Chapter 13

Singularities, Aims and Assumptions

In this chapter we define our terms in considering singularities. We also describe the assumptions that lend themselves to usage with as much as possible a description of the reason for their use. We also show in the case of the 'boundary approaches boundary assumption' (which we will define in due course) that there definitely exists support and initial surfaces satisfying the assumption so that we do not, in this case, reduce the problem to an empty class of problems.

13.1 Singularities

We begin by defining Singularities. This is done, as in Part I, by way of considering which points in the limit surface allow the flow to at least locally be extended. First we define what the limit surface is. As is implied by the term 'limit surface' these are the points that can be expressed as a limit of points in the flow as the time goes to T.

Definition 13.1.1.
Let (M_t) be a one-parameter family of sets in $\mathbb{R}^{n+1}$. We say that the sets (or flow in the case (M_t) is a flow) reaches $x_0 \in \mathbb{R}^{n+1}$ at time t_0 if there exists a sequence (x_j, t_j) with $t_j \nearrow t_0$ so that $x_j \in M_{t_j}$ and $x_j \to x_0$. We denote $\mathcal{M} = (M_t)$ reaching x_0 at time t_0 by $\mathcal{M} \to_{t_0} x_0$.

Remark: Due to the possible misunderstanding that $\mathcal{M} \to_{t_0} x_0$ implies that $\mathcal{M}$ degenerates to the point x_0 we point out that this is not at all implied. $\mathcal{M} \to_{t_0} x_0$ simply denotes that x_0 is one of, in general, many points that are reached by the flow.

Using this definition of points reached by the flow, we can make a definition of what the limit surface, M_T actually is. This needs to be done as a definition of M_T is not included in the definition of the mean curvature flow with Neumann free boundary conditions, $\mathcal{M}$. We therefore make the following definition.

Definition 13.1.2.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a mean curvature flow supported on the support surface Σ. Then we define the limit surface $M_T \subset \mathbb{R}^{n+1}$ by

$$M_T := \{x \in \mathbb{R}^{n+1} : \mathcal{M} \to_T x\}.$$

For the boundary we first define

$$\partial\mathcal{M} := (\partial M_t)_{t\in[0,T)}$$

so that we can define the limit boundary $\partial M_T \subset \Sigma$ by

$$\partial M_T = \{x \in \Sigma : \partial\mathcal{M} \to_T x_0\}.$$

Remark: We make this definition of the limit boundary and don't try to define ∂M_T by a term similar to $frontier(M_T)$ as with ∂M_T we wish to examine the behaviour of the flow on the support surface Σ. It is conceivable that extra frontier is developed at the first singular time away from the support surface (specifically in the singular set of $\mathcal{M}$ at time T). This extra boundary is not part of the set that we wish to consider under the term ∂M_T.

We note that we have already, in the form of Definition 11.4.3, made a definition relating to flows reaching points. Particularly as it becomes important in a proof of Global regularity we show the relationship between the two definitions. In summary, we show that they are in a sense equivalently describing the same situation, at least under the type I assumption. Independently of interest and necessary to show the mentioned equivalence, we note the following consequences of the type I assumption.

Proposition 13.1.1.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a mean curvature flow with Neumann free boundary conditions that reaches x_0 at time T and satisfies the Type I assumption. Then for $t < T$

$$|F(p,t) - x_0| \leq 2C_H\sqrt{n(T-t)}.$$

Proof:
We calculate

$$|F(p,t) - x_0| = \left|\int_t^T \frac{dF}{d\gamma}(p,\gamma)d\gamma\right| \leq \int_t^T |H(p,q)|d\gamma \leq \int_t^T \frac{C_H\sqrt{n}}{\sqrt{T-\gamma}}d\gamma = 2C_H\sqrt{n(T-T)}.$$

$\diamondsuit$

As a corollary to this proposition, we can state a result on how close a surface is to the point that it will reach. This result does not need the type I assumption (see Ecker [7] Corollary 3.6 where the sphere comparison principle is used), however, the proof is quicker using the type I assumption, an assumption that we will always have when needing this result. We therefore present the proof under the type I assumption.

