Appendix A

Direct Proof that dimA_ε = dim$\mathcal{A}_\varepsilon$ = 1

We present here the direct proof that $dimA_\varepsilon = dim\mathcal{A}_\varepsilon = 1$.

Theorem A.0.1.
$dimA_\varepsilon = dim\mathcal{A}_\varepsilon = 1$.

Proof:
First note that $\mathcal{H}^1(A_\varepsilon) \geq \mathcal{H}^1(\mathcal{A}_\varepsilon) \geq \mathcal{H}^1(\pi_x(\mathcal{A}_\varepsilon))$. From the definition of $\mathcal{B}^{\mathcal{A}_\varepsilon}$ it follows that

$$\mathcal{H}^1(\pi_x(\mathcal{A}_\varepsilon)) = \mathcal{H}^1(\pi_x(A_\varepsilon \sim \pi_x(\mathcal{B}^{\mathcal{A}_\varepsilon})) > \mathcal{H}^1([0,1]) - \frac{1}{2} > 0.$$

It follows that

$$\text{dim}A_\varepsilon \geq \text{dim}\mathcal{A}_\varepsilon \geq 1 \tag{A.1}$$

Now let $s > 0$ and $\delta > 0$. Then for any given $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that

$$\delta \in (2^{2-n}\varepsilon, 2^{3-n}\varepsilon]. \tag{A.2}$$

We note that the vertical height of the triangular caps in the n-th stage of construction of A_ε is 2^{1-n} so that $\delta \geq 2$ times the vertical height of the triangular caps in the nth construction stage. Since

$$A_\varepsilon \subset \bigcup_{i=1}^{2^n} T_{n,i}$$

any cover of $\cup_{i=1}^{2^n} T_{n,i}$ is also a cover of A_ε. By taking balls of radius δ with centers in A_n we note that we can take these balls along an $A_{n,i}$ such that the overlaps ensure that $A_{n,i}^{\delta/\sqrt{2}}$ is covered. By taking such a cover of $A_{n,i}$ for each i we have a cover consisting of balls of radius δ, $\mathcal{B}_\delta = \{B_\delta\}$ such that

$$\bigcup_{B_\delta \in \mathcal{B}_\delta} B_\delta \supset A_n^{\delta/\sqrt{2}} \supset A_n^{2^{1-n}}.$$

Since

$$A_n^{2^{1-n}} \supset \bigcup_{i=1}^{2^n} T_{n,i} \supset A_\varepsilon$$

we also have that $\mathcal{B}_\delta$ is a cover of A_ε. Since with such a cover no more than $\delta/\sqrt{2}$ of the radius of a ball in $\mathcal{B}_\delta$ will uniquely contribute to the cover of A_n, and since the inefficiencies of taking $A_{n,i}$'s that meet at non-uniform angles can not do any worse than forcing us to cover A_n twice, it follows that

$$\sum_{B_\delta \in \mathcal{B}_\delta} \delta \leq 2\sqrt{2}\mathcal{H}^1(A_n)$$

so that from Lemma 5 we have

$$\sum_{B_\delta \in \mathcal{B}_\delta} \delta \leq 2\sqrt{2}(1 + n16\varepsilon^2)^{1/2}.$$

Thus, from $(A.2)$ we have

$$\sum_{B_\delta \in \mathcal{B}_\delta} \delta^{1+s} \leq (2\varepsilon)^s 2\sqrt{2}\frac{(1 + n16\varepsilon^2)^{1/2}}{2^{ns}}.$$

As $\mathcal{B}_\delta$ is a cover of A_ε this means

$$\mathcal{H}_\delta^{1+s}(A_\varepsilon) \leq (2\varepsilon)^s 2\sqrt{2}\frac{(1 + n16\varepsilon^2)^{1/2}}{2^{ns}}$$

so that we have

$$\mathcal{H}^{1+s}(A_\varepsilon) = \lim_{\delta \to 0} \mathcal{H}_\delta^{1+s}(A_\varepsilon) \leq \lim_{n \to \infty} (2\varepsilon)^s 2\sqrt{2}\frac{(1 + n16\varepsilon^2)^{1/2}}{2^{ns}} = 0.$$

Since this is true for all $s > 0$ it follows that $\dim A_\varepsilon \leq 1$ and since $\mathcal{A}_\varepsilon \subset A_\varepsilon$ that $\dim \mathcal{A}_\varepsilon \leq 1$. Combining with $(A.1)$ gives the result. $\diamondsuit$

A.1 Notes

The proof presented here is our own.

