N := N(M, H) := \bigcap_{k,n\in\omega} M_n^k.$$

Since H is countable and by the arguments at the start of the proof, all the sets M_n^k, and N as well, are comeagre.

Let $h \in \bigcap_{k\in\omega} H_k$. Then we have always found a h-suitable comeagre subset M_{n+1}^k of M_n^k if $h = h_n$. We show that h is surjective on N. Let $y \in N$, so $y \in M_0^{k+1}$ for all $k \in \omega$. This means that $h^{-1}(y) \cap M_{n+1}^k$ is not empty and therefore comeagre in $h^{-1}(y)$ because M_{n+1}^k is suitable for h. But then $N \cap h^{-1}(y)$ is a countable intersection of sets comeagre in $h^{-1}(y)$ and, as a consequence, comeagre (non-empty!) itself. It is trivial that

$$h''N = h'' \bigcap_{k\in\omega} M_\omega^k \subset \bigcap_{k\in\omega} M_0^{k+1} = N.$$

Now let $h \in H$ which is not in H_k for some $k \in \omega$, i.e., there was a number $n \in \omega$ with $h = h_n$ and there was no comeagre subset of M_n^k suitable for h. Since N is comeagre, it cannot be suitable for h. So there must be an element $x \in N$ with $N \cap h^{-1}(h(x))$ not dense in $h^{-1}(h(x))$. $\square$

Remark 2.2.12. The careful reader might have already guessed, that in our applications of the First Reduction Lemma,

- the space X will be of the form $[T\restriction\alpha]$ for some countable limit α and
- the mappings $h \in H$ will be of the form $h = \overline{\rho} = \overline{\varphi} \circ \overline{\pi_{\equiv}}$ as in part b) of Lemma 2.2.9 with $S = T$. (We also allow that the t.e.r. $\equiv$ is just the identity, so φ can also be an automorphism of T.)

If N is suitable for h, then we can choose the next level of our tree in a way that guarantees that $\equiv$ extends to a decent t.e.r. on the new level and we can also extend φ as a tree isomorphism. If on the other hand N is not suitable for h, then we can choose T_α so that it contains a witness for this. But then $\equiv$ is no longer decent, and we do not have to reconsider it in later stages of the construction.

At this point we have collected enough of the representation theory for subalgebras of Souslin algebras to perform the first construction of a chain homogeneous algebra given in Section 2.3.3.

In the construction of a big Souslin algebra in Section 2.4 we will carry out an iteration of length ω_2. In the remainder of this section we prepare the treatment of certain subalgebras of the initial algebra of that iteration.

Recall from Section 1.3 that we call a t.e.r. $\equiv$ ∞-*nice* if it is nice and for all $\alpha < \beta < \mathrm{ht}(T)$ and all $s \in T_\beta$ the projections $t \mapsto t\restriction\alpha$ are ∞-to-one when restricted to the $\equiv$-class of s, i.e.,

for all $r \in (s\restriction\alpha)/\!\equiv$ the set $\{t \in s/\!\equiv \mid t\restriction\alpha = r\}$ is infinite.

A subalgebra $\mathbb{A}$ of $\mathbb{B}$ is called ∞-*nice* if $\mathbb{B}$ has a Souslinisation T which carries an ∞-nice t.e.r. $\equiv$ such that $\{\sum s/\equiv \mid s \in T\}$ is dense in $\mathbb{A}$.

The equivalence classes of ∞-nice t.e.r.s have an especially nice structure:

Proposition 2.2.13. *Let T be a normal, $\aleph_0$-branching tree of countable limit height carrying an ∞-nice t.e.r. $\equiv$. Then for every cofinal branch b of T the set*

$$\bigcup b/\!\equiv := \{s \in T \mid s \equiv (b\restriction\mathrm{ht}(s)\}$$

is an $\aleph_0$-branching, normal tree.

Proof. Unique limits are inherited from T, infinte branching follows from the ∞-condition, successors in every higher level follow from niceness. □

Recall Kurepa's Isomorphism Lemma 2.1.6 which states that $\aleph_0$-branching normal trees of the same countable height are isomorphic. We once again refine this result in the following proposition, which eventually enables us to produce the additional isomorphisms we need in the construction in Section 2.4.

Proposition 2.2.14. *Let T be a normal, $\aleph_0$-branching tree of limit height $\alpha < \omega_1$ and let $\equiv$ and $\sim$ be ∞-nice t.e.r.'s on T. Let γ be equal to α or else be a successor ordinal below α. Let φ' be an ismorphism between $(T\restriction\gamma)/\equiv$ and $(T\restriction\gamma)/\sim$. Then there is an automorphism φ of T that carries $\equiv$ to $\sim$, s.t. the induced map on $T/\equiv$ is an isomorphism onto $T/\sim$ that extends φ'.*

Proof. First use Lemma 2.1.6 to extend φ' to an isomorphism between $T/\equiv$ and $T/\sim$ in the case that $\gamma < \alpha$. So we can assume $\alpha = \gamma$.

We now give a back-and-forth-argument which lifts the isomorphism φ' to an automorphism φ of T. Enumerate T in order type ω by $s_0, s_1, \ldots$ For the induction step in the forth-direction let n be the minimal index i for which $\varphi(s_i)$ has not yet been defined. Our choice for $\varphi(s_i)$ has to respect the tree order and the equivalence relations. So let s be the predecessor of maximal height whose image under φ is already determined by the choices met earlier in the construction, where $s = s_i$ is allowed. In the case that $s <_T s_i$, we want to pick a node $t \in \varphi'(s_i/\equiv)$ above $\varphi(s)$ which exists by the niceness of $\sim$. We furthermore require that t has not yet been assigned as some $\varphi(s_j)$. This choice is possible due to the ∞-part in the ∞-niceness of $\sim$.

For the back-step we replace in the above argument all φ and φ' by φ^{-1} and φ'^{-1}. □

We now come to state the key lemma for the construction of a Souslin algebra with ∞-nice subalgebras. Note that, by an argument as in the proof of part a) of Proposition 2.2.8, given an ∞-nice t.e.r. $\equiv$ on $T\restriction\alpha$ and any $x \in [T\restriction\alpha]$, then the $\equiv$-class of x is a nowhere dense, perfect subset of the Polish space $[T\restriction\alpha]$ and thus a perfect Polish space itself.

Lemma 2.2.15 (Second Reduction Lemma)**.** *Let T be a countable, normal and $\aleph_0$-branching tree with an ∞-nice t.e.r. $\equiv$. The induced equivalence relation on $[T]$ will also be called $\equiv$. Let $M \subseteq [T]$ be comeagre. Then there is a comeagre subset $M' \subseteq M$ which is suitable for $\equiv$.*

Proof. Without loss of generality we assume that M is a G_δ set. So let $M = \bigcap U_n$, where all the U_n are dense open and define for $n \in \omega$

$$X_n = \bigcup \{x/\equiv \mid x/\equiv \cap U_n \text{ is not dense in } x/\equiv\}.$$

Note that for $x \notin X_n$ the set $x/{\equiv} \cap U_n$ is then open and dense in $x/{\equiv}$. We show that X_n is meagre for every $n < \omega$. Let then $M' := M \setminus \bigcup_{n\in\omega} X_n$. For $x \in M'$ we then have $(x/{\equiv}) \cap M' = (x/{\equiv}) \cap M$, which is comeagre in $x/{\equiv}$ by the choice of M'.

Fix n and let for $s \in T$

$$Y_s := \bigcup \{x/{\equiv} \mid s \equiv x\restriction(\operatorname{ht} s) \text{ and } x/{\equiv} \cap U_n \cap \hat{s} = \varnothing\},$$

to get $X_n = \bigcup_{s\in T} Y_s$. The basic open set $\hat{s}$ is the witness for the non-denseness of $x/{\equiv} \cap U_n$ in $x/{\equiv}$.

If we fix $s \in T_\gamma$ then of course $Y_s \subseteq \bigcup_{r\in T_\gamma} \hat{r} = [T]$. Now for $r \not\equiv s$ we clearly have $Y_s \cap \hat{r} = \varnothing$. On the other hand the set $Y_s \cap \hat{s} \subset \hat{s} \setminus U_n$ is nowhere dense. For $r \equiv s$ we can find by Proposition 2.2.14 a tree isomorphism $\varphi : T(s) \to T(r)$ that respects $\equiv$. But the homeomorphism $\overline{\varphi} : \hat{s} \to \hat{r}$ induced by that tree isomorphism maps $Y_s \cap \hat{s}$ onto $Y_s \cap \hat{r}$. This shows that the latter set is nowhere dense as well. But then we have established Y_s as well as X_n as a countable union of nowhere dense sets. □

In order to get, once and for all, rid of this kind of argument, we combine the First with the Second Reduction Lemma and get the result which is appropriate for our use in the construction in Section 2.4.