Corollary 13.1.1.
Let a mean curvature flow with Neumann free boundary conditions $\mathcal{M}$ reach x_0 at time T and $\mathcal{M}$ satisfy the Type I assumption. Then for $t < T$

$$d(x_0, M_t) \leq 2C_H\sqrt{n(T-t)}$$

Proof:
From Proposition 13.1.1 $|F(p,t) - x_0| \leq 2C_H\sqrt{n(T-t)}$. It follows that

$$d(x_0, M_t) \leq |F(p,t) - x_0| \leq 2C_H\sqrt{n(T-t)}.$$

$\diamondsuit$

We are now able to prove the equivalence of our definitions of a flow approaching a point.

Proposition 13.1.2.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, properly embedded solution to mean curvature flow with Neumann free boundary conditions supported on the support surface Σ. Let (M_t) be represented by the family of transformations $F(p,t)$ for $p \in M^n$ (a smooth orientable n-manifold) and $t \in [0,T)$ and let $\mathcal{M}$ satisfy the Type I assumption. $\mathcal{M}$ then reaches a point $x_0 \in \mathbb{R}^{n+1}$ if and only if there is a $p \in M^n$ such that x_0 is the limit point of p.

Proof:
If x_0 is a limit point of some $p \in M^n$ at time T then for any sequence of times $t_j \nearrow T$ we consider the sequence $x_j := F(p, t_j)$ for each $j \in \mathbb{N}$. We note $F(p,t_j) \in M_{t_j}$ and

$$\lim_{j\to\infty} x_j = \lim_{j\to\infty} F(p,t_j) = \lim_{t\nearrow\infty} F(p,t) = x_0,$$

thus $\mathcal{M}$ reaches x_0 at time T.

Now suppose that $\mathcal{M}$ reaches x_0 at time T. Consider the time sequence t_j such that for each $j \in \mathbb{N}$, $2nC_H\sqrt{T-t_j} < 2^{-(j+1)}$ and define $F_j := F^{-1}(\overline{B_{1/2^j}(x_0)} \cap M_{t_j}) \subset M^n$.

Since $\mathcal{M}$ reaches x_0 at time T, $F_j \neq \emptyset$ for all $j \in \mathbb{N}$. Further since, from Proposition 13.1.1 $|F(p,t_j) - \lim_{t\to T} F(p,t)| \leq 2nC_H\sqrt{T-t_j} < \frac{1}{2^{j+1}}$ we see that for all $p \in F_j$,

$$\lim_{t\to T} F(p,t) \in B_{3/2^{j+1}}(x_0) \text{ and that if } p \notin F_j, \quad \lim_{t\to T} F(p,t) \notin B_{1/2^{j+1}}(x_0)$$

so that (by the above and induction) for all $p \notin F_j$, $p \notin F_{j+n}$ for all $n \in \mathbb{N}$. That is, $\{F_j\}$ is a decreasing sequence of sets. Now, since $\mathcal{M}$ is properly embedded and smooth and $\overline{B_{1/2^j}(x_0)} \cap M_{t_j}$ is compact for sufficiently large j it follows that F_j is compact for each sufficiently large $j \in \mathbb{N}$. Thus $\cap_{j\in\mathbb{N}} F_j$ is a closed non-empty set. Take $p_0 \in \cap_{j\in\mathbb{N}} F_j$, then $|F(p_0,t_j) - x_0| < 3/2^{j+1}$ for each $j \in \mathbb{N}$. Thus

$$\lim_{t\to T} F(p_0,t) = \lim_{j\to\infty} F(p,t_j) = x_0;$$

and therefore x_0 is a limit point of p_0 at time T. $\diamondsuit$

Remark: The p_0 found to reach x_0 at time T following from $\mathcal{M}$ reaching x_0 at time T may not be unique.