Appendix B

Weak Flow Monotonicity

The usual generalisation of the smooth mean curvature flow is the weakened flow of varifolds by their mean curvature. Such flows are also know as Brakke flows due to the founding work showing their existence and some regularity for mean curvature not equal to zero (the mean curvature equal to zero, also called minimal surfaces, case had already been studied) which is due to Brakke [5].

The regularity theory due to Brakke is the weak version of (though not following from) the regularity theory presented in Ecker [7], that we have followed.

Allard, see for example [1] and [2], and Almgren [3] studied varifolds, particularly minimal varifolds. Allard [2] also looked at varifolds with a fixed boundary and developed a local regularity theory for such varifolds. Grüter and Jost [13] then translated these ideas to the case where the minimal varifold had a boundary satisfying the Neumann free boundary conditions. They also developed the tilde reflection function discussed in Chapter 11.

It would seem then that there should be a theory of varifolds moving by their mean curvature flow with Neumann free boundary conditions. We have not proven that this is the case. We present however here a formulation based on the work of Brakke [5] and Grüter and Jost [13] that we believe is appropriate for such flows. Although we have not shown existence, we show that under this definition, assuming the flow exists in a reasonable sense, then the monotonicity formula of Buckland translates to the weak flow.

We begin by defining a the Brakke flow integral, which serves in Brakke flow, a similar function to the integral of mean curvature in the smooth case. We then give our definition of mean curvature flow of varifolds satisfying the Neumann free boundary conditions. We then prove directly the monotonicity formula.

Definition B.0.1. (Brakke Flow Integral)
Let μ be a measure on a complete subset N of $\mathbb{R}^{n+1}$. Should μ be an n-rectifiable Radon measure we can define $V = \nu(M, \theta)$ to be the varifold associated with μ. We then use the notations
$\delta V =$ the first variation of V,
$|\delta V| =$ the total first variation of V.

Let $\phi \in C^2_c(N, \mathbb{R}^+)$. Then if

1. *$\mu|_{\{\phi>0\}}$ is not a Radon measure,*
2. *$|\delta V||_{\{\phi>0\}}$ is not a Radon measure,*

3. $|\delta V||_{\{\phi>0\}}$ *is singular with respect to* $\mu|_{\{\phi>0\}}$*, or*

4. $\int \phi|\vec{H}|^2 d\mu = \infty$*, where* $\vec{H} = \frac{d(\delta V)}{d\mu}|_{\{\phi>0\}}$

we define the Brakke flow integral $\mathcal{B}(\mu,\phi)$ *as*

$$\mathcal{B}(\mu,\phi) = -\infty.$$

Otherwise we define the Brakke flow integral $\mathcal{B}(\mu,\phi)$ *as*

$$\mathcal{B}(\mu,\phi) = \int -\phi|\vec{H}|^2 + \nabla\phi \cdot S^{\perp} \cdot \vec{H} d\mu < \infty$$

where $\nabla = \nabla^N$ *and* $S = S(x) = T_x\mu$ *for* $\mathcal{H}^n$*-a.e.* $x \in \{\phi > 0\}$.

To define Brakke flow with Neumann free boundary conditions for varifolds we need, as in the smooth case, a support surface. We use in the case the same Neumann free boundary support surface Σ as we did for the smooth case which we defined in Definition 11.1.2. With such a support surface we can define Brakke flow with Neumann free boundary conditions as follows:

Definition B.0.2. (Brakke flow with the Neumann free boundary condition)
Let $\Sigma := \partial G$ *be a free boundary support surface in* $\mathbb{R}^{n+1}$ *and let* $\{\mu_t\}_{t\geq 0}$ *be a 1-parameter family of measures on the complete subset* $\bar{G}$ *of* $\mathbb{R}^{n+1}$.