Corollary 2.2.16. *Let T be a countable, normal and $\aleph_0$-branching tree of limit height α. Let H be a countable set of triples $h = (c_h, \equiv_h, \varphi_h)$ where c_h is a club subset of α, $\equiv_h$ is a decent t.e.r. on $T\restriction c_h$ and $\varphi_h : (T\restriction c_h)/{\equiv_h} \to T\restriction c_h$ is an isomorphism. Let furthermore I be a countable set of pairs $i = (c_i, \equiv_i)$ such that c_i is club in α and $\equiv_i$ is an ∞-nice t.e.r. on $T \restriction c_i$. If M is a comeagre subset of $[T]$ then there is a comeagre subset N of M, such that*

1. *for all $h \in H$, M is either suitable for h (resp. $\equiv_h$) and $h''N = N$, or N is not suitable for h;*

2. *N is suitable for all $\equiv_i$ where $i \in I$.*

(We hope that the reader may forgive us the use of the name of a tuple for indexing its components.)

Proof. As in the proof of the First Reduction Lemma 2.2.11, but we enumerate the memebers $H \cup I$ in an $\aleph_0$-to-1 fashion, instead of those of H only. The Second Reduction Lemma states that we always find suitable subsets for the ∞-nice t.e.r.s $\equiv_i$ for $i \in I$. □

Our last technical lemma concerns nested ∞-nice t.e.r.s. With its aid we will see that constructing an increasing chain of subalgebras, the smaller ones ∞-nice in the larger ones, is not as hard as it might look like at first sight.

If $\equiv_1$ refines $\equiv_0$ as an equivalence relation on some set M, we define the equivalence relation $\equiv_{0/1}$ on $M/{\equiv_1}$ by

$$(x/{\equiv_1}) \equiv_{0/1} (y/{\equiv_1}) :\Longleftrightarrow \; x \equiv_0 y$$

for $x, y \in M$.

Lemma 2.2.17. *Assume we are given two ∞-nice t.e.r.'s $\equiv_0$ and $\equiv_1$ on T which is normal, $\aleph_0$-branching and of countable limit height, such that $\equiv_0$ is refined by $\equiv_1$ and the t.e.r. $\equiv_{0/1}$ on the quotient tree $T/{\equiv_1}$ induced by $\equiv_0$ is ∞-nice. Let furthermore M be a comeagre subset of $[T]$ that is suitable for both $\equiv_0$ and $\equiv_1$. Then $M/{\equiv_1}$ is suitable for $\equiv_{0/1}$.*

Proof. Assume the statement is false, i.e., that there is $s \in T$, such that

$$M/{\equiv_1} \cap (x/{\equiv_1})/{\equiv_{0/1}} \cap \widehat{s/{\equiv_1}} = \varnothing,$$

while $(x/{\equiv_1})/{\equiv_{0/1}} \cap \widehat{s/{\equiv_1}} \neq \varnothing$. Now choose

$$y \in \hat{s} \cap M \cap (x/{\equiv_0}) \supseteq \hat{s} \cap M \cap (x/{\equiv_1}) \neq \varnothing.$$

But then $(x/{\equiv_1}) \equiv_{0/1} (y/{\equiv_1})$, so $(y/{\equiv_1}) \in (M/{\equiv_1}) \cap (x/{\equiv_1})/{\equiv_{0/1}}$ which contradicts our choice of s. $\square$

2.3 Chain-homogeneous Souslin algebras

We give three $\diamondsuit^+$-constructions of chain-homogeneous Souslin algebras, the first one in this section and the other two in the next section. The second one is a variation of the first one and will be used as the starting point for the iteration of the third construction that eventually yields a big, chain homogeneous Souslin algebra.

2.3.1 The strategy

We assume $\diamondsuit^+$ and aim at constructing a Souslin tree order $T = (\omega_1, <_T)$ such that

- all generating chains of $\mathbb{B} = \mathrm{RO}\, T$ are pairwise order isomorphic and
- all complete and atomless subalgebras of $\mathbb{B}$ are isomorphic to $\mathbb{B}$.

In this sketch we only care about the generating chains and ignore the task of rendering all atomless, complete subalgebras isomorphic to $\mathbb{B}$. As in many other Souslin tree constructions, e.g. as performed in Example 2.1.13, we will have

$$T\restriction\alpha = \omega\alpha \text{ and } T_\alpha = \omega(\alpha+1) \setminus \omega\alpha$$

for all infinite, countable ordinals α. We will concentrate our efforts on tree levels T_α with $\alpha = \omega \cdot \alpha$ and choose arbitrarily in all other levels.

After the construction we want to exhibit an isomorphism between two generating chains, say K_0 and K_1, and therefore consider two wipers W_0, W_1 given by Lemma 2.2.4 living on the same club $C \subset C_0$. We can code this pair of wipers by a set $X \subset \omega_1$ in such a way that $X \cap \alpha$ is a code for the pair of wipers restricted to $C \cap \alpha$ for all relevant α. These sets $X \cap \alpha$ will appear in our $\diamondsuit^+$-sequence for α in a club D of ω_1. Our construction will yield an automorphism of $T\restriction(C \cap D)$ which carries W_0 to W_1.

We now take a look at a relevant construction stage $\alpha \in C \cap D$. Whenever $C \cap D \cap \alpha$ is bounded below α we do not know how W_0 and W_1 respectively are extended on T_α, so we only consider α in which $C \cap D \cap \alpha$ is cofinal. By the induction hypotheses we have already constructed $\phi_0 \in \mathrm{Aut}(T\restriction(C \cap D \cap \alpha_0))$ carrying W_0 to W_1 for some $\alpha_0 \leq \alpha$ such that the order type of $C \cap D \cap \alpha \setminus \alpha_0$ is 0 or ω. In the latter case we inductively extend ϕ_0 up to α which works by Proposition 2.2.6, say ϕ.

Then we have to choose a countable dense set of cofinal branches which is also closed under the action of the tree automorphism ϕ of $T\restriction(C \cap D \cap \alpha)$ associated to our pair of wipers. This is where the topological perspective on the cofinal branches of the countable tree is most effective: the tree automorphism ϕ induces a homeomorphism $\overline{\phi}$ on the Polish space of cofinal branches. If we are given any comeagre set of branches we easily find a comeagre subset which is closed under the action of this homeomorphism $\overline{\phi}$ from which we choose our countable, dense set. Finally we extend this set to its closure under the application of the homeomorphism. This has to be achieved simultaneously for all pairs of wipers proposed by the $\diamondsuit^+$-sequence on stage α.

2.3.2 The $\diamondsuit^+$-machinery

In the Souslin tree constructions to come we want to build additional objects (mappings, tree isomorphisms) on club sets of levels of the tree T to be constructed, that relate given objects (e.g. pairs of wipers on $T\upharpoonright C$) to each other. During the relevant construction steps, initial segments of the given objects are proposed by a $\diamondsuit^+$-sequence and in order to extend the additional object we need some pointer to indicate the ordinal stage up to which the recursive construction of the additional object has reached so far. The following definition is kept somewhat general to fit in also for later uses.

Definition 2.3.1. Fix a $\diamondsuit^+$-sequence $(S_\alpha)_{\alpha<\omega_1}$.

a) Let $C_0 := \{\alpha < \omega_1 \mid \omega\alpha = \alpha\}$ be the set of countable fixed points of the left-multiplication with ω.

b) Let $\alpha \in C_0$. For $x \in \mathcal{P}(\alpha)$ set

$$c(x) = \{\gamma \in C_0 \cap \alpha \mid x \cap \omega(\gamma+1) \setminus \omega\gamma \neq \varnothing\}.$$

c) The *set of relevant guesses for stage* α, G_α is the set of pairs $(x,d) \in S_\alpha \times S_\alpha$ such that $c(x)$, d and $c(x)\cap d$ are club in α and for $\gamma \in d$ the sets $x\cap\gamma$ and $d\cap\gamma$ are in S_γ.

d) For $(x,d) \in G_\alpha$ let

$$e_{x,d} := \{\gamma \in c(x)\cap d \mid \bigcup(\gamma\cap c(x)\cap d) = \gamma\}$$

be the Cantor-Bendixson derivative of $c(x)\cap d$, i.e., the set of its limit points.

e) Let $\varepsilon_{x,d} := \bigcup e_{x,d}$. (Note that $\varepsilon_{x,d} = 0$ if $\mathrm{ot}(c(x)\cap d) = \omega$.)

Now if for example $(x,d) \in G_\alpha$ and x codes a pair of wipers on $T\upharpoonright c(x)$, we know that up to stage $\varepsilon_{x,d}$ our recursive construction of the additional object — here: an isomorphism between the wipers given by x — has been invoked and therefore up to this stage this isomorphism has yet been constructed.

2.3.3 Homogeneity for all maximal chains

In this section we give the first full solution to our problem.