Having now explored the concept of reaching we are now in a position to define our singular and regular sets.

Definition 13.1.3.
Let $\mathcal{M} = (M_t)$ be a smooth, properly embedded solution of mean curvature flow (or a smooth properly embedded solution of mean curvature flow with Neumann free boundary conditions) in $U \times [0,T)$. We say that $x_0 \in U$ is a singular point of the solution at time T if $\mathcal{M}$ reaches x_0 at time T and has no smooth extension beyond time T in any neighbourhood of x_0. All other points (which includes those not reached by the solution) are called regular points. The singular set at time T is denoted by $sing_T\mathcal{M}$ and the regular set by $reg_T\mathcal{M}$.

13.2 Aims and Assumptions

In this section we give a more detailed statement of those results for which we are aiming. Having now introduced singularities we are able to delineate our aims more formally. After outlining our aims, we will describe the assumptions we will make on the surfaces being considered in order to prove our results.

It has already been mentioned that we are working in this part of the thesis towards local and global regularity. The local regularity gives conditions that will imply regularity for a given point, whereas global regularity says something about the total measure and/or dimension of the singularity set.

The quantity we will be looking at for local regularity is Gaussian density. Properly formally defined in Definition 16.1.2 we simply mention here that we can write the Gaussian density for a mean curvature flow with Neumann free boundary conditions on the support surface Σ around the point $(x_0, t_0) \in \Sigma \times (0, T]$ as

$$\Theta(\mathcal{M}, x_0, t_0) = \lim_{t \to t_0} e^{C\kappa_\Sigma^{2\delta}(t_0 - t)^\delta} \int_{M_t} \varphi_{\sigma,(x_0,t_0)} \eta_{x_0,t_0} \rho_{\kappa_\Sigma,(x_0,t_0)} d\mu_t$$

for any $\sigma > 0$ where $\varphi_{\sigma,(x_0,t_0)}$ is a localisation function similar to the boundaryless localisation function defined formally in Definition 14.1.1. As we shall see, in the smooth cases, the Gaussian density gives us the expected values that correspond exactly to usual measure densities. This fact brings us to the intent to show that the implication also holds in the other direction. That is, if the Gaussian density is approximately what one would expect in the smooth case, then the point in space time around which we took the Gaussian density should be regular. That is we expect a theorem to the effect of:

Theorem:
Suppose that $\mathcal{M} = (M_t)_{t \in [0,T)}$ is a mean curvature flow with Neumann free boundary conditions supported on the support surface Σ. Then there exists an $\varepsilon > 0$ such that if, for any given $(x_0, t_0) \in \Sigma \times (0, T]$

$$\Theta(\mathcal{M}, x_0, t_0) \leq \frac{1}{2} + \varepsilon$$

then x_0 is regular at time t_0.

Although we will need some slightly stronger assumptions, it is exactly this question that is addressed in the Chapter on Local Regularity.

For global regularity we look to give a result on the measure of the singularity set. It is well known that under certain additional assumptions the Hausdorff n-measure of the singularity set of an n-dimensional mean curvature flow without boundary is zero. We will show (under assumptions) the same result here. That is we show

Theorem:
Suppose that $\mathcal{M} = (M_t)_{t \in [0,T)}$ is an n-dimensional mean curvature flow with Neumann free boundary conditions supported on the support surface Σ satisfying the technical assumptions made in this thesis (yet to be described). Then

$$\mathcal{H}^n(sing_T \mathcal{M}) = 0.$$

One immediately asks if anything better holds on the boundary itself, $\mathcal{H}^{n-1}(sing_T \partial \mathcal{M}) = 0$ perhaps?. This is, however, unfortunately not to be hoped for as an entire Hausdorff $(n-1)$-dimensional singularity set can occur on the support surface in a very natural example of a mean curvature flow that we describe below:

