Chapter 11

Definitions and Background Results

In this chapter we define mean curvature flow with Neumann free boundary conditions (MCFwNfBC), and mention the necessary results already existing that will be needed to attack regularity theory. This includes the existence theory of Stahl, as well as his interior estimates. More often however, we will use the results of Buckland who developed the Monotonicity Formula for mean curvature flow with Neumann free boundary conditions analogous to Huisken's Monotonicity Formula.

11.1 Mean Curvature Flow with Neumann Free Boundary Conditions

We begin by giving the definition of mean curvature flow with Neumann free boundary conditions. In order to do this we first need to define the support surface.

Definition 11.1.1.
By saying that a smooth hypersurface $S \subset \mathbb{R}^{n+1}$ satisfies **the rolling ball condition with ball of radius** *r for some $r > 0$ we mean that for each $x \in S$ and $\rho \leq r$ there exists an $n+1$ dimensional ball B with radius ρ such that $\partial B \cap S = x$.*

With this definition we can define our support surface.

Definition 11.1.2. (Free boundary support surface)
Let G^c be a simply connected C^3 $(n+1)$-dimensional subset of $\mathbb{R}^{n+1}$. Let $\Sigma := \partial G^c$ satisfy the rolling ball condition for balls of maximal radius $1/\kappa_\Sigma$ and satisfy the condition on the second fundamental form, A_Σ of Σ

$$||A_\Sigma||^2 + ||\nabla A_\Sigma|| \leq \kappa_\Sigma^2 < \infty.$$

Σ is then said to be a **Neumann free boundary support surface***. Here, for a set G the notation G^c indicates the compliment of G.*

We will then let G be the domain inside of which a surface will be allowed to flow. The set G will always denote this domain in this section of the thesis. Similarly, Σ (when not understood to be signifying a summation) will always denote a Neumann free boundary support surface or simply, support surface. In general for any given result, the support surface will be an arbitrary but fixed Neumann free boundary support surface. This comment is especially important in Lemma 17.1.2 where the comment will be restated. We begin defining the flow with an initial surface.

Definition 11.1.3. (Initial Surface)
Let M^n denote a smooth orientable n-dimensional manifold with smooth, compact boundary ∂M^n and set

$$M_0 := F_0(M^n),$$

where F_0 is a smooth embedding satisfying

$$\begin{array}{rcll} \partial M_0 \equiv F_0(\partial M^n) & = & M_0 \cap \Sigma, \text{ and} & \\ < \nu_0, \nu_\Sigma \circ F_0 > (p) & = & 0 \quad \text{for all } \partial M^n, & \end{array} \tag{11.1}$$

for unit normal fields ν_0 to M_0 and ν_Σ to Σ.

With this initial set up we are now able to define how to let the initial surface flow along the support surface and thus define mean curvature flow with Neumann free boundary conditions.

Definition 11.1.4. (Mean Curvature Flow with Neumann Free Boundary Conditions)
Let Σ be a Neumann free boundary support surface. Let $I := [0,T)$ (for some $T \in (0,\infty)$) be an interval and let $F(\cdot,t) : M^n \to \mathbb{R}^{n+1}$ be a one-parameter family of smooth embeddings for all $t \in I$. The family of hypersurfaces $(M_t)_{t\in I}$, where $M_t = F_t(M^n)$, are said to be evolving by **mean curvature with Neumann free boundary conditions on** Σ *(or to be a* **mean curvature flow with Neumann free boundary conditions on** Σ*) if*

$$\begin{array}{rcll} \frac{\partial F}{\partial t}(p,t) & = & \vec{H}(p,t) & \text{for all } (p,t) \in M^n \times I, \\ F(\cdot,0) & = & F_0, & \\ F(p,t) & \subset & \Sigma & \text{for all } (p,t) \in \partial M^n \times I, \text{ and} \\ < \nu, \nu_\Sigma \circ F > (p,t) & = & 0 & \text{for all } (p,t) \in M^n \times I. \end{array} \tag{11.2}$$

Here $\vec{H}(p,t) = -H(p,t)\nu(p,t)$ denotes the mean curvature vector of the immersions M_t at $F(p,t)$, for a choice of unit normal ν for M_t.

Remark: We will, in general, suppress the notation referring to the embedding map, using rather only the position vector $x \in \mathbb{R}^{n+1}$ instead of $F(p,t)$. With this understanding, we may re-express the above equations geverning mean curvature flow with Neumann free boundary conditions as

$$\begin{array}{rcll} \frac{\partial x}{\partial t} & = & \vec{H}(x) & \text{for all } x \in M_t, \\ \partial M_t & \subset & \Sigma & \text{and} \\ < \nu, \nu_\Sigma > (x) & = & 0 & \text{for all } x \in \partial M_t. \end{array} \tag{11.3}$$

Moreover, we take ν_Σ to be the inner unit normal to G^c, that is the outer unit normal to G on Σ so that $\nu_\Sigma(x)$ corresponds to the **outer** unit normal to ∂M_t for all $x \in \partial M_t$ for each $t \in I$.

We have thus defined mean curvature flow with Neumann free boundary conditions. The proof that it is a reasonable definition in the sense that it is well defined and gives interesting solutions comes from the work of Stahl [29] that we consider in the next section.