Theorem 2.3.2. *Assuming $\diamondsuit^+$, there is a small Souslin algebra $\mathbb{B}$, such that for each pair $K, K' \in \operatorname{mc}\mathbb{B}$ there is an order isomorphism between K and K'.*

Proof. We inductively construct a tree-order $<_T$ level by level on the set ω_1, such that the resulting tree T will be a normal and $\aleph_0$-branching Souslin tree. The Souslin algebra to be constructed will be the regular open algebra of $T = (\omega_1, <_T)$.

We fix a $\diamondsuit$-sequence $(R_\nu)_{\nu<\omega_1}$, a $\diamondsuit^+$-sequence $(S_\nu)_{\nu<\omega_1}$ and a bijection $g : \omega_1 \to (2 \times \omega_1 \times \omega_1)$ with $g''\lambda = 2 \times \lambda \times \lambda$ for all limit ordinals λ. Let for $i \in 2$ be g_i the concatenation of g and the projection onto the fibre over i. We will use g for coding wipers and t.e.r.s as sets of ordinals.

Let 0 be the root of T and in the successor step fix $\aleph_0$ distinct direct successors for each maximal node in such a way that for every $\alpha < \omega_1 \setminus \{0\}$ the level T_α consists of the next ω many ordinals not yet used in the construction. So we have $T_1 = \omega \setminus \{0\}$, $T_n = \omega n \setminus \omega(n-1)$ for all natural numbers $n \geq 2$ and finally $T \restriction \alpha = \omega\alpha$ for all infinite, countable ordinals α.

Now let $\alpha < \omega_1$ be a limit ordinal. By the inductive assumption we have so far constructed a normal tree order $(T \restriction \alpha, <_T)$ on the supporting set $\omega\alpha$.

We consider the space $[T\restriction\alpha]$ of all cofinal branches of $T\restriction\alpha$ with the topology generated by the clopen sets $\hat{t} := \{z \in [T\restriction\alpha] : t \in z\}$ for $t \in T\restriction\alpha$. By Proposition 2.1.7 we know that this topology is Polish and perfect.

If $\alpha < \omega\alpha$ we simply choose a countable dense subset Q_α of $[T\restriction\alpha]$ and embed Q_α onto $\omega(\alpha+1) \setminus \omega\alpha$, i.e., we choose a bijection between Q_α and $\omega(\alpha+1) \setminus \omega\alpha$ and extend $<_T$ on $\omega(\alpha+1)$ in the obvious way.

If $\alpha = \omega\alpha$, we want to choose a countable dense subset Q_α of $[T\restriction\alpha]$, too, but this time our set also has to seal certain maximal antichains of $T\restriction\alpha$ and extend enough tree isomorphisms. In order to state the inductive assumption and the inductive claim we need some notation. Recall the definitions from the last section concerning the use of the $\diamondsuit^+$-sequence (G_α, $c(x)$, $\varepsilon_{x,d}$, etc.).

Consider the subset E_α of $S_\alpha \times S_\alpha$ of all pairs $(x, d) \in G_\alpha$ such that $g_0''x$ and $g_1''x$ are wipers $W_0 = \langle \prec_\gamma : \gamma \in c(x) \rangle$ and $W_1 = \langle <_\gamma : \gamma \in c(x) \rangle$ on $T\restriction c(x)$ respectively. (It would have been more correct to write $g_0{}''x = \bigcup W_0$ and $g_1{}''x = \bigcup W_1$.) So $(x, d) \in E_\alpha$ if x codes a pair of wipers on $T\restriction c(x)$ and is guessed correctly by the $\diamondsuit^+$-sequence along with a club d on the members of d itself. From now on, if $(x, d) \in G_\alpha$ is fixed, we write c for $c(x)$.

The guesses of the $\diamondsuit^+$-sequence for t.e.r.s are collected in the set F_α:

$$F_\alpha := \{(x, d) \in G_\alpha \mid x \text{ codes a decent t.e.r. on } c(x)\}.$$

Here we code e.g. with respect to the map g_0. For $(x,d) \in F_\alpha$ let $\equiv_x$ denote the decent t.e.r. which is coded by x. It induces the $\aleph_0$-branching, normal tree $(T\restriction c(x))/{\equiv_x}$.

Keep in mind that $c \cap d$ is unbounded in α if $(x,d) \in E_\alpha$ or F_α by the definition of the relevant stages set G_α.

The inductive hypothesis (**IH**) is, that $T\restriction\alpha$ is a countable and $\aleph_0$-branching tree of height α and

1. for every pair $(x,d) \in E_\alpha$ there is a $\subset$-chain $\langle \varphi_{x\cap\gamma,d\cap\gamma} : \gamma \in e_{x,d}\rangle$ and each $\varphi_{x\cap\gamma,d\cap\gamma}$ is a tree automorphism of $T\restriction(c \cap d \cap \gamma + 1)$ that was fixed in induction step $\gamma \in e_{x,d}$ and which carries $W^0_{x,d}\restriction c \cap d \cap \gamma + 1 = \langle \prec_\delta \colon \delta \in c \cap d \cap \gamma + 1\rangle$ to $W^1_{x,d}\restriction c \cap d \cap \gamma + 1 = \langle <_\delta \colon \delta \in c \cap d \cap \gamma + 1\rangle$. In short: $\varphi_{x\cap\gamma,d\cap\gamma}$ is an isomorphism between $W^0_{x,d}\restriction c \cap d \cap \gamma + 1$ and $W^1_{x,d}\restriction c \cap d \cap \gamma + 1$ for all $\gamma \in e_{x,d}$ and

2. for every pair $(x,d) \in F_\alpha$ there is a $\subset$-chain $\langle \psi_{x\cap\gamma,d\cap\gamma} : \gamma \in e_{x,d}\rangle$ of tree isomorphisms $\psi_{x\cap\gamma,d\cap\gamma} : T\restriction(c \cap d \cap \gamma + 1)/{\equiv_x} \to T\restriction(c \cap d \cap \gamma + 1)$ each of them fixed in induction step $\gamma \in e_{x,d}$.

(Recall that $c = c(x)$ and $e_{x,d}$ is the set of limit points of $c \cap d$.) For those (x,d) in E_α or in F_α but with $\varepsilon_{x,d} < \alpha$ we need to choose extensions for the maps granted by the IH as follows.

Fix $(x,d) \in E_\alpha$. If $\varepsilon_{x,d} = \alpha$ let

$$\varphi_{x,d} := \bigcup_{\gamma \in e_{x,d}} \varphi_{x\cap\gamma,d\cap\gamma}.$$

Otherwise extend $\varphi_{x\cap\varepsilon_{x,d},d\cap\varepsilon_{x,d}}$ by Proposition 2.2.6 to some isomorphism $\varphi_{x,d}$ between $W^0_{x,d}\restriction c \cap d$ and $W^1_{x,d}\restriction c \cap d$.

Let $(x,d) \in F_\alpha$ and set $\psi_{x,d} := \bigcup_\gamma \psi_{x\cap\gamma,d\cap\gamma}$ if $\varepsilon_{x,d} = \alpha$. Otherwise extend the union of the chain by Kurepa's Lemma 2.1.6 to some isomorphism

$$\psi_{x,d} : (T\restriction c \cap d)/{\equiv_x} \to T\restriction c \cap d.$$

Now Lemma 2.2.9 comes into play. Let

$$\pi_{x,d} : T\restriction c \cap d \to (T\restriction c \cap d)/\equiv_x, \quad s \mapsto s/\equiv_x$$

be the canonical mapping associated to $\equiv_x$ and define

$$\rho_{x,d} := \psi_{x,d} \circ \pi_{x,d} : T\restriction c \cap d \to T\restriction c \cap d.$$

Since $\equiv_x$ is decent and $\psi_{x,d}$ is an isomorphism, $\rho_{x,d}$ has the same properties as the map φ in Lemma 2.2.9. The induced continuous map

$$\overline{\rho}_{x,d} : [T\restriction\alpha] \to [T\restriction\alpha], \quad b \mapsto \{s \mid (\exists t \in b \cap (T\restriction c \cap d))s <_T \rho_{x,d}(t)\}$$

has a comeagre image in $[T\restriction\alpha]$ and is an open mapping when the range is restricted to $\overline{\rho}_{x,d}''[T\restriction\alpha]$.

The $\diamondsuit$-sequence $(R_\nu)_{\nu<\omega_1}$ proposes candidates for maximal antichains in the usual way. If R_α is a maximal antichain of $T\restriction\alpha$ then we have to ensure that each member of T_α is a $<_T$-successor of some element of R_α. That means Q_α has to be a subset of

$$M_\alpha = \{x \in [T\restriction\alpha] : \exists\gamma < \alpha\, x\restriction\gamma \in R_\alpha\}$$

which is itself an open dense subset of $[T\restriction\alpha]$, because R_α is a maximal antichain. If R_α is not a maximal antichain in $T\restriction\alpha$ we simply set $M_\alpha = [T\restriction\alpha]$.