Construction 13.2.1.
We construct Σ *as follows: Let* $B^2_2(0) \subset \Sigma$ *and let* Σ *be otherwise smooth and compact in such a way that* $B^3_2(0) \cap \{x \in \mathbb{R}^3 : x_3 > 0\} \subseteq G$*. We then place* M_t *on* Σ *by letting* M_0 *be half the torus* $S^1_1 \times S^1_{1/4}$ *with the* S^1_1 *corresponding to* $\partial B^2_1(0)$*. We see that the torus flows for* $t \in [0,T)$ *for some* $T < \sqrt{(5/4)^2 - 2n} = 25/32n$ *and that* M_t *flows to a circle* $S^1_r \subset B^2_2(0)$*. It foll0ws that* $sing_T\mathcal{M} \supseteq S^1_r$ *so that dim(*$sing_T\mathcal{M}$*)* $\geq 1 = n-1$*. We note that in this case* $sing_T\mathcal{M}$ *is approached on the boundary. That is for all* $x \in sing_T\mathcal{M}$*,* $\partial\mathcal{M} \to_T x$*.*

We now move on to the assumptions that we will be making.

Firstly, the proof of regularity relies on being able to compare limits (particularly limit surfaces) to a smooth regular situation. The smooth situation occurs when we are on a manifold and is thus everywhere locally 'plane like'. We clearly do not want to simply assume that the limit surface is everywhere locally 'plane like' as it would then be smooth in some sense and thus everywhere regular. We therefore only assume the weakest form of being mostly plane like. That is, rectifiable. The idea of rectifiability was explored in detail in the first part of the thesis. Here however, we only need a passing acquaintence. To motivate the mostly 'plane like' behaviour, we include in this part of the thesis the following definition.

Definition 13.2.1.
A set $M \subset \mathbb{R}^{n+k}$ *is said to be countably* n*-rectifiable if*

$$M \subset M_0 \cup \bigcup_{j=1}^{\infty} F_j(\mathbb{R}^n)$$

where $F_j : \mathbb{R}^n \to \mathbb{R}^{n+k}$ *are Lipschitz functions and* $\mathcal{H}^n(M_0) = 0$*.*

In order to be able to compare a limit surface to the limit of the flow in a reasonable way we need to be sure that there is only one surface and not two or more of them resting on each other (multiple sheets). For this reason we assume the unit density hypothesis. Although it is not known if we could flow to such a situation in mean curvature flow with Neumann free boundary conditions, there are certainly conceivable limit surfaces that we want to avoid. A deeper discussion of this problem can be found in Brakke [5] or Ecker [7]. In order to place an assumption on the density of a surface we also need to know what the density is. To this end we make the following definition.

Definition 13.2.2.
Let μ *be a measure on* $\mathbb{R}^{n+1}$*. Let* $M \subset \mathbb{R}^{n+1}$ *and* $x \in \mathbb{R}^{n+1}$*. Then the* n*-dimensional density of* M *with respect to* μ *at* x *is defined by*

$$\Theta^n(\mu, M, x) := \lim_{r \to 0} \frac{\mu(M \cap B_r(x))}{\omega_n r^n}$$

where ω_n *is the volume of the* n*-ball.*

We also need to examine the measure of limit surfaces, this clearly cannot be done when the limit of the measures has nothing to do with the limit measure. We therefore assume area continuity. We state the above three assumptions formally in the following definition.