11.2 Stahl's contribution and related results

Stahl's work is important here as it provides the proof that we are studying anything at all, and provides some technical PDE results that prove essential in later work. We first state Stahl's theorem stating that the mean curvature flow with Neumann free boundary conditions exists.

Theorem 11.2.1. (Stahl's existence Theorem)
For any Neumann free boundary support surface, Σ, *and initial surface* M_0 *there exists a unique solution to (*11.2*), the mean curvature flow with Neumann free boundary conditions, on a maximal time interval* $[0, T)$ *which is smooth for* $t > 0$ *and in the class* $C^{2+\alpha, 1+\alpha/2}$ *for* $t \geq 0$ *and any* $\alpha \in (0, 1)$. *Moreover, if* $T < \infty$ *then*

$$\sup\{|A|^2(x, t) : x \in M^n\} \to \infty \text{ as } t \to T. \tag{11.4}$$

Since the flow is smooth (that is M_t is smooth $t \in [0, T)$) it follows that there are no singularities in this set and thus we are interested in M_T.(M_T is the 'limit surface' which we define formally in Definition 13.1.2 which can be written as the set of all points $x = \lim_{j\to\infty} x_{t_j}$ where $x_{t_j} \in M_{t_j}$ for a sequence $t_j \nearrow T$, that is all points 'reached by the surface'.) For this to produce meaningful results we need $T < \infty$ which we will now assume to be the case for the remainder of this Thesis.

The direct PDE approach of Stahl in [29] and [28] lead to important technical results. In particular the interior derivative estimates of the second fundamental form. Which in our notation and set up, can be stated, as can be seen in Buckland [6], as follows.

Theorem 11.2.2.
Let $(M_t)_{t\in[0,T)}$ *be a mean curvature flow with Neumann free boundary conditions. Suppose that for some* $x_0 \in \mathbb{R}^{n+1}$, $R > 0$ $0 < t_1 \leq T$ *there exists a* $C_0 \in \mathbb{R}$ *such that*

$$|A(x, t)|^2 \leq \frac{C_0}{R^2}$$

for all $x \in B_R(x_0)$ *and* $t \in (t_1 - R^2, t_1)$ *then there exists, for each* $k \in \mathbb{N}$, *a constant* $C_k = C_k(n, k, C_0)$ *such that for all* $x \in M_t \cap B_{R/2(1+C(C_0))}(x_0)$ *and* $t \in (t_1 - R^2/4, t_1)$

$$|\nabla^k A(x, t)|^2 \leq \frac{C_k}{R^{2(k+1)}}.$$

The main use of the interior estimates is to be able to apply the Arzela-Ascoli Theorem to sequences of flows (in particular parabolic blowups of flows. What blowups are will be discussed later) in order to obtain a smooth limit flow. The Theorem of Arzela-Ascoli is actually a theorem in Analysis applying to functions. However, since mean curvature flow with Neumann free boundary conditions can be formulated as a system of equations for which the Arzela-Ascoli Theorem is applicable (See Stahl [29]) we can also apply it here. The form of the Arzela-Ascoli Theorem that we will be using will be stated following a definition of a type of convergence that is used in the theorem and in general will be important to us.

Definition 11.2.1.
Let $\{(M_t^j)_{t\in[t_j,0)}\}_{j\in\mathbb{N}}$ *be a sequence of mean curvature flows with Neumann free boundary conditions with* $t_j \searrow -\infty$ *as* $j \to \infty$. *Let* $(M_t)_{t\leq 0}$ *be a mean curvature flow with Neumann free boundary conditions. We then say that* (M_t^j) **converges smoothly to** (M_t) *if the respective families of embedding maps,* $\{F_t^j\}$, *representing the mean curvature flows with Neumann free boundary conditions converge to the limiting family of embedding maps* $\{F_t\}$ *in* C^∞.

We then say that (M_t^j) **converges weakly to** (M_t) *if for each* $t < 0$ *and* $\phi \in C_C^0(\mathbb{R}^{n+1}, \mathbb{R})$

$$\lim_{j\to\infty} \int_{M_t^j} \phi d\mathcal{H}^n = \int_{M_t} \phi d\mathcal{H}^n.$$

Remark: Note that smooth convergence implies weak convergence, a fact that we will regularly use in the remainder of the thesis.

We can now state the Arzela-Ascoli Theorem.

Theorem 11.2.3.
Let $\{(M_t)_{t\in[-t_j,0)}\}_{j\in\mathbb{N}}$ be a sequence of mean curvature flows with Neumann free boundary conditions supported respectively on the support surfaces Σ_j. Suppose also that there exists, for each $k \in \mathbb{N}$, constants C_k such that

$$|\nabla^k A_j|^2 \leq C_k$$

for all $t \in [t_j, 0)$ and $x \in U_j \subset \mathbb{R}^{n+1}$ (where U_j is open). Suppose also that $U_j \times [t_j, 0) \to \mathbb{R}^{n+1} \times (-\infty, 0)$ as sets. Then there exists a smooth limit flow $(M'_s)_{s\leq 0}$ to which the flows $(M_t^j)_{t\in[t_j,0)}$ converge smoothly.