$\{\mu_t\}_{t\geq 0}$ *is then said to be a Brakke flow with the Neumann free boundary condition if*

1. μ_t *is a measure on* $\bar{G}$ *for each* $t \geq 0$
2. $\overline{D_{\mu_t}}(\phi) \leq \mathcal{B}(\mu_t,\phi)$ *for all* $\phi \in C^2_c(\mathbb{R}^{n+1},\mathbb{R}^+)$ *and* $t \geq 0$*, and*
3. $\int div_{S(x)} X d\mu_t = -\int X \cdot \vec{H} d\mu_t$ *for all vector fields* $X \in C^1_c(\mathbb{R}^{n+1},\mathbb{R}^{n+1})$ *with* $X(x) \in T_x\Sigma$ *for each* $x \in \Sigma$.

where $S(x) = T_x\mu_t$ *wherever it exists for all* $t \geq 0$ *and*

$$\overline{D_{\mu_t}}(\phi) := \limsup_{s\to t} \frac{\mu_s(\phi) - \mu_t(\phi)}{s-t}$$

is the upper derivate of μ_t *with respect to time.*

Remark: Both Allard in [2] and Grüter and Jost in [13] retain a strict condition that the entire varifold remain within a ball of radius 1. I do not see at present a particular need for this assumption.

We will also make the following additional assumptions:

$$\theta_t(x) \in \{0,1\} \quad \mathcal{H}^n\text{-a.e.}x \in \mathbb{R}^{n+1}, \mathcal{H}^1\text{-a.e.}t \geq 0, \tag{B.1}$$

where θ_t is the multiplicity function associated with μ_t. This is not only a simplifying assumption but is also necessary in the regularity theorems of Brakke [5] 6.12 as well as those in [7] and [8]. Also

$$\sup_{x\in\mathbb{R}^{n+1}} \sup_{R>0} \frac{\mu_0(B_R(x))}{\omega_n R^n} \leq D < \infty. \tag{B.2}$$

which is the stronger of the locally finite measure assumptions needed in many of the referenced works on varifolds.

Under this definition of Brakke flow with Neumann free boundary we have the following monotonicity formula.

Theorem B.0.1. Monotonicity Formula for Weak Flows
On the assumption that Definition B.0.1 provides for a well defined flow of varifolds with Neumann free boundary conditions, for any Brakke flow with the Neumann free boundary conditions, $\{\mu_t\}_{t\geq 0}$, supported on Σ satisfying (B.1) and (B.2), for any $(x_0, s) \in \Sigma \times \mathbb{R}^+$, $\delta \in (0, 2/5]$, and all $s - \tau_0 \leq t < s$

$$\overline{D}_t \left(e^{C\kappa_\Sigma^{2\delta}\tau^\delta\delta^{-1}} \int \eta\rho_{\kappa_\Sigma}((x,t)d\mu_t \right) \leq -e^{C\kappa_\Sigma^{2\delta}\tau^\delta\delta^{-1}} \int \left| \vec{H} - \frac{S^\perp \cdot D\rho_{\kappa_\Sigma}}{\rho_{\kappa_\Sigma}} \right|^2 \eta\rho_{\kappa_\Sigma} d\mu_t$$

where, as in the smooth case we define

$$\eta(x,t) := \left(1 - \left(\frac{2\kappa_\Sigma}{(\kappa_\Sigma^2\tau)^\delta} \right)^2 (r_{x_0} - 40n\tau) \right)_+^4$$

$$\rho_{\kappa_\Sigma} := \frac{1}{(4\pi\tau)^{n/2}} e^{-\frac{r_{x_0}}{8(16(\kappa_\Sigma^2\tau)^\delta+1)\tau}}$$

$$r_{x_0} := |x - x_0|^2 + |\widetilde{x - x_0}|^2$$

$$\tilde{x} := x - 2(< x, Dd > -d)Dd$$

where d is the signed distance function from Σ, the Neumann boundary support surface,

$$\tau_0 := \frac{(3/160n)^{2/\delta}}{\kappa_\Sigma^2},$$

$S = T_x\mu$ wherever it is defined, $\tau(t) = s - t$ and $C = C(n)$.