Definition 13.2.3.
A smooth, properly embedded solution $\mathcal{M} = (M_t)_{t\in[0,T)}$ *of mean curvature flow with Neumann free boundary conditions in* $U \times [0,T)$ *(for some open* $U \subset \mathbb{R}^{n+1}$*) is said to satisfy the* **area continuity and unit density hypothesis at time** T *if the hypersurfaces* M_t *converge in the sense of measures*

to an $\mathcal{H}^n$-measurable, countably n-rectifiable subset $M_T \subset U$ of $\mathbb{R}^{n+1}$ of locally finite $\mathcal{H}^n -measure$. That is

$$\lim_{t \nearrow T} \int_{M_t} \phi d\mathcal{H}^n = \int_{M_T} \phi d\mathcal{H}^n \tag{13.1}$$

for all $\phi \in C_0^0(U)$.

Note that (13.1) implies in particular (see, for e.g. Ecker [7]) that for every $x_0 \in U$

$$\lim_{t \nearrow T} \mathcal{H}^n(M_t \cap B_\rho(x_0)) = \mathcal{H}^n(M_T \cap B_\rho(x_0))$$

for all but countably many $\rho > 0$ whenever $B_\rho(x_0) \subset U$.

The area continuity and unit density hypothesis is not a new assumption, being also made for the smooth case in Ecker [7]. A similar assumption is also made in the work of Brakke for the regularity theorem (Theorem 6.12) in Brakke [5]. We will however need additional assumptions about what happens on the boundary. These are not restricitions simply to make the proof easier, rather the desired regularity result is simply not true without some control over the boundary as we see with the following example:

Construction 13.2.2.
We construct the 2-dimensional support surface $\Sigma \subset \mathbb{R}^3$ in five pieces. Σ_i, $i = 1, 2, 3, 4, 5$.

$$\Sigma_1 := \partial B_1(0) \cap \{x \in \mathbb{R}^3 : x_3 > 1/4\},$$

a piece of a sphere above the 2 plane.

$$\Sigma_2 := B^2_{31/16}(0) \sim B^2_{1/2}(0),$$

an annulus in the 2-plane.

$$\Sigma_3 := \partial B_2(0) \cap \{x \in \mathbb{R}^3 : x_3 > 1/8\},$$

an outer piece of sphere. Σ_4 is then a smooth connection between Σ_1 and Σ_2 such that $\Sigma_4 \subset B_1(0)$ and Σ_5 is a smooth connection between Σ_2 and Σ_3 such that Σ_5 is a subset of the 1/8 neighbourhood of $\{x \in \mathbb{R}^2 : |x| = 2\}$.

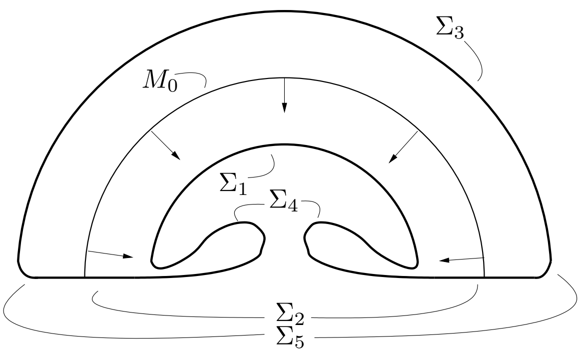

Figure 13.1: Cross section of Construction 13.2.2

We then set $M_0 := \partial B_{3/2}(0) \cap \{x \in \mathbb{R}^3 : x \geq 0\}$. We see that M_t is a homothetically shrinking half sphere for $t \in [0, 5/16)$. However, $\mathcal{M}$ reaches Σ_1 at $t = 5/16$ (in the sense that for each

$x \in \Sigma_1$, $\mathcal{M}$ reaches x at time $t = 5/16$). Since there is no smooth extension of the mean curvature flow with Neumann free boundary conditions supported on Σ in any neighbourhood of any $x \in \Sigma_1$, it follows that $\Sigma_1 \subset sing_{5/16}\mathcal{M}$ and since dim(Σ_1) $= 2 = n$ it follows that we can have n-dimensional singularities of positive $\mathcal{H}^n$ measure in the Neumann free boundary case.