11.3 Buckland's contribution and related results

After the ground setting work of Stahl, the next step of reseach on mean curvature flow with Neumann free boundary conditions in the classical sense was conducted by Buckland in his doctoral thesis [6]. The main contribution of his that we are interested in is the general Monotonicity Formula. We will in fact need a Localised Monotonicity Formula that we develop in the following chapters, but it is based on Bucklands general formula. Further, the properties of the distance and reflection functions of Grüter and Jost, and the properties of Buckland's test function η_{x_0,t_0} and the modified backward heat kernel developed by Buckland will become important to us.

The central idea of Buckland, and thus, the central idea of the Localised Monotonicity Formula presented in this Thesis, is based on the following expansion formula, proven in Buckland [6]

Proposition 11.3.1. (Buckland's Expansion Formula)
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a solution of (11.2) and U an open subset of $\mathbb{R}^{n+1}$ containing $\mathcal{M}$. For any functions $f, g : U \times [0,T) \to \mathbb{R}$, where $f \in C_0^2(U)$, $\frac{\partial f}{\partial t} \in C_0^0(0)$, $g \in C^2(U)$ and $\frac{\partial g}{\partial t} \in C^0(U)$, we have the following general expansion formula:

$$\begin{aligned} \tfrac{d}{dt}\textstyle\int_{M_t} fg d\mu_t &= -\textstyle\int_{M_t} fg\left|\vec{H} - \frac{D^\perp g}{g}\right|^2 d\mu_t + \textstyle\int_{M_t} fQ(g)d\mu_t + \textstyle\int_{M_t} g\left(\frac{d}{dt} - \triangle_{M_t}\right) f d\mu_t \\ &+ \textstyle\int_{\partial M_t}(g < Df, \nu_\Sigma > -f < Dg, \nu_\Sigma >)d\sigma_t, \end{aligned} \tag{11.5}$$

where here and henceforth the operator Q is defined by

$$Q(g) := \frac{\partial g}{\partial t} + div_{M_t} Dg + \frac{|D^\perp g|^2}{g}. \tag{11.6}$$

Remark: In general the measure μ_t is the Hausdorff n-measure restricted to the related M_t times the multiplicity of the surface. That is

$$\mu_t(A) = \int_{M_t\cap A} \theta d\mathcal{H}^n$$

where θ is the multiplicity function. Similarly σ_t will denote the Hausdorff $n-1$-measure restricted to the boundary of the associated surface ∂M_t times the multiplicity of the surface. As we will later impose for the classical case that the multiplicity be identified with 1 μ_t and σ_t will denote exactly

Hausdorff n-measure restricted to M_t and Hausdorff $n-1$-measure restricted to ∂M_t. The notation with $\mathcal{H}^n$, or $\mathcal{H}^{n-1}$ will be used interchangeably with μ_t and σ_t depending on notational convenience.

The general idea for drawing monotonicity from Proposition 11.3.1 is to choose f and g such that the last two terms of (11.5) disappear and that the second term is controlled by f and g themselves.

We will introduce Buckland's choices of f and g, that also become important here, shortly. In order to do so, however, we need to introduce Grüter and Jost's reflection function and some of its properties. So that we may introduce the reflection function we first need to discuss the distance function, where we are taking the distance from an arbitrary point to the support surface Σ.

Definition 11.3.1.
Let $\Sigma = \partial G$ be a Neumann free boundary support surface. For any point $x \in \mathbb{R}^{n+1}$, we denote the minimum distance of x to Σ by $d_\Sigma(x) := dist(x, \Sigma)$, whenever it is well defined. We then define the **signed distance function** *by*

$$d(x) := \begin{cases} -d_\Sigma(x) & \text{if } x \in G^c \\ d_\Sigma(x) & \text{if } x \in G \end{cases} \tag{11.7}$$

We note the following standard result concerning the distance function.

Proposition 11.3.2.
Let $G \subset \mathbb{R}^{n+1}$ with $\Sigma = \partial G \in C^k$ for some $k \geq 2$. Then there exists an $\varepsilon > 0$ such that $d \in C^k(\Sigma_\varepsilon)$, where Σ_ε is the ε-tubular neighbourhood of Σ given by

$$\Sigma_\varepsilon := \{x \in \mathbb{R}^{n+1} : d_\Sigma(x) < \varepsilon\}.$$

On this neighbourhood, the nearest point projection $\mathbb{R}^{n+1} \to \Sigma$ also exists. This projection will be denoted by π_Σ.

Remark: Due to the rolling ball condition with maximal radius κ_Σ^{-1}, it can be shown (see [6]) that for Σ a Neumann free boundary conditions support surface $\Sigma_\varepsilon \supset \Sigma_{1/\kappa_\Sigma}$ so that the distance function and nearest point projection functions are also well defined on Σ_{1/κ_Σ}.

We are mainly interested in the distance function only as it is necessary for the definition of the reflection function of Grüter and Jost. However it is also important to note the following.