The inductive claim (**IC**) is that there is a choice for Q_α, i.e., a countable and dense subset of M_α, scuh that

1. every tree automorphisms $\varphi_{x,d}$ of $T\restriction c \cap d$ for $(x, d) \in E_\alpha$ and
2. for every $(x, d) \in F_\alpha$, either $\equiv_x$ is no longer decent when extended to $T\restriction(c\cup\{\alpha\})$, or the tree isomorphism $\psi_{x,d} : (T\restriction c \cap d)/{\equiv_x} \to T\restriction c \cap d$ extends to the respective trees with the new top level T_α corresponding to Q_α (or $T_\alpha/{\equiv_x}$ respectively) added on.

We apply the First Reduction Lemma 2.2.11 to the sets $M = M_\alpha$ and

$$H = \{\overline{\varphi}_{x,d} \mid (x, d) \in E_\alpha\} \cup \{\overline{\rho}_{x,d} \mid (x, d) \in F_\alpha\}.$$

Since the E_α-part of H consists of homeomorphisms and the F_α-part is subject to Lemma 2.2.9, the hypotheses of the First Reduction Lemma are satisfied. The result is a comeagre subset N_α of M_α, such that

- for all $(x, d) \in E_\alpha$ we have $\overline{\varphi}''_{x,d} N_\alpha = N_\alpha$ and
- for each $(x, d) \in F_\alpha$, if N_α is suitable for $\overline{\rho}_{x,d}$ then $\overline{\rho}''_{x,d} N_\alpha = N_\alpha$.

(Recall that we say that a comeagre subset M of $[T\restriction\alpha]$ is suitable for $\overline{\rho}_{x,d}$ if for every branch $b \in [T]$ the intersection of the class $b/{\equiv_x}$ with M is either empty or dense in $b/{\equiv_x}$.)

Let F^*_α be the set of those $(x, d) \in F_\alpha$, such that N_α is suitable for $\equiv_x$. So we can choose for every $(x, d) \in F^*_\alpha$ a right inverse $\sigma_{x,d} : N_\alpha \to N_\alpha$ of $\overline{\rho}_{x,d}\restriction N_\alpha$. In general, these sections $\sigma_{x,d}$ will not be continuous, but this is no longer important.

To rule out the bad t.e.r.s, pick for each $(x, d) \in F_\alpha \setminus F^*_\alpha$ one witness $b_x \in N_\alpha$ for the fact that N_α is not suitable for $\equiv_x$, i.e., that $N_\alpha \cap b_x/{\equiv_x}$ is not dense in $b_x/{\equiv_x}$. As already noted in Remark 2.2.12, this choice impeaches every t.e.r. extending $\equiv_x$

from being a *decent* t.e.r., because it fails to satisfy the necessary denseness condition stated in Proposition 2.2.8.

Finally extend the set $\{b_x \mid (x,d) \in F_\alpha \setminus F^*_\alpha\}$ to a countable and dense subset Z_α of N_α. To obtain Q_α, form the hull of Z_α under the application of the mappings

$$\overline{\varphi}_{x,d} \text{ and } \overline{\varphi}^{-1}_{x,d} \text{ for } (x,d) \in E_\alpha$$

and

$$\overline{\rho}_{x,d} \text{ and } \sigma_{x,d} \text{ for } (x,d) \in F^*_\alpha.$$

Choose a bijection $j : Q_\alpha \to \omega(\alpha+1) \setminus \omega\alpha$ and extend $<_T$ in the obvious way.

To finish the induction step, we show that our choice of T_α admits extensions of the tree isomorphisms $\psi_{x,d}$ for all $(x,d) \in F^*_\alpha$. Fix $(x,d) \in F^*_\alpha$ and set $c' = (c(x) \cap d) \cup \{\alpha\}$. First of all extend $\equiv_x$ to T_α by letting $s \equiv_x t$ if and only if $s{\restriction}\gamma \equiv_x t{\restriction}\gamma$ for all $\gamma \in c(x)$. (This is the only t.e.r. extending $\equiv_x$ on T_α because of the normality requirement in the definition of t.e.r.)

We can identify T_α with Q_α via the bijection j. This in mind, we show that the unique extension ψ of $\psi_{x,d}$ to $(T{\restriction}c')/{\equiv_x}$ is an isomorphism onto $T{\restriction}c'$: for $s \in T_\alpha$ define

$$\psi'(s/{\equiv_x}) := \text{the unique } t \in T_\alpha \text{ with } \psi_{x,d}(s{\restriction}\gamma/{\equiv}) = t{\restriction}\gamma \text{ for all } \gamma \in c' \setminus \{\alpha\}$$

and let $\psi = \psi_{x,d} \cup \psi'$. We check the soundness of this definition. For $s \in T_\alpha$ there is $\psi(s/{\equiv_x}) = j \circ \overline{\rho}_{x,d} \circ j^{-1}(s)$. The normality of $T/{\equiv}$ guarantees that the definition of $\psi(s/{\equiv_x})$ is independent from the choice of the representative s. So it is clear that ψ is indeed a tree homomorphism.

Now, by the definition of $\equiv_x$ on T_α, if $s \not\equiv t$ for $s, t \in T_\alpha$, then there is some $\gamma \in c' \setminus \{\alpha\}$ with $s{\restriction}\gamma \not\equiv_x t{\restriction}\gamma$ and so $\psi(s/{\equiv_x}) \neq \psi(t/{\equiv_x})$. On the other hand, for every $s \in T_\alpha$ we have

$$\psi(j \circ \sigma_{x,d} \circ j^{-1}(s)/{\equiv_x}) = s.$$

So ψ is bijective also on the top level of $(T{\restriction}c')/{\equiv}$ and therefore a tree isomorphism.

For the reference in later induction steps we denote this tree isomorphism ψ by $\psi_{x,d}$. This completes the inductive construction.

It remains to show that the above construction yields a Souslin tree T whose regular open algebra $\mathbb{B}$ is chain homogeneous. We omit the standard argument proving that T is Souslin.

If we are given two generating chains K, K' of $\mathbb{B}$ let $X \subset \omega_1$ be a code with respect to g for a pair of wipers on the club $C \subset \{\alpha : \omega\alpha = \alpha\}$ inducing K and K'. Let D be a club set in ω_1 associated to X by $\diamondsuit^+$. Then for each $\alpha \in E = \{\gamma \in$

$C \cap D \mid \gamma = \bigcup(\gamma \cap C \cap D)\}$ the construction of the tree gives us a tree automorphism $\varphi_{X\cap\alpha, D\cap\alpha}$ of $T\restriction(C \cap D \cap \alpha)$, and the union of that increasing chain,

$$\varphi = \bigcup_{\alpha \in E} \varphi_{X\cap\alpha, D\cap\alpha},$$

extends to an automorphism φ of $\mathbb{B}$ that carries K to K'.

Now let $\mathbb{A}$ be a complete and atomless subalgebra of $\mathbb{B}$ represented by the decent t.e.r. $\equiv$ on $T\restriction C$. Let D be a club, such that the $\diamondsuit^+$-sequence guesses $D \cap \alpha$ and $X \cap \alpha$ for a code X for $\equiv$ for all $\alpha \in D$. Then the construction yields a tree isomorphism $\psi : (T\restriction C \cap D)/\equiv \to T\restriction C \cap D$ and therefore a Boolean isomorphism between $\mathbb{B}$ and $\mathbb{A}$.

So let K be a maximal chain of $\mathbb{B}$, $\mathbb{A} = \langle K \rangle^{\mathrm{cm}}$ and $\psi : \mathbb{B} \cong \mathbb{A}$. Now K is isomorphic to the generating chain $\psi^{-1\prime\prime}K$ and is thus of that unique order type. $\square$

2.3.4 Some features of chain homogeneous Souslin algebras

Proposition 2.3.3. *A small and chain homogeneous Souslin algebra $\mathbb{B}$ is homogeneous in the following strong sense. For every pair $\mathbb{A}_0, \mathbb{A}_1$ of Souslin subalgebras and $x \in \mathbb{A}_0$ and $y \in \mathbb{A}_1$ where $0 <_{\mathbb{B}} x, y <_{\mathbb{B}} 1$, there are $2^{\aleph_1}$ distinct isomorphisms $\varphi : \mathbb{A}_0 \to \mathbb{A}_1$ with $\varphi(x) = y$.*

Proof. We first argue that $\mathbb{B}$ is weakly homogeneous. $\mathbb{B}$ is assumed to be small. So for every pair a, b of non-zero elements of $\mathbb{B}$ there are generating chains K, K' of $\mathbb{B}$ with $a \in K$ and $b \in K'$. Each isomorphism φ between K and K' satisfies $\varphi(a) \cdot b \neq 0$ and extends to an automorphism of $\mathbb{B}$.

By a theorem of Koppelberg and Solovay (cf. [ŠR89, Theorem 18.4.1]), every complete and weakly homogeneous Boolean algebra is a power of a homogeneous factor, which in our case is isomorphic to $\mathbb{B}$, because of the c.c.c. satisfied by $\mathbb{B}$.