We address this problem firstly through the boundary approaches boundary assumption which states that the surface can only reach a point on Σ if it is reached by the boundary and so cannot collide with the support surface from the interior as it did in the above construction. This assumption is stated formally below.

Definition 13.2.4.
A smooth, properly embedded solution, $\mathcal{M} = (M_t)_{t\in[0,T)}$, of mean curvature flow with Neumann free boundary conditions supported on the support surface Σ is said to satisfy the **boundary approaches boundary assumption** *if for all $x_0 \in \Sigma$ $\mathcal{M} \to_T x_0$ implies $\partial\mathcal{M} \to_T x_0$.*

Remark: Note that the boundary approaches boundary assumption implies that $\partial M_T = M_T \cap \Sigma$.

This assumption alone is not sufficient. To make the assumptions sufficient we can take one of two additional assumptions. The two assumptions are in a way analogies of either the unit density hypothesis or the area continuity hypothesis. The first, similar to the unit density assumption, as we see in Chapter 18, leads very quickly to the global regularity theorem. It is due to the apparently trivialising simplicity that we also address a different additional assumption. This second additional assumption is weaker in the sense that more work is required to prove the global regularity theorem, but stronger in the sense that we need to assume type I curvature bounds. We describe the two assumption frameworks below.

We note firstly, however, that although the boundary approaches boundary assumption does not appear to be necessary in the global regularity proof under the regularity assumptions I (defined formally below) it is intrinsically necessary in that it is used in the proof of the Clearing Out Lemma, which in turn is important for global regularity.

Boundary Assumptions I:
Like the unit density assumption for the boundaryless case where the assumption is in principle preventing the congregation of multiple sheets, stating that there is in essence only one sheet of the flow reaching any point. Under the same principle of assuming that only one sheet reaches any boundary point, we would then be assuming half density, or unit multiplicity on a half plane. We state this assumption formally as follows.

Definition 13.2.5.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, properly embedded mean curvature flow with Neumann free boundary conditions supported on the support surface Σ. $\mathcal{M}$ is then said to satisfy the **unit multiplicity assumption** *if for $\mathcal{H}^n$-almost all $x \in \partial M_T$*

$$\Theta^n(\mathcal{H}^n, M_T, x) \in \left\{0, \frac{1}{2}\right\}.$$

Remark: Note that the flow in Construction 13.2.2 does not satisfy the unit multiplicity assumption since $\mathcal{H}^n(\Sigma_1) > 0$ and for each $x \in \Sigma_1$, $\Theta^n(\mathcal{H}^n, M_T, x) = 1$.

We bring all of our assumptions in the boundary assumptions I framework together in the following definition:

Definition 13.2.6. (Regularity Assumptions I)
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a mean curvature flow with Neumann free boundary conditions supported on a Neumann free boundary support surface Σ. Let $t_0 \in (0,T]$. Then $\mathcal{M}$ is then said to satisfy the **regularity assumptions I** *at time t_0 if $\mathcal{M}$ satisfies the area continuity and unit density hypothesis at time t_0 as well as the boundary approaches boundary and unit multiplicity assumptions.*

Boundary Assumptions II:
In the boundary assumptions II case we actually make two assumptions. The first is an analog of the area continuity for boundary.

Definition 13.2.7.
A smooth, properly embedded solution, $\mathcal{M} = (M_t)_{t\in[0,T)}$, of mean curvature flow with Neumann free boundary conditions in $U \times [0,T)$ (for some open $U \subset \mathbb{R}^{n+1}$) is said to satisfy the **boundary area continuity hypothesis at time** T *if the $n-1$-dimensional surfaces ∂M_t converge in the sense of measures to an $\mathcal{H}^n$-measurable, countably n-rectifiable subset $\partial M_T \subset U$ of $\mathbb{R}^{n+1}$. That is*

$$\lim_{t\nearrow T}\int_{\partial M_t}\phi d\mathcal{H}^{n-1} = \int_{\partial M_T}\phi d\mathcal{H}^{n-1} \tag{13.2}$$

for all $\phi \in C_0^0(U)$.