Proposition 11.3.3.
Let Σ be a Neumann free boundary support surface and let $x_0 \in \Sigma_\varepsilon$. Then

1. $Dd(x_0) = \nu_\Sigma^+(\pi_\Sigma(x_0))$,
2. $||D^2 d|| \leq \kappa_\Sigma(1 - d\kappa_\Sigma)^{-1}$, *and*
3. $||D^3 d|| \leq \kappa_\Sigma^2(1 - d\kappa_\Sigma)^{-3}$,

where $||\cdot||$ refers to the maximum norm and ν_Σ^+ is the unit normal to Σ that points in the direction of increasing signed distance. We will henceforth set ν_Σ to always denote ν_Σ^+. Further we will take $\nu_\Sigma(x)$ to mean $\nu_\Sigma^+(\pi_\Sigma(x))$. Since the vector function ν_Σ will not be used in any other way this simplifies notation without creating confusion.

We now introduce the reflection function of Grüter and Jost that is so important to our analysis. Just as importantly is the 'radius' function related to the reflection function that we also now define.

Definition 11.3.2.
For any $x \in \Sigma_{1/\kappa_\Sigma}$ *we define*

$$\tilde{x} := x - 2(< x, \nu_\Sigma - d_\Sigma)\nu_{Sigma}$$

to be the **tilde-reflection of** x **across** Σ *and set*

$$r := |x|^2 + |\tilde{x}|^2.$$

Furthermore, for any $x_0 \in \mathbb{R}^{n+1}$ *we define the translated* r *function*

$$r_{x_0} := |x - x_0|^2 + |\widetilde{x - x_0}|^2,$$

where the reflection $\widetilde{x - x_0}$ *is then actually the reflection of* $x - x_0$ *around* $\Sigma - x_0$*.*

Remarks:
1) Using the fact that $Dd = \nu_\Sigma$ we can also write

$$\tilde{x} = x - 2(< x, Dd > -d)Dd. \tag{11.8}$$

2) The motivation for the definition of this reflection function will not become naturally clear in this thesis. We note however, that the original very important property of this reflection is that on Σ (when $0 \in \Sigma$) $< x + \tilde{x}, \nu_\Sigma >= 0$, a fact that is not shared by the usual reflection function.

3) We note that in the special case of a planar support surface $\Sigma \ni 0$ $\tilde{x} = x$ and thus $r = 2|x|^2$. This reduction to a simple case, allows the boundaryless case to be retrieved from the boundary case by reflection around the boundary.

Those properties of the reflection function proven by Buckland that we need are as follows.

Proposition 11.3.4.
Let Σ *be a Neumann free boundary support surface. Then for any* $x \in \Sigma_{1/\kappa_\Sigma}$ *we have the following estimates:*

1. $\lim_{\kappa_\Sigma \to 0} \tilde{x} = x$
2. $|\tilde{x}| = |x - 2dDd|$
3. $|Dr|^2 \leq 8r + \frac{32|x|^3\kappa_\Sigma}{1-d\kappa_\Sigma} + \frac{16|x|^4\kappa_\Sigma^2}{(1-d\kappa_\Sigma)^2}$,
4. $|div_{M_t} Dr - 4n| \leq \frac{20n\kappa_\Sigma|x|}{1-d\kappa_\Sigma} + \frac{4n\kappa_\Sigma^2|x|^2}{(1-d\kappa_\Sigma)^3}$, *and*
5. $< Dr, \nu_\Sigma >= 0$ *for all* $x \in \Sigma$.

To this we add the following property that will become important in our area bounds.

Proposition 11.3.5.
Let Σ *be a Neumann free boundary support surface and let* $x_0 \in \Sigma$*. Then for all* $x \in \mathbb{R}^{n+1}$

$$\frac{|x - x_0|}{3} \leq |\widetilde{x - x_0}| \leq 3|x - x_0|.$$

It then also follows that

$$\frac{10|x - x_0|^2}{9} \leq r_{x_0} \leq 10|x - x_0|^2.$$

Proof:
From (2) in Proposition 11.3.4 and using that $|Dd| = 1$ we have $|\widetilde{x - x_0}| = |x - x_0 - 2dDd| \leq |x-x_0|+2|d|$. However, since $x_0 \in \Sigma$, $d(x) \leq |x-x_0|$, thus $|\widetilde{x - x_0}| \leq 3|x-x_0|$. Since $x-x_0 = \widetilde{\widetilde{x - x_0}}$ we have reciprocally that $|x-x_0| \leq 3|\widetilde{x - x_0}|$. The result follows from the definition of r_{x_0} and these two facts. $\diamondsuit$

We now introduce the two functions involved in the monotonicity formulas introduced by Buckland. The first is a time dependent localisation function on the support of which certain important properties hold.

Definition 11.3.3.
Let Σ be a Neumann free boundary support surface with $\kappa_\Sigma \geq 0$. Then for any $x_0 \in \Sigma$, $t_0 \in \mathbb{R}$ and $\delta \in (0, 2/5]$ we define **Buckland's localisation function** $\eta_{(x_0,t_0)} : \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}$ *by*

$$\eta_{(x_0,t_0)}(x,t) = \left(1 - \left(\frac{2\kappa_\Sigma}{(\kappa_\Sigma^2 \tau_{t_0})^\delta}\right)^2 (r_{x_0} - 40n\tau_{t_0})\right)_+^4, \tag{11.9}$$

where $\tau_{t_0} := t_0 - t$. Should (x_0, t_0) be understood, we write simply η. We further define the following number which proves to be critical:

$$\tau_0 := \frac{\left(\frac{3}{160n}\right)^{2/\delta}}{\kappa_\Sigma^2}.$$

The fundamental properties of Buckland's localisation function we present below are mostly due to Buckland. However, as we will need to have differing time centers and an additional lower bound, we present the properties we will need below.