By Lemma 2.2.5 there are $2^{\aleph_1}$ distinct generating chains. Chain homogeneity then implies that $\mathbb{B}$ has $2^{\aleph_1}$ automorphisms. It is then easy, granted that $\mathbb{A}_0, \mathbb{A}_1 \cong \mathbb{B}$, to construct a large family of isomorphisms as stated in the proposition. $\square$

Lemma 2.3.4. *a) If $\mathbb{A}$ is an atomless, complete subalgebra of the Souslin algebra $\mathbb{B}$, and $\mathbb{A}$ and $\mathbb{B}$ are isomorphic, then no atomless, complete subalgebra of $\mathbb{B}$ can be independent from $\mathbb{A}$.*

b) A chain homogeneous Souslin algebra has no independent pair of atomless and complete subalgebras.

Proof. To prove a), choose an isomorphism φ from $\mathbb{B}$ onto its complete subalgebra $\mathbb{A}$, and assume that $\mathbb{C}$ is an atomless, complete subalgebra of $\mathbb{B}$ and independent from $\mathbb{A}$. Then the image of $\mathbb{C}$ under φ is an isomorphic copy of $\mathbb{C}$ and independent from $\mathbb{C}$. So by choosing a Souslinization T of $\mathbb{C}$ we can by a standard argument construct an uncountable antichain in the subset $T \otimes \varphi''T$ of $\mathbb{B}$ contradicting the assumption that $\mathbb{B}$ is Souslin.

For the proof of b) note, that here we have not assumed $\mathbb{B}$ to be small. In the case of a small Souslin algebra, an application of part a) suffices to prove b). But also a big Souslin algebra $\mathbb{B}$ has only maximal chains, that completely generate subalgebras which are small Souslin algebras. So assume that $\mathbb{A}_0, \mathbb{A}_1$ form an independent pair of complete and atomless subalgebras of $\mathbb{B}$. Then there are maximal chains $K_0 \subset \mathbb{A}_0$ and $K_1 \subset \mathbb{A}_1$, which are isomorphic to each other. The isomorphism between the chains extends to an isomorphism between the two subalgebras that are completely generated by the chains:

$$\mathbb{C}_0 := \langle K_0 \rangle^{\mathrm{cm}} \subset \mathbb{A}_0 \text{ and } \mathbb{C}_1 := \langle K_1 \rangle^{\mathrm{cm}}.$$

But then again, we have an isomorphic pair of subalgebras $\mathbb{C}_0, \mathbb{C}_1$ of $\mathbb{B}$ that cannot be independent unless $\mathbb{B}$ fails to satisfy the countable chain condition. □

Concerning the hypothesis ($\diamondsuit^+$) met for our construction, we remark the following. In [AS93, Section 6] a model of $\mathsf{ZFC} + \neg\mathsf{SH}$ is constructed, in which there is no homogeneous Souslin tree. By Proposition 1.3.4 a (chain) homogeneous Souslin algebra always has a homogeneous Souslinization. So we have found a ZFC-model where Souslin's hypothesis fails, yet there are no chain homogeneous Souslin algebras.

It is open whether we could make do with less than $\diamondsuit^+$, e.g., if the assumption of $\diamondsuit$ is sufficient to guarantee the existence of a chain homogeneous Souslin algebra. However, there are clues that this is not the case.

In [DJ74, Section V.3], a Souslin tree with at least $\aleph_2$ automorphisms is constructed under $\diamondsuit^+$, and on p. 51 the authors remark: *It is doubtful whether this is provable form $\diamondsuit$.* It seems very likely, that the regular open algebra of that Souslin tree is also chain homogeneous as, for instance, it is not hard to see that it is homogeneous for generating chains.

This is reinforced by the fact that the small Souslin algebra constructed in Section 2.4 has stronger homogeneity properties of which we can prove, that they do not exist under the assumption of $\diamondsuit$ alone (cf. Corollary 2.4.6). And the methods used to implement these stronger homogeneity properties strongly resemble those used in the proof of Theorem 2.3.2.

So we join in the doubts of Devlin and Johnsbråten cited above and conjecture that there is a model of $\mathsf{ZFC} + \diamondsuit$ in which there are no chain homogeneous Souslin algebras.

2.4 A big and chain homogeneous Souslin algebra

In the present section we give a $\diamondsuit^+$-construction of a chain homogeneous Souslin algebra that has no dense subset of cardinality $\aleph_1$, i.e., it is big. This answers a question by Stevo Todorcevic asked on the occasion of a talk on chain homogeneous Souslin algebras the author gave at the Toposym 10 conference in Prague, 2006.

We start and construct in Section 2.4.1 a small, chain homogeneous Souslin algebra $\mathbb{B}_0$ in which a sequence of subalgebras of length ω_1 with certain favourable properties is already realized.

Then we construct the iteration sequence of length ω_2 while embedding each algebra of that iteration sequence in $\mathbb{B}_0$ onto one of the subalgebras built up in the construction of $\mathbb{B}_0$. The chain homogeneity implies that all the $\aleph_2$ algebras on the iteration sequence are isomorphic to $\mathbb{B}_0$. The final, big Souslin algebra will then be obtained as the direct limit (or simply the union) of this ω_2-sequence.

2.4.1 The footing

Before we state and prove the theorem of this section we introduce some notation.

Recall Proposition 2.2.14, which states that an isomorphism between two quotient trees $T/{\equiv_0}$ and $T/{\equiv_1}$ can be lifted to an automorphism of T once the t.e.r.s are ∞-nice. To get hold of isomorphisms between quotient trees we introduce the notion of an engaging relation. Consider two ∞-nice t.e.r.'s $\equiv_0$ and $\equiv_1$ on $T\restriction c$, with $c \subset \alpha$ club, and an isomorphism $\varphi : (T\restriction c)/{\equiv_0} \to (T\restriction c)/{\equiv_1}$ between the quotient trees. Then φ naturally induces a relation Φ on $T\restriction c$, that consists of the pairs $s, t \in T\restriction c$ with $\varphi(s/{\equiv_0}) = t/{\equiv_1}$. The properties of such a relation Φ are captured in the following definition.

Definition 2.4.1. We say that a relation Φ on a tree T satisfying points 1-4) below is *engaging*.

1. There is a set c_Φ such that for all $s, t \in T$ we have $s\Phi t$ only if $\mathrm{ht}(s) = \mathrm{ht}(t) \in c_\Phi$,
2. the left-induced relation $\Phi^0 := \{(s, s') \in (T\restriction c)^2 \mid (\exists t)s\Phi t \text{ and } s'\Phi t\}$ is an ∞-nice t.e.r. on $T\restriction c_\Phi$,
3. the right-induced relation $\Phi^1 := \{(t, t') \in (T\restriction c)^2 \mid (\exists s)s\Phi t \text{ and } s\Phi t'\} =\equiv_1$ also is an ∞-nice t.e.r. on c_Φ
4. Φ induces an isomorphism φ_Φ between $(T\restriction c)/\Phi^0$ and $(T\restriction c)/\Phi^1$ via $\varphi_\Phi(s/\Phi^0) = t/\Phi^1$ for any t with $s\Phi t$.

It is clear that the relation Φ considered above the last definition is indeed engaging with $c_\Phi = c$, and Φ^0 is $\equiv_0$ while Φ^1 is $\equiv_1$.

Next we define a certain sequence of club sets of ω_1. Recall the definition of C_0, the set of infinite fixed points of the left hand ordinal multiplication with ω in ω_1:

$$C_0 := \{\alpha < \omega_1 \mid \alpha \neq 0,\, \omega\alpha = \alpha\}.$$

Inductively define C_{i+1} for $i < \omega_1$ to be the Cantor-Bendixson-derivative of C_i, and for limit ordinals i let C_i be the intersection of the C_j defined so far:

$$C_{i+1} := \{\alpha \in C_i \mid \sup(C_i \cap \alpha) = \alpha\} \quad \text{and} \quad C_i = \bigcap_{j<i} C_j \text{ for limit } i.$$

We list some properties of the sequence $(C_i)_{i<\omega_1}$ used in the construction below.

(a) all the C_i are club in ω_1,

(b) the sequence is continuously decreasing and has an empty intersection, hence there is for every $\alpha \in C_0$ a unique $i = i(\alpha)$ with $\alpha \in C_i \setminus C_{i+1}$,

(c) every $\alpha \in C_0$ has a direct predecessor in $C_{i(\alpha)}$, call it α^-,

(d) for limit $i < \omega_1$, the minimum of C_i is the supremum of the minima of the C_j for $j < i$.