Proof:
We adapt Bucklands proof to varifolds. In order to do this we list some necessary identities. We need Brakke 5.8 and 3.5 from [5]. We will need the fact that $\int div_S X d\mu_t = -\int X \cdot \vec{H} d\mu_t$ for $X \in C_c^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ tangent to Σ on Σ from our definition. We note that from Proposition 11.3.4 (5), that $< Dr_{x_0}, \nu_\Sigma > \equiv 0$. We can then calculate that

$$D\eta = 4\left(\frac{2\kappa_\Sigma}{(\kappa_\Sigma^2\tau)^\delta} \right)^2 \left(1 - \left(\frac{2\kappa_\Sigma}{\kappa_\Sigma^2\tau)^\delta} \right)^2 (r_{x_0} - 40n\tau) \right)_+^3 Dr_{x_0} =: \eta^* Dr_{x_0}$$

which is compactly supported, that

$$D\rho_{\kappa_\Sigma} = -\frac{1}{(4\pi\tau)^{n/2}} \frac{1}{8(16(\kappa_\Sigma^2\tau)^\delta + 1)\tau} e^{-\frac{r_{x_0}}{8(16(\kappa_\Sigma^2\tau)^\delta+1)\tau}} Dr_{x_0} =: \rho^* Dr_{x_0}$$

which is not compactly supported and that

$$D(\eta\rho_{\kappa_\Sigma}) = \eta D\rho_{\kappa_\Sigma} + \rho_{\kappa_\Sigma} D\eta$$

which is compactly supported. Together the above equations give

$$< D\eta, \nu_\Sigma > = < \eta^* Dr_{x_0}, \nu_\Sigma > = \eta^* < Dr_{x_0}, \nu_\Sigma > = 0,$$

$$< D\rho_{\kappa_\Sigma}, \nu_\Sigma > = < \rho^* Dr_{x_0}, \nu_\Sigma > = \rho^* < Dr_{x_0}, \nu_\Sigma > = 0,$$

and thus

$$< D(\eta\rho_{\kappa_\Sigma}), \nu_\Sigma > = < \eta D\rho_{\kappa_\Sigma}, \nu_\Sigma > + < \rho_{\kappa_\Sigma} D\eta, \nu_\Sigma > = 0.$$

Since $D(\eta\rho_{\kappa_\Sigma}), D\eta \in C_c^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ which implies that

$$\int div_S D(\eta\rho_{\kappa_\Sigma}) d\mu_t + D(\eta\rho_{\kappa_\Sigma}) \cdot \vec{H} d\mu_t = 0,$$

and

$$\int div_S D(\eta) d\mu_t + D(\eta) \cdot \vec{H} d\mu_t = 0.$$

We see also that $< \rho_{\kappa_\Sigma} D\eta, \nu_\Sigma >= \rho_{\kappa_\Sigma} < D\eta, \nu_\Sigma >= 0$ and that $\rho_{\kappa_\Sigma} D\eta \in C_c^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ so that

$$\int div_S(\rho_{\kappa_\Sigma} D\eta) d\mu_t + (\rho_{\kappa_\Sigma} D\eta) \cdot \vec{H} d\mu_t = 0.$$

Additionally, we note the identities that for appropriately smooth functions and vector spaces

$$\frac{d}{dt} g = \frac{\partial}{\partial t} g + < Dg, \vec{H} >, \tag{B.3}$$

$$div_S(gX) = g div_S Df + < \nabla_S g, X >, \tag{B.4}$$

$$\triangle_S fg = f \triangle_S g + g \triangle_S f + 2 < \nabla_S f, \nabla_S g > \tag{B.5}$$

and

$$\triangle_S g = div_S Dg + < Dg, \vec{H} > . \tag{B.6}$$

We will also use, as in the smooth case, from Proposition 11.3.6 and Theorem 11.3.1 that

$$\left(\frac{d}{dt} - \triangle_S \right) \eta \le 0$$

and

$$Q(\rho_{\kappa_\Sigma}) \le C \rho_{\kappa_\Sigma} \kappa_\Sigma^{2\delta} \tau^{\delta - 1},$$

where C is dependent only on n.

Now, by Brakke 3.5, the definition $\bar{D}_t(\phi) \le \mathcal{B}(\mu_t, \phi)$ implies that for any nonnegative test function $\phi = \phi(x,t) \in C_c^1(\mathbb{R}^{n+1} \times \mathbb{R}^+, \mathbb{R}^+)$ and $t \in \mathbb{R}^+$

$$\bar{D}_t \mu_t(\phi) \le \int -\phi |\vec{H}|^2 + D\phi \cdot S \cdot \vec{H} + \frac{\partial}{\partial t} \phi d\mu_t$$

We note then firstly that $\eta\rho_{\kappa_\Sigma} \in C_c^1(\mathbb{R}^{n+1} \times \mathbb{R}^+, R^+)$ and thus is a valid test function to be used.