Remark: 1) Note that we do not assume that the boundary stays in its own dimension as we did for the surface as a whole. We want to prove that this will follow in any case.

2) Note also that the boundary approaches boundary assumption is not a consequence of the boundary area continuity hypothesis, since the boundary area hypothesis allows for sets reaching the boundary from the interior with dimension no greater than $n-2$, whereas the boundary approaches boundary assumption allows for no set to reach the boundary from the interior at all.

The second additional assumption we make is simply that the flow be a type I flow. A condition that has already been described in Definition 11.4.2. We bring all of our assumptions in the boundary assumptions II framework together in the following definition:

Definition 13.2.8. (Regularity Assumptions II)
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a mean curvature flow with Neumann free boundary conditions supported on a Neumann free boundary support surface Σ. Let $t_0 \in (0,T]$. $\mathcal{M}$ is then said to satisfy the **regularity assumptions II** *at time t_0 if $\mathcal{M}$ satisfies the area continuity and unit density hypothesis at time t_0, the boundary area continuity hypothesis at time t_0, the boundary approaches boundary assumption at time t_0 and the Type I assumption at time t_0.*

13.3 A Class of Boundary Approaches Boundary Support Surfaces

Although it is fairly clear that there are at least some coincidental examples that happen to satisfy the boundary approaches boundary assumption, we would like to know that there is at least some fixed class of mean curvature flows with Neumann free boundary conditions that always satisfies the boundary approaches boundary assumption. In this way, we are then certain that we are proving results about some definite existent set of flows.

We prove just such a result by showing that should the support surface Σ be convex, then the boundary approaches boundary assumption is satisfied. We define what we mean by the support surface being convex as follows.

Definition 13.3.1.
Let $\Sigma := \partial G$ be a Neumann free boundary support surface. This surface is said to be **strictly convex** *if for every distinct pair of points $x, y \in \Sigma$*

$$\{tx + (1-t)y : t \in (0,1)\} \subset \bar{G} \sim \partial G.$$

We then show that mean curvature flows with Neumann free boundary conditions supported on such support surfaces Σ satisfy the boundary approaches boundary assumption.

Proposition 13.3.1.
Suppose $\mathcal{M} = (M_t)_{t\in[0,T)}$ is a mean curvature flow with Neumann free boundary conditions supported on the support surface Σ. Suppose Σ is strictly convex and that $\mathcal{M}$ reaches $x_0 \in \Sigma$ at time $t_0 \leq T$. Then $\partial\mathcal{M}$ reaches x_0 at time t_0. That is, we can choose $(x_j, t_j) \to (x_0, t_0)$ so that $x_j \in \partial M_{t_j}$ for each j.

Proof:
Suppose that it is not possible to reach x_0 on Σ, then it can be easily checked that there exists a $\rho > 0$ and $\tau > 0$ such that $B_\rho(x_0) \cap \partial M_t = \emptyset$ for all $t \in (t_0 - \tau, t_0)$.

We choose and fix such a pair ρ and τ.

We now re-orient our surfaces (which we may do due to the invariance under orthogonal transformations of the relevant quantities) so that $x_0 = 0$ and $T_{x_0}\Sigma = \mathbb{R}^n$. By the strict convexity of Σ we may also choose our orientation so that $(x_0)_{n+1} = \max\{x_{n+1} : x \in \Sigma\}$. By our definition of strictly convex it also follows that $x_{n+1} \leq 0$ for all $x \in \mathcal{M}$ and that there is a $h_0 < 0$ such that for all $x \in \Sigma \sim B_\rho(x_0)$ $x_{n+1} < h_0$.