Theorem 2.4.2. *Assume $\diamondsuit^+$. There is a small Souslin algebra $\mathbb{B}$ with an increasing sequence $(\mathbb{A}_i \mid i < \omega_1)$ of subalgebras, such that the following hold.*

(i) $\mathbb{B}$ is chain homogeneous;

(ii) for all pairs $\mathbb{A}, \mathbb{A}'$ of ∞-nice subalgebras and every isomorphism $\varphi : \mathbb{A} \cong \mathbb{A}'$ there is a $\tilde{\varphi} \in \operatorname{Aut} \mathbb{B}$ with $\tilde{\varphi}\restriction\mathbb{A} = \varphi$;

(iii) the members $\mathbb{A}_i$ of the sequence are ∞-nice in $\mathbb{B}$;

(iv) for $i < j < \omega_1$ we have $\mathbb{A}_i$ is an ∞-nice subalgebra of $\mathbb{A}_j$;

(v) the sequence $(\mathbb{A}_i)_{i<\omega_1}$ is continuous in the sense that for a countable limit ordinal i the union $\bigcup_{j<i} \mathbb{A}_j$ completely generates $\mathbb{A}_i$;

(vi) if λ is a countable limit ordinal and $(j_\nu \mid \nu < \lambda)$ a normal sequence of countable ordinals with supremum i, then there is an isomorphism $\varphi : \mathbb{A}_\lambda \to \mathbb{A}_i$ with $\varphi''\mathbb{A}_\nu = \mathbb{A}_{j_\nu}$ for all $\nu < \lambda$.

(vii) *$\mathbb{B}$ is the direct limit of the sequence of the $\mathbb{A}_i$, i.e., $\bigcup_{i<\omega_1} \mathbb{A}_i = \mathbb{B}$, as $\mathbb{B}$ satisfies the c.c.c.*

It is hard to think of any realisable homogeneity property that is not achieved in this Souslin algebra. Note also that the homogeneity property of *(ii)* cannot be extended to include large subalgebras $\mathbb{A}$, $\mathbb{A}'$ (and of course not to the case $\mathbb{A} < \mathbb{A}' = \mathbb{B}$ as well). Consider for example $\mathbb{A}$ such that $\langle \mathbb{A}' \cup \{a\}\rangle^{\text{cm}} = \mathbb{B}$ for some $a \in \mathbb{B}$ while $\langle \mathbb{A} \cup \{b\}\rangle^{\text{cm}} \neq \mathbb{B}$ for all $b \in \mathbb{B}$. The existence of an automorphism φ of $\mathbb{B}$ mapping $\mathbb{A}$ to $\mathbb{A}'$ would imply that $\langle \mathbb{A} \cup \{\varphi^{-1}(a)\}\rangle^{\text{cm}} = \mathbb{B}$ which is impossible by the choice of $\mathbb{A}$.

Proof. We will essentially explain how to modify the construction in Section 2.3.3 on pages 83 ff. to achieve a Souslin algebra that has properties *(ii)* through *(vii)* of the theorem.

Let $\alpha > 0$ satisfy $\alpha = \omega\alpha$ and recall the $\diamondsuit^+$-machinery as introduced in Section 2.3.2 and used in the former construction of a chain homogeneous Souslin algebra.

First we take care of the homogeneity property stated above as *(ii)*. We define

$$E'_\alpha := \{(x, d) \in G_\alpha \mid x \text{ codes an engaging relation } \Phi_x \text{ on } T{\restriction}c(x)\}$$

and (in the notation of Definition 2.4.1) use Proposition 2.2.14 to lift φ_{Φ_x} to an automorphism $\varphi_{x,d}$ of $T{\restriction}c(x) \cap d$ carrying Φ^0 to Φ^1. (As in Section 2.3.3 we will finally choose T_α to be a subset of $[T{\restriction}\alpha]$ that is among other closed under the induced homeomorphisms $\overline{\varphi}_{x,d}$ for $(x, d) \in E'_\alpha$.)

We now describe how to embed the increasing ω_1-sequence of ∞-nice subalgebras $\mathbb{A}_i$ in $\mathbb{B}$. Recall the definition of the decreasing sequence $(C_i \mid i \in \omega_1)$ of club subsets of ω_1 as well as the derived definitions of $i(\alpha)$ and α^-. The ∞-nice t.e.r. $\equiv_i$ representing $\mathbb{A}_i$ will be defined on $T{\restriction}C_i \cup \{0\}$ in the course of the construction of T. The main requirements to meet are:

1. $\equiv_i$ is ∞-nice for all $i < \omega_1$,

2. for $i > j$ the restriction of $\equiv_j$ to $T{\restriction}C_i$ is refined by $\equiv_i$ in a way such that the induced t.e.r. $\equiv_{j/i}$ on the normal Souslin tree $(T{\restriction}C_i)/\equiv_i$ is ∞-nice,

3. for limit $i < \omega_1$ we want to have $\mathbb{A}_i = \langle \bigcup_{j<i} \mathbb{A}_j\rangle^{\text{cm}}$, so $\equiv_i$ shall be the conjunction of the $\equiv_j$ for $j < i$ in this case:

$$s \equiv_i t :\Leftrightarrow (\forall j < i) s \equiv_j t.$$

For any i, on level $T_0 = \{\mathbf{root}\}$ the relation $\equiv_i$ is of course trivial. On level $T_{\min C_i}$ we define $\equiv_i$ to be the identity, i.e., $s \equiv_i t$ if and only if $s = t$ for $s, t \in T_{\min C_i}$. This is a minor violation of the ∞-niceness requirement we posed on $\equiv_i$. But this is easily remedied by deleting $\min C_i$ from the club set C_i. On the other hand, by this convention we directly see that in the end $\bigcup \mathbb{A}_i$ will be a dense subset of $\mathbb{B}$, because $\{\min C_i \mid i < \omega_1\}$ is unbounded in ω_1.

In lavel $\alpha \in C_0$ we have that for all $j < i := i(\alpha)$ the set $C_j \cap \alpha$ is club in α and the t.e.r. $\equiv_j$ on $T \upharpoonright (C_j \cap \alpha)$ has by normality of the quotient tree a unique t.e.r.-extension to T_α. So, to satisfy the niceness condition for the t.e.r.s $\equiv_j$ with $j < i(\alpha)$, level T_α has to be chosen carefully with the aid of the Second Reduction Lemma 2.2.15. In the case where $i(\alpha)$ is a limit ordinal, we even have $\equiv_{i(\alpha)}$ on $[T\upharpoonright\alpha]$ at hand before we choose T_α, by the requirement (3) above. It is easily seen that, since for a limit ordinal i the minimum of C_i is just the supremum of the minima of the C_j for $j < i$, this is consistent with our appointment that $\equiv_i$ be the identity on $T_{\min C_i}$.

However, for successor $i(\alpha)$ the definition of $\equiv_{i(\alpha)}$ on T_α involves the choice of T_α. So we continue by giving the rules for the choice of T_α in the successor case $i = i(\alpha) = j + 1$. Let the sets E_α and F_α be defined as in the construction in Section 2.3.3, i.e., E_α consists of pairs (x, d), where x codes a pair of wipers while the x of a pair $(x, d) \in F_\alpha$ codes a decent t.e.r. $\equiv_x$. As in Section 2.3.3 we assume that we have so far constructed

- tree endomorphisms $\rho_{x,d}$ of T induced by isomorphisms between the quotient by $\equiv_x$ and $T\upharpoonright(c(x) \cap d)$ itself for $(x, d) \in F_\alpha$,
- tree automorphisms $\varphi_{x,d}$ for $(x, d) \in E_\alpha$ and furthermore
- tree automorphisms $\varphi_{x,d}$ of $T\upharpoonright(c(x) \cap d)$ for $(x, d) \in E'_\alpha$.

We have to choose the subset Q_α of $[T\upharpoonright\alpha]$ corresponding to T_α in a way that guarantees that

(a) Q_α is closed under the application of the induced homeomorphisms $\overline{\varphi}_{x,d}$ for $(x, d) \in E_\alpha \cup E'_\alpha$,

(b) for each pair $(x, d) \in F_\alpha$ the mapping $\rho_{x,d}$ extends to the new top level if $\equiv_x$ remains decent and

(c) finally we have to care for the various t.e.r.s $\equiv_j$, and guarantee that they remain nice when extended to T_α, so Q_α has to be suitable for all the $\equiv_j$ for $j < i$.

Consider for $j < i$ the induced equivalence relation on the set $[T\restriction\alpha]$ which we denote by the same symbol,

$$x \equiv_j y :\Longleftrightarrow (\forall\gamma \in C_j \cap \alpha) x\restriction\gamma \equiv_j y\restriction\gamma.$$

We apply combined reduction as stated in Corollary 2.2.16 to the set H of the maps given by $E_\alpha\, E'_\alpha$ and F_α, the collection $I := \{(C_j \cap \alpha, \equiv_j) \mid j < i\}$ of our ∞-nice t.e.r.s and the dense open set $M = \bigcap_{s\in R_\alpha} \hat{s}$ if R_α is a maximal antichain (where $(R_\nu \mid \nu < \omega_1)$ is our fixed $\diamondsuit$-sequence), or $M = [T\restriction\alpha]$ otherwise. The result is a comeagre subset N of M, which is suitable for $\equiv_j$ for all $j < i$, closed under the application of the homeomorphisms $\overline{\varphi}_{x,d}$ and their inverses for all pairs $(x, d) \in E_\alpha \cup E'_\alpha$ and which for each $(x, d) \in F_\alpha$ is not suitable for $\overline{\rho}_{x,d}$ if it does not satisfy $\overline{\rho}''_{x,d}N = N$.