Proposition 11.3.6.
Let Σ be a Neumann free boundary conditions support surface. Let $x_0 \in \Sigma$, $\delta \in (0, 2/5]$ and $t_1 \in \mathbb{R}$. Then for each $t \in [t_1 - \tau_0, t_1)$ we have

$$\eta_{x_0,t_1} \leq 256,$$

$$\textit{spt}\ \eta_{x_0,t_1} \subset \{x \in \mathbb{R}^{n+1} : |x - x_0|\kappa_\Sigma \leq (\kappa_\Sigma^2 \tau)^\delta\} \cap \Sigma_{1/\kappa_\Sigma},$$

$$\frac{1}{1 - d\kappa_\Sigma} \leq 2 \ \textit{on spt}\ \eta_{x_0,t_1},$$

$$\left(\frac{d}{dt} - \triangle_{M_t}\right)\eta_{x_0,t_1} \leq 0,$$

$$\textit{spt}\ \eta_{x_0,t_1} \to \mathbb{R}^{n+1} \ \textit{as}\ \kappa_\Sigma \to 0.$$

Further, for each $t \in [0, T)$ we have

$$\eta \to 1 \ \textit{as}\ \kappa_\Sigma \to 0.$$

Additionally,

$$\eta_{x_0,t_1} \geq \frac{1}{256} \ \textit{on}\ B_{(\tau_0/2)^{1/2}}(x_0) \times (t_1 - \tau_0, t_1 - \tau_0/2].$$

Proof:
All of the properties except the last,

$$\eta_{x_0,t_1} \geq \frac{1}{256} \text{ on } B_{(\tau_0/2)^{1/2}}(x_0) \times (t_1 - \tau_0, t_1 - \tau_0/2],$$

follow from Bucklands Proposition 4.1.1 in [6]. It only remains to show this last property. For this we note, from Proposition 11.3.5, that $|\widetilde{x - x_0}| \leq 3|x - x_0|$. We then calculate directly as follows:

$$\begin{aligned}\eta_{x_0,t_1}(x,t) &= \left(1 - \left(\frac{2\kappa_\Sigma}{(\kappa_\Sigma^2\tau)^\delta}\right)^2 (r_{x_0} - 40n\tau)\right)_+^4 \\ &\geq \left(1 - \frac{2^{2+2\delta}10|x - x_0|^2\kappa_\Sigma^2}{(3/160n)^4}\right)_+^4 \\ &\geq (1 - 35(3/160n)^{(2/\delta)-4})_+^4 \\ &\geq \frac{1}{256}.\end{aligned}$$

$\diamondsuit$

We now look at the backward heat kernel. We define first the usual backward heat kernel as used in Huisken's original monotonicity formula.

Definition 11.3.4.
We define the **usual backward heat kernel** $\rho : \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}$ *by*

$$\rho(x,t) = \frac{1}{(-4\pi t)^{n/2}} e^{\frac{|x|^2}{4t}}.$$

Also the translate around $(x_0, t_0) \in \mathbb{R}^{n+1} \times \mathbb{R}$ *of* ρ *is defined by*

$$\rho_{(x_0,t_0)}(x,t) = \frac{1}{(4\pi(t_0 - t))^{n/2}} e^{-\frac{|x - x_0|^2}{4(t_0 - t)}}, \quad \textit{for } t < t_0.$$

An important property of the usual backward heat kernel is that $Q(\rho) = 0$. We state this formally below. The proof is simply direct calculation.

Proposition 11.3.7.
Let (M_t) *be a mean curvature flow with Neumann free bounday conditions and* $(x_0, t_0) \in \mathbb{R}^{n+1} \times \mathbb{R}$. *Then for each* $t \in [0, T)$ *and* $x \in M_t \sim \partial M_t$

$$Q(\rho_{(x_0,t_0)}) = 0.$$

The backward heat kernel needs to be altered to be used in the boundary case. The main two properties we need are that Q of the backward heat kernel be controllable, and that its derivative is perpendicular to ν_Σ so as to get rid of the boundary term. To do this we define the following modified backward heat kernel.

Definition 11.3.5.
Let Σ *be a Neumann free boundary support surface and* $\delta > 0$. *We then define the* **modified backward heat kernel** $\rho_{\kappa_\Sigma} : \mathbb{R}^{n+1} \times (-\infty, 0] \to \mathbb{R}$ *by*

$$\rho_{\kappa_\Sigma}(x,t) := \frac{1}{(-4\pi t)^{n/2}} e^{\frac{r}{8(16(-t\kappa_\Sigma^2)^\delta + 1)t}}.$$

Furthermore for any $x_0 \in \mathbb{R}^{n+1}$ *and* $t_0 \in \mathbb{R}$ *we define the translates*

$$\rho_{\kappa_\Sigma,x_0,t_0}(x,t) := \frac{1}{(4\pi\tau_{t_0})^{n/2}} e^{-\frac{r}{8(16(\kappa_\Sigma^2\tau_{t_0})^\delta + 1)\tau_{t_0}}},$$

where $\tau_{t_0} = t_0 - t$. *Whenever the* x_0 *and* t_0 *are understood we will write simply* ρ_{κ_Σ} *for* $\rho_{\kappa_\Sigma,x_0,t_0}$.

