Chapter 12

Interior Considerations

In this chapter we simply note that the local results applicable to mean curvature flow without boundary, as in Ecker [7], hold on small enough balls around points in the interior. This is due to the application of the Localised Monotonicity Formula of Ecker.

We show that the Localised Monotonicity Formula holds in the Neumann free boundary case, and state the relevant local results that we will need, or find interesting in the Neumann free boundary case. First we give the definition of the localising test function.

Definition 12.0.1.
Let $(x_0, t_0) \in \mathbb{R}^{n+1} \times \mathbb{R}$. We then define for each $\sigma > 0$ $\psi_{(x_0,t_0),\sigma} : \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}$ by

$$\psi_{(x_0,t_0),\sigma}(x,t) := \left(1 - \frac{|x - x_0|^2 + 2n(t - t_0)}{\sigma^2}\right)_+^3 .$$

This will be called the localisation function (or when needing to differentiate from other localisation functions, the boundaryless localisation function).

We note the following property of $\psi_{(x_0,t_0),\sigma}$ proven in Ecker [7] (Equation 4.4)

Proposition 12.0.1.
Let $\psi_{(x_0,t_0),\sigma} : \mathbb{R}^{n+1} \to \mathbb{R}$ be the test function defined in Definition 12.0.1. Then

$$\left(\frac{d}{dt} - \triangle_{M_t}\right) \psi_{(x_0,t_0),\sigma} \leq 0.$$

We combine this with Buckland's Expansion Formula 11.5 to show Ecker's Local Monotonicity Formula for the interior of mean curvature flows with Neumann free boundary conditions.

Proposition 12.0.2.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a mean curvature flow with Neumann free boundary conditions in the open set $U \subset \mathbb{R}^{n+1}$ supported on the support surface Σ. Let $t_0 \in (0, T]$. For every $x_0 \in U \sim \Sigma$, let $d_{x_0} > 0$ be a number such that $B_d(x_0) \subset U$ and $B_d(x_0) \cap \Sigma = \emptyset$ for each $d \in (0, d_{x_0}]$. Set $\rho_0 = d_{x_0}/(2\sqrt{1+2n})$. Then for all $x_0 \in \Sigma$, $\sigma \in (0, \rho_{x_0}]$ and $t \in (t_0 - \sigma^2, t_0)$ we have

$$spt\ \psi_{(x_0,t_0),\sigma}(\cdot, \sigma) \subset U \sim \Sigma$$

and

$$\frac{d}{dt}\int_{M_t}\rho_{(x_0,t_0)}\psi_{(x_0,t_0),\sigma}d\mu_t \leq -\int_{M_t}\left|\vec{H}-\frac{D^\perp\rho_{(x_0,t_0)}}{\rho_{(x_0,t_0)}}\right|^2\rho_{(x_0,t_0)}\psi_{(x_0,t_0),\sigma}d\mu_t.$$

Proof:
We observe that for each $\sigma \in (0, \rho_{x_0}]$ and $t \in (t_0 - \sigma^2, t_0)$

$$\text{spt }\psi_{(x_0,t_0),\sigma} \subset \left\{x \in \mathbb{R}^{n+1} : \frac{|x-x_0|^2+2n(t-t_0)}{\rho_0^2} \leq 1\right\} = B_{\rho_0\sqrt{1+2n}}(x_0) \subset U \sim \Sigma \qquad (12.1)$$

Then from (11.5)

$$\begin{aligned}\frac{d}{dt}\int_{M_t}\psi_{(x_0,t_0),\sigma}\rho_{(x_0,t_0)}d\mu_t &= -\int_{M_t}\psi_{(x_0,t_0),\sigma}\rho_{(x_0,t_0)}\left|\vec{H}-\frac{D^\perp\rho_{(x_0,t_0)}}{\rho_{(x_0,t_0)}}\right|^2 d\mu_t \\ &+\int_{M_t}\psi_{(x_0,t_0),\sigma}Q(\rho_{(x_0,t_0)})d\mu_t \\ &+\int_{M_t}\rho_{(x_0,t_0)}\left(\frac{d}{dt}-\triangle_{M_t}\right)\psi_{(x_0,t_0),\sigma}d\mu_t \\ &+\int_{\partial M_t} < \rho_{(x_0,t_0)}D\psi_{(x_0,t_0),\sigma} - \psi_{(x_0,t_0),\sigma}D\rho_{(x_0,t_0)}, \nu_\Sigma > d\sigma_t,\end{aligned}$$

By Proposition 11.3.7 $Q(\rho_{(x_0,t_0)}) = 0$, by Proposition 12.0.1, $\left(\frac{d}{dt}-\triangle_{M_t}\right)\psi_{(x_0,t_0),\sigma} \leq 0$ and by Equation (12.1) $D\psi_{(x_0,t_0),\sigma} = 0$ and $\psi_{(x_0,t_0),\sigma} = 0$ on ∂M_t for each $t \in (t_0 - \sigma^2, \sigma)$. It therefore follows that for each $t \in (t_0 - \sigma^2, t_0)$

$$\frac{d}{dt}\int_{M_t}\rho_{(x_0,t_0)}\psi_{(x_0,t_0),\sigma}d\mu_t \leq -\int_{M_t}\left|\vec{H}-\frac{D^\perp\rho_{(x_0,t_0)}}{\rho_{(x_0,t_0)}}\right|^2\rho_{(x_0,t_0)}\psi_{(x_0,t_0),\sigma}d\mu_t$$

as required. ◇

We now state simply that the interior behaves as in the boundaryless case. Each result, as with that above can be shown by sufficiently localising to remove the boundary from influence.

Corollary 12.0.1.
The regularity results presented in Ecker [7] for mean curvature flow without boundary hold on the interior of a mean curvature flow with Neumann free boundary conditions.

12.1 Notes

The boundaryless localisation function, defined in Definition 12.0.1 has been used by both Brakke [5] and Ecker [7]. The related result, Proposition 12.0.1 is also due to Ecker [7]. Proposition 12.0.2 is due to Ecker [7]. Our own variation to Proposition 12.0.2 to make it worth presenting here is only choosing the radius small enough to ensure spt $\psi_{(x_0,t_0),\sigma} \cap \Sigma = \emptyset$.