Again, we denote by F^*_α the set of indices (x, d) in F_α of mappings $\overline{\rho}_{x,d}$ for which N is suitable. For $(x, d) \in F^*_\alpha$ we choose a right inverse $\sigma_{x,d}$ of $\overline{\rho}_{x,d}\restriction N$.

Now it is routine to choose a countable and dense subset Q_α of N which has the properties (a-c) listed above.

Next we define $\equiv_i$ on T_α in the case, that $i = j + 1$. Letting α^- be the maximum of $\alpha \cap C_i$, $t \in T_\alpha$ and $r := tuhr\alpha$, we easily see that we have an infinite set $\hat{r} \cap t/{\equiv_j}$ (here we view T_α as a dense subset of $[T\restriction\alpha]$) which we partition into $\aleph_0$ infinite sets $P(t/{\equiv_j}, r, n)$, each of them dense in $\hat{r} \cap t/{\equiv_j}$. Note, that the definition of P has to be independent from the choice of t. Finally we define $\equiv_i$ on T_α by letting $s \equiv_i t$ if and only if

$$s\restriction\alpha^- \equiv_i t\restriction\alpha^- \text{ and } (\exists n \in \omega) s \in P(s/{\equiv_j}, s\restriction\alpha^-, n) \text{ and } t \in P(t/{\equiv_j}, t\restriction\alpha^-, n).$$

Then by construction, $\equiv_i$ is ∞-nice as well as $\equiv_{j/i}$, and the "$i(\alpha)$=successor" step is complete.

Let now $\alpha \in C_0$ be such that $i = i(\alpha)$ is a limit ordinal. For $j < i$ define $\equiv_j$ on $[T\restriction\alpha]$ as above and let for $x, y \in [T\restriction\alpha]$

$$x \equiv_i y :\Longleftrightarrow (\forall j < i) x \equiv_j y.$$

This is the coarsest possibility to extend $\equiv_i$ to T_α and the only one by our requirement that $\bigcup_{j<i} \mathbb{A}_j$ completely generate $\mathbb{A}_i$.

We finally show that the $\equiv_i$-classes are perfect subsets of $[T\restriction\alpha]$ and that we can use the Second Reduction Lemma 2.2.15 on them. Then the same choice-procedure for T_α as in the first case can be adopted, only that we add $(C_i, \equiv_i)$ in the collection I of the t.e.r.s to be considered.

We construct an ∞-nice t.e.r. on some club below α which induces $\equiv_i$. Fix δ_0 in C_0 between α^- and α, so $j_0 = i(\delta_0) < i$. Then define

$$j_\nu := i(\delta_\nu) \text{ and } \delta_{\nu+1} := \min C_{j_\nu+1} \setminus \delta_\nu = \min C_{j_\nu+1} \setminus \delta_0$$

and $\delta_\mu = \sup\{\delta_\nu \mid \nu < \mu\}$ for limit μ with $i(\delta_\nu) < i$ for all $\nu < \mu$. Then the $i(\nu)$ are the ordinals from j_0 up to i. The final δ_μ is just α, and this ordinal $\mu = \mathrm{ot}(i \setminus j_0)$ is a limit.

The set of the δ_ν joined with $\{0\}$ is the club set of α on which we now define the t.e.r. $\simeq$. For $s, t \in T_{\delta_\nu}$ define

$$s \simeq t :\Longleftrightarrow\ s \equiv_{j_\nu} t \text{ and } (\forall \beta < \delta_0)(i(\beta) \leq i \Rightarrow s{\restriction}\beta \equiv_{i(\beta)} t{\restriction}\beta).$$

Then $\simeq$ is an ∞-nice t.e.r.: fix $s \simeq t$ on level δ_ξ and consider $s' > s$, where $s \in T_{\delta_\nu}$. In the successor case, letting $\nu = \nu^- + 1$, i.e. $j_\nu = j_{\nu^-} + 1$, this follows from the definition of $\equiv_{j_\nu}$ having only classes that are dense subsets of $\equiv_{j_{\nu^-}}$-classes. To argue for niceness in the limit case we refer to the choice of T_{δ_ν} which assures, that the $\equiv_{j_\nu}$-classes lie densely in the $\equiv_{j_\nu}$-classes while the ∞-part of ∞-niceness is trivially satisfied on limit stages when satisfied everywhere below. Since the j_ν are cofinal in i, the ∞-nice t.e.r. $\simeq$ induces $\equiv_i$ on $[T{\restriction}\alpha]$.

We now choose Q_α as in the case where i was a successor, only that we add $\equiv_i$ to the collection I of t.e.r.s that shall remain ∞-nice.

Now that the construction of T and the sequence $(\equiv_i \mid i < \omega_1)$ is completed, we argue that it satisfies points *(v)* and *(vi)* of the statement of Theorem 2.4.2. So let i be a countable limit ordinal. Then, by our definition of $\equiv_i$ (in the limit case), for every node $s \in T{\restriction}C_i$, its $\equiv_i$-class is just the intersection over the family $(s/{\equiv_j})_{j<i}$. This shows that $\mathbb{A}_i = \langle \bigcup_{j<i} \mathbb{A}_j \rangle^{\mathrm{cm}}$. For the proof of *(vi)* let λ, $(j_\nu \mid \nu < \lambda)$ and i be given. By chain homogeneity of $\mathbb{B}$ (cf. Proposition 2.3.3) we can choose $\varphi_0 : \mathbb{A}_0 \cong \mathbb{A}_{j_0}$, and then inductively extend the given $\varphi_\nu : \mathbb{A}_\nu \to \mathbb{A}_{j_\nu}$ to $\varphi_{nu+1} : \mathbb{A}_{\nu+1} \to \mathbb{A}_{j_{\nu+1}}$ by virtue of condition *(ii)*. In limit stages μ, the argument given in the proof of *(v)* above shows that

$$\left\langle \bigcup_{\nu<\mu} \mathbb{A}_{j_\nu} \right\rangle^{\mathrm{cm}} = \mathbb{A}_{j_\mu} = \left\langle \bigcup_{\nu<\mu} \varphi_\nu'' \mathbb{A}_\nu \right\rangle^{\mathrm{cm}}.$$

So there is a unique extension φ of $\bigcup_{\nu<\mu} \varphi_\nu$ with domain $\mathbb{A}_\lambda$. □

2.4.2 The iteration

We realise the big and chain homogeneous Souslin algebra as the direct limit of an increasing chain of small, chain homogeneous Souslin algebras. All the small algebras on this chain are isomorphic to the initial algebra $\mathbb{B}_0$, which we take from the last section.

Theorem 2.4.3. *Assume $\diamondsuit^+$. Then there is a big, chain homogeneous Souslin algebra.*

Proof. Let $\mathbb{B}_0$ be the algebra constructed in the proof of Theorem 2.4.2 with its continuously increasing chain $(\mathbb{A}_i \mid i < \omega_1)$ of ∞-nice subalgebras. Choose an isomorphism $\varphi_0 : \mathbb{B}_0 \cong \mathbb{A}_0$. Next choose a Souslin algebra $\mathbb{B}_1$ extending $\mathbb{B}_0$, such that there is an extension φ_1 of φ_0 witnessing $\mathbb{B}_1 \cong \mathbb{B}_0$. Given $\varphi_\nu : \mathbb{B}_\nu \cong \mathbb{A}_0$ for some $\nu < \omega_2$ choose in the same manner $\varphi_{\nu+1} : \mathbb{B}_{\nu+1} \cong \mathbb{B}_0$.

Now for the limit cases. If $\lambda < \omega_2$ is of countable cofinality, we choose a normal sequence $(i_\nu \mid \nu < \mu)$ with $\sup_{\nu<\mu} i_\nu = \lambda$ for some countable limit ordinal μ.

We inductively construct a chain $(\psi_\nu \mid \nu < \mu)$ of isomorphisms $\psi_\nu : \mathbb{A}_\nu \cong \mathbb{B}_{i_\nu}$ where for $\nu < \nu'$ the map $\psi_{\nu'}$ extends ψ_ν. Then $\mathbb{B}_\lambda$ is a super-algebra of

$$\bigcup_{\nu<\mu} \mathbb{B}_{i_\nu} = \bigcup_{\alpha<\lambda} \mathbb{B}_\alpha$$

isomorphic to $\mathbb{A}_\mu$ by some extension ψ_μ of $\bigcup_{\nu<\mu} \psi_\nu$.