With this modified backward heat kernel we do not have $Q(\rho_{\kappa_\Sigma}) = 0$ in analogy to the $Q(\rho) = 0$ result for the usual backward heat kernel. However we continue to have what proves to be sufficient control, as embodied in the following theorem. The Theorem is essentially due to Buckland, we show here, however, that the same theorem holds for a larger range of times, which becomes important to the regularity results, in particular through Theorem 14.2.1 and the Clearing Out Lemma (Lemma 16.2.1) later on.

Theorem 11.3.1.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a mean curvature flow with Neumann free boundary conditions supported on the support surface Σ. Let $x_0 \in \Sigma$, $\delta \in (0, 2/5]$ and $t_1, t_2 \in \mathbb{R}$. Then for all $t \in [t_1 - \tau_0)$, over the support of $\eta_{x_0,t_2}(x,t)$, we have

$$Q(\rho_{\kappa_\Sigma,x_0,t_1}) \le C(n)\rho_{\kappa_\Sigma,x_0,t_1}\kappa_\Sigma^{2\delta}\tau^{\delta-1}$$

where $\tau = t_1 - t$.

Proof:
We proceed as in [6]. Without loss of generality we let $x_0 = 0$. As in [6] we then calculate for the broader class of perturbed backward heat kernels $\rho := \rho_{a,0,t_1}$ ($a = a(\tau)$)-where in general for $x_0 \in R^{n+1}$ and $t_0 \in \mathbb{R}$

$$\rho_{a,x_0,t_0}(x,t) := \frac{1}{(4\pi(t_0-t)^{n/2}} e^{-\frac{r_{x_0}}{8(a+1)(t_0-t)}}$$

$$Q(\rho) \le \rho\left(\frac{n}{2\tau} - \frac{a'r}{8(a+1)^2\tau} - \frac{r}{8(a+1)\tau^2} - \frac{div_{M_t}Dr}{8(a+1)\tau} + \frac{|Dr|^2}{(8(a+1)\tau)^2}\right)$$

Using Proposition 11.3.4 and the fact that $(1 - d\kappa_\Sigma)^{-1} < 2$ on spt η_{x_0,t_2} it follows that

$$Q(\rho) \le \rho\left(\frac{an}{2(a+1)\tau} - \frac{a'r}{8(a+1)^2\tau} + \frac{10n\kappa_\Sigma|x| + 8n\kappa_\Sigma^2|x|^2)}{2(a+1)\tau}\right) + \rho\left(\frac{64|x|^3\kappa_\Sigma + 64|x|^4\kappa_\Sigma^2 - 8ar}{(8(a+1)\tau)^2}\right)$$

Now choosing, $a(\tau) = c(\kappa_\Sigma^2\tau)^\delta$, where $c > 0$ is to be chosen later, and noting $a'(\tau) \ge 0$, that we are in spt η_{x_0,t_2} and $(\kappa_\Sigma^2\tau)^\delta < 1$ for $\tau \le \tau_0$, we have

$$Q(\rho) \le \frac{\rho(\kappa_\Sigma^2\tau)^\delta}{(a+1)\tau}\left(\frac{cn}{2} + 9n + \frac{|x|^2(16-c)}{8(a+1)\tau}\right).$$

Hence, choosing $c = 16$ gives $\rho_{a,0,t_1} = \rho_{\kappa_\Sigma,0,t_1}$ and $Q(\rho) \le 17n\rho\kappa_\Sigma^{2\delta}\tau^{\delta-1}$. ◇

One additional important property of Bucklands localisation function and the modified backward heat kernel has not yet been mentioned. It is this property that allows for the boundary terms in (11.5) to be nullified.

Proposition 11.3.8.
Let Σ be a Neumann free boundary support surface, $x_0 \in \Sigma$ and $t_0 \in \mathbb{R}$. Then for all $x \in \Sigma$

1. $< D\eta_{x_0,t_0}, \nu_\Sigma >= 0$, *and*

2. $< D\rho_{\kappa_\Sigma,x_0,t_0}, \nu_\Sigma >= 0$.

Before continuing to other background results we state Bucklands Monotonicity Formula, which, although not directly used in this thesis, forms the basis of our local monotonicity formulas proved and used later in the Thesis.