Brakke, 5.8, proved the fact that $S^\perp \cdot \vec{H} = \vec{H}$ μ-a.e. whenever $|\delta V_\mu|$ is a Radon measure. Therefore either for each t $\bar{D}_t \mu_t(\phi) = -\infty$ so that $\bar{D}_t(\mu_t(\eta\rho_{\kappa_\Sigma})) = -\infty$ completing the proof or

$$\bar{D}_t \mu_t(\phi) \le \int -\phi |\vec{H}|^2 + D\phi \cdot \vec{H} + \frac{\partial}{\partial t} \phi d\mu_t = \int -\phi |\vec{H}|^2 + \frac{d}{dt} \phi d\mu_t.$$

We now seperate our ϕ by choosing specifically choosing $\phi = \eta\rho_{\kappa_\Sigma}$ so that

$$\bar{D}_t \mu_t(\eta\rho_{\kappa_\Sigma}) = \int -\eta\rho_{\kappa_\Sigma} |\vec{H}|^2 + \frac{d}{dt} \eta\rho_{\kappa_\Sigma} d\mu_t.$$

Recalling that $< D\eta\rho_{\kappa_\Sigma}, \nu_\Sigma >= 0$ for all $x \in \Sigma$ so that $D\eta\rho_{\kappa_\Sigma}$ is an appropriate vector field for our divergence theorem which we may use since $\{\mu_t\}_{t\in\mathbb{R}^+}$ is a Brakke flow for the Neumann free boundary condition. That is we know

$$\int div_{S(x)} D\eta\rho_{\kappa_\Sigma} d\mu_t = -\int D\eta\rho_{\kappa_\Sigma} \cdot \vec{H} d\mu_t$$

and in a similar fashion to the previous discussion we have

$$\int div_S(\rho_{\kappa_\Sigma} D\eta) d\mu_t + (\rho_{\kappa_\Sigma} D\eta) \cdot \vec{H} d\mu_t = 0.$$

so that using identities $(B.3)$ to $(B.6)$ and Proposition 11.3.6 we have

$$\begin{aligned}
\bar{D}_t\mu_t(\eta\rho_{\kappa_\Sigma}) &= \int -\eta\rho_{\kappa_\Sigma}|\vec{H}|^2 + \frac{d}{dt}\eta\rho_{\kappa_\Sigma} d\mu_t \\
&= \int -\eta\rho_{\kappa_\Sigma}|\vec{H}|^2 + \frac{d}{dt}\eta\rho_{\kappa_\Sigma} + div_S D\eta\rho_{\kappa_\Sigma} + D\eta\rho_{\kappa_\Sigma} \cdot \vec{H} \\
&\quad -2(div(\rho_{\kappa_\Sigma} D\eta) + (\rho_{\kappa_\Sigma} D\eta) \cdot \vec{H}) d\mu_t \\
&= \int -\eta\rho_{\kappa_\Sigma}|\vec{H}|^2 + \frac{d}{dt}\eta\rho_{\kappa_\Sigma} + \triangle_S \eta\rho_{\kappa_\Sigma} \\
&\quad -2(\rho_{\kappa_\Sigma} div(D\eta) + \rho_{\kappa_\Sigma}(D\eta) \cdot \vec{H} + < \nabla_S \rho_{\kappa_\Sigma}, D\eta >) d\mu_t \\
&= \int -\eta\rho_{\kappa_\Sigma}|\vec{H}|^2 + \frac{d}{dt}\eta\rho_{\kappa_\Sigma} + \rho_{\kappa_\Sigma}\triangle_S \eta + \eta\triangle_S \rho_{\kappa_\Sigma} \\
&\quad -2(\rho_{\kappa_\Sigma}\triangle_S \eta + < \nabla_S \rho_{\kappa_\Sigma}, D\eta >) d\mu_t \\
&= \int -\eta\rho_{\kappa_\Sigma}|\vec{H}|^2 + \frac{d}{dt}\eta\rho_{\kappa_\Sigma} - \rho_{\kappa_\Sigma}\triangle_S \eta + \eta\triangle_S \rho_{\kappa_\Sigma} d\mu_t \\
&= \int -\eta\rho_{\kappa_\Sigma}|\vec{H}|^2 + \eta\left(\frac{d}{dt} + \triangle_S\right)\rho_{\kappa_\Sigma} + \rho_{\kappa_\Sigma}\left(\frac{d}{dt} - \triangle_S\right)\eta d\mu_t \\
&\leq \int -\eta\rho_{\kappa_\Sigma}|\vec{H}|^2 + \eta\left(\frac{d}{dt} + \triangle_S\right)\rho_{\kappa_\Sigma} d\mu_t \\
&= \int -\eta\rho_{\kappa_\Sigma}|\vec{H}|^2 + \eta\left(\frac{d}{dt}\rho_{\kappa_\Sigma} + div_S D\rho_{\kappa_\Sigma} + D\rho_{\kappa_\Sigma} \cdot \vec{H}\right)
\end{aligned}$$