If λ is of uncountable cofinality we simply let $\mathbb{B}_\lambda = \bigcup_{\alpha<\lambda} \mathbb{B}_\alpha$, the direct limit. Then we see that $\mathbb{B}_\lambda \cong \mathbb{B}_0$ for $\lambda < \omega_2$ by choosing a cofinal sequence $(i_\nu \mid \nu < \omega_1)$ and recursively choosing a chain of isomorphisms $\psi_\nu : \mathbb{A}_\nu \cong \mathbb{B}_{i_\nu}$. This goes through limit stages $\nu < \omega_1$ by property *(vi)* of Theorem 2.4.2. The direct limit $\mathbb{B}_{\omega_2}$ of the increasing chain of small Souslin algebras is Souslin as well: the c.c.c. is preserved by direct limits (i.e. finite support iterations or here simply the union of the $\mathbb{B}_\alpha$). And then the same holds for distributivity, for if $(a_{ij} \mid i, j < \omega)$ is any family of members of $\mathbb{B}_{\omega_2}$ then there is some $\alpha < \omega_2$ with $a_{ij} \in \mathbb{B}_\alpha$ for all $i, j < \omega$, and $\mathbb{B}_\alpha$ witnesses that the distributivity law is preserved. □

Remark 2.4.4. As far as we know, this is the first construction of a big Souslin algebra assuming the principle $\diamondsuit^+$ only. Jensen's constructions use $\diamondsuit$ and $\square$. In [Jec73, §5] gives a forcing that adjoins a so-called Souslin mess. This is a partial order of partial functions generalising the notion of a Souslin tree. The regular open algebra of a large enough Souslin mess is a big Souslin algebra. Laver has constructed a Souslin mess, only using $\diamondsuit$ and Silver's principle W. (W is a strengthening of Kurepa's Hypothesis KH cf. [Jec97, (24.16)].) We do not know yet, if and how the principles $\diamondsuit$+ W and $\diamondsuit^+$ are correlated.

Proposition 2.4.5. *Forcing with $\mathbb{B}_{\omega_2}$ turns any ground model Souslin tree T, which is regularly embeddable into $\mathbb{B}_{\omega_2}$, into a Kurepa tree.*

Proof. Let $G \subset \mathbb{B}_{\omega_2}$ be a V-generic filter, and let $\mathbb{A} := \langle T \rangle^{\mathrm{cm}}$. For $\alpha < \omega_2$ fix in V an isomorphism $\varphi_\alpha : \mathbb{B}_\alpha \cong \mathbb{A}$ and in $V[G]$ define $G_\alpha := \mathbb{B}_\alpha \cap G$.

Staying in $V[G]$, for $\alpha < \omega_2$, the set

$$b_\alpha := \{t \in T \mid (\exists g \in G_\alpha\})\varphi_\alpha(g) \leq_{\mathbb{A}} t\}$$

is an ω_1-branch of T, and we have $V[G_\alpha] = V[b_\alpha]$ for all $\alpha < \omega_2$. Finally, if $\gamma < \beta < \omega_2$ then $b_\beta \in V[b_\beta] \backslash V[b_\gamma]$, because the map φ_γ is in V. In particular we have $b_\beta \neq b_\gamma$. □

Since the Iteration Theorem 2.4.3 only assumes the strong properties of the Souslin algebra of Theorem 2.4.2 we have the following result on the hypotheses used.

Corollary 2.4.6. *Assume that there is a Mahlo cardinal. Then there is a model of* ZFC + $\diamondsuit$ *in which there is no Souslin algebra with the properties stated in theorem 2.4.2.*

Proof. In [Jen] Jensen considered the generic Kurepa hypothesis GKH:= "there is a c.c.c. forcing that forces KH in the generic extension". He shows that if κ is a Mahlo cardinal then GKH is false in the Levy-style generic extension collapsing κ to become $\aleph_2$. This partial order always forces $\diamondsuit$ (cf. [Kun80, Exercises VIII.J.5/6]). Finally, $\mathbb{B}_{\omega_2}$ is c.c.c. and forces, by the last Proposition, KH. □

Zusammenfassung der Ergebnisse

Das Thema dieser Dissertation ist im Bereich der kombinatorischen Mengenlehre und der Theorie der vollständigen Booleschen Algebren angesiedelt. Als Rahmentheorie setzen wir die Axiome der Zermelo-Fraenkel-Mengenlehre mit Auswahlaxiom, kurz ZFC voraus.

Wir wollen eine Teilmenge K einer vollständigen Booleschen Algebra $\mathbb{B}$ als *Kette* bezeichnen, falls K von der kanonischen partiellen Ordnung $<_{\mathbb{B}}$ auf $\mathbb{B}$ linear angeordnet wird. Die Menge der Ketten von $\mathbb{B}$ ist durch die Teilmengenbeziehung induktiv geordnet, so dass aus einer einfachen Anwendung des Zornschen Lemmas sofort die Existenz maximaler Ketten folgt. Wir wollen eine Boolesche Algebra als *kettenhomogen* bezeichnen, wenn alle ihre maximalen Ketten zueinander isomorph sind.

Die Ausgangsfrage für diese Dissertation war, ob es unter geeigneten mengentheoretischen Voraussetzungen eine atomlose und kettenhomogene, vollständige Boolesche Algebra $\mathbb{B}$ gibt, deren maximale Ketten aber nicht isomorph zum reellen Einheitsintervall sind.

Dieser Frage liegt die Beobachtung zugrunde, dass jede maximale Kette K einer vollständigen, kettenhomogenen Booleschen Algebra $\mathbb{B}$ stets eine vollständige, lineare Ordnung mit Endpunkten ist und die abzählbare Antikettenbedingung c.c.c. erfüllt, d.h. jede Familie paarweise disjunkter, offener Intervalle von K ist abzählbar.

Somit ist unser Problem eng mit Souslins Hypothese SH verknpft, welche postuliert, dass jede vollständige und dichte, lineare Ordnung, welche die c.c.c. erfüllt, ordnungsisomorph zu einem Intervall der reellen Zahlen ist. Wenn wir SH annehmen, folgt sofort, dass die Antwort auf unsere Ausgangsfrage negativ ausfällt.

In den 1960'er Jahren wurde aber bewiesen, dass SH unabhängig von ZFC ist, d.h. im Falle der Konsistenz von ZFC ist sowohl $\mathsf{ZFC} + \mathsf{SH}$ als auch $\mathsf{ZFC} + \neg\mathsf{SH}$ konsistent. Ein Gegenbeispiel zu Souslins Hypothese nennen wir eine *Souslin-Gerade*, die assoziierten vollständigen Booleschen Algebren *Souslin-Algebren*. Das wichtigste technische Hilfsmittel sind hier die sog. *Souslin-Bäume*. Ist $\mathbb{B}$ eine Souslin-Algebra, so sind alle maximalen Ketten von $\mathbb{B}$ Souslin-Geraden. Unsere Ausgangsfrage lässt sich nun wie folgt umformulieren:

> Ist es konsistent relativ zu ZFC, dass es eine kettenhomogene Souslin-Algebra gibt?

Als Hauptresultat meiner Dissertation ist die positive Beantwortung dieser Frage anzusehen, siehe Theoreme 2.3.2, 2.4.2 und 2.4.3 des Kapitels *Maximal chains in Souslin algebras*. Unter Annahme der Theorie ZFC + $\diamondsuit^+$, einer Erweiterung, die konsistent ist relativ zu ZFC, gebe ich u.a. die Konstruktion einer kettenhomogenen Souslin-Algebra mit maximalen Homogenitätseigenschaften und auch die Konstruktion einer *großen*, kettenhomogenen Souslin-Algebra $\mathbb{B}$ an, d.h. $\mathbb{B}$ wird von keiner Teilmenge der Mächtigkeit $\aleph_1$ vollständig erzeugt.

Um diese Konstruktionen zu ermöglichen, habe ich die bestehende Darstellungstheorie für Souslin-Algebren ausgebaut und konnte letztere für die folgenden weiteren Ergebnisse anwenden.

1. In den Abschnitten 1.2-1.4 des Kapitels *Subalgebras of small Souslin algebras* wird die Strukturtheorie für reguläre Einbettungen zwischen Souslin-Algebren entwickelt.

2. Für die Klasse der sog. *stark homogenen* Souslin-Bäume konnte ich ein bestehendes Dekompositionsresultat erweitern (siehe Theorem 1.5.10 in *Subalgebras of small Souslin algebras*) und ein neues angeben (Theorem 1.5.3, ebd.).

3. Schließlich konnte ich diese Zerlegungen verwenden, um Beispiele von Souslin-Bäumen zu konstruieren, die gewisse Starrheitseigenschaften für Souslin-Bäume voneinander isolieren und damit eine Frage von Fuchs und Hamkins beantworten, vgl. [FH06, Question4.1].

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Lebenslauf

Aus Gründen des Datenschutzes
ist in der Online-Version
kein Lebenslauf enthalten.

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