Theorem 11.3.2.
Let M_t be a family of hypersurfaces evolving by mean curvature with Neumann free boundary conditions supported on the support surface Σ for all $t \in [0,T)$ and $\delta \in (1/3, 2/5]$. Then for all $t \in [T-\tau_0, T)$ and any $x_0 \in \Sigma$ we have

$$\frac{d}{dt}\left(e^{C\kappa_\Sigma^{2\delta}\tau_T^\delta}\int_{M_t}\eta\rho_{\kappa_\Sigma}d\mu_t\right) \leq -e^{C\kappa_\Sigma^{2\delta}\tau_T^\delta}\int_{M_t}\left|\vec{H} - \frac{D^\perp\rho_{\kappa_\Sigma}}{\rho_{\kappa_\Sigma}}\right|^2 \eta\rho_{\kappa_\Sigma}d\mu_t,$$

where η and ρ_{κ_Σ} are translated around (x_0, T) and C is a constant depending only on n.

11.4 Parabolic Rescaling

The observance of the behaviour of the flows under parabolic rescaling is also found to be very useful in proving our results. It allows us to work with the asymptotic behaviour of the flows without losing all of the mass of the surface.

Definition 11.4.1.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a solution of mean curvature flow with Neumann free boundary conditions on the Neumann free boundaary support surface Σ. Let $x_0 \in \mathbb{R}^{n+1}$ and $t_0 \in (0,T]$. Then for $(x,t) \in M_t \cup \Sigma \times [0,t_0)$ and any $\lambda > 0$ we define the **parabolic rescaling of $\mathcal{M}$ by a factor of λ**, *or simply the parabolic rescale (or blow up), to be the change of variables $(x,t) \mapsto (y,s)$ given by*

$$y = \lambda^{-1}(x - x_0) \quad \textit{and} \quad s = \lambda^{-2}(t - t_0).$$

This definition implies the following equivalences

$$x \in M_t \Leftrightarrow y \in \lambda^{-1}(M_{\lambda^2 s + t_0} - x_0) \equiv M_s^{(x_0,t_0),\lambda}$$

and

$$x \in \Sigma \Leftrightarrow y \in \lambda^{-1}(\Sigma - x_0) \equiv \Sigma_\lambda^{x_0};$$

$M_s^{(x_0,t_0),\lambda}$ and $\Sigma_\lambda^{x_0}$ will be written simply as M_s^λ and Σ_λ respectively when the x_0 and t_0 are understood.

In the parabolically rescaled setting we have

Proposition 11.4.1.
Under parabolic rescaling the hypersurface $\Sigma_\lambda^{x_0}$ is a Neumann free boundary support surface and the resclaed flow $M_s^{(x_0,t_0),\lambda}$ is a mean curvature flow with Neumann free boundary conditions supported on the Neumann free boundary support surface $\Sigma_\lambda^{x_0}$. Further the induced measures μ_t and μ_s of the surfaces M_t and $M_s^{(x_0,t_0),\lambda}$ are related by

$$d\mu_t(x) = \lambda^n d\mu_s(y).$$

Proposition 11.4.2.
Under parabolic rescaling we have

$$\eta_{x_0,t_0} \mapsto \hat{\eta} \quad \textit{and} \quad \rho_{\kappa_\Sigma,x_0,t_0} \mapsto \lambda^{-n}\hat{\rho}_{\lambda\kappa_\Sigma}$$

where

$$\hat{\eta}(y,s) = \left(1 - \left(\frac{2(\lambda\kappa_\Sigma)^{1-2\delta}}{(-s)^\delta}\right)^2 (r(y) + 40ns)\right)_+^4$$

and

$$\hat{\rho}_{\lambda\kappa_\Sigma}(y,s) = \frac{1}{(-4\pi s)^{n/2}} e^{\frac{r(y)}{8(1+16(-(\lambda\kappa_\Sigma)^2 s)^\delta)s}}.$$

Here $r(y) = |y|^2 + |\tilde{y}|^2$ *as in Definition* 11.3.2 *where the reflection* $\tilde{y}$ *is taken with respect to the rescaled support surface* $\Sigma_\lambda^{x_0}$.

Proposition 11.4.3.
Let Σ *be a Neumann free boundary support surface and* $\delta \in (1/3, 2/5]$. *Then for each* $s < 0$ *we have*

1. $\tilde{y} \to y$,
2. $\hat{\rho}_{\lambda\kappa_\Sigma}(y,s) \to \rho(y,s)$, *and*
3. $\hat{\eta} \to 1$

uniformly on compact sets as $\lambda \to 0$, *where* $\tilde{y}$ *is the tilde reflection of* y *around* Σ_λ. *Furthermore, as* $\lambda \to 0$ *we have spt* $\hat{\eta} \to \mathbb{R}^{n+1}$.

All previous results also clearly allow for rescaled versions, since parabolic rescaling is only really a change of variables. In particular we can write the rescaled version of Buckland's monotonicity formula as follows.

Theorem 11.4.1.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ *be a mean curvature flow with Neumann free boundary conditions supported on the support surface* Σ. *Then for any* $\lambda > 0$, $t_0 \in (0,T]$ *and* $x_0 \in \Sigma$ *we have for all* $s \in [-\tau_0\lambda^{-1}, 0)$

$$\frac{d}{ds}\left(e^{C(-(\lambda\kappa_\Sigma)^2 s)^\delta} \int_{M_s^{(x_0,t_0),\lambda}} \hat{\eta}\hat{\rho}_{\lambda\kappa_\Sigma} d\mu_s\right) \leq -e^{C(-(\lambda\kappa_\Sigma)^2 s)^\delta} \int_{M_s^{(x_0,t_0),\lambda}} \left|\vec{H} - \frac{D^\perp \hat{\rho}_{\lambda\kappa_\Sigma}}{\hat{\rho}_{\lambda\kappa_\Sigma}}\right|^2 \hat{\eta}\hat{\rho}_{\lambda\kappa_\Sigma} d\mu_s,$$

where $\vec{H}$ *is the mean curvature of the surfaces* $M_s^{(x_0,t_0),\lambda}$ *and* $C = C(n)$.