Next, using $(B.3)$ and Brakke 5.8 we have

$$\begin{aligned}
\bar{D}_t\mu_t(\eta\rho_{\kappa_\Sigma}) &\leq \int -\eta\rho_{\kappa_\Sigma}|\vec{H}|^2 + \eta(\frac{d}{dt}\rho_{\kappa_\Sigma} + div_S D\rho_{\kappa_\Sigma} + D\rho_{\kappa_\Sigma} \cdot \vec{H}) \\
&= \int \eta\left(-\rho_{\kappa_\Sigma}|\vec{H}|^2 + \frac{\partial}{\partial t}\rho_{\kappa_\Sigma} + div_S D\rho_{\kappa_\Sigma} + 2D\rho_{\kappa_\Sigma} \cdot \vec{H}\right) d\mu_t \\
&= \int \eta(-\rho_{\kappa_\Sigma}|\vec{H}|^2 + \frac{\partial}{\partial t}\rho_{\kappa_\Sigma} + div_S D\rho_{\kappa_\Sigma} + 2D^\perp\rho_{\kappa_\Sigma} \cdot \vec{H}) d\mu_t \\
&= \int \eta(-\rho_{\kappa_\Sigma}|\vec{H}|^2 + \frac{\partial}{\partial t}\rho_{\kappa_\Sigma} + div_S D\rho_{\kappa_\Sigma} + 2D^\perp\rho_{\kappa_\Sigma} \cdot \vec{H} \\
&\quad -\frac{|D^\perp\rho_{\kappa_\Sigma}|^2}{\rho_{\kappa_\Sigma}} + \frac{|D^\perp\rho_{\kappa_\Sigma}|^2}{\rho_{\kappa_\Sigma}} d\mu_t \\
&= \int \eta(\frac{\partial}{\partial t}\rho_{\kappa_\Sigma} + div_S D\rho_{\kappa_\Sigma} + \frac{|D^\perp\rho_{\kappa_\Sigma}|^2}{\rho_{\kappa_\Sigma}}
\end{aligned}$$

$$
\begin{aligned}
&\quad -\left|\vec{H}-\frac{D^\perp\rho_{\kappa_\Sigma}}{\rho_{\kappa_\Sigma}}\right|^2)d\mu_t\\
&= \int\eta\left(Q(\rho_{\kappa_\Sigma})-\left|\vec{H}-\frac{D^\perp\rho_{\kappa_\Sigma}}{\rho_{\kappa_\Sigma}}\right|^2\right)d\mu_t\\
&\leq C\kappa_\Sigma^{2\delta}\tau^{\delta-1}\int\eta\rho_{\kappa_\Sigma}d\mu_t-\int\left|\vec{H}-\frac{D^\perp\rho_{\kappa_\Sigma}}{\rho_{\kappa_\Sigma}}\right|^2 d\mu_t.
\end{aligned}
$$

We now write

$$
e(t)=e^{C\kappa_\Sigma^{2\delta}\tau^\delta\delta^{-1}},\quad t\in\mathbb{R}^+,\quad \text{and}\quad i(t)=\int\eta\rho_{\kappa_\Sigma}(x,t)d\mu_t\quad t\in\mathbb{R}^+
$$

and calculate

$$
\begin{aligned}
\bar{D}_t\left(e^{C\kappa_\Sigma^{2\delta}\tau^\delta\delta^{-1}}\right)\int\eta\rho_{\kappa_\Sigma}(x,t)d\mu_t &= \limsup_{r\to t}\frac{e(r)i(r)-e(t)i(t)}{r-t}\\
&= \limsup_{r\to t}\frac{e(r)i(r)-e(t)i(r)+e(t)i(r)-e(t)i(t)}{r-t}.
\end{aligned}
$$