Since (M_t) is a properly embedded manifold in G for each $t \in [0,T)$ we know $\sup_{K\cap M_{t_0-\tau}} x_{n+1} = \max_{K\cap M_{t_0-\tau}} x_{n+1}$ for any compact set $K \subset \mathbb{R}^{n+1}$.

Suppose now that

$$\sup_{\overline{B_\rho(x_0)}\cap M_{t_0-\tau}} x_{n+1} = 0, \text{ and thus } \max_{\overline{B_\rho(x_0)}\cap M_{t_0-\tau}} x_{n+1} = 0.$$

Since $M_t \subset \bar{G}$ for all $t \in [0,T)$ and $\bar{G} \cap \overline{B_\rho(x_0)} \cap \{x : x_{n+1} = 0\} = x_0$ it follows that $x_0 \in M_{t_0-\tau}$. As $x_0 \in \Sigma \cap B_\rho(x_0)$ this implies $\partial M_{t_0-\tau} \cap B_\rho(x_0) \supset \{x_0\} \neq \emptyset$. This condition tells us that $\sup_{\overline{B_\rho(x_0)}\cap M_{t_0-\tau}} x_{n+1} < 0$ and therefore there exists a $h_1 < 0$ such that $\sup_{\overline{B_\rho(x_0)}\cap M_{t_0-\tau}} x_{n+1} < h_1$.

Additionally, since $M_t \subset \bar{G}$ for each t it follows that for all $x \in M_{t_0-\tau}$

$$x_{n+1} \leq \max\{\max\{x_{n+1} : x \in \mathcal{M} \cap \overline{B_\rho(x_0)}\}, \max\{x_{n+1} : x \in \Sigma \sim B_\rho(x_0)\}\} \leq \max\{h_0, h_1\} =: h < 0.$$

Next, we consider $u(x,t) := x_{n+1}$ so that then

$$\left(\frac{d}{dt} - \triangle_{M_t}\right) u = 0 \leq 0.$$

Also $\max_{M_{t_0-\tau}} u < h$. By the maximum principle any new maximum must then be reached on the boundary. That is if $\max_{M_{t_1}} u > \max_{M_t} u$ for all $t \in [t_0 - \tau, t_1)$ for some $t_1 \in (t_0 - \tau, t_0)$, then $\max_{M_{t_1}} u = \max_{\partial M_{t_1}} u$ and therefore

$$\begin{aligned} \sup\{u(x,t) : t \in [t_0-\tau, t_0), x \in M_t\} &\leq \sup\{u(x,t) : t \in [t_0-\tau, t_0), x \in \partial M_t\} \\ &\leq \sup\{x_{n+1} : x \in \Sigma \sim B_\rho(x_0)\} \\ &< h. \end{aligned}$$

Should such a t_1 not exist, then $\max_{M_t} u \leq \max_{M_{t_0-\tau}} u < h$. It follows that for all $t \in [t_0 - \tau, t_0)$, $M_t \cap B_h(x_0) = \emptyset$ and therefore that M_t does not reach x_0. This contradiction proves the result. $\diamondsuit$

13.4 Notes

The definitions, Definition 13.1.1 and Definition 13.1.2, of flows reaching points and limit flows are extensions of that to be found in Ecker [7]. Of the results showing the consequences of type I with respect to the limit point function, Proposition 13.1.1 is due to Buckland [6] while Corollary 13.1.1 and Proposition 13.1.2 are our own. The definition, Definition 13.1.3, of singular sets follows Ecker [7]. Gaussian density will be further discussed in the chapter addressing Gaussian density in more detail, as will the Theorems for which we are aiming. Constructions 13.2.1 and 13.2.2 are our own. A good discussion of rectifiability and density can be found in Simon [25] or Evans and Gariepy [10]. The area continuity and unit density hypothesis is stated as in Ecker [7] and the boundary area continuity is our extension of this definition. Proposition 13.3.1 is our own which uses the maximum principle. A good source for information on the maximum principle is Evans [9].