The main interest in parabolic rescaling is usually the limiting surface or flow (when it exists) arising from letting λ go to zero. In particular, our last preliminary result gives the existence of such a limit surface, ensuring also that it is non-empty provided that the Type I assumption is satisfied.

Definition 11.4.2.
A mean curvature flow with Neumann free boundary conditions, $(M_t)_{t\in[0,T)}$ *is said to satisfy the* **type I assumption** *at time* $t_0 \in (0,T]$ *if*

$$|A(\cdot,t)| \leq \frac{C_H}{(t_0 - t)^{1/2}} \tag{11.10}$$

on the surfaces M_t *for all* $t \in [0,T)$ *and some constant* $C_H > 0$. *For the remainder of the thesis* C_H *will always be understood to be a type I constant.*

The type I assumption in this case is necessary because it ensures that the limit flow will not be empty. In particular it assures the existence of the below defined limit point.

Definition 11.4.3.
We define the limit point function $\Upsilon : M^n \to \mathbb{R}^{n+1}$ *by*

$$\Upsilon(p) := \lim_{t\to T} F(p,t),$$

for any point $p \in M^n$. *The existence of this limit follows directly from the type I assumption and the fact that*

$$\frac{dF}{dt} = \vec{H}.$$

We then say that $\Upsilon(p)$ *is the limit point of* p.

With these definitions we can now state a limit existence theorem.

Theorem 11.4.2.
Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth properly embedded mean curvature flow with Neumann free boundary conditions supported on the support surface Σ satisfying the type I assumption at time $t_0 \in (0,T]$ and let $x_0 = \Upsilon(p)$ for some $p \in \mathcal{M}^n$. Then for every sequence $\lambda_j \searrow 0$, corresponding to $t \nearrow T$, there is a subsequence $\{\lambda_{j_k}\}$ such that the rescaled surfaces $M_s^{\lambda_{j_k}}$ converge smoothly to a non-empty embedded limit surface, $\mathcal{M}' = (M_s')_{s<0}$ such that

1. *(M_s') evolves by mean curvature flow for $s < 0$;*
2. *If $p \notin \partial M^n$ then M_s' has no boundary;*
3. *If $p \in \partial M^n$ then M_s' has boundary $\partial M_s' \subset \Sigma_{x_0}'$, where Σ_{x_0}' is a hyperplane through the origin $y = 0$ and $< \hat{\nu}, \widehat{\nu_{\Sigma_{x_0}'}} >= 0$ on $\partial M_s'$.*

11.5 Notes

The problem of mean curvature flow with Neumann free boundary conditions, as has been made apparent is not new, though Definitions 11.1.2-11.1.4 follow Buckland's presentation [6]. Theorem 11.2.1 is due to Stahl in his ground setting works [29] and [28]. Theorem 11.2.2 is due to Stahl [29] formulated (albeit already in rescaled form) in the presented way by Buckland [6]. Smooth convergence in the sense of Definition 11.2.1 is a standard differential geometric concept, just as weak convergence is a standard measure theoretic concept. (See for example Ecker [7] or Evans and Gariepy [10].) The Arzela-Ascoli Theorem, Theorem 11.2.3, is actually a functional analysis result that can be applied here due to the formulability of MCFwNfBC in terms of functions. Discussion of the form used here can be found in, for example, White [32]. Proposition 11.3.1 follows from standard differential geometric identities, a proof can be found in Buckland [6]. Although the reflection function, Definition 11.3.2, is due to Grüter and Jost [13], we follow Buckland's [6] treatment for Definitions 11.3.1 and 11.3.2 as well as Propositions 11.3.2, 11.3.3 and 11.3.4. Proposition 11.3.5 is our own observation. Buckland's localisation function is due to Buckland [6] as are all points of Proposition 11.3.6 excepting the last which is our own observation. The usual backward heat kernel was first used effectively in mean curvature flows by Huisken [14], though the modified backward heat kernel, Definition 11.3.5, is due to Buckland [6]. Theorem 11.3.1 comes in principle from Buckland, the separation of time centres is, however, our own variation. Proposition 11.3.8 and Theorem 11.3.2 are due to Buckland [6]. The principle of parabolic rescaling is standard in the treatment of mean curvature flow, see for example Huisken [14], Ecker [7] and Buckland [6]. The particular implications in the MCFwNfBC case as presented in Propositions 11.4.1, 11.4.2 and 11.4.3 as well as Theorem 11.4.1 are, however, due to Buckland [6]. For previous applications of the type I assumption one could look to Huisken [14], Stone [30], and Buckland [6] amongst others. The limit flow existence theorem, Theorem 11.4.2, is due to Buckland [6].