We note that

$$
\begin{aligned}
\limsup_{r\to t}\frac{e(r)i(r)-e(t)i(r)}{r-t} &= i(r)\limsup_{r\to t}\frac{e(r)-e(t)}{r-t}\\
&= i(t)\frac{d}{dt}e(t)\\
&= i(t)-C\kappa_\Sigma^{2\delta}\tau^{\delta-1}e^{C\kappa_\Sigma^{2\delta}\tau^\delta\delta^{-1}}\\
&< \infty.
\end{aligned}
$$

Also

$$
\begin{aligned}
\limsup_{r\to t}\frac{e(t)i(r)-e(t)i(t)}{r-t} &= e(t)\limsup_{r\to t}\frac{i(r)-i(t)}{r-t}\\
&= e(t)\bar{D}_t\mu_t(\eta\rho_{\kappa_\Sigma})\\
&\leq e(t)\left(C\kappa_\Sigma^{2\delta}\tau^{\delta-1}\int\eta\rho_{\kappa_\Sigma}d\mu_t-\int\left|\vec{H}-\frac{D^\perp\rho_{\kappa_\Sigma}}{\rho_{\kappa_\Sigma}}\right|^2 d\mu_t\right)
\end{aligned}
$$

which is finite unless $\int|\vec{H}|^2 d\mu_t=-\infty$ in which case $\bar{D}_t\mu_t(\phi)=-\infty$ by definition for all $\phi\in C_c^2(\mathbb{R}^{n+1},\mathbb{R}^+)$ which would in any case complete the proof. In the former case, we can then take the limiting supremums seperately to find

$$
\begin{aligned}
\bar{D}_t\big(e^{C\kappa_\Sigma^{2\delta}\tau^\delta\delta^{-1}}\int\eta\rho_{\kappa_\Sigma}(x,t)d\mu_t\big) &\leq -C\kappa_\Sigma^{2\delta}\tau^{\delta-1}e^{C\kappa_\Sigma^{2\delta}\tau^\delta\delta^{-1}}\int\eta\rho_{\kappa_\Sigma}d\mu_t+C\kappa_\Sigma^{2\delta}\tau^{\delta-1}e^{C\kappa_\Sigma^{2\delta}\tau^\delta\delta^{-1}}\int\eta\rho_{\kappa_\Sigma}d\mu_t\\
&\quad +e^{C\kappa_\Sigma^{2\delta}\tau^\delta\delta^{-1}}\int\left|\vec{H}-\frac{D^\perp\rho_{\kappa_\Sigma}}{\rho_{\kappa_\Sigma}}\right|^2 d\mu_t\\
&= e^{C\kappa_\Sigma^{2\delta}\tau^\delta\delta^{-1}}\int\left|\vec{H}-\frac{D^\perp\rho_{\kappa_\Sigma}}{\rho_{\kappa_\Sigma}}\right|^2 d\mu_t.
\end{aligned}
$$

$\diamondsuit$

B.1 Notes

The Brakke flow integral, Definition $B.0.1$ was first introduced by Brakke [5], though an excellent discussion can be found in Ilmanen [16]. The definition of Brakke flow with Neumann free boundary conditions is a combination of the features of the Brakke flow, see Brakke [5] or Ilmanen [16] and those of the minimal varifold definitions with boundary in Allard [2] and Grüter and Jost [13]. For general background theory on rectifiable varifolds and Radon measures, an excellent source is Simon [25]. The same theory specifically oriented to varifolds flowing by Brakke flow is discussed in Brakke [5] and Ilmanen [16]. The monotonicity formula, Theorem $B.0.1$ is our own. Ideas for Theorem $B.0.1$ were drawn from Buckland [6] who translated Huisken's ([14]) original monotonicity formula to the smooth mean curvature flow with Neumann free boundary conditions and from Ilmanen [17] who translated Huiskens original monotonicity formula to Brakke flows without boundary.